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MARCINKIEWICZ ESTIMATES FOR DEGENERATE PARABOLIC EQUATIONS WITH MEASURE DATA

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ABSTRACT. We consider solutions to degenerate parabolic equations with measurable coefficients, having on the right-hand side a measure satisfying a suitable density condition; we prove integrability results for the gradient in the Marcinkiewicz scale.

1. INTRODUCTION

The aim of this paper is to give a natural integrability result for solutions of degenerate parabolic equations of the type

$$u_t - \operatorname{div} a(x,t,Du) = \mu \qquad \text{in } \Omega_T := \Omega \times (-T,0), \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, and the vector field $a(\cdot)$ satisfies only minimal measurability and monotonicity assumptions of Ladyzhenskaya-Uralt'seva type, see the forthcoming Section 2 for more details about the precise assumptions made and the notation adopted in this paper. Here, as in the forthcoming pages, μ is a signed Borel measure with finite total mass which in general does not belong to the dual of the energy space naturally associated to the operator on the left-hand side. The most prominent model we have in mind for (1.1) is the degenerate parabolic p-Laplace equation with measurable coefficients, i.e.

$$a(x,t,Du) = A(x,t)\big(s^2 + |Du|^2\big)^{\frac{p-2}{2}} Du, \qquad p \geq 2, \tag{1.2}$$

where $A(\cdot)$ is a measurable, bounded and uniformly elliptic matrix and $s \geq 0$ a parameter used to discern the degenerate case ($s = 0$) from the non-degenerate one ($s > 0$).

The phenomenon object of this investigation is *the improvement of integrability for the gradient* of solutions of (1.1) in the case the measure on the right-hand side satisfies a quantitative density type condition that, roughly speaking, says that the measure does not concentrate on sets with small parabolic Hausdorff dimension. This general fact, first noted in [26, 27] for the elliptic case, is here found to hold in the parabolic degenerate one, which on the other hand needs very different techniques. To be more specific, we shall consider, for a Borel measure μ of finite total mass, a Morrey type condition as follows:

$$\sup_{Q_R(z_0) \subset \Omega_T} \frac{|\mu|(Q_R(z_0))}{R^{N-\vartheta}} \leq c_d \qquad \Longleftrightarrow \qquad \mu \in L^{1,\vartheta}(\Omega_T), \tag{1.3}$$

where $Q_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$ is the standard parabolic cylinder and $N := n+2$ denotes the parabolic dimension. The improved integrability of the gradient is naturally formulated in terms of Marcinkiewicz spaces, which are the natural and optimal

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ones to be used when dealing with measure data problems, see for instance [6, 13, 26, 28]: we indeed prove that

$$\mu \in L^{1,\vartheta}(\Omega_T), \quad \vartheta_c < \vartheta \le N \qquad \Longrightarrow \qquad |Du| \in \mathcal{M}^{p-1+\frac{1}{\vartheta-1}}_{\rm loc}(\Omega_T). \tag{1.4}$$

The number $\vartheta_c \in (1,2)$ is a threshold depending on the data of the problem, i.e. upon n, p, ν, L – see the next Section and in particular (2.6) – and it is linked with the higher integrability exponent for homogeneous problems. We recall the reader that Marcinkiewicz spaces (also called weak-Lebesgue spaces) $\mathcal{M}^m(\Omega_T)$ are defined via the decay condition on level sets (for $f : \Omega_T \to \mathbb{R}$ to be a measurable map)

$$\sup_{\lambda>0} \lambda^m \big|\{z \in \Omega_T : |f(z)| > \lambda\}\big| =: \|f\|^m_{\mathcal{M}^m(\Omega_T)} < \infty;$$

their local variant is defined in the usual way.

The existence theory for measure data problems has been settled by Boccardo and Gallouët in [8] for elliptic equations of p-Laplacian type; for parabolic operators, their approach has been developed in [7, 10], where it is shown that in the general case where μ is a Borel measure with finite mass, there exists a distributional solution (in the sense of Definition 2.1) to (1.1) such that

$$|Du| \in L^q(\Omega_T) \qquad \text{for all } q \in \Big[1, p-1+\frac{1}{N-1}\Big). \tag{1.5}$$

The strategy of these papers is based on an approximation procedure leading to a *particular type of solution* usually called *Solution Obtained as Limit of Approximations*, *SOLA* in short. Note indeed that it turns out that different notions of solution can be given when dealing with measure data, for example entropy solutions [6, 9] or renormalized ones [13]; uniqueness in general holds only in very particular cases, see [12]. In this paper we shall focus on the solutions obtained by approximation and we shall briefly recall the procedure we follow in the next Section 2; note on the other hand that in the elliptic case, for non-negative measures, all these notions of solution collapse, see [20]; this suggests that our results holds also for the solutions considered in [6, 13], in the case the measure is non-negative. The improvement of integrability in the case of Morrey data for the elliptic case has been presented in [26]:

$$\mu \in L^{1,\vartheta}(\Omega), \quad 2 \le \vartheta \le n \qquad \Longrightarrow \qquad |Du| \in \mathcal{M}^{\frac{\vartheta(p-1)}{\vartheta-1}}_{\rm loc}(\Omega).$$

In the general case $\vartheta = n$ without density properties, this gives back the classic, sharp result

$$\mu \in L^{1,n}(\Omega) \equiv \mathcal{M}_b(\Omega) \qquad \Longrightarrow \qquad |Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega),$$

whose origins we reassume by citing [6, 13] for the case $p < n$, in a slightly different setting; the borderline case $p = n$ requires different techniques and has been settled in [16]. Notice that $L^{1,n}(\Omega)$ is nothing else than the full space of signed Borel measures with finite total mass $\mathcal{M}_b(\Omega)$.

More comments are now in order. The result in (1.4) has been proved for the linear growth case $p = 2$ in [5] and extended up to the boundary in [15]; the degenerate case $p \neq 2$ needs completely different techniques and the use, in the context of measure data problems, of the "intrinsic scaling" method first pioneered by DiBenedetto in [14]; see also the more recent contributions [1, 23, 24, 32]. We note that the condition and the notation adopted in (1.3) in fact involves the notion of Morrey space, typically adopted for functions in the non-degenerate case, and in this paper we show how to deal with such a condition in the setting of degenerate evolutionary equations, and getting corresponding results; very recently, similar conditions have been considered by Misawa [30], who analyzed the relation between Morrey spaces conditions and regularity of solutions in the setting of degenerate parabolic equations.

Let us moreover note that (1.4) actually sharpens, at least locally, the result in [7, 10]: in this case indeed (1.4) reads as

$$\mu \in \mathcal{M}_b(\Omega_T) \qquad \Longrightarrow \qquad |Du| \in \mathcal{M}^{p-1+\frac{1}{N-1}}_{\rm loc}(\Omega_T) \tag{1.6}$$

and we note that $\mathcal{M}^{p-1+1/(N-1)} \subset L^q$ for all q as in (1.5). A preliminary form of (1.6) can be found in [2], where only the case $2 \le p < n$ is considered, due to the technique of the proof; on the other hand the result therein is global and requires no regularity of $\partial\Omega$, while it can be expected that the local estimates (1.4) and (1.6) could be extended up to the boundary similarly as in [4], assuming a (mild) capacitary condition on the complement of Ω. In this paper we extend (1.6) to all the exponents $p \ge 2$ with an approach which, resting directly on estimates for the gradient, allows for a unitary treatment of the full range $p \in [2, \infty)$.

Now, a few words on the techniques used in this paper. As mentioned above, treating the parabolic case $p \neq 2$ needs a different approach than the one used in [26] for the elliptic and adapted in [5] to parabolic equations with linear growth, based on maximal operators and Calderón-Zygmund coverings. Indeed equation (1.1), already in the case $\mu = 0$, does not have a universal scaling property, and as consequence estimates do not show homogeneity, see for instance Proposition 4.1; therefore Harmonic Analysis tools are difficult, if not impossible, to apply. We overcome this difficulty by using an intrinsic geometry viewpoint which finds its roots in [1] and in turn uses tools from [11, 19, 21], arguing directly on certain covering arguments to get estimates as

$$\begin{aligned}(H\lambda)^m\big|\{z \in Q_R : |Du(z)| > H\lambda\}\big| \\ \le \frac{1}{2}\lambda^m\big|\{z \in Q_{2R} : |Du(z)| > \lambda\}\big| + c\,|\mu|(Q_{2R})\end{aligned} \tag{1.7}$$

for an *a priori given* constant $H \gg 1$ depending only upon the data of the problem. This is sufficient to prove (1.4), after reabsorbing the term on the right-hand side with standard iteration arguments. The main point in proving the estimate in the previous display is, given a fixed super-level set $\{|Du| > \lambda\}$, to build a covering of *intrinsic cylinders* with scaling parameter λ. These are cylinders of the type

$$Q^\lambda_R(z) := B_R(x) \times (t - \lambda^{2-p}R^2, t + \lambda^{2-p}R^2).$$

where $|Du| \approx \lambda$, in some integral sense; these cylinders are called intrinsic since the parameter λ comes into play both in the definition of the cylinder itself and in the value of the average of Du over it. As we are working with λ fixed, the construction of such a covering is done using a refined exit-time argument on the function $R \to \fint_{Q^\lambda_R} |Du|^{p-1}\,dz$. The heuristic under the use of these cylinders for the parabolic p-Laplacian equation is the following: by the fact $|Du| \approx \lambda$ the equation $\partial_t u = \mathrm{div}[|Du|^{p-2}Du]$ behaves like the λ-heat equation $u_t = \lambda^{p-2}\Delta u$, and by their particular form we can rebalance the occurring multiplicative factor λ^{p-2} by rescaling u in time by a factor λ^{2-p}. What we explained here in a naïf way can be made rigorous, and estimates on these cylinders recover the homogeneous character needed for covering arguments, see for instance Proposition 4.6 and Corollary 4.7 in our case. For some more (again heuristic) details about the implementation of this technique in the setting of measure data problems, see the last part of Section 2.

Finally, the techniques developed in this paper would allow to treat also the case where μ is not anymore just a measure but a Lebesgue function, therefore fully extending the paper [5] to the degenerate parabolic case; this will be the content of the forthcoming [3]. Indeed estimates on level sets as (1.7) allow for bounds in all rearrangement invariant spaces: in connection we suggest the reader to look at [27], which contains Adams type theorem for singular and degenerate elliptic equations of p-Laplacian type.

