STABLE SOLUTIONS OF ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. This paper is devoted to the study of rigidity properties for special solutions of nonlinear elliptic partial differential equations on smooth, boundaryless Riemannian manifolds. As far as stable solutions are concerned, we derive a new weighted Poincaré inequality which allows to prove Liouville type results and the flatness of the level sets of the solution in dimension 2, under suitable geometric assumptions on the ambient manifold.

Contents

Notation 1
1. Main results 3
2. A Preliminary result 5
3. A geometric inequality 6
4. Flatness lemmata 6
5. Proof of Theorem 1 8
6. Proof of Theorem 2 8
7. Proof of Theorem 3 9
8. A useful cutoff 10
9. Proof of Theorem 4 11
10. Proof of Theorem 5 11
References 11

Notation

Throughout this paper, M will denote a complete, connected, smooth, n-dimensional, manifold without boundary, endowed with a smooth Riemannian metric $g = \{g_{ij}\}$.

As customary, we consider the volume term induced by g, that is, in local coordinates,

$$dV_g = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n, \tag{1}$$

where $\{dx^1, \ldots, dx^n\}$ is the basis of 1-forms dual to the vector basis $\{\partial_i, \ldots, \partial_n\}$, and $|g| = \det(g_{ij}) \geqslant 0$.

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We denote by $\mathrm{div}_g X$ the divergence of a smooth vector field X on M, that is, in local coordinates

$$\mathrm{div}_g X = \frac{1}{\sqrt{|g|}}\partial_i\Big(\sqrt{|g|}X^i\Big),$$

where the Einstein summation convention is understood.

We also denote by ∇_g the Riemannian gradient and by Δ_g the Laplace-Beltrami operator, that is, in local coordinates,

$$(\nabla_g\phi)^i = g^{ij}\partial_j\phi$$

and

$$\Delta_g\phi = \mathrm{div}_g(\nabla_g\phi) = \frac{1}{\sqrt{|g|}}\partial_i\Big(\sqrt{|g|}g^{ij}\partial_j\phi\Big), \tag{2}$$

for any smooth function $\phi : M \to \mathbb{R}$.

Due to this divergence structure (see, for example, page 184 of [GHL90]), we have that

$$\int_M \phi\Delta_g\psi\, dV_g = -\int_M \langle\nabla_g\phi, \nabla_g\psi\rangle\, dV_g, \tag{3}$$

for any smooth ϕ, $\psi : M \to \mathbb{R}$, with either ϕ or ψ compactly supported, where $\langle\cdot,\cdot\rangle$ is the scalar product induced by g (no confusion should arise with the standard Euclidean dot product).

In fact, by approximation, we have that (3) also holds when ϕ is compactly supported and Lipschitz continuous with respect to the metric structure induced by g.

Given a vector field X, we also denote

$$|X| = \sqrt{\langle X, X\rangle}.$$

Also (see, for instance Definition 3.3.5 in [Jos98]), it is customary to define the Hessian of a smooth function ϕ as the symmetric 2-tensor given in a local patch by

$$(H_\phi)_{ij} = \partial^2_{ij}\phi - \Gamma^k_{ij}\partial_k\phi,$$

where Γ^k_{ij} are the Christoffel symbols, namely

$$\Gamma^k_{ij} = \frac{1}{2}g^{hk}\left(\partial_i g_{hj} + \partial_j g_{ih} - \partial_h g_{ij}\right).$$

Given a tensor A, we define its norm by $|A| = \sqrt{AA^*}$, where A^* is the adjoint.

The above quantities are related to the Ricci tensor Ric_g via the Bochner-Weitzenböck formula (see, for instance, [BGM71, Wan05] and references therein):

$$\frac{1}{2}\Delta_g|\nabla_g\phi|^2 = |H_\phi|^2 + \langle\nabla_g\Delta_g\phi, \nabla_g\phi\rangle + \mathrm{Ric}_g(\nabla_g\phi, \nabla_g\phi). \tag{4}$$

We say that M is parabolic if for any $p \in M$ there exists a precompact neighborhood U_p of p in M such that for any $\epsilon > 0$ there exists $\phi_\epsilon \in C^\infty_0(M)$ for which $\phi_\epsilon(q) = 1$ for any $q \in U_p$ and

$$\int_M |\nabla\phi_\epsilon|^2\, dV_g \leqslant \epsilon. \tag{5}$$

We refer to [Roy52, LS84, GT99] for further comments on parabolicity.

