Local and non local continuum limits of Ising type energies for spin systems

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Abstract

We study, through a Γ-convergence procedure, the discrete to continuum limit of Ising type energies of the form

$$F_\varepsilon(u) = -\sum_{i,j} c_{i,j}^\varepsilon u_i u_j,$$

where u is a spin variable defined on a portion of a cubic lattice $\varepsilon\mathbb{Z}^d \cap \Omega$, Ω being a regular bounded open set, and valued in $\{-1, 1\}$. If the constants $c_{i,j}^\varepsilon$ are non negative and satisfies suitable coercivity and decay assumptions, we show that all possible Γ-limits of surface scalings of the functionals F_ε are finite on $BV(\Omega; \{-1, 1\})$ and of the form

$$\int_{S_u} \varphi(x, \nu_u) \, d\mathcal{H}^{d-1}.$$

If such decay assumptions are violated, we show that we may approximate non local functionals of the form

$$\int_{S_u} \varphi(\nu_u) \, d\mathcal{H}^{d-1} + \int_\Omega \int_\Omega K(x, y) g(u(x), u(y)) \, dxdy.$$

We focus on the approximation of two relevant examples: fractional perimeters and Ohta-Kawasaki type energies. Eventually, we provide a general criterion for a ferromagnetic behavior of the energies F_ε even when the constants

$c_{i,j}^{\varepsilon}$ change sign. If such criterion is satisfied, the ground states of F_ε are still the uniform states 1 and -1 and the continuum limit of the scaled energies is an integral surface energy of the form above.

2010 Mathematics Subject Classification: 49J45.

Keywords: Atomistic-to-continuum limit; Γ-convergence; spin systems; surface energies.

1   Introduction

The description and prediction of mesoscopic pattern formation have been and continue to be an object of extensive research both in the physical and in the mathematical literature. A possible approach in pursuing this task, usually adopted in Statistical Mechanics, is the analysis of discrete models, which aims at predicting such patterns starting from discrete systems of particles in interaction. A different but somehow complementary approach is the analysis of continuum models, which very often correspond to a coarse-graining description of the previous ones, which averages or smooths away fine details but is capable to capture some of the main features of the original problems.

Among the discrete models an efficient and celebrated example is given by the Ising model, introduced originally to model ferromagnetism in Statistical Mechanics but then applied successfully in many other contests. In its original form it amounts to consider an hamiltonian of the form

$$F(u) = -\sum_{n.n.} u_i u_j \tag{1.1}$$

which is the energy of a system of interacting spins through their nearest-neighbor (n.n.) bonds. Here the spin variable u is defined on a portion $\mathcal{L}$ of a periodic lattice, say it $\mathbb{Z}^d$, and takes values in $\{-1, 1\}$. By scaling $\mathbb{Z}^d$ by a small parameter $\varepsilon > 0$ and identifying $\mathcal{L}$ with $\varepsilon \mathbb{Z}^d \cap \Omega$, where Ω is an open bounded open set of $\mathbb{R}^d$, it can be proved that the discrete-to continuum limit of suitable power scaling of E, as $\varepsilon \to 0$, resembles that of a Cahn-Hilliard type functional. Namely, it leads in the limit as $\varepsilon \to 0$ to the surface energy $\int_{S_u} |\nu_u|_1 \, d\mathcal{H}^{d-1}$ for $u \in BV(\Omega; \{-1, 1\}$, where S_u is the jump set of u, that is the interface between the $\{u = 1\}$ phase and the $\{u = -1\}$ phase, ν_u is the unit normal to S_u and $|\cdot|_1$ is the l^1-norm in $\mathbb{R}^d$.

Ising type energies are more generally written in the form

$$F(u) = -\sum_{i \neq j \in \mathcal{L}} c_{i,j} u_i u_j. \tag{1.2}$$

Here, according to the sign of the constants $c_{i,j}$, we speak of *ferromagnetic interactions*, corresponding to $c_{i,j} > 0$ and favoring uniform pairs $u_i = u_j$, or of *anti-ferromagnetic interactions*, corresponding to $c_{i,j} < 0$ and preferring instead

alternating pairs $u_i = -u_j$. Many studies have shown that models involving energies of the form (1.2), for a suitable choice of the range of interactions and of the sign of the constants $c_{i,j}$, can be approximated in the continuum by models involving surface energies of the form above, where the limiting interfaces may represent phase or anti-phase boundaries (see [2] for some relevant examples in a variational framework).

A general question is, then, whether an approximation by surface energies can be used. Purpose of this paper is to answer to this question and, more precisely, to determine a large class of energies of the form (1.2) whose discrete-to-continuum limits can be described by surface functionals.

We start our analysis by considering a bulk scaling of the functionals in (1.2) and letting the constant $c_{i,j}$ depend also on the small parameter ε. Therefore we consider energies of the form

$$F_\varepsilon(u) = -\sum_{i \neq j \in \Omega_\varepsilon} \varepsilon^d c_{i,j}^\varepsilon u_i u_j, \qquad (1.3)$$

defined for $u : \varepsilon \mathbb{Z}^d \cap \Omega \to \{-1, 1\}$, where we have used the notation $\Omega_\varepsilon := \mathbb{Z}^d \cap (\varepsilon^{-1}\Omega)$, $u_i := u(\varepsilon i)$. The asymptotic behavior as $\varepsilon \to 0$ of energies of the form (1.3) has been studied in [4] and leads in the continuum limit to a class of integral functionals of the type

$$\int_\Omega f(x, u(x)) \, dx,$$

where the limit variable u represents the *magnetization* of the spin variable u_ε, that is, loosely speaking, $u(x)$ is the average value of u_ε around x. We underline that, with this scaling, the asymptotic analysis gives some insight in the structure of the ground states only when the constants $c_{i,j}^\varepsilon$ change sign, that is there is a competition between ferromagnetic and anti-ferromagnetic interactions.

Let us focus instead on the purely ferromagnetic case, that is when $c_{i,j}^\varepsilon \geq 0$ for all $i, j \in \Omega_\varepsilon$. Note that with this assumption the ground states of F_ε are trivially the constant functions $u \equiv 1$ and $u \equiv -1$. Nevertheless, if boundary conditions or additional constraints are added, minimizers are not trivial and it is interesting to describe their behavior as $\varepsilon \to 0$, that is when the number of indices diverges. To this end, a higher order description of F_ε is needed.

Let, then, $m_\varepsilon := \min F_\varepsilon = F_\varepsilon(\pm 1)$ and consider a surface scaling of F_ε given by

$$E_\varepsilon(u) = \frac{F_\varepsilon(u) - m_\varepsilon}{\varepsilon} = \sum_{i \neq j \in \Omega_\varepsilon} \varepsilon^{d-1} c_{i,j}^\varepsilon (1 - u_i u_j). \qquad (1.4)$$

Let us identify the functions u with their piecewise constant interpolations on the cells of the lattice $\varepsilon \mathbb{Z}^d$. Note that, with this identification, if we split E_ε into the contribution accounting for the interactions between nearest neighbors and between all the other pairs we may rewrite it as

$$E_\varepsilon(u) = \int_{S_u} a_\varepsilon(x) \, d\mathcal{H}^{d-1} + \int_\Omega \int_\Omega K_\varepsilon(x, y)|u(x) - u(y)| \, dxdy + O(1), \qquad (1.5)$$

for suitable functions $a_\varepsilon(\cdot)$ and $K_\varepsilon(\cdot, \cdot)$, where the additional $O(1)$ term is due to the energetic contribution of the interactions near $\partial\Omega$.

The asymptotic behavior as $\varepsilon \to 0$ of the energies E_ε can be described by using the methods of Γ-convergence (see [9, 12]). We make assumptions of two types on the constants $c_{i,j}^\varepsilon$: a coerciveness hypothesis on nearest neighbors, namely that $c_{i,j}^\varepsilon \geq c > 0$ when $|j - 1| = 1$, which ensures that the limit is finite only on $BV(\Omega; \{-1, 1\}$, and a decay assumption as $|j - i| \to +\infty$, that allows to neglect very long-range interactions. We underline that in this case the non-local term on the right-hand side of (1.5) gives, as $\varepsilon \to 0$, only a contribution to the surface energy. Indeed, under these conditions we show that, up to passing to a subsequence, the functionals E_ε Γ-converge to an integral surface functional defined on $BV(\Omega; \{-1, 1\})$ and of the form

$$\int_{S_u} \varphi(x, \nu_u) \, d\mathcal{H}^{d-1} \tag{1.6}$$

(see Theorem 4.2).

To perform our analysis, we apply an abstract method, originally exploited in the homogenization theory for multiple integrals and then adapted to a discrete setting in the continuum approximation, by Γ-convergence, of discrete models in nonlinear elasticity (see [3], [5]). It amounts to apply a localization argument which allows to regard our functionals and their Γ-limits as defined on pairs function-set and then to prove that all the hypotheses of an integral representation result of [8] (see Theorem 2.2) are fulfilled.

In the case of periodic interactions, corresponding to have $c_{i,j}^\varepsilon = c(i, j - i)$, with $c(\cdot, \xi)$ $[0, k)^d$-periodic for some $k \in \mathbb{N}$ and for any $\xi \in \mathbb{Z}^d$, we may derive, as a consequence of our integral representation result, an alternative proof of the homogenization result proved in [11]. Indeed, in this case the whole family of functionals E_ε Γ-converges to an homogenized surface energy of the form

$$\int_{S_u} \varphi_{hom}(\nu_u) \, d\mathcal{H}^{d-1}, \tag{1.7}$$

with φ_{hom} defined by a suitable asymptotic formula (see Theorem 4.7).

As a consequence of our Γ-convergence results, in Section 4.2 we derive the convergence of mimimum problems involving discrete energies as in (1.4) to the corresponding minimum problems involving the limiting energy. We focus on boundary value problems and minimum problems with prescribed volume fraction.

We underline that we limit our analysis to square lattices, but our results can be easily extended to any Bravais lattice or multi-lattice (see also [6] for an extension of these results to stochastic lattices). Moreover, we provide an extension of our analysis to the case of energies accounting for multi-body interactions of the form, for a given $M \in \mathbb{N}$,

$$E_\varepsilon^M(u) = \sum_{i_1, \ldots, i_M \in \Omega_\varepsilon} \varepsilon^{d-1} \psi_\varepsilon(i_1, \ldots, i_M, u_{i_1}, \ldots, u_{i_M}). \tag{1.8}$$

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Here the ferromagnetic behavior of E_ε^M is ensured by the assumption that the potentials ψ_ε are non negative and equal 0 only on the uniform configurations $u_1 = u_2 = \ldots u_M$. Under coercivity and decay assumptions analogous to those given in the case of pairwise interactions and applying the same argument exploited in that case, it is possible to show that the same integral representation and homogenization results hold for the Γ-limits of energies of the form (1.8) (see Theorems 4.13 and 4.14). Moreover, we provide an example of energies accounting for nearest-neighbors 3-body interactions, obtained by adding a term in the potentials of the energies in (1.1) which penalizes changes of direction in the interfaces. The interesting effect is that such additional term does not give any contribution to the limiting energy, but acts as a selector of the ground states of E_ε (see Example 4.15).