2. Assumptions and statement of the results

The most natural and classic way to generalize (1.2) is to consider Carathéodory regular vector fields $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following ellipticity and growth conditions:

$$\begin{cases} \langle a(x,t,\xi_1) - a(x,t,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu \big(s^2 + |\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \\ |a(x,t,\xi)| \leq L\big(s^2 + |\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.1}$$

for $p \geq 2$, for almost every $(x,t) \in \Omega_T$, every $\xi_1, \xi_2, \xi \in \mathbb{R}^n$, with $0 < \nu \leq 1 \leq L$ and for $s \geq 0$. In the case here considered where the data do not belong to the natural energy space, it is customary to consider solutions not lying in the energy space; these solutions are usually called *very weak solutions*:

Definition 2.1. *A very weak solution to* (1.1) *is a function $u \in L^{p-1}_{\text{loc}}(-T, 0; W^{1,p-1}_{\text{loc}}(\Omega))$ such that*

$$\int_{\Omega_T} \Big[-u\, \varphi_t + \langle a(x,t,Du), D\varphi \rangle \Big]\, dz = \int_{\Omega_T} \varphi\, d\mu, \tag{2.2}$$

for every $\varphi \in C^\infty(\Omega_T)$ vanishing in a neighborhood of $\partial_{\mathcal{P}}\Omega_T$.

In general, since the right-hand side in (1.1) does not belong to the dual of the energy space, usual monotonicity methods do not apply and therefore it is not clear how to find a solution, in the general case, to the distributional formulation (2.2). A solution can be obtained via an approximation procedure, see [7, 10], once we consider equation (1.1) together with null Cauchy-Dirichlet data. Indeed we briefly recall here to the reader the argument proposed in [7, 10] to solve the problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{2.3}$$

where the vector field satisfies (2.1). One considers regular functions $L^\infty(\Omega_T) \ni f_k \rightharpoonup \mu$, $k \in \mathbb{N}$, converging to μ in the weak sense of measures, such that

$$\|f_k\|_{L^{1,\vartheta}(\Omega_T)} \leq \|\mu\|_{L^{1,\vartheta}(\Omega_T)}, \qquad \|f_k\|_{L^{1,\vartheta}(Q_{2(R+1/k)})} \leq \|\mu\|_{L^{1,\vartheta}(Q_{2R})}, \qquad k \gg 1$$

for $Q_{2R} \Subset \Omega_T$ and solutions u_k to (2.3) where in place of μ we consider as right-hand sides the regular functions f_k; therefore

$$u_k \in L^p(-T, 0; W^{1,p}_0(\Omega)) \cap C^0(-T, 0; L^2(\Omega)) \qquad \text{and} \qquad Du \in L^{p\chi}_{\text{loc}}(\Omega_T),$$

see [19], and they solve (2.2) with $d\mu = f_k\, dz$. Existence of such solutions is classic since the datum is regular; a.e. pointwise convergence of both u and of Du, up to subsequences, proved in [10], allows to pass to the limit in (2.2) therefore yielding existence of a very weak solutions satisfying (1.5). These solutions are called, as we already said, SOLAs. Note that in this class of solutions uniqueness does not hold in the case μ is not more than a measure, in the sense that if one considers a different regular sequences $\{\bar{f}_k\}$ converging to μ, it could happen that the limit $\bar{u}$ of the corresponding approximated solutions $\bar{u}_k$ is different from u; this will not be the case, for instance, when μ belongs to $L^1(\Omega_T)$, see [12].

After this introduction, we can state our main result (1.4) in a more precise form:

Theorem 2.2. *There exists a constant $\vartheta_c \in (1,2)$ depending on n, p, ν, L such that if u is a SOLA to equation* (2.3) *with $\mu \in L^{1,\vartheta}(\Omega_T)$, for $\vartheta_c < \vartheta \leq N$, then*

$$Du \in \mathcal{M}^m_{\text{loc}}(\Omega_T, \mathbb{R}^n), \qquad \textit{where} \quad m := p - 1 + \frac{1}{\vartheta - 1}. \tag{2.4}$$

Moreover for any parabolic cylinder $Q_{2R} \equiv Q_{2R}(z_0) \Subset \Omega_T$ there exists a constant depending on $n, p, \nu, L, c_d, \vartheta$ such that the following local estimate holds:

$$\|s + |Du|\|^m_{\mathcal{M}^m(Q_R)} \le c\,R^N \left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|}\right]^{p-1} + c\,R^N \left[\fint_{Q_{2R}} (|Du| + s + 1)^{p-1}\, dz\right]^m. \tag{2.5}$$

We here point out that the critical value ϑ_c is the solution of the equation

$$p - 1 + \frac{1}{\vartheta_c - 1} = p\chi \tag{2.6}$$

where $\chi \equiv \chi(n, p, \nu, L) > 1$ is the higher integrability exponent for homogeneous problems $v_t - \operatorname{div} a(x, t, Du) = 0$, see Corollary 4.8. The point here is clear: for what concerns regularity for Du, we obviously cannot overcome the maximal regularity we can get when $\mu \equiv 0$, which is $|Du| \in L^{p\chi}_{\mathrm{loc}}(\Omega_T)$. Things change when considering more regular vector fields, see the last Section for the precise statements. In this case we have the following

Corollary 2.3. *Let u be a SOLA to (2.3) , where the vector field $a(\cdot)$ is more regular in the sense that (7.1)-(7.2) or (7.3) holds. Then, if $\mu \in L^{1,\vartheta}(\Omega_T)$ for $1 < \vartheta \le N$, the conclusions of Theorem 2.2 hold.*

Here we can see that if μ satisfies the condition $|\mu|(Q_R) \le c_d R^{N-1}$, then $|Du| \in L^q_{\mathrm{loc}}(\Omega_T)$ for all $q > 1$. Compare with [24, Theorem 1.5] which states that

$$|\mu|(Q_R) \le c_d R^{N-1+\varepsilon} \quad \text{for some } \varepsilon > 0 \quad \Longrightarrow \quad Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$$

for some $\beta \in (0, 1)$ depending on $n, p, \nu, L, \varepsilon$, and moreover the improved borderline case in [25]

$$|\mu|(Q_R) \le c_d R^{N-1} h(R) \qquad \Longrightarrow \quad Du \in C^0(\Omega_T, \mathbb{R}^n),$$

where $h(\rho)$ is Dini continuous, that is $\int_0^1 h(\rho)/\rho\, d\rho < \infty$ (actually in [24] only the case with no coefficients is considered).

Proof techniques. We finally want to spend here a few words to explain in which sense we want to mean here "$|Du| \approx \lambda$" in Q^λ_R, since there are two possibilities which need to be accommodated. Indeed, starting from the pioneering work of Iwaniec [18], local comparisons estimates with homogeneous problems have become an essential tool in proving regularity results (in particular Calderón-Zygmund estimates) for non-linear problems, lacking of representation formulae. The correct cylinders to use in order to approach questions regarding homogeneous problems are energy ones, i.e.

$$\fint_{Q^\lambda_R} \big(|Dv| + s\big)^p\, dz \approx \lambda^p,$$

see (4.18) and also the Caccioppoli estimate Lemma 4.3; this could be seen as the *energy geometry*. On the other hand, since we are dealing with *very weak solutions* to equation (1.1), these functions might in general even have infinite energy (also comparison estimate Lemma 5.1 does not hold for the natural exponent p). Therefore, the only meaningful definition for intrinsic cylinders for u is

$$\fint_{Q^\lambda_R} \big(|Du| + s\big)^{p-1}\, dz \approx \lambda^{p-1} \quad \text{implying} \quad \fint_{Q^\lambda_R} \big(|Dv| + s\big)^{p-1}\, dz \approx \lambda^{p-1}$$

via comparison, see (6.6) and (6.13); we could call this the *weak geometry*. It will turn out in Section 5 that the right geometry for the problem is the unique possible, the *weak* one, as one could expect; the regularity of solutions to homogeneous problems (in particular the sup bound for v) will allow to show that the two geometries for v are actually equivalent, as proved in Proposition 5.5.

3. Notation and technical Lemmas

This section is devoted to fix the notation we will use in the rest of the paper. We denote by $Q_{r,s}(z_0)$ the generic cylinder in $\mathbb{R}^{n+1}$

$$Q_{r,s}(z_0) := B_r(x_0) \times (t_0 - s, t_0 + s)$$

having "vertex" $z_0 = (x_0, t_0) \in \Omega_T$, for $r, s > 0$. Particular cylinders will be of special importance: these are the cylinders $Q_r^\lambda(z_0)$ defined starting from a radius r and a parameter $\lambda \geq 1$ in the following way:

$$Q_R^\lambda(z_0) := B_R(x_0) \times \Lambda_R^\lambda(t_0) =: B_R(x_0) \times (t_0 - \lambda^{2-p} R^2, t_0 + \lambda^{2-p} R^2). \quad (3.1)$$

Such cylinders will be called "intrinsic" if some intrinsically defined relation between λ, Du and μ will hold, see for example (6.6). When dealing with families of cylinders with the same "vertex" (respectively of balls, of intervals), we will often avoid to denote it. Accordingly with the parabolic metric, for $\alpha > 0$ we shall write $\alpha\Lambda_R^\lambda := \Lambda_{\alpha R}^\lambda(t_0) = (t_0 - \lambda^{2-p}(\alpha R)^2, t_0 + \lambda^{2-p}(\alpha R)^2)$. The same will hold for the cylinders $\alpha Q_R^\lambda(x_0, t_0) := B_{\alpha R}(x_0) \times \Lambda_{\alpha R}^\lambda(t_0)$.

$\mathbb{R}^{n+1}$ will always be thought as $\mathbb{R}^n \times \mathbb{R}$, so a point $z \in \mathbb{R}^{n+1}$ will be often also denoted as (x, t), z_0 also as (x_0, t_0), and so on. By parabolic boundary of a cylinder $C := A \times J$, with $A \subset \Omega$, $J \subset (-T, 0)$, we will mean $\partial_{\mathcal{P}} C := A \times \{\inf J\} \cup \partial A \times J$. Being $\mathfrak{S} \in \mathbb{R}^m$ a measurable set with positive measure and $f : \mathfrak{S} \to \mathbb{R}^k$ an integrable map, with $m, k \geq 1$, we denote with $(f)_{\mathfrak{C}}$ the averaged integral

$$(f)_{\mathfrak{S}} := \fint_{\mathfrak{S}} f(\xi)\, d\xi := \frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} f(\xi)\, d\xi.$$

Moreover for a measurable function $g : \mathfrak{C} = \mathfrak{B} \times \mathfrak{I} \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$ over a cylinder we will use the notation

$$(g)_{\mathfrak{B}}(t) := \fint_{\mathfrak{B}} g(x,t)\, dx \qquad \text{for all } t \in \mathfrak{I}.$$

Note that if $g \in L^p(\mathfrak{I}, W^{1,p}(\mathfrak{B}))$, $p \geq 1$, $\mathfrak{B}, \mathfrak{I}$ as above, then $x \mapsto g(x,t) \in W^{1,p}(\mathfrak{B})$ for a.e. $t \in \mathfrak{I}$. Hence we will be allowed to use Poincaré's inequality slice-wise. As the reader may have already noticed, the expression $|\mathfrak{S}|$ will denote the m-dimensional Lebesgue measure $\mathcal{L}^m$ depending on where $\mathfrak{S}$ lies: $|\mathfrak{S}| = \mathcal{L}^m(\mathfrak{S})$ if $\mathfrak{S} \subset \mathbb{R}^m$.