During the course of the paper, we will often use normal coordinates at some fixed point $p_o \in M$ (see, for example, page 93 of [GHL90]); that is we suppose that

$$g_{ij}(p_o) = \delta_{ij}, \qquad \partial_k g_{ij}(p_o) = 0, \qquad \text{and} \qquad \Gamma^i_{jk}(p_o) = 0. \tag{6}$$

This paper will deal with solutions $u \in C^3(M)$ of

$$-\Delta_g u = f(u), \tag{7}$$

where $f \in C^1(\mathbb{R})$.
We say that a solution u is stable if

$$\int_M |\nabla_g \xi|^2 - f'(u)\xi^2 \, dV_g \geqslant 0 \tag{8}$$

for every $\xi \in C_0^\infty(M)$.
Such a stability condition is customary in the calculus of variations (see, for example, [MP78, FCS80, AAC01]), and it states that the second variation of the (possibly formal) energy functional associated to (7) is nonnegative (for instance, local minima of the energy are stable solutions).

1. Main results

We give the following Liouville type and flatness results:

Theorem 1. *Let M be a connected Riemannian manifold.*
Let u be a stable solution of (7). Suppose that

- *either M is compact*
- *or M is complete and parabolic, and $|\nabla_g u| \in L^\infty(M)$.*

Assume also that the Ricci curvature is nonnegative and that Ric_g does not vanish identically. Then u is constant.

Note that the conclusion of Theorem 1 is sharp for parabolic manifolds. Indeed, $\mathbb{R}^2$ endowed with its usual flat metric is parabolic (with identically zero Ricci tensor). The function

$$u(x_1, x_2) = \tanh\left(\frac{x_1}{\sqrt{2}}\right) \tag{9}$$

is a stable non-constant solution of the two-dimensional Allen-Cahn equation, namely

$$-\Delta u = u - u^3.$$

On the other hand, for compact manifolds, Theorem 1 can be sharpened, since the rigidity result holds also when Ric_g vanishes identically, as next result points out:

Theorem 2. *Let M be a compact, connected Riemannian manifold and suppose that Ric_g vanishes identically. Let u be a stable solution of (7). Then u is constant.*

The compact case was also considered in the pioneer work of [Jim84]. The example in (9) also motivates the following result, which provides a rigidity property for stable solutions of (7) when $n = 2$.

Theorem 3. *Let M be a complete, connected Riemannian surface (that is, a complete, connected Riemannian manifold with $\dim M = 2$).*
Let u be a stable solution of (7), with $|\nabla_g u| \in L^\infty(M)$. Assume also that Ric_g vanishes identically. Then, any connected component of the level set of u on which $\nabla_g u$ does not vanish is a geodesic.

Of course, as well known, in dimension $n = 2$, Ricci flat surfaces are just surfaces with zero Gaussian curvature, thence, in Theorem 3, the assumption that Ric_g vanishes identically may be equivalently stated by requiring the Gaussian curvature to vanish identically.

Also, Theorem 3 does not hold in high dimensions $n \geqslant 9$, as shown in [dPKW08] for the Allen-Cahn equation in $\mathbb{R}^n$ endowed with its standard flat metric. More precisely, in $\mathbb{R}^9$ (with flat metric), one can construct monotone (hence stable, see Corollary 4.3 in [AAC01]) solutions whose level sets are not totally geodesic. This latter fact suggests that the parabolicity assumption in Theorem 3 (which is hidden in the two-dimensional character of M) seems to be necessary to obtain rigidity results on stable solutions of equation (7).

On the other hand, for rigidity results in negative curvature manifolds, we would like to recall [BM09], where the hyperbolic space was considered.

The proofs of our main results are based on a geometric formula, which will be given in Theorem 7 below, and which can be considered as a weighted Poincaré inequality.

The use of such a type of formula in the Euclidean setting was started in [SZ98a, SZ98b] and its importance for symmetry results was explained in [Far02]. Further applications to PDEs have been given in [FSV08, SV09, FV09, FV10, FSV10].