Our decay assumptions on the constants $c_{i,j}^\varepsilon$ are somehow the most general to guarantee the locality of the continuum limit. Indeed, in Section 5 we show that, if they are violated, we can approximate non local functionals of the form

$$\int_{S_u} \varphi(x,\nu)\, d\mathcal{H}^{d-1} + \int_\Omega \int_\Omega K(x,y) g(u(x), u(y))\, dx dy, \qquad (1.9)$$

which resembles the representation of E_ε in (1.5). We focus on the approximation of two relevant examples: fractional perimeters and Ohta-Kawasaki type energies. The first ones correspond to a non local term on the right-hand side of (1.9) of the form

$$\int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+s}}\, dx\, dy,$$

for $s \in (0,1)$, which is one quarter of the fractional Sobolev space seminorm $\frac{1}{4}|u|_{W^{s,1}(\Omega)}$. The second ones correspond to a non local term of the form

$$\gamma_0 \int_\Omega \int_\Omega G(x,y) u(x) u(y)\, dx dy,$$

where $\gamma_0 > 0$ and G is the Green's function for $-\Delta$. The original model proposed by Ohta-Kawasaki in [16] in the studies of energy-driven pattern forming systems is based on energies of the form

$$\mathcal{OK}_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2\, dx + \frac{1}{\varepsilon} \int_\Omega (1 - u^2)^2\, dx + \gamma_0 \int_\Omega \int_\Omega G(x,y) u(x) u(y)\, dx dy \quad (1.10)$$

and in the analysis of their minima subject to prescribed volume fraction. The first two integrals in (1.10) form the so called 'Modica-Mortola' energy which Γ-converges as $\varepsilon \to 0$ to the perimeter functional. It can be easily shown that the non-local term in (5.14) is an L^1 continuous perturbation of the Modica-Mortola energy. As a consequence, the functionals $\mathcal{OK}_\varepsilon$ Γ-converge as $\varepsilon \to 0$ with respect to the $L^1(\Omega)$ norm to the functional finite on $BV(\Omega; \{-1,1\}$ and defined by

$$\mathcal{OK}(u) := \frac{8}{3} \mathcal{H}^{d-1}(S_u) + \gamma_0 \int_\Omega \int_\Omega G(x,y) u(x) u(y)\, dx\, dy. \qquad (1.11)$$

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We provide a variational approximation of an anisotropic version of the functional in (1.11), given by

$$\int_{S_u} |\nu_u|_1 \, d\mathcal{H}^{d-1}(S_u) + \gamma_0 \int_\Omega \int_\Omega G(x,y)u(x)u(y) \, dx \, dy. \qquad (1.12)$$

The idea is that the first term can be approximated by the discrete functionals in (1.1), while the non local term is approximated just by 'discretizing' the double integral. Hence we show that the functional in (1.12) is the Γ-limit of the sequence of functionals defined by

$$\hat{E}_\varepsilon(u) = - \sum_{|j-i|=1} \varepsilon^d u_i u_j + \gamma_0 \sum_{i,j \in \Omega_\varepsilon} \varepsilon^{2d+1} G(\varepsilon i, \varepsilon j) u_j u_i. \qquad (1.13)$$

The result can be also extended to the periodic case, when $\Omega = \mathbb{T}^d$, and to functions satisfying a mean constraint. By virtue of a result proved in [1] and as a consequence of our approximation, we deduce that lamellar configurations are nearby local minimizers of $\hat{E}_\varepsilon$ (see Theorem 5.9). We underline that the periodicity of minimizers of energies for spin systems with competing interactions and different scalings, which resembles the behavior of Ohta-Kawasaki type energies, has been investigated in many papers (see for instance [14, 15]). It would be interesting, in our opinion, to recast the analysis of that models in our framework.

The last question we address is, in the case there is a competition between ferromagnetic and antiferromagnetic interactions, that is there is no constraint on the sign of the constants $c_{i,j}^\varepsilon$, whether energies of type (1.3) still have a ferromagnetic behavior, that is the ground states are still the uniform states $u \equiv 1$ and $u \equiv -1$ and the continuum limit of the scaled energies is an interfacial energy of the form (1.6). We restrict our analysis to the homogeneous and 'short range' case, that is $c_{i,j}^\varepsilon = c_{j-i}$ for all i and j and $c_\xi = 0$ if $|\xi| > R$, for some $R > 0$. We show, then, that, given $M \in \mathbb{N}$, $M \geq R$ and setting $C_M = [0, M]^d \cap \mathbb{Z}^d$, the energies in (1.3) can be rewritten as

$$F_\varepsilon(u) = \sum_{i \in \Omega_\varepsilon} \varepsilon^d F_{cell}(u, i + C_M) + O(\varepsilon)$$

for a suitable function F_{cell} accounting for the interactions within the cells $i + C_M$. Then the condition which ensures a ferromagnetic behavior is that F_{cell} is minimized only by the uniform states 1 and −1 (see Definition 6.3). Under this condition we study the Γ-limit of the scaled energies E_ε defined as in (1.4). Since it is not our purpose in this paper to investigate boundary layer effects, we limit our analysis to the case in which Ω is a torus. Then we show that the Γ-limit of E_ε still has an integral representation of the form (1.7), with φ defined by a suitable asymptotic formula (see Theorem 6.5). Eventually, we provide some one dimensional examples to better explain our analysis and to show its applicability.

2 Notation and preliminary results

In what follows, given $x, y \in \mathbb{R}^d$ we denote by (x, y) the usual scalar product in $\mathbb{R}^d$ and we set $|x| = \sqrt{(x, x)}$. Moreover we denote by $| \cdot |_1$ the l^1-norm in $\mathbb{R}^d$ defined as $|x|_1 = |x_1| + \ldots + |x_d|$. Given $t > 0$, we will denote by $[t]$ the integer part of t. For any measurable $E \subset \mathbb{R}^d$ we denote by $|E|$ the d-dimensional Lebesgue measure of E and by $\mathcal{H}^{d-1}(E)$ its $d-1$-dimensional Hausdorff measure. Given $x_0 \in \mathbb{R}^d$, $\nu \in S^{d-1}$ and $\rho > 0$, we denote by $Q_\nu(x_0, \rho)$ a cube centered in x_0, with side length ρ and one face orthogonal to ν. We drop the dependence on ν, x_0 or ρ whenever $\nu = e_i$, for $i \in \{1, 2, \ldots, d\}$, $x_0 = 0$ or $\rho = 1$, respectively, and we set $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^d$.

Next we recall some basic properties of BV functions with values in $\{-1, 1\}$ (see [7] for a general exposition of the subject). Let A be an open subset of $\mathbb{R}^d$. We denote by $BV(A; \{-1, 1\})$ the set of measurable functions $u : A \to \{-1, 1\}$ whose distributional derivative Du is a measure with bounded total variation. Note that if $u \in BV(A; \{-1, 1\})$, then $E := \{x \in A : u(x) = 1\}$ is a set of finite perimeter in A. We denote by $S(u)$ the jump set of u and by $\nu_u(x)$ the measure theoretic inner normal to $S(u)$ at x, which is defined for $\mathcal{H}^{d-1}$ a.e. $x \in S(u)$.

For the reader's convenience we recall the following compactness result (see [7]).

Theorem 2.1 *Let $u_k \in BV(A; \{-1, 1\})$ such that*

$$\sup_k \mathcal{H}^{d-1}(S(u_k)) < +\infty.$$

Then there exists a subsequence (not relabelled) and $u \in BV(A; \{-1, 1\})$ such that $u_k \to u$ in the L^1 convergence.

A main tool in the sequel will be the following representation result obtained in [8]. We state here the result only in the particular case of functionals defined on $BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$ and satisfying an additional symmetry property, although a more general theorem holds. Let Ω be a bounded open subset of $\mathbb{R}^d$ and denote by $\mathcal{A}(\Omega)$ the family of open subsets of Ω.

Theorem 2.2 *Let $\mathcal{F} : BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ satisfy for every $(u, A) \in BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$ the following hypotheses:*

  (i) *$\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;*

  (ii) *$\mathcal{F}(u, A) = \mathcal{F}(v, A)$ whenever $u = v$ a.e. on $A \in \mathcal{A}(\Omega)$;*

  (iii) *$\mathcal{F}(\cdot, A)$ is $L^1(\Omega)$ lower semicontinuous;*

  (iv) *there exists $C > 0$ such that*

$$\frac{1}{C} \mathcal{H}^{d-1}(S_u \cap A) \leq \mathcal{F}(u, A) \leq C \mathcal{H}^{d-1}(S_u \cap A);$$

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(v) $\mathcal{F}(u, A) = \mathcal{F}(-u, A)$.

Then for every $u \in BV(\Omega; \{\pm 1\})$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}(u, A) = \int_{S_u \cap A} \varphi(x, \nu_u) \, d\mathcal{H}^{d-1},$$

with

$$\varphi(x_0, \nu) = \limsup_{\rho \to 0} \frac{m(u_{x_0, \nu}, Q_\nu(x_0, \rho))}{\rho^{d-1}}, \qquad (2.1)$$

where

$$u_{x_0, \nu}(x) := \begin{cases} 1 & \text{if } \langle x - x_0, \nu \rangle \geq 0 \\ -1 & \text{otherwise}, \end{cases} \qquad (2.2)$$

and for any $(v, A) \in BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$ we set

$m(v, A) = \inf\{\mathcal{F}(w, A) : w \in BV(A; \{\pm 1\}), \ w = v \text{ in a neighborhood of } \partial A\}.$

The sets of BV functions are also defined in a periodic setting: we denote by $\mathbb{T}^d$ the d-dimensional flat torus of unit volume, which can be identified with the semi-open unit cube $[0, 1)^d$, and by $BV(\mathbb{T}^d; \{-1, 1\}$ the set of functions $u \in BV_{loc}(\mathbb{R}^d; \{-1, 1\}$ which are $[0, 1)^d$-periodic. Moreover, with a slight abuse of notation, for such functions u we identify $S(u)$ with $S(u) \cap [0, 1)^d$.

Eventually we recall a result proved by Kohn and Sternberg in [13] on the approximation of local minimizers by Γ-convergence.

Theorem 2.3 *Let (X, d) be a metric space and let $(F_n)_n$ be a sequence of lower semicontinuous and equi-coercive functionals from X to $\mathbb{R} \cup \{+\infty\}$. Assume that F_n Γ-converge to F and let x be an isolated local minimizer of F. Then there exist $x_n \to x$ such that x_n is a local minimizer of F_n for n large enough.*

3 Setting

In what follows Ω will denote a bounded open set of $\mathbb{R}^d$ with Lipschitz boundary. For fixed $\varepsilon > 0$ we denote by $\mathcal{PC}_\varepsilon(\Omega)$ the set of functions

$$\mathcal{PC}_\varepsilon(\Omega) := \{u : \varepsilon \mathbb{Z}^d \cap \Omega \to \{-1, 1\}\}.$$

In order to carry on our analysis it is convenient to regard $\mathcal{PC}_\varepsilon(\Omega)$ as a subset in $L^1(\Omega)$. To this aim we will identify a function $u \in \mathcal{PC}_\varepsilon(\Omega)$ with its piecewise-constant interpolation on the ε-cubes centered in the lattice, still denoted by u. More precisely, we set $u(z) = 0$ if $z \in \varepsilon \mathbb{Z}^d \setminus \Omega$ and $u(x) = u(z_x^\varepsilon)$, where $z_x^\varepsilon \in \mathbb{Z}^d$ is the closest point to x (which is uniquely defined up to a set of zero measure). Other similar interpolations could be taken into account, actually not affecting our asymptotic analysis.

Thus, with fixed $\varepsilon > 0$, we consider energies $F_\varepsilon : \mathcal{P}C_\varepsilon(\Omega) \to \mathbb{R}$ of the form

$$F_\varepsilon(u) = - \sum_{i,j \in \Omega_\varepsilon} c_{i,j}^\varepsilon \varepsilon^d u_i u_j,$$

where we use the notation $\Omega_\varepsilon := \mathbb{Z}^d \cap (\varepsilon^{-1}\Omega)$, $u_i = u(\varepsilon i)$. Up to the change of variable $\xi = j - i$, setting $c_{i,\xi}^\varepsilon = c_{i,i+\xi}^\varepsilon$, we may equivalently write F_ε as

$$F_\varepsilon(u) = - \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} c_{i,\xi}^\varepsilon \varepsilon^d u_i u_{i+\xi}. \tag{3.1}$$

In the sequel we will perform an asymptotic analysis of energies suitably scaled, focusing first on the case of 'ferromagnetic type' pairwise interactions.

4 Ferromagnetic case: integral representation of the continuum limits

In this section we treat the case $c_{i,\xi}^\varepsilon \geq 0$ for all i, ξ and ε. Under this hypothesis uniform states are groundstates. Denoting $m_\varepsilon := \min F_\varepsilon(u) = F_\varepsilon(\pm 1)$ we then consider the scaled energies

$$E_\varepsilon(u) = \frac{F_\varepsilon(u) - m_\varepsilon}{\varepsilon} = \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} c_{i,\xi}^\varepsilon \varepsilon^{d-1}(1 - u_i u_{i+\xi}). \tag{4.1}$$

Note that E_ε can be rewritten as

$$
\begin{aligned}
E_\varepsilon(u) &= \frac{1}{2} \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} c_{i,\xi}^\varepsilon \varepsilon^{d-1}(u_{i+\xi} - u_i)^2 \\
&= \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} c_{i,\xi}^\varepsilon \varepsilon^{d-1}|u_{i+\xi} - u_i| \\
&= \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} c_{i,\xi}^\varepsilon |\xi| \varepsilon^d \frac{|u_{i+\xi} - u_i|}{\varepsilon|\xi|}
\end{aligned}
$$

i.e., our energies depend linearly on the difference quotients.