Finally, we will denote with c a generic constant greater than one, possibly varying from line to line. Constants we need to recall will be denoted with special symbols, such as $c_1, c_2, \tilde{c}, c_*$. Relevant dependencies will be highlighted between parentheses or after the equations; when non essential, the dependence on a parameter will be suppressed (this will be the case, for instance, in (5.4), where the constant depends only on p and not on q).

Observe that our definition of $L^{1,\vartheta}$ slightly differs from the classical one; however, since we are treating inner local regularity, information near the boundary $\partial\Omega_T$ will play no role. Therefore in the definition of Morrey density we shall consider only cylinders not intersecting the boundary.

The following lemma is a standard iteration lemma and can for instance be found in [17, Lemma 6.1].

Lemma 3.1. *Let $\phi : [R, 2R] \to [0, \infty)$ be a function such that*

$$\phi(r_1) \leq \frac{1}{2}\phi(r_2) + \mathcal{A} + \frac{\mathcal{B}}{(r_2 - r_1)^\beta} \qquad \text{for every } R \leq r_1 < r_2 \leq 2R,$$

where $\mathcal{A}, \mathcal{B} \geq 1$ and $\beta > 0$. Then

$$\phi(R) \leq c(\beta) \left[\mathcal{A} + \frac{\mathcal{B}}{R^\beta}\right].$$

The following, on the other hand, is less known but allows to lower the exponents on the right-hand sides in our particular situation, encoding the self-improving property of reverse-Hölder inequalities; see [22, Lemma 5.1] for its proof.

Lemma 3.2. *Let ν be a non-negative Borel measure with finite total mass. Let moreover $1 < q < p < \infty$ and $\xi \geq 0$, and let $\{\theta U\}$ be a family of open sets with the property*

$$\theta_1 U \subset \theta_2 U \subset 1U = U \tag{3.2}$$

whenever $0 < \theta_1 < \theta_2 < 1$. If $w \in L^q(U)$ is a non-negative function satisfying

$$\left(\int_{\theta_1 U} w^p \, d\nu\right)^{1/p} \leq \frac{c_0}{(\theta_2 - \theta_1)^\xi}\left(\int_{\theta_2 U} w^q \, d\nu\right)^{1/q} \tag{3.3}$$

for all $1/2 \leq \theta_1 < \theta_2 \leq 1$, then there is a positive constant $c \equiv c(c_0, \xi, p, q)$ such that

$$\left(\int_{\theta U} w^p \, d\nu\right)^{1/p} \leq \frac{c}{(1-\theta)^{\xi'}} \int_U w \, d\nu, \tag{3.4}$$

for all $0 < \theta < 1$, where $\xi' = \xi q(p-1)/(p-q)$.

4. Estimates for homogeneous problems

Let us consider $v \in C^0(I; L^2(A)) \cap L^p(I; W^{1,p}(A))$ a solution to the problem

$$v_t - \operatorname{div} a(x,t,Dv) = 0 \qquad \text{in } A \times I \subset \Omega_T, \tag{4.1}$$

being A, I open sets and with $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}$ the vector field appearing in (1.1), therefore satisfying (2.1) and

$$\langle a(x,t,\xi), \xi\rangle \geq c\left(s^2 + |\xi|^2\right)^{\frac{p-2}{2}} |\xi|^2 - c^{-1} L s^p, \tag{4.2}$$

with $c \equiv c(p, \nu, L)$, by a simple use of Young's inequality. In this section we collect some regularity results for weak solutions to (4.1).

The following is the $\sup$ bound for solutions to degenerate parabolic equations. It can be found in [31], see obviously also [14, Chapter V, Theorems 3.1 & 4.1]. Some modifications of the proofs are needed; in particular we followed the proof of the homogeneous case and the last term in (4.3) is due to the fact that in (4.2) the term containing s^p appears. Moreover we introduced the parameter ε: following [14, §12, Proof of Theorem 4.1], once we fix $\varepsilon \in (0,1)$, we just need in equation (12.1) of the aforementioned proof to take k so big that $k \geq \varepsilon(\rho^p/\theta)^{1/(p-2)}$. Note that DiBenedetto's book notation differs from ours (in particular $\theta \leftrightarrow \sigma$). Recall that a sub-solution is a function such that the left-hand side of the weak formulation of (4.1) is negative, for every *positive* test function.

Proposition 4.1. *Any positive sub-solution w to (4.1) in $A \times I$ is locally bounded with the following quantitative estimate: for $Q_{\rho,\sigma}(z_0) \equiv Q_{\rho,\sigma} \subset A \times I$, $\theta \in (0,1)$ and for every $\varepsilon \in (0,1]$ there holds*

$$\sup_{Q_{\theta\rho,\theta\sigma}} w \leq c\left(\frac{1}{(1-\theta)\varepsilon}\right)^{\frac{n+p}{2}} \left(\frac{\sigma}{\rho^p}\right)^{\frac{1}{2}} \left(\fint_{Q_{\rho,\sigma}} w^p \, dz\right)^{1/2} + \varepsilon\left(\frac{\rho^p}{\sigma}\right)^{1/(p-2)} + s\,\rho \tag{4.3}$$

for a constant depending only on n, p, ν, L.

Starting on the other hand from (4.3) and following exactly [14, §12, Proof of Theorem 4.1] we can lower the exponent appearing on the right-hand side, therefore getting the following corollary:

Corollary 4.2. *Let w be as in Proposition 4.1; then for $Q_{\rho,\sigma}, \theta, \varepsilon$ as above there holds*

$$\sup_{Q_{\theta\rho,\theta\sigma}} w \le c\left(\frac{1}{(1-\theta)\varepsilon}\right)^{n+p}\frac{\sigma}{\rho^p}\fint_{Q_{\rho,\sigma}} w^{p-1}\,dz + \varepsilon\left(\frac{\rho^p}{\sigma}\right)^{1/(p-2)} + s\,\rho \tag{4.4}$$

with $c \equiv c(n,p,\nu,L)$.

The next Lemma is a standard Caccioppoli's inequality on generic cylinders; for its proof see [19, Lemma 3.2].

Lemma 4.3 (Caccioppoli's inequality)**.** *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (4.1) and let $Q_{\rho_2,\sigma_2} \equiv Q_{\rho_2,\sigma_2}(z_0) \subset A \times I$ be a cylinder. Moreover let $k \in \mathbb{R}$. Then the following estimate holds:*

$$\begin{aligned}\int_{Q_{\rho_1,\sigma_1}} &\left(|Dv|^2+s^2\right)^{\frac{p-2}{2}}|Dv|^2\,dz + \sup_{t\in(t_o-\sigma_1,t_o+\sigma_1)}\int_{B_{\rho_1}(x_o)}|v(\cdot,t)-k|^2\,dx\\ &\le \frac{c}{\sigma_2-\sigma_1}\int_{Q_{\rho_2,\sigma_2}}|v-k|^2\,dz\\ &\qquad + \frac{c}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2,\sigma_2}}|v-k|^p\,dz + c\,s^p|Q_{r_2,s_2}|,\end{aligned} \tag{4.5}$$

for all concentric cylinders $Q_{\rho_1,\sigma_1} \equiv Q_{\rho_1,\sigma_1}(z_0) \Subset Q_{\rho_2,\sigma_2}(z_0)$ and with a constant depending on n, p, ν, L.

The next is [19, Lemma 3.1]:

Lemma 4.4. *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (4.1). If $Q_{\rho_2,\sigma}(z_0) \subset A \times I$ and $\rho_1 < \rho_2$, then there exist a radius $\hat\rho \in (\rho_1, \rho_2)$ and a constant c depending on n, p, L such that*

$$|(v)_{B_{\hat\rho}(x_o)}(t_1) - (v)_{B_{\hat\rho}(x_o)}(t_2)| \le \frac{c\,\sigma}{\rho_2-\rho_1}\fint_{Q_{\rho_2,\sigma}}(|Dv|+s)^{p-1}\,dz \tag{4.6}$$

for a.e. $t_1, t_2 \in (t_0-\sigma, t_0+\sigma)$.

The following is instead a Sobolev-type inequality. See again [19, Lemma 3.3].

Lemma 4.5. *Let $1 \le q < \infty$ and suppose $v \in L^q(t_0-\sigma_2, t_0+\sigma_2; W^{1,q}(B_{\rho_2}(x_0)))$. Then, for a constant depending only on n, q, there holds*

$$\begin{aligned}\int_{Q_{\rho_1,\sigma_1}}\left|v-(v)_{B_{\rho_1}}(t)\right|^{q(1+2/n)}dz &\le c\left(\frac{R}{\rho_2-\rho_1}\right)^q\int_{Q_{\rho_2,\sigma_2}}|Dv|^q\,dz\,\times\\ &\times\left(\sup_{t\in(t_o-\sigma_2,t_o+\sigma_2)}\int_{B_{\rho_2}}\left|v(\cdot,t)-(v)_{B_{\rho_2}}(t)\right|^2dx\right)^{q/n}\end{aligned}$$

for every couple of radii $R/2 \le \rho_1 < \rho_2 \le R$. Q_{ρ_1,σ_1} and Q_{ρ_2,σ_2} share the vertex $z_0 = (x_0, t_0)$.

Finally a reverse-Hölder's inequality for solutions to (4.1). Since this proof is very similar to that of [19, Lemma 3.4], in some points the arguments are only sketched. We refer to the aforementioned papers for the missing details.