We now give two additional results in the spirit of Theorem 1, under a sign assumption on the nonlinearity and on the growth of the volume of the geodesic balls.

For this, we denote $\mathcal{B}_R$ the (open) geodesic ball of radius $R > 0$, centered at a given point of M.

We denote by $\mathcal{V}_R$ the volume of $\mathcal{B}_R$, computed with respect to the volume element dV_g in (1).

We obtain the following results:

Theorem 4. *Let M be a complete, connected Riemannian manifold and let u be a bounded stable solution of* (7)*.*

Suppose that

$$f(r) \geqslant 0 \text{ for any } r \in \mathbb{R} \tag{10}$$

and that

$$\liminf_{R\to+\infty} R^{-4}\mathcal{V}_R = 0. \tag{11}$$

Assume also that the Ricci curvature of M is nonnegative and that Ric_g *does not vanish identically.*

Then u is constant.

Theorem 5. *Let M be a complete, connected Riemannian manifold and let u be a stable solution of* (7)*.*

Suppose that

$$\liminf_{R\to+\infty} R^{-2}\mathcal{V}_R\Big(\sup_{\mathcal{B}_R}|\nabla_g u|\Big)^2 = 0. \tag{12}$$

Assume also that the Ricci curvature of M is nonnegative and that Ric_g *does not vanish identically.*

Then u is constant.

We recall that, for complete, connected, n-dimensional Riemannian manifolds with nonnegative Ricci curvature, one controls $\mathcal{V}_R$ with R^n (see [BC64]). Therefore, (11) always holds when $n \leqslant 3$.

The paper is organized as follows. In § 2, we make an observation about the positivity of an interesting geometric quantity. In § 3 we discuss the weighted Poincaré inequality which will be the keystone of the techniques presented here. From that, useful flatness results are obtained in § 4.

The proofs of the main results are given in § 5–10.

2. A Preliminary result

From now on, M will always denote a complete, connected Riemannian manifold.

Lemma 6. *For any smooth $\phi : M \to \mathbb{R}$, we have that*

$$|H_\phi|^2 \geqslant \big|\nabla_g |\nabla_g \phi|\big|^2 \qquad \textit{almost everywhere.} \tag{13}$$

Also, equality holds at $p \in M \cap \{\nabla_g \phi \neq 0\}$ if and only if for any $k = 1, \dots, n$ there exists $\kappa^k : M \to \mathbb{R}$ such that

$$\nabla_g \big(\nabla_g \phi\big)^k (p) = \kappa^k(p) \nabla_g \phi(p). \tag{14}$$

Proof. From Stampacchia's Theorem (see, for instance, Theorem 6.19 in [LL97]), we know that $\nabla_g |\nabla_g \phi| = 0$ on $\{\nabla_g \phi = 0\}$ up to a null-measure set.

Therefore, we can now concentrate on points in $M \cap \{\nabla_g \phi \neq 0\}$.

Fix $p \in M \cap \{\nabla_g \phi \neq 0\}$, with $\nabla_g \phi(p) \neq 0$. Recalling (6), we use normal coordinates at p. Therefore $(H_\phi)_{ij}(p) = \partial^2_{ij} \phi(p)$ and so

$$|H_\phi|^2(p) = \sum_{1 \leqslant i,j \leqslant n} \Big(\partial^2_{ij} \phi(p)\Big).$$

Analogously, we have

$$|\nabla_g \psi\big|(p) = |\nabla \psi(p)|,$$

for any $\psi : M \to \mathbb{R}$ smooth in the vicinity of p. As a consequence, taking $\psi = |\nabla_g \phi|$, one gets

$$\begin{aligned} \big|\nabla_g |\nabla_g \phi|\big|(p) &= \big|\nabla |\nabla_g \phi|\big|(p) \\ &= \left| \frac{\nabla_g \phi}{|\nabla_g \phi|} \cdot \nabla(\nabla_g \phi) \right| (p) = \left| \frac{\nabla \phi}{|\nabla \phi|} \cdot \nabla(\nabla_g \phi) \right| (p). \end{aligned}$$