We fix the following set of hypotheses on $(c_{i,\xi}^\varepsilon)$:

(H1) (*coerciveness*) $c_{i,e_k}^\varepsilon \geq c > 0$ for all $\varepsilon > 0$, $i \in \mathbb{Z}^d$ and $k \in \{1, \ldots, d\}$;

(H2) (*growth*) set $c_\xi^\varepsilon = \sup_i c_{i,\xi}^\varepsilon$, $\limsup_{\varepsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} |\xi| c_\xi^\varepsilon < +\infty$;

(H3) (*decay*) $\forall \, \delta > 0$ there exists $R_\delta > 0$ such that $\limsup_{\varepsilon \to 0} \sum_{|\xi| > R_\delta} |\xi| c_\xi^\varepsilon < \delta$.

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With a little abuse of notation, we identify the functionals E_ε by their extension on $L^1(\Omega)$ by setting

$$E_\varepsilon(u) = \begin{cases} \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} c_{i,\xi}^\varepsilon \varepsilon^{d-1} |u_{i+\xi} - u_i| & \text{if } u \in \mathcal{PC}_\varepsilon(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \qquad (4.2)$$

We also define a localized version of our energies on 'regular' open sets. Let $\mathcal{A}^{reg}(\Omega)$ be the subfamily of open subsets of Ω with Lipschitz boundary. For any $A \in \mathcal{A}^{reg}(\Omega)$, we isolate the contributions due to interactions within the set A by defining

$$E_\varepsilon(u, A) = \begin{cases} \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in A_\varepsilon} c_{i,\xi}^\varepsilon \varepsilon^{d-1} |u_{i+\xi} - u_i| & \text{if } u \in \mathcal{PC}_\varepsilon(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \qquad (4.3)$$

where $A_\varepsilon = \mathbb{Z}^d \cap (\varepsilon^{-1}A)$. Moreover we will denote by $E'(\cdot, A)$ and $E''(\cdot, A)$ the $\Gamma\text{-}\liminf_{\varepsilon \to 0}$ and the $\Gamma\text{-}\limsup_{\varepsilon \to 0}$ of $E_\varepsilon(\cdot, A)$, respectively.

Remark 4.1 *We could have also considered a localized version of E_ε on any open set $A \in \mathcal{A}(\Omega)$. In such a case, in order to have an upper bound with a surface energy of type $\mathcal{H}^{d-1}(S_u \cap A)$ (see Proposition (4.4)), we should have slightly modified the definition of $E_\varepsilon(\cdot, A)$ accounting only for pairwise contributions indexed by couples $(i, i+\xi)$ such that the whole segment joining $\varepsilon i, \varepsilon(i+\xi)$ lies in A. Note that this late definition of $E_\varepsilon(\cdot, A)$ coincides with that in 4.3 for ε small enough in case of an open set A with Lipschitz boundary. For the sake of simplicity we prefer to deal with the class $\mathcal{A}^{reg}(\Omega)$, extending suitably $E'(\cdot, A), E''(\cdot, A)$ to $\mathcal{A}(\Omega)$ as a last step.*

We now state the main result of this section.

Theorem 4.2 *Let $(c_{i,\xi}^\varepsilon)$ satisfy (H1)–(H3) and let E_ε be defined by (4.2). Then for any sequence $\varepsilon_n \to 0$ there exist a subsequence (ε_{n_k}) and a function $\varphi : \Omega \times S^{d-1} \to [0, +\infty)$ such that the functionals $E_{\varepsilon_{n_k}}$ Γ-converge with respect to the $L^1(\Omega)$ strong topology to the functional $E : L^1(\Omega) \to [0, +\infty]$ defined by*

$$E(u) = \begin{cases} \int_{S_u} \varphi(x, \nu_u) \, \mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise.} \end{cases} \qquad (4.4)$$

Moreover for any $u \in BV(\Omega; \{\pm 1\})$ and $A \in \mathcal{A}^{reg}(\Omega)$ there holds

$$\Gamma\text{-}\lim_k E_{\varepsilon_{n_k}}(u, A) = E(u, A) := \int_{S_u \cap A} \varphi(x, \nu_u) \, \mathcal{H}^{d-1}. \qquad (4.5)$$

We postpone the proof of Theorem 4.2, as it will be a consequence of some propositions which show that our limit functionals satisfy all the hypotheses of Theorem 2.2.

In the next two Propositions we show that, thanks to hypotheses (H1) and (H2), for any $A \in \mathcal{A}^{reg}(\Omega)$, $E'(u, A)$ and $E''(u, A)$ are finite only on $BV(A; \{\pm 1\})$ and satisfy hypothesis (4) of Theorem 2.2.

Proposition 4.3 *Let $(c_{i,\xi}^{\varepsilon})$ satisfy (H1) and $A \in \mathcal{A}^{reg}(\Omega)$. If $u \in L^1(\Omega)$ is such that $E'(u, A)$ is finite, then $u \in BV(A; \{\pm 1\})$ and*

$$E'(u, A) \geq C \, \mathcal{H}^{d-1}(S_u \cap A),$$

for some positive constant C independent on u and A.

PROOF. Let G_ε the functional accounting only for nearest neighbors interactions defined by

$$G_\varepsilon(u, A) = \begin{cases} \displaystyle\sum_{k=1}^{d} \sum_{i, i+e_k \in A_\varepsilon} \varepsilon^{d-1} |u_{i+e_k} - u_i| & \text{if } u \in \mathcal{PC}_\varepsilon(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.6}$$

In [2] it was proved that the $G(\cdot, A) := \Gamma\text{-}\lim_{\varepsilon \to 0} G_\varepsilon(\cdot, A)$ is finite only on $BV(A; \{\pm 1\})$ and

$$G(u, A) = \int_{S_u \cap A} |\nu_u|_1 \, d\mathcal{H}^{d-1}, \quad u \in BV(A; \{\pm 1\}),$$

where $|\cdot|_1$ is the l^1-norm in $\mathbb{R}^d$. Since by (H1) we have that $E_\varepsilon(\cdot, A) \geq C G_\varepsilon(\cdot, A)$, the conclusion easily follows. $\qquad\square$

Proposition 4.4 *Let $(c_{i,\xi}^{\varepsilon})$ satisfy (H2). Then for every $u \in BV(\Omega; \{\pm 1\})$ and for every $A \in \mathcal{A}^{reg}(\Omega)$ there holds*

$$E''(u, A) \leq C \, \mathcal{H}^{d-1}(S_u \cap A), \tag{4.7}$$

for some positive constant C independent on u and A.

PROOF. By a density argument, it suffices to prove (4.7) for u such that S_u is a polyhedral set. Up to a localization argument we can further reduce to the case when S_u is an hyperplane, that is $u = u_{0,\nu}$ defined in (2.2). Thus, defined u_ε as the pointwise interpolation of u, i.e., $(u_\varepsilon)_i := u(\varepsilon i)$, we have that $u_\varepsilon \to u$ strongly in $L^1(\Omega)$. Moreover, by the assumption regularity on ∂A, for ε small enough, any pair $\varepsilon i, \varepsilon(i + \xi)$ such that u attains opposite values at the nodes, being contained in the interior part of A, crosses the jump set S_u. Hence, the energy contribution can be estimated by

$$E_\varepsilon(u_\varepsilon, A) \leq C \sum_{\xi \in \mathbb{Z}^d} \varepsilon^{d-1} c_\xi^\varepsilon \# (A_\varepsilon^\xi \cap \varepsilon \mathbb{Z}^d),$$

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where $A_\varepsilon^\xi := \{x = y + t\xi : \ y \in S_u, \ -\varepsilon \leq t \leq 0\} \cap A$. Since, by regularity of S_u,

$$\#(A_\varepsilon^\xi \cap \varepsilon \mathbb{Z}^d) \leq C\varepsilon^{-d}|A_\varepsilon^\xi| \leq C\varepsilon^{1-d}|\xi|\mathcal{H}^{d-1}(S_u \cap A),$$

we eventually get

$$E_\varepsilon(u_\varepsilon, A) \leq C\mathcal{H}^{d-1}(S_u \cap A) \sum_{\xi \in \mathbb{Z}^d} |\xi| c_\xi^\varepsilon.$$

Passing to the limit as $\varepsilon \to 0$, by (H2) we get the conclusion. $\qquad\square$

Proposition 4.5 *Let $(c_{i,\xi}^\varepsilon)$ satisfy (H1)–(H3) and let $A, B, A' \in \mathcal{A}^{reg}(\Omega)$ given with $A' \subset\subset A$. Then for any $u \in BV(\Omega; \{\pm 1\})$*

$$E''(u, A' \cup B) \leq E''(u, A) + E''(u, B). \tag{4.8}$$

PROOF. It is not restrictive to assume as a further hypothesis that A' is an open set with smooth boundary. Indeed, being $E''(u, \cdot)$ an increasing set function on $\mathcal{A}^{reg}(\Omega)$, it is enough to notice that there exists an open set C with smooth boundary such that $A' \subset\subset C \subset\subset A$. Such a set can be actually selected by standard convolution and selection of level sets arguments.

Without loss of generality we may also suppose $E''(u, A)$ and $E''(u, B)$ finite. Let $u_\varepsilon, v_\varepsilon$ both converge to u in $L^1(\Omega)$ and be such that

$$\limsup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon, A) = E''(u, A), \quad \limsup_{\varepsilon \to 0} E_\varepsilon(v_\varepsilon, B) = E''(u, B).$$

Fix $\delta > 0$ and let R_δ provided by hypothesis (H3). As A' has smooth boundary, the distance function $d(x) := \mathrm{dist}\,(x, A')$ inherits the same regularity in a suitable neigbourhood U of A'. Set

$$\overline{d} := \mathrm{dist}\,(A', U^c), \quad N_\varepsilon := \left[\frac{\overline{d}}{\varepsilon R_\delta}\right].$$

For $t \in [0, \overline{d}]$ denote also

$$A^t := \{x \in A : \mathrm{dist}\,(x, A') \leq t\}.$$

Plugging classical Sard Theorem in the Fleming-Rishel coarea formula, for $\mathcal{L}^1$-a.e. $t \in [0, \overline{d}]$ A^t is a set of finite perimeter with smooth boundary. Moreover, for any fixed integer $k \in \{1, \ldots, N_\varepsilon\}$, we can select values $t_k \in (k\varepsilon R_\delta, (k+1)\varepsilon R_\delta)$ such that A^{t_k} is a smooth set and satisfies

$$\varepsilon R_\delta \mathcal{H}^{d-1}(\partial A^{t_k}) \leq \int_{k\varepsilon R_\delta}^{(k+1)\varepsilon R_\delta} \mathcal{H}^{d-1}(\partial^* A^t)\, dt \tag{4.9}$$

$$\leq \int_{\{k\varepsilon R_\delta \leq d(x) \leq (k+1)\varepsilon R_\delta\}} |\nabla d(x)|\, dx = |A^{(k+1)\varepsilon R_\delta} \setminus A^{k\varepsilon R_\delta}|. \tag{4.10}$$

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For any $k \in \{1, \dots, N_\varepsilon\}$ consider then the family of functions w_ε^k still converging to u in $L^1(\Omega)$ defined by

$$w_\varepsilon^k := \chi_{A^{t_k}} u_\varepsilon + (1 - \chi_{A^{t_k}}) v_\varepsilon.$$

By taking the regularity of A^{t_k} into account, we easily compute

$$
\begin{aligned}
E_\varepsilon(w_\varepsilon^k, A' \cup B) &\leq E_\varepsilon(u_\varepsilon, A^{t_k} \cap (A' \cup B)) + E_\varepsilon(v_\varepsilon, (A^{t_k})^c \cap (A' \cup B)) \\
&\quad + \sum_{\xi \in \mathbb{Z}^d} \sum_{i \in S_\varepsilon^{k,\xi}} c_{i,\xi}^\varepsilon \, \varepsilon^{d-1} |(w_\varepsilon^k)_{i+\xi} - (w_\varepsilon^k)_i| \\
&\leq E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(v_\varepsilon, B) \\
&\quad + \sum_{\xi \in \mathbb{Z}^d} \sum_{i \in S_\varepsilon^{k,\xi}} c_{i,\xi}^\varepsilon \, \varepsilon^{d-1} |(w_\varepsilon^k)_{i+\xi} - (w_\varepsilon^k)_i|, \quad (4.11)
\end{aligned}
$$

where $S_\varepsilon^{k,\xi} := (\varepsilon^{-1}\{x = y + t\xi, \ y \in \partial A^{t_k}, \ -\varepsilon \leq t \leq 0\} \cap (A' \cup B)) \cap \mathbb{Z}^d$. In order to get the desired estimate we will argue by splitting the last term in (4.11) in interactions in $S_\varepsilon^{k,\xi}$ with $|\xi| > R_\delta$ and with $|\xi| \leq R_\delta$, respectively. In the two cases we will provide different estimates for the energy contribution.