Proposition 4.6. *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (4.1) and let $Q_R^\lambda(z_0) \subset A \times I$ be a cylinder such that*

$$\left(\frac{\lambda}{\kappa}\right)^p \le \fint_{Q^\lambda_{R/2}}(|Dv|+s)^p\,dz \quad \text{and} \quad \fint_{Q^\lambda_R}(|Dv|+s)^p\,dz \le (\kappa\lambda)^p \tag{4.7}$$

hold for a constant $\kappa \geq 1$. Then there exist constants c depending on n, p, ν, L, κ and an exponent $\xi \equiv \xi(n,p)$ such that

$$\left(\fint_{Q^{\lambda}_{r_1}} \left(|Dv| + s\right)^p \, dz \right)^{1/p} \leq c \left(\frac{R}{r_2 - r_1} \right)^{\xi} \left(\fint_{Q^{\lambda}_{r_2}} \left(|Dv| + s\right)^q \, dz \right)^{1/q} \tag{4.8}$$

for all $R/2 \leq r_1 < r_2 \leq R$ and with $q := \max\left\{p-1, np/(n+2)\right\} < p$. Here $Q^{\lambda}_{r_1}$ and $Q^{\lambda}_{r_2}$ are concentric cylinders having the same vertex as Q^{λ}_R.

Proof. For ease of notation from now on we suppose $z_0 = 0$. If this were not the case, a simple translation would be sufficient to recover this situation.

We begin by defining $r_3 := (r_2 - r_1)/5$. Caccioppoli's inequality (4.5) applied with $Q_{\rho_1,\sigma_1} \equiv Q^{\lambda}_{r_1}$, $Q_{\rho_2,\sigma_2} \equiv B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3} = B_{r_1+r_3} \times (-\lambda^{2-p}(r_1+2r_3)^2, \lambda^{2-p}(r_1+2r_3)^2)$ gives, for $\varepsilon \in (0,1)$ to be chosen

$$\begin{aligned}\fint_{Q^{\lambda}_{r_1}} \left(|Dv|^2 + s^2\right)^{\frac{p-2}{2}} |Dv|^2 \, dz &\leq \frac{c}{r_3^p} \fint_{B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3}} |v-k|^p \, dz \\ &\quad + \frac{c\,\lambda^{p-2}}{r_3^2} \fint_{B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3}} |v-k|^2 \, dz + c\, s^p \\ &\leq \frac{\varepsilon}{2}\lambda^p + \frac{c_\varepsilon}{r_3^p} \fint_{B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3}} |v-k|^p \, dz + c\, s^p, \end{aligned} \tag{4.9}$$

with the constant c_ε depending on n, p, ν, L and on ε. In the last line we used Young's and Hölder's inequalities. Note moreover that we can take averages in (4.5) since $|B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3}|/|Q^{\lambda}_{r_1}| \leq c(n)$. Now, denoting $\hat{Q} := B_{r_1+\hat{r}} \times \Lambda^{\lambda}_{r_1+2r_3} =: \hat{B} \times \Lambda^{\lambda}_{r_1+2r_3}$, we choose

$$k \equiv (v)_{\hat{Q}} = \fint_{B_{r_1+\hat{r}} \times \Lambda^{\lambda}_{r_1+2r_3}} v \, dz,$$

where $\hat{r} \in (r_3, 2r_3)$ is such that $r_1 + \hat{r}$ is the radius $\hat{\rho}$ of Lemma 4.4 with $\rho_1 \equiv r_1 + r_3$, $\rho_2 \equiv r_1 + 2r_3$ and $\sigma \equiv \lambda^{2-p}(r_1+2r_3)^2$. Note that

$$B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3} \subset B_{r_1+\hat{r}} \times \Lambda^{\lambda}_{r_1+2r_3} \subset Q^{\lambda}_{r_1+2r_3}$$

and their measures differ only by a constant $c(n)$. The goal of the proof now is to estimate the second term of the right-hand side of (4.9); using triangle's inequality

$$\begin{aligned}\fint_{B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3}} \left|v - (v)_{\hat{Q}}\right|^p dz &\leq c(p) \fint_{B_{r_1+r_3} \times \Lambda^{\lambda}_{r_1+2r_3}} \left|v - (v)_{\hat{B}}(t)\right|^p dz \\ &\quad + c(p) \sup_{t \in \Lambda^{\lambda}_{r_1+2r_3}} \left|(v)_{\hat{B}}(t) - (v)_{\hat{Q}}\right|^p, \end{aligned} \tag{4.10}$$

where sup is to be understood in the sense of essential supremum. Applying (4.6) from Lemma 4.4 we have

$$|(v)_{\hat{B}}(t_1) - (v)_{\hat{B}}(t_2)| \leq c\, \frac{\lambda^{2-p}(r_1+2r_3)^2}{r_3} \fint_{Q^{\lambda}_{r_1+2r_3}} \left(|Du| + s\right)^{p-1} dz \tag{4.11}$$

for a.e. $t_1, t_2 \in \Lambda^{\lambda}_{r_1+2r_3}$. This will be useful to estimate the *second term in* (4.10): indeed for a.e. $t \in \Lambda^{\lambda}_{r_1+2r_3}$ by (4.11), using the intrinsic estimate (4.7), we have

$$\begin{aligned}|(v)_{\hat{B}}(t) - (v)_{\hat{Q}}| &\leq c\, \frac{\lambda^{2-p}(r_1+2r_3)^2}{r_3} \fint_{Q^{\lambda}_{r_1+2r_3}} \left(|Dv| + s\right)^{p-1} dz \\ &\leq c\, \frac{R^2}{r_3} \left(\fint_{Q^{\lambda}_{r_2}} \left(|Dv| + s\right)^{p-1} dz \right)^{1/(p-1)} \end{aligned} \tag{4.12}$$

for a constant depending on n, p, L, κ. Note again that $|Q^\lambda_{r_1+2r_3}| \approx |Q^\lambda_{r_2}| \approx |Q^\lambda_R|$, up to a constant depending on n.

Now we estimate the *first term of* (4.10). Using Sobolev's estimate of Lemma 4.5 with $q = np/(n+2)$, $Q_{\rho_1,\sigma_1} \equiv \hat{Q} = B_{r_1+\hat{r}} \times \Lambda^\lambda_{r_1+2r_3}$, $Q_{\rho_2,\sigma_2} \equiv B_{r_1+3r_3} \times \Lambda^\lambda_{r_1+2r_3}$ we gain

$$\begin{aligned}\fint_{B_{r_1+r_3}\times\Lambda^\lambda_{r_1+2r_3}} \left|v-(v)_{\hat{B}}(t)\right|^p dz &\le c \fint_{\hat{B}\times\Lambda^\lambda_{r_1+2r_3}} \left|v-(v)_{\hat{B}}(t)\right|^p dz \\ &\le c\left(\frac{R}{r_3}\right)^{np/(n+2)} \fint_{B_{r_1+3r_3}\times\Lambda^\lambda_{r_1+2r_3}} |Dv|^{np/(n+2)}\, dz \times \\ &\quad \times \left(\sup_{t\in\Lambda^\lambda_{r_1+3r_3}} \int_{B_{r_1+3r_3}} \left|v(\cdot,t)-(v)_{\check{Q}}\right|^2 dx\right)^{p/(n+2)}, \qquad (4.13)\end{aligned}$$

where $\check{Q} := \check{B} \times \Lambda^\lambda_{r_1+5r_3} := B_{\hat{\rho}} \times \Lambda^\lambda_{r_2}$ is the ball of Lemma 4.4 for the choice $\rho_1 = r_1 + 4r_3$, $\rho_2 = r_1 + 5r_3 = r_2$; note that we indeed used the elementary property

$$\int_{B_{r_1+3r_3}} |v(\cdot,t)-(v)_{B_{r_1+3r_3}}(t)|^2\, dx \le 4\int_{B_{r_1+3r_3}} |v(\cdot,t)-\ell|^2\, dx \qquad (4.14)$$

valid for any $\ell \in \mathbb{R}$. We now need to further estimate the supremum in the right-hand side of (4.13). Using the Caccioppoli's Lemma 4.3, this time with $Q_{\rho_1,\sigma_1} \equiv Q^\lambda_{r_1+3r_3}$, $Q_{\rho_2,\sigma_2} \equiv \check{Q}$ and $k \equiv (v)_{\check{Q}}$, after taking averages we have, since $\hat{\rho} - (r_1+3r_3) \ge r_3$

$$\begin{aligned}&\frac{1}{|\check{Q}|}\sup_{t\in\Lambda^\lambda_{r_1+3r_3}} \int_{B_{r_1+3r_3}} \left|v(\cdot,t)-(v)_{\check{Q}}\right|^2 dx \\ &\qquad \le \frac{c}{r_3^p}\fint_{\check{Q}} \left|v-(v)_{\check{Q}}\right|^p dz + \frac{c\lambda^{p-2}}{r_3^2}\fint_{\check{Q}} \left|v-(v)_{\check{Q}}\right|^2 dz + c\, s^p, \qquad (4.15)\end{aligned}$$

the constant depending upon n, p, ν, L. The goal now is to estimate the two averages on the right-hand side with appropriate powers of λ. We have

$$\fint_{\check{Q}} |v-(v)_{\check{Q}}|^2\, dz \le 4\fint_{\check{Q}} \left|v-(v)_{\check{B}}(t)\right|^2 dz + 4\sup_{t\in\Lambda^\lambda_{r_2}} \left|(v)_{\check{B}}(t)-(v)_{\check{Q}}\right|^2. \quad (4.16)$$

At this point Poincaré's inequality applied slicewise gives

$$\fint_{\check{Q}} \left|v-(v)_{\check{B}}(t)\right|^2 dz \le c(n)\,(r_1+5r_3)^2 \fint_{\check{Q}} |Dv|^2\, dz \le c(n,\kappa)\, r_2^2\lambda^2,$$

using also Hölder's inequality and (4.7). As for the second term in (4.16) we deduce, similarly as done in (4.10)-(4.11), using Lemma 4.4 with $\rho_1 \equiv r_1 + 4r_3$, $\rho_2 \equiv r_1 + 5r_3$ and $\sigma = \lambda^{2-p} r_2^2$

$$\begin{aligned}&\sup_{t\in\Lambda_{r_1+4r_3}} \left|(v)_{\check{B}}(t)-(v)_{\check{Q}}\right|^2 \\ &\qquad \le c\left[\frac{\lambda^{2-p}r_2^2}{r_3}\left(\fint_{Q^\lambda_{r_2}} (|Dv|+s)^p\, dz\right)^{\frac{p-1}{p}}\right]^2 \le c\left(\frac{r_2^2}{r_3}\right)^2\lambda^2\end{aligned}$$

with $c \equiv c(n,p,L,\kappa)$, so merging the last two inequalities into (4.16) gives

$$\fint_{\check{Q}} \left|v-(v)_{\check{Q}}\right|^2 dz \le c\, r_2^2\left(1+\frac{r_2}{r_3}\right)^2\lambda^2 \le c\, R^2\left(\frac{R}{r_3}\right)^2\lambda^2$$

c having the same dependencies as the constant above. Similar estimates give

$$\fint_{\check{Q}} \left|v-(v)_{\check{Q}}\right|^p dz \le c\, R^p\left(\frac{R}{r_3}\right)^p\lambda^p;$$

putting these two estimates into (4.15) and noting that $s \le \kappa\lambda$ gives

$$\frac{1}{|\check{Q}|} \sup_{t\in\Lambda^{\lambda}_{r_1+3r_3}} \int_{B_{r_1+3r_3}} \left|v(\cdot,t) - (v)_{\check{Q}}\right|^2 dx \le c\left(\frac{R}{r_3}\right)^{2p} \lambda^p + c\, s^p \le c\left(\frac{R}{r_3}\right)^{2p}\lambda^p,$$

that is

$$\left(\sup_{t\in\Lambda^{\lambda}_{r_1+3r_3}} \int_{B_{r_1+3r_3}} \left|v(\cdot,t) - (v)_{\check{Q}}\right|^2 dx\right)^{p/(n+2)} \le c\, R^p \left(\frac{R}{r_3}\right)^{2p^2/(n+2)} \lambda^{2p/(n+2)}.$$