Since, by (6),

$$\partial_i \big(\nabla_g \phi\big)^h (p) = \partial_i \big(g^{hk} \partial_k \phi\big)(p) = \delta^{hk} \partial^2_{ik} \phi(p),$$

we thus obtain

$$\nabla \phi \cdot \nabla \big(\nabla_g \phi\big)^h (p) = \sum_{1 \leqslant i \leqslant n} \partial_i \phi \partial^2_{ih} \phi(p).$$

Accordingly,

$$\begin{aligned} \big|\nabla_g |\nabla_g \phi|\big|^2(p) &= \frac{1}{|\nabla \phi|^2} \sum_{1 \leqslant h \leqslant n} \Big(\partial_i \phi \partial^2_{ih} \phi\Big)^2 = \frac{1}{|\nabla \phi|^2} \sum_{1 \leqslant h \leqslant n} \Big(\nabla \phi \cdot \nabla(\partial_h \phi)\Big)^2 \\ &\leqslant \sum_{1 \leqslant h \leqslant n} \Big|\nabla(\partial_h \phi)\Big|^2 = |H_\phi|^2(p), \end{aligned}$$

with equality if and only if $\nabla \phi$ and $\nabla(\partial_k \phi)$ are parallel, for any $k = 1, \dots n$. This gives the desired result. □

3. A geometric inequality

Theorem 7. *Let u be a stable solution of* (7)*. Then,*

$$\int_M \Big(\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_u|^2 - \big|\nabla_g|\nabla_g u|\big|^2\Big)\phi^2\, dV_g \leqslant \int_M |\nabla_g u|^2 |\nabla_g \phi|^2\, dV_g, \tag{15}$$

for any $\phi \in C_0^\infty(M)$.

Proof. We take $\phi \in C_0^\infty(M)$ and $\xi = |\nabla_g u|\phi$ in (8) (note that this choice is possible in the light of a density argument). We thus obtain

$$\begin{aligned}\int_M f'(u)|\nabla_g u|^2\phi^2\, dV_g &\leqslant \int_M \big|\nabla_g|\nabla_g u|\big|^2\phi^2 + |\nabla_g u|^2|\nabla_g\phi|^2 + 2\phi|\nabla_g u|\langle\nabla_g\phi, \nabla_g|\nabla_g u|\rangle\, dV_g \\ &= \int_M \big|\nabla_g|\nabla_g u|\big|^2\phi^2 + |\nabla_g u|^2|\nabla_g\phi|^2 + \frac{1}{2}\langle\nabla_g\phi^2, \nabla_g|\nabla_g u|^2\rangle\, dV_g.\end{aligned}$$

Therefore, recalling (3) and (4),

$$\begin{aligned}\int_M f'(u)|\nabla_g u|^2\phi^2\, dV_g &\leqslant \int_M \big|\nabla_g|\nabla_g u|\big|^2\phi^2 + |\nabla_g u|^2|\nabla_g\phi|^2 - \frac{1}{2}\phi^2\Delta_g|\nabla_g u|^2\, dV_g \\ &= \int_M \big|\nabla_g|\nabla_g u|\big|^2\phi^2 + |\nabla_g u|^2|\nabla_g\phi|^2 \\ &\qquad -\phi^2\Big(|H_u|^2 + \langle\nabla_g\Delta_g u, \nabla_g u\rangle + \mathrm{Ric}_g(\nabla_g u, \nabla_g u)\Big)\, dV_g.\end{aligned}$$

Since, by differentiating (7), we have that

$$-\nabla_g\Delta_g u = f'(u)\nabla_g u,$$

we obtain

$$0 \leqslant \int_M \big|\nabla_g|\nabla_g u|\big|^2\phi^2 + |\nabla_g u|^2|\nabla_g\phi|^2 - \phi^2\Big(|H_u|^2 + \mathrm{Ric}_g(\nabla_g u, \nabla_g u)\Big)\, dV_g,$$

which gives (15). □

4. Flatness lemmata

Lemma 8. *Let u be a smooth function on M.*
Assume that

$$\text{the Ricci curvature is nonnegative.} \tag{16}$$

Suppose also that for any $p \in M$ there exists a neighborhood V_p of p in M such that

$$\int_{V_p} \Big(\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_u|^2 - \big|\nabla_g|\nabla_g u|\big|^2\Big)\, dV_g \leqslant 0. \tag{17}$$