In the first case, by arguing as in the proof of Proposition 4.4, thanks to the regularity of A^{t_k}, we get that

$$\sum_{|\xi| > R_\delta} \sum_{i \in S_\varepsilon^{k,\xi}} c_{i,\xi}^\varepsilon \, \varepsilon^{d-1} |(w_\varepsilon^k)_{i+\xi} - (w_\varepsilon^k)_i| \leq C \, \mathcal{H}^{d-1}(\partial A^{t_k}) \sum_{|\xi| > R_\delta} |\xi| c_\xi^\varepsilon.$$

By using the mean property (4.9) of A^{t_k} we may refined the inquality above

$$\sum_{|\xi| > R_\delta} \sum_{i \in S_\varepsilon^{k,\xi}} c_{i,\xi}^\varepsilon \, \varepsilon^{d-1} |(w_\varepsilon^k)_{i+\xi} - (w_\varepsilon^k)_i| \leq C\delta \frac{|A^{(k+1)\varepsilon R_\delta} \setminus A^{k\varepsilon R_\delta}|}{\varepsilon R_\delta}. \quad (4.12)$$

We are left with the estimate of the energy accounting for the interactions in $S_\varepsilon^{k,\xi}$ when $|\xi| \leq R_\delta$. Note that in this case, by the definition of $A^{k\varepsilon R_\delta}$, we have that $S_\varepsilon^{k,\xi} \subseteq (A^{(k+1)\varepsilon R_\delta} \setminus A^{(k-1)\varepsilon R_\delta}) \cap B \cap \varepsilon \mathbb{Z}^d =: S_\varepsilon^k$. Moreover, we easily get that

$$|(w_\varepsilon^k)_{i+\xi} - (w_\varepsilon^k)_i| \leq 2(|(u_\varepsilon)_{i+\xi} - (u_\varepsilon)_i| + |(v_\varepsilon)_{i+\xi} - (v_\varepsilon)_i| + |(v_\varepsilon)_i - (u_\varepsilon)_i|);$$

hence,

$$
\begin{aligned}
&\sum_{|\xi| \leq R_\delta} \sum_{i \in S_\varepsilon^{k,\xi}} c_{i,\xi}^\varepsilon \, \varepsilon^{d-1} |(w_\varepsilon^k)_{i+\xi} - (w_\varepsilon^k)_i| \\
&\leq \quad C(E_\varepsilon(u_\varepsilon, S_\varepsilon^k) + E_\varepsilon(v_\varepsilon, S_\varepsilon^k)) + C_\delta \sum_{i \in S_\varepsilon^k} \varepsilon^{d-1} |(v_\varepsilon)_i - (u_\varepsilon)_i|, \quad (4.13)
\end{aligned}
$$

for some constant C_δ depending only on R_δ.

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Therefore, summing over $k \in \{1, \ldots, N_\varepsilon - 1\}$ and averaging, by (4.11), (4.12) and (4.13), we get

$$
\begin{aligned}
I_\varepsilon \quad := \quad & \frac{1}{N_\varepsilon - 1} \sum_{k=1}^{N_\varepsilon - 1} \Big(E_\varepsilon(w_\varepsilon^k, A' \cup B) \Big) \leq E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(v_\varepsilon, B) \\
+ \quad & \frac{2}{N_\varepsilon - 1}(E_\varepsilon(u_\varepsilon, A \cap B) + E_\varepsilon(v_\varepsilon, A \cap B)) + C'_\delta \sum_{i \in \Omega_\varepsilon} \varepsilon^d |(v_\varepsilon)_i - (u_\varepsilon)_i| + C\delta|A \setminus A'| \\
\leq \quad & \frac{C}{N_\varepsilon} + C\delta + C_\delta \|v_\varepsilon - u_\varepsilon\|_{L^1(\Omega)}, \quad\quad\quad\quad (4.14)
\end{aligned}
$$

where we have also used that $\cup_{k=0}^{N_\varepsilon} A^{(k+1)\varepsilon R_\delta} \setminus A^{k\varepsilon R_\delta} = \overline{U} \setminus A' \subseteq A \setminus A'$.

Eventually, for any $\varepsilon > 0$ let $k_\varepsilon \in \{1, \ldots, N_\varepsilon - 1\}$ be such that

$$
E_\varepsilon(w_\varepsilon^{k_\varepsilon}, A' \cup B) \leq I_\varepsilon. \quad\quad\quad\quad (4.15)
$$

Setting $w_\varepsilon := w_\varepsilon^{k_\varepsilon}$, we have that $w_\varepsilon \to u$ strongly in $L^1(\Omega)$ and, by (4.14), (4.15), passing to the limit as $\varepsilon \to 0$ we infer also that

$$
E''(u, A' \cup B) \leq \limsup_{\varepsilon \to 0} E_\varepsilon(w_\varepsilon, A' \cup B) \leq E''(u, A) + E''(v, B) + C\delta.
$$

The conclusion follows by the arbitrariness of $\delta > 0$. $\qquad\qquad\square$

In the following proposition we show that $E''(\cdot, \cdot)$ satisfies hypothesis (ii) of Theorem 2.2. The argument we use for the proof is the same exploited to prove the previous proposition.

Proposition 4.6 *Let $(c_{i,\xi}^\varepsilon)$ satisfy (H1)–(H3). Then for any $A \in \mathcal{A}^{reg}(\Omega)$ and for any $u, v \in BV(\Omega; \{\pm 1\})$ such that $u = v$ a.e. in A there holds*

$$
E''(u, A) = E''(v, A). \quad\quad\quad\quad (4.16)
$$

PROOF. Let $A \in \mathcal{A}^{reg}(\Omega)$ and $u, v \in BV(\Omega; \{\pm 1\})$ be fixed with $u = v$ a.e. in A. By reversing the role of u and v it is enough to prove that

$$
E''(v, A) \leq E''(u, A). \quad\quad\quad\quad (4.17)
$$

To this aim we used the same argument and constructions of the previous proposition.

With $\delta > 0$ fixed, let $A_\delta \in \mathcal{A}^{reg}(\Omega)$ such that $A_\delta \subset\subset A$ and

$$
\mathcal{H}^{d-1} \llcorner S_u(A \setminus \overline{A_\delta}) < \delta.
$$

Note that $A \setminus \overline{A_\delta} \in \mathcal{A}^{reg}(\Omega)$ and by Proposition 4.4, there holds

$$
E''(v, A \setminus \overline{A_\delta}) \leq C\mathcal{H}^{d-1} \llcorner S_u(A \setminus \overline{A_\delta}) \leq C\delta. \quad\quad\quad\quad (4.18)
$$

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Let u_ε and v_ε be converging to u and v in $L^1(\Omega)$, respectively, and such that

$$
\begin{aligned}
\limsup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon, A) &= E''(u, A), \\
\limsup_{\varepsilon \to 0} E_\varepsilon(v_\varepsilon, A \setminus \overline{A_\delta}) &= E''(v, A \setminus \overline{A_\delta}).
\end{aligned}
$$

Arguing as in the proof of Proposition 4.5, with $A' = A_\delta, B = A \setminus \overline{A_\delta}$, we may choose for any $\varepsilon > 0$ a set $A^{t_{k_\varepsilon}}$ and the relative interpolated function

$$
w_\varepsilon^{k_\varepsilon} := \chi_{A^{t_{k_\varepsilon}}} u_\varepsilon + (1 - \chi_{A^{t_{k_\varepsilon}}}) v_\varepsilon
$$

such that $w_\varepsilon^{k_\varepsilon}$ converges to v in $L^1(\Omega)$ and, by passing to the limsup as $\varepsilon \to 0$, it holds

$$
E''(v, A) \le \limsup_{\varepsilon \to 0} E_\varepsilon(w_\varepsilon^{k_\varepsilon}, A) \le E''(u, A) + E''(v, A \setminus \overline{A_\delta}) + C\delta.
$$

Hence (4.17) follows by (4.18) and the arbitrariness of $\delta > 0$. $\qquad\square$

Proof of theorems 4.2. As a consequence of Propositions 4.4 and 4.5, a standard argument shows that for any $u \in BV(\Omega; \{\pm 1\})$ $E''(u, \cdot)$ is inner regular and subadditive on $\mathcal{A}^{reg}(\Omega)$. By the compactness property of the Γ-convergence and arguing as in [10, Theorem 10.3], we can easily show that there exists a subsequence (ε_{n_k}) such that, for any $(u, A) \in BV(\Omega; \{\pm 1\}) \times \mathcal{A}^{reg}(\Omega)$ there exists

$$
\Gamma\text{-}\lim_k E_{\varepsilon_{n_k}}(u, A) =: E(u, A).
$$

Moreover we can extend $E(u, \cdot)$ to $\mathcal{A}(\Omega)$ by setting $E(u, A) := \sup\{E(u, A') : A' \in \mathcal{A}^{reg}(\Omega), \ A' \subset A\}$ and easily verify that all the properties enjoyed by $E''(\cdot, \cdot)$ stated in Propositions 4.3, 4.4, 4.5 and 4.6 still hold true for $E(\cdot, \cdot)$. Moreover, by Proposition 4.3,

$$
\Gamma\text{-}\lim_k E_{\varepsilon_{n_k}}(u) = +\infty
$$

for $u \in L^1(\Omega) \setminus BV(\Omega; \{\pm 1\})$. So far, it suffices to check that, for every $(u, A) \in BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$, $E(u, A)$ satisfies all the hypotheses of Theorem 2.2. In fact, it can be easily seen that the superadditivity property of $E_\varepsilon(u, \cdot)$ is conserved in the limit. Thus, thanks to the De Giorgi-Letta criterion (see [10, Theorem 5.2]), hypotheses (i), (ii), and (iv) hold true. By the lower semicontinuity property of the Γ-limit, also hypothesis (iii) is fulfilled and finally hypothesis (v) holds since it is satisfied by $E_\varepsilon(\cdot, A)$ for any $\varepsilon > 0$.

The previous argument actually provides also a proof of (4.5). $\qquad\square$

4.1 Homogenization

As a consequence of Theorem 4.2, if the interaction potentials are periodic in the independent variable we may give an alternative proof of an homogenization

result proved in [11]. More precisely, assume that $c_{i,\xi}^\varepsilon = c_\xi(i)$ where, for any $\xi \in \mathbb{Z}^d$, $c_\xi : \mathbb{Z}^d \to [0, +\infty)$ is $[0,l]^d$-periodic for some $l \in \mathbb{N}$. In this case hypotheses (H1)–(H3) reduce to:

(H1') $c_{e_k}(i) \geq c > 0$ for all $i \in \{0, \ldots, l-1\}^d$ and $k \in \{1, \ldots, d\}$;

(H2') $\sum_{\xi \in \mathbb{Z}^d} |\xi| c_\xi(i) < +\infty$ for all $i \in \{0, \ldots, l-1\}^d$.

Then the following theorem holds (see [11], Theorem 2.4 and Remark 2.5).

Theorem 4.7 (homogenization) *Let E_ε be defined by (4.2) and let $c_{i,\xi}^\varepsilon = c_\xi(i)$, where, for any $\xi \in \mathbb{Z}^d$, $c_\xi : \mathbb{Z}^d \to [0, +\infty)$ is $[0,l]^d$-periodic for some $l \in \mathbb{N}$. If (H1') and (H2') hold, then the functionals E_ε Γ-converge with respect to the $L^1(\Omega)$ strong topology, as $\varepsilon \to 0$, to the homogenized functional $E_{hom} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$E_{hom}(u) = \begin{cases} \displaystyle\int_{S_u} \varphi_{hom}(\nu_u)\, \mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.19}$$

where

$$\varphi_{hom}(\nu) = \lim_{T \to +\infty} T^{1-d} \inf \left\{ \sum_{\xi \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d \cap TQ_\nu} c_\xi(i) u_i u_{i+\xi} : u \in \mathcal{PC}_1(\mathbb{R}^d), \right.$$
$$\left. u(i) = u_{0,\nu}(i) \text{ for } i \notin TQ_\nu \right\}.$$

Sketch of the proof. The proof is based on a standard argument in homogenization theory. Here we highlight only its main steps for the reader's convenience.