Plugging in turn this estimate into (4.13) and with the help of Young's inequality, ε being the same quantity already chosen as in (4.9), finally gives an estimate for the *first term* of (4.10)

$$\begin{aligned}
&\frac{1}{r_3^p} \fint_{B_{r_1+r_3}\times\Lambda^{\lambda}_{r_1+2r_3}} \left|v - (v)_{\hat{B}}(t)\right|^p dz \\
&\quad\le c\,\lambda^{2p/(n+2)} \left(\frac{R}{r_3}\right)^{2p(n+p+1)/(n+2)} \fint_{Q^{\lambda}_{r_2}} |Dv|^{np/(n+2)}\, dz \\
&\quad\le \frac{\varepsilon}{2}\lambda^p + c_\varepsilon \left(\frac{R}{r_3}\right)^{2p(n+p+1)/n} \left(\fint_{Q^{\lambda}_{r_2}} |Dv|^{np/(n+2)}\, dz\right)^{(n+2)/n}
\end{aligned}$$

with $c_\varepsilon \equiv c_\varepsilon(n,p,\nu,L,\kappa,\varepsilon)$; putting this estimate inside (4.10) together with (4.12) and in turn the result into (4.9)

$$\begin{aligned}
&\fint_{Q^{\lambda}_{r_1}} \left(|Dv|^2 + s^2\right)^{\frac{p-2}{2}} |Dv|^2\, dz \\
&\le \varepsilon\lambda^p + c_\varepsilon \left(\frac{R}{r_3}\right)^{2p} \left(\fint_{Q^{\lambda}_{r_2}} \left(|Dv| + s\right)^{p-1} dz\right)^{p/(p-1)} \\
&\qquad + c_\varepsilon \left(\frac{R}{r_3}\right)^{2p(n+p+1)/n} \left(\fint_{Q^{\lambda}_{r_2}} |Dv|^{np/(n+2)}\, dz\right)^{(n+2)/n},
\end{aligned}$$

c_ε depending upon n, p, ν, L, κ and obviously ε. Now we only need some algebraic manipulations to get (4.8). In particular we first recall the definition of r_3 and that $q < p$, we then estimate from below the left-hand side and we sum to both sides s. Finally

$$\lambda^p \le \kappa^p \fint_{Q^{\lambda}_{R/2}} \left(|Dv| + s\right)^p dz \le \kappa^p\, 2^{n+2} \fint_{Q^{\lambda}_{r_1}} \left(|Dv| + s\right)^p dz;$$

choosing ε, depending on n, p, κ, small enough to make reabsorption possible finishes the proof. □

Matching the previous Proposition with Lemma 3.2 immediately implies the following

Corollary 4.7. *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (4.1) and let $Q^{\lambda}_R(z_0) \subset A \times I$ be a cylinder such that*

$$\left(\frac{\lambda}{\kappa}\right)^p \le \fint_{Q^{\lambda}_{R/2}} \left(|Dv| + s\right)^p dz \qquad \textit{and} \qquad \fint_{Q^{\lambda}_{R}} \left(|Dv| + s\right)^p dz \le (\kappa\lambda)^p$$

hold for a constant $\kappa \geq 1$. Then there exists a constant c depending on n, p, ν, L, κ such that

$$\left(\fint_{Q^\lambda_{\theta R}} (|Dv|+s)^p\,dz\right)^{1/p} \leq \frac{c}{(1-\theta)^{\xi'}}\left(\fint_{Q^\lambda_R} (|Dv|+s)^q\,dz\right)^{1/q}, \tag{4.17}$$

for any $\theta \in (0,1)$, $q \in [1,p]$ and with $\xi' \equiv \xi'(n,p,q)$.

Proof. We apply Lemma 3.2 with the choices

$$\nu = \frac{1}{|Q^\lambda_R|}\mathcal{L}^{n+1}, \qquad U = Q^\lambda_R(z_0), \qquad \sigma_i = \frac{r_i}{R}, \quad i=1,2,$$

so that $\sigma_i U = Q^\lambda_{r_i}(z_0)$, $i=1,2$. (3.2) obviously holds. With these agreements (4.8) looks exactly like (3.3), apart from a constant depending on n and p (the reader might recall that $|Q^\lambda_R| \approx |Q^\lambda_{r_1}| \approx |Q^\lambda_{r_2}|$). Then (4.17) follows straight from (3.4). □

Finally we can state the higher integrability result for the parabolic p-Laplacian of [19] in the homogeneous form we need. With all the preceding results at hand, its derivation is immediate.

Corollary 4.8. *Let v as in Proposition 4.6 and in particular let*

$$\left(\frac{\lambda}{\kappa}\right)^p \leq \fint_{Q^\lambda_{R/2}} (|Dv|+s)^p\,dz \quad \text{and} \quad \fint_{Q^\lambda_R} (|Dv|+s)^p\,dz \leq (\kappa\lambda)^p \tag{4.18}$$

hold in some cylinder Q^λ_R, with $\lambda \geq 1$ and for a constant $\kappa \geq 1$. Then there exists $\chi \equiv \chi(n,p,\nu,L) > 1$ such that

$$\left(\fint_{Q^\lambda_{R/2}} (|Dv|+s)^{p\chi}\,dz\right)^{1/(p\chi)} \leq c\left(\fint_{Q^\lambda_R} (|Dv|+s)^q\,dz\right)^{1/q} \tag{4.19}$$

for any $q \in [1,p]$, the constant c depending on n, p, ν, L, κ.

Proof. The estimate

$$\left(\fint_{Q^\lambda_{R/2}} (|Dv|+s)^{p\chi}\,dz\right)^{1/(p\chi)} \leq c\left(\fint_{Q^\lambda_{3R/4}} (|Dv|+s)^p\,dz\right)^{1/p}$$

is deduced starting from [19, Proposition 4.1] and (4.18) very similarly to [1, Lemma 3]. At this point using (4.17) in the previous display gives (4.19). □

5. Comparison lemmata & merging the geometries

In this section we approach the proof of Theorem 2.2, first collecting some comparison results and then showing how to accommodate the two geometries of the problem, as explained in the end of Section 2.

First of all, from now on we will choose as the set $A \times I$ a cylinder $Q^\lambda_R(z_0) \subset \Omega_T$ and we will introduce therein the comparison function solution to the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) = 0 & \text{in } Q^\lambda_R, \\ v = u & \text{on } \partial_{\mathcal{P}} Q^\lambda_R, \end{cases} \tag{5.1}$$

where u is a solution to (1.1). Recall we are dealing with approximating, regular solutions $u \equiv u_k$; therefore existence and uniqueness of v are well known arguments (see [14]) and so it is the fact that $v \in u + C^0\big(\Lambda; L^2(B)\big) \cap L^p\big(\Lambda; W^{1,p}_0(B)\big)$ if $Q^\lambda_R = \Lambda \times B$. The following comparison result is [23, Lemma 4.1].

Lemma 5.1. *Let u be a weak solution to (1.1) and let v be the comparison function defined in (5.1). Then*

$$\left(\fint_{Q_R^\lambda} |Du - Dv|^q \, dz\right)^{1/q} \le c \left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}} \tag{5.2}$$

for every $q \in \left[1, p-1+\dfrac{1}{N-1}\right)$ and for a constant $c \equiv c(n,p,\nu,q)$.

We note that this comparison estimate shows a non-homogeneous character, differently from the elliptic corresponding ones, see [29, Lemma 9.5]; however it perfectly fits our situation, once having intrinsic relations at hand, as we will see several times in the sequel. The first one is the following

Lemma 5.2. *Let u be a weak solution to (1.1) and let the density condition (1.3) for some $1 < \vartheta \le N$ hold. Moreover suppose that the intrinsic relation*

$$\left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|}\right]^{\frac{1}{m}} \le \epsilon\lambda \tag{5.3}$$

holds true for some constant $\epsilon \in (0,1)$, where m is defined in (2.4). Then

$$\left(\fint_{Q_R^\lambda} |Du - Dv|^q \, dz\right)^{1/q} \le c_1 \, \epsilon^{\frac{N}{A}} \, \lambda$$

for all $q \in \left[1, p-1+\dfrac{1}{N-1}\right)$, for a constant c_1 depending on n,p,ν,c_d,ϑ,q. Here $A := (N-1)(p-1)+1$ in short.

Proof. From (5.3) and (1.3) we obviously have

$$\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|} \le (\epsilon\lambda)^m \qquad \text{and} \qquad \frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|} \le c_d \lambda^{p-2} R^{-\vartheta}.$$

Hence by Lemma 5.1 for q as in the statement ,we deduce, using the previous relations and recalling the definition of A

$$\begin{aligned}\left(\fint_{Q_R^\lambda} |Du - Dv|^q \, dz\right)^{1/q} &\le c \left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|^{\frac{N-1}{N}}}\right]^{\frac{N}{A}} = c\left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|}\right]^{(\alpha+\beta)\frac{N}{A}} |Q_R^\lambda|^{\frac{1}{A}} \\ &\le c\, \epsilon^{\alpha m \frac{N}{A}} \, \lambda^{\alpha m \frac{N}{A} + \beta(p-2)\frac{N}{A} + \frac{2-p}{A}} R^{-\beta\vartheta\frac{N}{A} + \frac{N}{A}},\end{aligned}$$

where α, β are two positive constants such that $\alpha + \beta = 1$. If we choose β so that the exponent of R is zero, i.e. $\beta = 1/\vartheta$, then $\alpha = 1 - 1/\vartheta$ is such that a simple computation shows that the exponent of λ is 1, i.e.

$$\left(\frac{\vartheta - 1}{\vartheta} m + \frac{(p-2)}{\vartheta}\right)\frac{N}{A} + \frac{2-p}{A} = 1.$$

At this point again a direct calculation shows that $\alpha m \ge 1$. □

At this point we need a Poincaré-type estimate for the function v. Despite its proof is standard, we prefer to propose it, being quite short.