Then,

$$|H_u|^2(p) = \big|\nabla_g|\nabla_g u|\big|^2(p) \qquad \text{for any } p \in M \cap \{\nabla_g u \neq 0\}, \tag{18}$$

and

$$\mathrm{Ric}_g(\nabla_g u, \nabla_g u)(p) = 0 \qquad \text{for any } p \in M. \tag{19}$$

Furthermore, for any $k = 1, \dots, n$ there exist $\kappa^k : M \to \mathbb{R}$ such that

$$\nabla_g\big(\nabla_g u\big)^k(p) = \kappa^k(p)\nabla_g u(p) \qquad \text{for any } p \in M \cap \{\nabla_g u \neq 0\}. \tag{20}$$

Proof. We fix $p \in M$ and we show that (19) holds at p, and that (18) and (20) hold at p too if $\nabla_g u(p) \neq 0$.

From (13), (16) and (17), we have that

$$\int_{V_p} \mathrm{Ric}_g(\nabla_g u, \nabla_g u)\, dV_g = 0 = \int_{V_p} |H_u|^2 - \big|\nabla_g |\nabla_g u|\big|^2\, dV_g.$$

Accordingly,

$$\mathrm{Ric}_g(\nabla_g u, \nabla_g u) = 0 = |H_u|^2 - \big|\nabla_g |\nabla_g u|\big|^2 \quad \text{almost everywhere in } V_p. \tag{21}$$

Since Ric_g is continuous, (21) implies that $\mathrm{Ric}_g(\nabla_g u, \nabla_g u) = 0$ everywhere in V_p and so (19) holds at p.

In addition, if $p \in \{\nabla_g u \neq 0\}$, we have that the map $|H_u|^2 - \big|\nabla_g |\nabla_g u|\big|^2$ is continuous in the vicinity of p, and so (21) says that (18) holds at p in this case.

Finally, (18) and (14) give (20). □

Lemma 9. *Let u be a stable solution of* (7) *and let the Ricci curvature of M be nonnegative. Suppose that*

- *either M is compact*
- *or M is complete and parabolic, and $|\nabla_g u| \in L^\infty(M)$.*

Then, (18), (19) *and* (20) *hold true.*

Proof. We claim that there exists a neighborhood V_p of p in M such that (17) holds.

Indeed, if M is compact we can use Theorem 7, by taking $\phi = 1$ in (15), obtaining

$$\int_M \mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_u|^2 - \big|\nabla_g |\nabla_g u|\big|^2\, dV_g \leqslant 0.$$

This gives (17), with $V_p = M$.

If, on the other hand, M is parabolic and $|\nabla_g u|$ is bounded, we fix $p \in M$, we recall Theorem 7 once more, we take ϕ_ϵ as in (5) and we plug it in (15): we recall (13) and so we obtain

$$\begin{aligned}
&\int_{U_p} \Big(\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_u|^2 - \big|\nabla_g |\nabla_g u|\big|^2\Big)\, dV_g \\
\leqslant\ & \int_M \Big(\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_u|^2 - \big|\nabla_g |\nabla_g u|\big|^2\Big)\phi_\epsilon^2\, dV_g \\
\leqslant\ & \int_M |\nabla_g u|^2 |\nabla_g \phi_\epsilon|^2\, dV_g \\
\leqslant\ & \|\nabla_g u\|^2_{L^\infty(M)} \int_M |\nabla_g \phi_\epsilon|^2\, dV_g \\
\leqslant\ & \|\nabla_g u\|^2_{L^\infty(M)}\,\epsilon.
\end{aligned}$$

By taking ϵ arbitrarily small, we obtain (17), with $V_p = U_p$ in this case.

The desired result then follows from Lemma 8. □

Lemma 10. *Suppose that the Ricci curvature of M is nonnegative and that* Ric_g *does not vanish identically.*

Let u be a solution of (7), *with*

$$\mathrm{Ric}_g(\nabla_g u, \nabla_g u)(p) = 0 \text{ for any } p \in M. \tag{22}$$

Then, u is constant.

Proof. Since Ric_g is nonnegative definite and it does not vanish identically, we have that Ric_g is positive definite in a suitable open subset of M.

Consequently, (22) implies that $\nabla_g u(p) = 0$ for p in a suitable open subset of M.