By Theorem 4.13, given a sequence $\varepsilon_n \to 0$ we can extract a subsequence ε_{n_k} such that the functionals $E_{\varepsilon_{n_k}}$ Γ-converge to a functional E defined as in (4.29) and such that for any $u \in BV(\Omega; \{\pm 1\})$ and $A \in \mathcal{A}^{reg}(\Omega)$ there holds

$$\Gamma\text{-}\lim_k E_{\varepsilon_{n_k}}(u, A) = \int_{S_u \cap A} \varphi(x, \nu_u)\, \mathcal{H}^{d-1}.$$

The theorem is proved if we show that φ does not depend on x and $\varphi \equiv \varphi_{hom}$. To prove the first claim, by using the periodicity assumption, one can easily show that if we set $E(u, A) = \int_{S_u \cap A} \varphi(x, \nu_u)\, \mathcal{H}^{d-1}$, then

$$E(u_{x,\nu}, B(x, \rho)) = E(u_{z,\nu}, B(z, \rho)) \tag{4.20}$$

for any $\nu \in S^{d-1}$, $x, z \in \Omega$ and $\rho > 0$ such that $B(x, \rho) \cup B(z, \rho) \subseteq \Omega$. Thus, the independence of φ on x easily follows from (4.20).

So far, assuming for the sake of simplicity that $0 \in \Omega$, by using the characterization of φ in (2.1), we have

$$\varphi(\nu) = \limsup_{\rho \to 0} \frac{m(u_{0,\nu}, \rho Q_\nu)}{\rho^{d-1}},$$

where for any $(v, A) \in BV(\Omega; \{\pm 1\}) \times \mathcal{A}(\Omega)$

$$m(v, A) = \inf\{E(w, A) : w \in BV(A; \{\pm 1\}),\ w = v \text{ in a neighborhood of } \partial A\}.$$

By using the convergence of boundary value problems stated in Theorem 4.9 and taking into account Remark 4.10, we infer that

$$\varphi(\nu) = \lim_{k \to +\infty} T_k^{1-d} \inf \left\{ \sum_{\underline{\xi} \in \mathbb{Z}^{d(M-1)}} \sum_{i \in \mathbb{Z}^d \cap T_k Q_\nu} c_\xi(i) u_i u_{i+\xi} :\ u \in \mathcal{P}C_1(\mathbb{R}^d), \right.$$

$$\left. u(i) = u_{0,\nu}(i) \text{ for } i \notin T_k Q_\nu \right\}, \quad (4.21)$$

for some suitable sequence $T_k \to +\infty$. A standard argument in homogenization theory, based on the periodicity assumption, shows that the limit in (4.32) exists and is finite for any $\nu \in S^{d-1}$. Hence, by (4.21), we immediately conclude that $\varphi \equiv \varphi_{hom}$. $\qquad\square$

As a straightforward application of Theorem 4.2, in the following example we show that if the constants $c_{i,\xi}^\varepsilon$ do not satisfy the periodicity assumptions before, we may obtain in the continuum limit more general not homogeneous surface functionals.

Example 4.8 *Let $\Omega = (-1, 1)^2$ and let $c : [-1, 1]^2 \to \mathbb{R}$ be defined as*

$$c(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \times [-1, 1] \\ \\ 2 & \text{otherwise in } [-1, 1]^2 . \end{cases}$$

Let $E_\varepsilon : L^1(\Omega) \to [0, +\infty]$ be defined as in (4.2) with $d = 2$, $c_{i,\xi}^\varepsilon = 0$ if $\xi \notin \{e_1, e_2\}$ and $c_{i,e_1}^\varepsilon = c_{i,e_2}^\varepsilon = c(\varepsilon i)$. Then the functionals E_ε Γ-converge with respect to the $L^1(\Omega)$ strong topology to the functional $E(u)$ defined as in (4.4) with $\varphi(x, \nu) = |\nu|_1$ if $x \in [0, 1) \times (-1, 1)$ and $\varphi(x, \nu) = 2|\nu|_1$ if $x \in (-1, 0) \times (-1, 1)$. Indeed, by Theorem 4.2, it suffices to show that $\Gamma\text{-}\lim_{\varepsilon \to 0} E_\varepsilon(u_{x_0, \nu}) = E(u_{x_0, \nu})$ for every $(x_0, \nu) \in \Omega \times S^{d-1}$. The $\Gamma\text{-}\liminf$ inequality follows by a standard slicing argument, while, for the $\Gamma\text{-}\limsup$ inequality, a recovery sequence for $u_{x_0, \nu}$ is provided by an infinitesimal right translation of the pointwise interpolation of $u_{x_0, \nu}$, defined by $(u_\varepsilon)_i = u_{x_0, \nu}(\varepsilon i - \varepsilon e_1)$.

More generally, an analogous convergence result could be inferred in any dimension, considering a finite coolection of disjoint sets $\{A_k\}_{k=1}^N$ in Ω with $A_k \in \mathcal{A}^{reg}(\Omega)$ and positive constants $c_1, \ldots, c_N$. Indeed, set $c(x) = \sum_{k=1}^N c_k \chi_{A_k}(x)$ and let E_ε be defined as in (4.2) with $c_{i,\xi}^\varepsilon = 0$ if $\xi \notin \{e_1, \ldots, e_d\}$ and $c_{i,e_1}^\varepsilon = \ldots = c_{i,e_d}^\varepsilon = c(\varepsilon i)$. Then as a consequence of Theorem 4.2 and applying the argument above, we could deduce that the functionals E_ε Γ-converge to with respect to the $L^1(\Omega)$ strong topology to the functional $E(u)$ defined as in (4.4) with $\varphi(x, \nu) = a(x)|\nu|_1$ and $a : \Omega \to (0, +\infty)$ defined by $a(x) = \sum_{k=1}^N c_k \chi_{A_k}(x) + \sum_{k,h=1}^N c_k \wedge c_h \chi_{\partial A_k \cap \partial A_h}(x)$.

4.2 Convergence of minimum problems

In this section we show that, as a consequence of the Γ-convergence result stated in Theorem 4.2, we may derive the convergence of minimum problems involving discrete energies as in (4.2) to the corresponding minimum problem involving the limiting energy. We focus on boundary value problems and minimum problems with prescribed volume fraction.

In order to state boundary value problems, let $A \in \mathcal{A}^{reg}(\Omega)$ and assume $u_0 \in BV_{loc}(\mathbb{R}^d; \{-1, 1\})$ to be fixed with S_{u_0} polyhedral and transversal to A, that is $\mathcal{H}^{d-1}(S_u \cap \partial A) = 0$. Actually, more general boundary data could be also taken into account. Here for the sake of simplicity we prefer to restrict our analysis for a boundary datum u_0 as above.

For any $\varepsilon, \eta > 0$ set $A^\eta = \{x \in A : \text{dist}\,(x, \partial A) < \eta\}$,

$$\mathcal{P}C_\varepsilon^{u_0, \eta}(A) := \{u \in \mathcal{P}C_\varepsilon(\Omega) : u(\varepsilon i) = u_0(\varepsilon i) \text{ for any } i \in \mathbb{Z}^d \cap (\varepsilon^{-1} A^\eta)\},$$

and

$$m_\varepsilon^\eta(u_0, A) := \inf\{E_\varepsilon(u, A) : u \in \mathcal{P}C_\varepsilon^{u_0, \eta}(A)\} \tag{4.22}$$

It is easy to show that $m_\varepsilon^\eta(u_0)$ is increasing in η and bounded uniformly in η and ε.

Theorem 4.9 *Let $(c_{i,\xi}^\varepsilon)$ satisfy (H1)–(H3) and let E_ε and $m_\varepsilon^\eta(u_0)$ be defined by (4.1) and (4.22), respectively. Given a sequence $\varepsilon_n \to 0$ let (ε_{n_k}) and $E(u, A)$ be as in Theorem 4.2. Then*

$$\lim_{\eta \to 0} \limsup_{k \to +\infty} m_{\varepsilon_{n_k}}^\eta (u_0, A) = m(u_0, A), \tag{4.23}$$

where

$$m(u_0, A) := \inf\{E(u, A) : u \in BV(A; \{-1, 1\}), \ u(x) = \tilde{u}_0(x)$$

$$\text{in a neighborhood of } \partial A\}.$$

PROOF. Let $\eta > 0$ be fixed and let $u_k \in \mathcal{P}C_{\varepsilon_{n_k}}^{u_0, \eta}(A)$ such that $E_{\varepsilon_{n_k}}(u_k, A) \leq m_{\varepsilon_{n_k}}^\eta (u_0, A) + k^{-1}$. Since, by the coerciveness assumption (H1), $\mathcal{H}^{d-1}(S_{u_k} \cap A) \leq C$, then, by Theorem 2.1, up to extracting a subsequence u_k converge strongly in $L^1(A)$ to some $u \in BV(A; \{-1, 1\})$. Moreover it is easy to show that $u = u_0$ on A^η. Hence, by Theorem 4.2, we get

$$m(u_0, A) \leq E(u, A) \leq \liminf_k E_{\varepsilon_{n_k}}(u_k, A) \leq \liminf_k m_{\varepsilon_{n_k}}^\eta (u_0, A).$$

By the arbitrariness of η it follows that

$$m(u_0, A) \leq \lim_{\eta \to 0} \limsup_{k \to +\infty} m_{\varepsilon_{n_k}}^\eta (u_0, A).$$

It remains to prove the opposite inequality. To this end, given $\delta > 0$, let $u \in BV(A; \{-1, 1\}$ such that $u = u_0$ in a neighborhood of ∂A and $E(u, A) \leq m(u_0, A) + \delta$. Let $u_k \to u$ strongly in $L^1(A)$ such that $\lim_k E_{\varepsilon_{n_k}}(u_k, A) = E(u, A)$. Then, by applying the argument used in the proof of Proposition 4.5, choosing suitable convex combinations of u_k and $\tilde{u}_0$, it is possible to construct functions $v_k \in \mathcal{PC}_{\varepsilon_{n_k}}^{u_0, \eta}(A)$ for some $\eta > 0$ such that $\lim_k E_{\varepsilon_{n_k}}(v_k, A) = E(u, A)$. Hence we get

$$\limsup_{k \to +\infty} m_{\varepsilon_{n_k}}^{\eta}(u_0, A) \leq \lim_{k \to +\infty} E_{\varepsilon_{n_k}}(v_k, A) = E(u, A) \leq m(u_0, A) + \delta.$$

The conclusion follows by the arbitrariness of η and δ. $\qquad\square$

Remark 4.10 *It can be shown that if the density function $\varphi(x, \nu)$ in Theorem 4.2 is continuous in $\overline{A} \times S^{d-1}$, then*

$$m(u_0, A) = \inf \left\{ \int_{S_{u_{A, u_0}} \cap \overline{A}} \varphi(x, \nu_{u_{A, u_0}}) \, d\mathcal{H}^{d-1} : \ u \in BV(A; \{-1, 1\}) \right\},$$

where

$$u_{A, u_0}(x) := \begin{cases} u(x) & \text{if } x \in A \\ u_0(x) & \text{if } x \notin A. \end{cases}$$

Moreover in this case it can be also shown that

$$m(u_0, A) = \lim_{k \to +\infty} \min\{E_{\varepsilon_{n_k}}^{u_0}(u, A) : u \in \mathcal{PC}_{\varepsilon_{n_k}}(A)\}$$

where

$$E_{\varepsilon}^{u_0}(u, A) := \sum_{\xi \in \mathbb{Z}^d} \sum_{\varepsilon i \in A_\varepsilon} c_{i, \xi}^{\varepsilon} \varepsilon^{d-1} |u_{A, u_0}(\varepsilon(i + \xi)) - u_{A, u_0}(\varepsilon i)|$$

(see Theorems 4.2 and 4.4 in [6]).

On the other hand Example 4.8 shows that $\varphi(x, \nu)$ is lower semicontinuous but not necessarily continuous with respect to the variable x.

We conclude the section presenting the convergence result of minimum problems with prescribed volume fraction. More precisely we consider minimum problems of the type

$$m_\varepsilon := \min\{E_\varepsilon(u) : \ \#\{i \in \Omega_\varepsilon : \ u_i = 1\} = l_\varepsilon\},$$

where $l_\varepsilon \in \{0, \dots, \#\Omega_\varepsilon\}$. We suppose that there exists $p \in [0, 1]$ such that

$$\lim_{\varepsilon \to 0} l_\varepsilon (\#\Omega_\varepsilon)^{-1} = p \tag{4.24}$$

(this is not restrictive, up to passing to a subsequence). Set

$$\mathcal{PC}_\varepsilon^{l_\varepsilon}(\Omega) := \{u \in \mathcal{PC}_\varepsilon(\Omega) : \ \#\{i \in \Omega_\varepsilon : \ u_i = 1\} = l_\varepsilon\}$$

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and let $E_\varepsilon^{l_\varepsilon} : L^1(\Omega) \to [0, +\infty]$ be defined by

$$E_\varepsilon^{l_\varepsilon}(u) = \begin{cases} E_\varepsilon(u) & \text{if } u \in \mathcal{PC}_\varepsilon^{l_\varepsilon}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{4.25}$$

with E_ε as in (4.1). The following Γ-convergence theorem holds.