Proposition 5.3. *Let v be as in (5.1) and let $Q_R^\lambda \equiv Q_R^\lambda(z_0) \subset A \times I$ be a parabolic cylinder, not necessarily intrinsic. Then*

$$\begin{aligned}\fint_{Q_R^\lambda} \left|\frac{v - (v)_{Q_R^\lambda}}{R}\right|^q dz &\le c \fint_{Q_R^\lambda} (|Dv| + s)^q \, dz \\ &\quad + c\, \lambda^{q(2-p)} \left(\fint_{Q_R^\lambda} (|Dv| + s)^{p-1} \, dz\right)^q \end{aligned} \tag{5.4}$$

for all $1 \le q \le p$ and for a constant c depending only on n,p,L.

Proof. Take a positive weight function $\eta \in C_c^\infty(B_R)$ satisfying

$$\fint_{B_R} \eta \, dx = 1, \qquad \eta(x) + R|D\eta(x)| \le c(n) \quad \text{for all } x \in B_R$$

and define the weighted mean of $v(\cdot, t)$ on B_R by

$$(v)^\eta_{B_R} := \fint_{B_R} u(\cdot, t)\eta \, dx.$$

Now we split the integral on the left-hand side of (5.4) in the following way:

$$\begin{aligned}
\fint_{Q_R^\lambda} \left| \frac{v - (v)_{Q_R^\lambda}}{R} \right|^q dz \le c(q) \fint_{Q_R^\lambda} \left| \frac{v - (v)^\eta_{B_R}(t)}{R} \right|^q dz \\
+ \frac{c(q)}{R^q} \fint_{\Lambda_R^\lambda} \left| (v)^\eta_{B_R}(t) - \fint_{\Lambda_R^\lambda} (v)^\eta_{B_R}(\tau) \, d\tau \right|^q dt \\
+ \frac{c(q)}{R^q} \left| \fint_{\Lambda_R^\lambda} (v)^\eta_{B_R}(\tau) \, d\tau - (v)_{Q_R^\lambda} \right|^q = I + II + III.
\end{aligned}$$

We have, by a standard variation of Poincaré's inequality applied slice-wise

$$III \le I \le c(n,p) \fint_{Q_R^\lambda} |Dv|^q \, dz.$$

To estimate II we rather use the equation. Test indeed directly $(5.1)_1$ with the test function η independent of time: for a.e. $t_1, t_2 \in \Lambda_R^\lambda$

$$\begin{aligned}
|(v)^\eta_{B_R}(t_1) - (v)^\eta_{B_R}(t_2)| &= \left| \int_{t_1}^{t_2} \partial_t \left[(v)^\eta_{B_R}(t) \right] dt \right| = \left| \int_{t_1}^{t_2} \fint_{B_R} \partial_t v \, \eta \, dx \, dt \right| \\
&= \left| \int_{t_1}^{t_2} \fint_{B_R} \langle a(\cdot, Dv), D\eta \rangle \, dx \, dt \right| \\
&\le \frac{c(n)L}{R} \int_{\Lambda_R^\lambda} \fint_{B_R} \left(s^2 + |Dv|^2 \right)^{(p-1)/2} dz \\
&\le c(n,L) \, \lambda^{2-p} R \fint_{Q_R^\lambda} \left(|Dv| + s \right)^{p-1} dz,
\end{aligned}$$

so

$$II \le c(n,L,q) \lambda^{q(2-p)} \left(\fint_{Q_R^\lambda} \left(|Dv| + s \right)^{p-1} dz \right)^q.$$

Note that the previous estimate is just formal: a precise proof can be done using a regularizing procedure in time, for example Steklov's averaging. Merging together the estimates for I, II, III gives (5.4). □

The previous proposition immediately translates in the following corollary, once we know that the cylinder Q_R^λ is intrinsic:

Corollary 5.4. *Let v be as in Proposition 5.3 and moreover let us suppose the intrinsic relation*

$$\fint_{Q_R^\lambda} \left(|Dv| + s \right)^{p-1} dz \le (\kappa\lambda)^{p-1}$$

satisfied for some $\kappa \ge 1$. Then

$$\fint_{Q_R^\lambda} \left| \frac{v - (v)_{Q_R^\lambda}}{R} \right|^{p-1} dz \le c \fint_{Q_R^\lambda} \left(|Dv| + s \right)^{p-1} dz$$

for a constant c depending on n, p, L, κ.

The following Proposition finally shows that the *weak* geometry for Dv is equivalent to the standard one.

Proposition 5.5. *Let v be the weak solution to (4.1). If*

$$\left(\frac{\lambda}{\kappa}\right)^{p-1} \le \fint_{Q^\lambda_{R/4}} (|Dv|+s)^{p-1}\,dz, \qquad \fint_{Q^\lambda_R} (|Dv|+s)^{p-1}\,dz \le (\kappa\lambda)^{p-1} \tag{5.5}$$

hold for a constant $\kappa \ge 1$, then

$$\left(\frac{\lambda}{\kappa}\right)^{q} \le \fint_{Q^\lambda_{R/4}} (|Dv|+s)^{q}\,dz, \qquad \fint_{Q^\lambda_{R/2}} (|Dv|+s)^{q}\,dz \le c\,(\kappa\lambda)^{q} \tag{5.6}$$

hold for every $q \in [p-1,p]$, with c depending on n,p,ν,L.

Proof. The first inequality of (5.6) follows immediately from $(5.5)_1$. For the second one, we use Caccioppoli's inequality together with Hölder's and Young's inequalities to infer

$$\begin{aligned}
\fint_{Q^\lambda_{R/2}} (|Dv|+s)^{q}\,dz &\le c\left[\lambda^{p-2}\left(\fint_{Q^\lambda_{3R/4}}\left|\frac{v-(v)_{Q^\lambda_{3R/4}}}{R}\right|^p dz\right)^{2/p}\right. \\
&\qquad\left.+\fint_{Q^\lambda_{3R/4}}\left(\left|\frac{v-(v)_{Q^\lambda_{3R/4}}}{R}\right|^p + s^p\right)dz\right]^{q/p} \\
&\le c\left[\lambda^p + \fint_{Q^\lambda_{3R/4}}\left(\left|\frac{v-(v)_{Q^\lambda_{3R/4}}}{R}\right|^p + s^p\right)dz\right]^{q/p} \\
&\le \frac{c}{R^q}\left[\operatorname*{osc}_{Q^\lambda_{3R/4}} v\right]^q + c\,s^q + c\,(\kappa\lambda)^q,
\end{aligned} \tag{5.7}$$

where $c \equiv c(n,p,\nu,L)$. Now we estimate the oscillation using Corollary 4.2: we indeed apply (4.4) with $\varepsilon = 1$, $Q_{\rho,\sigma} \equiv Q^\lambda_R$, $\theta \equiv 3/4$, to the positive sub-solutions $w = (v-(v)_{Q^\lambda_{3R/4}})_\pm$; then we sum up the resulting inequalities and, noting that $Q^\lambda_{3R/4} \subset Q_{3R/4,3\lambda^{2-p}R^2/4}$, we infer

$$\operatorname*{osc}_{Q^\lambda_{3R/4}} v \le c\,\lambda^{2-p}R\fint_{Q^\lambda_R}\left|\frac{v-(v)_{Q^\lambda_R}}{R}\right|^{p-1} dz + 2\lambda R + 2sR.$$

with $c \equiv c(n,p,\nu,L)$. Therefore, using Poincaré's inequality Corollary 5.4, and this is allowed since $(5.5)_2$ holds, we infer

$$\frac{1}{R}\operatorname*{osc}_{Q^\lambda_{3R/4}} v \le c\,\lambda^{2-p}\fint_{Q^\lambda_R}(|Dv|+s)^{p-1}\,dz + 2\lambda + 2s \le c\,\kappa\lambda, \tag{5.8}$$

using again $(5.5)_2$, which also yields $s \le \kappa\lambda$. Using again this fact together with (5.8) into (5.7) concludes the proof. □

6. Proof of Theorem 2.2

Finally we come to the proof of the Theorem. We take a parabolic cylinders $Q_{2R} \equiv Q_{2R}(\mathfrak{z}_0) \subset \Omega_T$, $R > 0$, and following [29], letting $M \ge 1$ be a free parameter to be chosen, we define the Calderón-Zygmund functional

$$CZ(\mathfrak{Q}) := \left(\fint_{\mathfrak{Q}/20}(|Du|+s)^{p-1}\,dz\right)^{\frac{1}{p-1}} + \left[M\frac{|\mu|(\mathfrak{Q}/20)}{|\mathfrak{Q}/20|}\right]^{\frac{1}{m}}$$

for cylinders $\mathfrak{Q} \equiv \mathfrak{Q}(z_0) \subset \Omega_T$, where

$$m := p-1+\frac{1}{\vartheta-1} > p-1.$$

Now, after fixing two radii $R \le r_1 < r_2 \le 2R$, we define

$$\lambda_0^{\frac{1}{p-1}} := \left(\fint_{Q_{r_2}} \big(|Du| + s\big)^{p-1}\, dz\right)^{\frac{1}{p-1}} + \left[M\frac{|\mu|(Q_{r_2})}{|Q_{r_2}|}\right]^{\frac{1}{m}} + 1, \tag{6.1}$$

we take λ such that

$$\lambda > B\lambda_0 \qquad \text{where} \qquad B := \left(\frac{800 r_2}{r_2 - r_1}\right)^N \ge 1 \tag{6.2}$$

and first consider radii satisfying

$$\frac{r_2 - r_1}{40} \le r \le \frac{r_2 - r_1}{2}. \tag{6.3}$$

Note that due to such a bound $Q_r^\lambda(z) \Subset Q_{r_2}$ for any $z \in Q_{r_1}$ and for all r satisfying (6.3). Hence by (6.1) and (6.2), enlarging the domain of integration, we have

$$\begin{aligned} CZ(Q_r^\lambda(z)) \le \left[\frac{|Q_{r_2}|}{|Q^\lambda_{r/20}|}\right]^{\frac{1}{p-1}} \lambda_0^{\frac{1}{p-1}} &< \lambda^{\frac{p-2}{p-1}}\left(\frac{20 r_2}{r}\right)^{\frac{N}{p-1}} \lambda^{\frac{1}{p-1}} B^{-\frac{1}{p-1}} \\ &\le \lambda < 4\lambda. \end{aligned} \tag{6.4}$$