Thence, u is constant on such a subset. By Unique Continuation Principle (see Theorem 1.8 of [Kaz88]), we have that u is constant on M. □

5. Proof of Theorem 1

From Lemma 9, we have that (19) holds true.

This makes it possible to use Lemma 10 and thus complete the proof of Theorem 1.

6. Proof of Theorem 2

Since M is compact, we have that u is compactly supported, and so we may apply the Bochner-Weitzenböck formula in (4) with $\phi := u$. Since Ric_g vanishes identically, this gives that

$$\frac{1}{2}\Delta_g |\nabla_g u|^2 = |H_u|^2 + \langle \nabla_g \Delta_g u, \nabla_g u \rangle.$$

By integrating over M and recalling (2), (7) and (13), we obtain

$$\begin{aligned} & \int_M \big|\nabla_g |\nabla_g u|\big|^2 - f'(u)\,|\nabla_g u|^2\,dV_g \\ \leqslant\ & \int_M |H_u|^2 + \langle \nabla_g \Delta_g u, \nabla_g u\rangle\,dV_g \\ =\ & \frac{1}{2}\int_M \Delta_g u\,dV_g \\ =\ & \frac{1}{2}\int_M \mathrm{div}_g(\nabla_g \phi)\,dV_g, \end{aligned}$$

which is zero, by the Divergence Theorem, since $\partial M = \varnothing$.

This, and the fact that u is stable, gives that

$$\begin{aligned} & \int_M \big|\nabla_g |\nabla_g u|\big|^2 - f'(u)\,|\nabla_g u|^2\,dV_g \\ & \qquad \leqslant 0 \leqslant \int_M |\nabla_g \phi|^2 - f'(u)\,\phi^2\,dV_g \end{aligned}$$

for any $\phi \in C^\infty(M)$ – and, by density, for any $\phi \in H^1(M)$. That is, $v := |\nabla_g u|$ minimizes the functional

$$\mathcal{S}(\phi) := \int_M |\nabla_g \phi|^2 - f'(u)\,\phi^2\,dV_g$$

for $\phi \in H^1(M)$. Accordingly, $\Delta_g v + f'(u)v = 0$. On the other hand, by the compactness of M, there esists $p_o \in M$ for which

$$u(p_o) = \max_{p \in M} u(p),$$

hence $\nabla_g u(p_o) = 0$, and so $v(p_o) = 0 \leqslant v(p)$ for any $p \in M$. Therefore, by the Strong Maximum Principle, v vanishes identically, that is $\nabla_g u$ vanishes identically, which proves Theorem 2.

7. Proof of Theorem 3

First of all, we observe that M has nonnegative Gaussian curvature, since it has nonnegative Ricci curvature and $\dim M = 2$.

Therefore, from[1] Theorem 15 of [Hub57] (see also [CY75, Var81]), we get that

$$M \text{ is parabolic.} \tag{23}$$

Take any connected component $\mathcal{C}$ of $\{u = c\} \cap \{\nabla_g u \neq 0\}$. Then, $\mathcal{C}$ is a smooth curve.

Thus, we take $\gamma : \mathbb{R} \to M$ to be $\mathcal{C}$ traveled with unit speed (with respect to the metric g), that is

$$|\dot\gamma|^2 = 1. \tag{24}$$

With this notation, Theorem 3 is proved once we show that

$$\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0. \tag{25}$$

To prove (25), we take any $t_o \in \mathbb{R}$ and we show that (25) holds at t_o. For this, we choose a normal coordinate frame at $p_o = \gamma(t_o)$.

Then, from (24),

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\Big(g_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\Big) \\ &= \frac{1}{2}\partial_k g_{ij}(\gamma(t))\dot\gamma^k(t)\dot\gamma^i(t)\dot\gamma^j(t) + g_{ij}(\gamma(t))\dot\gamma^i(t)\ddot\gamma^j(t). \end{aligned}$$

Consequently, from (6), we have

$$0 = \dot\gamma(t_o)\cdot\ddot\gamma(t_o). \tag{26}$$

Moreover, since $u(\gamma(t)) = c$, we also have

$$0 = \frac{d}{dt}u(\gamma(t)) = \partial_i u(\gamma(t))\dot\gamma^i(t). \tag{27}$$

By differentiating (27) once more time, one gets

$$0 = \frac{d}{dt}\Big(\partial_i u(\gamma(t))\dot\gamma^i(t)\Big) = \partial_{ij}u(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) + \partial_i u(\gamma(t))\ddot\gamma^i(t). \tag{28}$$

We now observe that (23) and Lemma 9 make it possible to use (20) here.