Theorem 4.11 *Let $(c_{i,\xi}^\varepsilon)$ satisfy (H1)–(H3) and let l_ε satisfy (4.24). Given any sequence $\varepsilon_n \to 0$, let (ε_{n_k}) and $E(u)$ be as in Theorem 4.2. Then the functionals $E_{\varepsilon_{n_k}}^{l_{\varepsilon_{n_k}}}$ Γ-converge with respect to the $L^1(\Omega)$ strong topology to the functional $E^p : L^1(\Omega) \to [0, +\infty]$ defined as*

$$E^p(u) = \begin{cases} E(u) & \text{if } |\{u = 1\}| = p|\Omega| \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.26}$$

PROOF. For the sake of notation, in what follows we drop the dependence on the sequence n_k. Let, then, $u_\varepsilon \in \mathcal{PC}_\varepsilon^{l_\varepsilon}(\Omega)$ converge to u strongly in $L^1(\Omega)$ such that $\liminf_\varepsilon E_\varepsilon(u_\varepsilon) < +\infty$. By (4.24) and taking into account that $\varepsilon^d \#\Omega_\varepsilon \to |\Omega|$, we immediately infer that $|\{u = 1\}| = p|\Omega|$. As a straightforward consequence of Theorem 4.2, we get

$$\liminf E_\varepsilon^{l_\varepsilon}(u_\varepsilon) \geq E^p(u).$$

It remains to prove the Γ-limsup inequality. Thanks to the density result stated in Theorem 7.1 and Remark 7.2, it is enough to consider $u \in BV(\Omega; \{-1, 1\})$ with $|\{u = 1\}| = p|\Omega|$ and S_u a polyhedral set. Let u_ε be a recovery sequence for $E(u)$. Note that, arguing as in the proof of Proposition 4.5 and taking into account that $E_\varepsilon(u, A)$ is negligible as $\varepsilon \to 0$ for any open set $A \subset \Omega \setminus S_u$, we could replace u_ε by $\chi_{A_\varepsilon} u_\varepsilon + (1 - \chi_{A_\varepsilon})u$, where A_ε is a suitable neighborhood of S_u, still obtaining a recovery sequence for $E(u)$. Hence, it is not restrictive to assume $u_\varepsilon = u$ in $\Omega \setminus \overline{S}_\delta$, where S_δ is a δ-neighborhood of S_u for some suitable $\delta > 0$ such that $|\Omega \setminus \overline{S}_\delta| > 0$.

Having set $\tilde{l}_\varepsilon = \#\{i \in \Omega_\varepsilon : u_\varepsilon(\varepsilon i) = 1\}$, we have that $\lim_{\varepsilon \to 0} |\tilde{l}_\varepsilon - l_\varepsilon|\varepsilon^d = 0$. If $\tilde{l}_\varepsilon = l_\varepsilon$ for infinite infinitesimal values of ε, there is nothing left to prove. Assume on the contrary that $\tilde{l}_\varepsilon > l_\varepsilon$ (the other case being totally simmetric). Let $h_\varepsilon := (\tilde{l}_\varepsilon - l_\varepsilon)^{\frac{1}{d}}$, $\overline{h}_\varepsilon := [h_\varepsilon]$, where we denote by $[t]$ the integer part of $t \in \mathbb{R}$. Note that $\varepsilon h_\varepsilon \to 0$ and $0 < h_\varepsilon^d - \overline{h}_\varepsilon^d \leq dh_\varepsilon^{d-1}$. Set $G_\varepsilon^\delta := \{u = 1\} \cap (\Omega \setminus \overline{S}_\delta)$, we choose $i_\varepsilon \in \Omega_\varepsilon$ such that $Q_\varepsilon := \varepsilon i_\varepsilon + [0, \overline{h}_\varepsilon \varepsilon]^d \subset G_\varepsilon^\delta$ and let $J_\varepsilon \subset (G_\varepsilon^\delta \setminus Q_\varepsilon) \cap \varepsilon \mathbb{Z}^d$ be such that $\#J_\varepsilon = h_\varepsilon^d - \overline{h}_\varepsilon^d$. We then set

$$v_\varepsilon(\varepsilon i) = \begin{cases} -1 & \text{if } \varepsilon i \in Q_\varepsilon \cup J_\varepsilon \\ u_\varepsilon(\varepsilon i) & \text{otherwise.} \end{cases}$$

Then, by construction, $v_\varepsilon \in \mathcal{PC}_\varepsilon^{l_\varepsilon}(\Omega)$, $v_\varepsilon \to u$ strongly in $L^1(\Omega)$ and moreover

$$E_\varepsilon(v_\varepsilon) \leq E_\varepsilon(u_\varepsilon) + C\varepsilon^{d-1}(\#(\partial Q_\varepsilon \cap \varepsilon\mathbb{Z}^d) + \#J_\varepsilon) \leq E_\varepsilon(\tilde{v}_\varepsilon) + C(\varepsilon h_\varepsilon)^{d-1},$$

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from which the conclusion for $d \geq 2$. For $d = 1$ we may assume that there exist $i_\varepsilon \in \Omega_\varepsilon$ and $\eta > 0$ independent of ε such that $\varepsilon i_\varepsilon \in S_{u_\varepsilon}$ and $(\varepsilon i_\varepsilon - \eta, \varepsilon i_\varepsilon] \subset \{u_\varepsilon = 1\}$ or $[\varepsilon i_\varepsilon, \varepsilon i_\varepsilon + \eta) \subset \{u_\varepsilon = 1\}$. Hence a recovery sequence $v_\varepsilon \in \mathcal{PC}_\varepsilon^{l_\varepsilon}(\Omega)$ is given by $v_\varepsilon := u_\varepsilon - 2\chi_{((i_\varepsilon - \tilde{l}_\varepsilon + l_\varepsilon)\varepsilon, \varepsilon i_\varepsilon]}$ in the first case or $v_\varepsilon := u_\varepsilon - 2\chi_{[\varepsilon i_\varepsilon, (i_\varepsilon + \tilde{l}_\varepsilon - l_\varepsilon)\varepsilon)}$ in the second case. $\qquad\square$

As a straightforward consequence of Theorem 4.11, the compactness result stated in Theorem 2.1 and the standard properties of Γ-convergence, we eventually get the following result.

Theorem 4.12 *Under the hypotheses of Theorem 4.11, we get*

$$\lim_k \min\{E_{\varepsilon_{n_k}} : \#\{i \in \Omega_{\varepsilon_{n_k}} : u_i = 1\} = l_{\varepsilon_{n_k}}\} = \min\{E(u) : |\{u = 1\}| = p|\Omega|\}$$

Moreover if (u_k) is a converging sequence such that $\#\{i \in \Omega_{\varepsilon_{n_k}} : u_i = 1\} = l_{\varepsilon_{n_k}}$ and

$$\lim_k E_{\varepsilon_{n_k}}(u_k) = \lim_k \min\{E_{\varepsilon_{n_k}} : \#\{i \in \Omega_{\varepsilon_{n_k}} : u_i = 1\} = l_{\varepsilon_{n_k}}\},$$

then its limit is a minimizer to $\min\{E(u) : |\{u = 1\}| = p|\Omega|\}$.

4.3 Generalization: multi-body interactions

In this section we extend the previous results to the case of energies accounting for multi-body interactions. More precisely, given $M \in \mathbb{N}$, we consider energies of the form

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \displaystyle\sum_{i_1, \ldots, i_M \in \Omega_\varepsilon} \varepsilon^{d-1} \psi_\varepsilon(i_1, \ldots, i_M, u_{i_1}, \ldots, u_{i_M}) & \text{if } u \in \mathcal{PC}_\varepsilon(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.27}$$

Here $\psi : \mathbb{Z}^{dM} \times \{-1, 1\}^M \to [0, +\infty)$ satisfies the following hypothesis:

(HM0) $\min_{\underline{u}} \psi_\varepsilon(\underline{i}, \underline{u}) = \psi_\varepsilon(\underline{i}, \pm\underline{1}) = 0$ for any $\underline{i} \in \mathbb{Z}^{dM}$ and either $\psi_\varepsilon(\underline{i}, \cdot) \equiv 0$ or $\psi_\varepsilon(\underline{i}, \underline{u}) > 0$ for all $\underline{u} \neq \pm\underline{1}$, where we have used the notation

$$\underline{i} := (i_1, \ldots, i_M), \quad \underline{u} := (u_1, \ldots, u_M), \quad \underline{1} = (1, \ldots, 1).$$

Hypothesis (HM0) implies the $\mathcal{E}_\varepsilon$ has a *ferromagnetic behavior*, in the sense that

$$\min \mathcal{E}_\varepsilon = \mathcal{E}_\varepsilon(\underline{u}) = 0 \quad \text{if and only if} \quad \underline{u} \equiv \pm 1.$$

As for pairwise interactions, through the change of variables $i = i_1$, $\xi_l = i_{l+1} - i$, $l \in \{1, \ldots, M-1\}$, we find convenient to rewrite $E_\varepsilon(u)$ for $u \in \mathcal{PC}_\varepsilon(\Omega)$ as

$$\mathcal{E}_\varepsilon(u) = \sum_{\underline{\xi} \in \mathbb{Z}^{d(M-1)}} \sum_{i, i+\xi_1, \ldots, i+\xi_{M-1} \in \Omega_\varepsilon} \varepsilon^{d-1} \psi_\varepsilon(i, i + \underline{\xi}, u_i, u_{i+\underline{\xi}}),$$

where we have used the notation

$$\underline{\xi} := (\xi_1, \ldots, \xi_{M-1}), \quad i+\underline{\xi} := (i+\xi_1, \ldots, i+\xi_{M-1}), \quad u_{i+\underline{\xi}} = (u_{i+\xi_1}, \ldots, u_{i+\xi_{M-1}}).$$

The analogue of assumptions (H1)–(H3) is given by the following set of hypotheses:

(HM1) (*coerciveness*) $\min\{\psi_\varepsilon(i, i+\underline{\xi}, \underline{u}) : \exists\, l \in \{1, \ldots, M-1\}$ with $\xi_l = e_k, \ u_{l+1} = -u_1\} \geq c > 0$ for all $\varepsilon > 0$ and $k \in \{1, \ldots, d\}$;

(HM2) (*growth*) set $c_{\varepsilon,\underline{\xi}} = \sup_{i,\underline{u}} \psi_\varepsilon(i, \underline{\xi}, \underline{u})$ it holds $\limsup\limits_{\varepsilon \to 0} \sum\limits_{\underline{\xi} \in \mathbb{Z}^{d(M-1)}} |\underline{\xi}| c_{\varepsilon,\underline{\xi}} < +\infty$;

(HM3) (*decay*) $\forall\, \delta > 0$ there exists $R_\delta > 0$ such that $\limsup\limits_{\varepsilon \to 0} \sum\limits_{|\underline{\xi}| > R_\delta} |\underline{\xi}| c_{\varepsilon,\underline{\xi}} < \delta$,

where we denote by $|\underline{\xi}|$ the standard euclidean norm of $\underline{\xi}$ as a vector in $\mathbb{R}^{d(M-1)}$.

We define also a local version of the functionals in (4.27), by setting for any $A \in \mathcal{A}(\Omega)$

$$\mathcal{E}_\varepsilon(u, A) = \begin{cases} \sum\limits_{i_1, \ldots, i_M \in A_\varepsilon} \varepsilon^{d-1} \psi_\varepsilon(i_1, \ldots, i_M, u_{i_1}, \ldots, u_{i_M}) & \text{if } u \in \mathcal{P}C_\varepsilon(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.28}$$

By applying the abstract method exploited in the previous section, we can show the all possible Γ-limits of $\mathcal{E}_\varepsilon(u, A)$ satisfy the hypotheses of Theorem 2.2 and prove the following theorem, which is the analogue of Theorems 4.2. We omit its proof, since it follows, step by step, the proof of Theorem 4.2 without any significant changes.