Now we prove by Lebesgue's theorem that the converse inequality holds for cylinders centered in points where the gradient takes big values. More precisely, for $\lambda > 0$ and radii $\gamma \in [R, 2R]$, define the level sets

$$E(\lambda,\gamma) := \Big\{z \in Q_\gamma(z_0) : |Du(z)| + s > \lambda\Big\}. \tag{6.5}$$

Note that the cylinder $Q_\gamma(\mathfrak{z}_0)$ have the same "vertex" as Q_R and Q_{2R}. Take then a point $z \in E(4\lambda, r_1)$ with $\lambda > B\lambda_0$. By Lebesgue's differentiation theorem, for almost every such points it holds

$$\lim_{r\searrow 0} CZ(Q_r^\lambda(z)) \ge \lim_{r\searrow 0}\left(\fint_{Q^\lambda_{r/20}(z)} \big(|Du| + s\big)^{p-1}\, dz\right)^{\frac{1}{p-1}} > 4\lambda.$$

Hence for small radii $0 < r \ll 1$ we have by continuity $CZ(Q_r^\lambda(z)) > 4\lambda$. From this consideration and the fact that (6.4) holds, together with the absolute continuity of the integral, we infer the existence of a maximal radius r_z such that

$$CZ(Q^\lambda_{r_z}) = \left(\fint_{Q^\lambda_{r_z/20}} \big(|Du| + s\big)^{p-1}\, dz\right)^{\frac{1}{p-1}} + \left[M\frac{|\mu|(Q^\lambda_{r_z/20})}{|Q^\lambda_{r_z/20}|}\right]^{\frac{1}{m}} = 4\lambda. \tag{6.6}$$

The word "maximal" refers to the fact that for all radii $\tilde r \in (r_z, (r_2 - r_1)/2]$ the inequality $CZ(Q^\lambda_{\tilde r})(z) < 4\lambda$ holds. In particular for $\tilde r = 20 r_z$ we have

$$\left(\fint_{Q^\lambda_{r_z}} \big(|Du| + s\big)^{p-1}\, dz\right)^{\frac{1}{p-1}} + \left[M\frac{|\mu|(Q^\lambda_{r_z})}{|Q^\lambda_{r_z}|}\right]^{\frac{1}{m}} = CZ\big(Q^\lambda_{20 r_z}\big) < 4\lambda. \tag{6.7}$$

Note also that obviously $r_z < (r_2 - r_1)/40$ and hence $Q^\lambda_{20 r_z}(z) \subset Q_{r_2}$.

A favorable case. Now we single out an intrinsic cylinder $Q^\lambda_{r_z}(z)$, $z \in E(4\lambda, r_1)$, $\lambda > B\lambda_0$, where (6.6) holds. For ease of notation, from now on let denote $Q := Q^\lambda_{r_z}(z)$. Assume that, in addition to condition (6.6), also

$$(2\lambda)^{p-1} \le \fint_{Q/20} \big(|Du| + s\big)^{p-1}\, dz \tag{6.8}$$

holds true. The reason for this additional assumption will become clear in the remainder of the proof.

We introduce the comparison function solution to the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) = 0 & \text{in } Q, \\ v = u & \text{on } \partial_{\mathcal{P}} Q; \end{cases} \tag{6.9}$$

applying Lemma 5.2 with $\epsilon = 4^m/M$ (we can here assume $M \geq 4^m$), we get

$$\left(\fint_Q |Du - Dv|^q \, dz \right)^{1/q} \leq \frac{4^{m\frac{N}{A}} c_1}{M^{\frac{N}{A}}} \lambda = \frac{c_*}{M} \lambda \leq c\,\lambda, \tag{6.10}$$

with c_* and c depending on $n, p, \nu, c_d, \vartheta, q$, for all $q \in \left[1, p-1+\frac{1}{N-1}\right)$. The previous estimate (6.10) holds in particular for the choice $q = p-1$. Hence first we have

$$\begin{aligned} \fint_Q (|Dv|+s)^{p-1} \, dz &\leq 2^{p-2} \left[\fint_Q (|Du|+s)^{p-1} \, dz + \fint_Q |Du - Dv|^{p-1} \, dz \right] \\ &\leq c\,\lambda^{p-1} \end{aligned} \tag{6.11}$$

by (6.7) and (6.10); then

$$\begin{aligned} \fint_{Q/4} (|Dv|+s)^{p-1} \, dz &\geq \frac{1}{2^{p-2} 5^N} \fint_{Q/20} (|Du|+s)^{p-1} \, dz \\ &\qquad\qquad - 4^N \fint_Q |Du - Dv|^{p-1} \, dz \\ &\geq \frac{2}{5^N} \lambda^{p-1} - \frac{4^N c_*^{p-1}}{M^{\frac{N(p-1)}{A}}} \lambda^{p-1}, \end{aligned}$$

since we are assuming (6.8) and we can use (6.10). Now we impose that $M \geq 4^m$ is so large that also

$$\frac{4^N c_*^{p-1}}{M^{\frac{N(p-1)}{A}}} \leq \frac{1}{5^N} \tag{6.12}$$

is satisfied and this, making M depend on $n, p, \nu, c_d, \vartheta$, finally yields, together with (6.11)

$$\left(\frac{\lambda}{c}\right)^{p-1} \leq \fint_{Q/4} (|Dv|+s)^{p-1} \, dz, \qquad \fint_Q (|Dv|+s)^{p-1} \, dz \leq c\,\lambda^{p-1}; \tag{6.13}$$

therefore we can apply Proposition 5.5 which gives

$$\left(\frac{\lambda}{c}\right)^q \leq \fint_{Q/4} (|Dv|+s)^q \, dz, \qquad \fint_{Q/2} (|Dv|+s)^q \, dz \leq c\lambda^q \tag{6.14}$$

for all $q \in [p-1, p]$ and with a constant c depending on $n, p, \nu, L, c_d, \vartheta$. Since in particular (6.14) holds for $q = p$, we are finally in a position to apply Corollary 4.8:

$$\left(\fint_{Q/4} (|Dv|+s)^{p\chi} \, dz \right)^{1/(p\chi)} \leq c \left(\fint_{Q/2} (|Dv|+s)^{p-1} \, dz \right)^{1/(p-1)} \leq c\,\lambda, \tag{6.15}$$

for $\chi \equiv \chi(n, p, \nu, L) > 1$ and the constants c at this point depending only upon $n, p, \nu, L, c_d, \vartheta$. We moreover have, for any $q \in \left[1, p-1+\dfrac{1}{N-1}\right)$, using the previous (6.15) and (6.10)

$$\begin{aligned} \left(\fint_{Q_{R/4}} (|Du|+s)^q \, dz \right)^{1/q} &\leq \left(\fint_{Q_{R/4}} (|Dv|+s)^q \, dz \right)^{1/q} \\ &\quad + \left(\fint_{Q_{R/4}} |Du - Dv|^q \, dz \right)^{1/q} \leq c\,\lambda. \end{aligned} \tag{6.16}$$

Splitting the intrinsic cylinder – a density estimate. We here show when the additional assumption (6.8) can be assumed and, at the same time, we make use of the results of the preceding section. Clearly, by the definition of the CZ operator and by (6.6), one of the following two inequalities must hold true:

$$(2\lambda)^{p-1} \leq \fint_{Q/20} \big(|Du| + s\big)^{p-1}\, dz \qquad \text{or} \qquad (2\lambda)^m \leq M\frac{|\mu|(Q/20)}{|Q/20|}. \tag{6.17}$$

Suppose **we are in the first case**, so we can use the results of the previous section: we split the integral, observing that $Q/20 \equiv Q^{\lambda}_{r_z/20}(z) \subset Q_{r_2}$ and we use Hölder's inequality to infer

$$\begin{aligned}\fint_{Q/20} \big(|Du| + s\big)^{p-1}\, dz &\leq \frac{|Q/20 \setminus E(\lambda, r_2)|}{|Q/20|}\lambda^{p-1} \\ &\qquad + \frac{1}{|Q/20|}\int_{Q/20\cap E(\lambda,r_2)} \big(|Du| + s\big)^{p-1}\, dz \\ &\leq \lambda^{p-1} + c\left(\frac{|Q/20 \cap E(\lambda, r_2)|}{|Q|}\right)^{1-\frac{p-1}{\tilde q}} \left(\fint_{Q/20} \big(|Du| + s\big)^{\tilde q}\, dz\right)^{\frac{p-1}{\tilde q}},\end{aligned} \tag{6.18}$$

for $c \equiv c(n, p)$ and for the exponent

$$\tilde q := p - 1 + \frac{1}{2(N-1)} \in \Big(p - 1, p - 1 + \frac{1}{N-1}\Big). \tag{6.19}$$

Plugging the density estimate (6.16) into (6.18) (the reader might recall now (6.19)) and taking into account the fact that $(6.17)_1$ holds, we infer

$$(2\lambda)^{p-1} \leq \fint_{Q/20} \big(|Du|+s\big)^{p-1}\, dz \leq \lambda^{p-1}\left[1+c\left(\frac{|Q/20 \cap E(\lambda, r_2)|}{|Q|}\right)^{1-\frac{p-1}{\tilde q}}\right];$$

in turn, dividing by λ^{p-1} and reabsorbing the first term, we get

$$\frac{1}{c} \leq \frac{|Q/20 \cap E(\lambda, r_2)|}{|Q|},$$

with $c \equiv c(n, p, \nu, L)$. Merging this estimate with **the second alternative** $(6.17)_2$, we get finally the estimate for $|Q|$ we were looking for:

$$|Q| \leq c\,|Q/20 \cap E(\lambda, r_2)| + c\,\frac{M}{\lambda^m}|\mu|(Q/20). \tag{6.20}$$

We recall that $Q \equiv Q^{\lambda}_{r_z}(z)$.

A covering argument. The following covering argument has been firstly developed in [19], but the "weighted" version we use here is very similar to the improved version which has been described in [1]. We saw in the preceding step of the proof that, once we fix $\lambda > B\lambda_0$, then for every $z \in E(4\lambda, r_1)$ we can find a cylinder $Q^{\lambda}_{r_z}(z)$ such that (6.6) and subsequently (6.20) hold.