Accordingly, from (20) and (6) we obtain, for any $j = 1, \dots, n$,

$$\partial_j \nabla u(p_o) = \kappa_j(p_o)\nabla u(p_o),$$

for some $\kappa_j(p_o) \in \mathbb{R}$.

This and (28) give that

$$\begin{aligned} 0 &= \partial_{ij}u(p_o)\dot\gamma^i(t_o)\dot\gamma^j(t_o) + \partial_i u(p_o)\ddot\gamma^i(t_o) \\ &= \kappa_j(p_o)\partial_i u(p_o)\dot\gamma^i(t_o)\dot\gamma^j(t_o) + \partial_i u(p_o)\ddot\gamma^i(t_o) \\ &= \Big(\kappa_j(p_o)\dot\gamma^j(t_o)\Big)\Big(\partial_i u(p_o)\dot\gamma^i(t_o)\Big) + \partial_i u(p_o)\ddot\gamma^i(t_o). \end{aligned}$$

Hence, employing (27),

$$0 = \partial_i u(p_o)\ddot\gamma^i(t_o). \tag{29}$$

[1] We remark that we are using here in a crucial way the fact that M has nonnegative Gauss curvature to obtain (23), since there are examples of hyperbolic Riemannian surfaces (or, even, hyperbolic two-dimensional graphs): see [Oss56a, Oss56b, Mil77].

By (26) and (29), we see that $\ddot\gamma(t_o)$ is orthogonal (in the Euclidean sense) both to $\dot\gamma(t_o)$, which is tangent to $\{u=c\}$ at p_o, and to $\nabla u(p_o)$, which is normal to $\{u=c\}$ at p_o.
Therefore, $\ddot\gamma(t_o)=0$.
As a consequence, from (6),

$$\ddot\gamma^k(t_o)+\Gamma^k_{ij}(p_o)\dot\gamma^i(t_o)\dot\gamma^j(t_o)=\ddot\gamma^k(t_o)+0=0.$$

This proves (25) at the generic time $t=t_o$ and it thus completes the proof of Theorem 3.

8. A useful cutoff

For the proof of Theorems 4 and 5, it is useful to introduce the following cutoff function.
Let d_g be the geodesic distance. Then $\mathcal{B}_R=\{p\in M \text{ s.t. } d_g(p)<R\}$.
Fix $\tau\in C^\infty_0([-2,2],[0,1])$ with $\tau(t)=1$ for any $t\in[-1,1]$.
Given $R>0$, for any $p\in M$, we define

$$\tau_R(p)=\tau\left(\frac{d_g(p)}{R}\right). \tag{30}$$

Then,

$$\begin{aligned}&\tau_R(p)=1 \text{ for any } p\in\mathcal{B}_R,\ \tau_R(p)=0 \text{ for any } p\in M\setminus\mathcal{B}_{2R}, \text{ and}\\ &|\nabla_g\tau_R(p)|\leqslant\frac{C_o}{R}\chi_{\mathcal{B}_{2R}\setminus\mathcal{B}_R}(p)\qquad \text{for any } p\in M.\end{aligned} \tag{31}$$

Then, we have:

Lemma 11. *Suppose that the Ricci curvature of M is nonnegative and that Ric_g does not vanish identically.*
Let u be a stable solution of (7) such that

$$\liminf_{R\to+\infty}\int_M|\nabla_g u|^2|\nabla_g\tau_R|^2\,dV_g=0. \tag{32}$$

Then u is constant.