Theorem 4.13 *Let ψ_ε satisfy (HM0)–(HM3) and let $\mathcal{E}_\varepsilon$ be defined by (4.27). Then for any sequence $\varepsilon_n \to 0$ there exist a subsequence (ε_{n_k}) and a function $\varphi : \Omega \times S^{d-1} \to [0, +\infty)$ such that the functionals $\mathcal{E}_{\varepsilon_{n_k}}$ Γ-converge with respect to the $L^1(\Omega)$ strong topology to the functional $\mathcal{E} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$\mathcal{E}(u) = \begin{cases} \int_{S_u} \varphi(x, \nu_u)\, \mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.29}$$

Moreover for any $u \in BV(\Omega; \{\pm 1\})$ and $A \in \mathcal{A}^{reg}(\Omega)$ there holds

$$\Gamma\text{-}\lim_k \mathcal{E}_{\varepsilon_{n_k}}(u, A) = \mathcal{E}(u, A) := \int_{S_u \cap A} \varphi(x, \nu_u)\, \mathcal{H}^{d-1}. \tag{4.30}$$

Under periodicity assumptions on the interaction potentials, the analogue of the homogenization result stated in Theorem 4.7 holds true. More precisely, assume that

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(HMP) $\psi_\varepsilon(i, i + \underline{\xi}, \underline{u}) = \psi(i, \underline{\xi}, \underline{u})$, where, for any $\underline{\xi} \in \mathbb{Z}^{d(M-1)}$ and $\underline{u} \in \{-1, 1\}^M$, $\psi(\cdot, \underline{\xi}, \underline{u})$ is $[0, l]^d$-periodic for some $l \in \mathbb{N}$.

In this case assumptions (HM2) and (HM3) reduce to:

(HM2') $\displaystyle\sum_{\underline{\xi} \in \mathbb{Z}^{d(M-1)}} |\xi| \sup_{\underline{u}} \psi(i, \underline{\xi}, \underline{u}) < +\infty$ for any $\underline{i} \in \{0, \ldots, l-1\}^d$.

Theorem 4.14 *Let ψ_ε satisfy (HM0), (HM1), (HMP) and (HM2') and let $\mathcal{E}_\varepsilon$ be defined by (4.27). Then the functionals $\mathcal{E}_\varepsilon$ Γ-converge with respect to the $L^1(\Omega)$ strong topology, as $\varepsilon \to 0$, to the homogenized functional $\mathcal{E}_{hom} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$\mathcal{E}_{hom}(u) = \begin{cases} \displaystyle\int_{S_u} \varphi_{hom}(\nu_u) \, \mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.31}$$

where

$$\varphi_{hom}(\nu) = \lim_{T \to +\infty} T^{1-d} \inf \Bigg\{ \sum_{\underline{\xi} \in \mathbb{Z}^{d(M-1)}} \sum_{i \in \mathbb{Z}^d \cap TQ_\nu} \psi(i, \underline{\xi}, u_i, u_{i+\underline{\xi}}) : \ u \in PC_1(\mathbb{R}^d),$$
$$u(i) = u_{0,\nu}(i) \text{ for } i \notin TQ_\nu \Bigg\} \tag{4.32}$$

Again the proof follows step by step that of Theorem 4.7, taking into account that the convergence of boundary value problems stated in Theorem 4.9 holds also in the present case of multi-body interactions.

The following example provides an approximation of the usual l^1-anisotropic perimeter in $\mathbb{R}^2$ by means of energies accounting for nearest-neighbor 2-body and 3-body interactions. The related asymptotics highlights how the presence of non pairwise interaction potentials may induce formation of special optimal patterns among those arising in the simpler 'central' case.

Example 4.15 *Let $\overline{\mathcal{E}}_\varepsilon(u)$ be defined as in (4.27) with $d = 2$ and*

$$\psi_\varepsilon(i, j, k, u_i, u_j, u_k) = |u_i - u_j| + |u_i - u_k| + |u_i - u_j||u_i - u_k|$$

if $(i, j, k) = (i, i + e_1, i + e_2)$ and 0 otherwise. Note that $\overline{\mathcal{E}}_\varepsilon$ is obtained by adding the 3-body interaction potential $|u_i - u_{i+e_1}||u_i - u_{i+e_2}|$ in the definition (4.6) of the functionals G_ε. In fact $\overline{\mathcal{E}}_\varepsilon$ and G_ε share the same Γ-limit, that is $\overline{\mathcal{E}}_\varepsilon(u)$ Γ-converge to the functional $\overline{\mathcal{E}}(u)$ defined as in (4.31), with $\varphi_{hom}(\nu) = |\nu|_1$. In order to prove this result, since the functionals $\overline{\mathcal{E}}_\varepsilon$ satisfy the hypotheses of Theorem 4.14, it suffices to check that $\Gamma\text{-}\lim_{\varepsilon \to 0} \overline{\mathcal{E}}_\varepsilon(u_{0,\nu}) = \overline{\mathcal{E}}(u_{0,\nu})$ for any $\nu \in S^{d-1}$. The Γ-liminf inequality follows at once by neglecting the 3-body interaction potential, that is, using the inequality

$$\mathcal{E}_\varepsilon(u) \geq G_\varepsilon(u).$$

Conversely, it is easy to check that, given $N_\varepsilon \in \mathbb{N}$ such that $\lim_\varepsilon \varepsilon N_\varepsilon = 0$ and set $\delta_\varepsilon := \varepsilon N_\varepsilon$, then the functions $u_\varepsilon \in \mathcal{PC}_{\delta_\varepsilon}(\Omega) \subset \mathcal{PC}_\varepsilon(\Omega)$ defined by $u_\varepsilon(\delta_\varepsilon i) = u_{0,\nu}(\delta_\varepsilon i)$, $i \in \mathbb{Z}^d$, is a recovery sequence for G_ε at $u_{0,\nu}$. A direct computation shows that the 3-body potential gives a positive contribution to $\mathcal{E}_\varepsilon(u_\varepsilon)$ only on the 'corners' of the interface of u_ε, that is when the interface of u_ε changes direction. Hence, whenever $\nu \notin \{e_1, e_2\}$ we get

$$\mathcal{E}_\varepsilon(u_\varepsilon) = G(u_\varepsilon) + CN_\varepsilon^{-1}.$$

Thus, u_ε is a recovery sequence for $\mathcal{E}_\varepsilon$ at $u_{0,\nu}$ if and only if $\lim_{\varepsilon \to 0} N_\varepsilon = +\infty$. The computation above shows that the 3-body potential does not give any contribution to the limiting energy but acts as a selector of the optimal states of G_ε.

5 Approximation of non local continuum functionals

In this section we show if hypothesis (H3) is violated, then energies as in (4.1) can approximate non-local functionals. We will focus on the approximation of two relevant examples of non-local functionals: fractional perimeters and Ohta-Kawasaki type energies.

5.1 Fractional perimeters

We first recall the definition of fractional perimeters.

Definition 5.1 *Let $E \subset \Omega$ and $0 < s < 1$. The s-fractional perimeter $P_s(E; \Omega)$ of E is defined as follows*

$$P_s(E, \Omega) := \int_E \int_{E^c \cap \Omega} \frac{1}{|y - x|^{d+s}} \, dx \, dy. \tag{5.1}$$

Note that $P_s(E, \Omega)$ is simply half of the fractional Sobolev space seminorm $|\chi_E|_{W^{s,1}(\Omega)}$, where χ_E denotes the characteristic function of E. Moreover, if $u := \chi_E - \chi_{E^c}$ then

$$P_s(E, \Omega) = \frac{1}{4}|u|_{W^{s,1}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+s}} \, dx \, dy. \tag{5.2}$$

We recall that if E is a set of finite perimeter in Ω, then $P_s(E, \Omega)$ is also finite, since in general $BV(\Omega) \subset W^{s,1}(\Omega)$ for any $s \in (0, 1)$. We will approximate functionals defined on $BV(\Omega; \{\pm 1\})$ and of the form

$$\int_{S_u} \varphi(\nu_u) \, \mathcal{H}^{d-1} + \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+s}} \, dx \, dy. \tag{5.3}$$

By the results in [2], we may approximate the first term in (5.3) by ferromagnetic energies accounting for example only for nearest neighbors interactions. In order to approximate the second term in (5.3), the idea is simply to discretize it. In fact, setting $Q_i^\varepsilon = \varepsilon i + [0, \varepsilon)^d$ for $i \in \mathbb{Z}^d$, we have

$$\int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+s}} \, dx \, dy = \sum_{i,j} \int_{(\Omega \cap Q_i^\varepsilon) \times (\Omega \cap Q_j^\varepsilon)} \frac{|u(y) - u(x)|}{|y - x|^{d+s}} \, dx \, dy$$

$$= \sum_{i,j \in \Omega_\varepsilon} \varepsilon^{2d} \frac{|u(\varepsilon j) - u(\varepsilon i)|}{|\varepsilon(j - i)|^{d+s}} + o(1) = \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{d-1} c_\xi^\varepsilon |u_{i+\xi} - u_i| + o(1),$$

where $c_\xi^\varepsilon := \frac{\varepsilon^{1-s}}{|\xi|^{d+s}}$.

Let, then, $\overline{E}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ be defined by

$$\overline{E}_\varepsilon(u) := \begin{cases} \displaystyle\sum_{|j-i|=1} \varepsilon^{d-1} |u_j - u_i| + \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{d-s} \frac{|u_{i+\xi} - u_i|}{|\xi|^{d+s}} \\ =: E_\varepsilon^{loc}(u) + E_\varepsilon^{nl}(u) & \text{if } u \in \mathcal{PC}_\varepsilon(\Omega) \\ \\ +\infty & \text{otherwise} \end{cases} \quad (5.4)$$

In the next proposition we show that E_ε^{nl} is essentially a 'continuous' perturbation of E_ε^{loc}.

Proposition 5.2 *Let* $u_\varepsilon : \Omega_\varepsilon \to \{\pm 1\}$ *be such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$ *and* $\sup_\varepsilon \mathcal{H}^{d-1}(S_{u_\varepsilon}) < +\infty$. *Then* $u \in BV(\Omega; \pm 1)$ *and*

$$\lim_{\varepsilon \to 0} E_\varepsilon^{nl}(u_\varepsilon) = \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+s}} \, dx \, dy. \quad (5.5)$$

PROOF.

Let us extend u_ε outside Ω_ε by setting $u_\varepsilon(\varepsilon i) = 1$ if $i \in \mathbb{Z}^d \setminus \varepsilon^{-1}\Omega$. Since $\partial\Omega$ is Lipschitz, we still have that $\sup_\varepsilon \mathcal{H}^{d-1}(S_{u_\varepsilon}) < +\infty$, which implies that for any $h \in \mathbb{R}^d$ it holds

$$\int_{\mathbb{R}^d} |u_\varepsilon(x + h) - u_\varepsilon(x)| \, dx \leq C|h|, \quad (5.6)$$

with $C > 0$ independent of ε and h. Given $\delta > 0$, we split $E_\varepsilon^{nl}(u_\varepsilon)$ into two terms accounting for the interactions at distance greater and less than δ, respectively, i.e.

$$\begin{aligned} E_\varepsilon^{nl}(u_\varepsilon) &= \sum_{\xi \in \mathbb{Z}^d \cap B_{\delta/\varepsilon}} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{d-s} \frac{|(u_\varepsilon)_{i+\xi} - (u_\varepsilon)_i|}{|\xi|^{d+s}} \\ &+ \sum_{\xi \in \mathbb{Z}^d \setminus B_{\delta/\varepsilon}} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{d-s} \frac{|(u_\varepsilon)_{i+\xi} - (u_\varepsilon)_i|}{|\xi|^{d+s}} =: I_{\varepsilon,\delta}^1 + I_{\varepsilon,\delta}^2 \quad (5.7) \end{aligned}$$

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It is easy to show that

$$I_{\varepsilon,\delta}^2 = \int_{\Omega \times \Omega \cap \{|y-x|>\delta\}} \frac{|u_\varepsilon(y) - u_\varepsilon(x)|}{|y-x|^{d+s}} \, dx \, dy + o(1),$$

from which we get that

$$\lim_{\varepsilon \to 0} I_{\varepsilon,\delta}^2 = \int_{\Omega \times \Omega \cap \{|y-x|>\delta\}} \frac{|u(y) - u(x)|}{|y-x|^{d+s}} \, dx \, dy. \tag{5.8}$$

We proceed now by estimating $I_{\varepsilon,\delta}^1$. It can be easily shown that

$$I_{\varepsilon,\delta}^1 \le \sum_{\xi \in \mathbb{Z}^d \cap B_{\delta/\varepsilon}} \frac{\varepsilon^{-s}}{|\xi|^{d+s}} \int_{\mathbb{R}^d} |u_\varepsilon(x + \varepsilon\xi) - u_\varepsilon(x)| \, dx$$

Hence, by (5.6), we get

$$I_{\varepsilon,\delta}^1 \le C\varepsilon^{1-s} \sum_{\xi \in \mathbb{Z}^d \cap B_{\delta/\varepsilon}} \frac{1}{|\xi|^{d+s-1}}. \tag{5.9}$$

Note that

$$\sum_{\xi \in \mathbb{Z}^d \cap B_{\delta/\varepsilon}} \frac{1}{|\xi|^{d+s-1}} = \sum_{k=1}^{[\delta/\varepsilon]-1} \sum_{\xi \in \mathbb{Z}^d \cap (B_{k+1} \setminus B_k)} \frac{1}{|\xi|^{d+s-1}}$$

$$\le \sum_{k=1}^{[\delta/\varepsilon]-1} \frac{\#(\mathbb{Z}^d \cap (B_{k+1} \setminus B_k))}{k^{d+s-1}} \le C \sum_{k=1}^{[\delta/\varepsilon]-1} \frac{1}{k^s} \le C\left(\frac{\delta}{\varepsilon}\right)^{1-s}.$$

Thus, from (5.9) we deduce that

$$I_{\varepsilon,\delta}^1 \le C\delta^{1-s}. \tag{5.10}$$

Eventually, by (5.7), (5.8) and (5.10), we infer that

$$\lim_{\varepsilon \to 0} E_\varepsilon^{nl}(u_\varepsilon) = \int_{\Omega \times \Omega \cap \{|y-x|>\delta\}} \frac{|u(y) - u(x)|}{|y-x|^{d+s}} \, dx \, dy + O(\delta^{1-s})$$

and the conclusion follows by the arbitrariness of $\delta > 0$. $\qquad\square$

As a straightforward consequence of Proposition 5.2 and the results in [2], we derive the following Γ-convergence result.