Then we consider the collection of all such cylinders $\mathcal{E}_\lambda := \{Q^{\lambda}_{r_z/20}(z)\}_{z\in E(4\lambda,r_1)}$ and, by a Vitali type argument, we extract a countable sub-collection $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda$ such that the 5-times enlarged cylinders cover almost all $E(4\lambda, r_1)$ and the cylinders are pairwise disjoints. I.e., if we denote the cylinders of $\mathcal{F}_\lambda$ by $Q^0_i := Q^{\lambda}_{r_{z_i}/20}(z_i)$, for $i \in \mathcal{I}_\lambda$, being eventually $\mathcal{I}_\lambda = \mathbb{N}$, and with $z_i \in E(4\lambda, r_1)$, we have

$$Q^0_i \cap Q^0_j = \emptyset \quad \text{whenever } i \neq j \quad \text{and} \quad E(4\lambda, r_1) \subset \bigcup_{i\in\mathcal{I}_\lambda} Q^1_i \cup \mathcal{N}, \tag{6.21}$$

with $|\mathcal{N}| = 0$ and where we denoted $Q_i^1 := 5Q_i^0 = Q^\lambda_{r_{z_i}/4}(z_i)$; note that by (6.3) we have the inclusion $Q_i^1 \subset Q_{r_2}$ for all $i \in \mathcal{I}_\lambda$. We now fix $H \geq 4$ to be chosen later and we estimate

$$|E(H\lambda, r_1)| \leq \sum_{i\in\mathcal{I}_\lambda} |Q_i^1 \cap E(H\lambda, r_2)|. \tag{6.22}$$

We split every term in the following way:

$$\begin{aligned} |Q_i^1 \cap E(H\lambda, r_2)| &= \left|\{z \in Q_i^1 : |Du(x)| + s > H\lambda\}\right| \\ &\leq \left|\{z \in Q_i^1 : |Du(x) - Dv_i(x)| > H\lambda/2\}\right| \\ &\quad + \left|\{z \in Q_i^1 : |Dv_i(x)| + s > H\lambda/2\}\right| =: I_i + II_i. \end{aligned} \tag{6.23}$$

Here v_i is the comparison function, solution to (6.9) with $Q \equiv Q_i^2 \equiv Q^\lambda_{r_{z_i}} = 4Q_i^1$. We estimate separately the two pieces: for the first one we use (6.10) and subsequently (6.20) to infer

$$\begin{aligned} I_i &\leq \left(\frac{2}{H\lambda}\right)^{p-1} \int_{Q_i^2} |Du - Dv_i|^{p-1}\, dz \leq \frac{c}{(H\lambda)^{p-1}M} |Q_i^2|\lambda^{p-1} \\ &\leq \frac{c}{H^{p-1}}\left[\frac{|Q_i^0 \cap E(\lambda, r_2)|}{M} + \frac{|\mu|(Q_i^0)}{\lambda^m}\right], \end{aligned} \tag{6.24}$$

since $N(p-1) \geq A = (N-1)(p-1)+1$. On the other hand we use higher integrability (6.15) to get

$$\begin{aligned} II_i &\leq \left(\frac{2}{H\lambda}\right)^{p\chi} \int_{Q_i^1} \big(|Dv_i| + s\big)^{p\chi}\, dz \leq \frac{c}{(H\lambda)^{p\chi}} |Q_i^2|\lambda^{p\chi} \\ &\leq \frac{c}{H^{p\chi}}\left[|Q_i^0 \cap E(\lambda, r_2)| + M\frac{|\mu|(Q_i^0)}{\lambda^m}\right]. \end{aligned} \tag{6.25}$$

Connecting the two estimates (6.24) and (6.25) and plugging the result into (6.23), taking into account that $H \geq 1$, gives

$$|E(H\lambda, r_2) \cap Q_i^1| \leq \left[\frac{c_*}{H^{p-1}M^{p-1}} + \frac{c_*}{H^{p\chi}}\right]|Q_i^0 \cap E(\lambda, r_2)| + c\, M\frac{|\mu|(Q_i^0)}{\lambda^m}.$$

At this point, since the $\{Q_i^0\}$ are disjoint, see (6.21), summing up and multiplying both sides of the previous inequality by $(H\lambda)^m$, see also (6.22), gives

$$\begin{aligned} (H\lambda)^m|E(H\lambda, r_1)| \leq \left[\frac{c_*}{H^{p-1-m}M} + \frac{c_*}{H^{p\chi-m}}\right]\lambda^m|E(\lambda, r_2)| \\ + c\, MH^m|\mu|(Q_{2R}) \end{aligned} \tag{6.26}$$

Finally we perform the choice of M and H: recall that $p-1 < m < p\chi$ by (2.6). First choose H so big that

$$\frac{c_*}{H^{p\chi-m}} \leq \frac{1}{4} \qquad \text{and} \qquad H \geq 4.$$

Then at this point, having fixed $H \equiv H(n, p, \nu, L, c_d, \vartheta)$, choose $M \geq 4^m$, satisfying (6.12) and such that

$$\frac{c_*}{M} \leq \frac{1}{4}H^{p-1-m}.$$

This choice makes also M depend on $n, p, \nu, L, c_d, \vartheta$. Having such choices at hand, after taking the supremum with respect to $\lambda > B\lambda_0$, (6.26) rewrites as

$$\begin{aligned} \sup_{\lambda > HB\lambda_0} \lambda^m|E(\lambda, r_1)| &\leq \frac{1}{2}\sup_{\lambda > B\lambda_0} \lambda^m|E(\lambda, r_2)| + c\,|\mu|(Q_{2R}) \\ &\leq \frac{1}{2}\||Du| + s\|^m_{\mathcal{M}^m(Q_{r_2})} + c\,|\mu|(Q_{2R}) \end{aligned} \tag{6.27}$$

and therefore, recalling that $E(\lambda,\gamma)$ denotes the super-level set (6.5)

$$\||Du|+s\|^m_{\mathcal{M}^m(Q_{r_1})}\le \frac{1}{2}\||Du|+s\|^m_{\mathcal{M}^m(Q_{r_2})}+c\left[B\lambda_0\right]^m R^N+c\,|\mu|(Q_{2R})$$

for all $R\le r_1<r_2\le 2R$, since $B\lambda_0\ge 1$. We now, recalling the definitions of λ_0 and B, apply Lemma 3.1 with $\phi(r):=\||Du|+s\|^m_{\mathcal{M}^m(Q_r)}$, $\mathcal{A}:=c\,|\mu|(Q_{2R})$,

$$\mathcal{B}=c\,R^{N(m+1)}\left(\left[\fint_{Q_{2R}}(|Du|+s)^{p-1}\,dz\right]^m+\left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|}\right]^{p-1}+1\right)$$

and $\beta=Nm$. Note that this is possible since we are dealing with approximate energy solutions; since $Du\in L^{p\chi}_{\rm loc}(\Omega_T)$, we have $\||Du|+s\|_{\mathcal{M}^m(Q_{2R})}<\infty$ for $m<p\chi$. This yields, using also Young's inequality with conjugate exponents $p-1$ and $(p-1)/(p-2)$

$$\||Du|+s\|^m_{\mathcal{M}^m(Q_R)}\le c\left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|^{\frac{p-2}{p-1}}}\right]^{p-1} +c\,R^N+c\,R^N\left[\fint_{Q_{2R}}(|Du|+s)^{p-1}\,dz\right]^m$$

which finally gives (2.5).

7. More regular vector fields

In this section we show in detail what we meant when we mentioned "more regular vector fields" and we show how to modify the proof to match the different assumptions. We here can consider vector field of the form $a(x,t,\xi)=b(x)\tilde a(t,\xi)$ with $\tilde a(\cdot)$ a Carathéodory map differentiable with respect to the variable ξ, with $(t,\xi)\mapsto\partial_\xi\tilde a(t,\xi)$ Carathéodory regular and satisfying

$$\begin{cases}\langle\partial_\xi\tilde a(t,\xi)\tilde\xi,\tilde\xi\rangle\ge\sqrt{\nu}\big(s^2+|\xi|^2\big)^{\frac{p-2}{2}}|\tilde\xi|^2,\\ |a(t,\xi)|+|\partial_\xi a(t,\xi)|\big(s^2+|\xi|^2\big)^{\frac12}\le\sqrt{L}\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}},\end{cases}\tag{7.1}$$

for all $t\in(-T,0)$, $\xi,\tilde\xi\in\mathbb{R}^n$, $p\ge 2$ and with ν,L,s as in (2.1). We moreover suppose $b:\Omega\to\mathbb{R}$ bounded and VMO regular, i.e. $\sqrt{\nu}\le b(\cdot)\le\sqrt{L}$ and

$$\lim_{R\searrow 0}\omega(R)=0,\quad\text{where}\quad \omega(R):=\sup_B\fint_B|b-(b)_B|\,dx,\tag{7.2}$$

where the supremum is taken with respect to all the balls $B=B_\rho(x_0)\Subset\Omega$ with $0<\rho\le R$.

We could also consider the case where the dependence with respect to the x variable is continuous, i.e. vector fields $a(x,t,\xi)$ satisfying

$$\begin{cases}\langle\partial_\xi a(x,t,\xi)\tilde\xi,\tilde\xi\rangle\ge\nu\big(s^2+|\xi|^2\big)^{\frac{p-2}{2}}|\tilde\xi|^2,\\ |a(x,t,\xi)|+|\partial_\xi a(x,t,\xi)|\big(s^2+|\xi|^2\big)^{\frac12}\le L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}},\\ |a(x,t,\xi)-a(x_0,t,\xi)|\le L\tilde\omega(|x-x_0|)\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}},\end{cases}\tag{7.3}$$

for all $x,x_0\in\Omega,t\in(-T,0),\xi,\tilde\xi\in\mathbb{R}^n$, with $p\ge 2$ and ν,L,s as above. We suppose $\tilde\omega:[0,\infty)\to[0,1)$ a concave modulus of continuity such that $\lim_{R\searrow 0}\tilde\omega(R)=0$.

In both these cases higher integrability holds for every $\chi>1$, see [5, Theorem 5.7], with the constant appearing on the right-hand side depending critically upon χ. Therefore an argument similar to the one carried in Section 4 can be performed, in order to get that Corollary 4.8 holds for every $\chi>1$ and with the constant depending also on χ. Now the only different point in Section 6 is that in (6.25) now we can choose, given $\vartheta > 1$, $\chi \equiv \chi(p, \vartheta)$ such that $p\chi = m + 1$; this reflects in the critical dependence of the constant upon ϑ, as $\vartheta \to 1$. The same Theorems justify the reabsorption after (6.27): since the data for the approximating problems are regular, then the energy solutions u_k, under assumptions (7.1)-(7.2) or (7.3), are as integrable as needed.

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