Proof. From (13), (31), (32) and Theorem 7,

$$\begin{aligned}&\int_M\Big(\mathrm{Ric}_g(\nabla_g u,\nabla_g u)+|H_u|^2-\big|\nabla_g|\nabla_g u|\big|^2\Big)\,dV_g\\
=\;&\liminf_{R\to+\infty}\int_{\mathcal{B}_R}\Big(\mathrm{Ric}_g(\nabla_g u,\nabla_g u)+|H_u|^2-\big|\nabla_g|\nabla_g u|\big|^2\Big)\,dV_g\\
\leqslant\;&\liminf_{R\to+\infty}\int_M\Big(\mathrm{Ric}_g(\nabla_g u,\nabla_g u)+|H_u|^2-\big|\nabla_g|\nabla_g u|\big|^2\Big)\tau_R^2\,dV_g\\
\leqslant\;&\liminf_{R\to+\infty}\int_M|\nabla_g u|^2|\nabla_g\tau_R|^2\,dV_g\\
=\;&0.\end{aligned}$$

Hence, (17) holds true with $V_p=M$.
Therefore, by Lemma 8, $\mathrm{Ric}_g(\nabla_g u,\nabla_g u)$ vanishes identically on M.
Hence, the desired result follows from Lemma 10. □

9. Proof of Theorem 4

Let m_-, $m_+ \in \mathbb{R}$ be such that $m_- \leqslant u(p) \leqslant m_+$ for any $p \in M$. Let also τ_R as in (30). Making use of (3) and (10), we see that

$$\begin{aligned} 0 &\geqslant \int_M f(u)(u-m_+)\tau_R^2\,dV_g \\ &= \int_M \langle \nabla_g u, \nabla_g\big((u-m_+)\tau_R^2\big)\rangle\,dV_g \\ &= \int_{\mathcal{B}_{2R}} |\nabla_g u|^2\tau_R^2\,dV_g + 2\int_{\mathcal{B}_{2R}} \langle \nabla_g u, \nabla_g \tau_R\rangle \tau_R(u-m_+)\,dV_g \\ &\geqslant \int_{\mathcal{B}_{2R}} |\nabla_g u|^2\tau_R^2\,dV_g - 2(m_+-m_-)\int_{\mathcal{B}_{2R}} |\nabla_g u|\,|\nabla_g\tau_R|\,\tau_R\,dV_g. \end{aligned}$$

Therefore, by Cauchy-Schwarz inequality,

$$0 \geqslant \frac{1}{2}\int_{\mathcal{B}_{2R}} |\nabla_g u|^2\tau_R^2\,dV_g - C_\star \int_{\mathcal{B}_{2R}} |\nabla_g \tau_R|^2\,dV_g$$

for a suitable $C_\star > 0$, possibly depending on m_- and m_+, and so, recalling (31),

$$\begin{aligned} \int_{\mathcal{B}_R} |\nabla_g u|^2\,dV_g &\leqslant \int_{\mathcal{B}_{2R}} |\nabla_g u|^2\tau_R^2\,dV_g \\ &\leqslant 2C_\star \int_{\mathcal{B}_{2R}} |\nabla_g\tau_R|^2\,dV_g \leqslant \frac{\bar{C}}{R^2}\mathcal{V}_{2R}, \end{aligned} \tag{33}$$

for some $\bar{C} > 0$ which does not depend on R.

From (11), (31) and (33), we conclude that

$$\liminf_{R\to+\infty}\int_M |\nabla_g u|^2|\nabla_g\tau_R|^2\,dV_g \leqslant \liminf_{R\to+\infty}\frac{C_o^2}{R^2}\int_{\mathcal{B}_{2R}} |\nabla_g u|^2\,dV_g \leqslant \liminf_{R\to+\infty}\frac{C_o^2\bar{C}}{4R^4}\mathcal{V}_{4R} = 0.$$

Then, we use Lemma 11 to end the proof of Theorem 4.

10. Proof of Theorem 5

We take τ_R as in (30). Then, from (12) and (31),

$$\begin{aligned} \liminf_{R\to+\infty}\int_M |\nabla_g u|^2|\nabla_g\tau_R|^2\,dV_g &= \liminf_{R\to+\infty}\int_{\mathcal{B}_{2R}} |\nabla_g u|^2|\nabla_g\tau_R|^2\,dV_g \\ &\leqslant \liminf_{R\to+\infty}\Big(\sup_{\mathcal{B}_{2R}}|\nabla_g u|\Big)^2\frac{C_o^2}{R^2}\int_{\mathcal{B}_{2R}} dV_g \\ &= 0. \end{aligned}$$

Then, the proof of Theorem 5 is ended via Lemma 11.

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