Theorem 5.3 *The functionals $\overline{E}_\varepsilon$, defined in (5.4), Γ-converge with respect to the $L^1(\Omega)$ strong topology to the functional $\overline{E} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$\overline{E}(u) = \begin{cases} \displaystyle\int_{S_u} |\nu_u|_1 \, \mathcal{H}^{d-1} + \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y-x|^{d+s}} \, dx \, dy & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.11}$$

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PROOF. Notice that $E_\varepsilon^{loc}(u) = G_\varepsilon(u)$ for $u \in \mathcal{P}C_\varepsilon(\Omega)$, where G_ε is defined in (4.6), and we have already recalled that in [2] it was proved that

$$\Gamma\text{-}\lim_{\varepsilon \to 0} G_\varepsilon(u) = \begin{cases} \displaystyle\int_{S_u} |\nu_u|_1 \, \mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.12}$$

Hence, the thesis follows by Proposition 5.2.

$\square$

Remark 5.4 *Note that the functional $\overline{E}_\varepsilon$ defined in (5.4) is of the form (4.1) with $c_{i,e_k}^\varepsilon = 1 + \varepsilon^{1-s}$, for $k \in \{1, \dots, d\}$, $c_{i,\xi}^\varepsilon = \frac{\varepsilon^{1-s}}{\xi^{d+s}}$ if $\xi \in (\mathbb{Z}^d \cap B_{R/\varepsilon}) \setminus \{e_1, \dots, e_d\}$ and $c_{i,\xi}^\varepsilon = 0$ otherwise, where $R := \operatorname{diam}(\Omega)$. A direct computation shows that for any $R > 0$*

$$\limsup_{\varepsilon \to 0} \sum_{\xi \in \mathbb{Z}^d \cap B_{R/\varepsilon}} \frac{\varepsilon^{1-s}}{|\xi|^{d+s}} < +\infty; \quad \limsup_{\varepsilon \to 0} \sum_{\xi \in \mathbb{Z}^d \cap (B_{R/\varepsilon} \setminus B_{R/2\varepsilon})} \frac{\varepsilon^{1-s}}{|\xi|^{d+s}} \geq C > 0. \tag{5.13}$$

Hence hypotheses (H1), (H2) are satisfied, while (H3) is violated.

Remark 5.5 (Generalization) *The argument used to prove Proposition 5.2 and Theorem 5.3 can be easily adapted to approximate more general non-local functionals. More precisely, let $K : \Omega \times \Omega \to [0, +\infty)$ be such that $K \in C^1(\Omega \times \Omega \setminus \{x = y\})$ and $K(x, y) \leq C|y - x|^{-d-s}$ in a neighborhood of $\{x = y\}$ for some $s \in (0, 1)$. Moreover let $H_\varepsilon : L^1(\Omega) \to [0, +\infty]$ be defined by*

$$H_\varepsilon(u) := \begin{cases} \displaystyle\sum_{|j-i|=1} \varepsilon^{d-1}|u_j - u_i| \\ \quad + \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{2d} K(\varepsilon i, \varepsilon(i+\xi))|u_{i+\xi} - u_i| & \text{if } u \in \mathcal{P}C_\varepsilon(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and let $H : L^1(\Omega) \to [0, +\infty]$ defined by

$$H(u) = \begin{cases} \displaystyle\int_{S_u} |\nu_u|_1 \, \mathcal{H}^{d-1} + \int_\Omega \int_\Omega K(x, y)|u(y) - u(x)| \, dx \, dy & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases}$$

Then Theorem 5.3 still holds with H_ε in place of $\overline{E}_\varepsilon$ and H in place of $\overline{E}$.

5.2 Ohta-Kawasaki type energies

A canonical mathematical model in the studies of energy-driven pattern forming systems is based on the following energy first proposed by Ohta-Kawasaki, see [16]:

$$\mathcal{E}_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (1 - u^2)^2 \, dx + \gamma_0 \int_\Omega \int_\Omega G(x, y) u(x) u(y) \, dx dy. \tag{5.14}$$

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Here u is an $H^1(\Omega)$ phase parameter describing the density distribution of the components ($u = -1$ stands for one phase, $u = +1$ for the other), subject to the constraint $m = \fint_\Omega u \, dx$, that is the difference of the phases' volume fractions is prescribed, and G is the Green's function for $-\Delta$. The first two integrals in (5.14) form the so called 'Modica-Mortola' energy which Γ-converges as $\varepsilon \to 0$ to the perimeter functional. It can be easily shown that the non-local term in (5.14) is an L^1 continuous perturbation of the Modica-Mortola energy. As a consequence, the functionals $\mathcal{E}_\varepsilon$ Γ-converge with respect to the $L^1(\Omega)$ norm to the functional $\mathcal{E} : L^1(\Omega) \to [0, +\infty]$ given by

$$
\mathcal{E}(u) := \begin{cases} \dfrac{8}{3}\mathcal{H}^{d-1}(S_u) + \gamma_0 \displaystyle\int_\Omega \int_\Omega G(x,y)u(x)u(y)\,dx\,dy & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.15}
$$

We provide a variational approximation of an anisotropic version of the functional in (5.15), that is functionals finite on $BV(\Omega; \{\pm 1\})$ and of the type

$$
\int_{S_u} \varphi(\nu_u)\,d\mathcal{H}^{d-1}(S_u) + \gamma_0 \int_\Omega \int_\Omega G(x,y)u(x)u(y)\,dx\,dy.
$$

We may 'discretize' the non-local as in the previous section, that is

$$
\begin{aligned}
\int_\Omega \int_\Omega G(x,y)u(x)u(y)\,dx\,dy &= \sum_{i,j} \int_{(\Omega \cap Q_i^\varepsilon) \times (\Omega \cap Q_j^\varepsilon)} G(x,y)u(x)u(y)\,dx\,dy \\
&= \sum_{i,j \in \Omega_\varepsilon} \varepsilon^{2d} G(\varepsilon i, \varepsilon j)u(\varepsilon i)u(\varepsilon j) + o(1) \\
&= \sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{2d} G(\varepsilon i, \varepsilon(i + |\xi|))u_{i+\xi}u_i + o(1).
\end{aligned} \tag{5.16}
$$

We note that the Green's function G satisfies

$$
|G(x,y)| \leq C|x-y|^{d-1}
$$

and in particular

$$
\int_\Omega \int_\Omega |G(x,y)|\,dx\,dy < +\infty. \tag{5.17}
$$

A discrete version of (5.17) is provided by the following estimate

$$
\begin{aligned}
\sum_{\xi \in \mathbb{Z}^d} \sum_{i, i+\xi \in \Omega_\varepsilon} \varepsilon^{2d} G(\varepsilon i, \varepsilon(i + |\xi|)) &\leq C \sum_{\xi \in \mathbb{Z}^d \cap B_{r/\varepsilon}} \varepsilon^d \frac{1}{|\varepsilon \xi|^{d-1}} \\
&= C\varepsilon \sum_{\xi \in \mathbb{Z}^d \cap B_{r/\varepsilon}} \frac{1}{|\xi|^{d-1}} \leq C, \tag{5.18}
\end{aligned}
$$

where we have set $r := \text{diam}(\Omega)$.

Let, then, $\hat{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ be defined by

$$
\hat{F}_\varepsilon(u) := \begin{cases} -\displaystyle\sum_{|j-i|=1} \varepsilon^d u_i u_j + \gamma_0 \sum_{\xi \in \mathbb{Z}^d} \sum_{i,i+\xi \in \Omega_\varepsilon} \varepsilon^{2d+1} G(\varepsilon i, \varepsilon(i+|\xi|)) u_{i+\xi} u_i \\ =: \hat{F}_\varepsilon^{loc}(u) + \hat{F}_\varepsilon^{nl}(u) \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } u \in \mathcal{PC}_\varepsilon(\Omega) \\[2mm] +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}
$$

Note that $\hat{F}_\varepsilon$ is of the form (3.1). Moreover, we underline that, since $G(\cdot, \cdot)$ changes sign, $\hat{F}_\varepsilon$ mixes ferromagnetic and anti-ferromagnetic interactions. Hence the uniform states $u \equiv \pm 1$ are not absolute minimizers of $\hat{F}_\varepsilon$. Neverthless, estimate (5.18) yields that $\hat{F}_\varepsilon^{nl}(u)$ vanishes uniformly w.r.t. u as ε goes to 0, and then

$$
\hat{F}_\varepsilon(\pm 1) = \min \hat{F}_\varepsilon + O(\varepsilon) = \hat{F}_\varepsilon^{loc}(\pm 1) + O(\varepsilon).
$$

We then consider the scaled energies

$$
\hat{E}_\varepsilon(u) := \frac{\hat{F}_\varepsilon(u) - \hat{F}_\varepsilon^{loc}(\pm 1)}{\varepsilon},
$$

which can be written as

$$
\hat{E}_\varepsilon(u) = \begin{cases} \displaystyle\sum_{|j-i|=1} \varepsilon^{d-1} |u_j - u_i| + \gamma_0 \sum_{\xi \in \mathbb{Z}^d} \sum_{i,i+\xi \in \Omega_\varepsilon} \varepsilon^{2d} G(\varepsilon i, \varepsilon(i+|\xi|)) u_{i+\xi} u_i \\ =: \hat{E}_\varepsilon^{loc}(u) + \hat{E}_\varepsilon^{nl}(u) \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } u \in \mathcal{PC}_\varepsilon(\Omega) \\[2mm] +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \qquad (5.19)
$$

Arguing as in the proof of Proposition 5.2, it can be proved the following result.

Proposition 5.6 *Let $u_\varepsilon : \Omega_\varepsilon \to \{\pm 1\}$ be such that $u_\varepsilon \to u$ in $L^1(\Omega)$. Then*

$$
\lim_{\varepsilon \to 0} \hat{E}_\varepsilon^{nl}(u_\varepsilon) = \int_\Omega \int_\Omega G(x, y) u(x) u(y) \, dx \, dy. \qquad (5.20)
$$

As a consequence of Proposition 5.6 and the result of [2] stated in (5.12), we derive the following Γ-convergence result. Its proof is straightforward.

Theorem 5.7 *The functionals $\hat{E}_\varepsilon$, defined in (5.19), Γ-converge with respect to the $L^1(\Omega)$ strong topology to the functional $\hat{E} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$
\hat{E}(u) := \begin{cases} \displaystyle\int_{S_u} |\nu_u|_1 \, d\mathcal{H}^{d-1}(S_u) + \gamma_0 \int_\Omega \int_\Omega G(x, y) u(x) u(y) \, dx \, dy \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } u \in BV(\Omega; \{\pm 1\}) \quad (5.21) \\[2mm] +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}
$$

The result in Theorem 5.7 can be easily extended to the periodic case, when $\Omega = \mathbb{T}^d$, and to functions satisfying a mean constraint. More precisely, assume $\varepsilon = \frac{1}{n}$ and, given $l_n \in \{0, \ldots n^d\}$, set

$$\mathcal{P}C_n^{l_n}(\mathbb{T}^d) := \{u : \frac{1}{n}\mathbb{Z}^d \to \{-1,1\}, \quad u \ [0,1)^d\text{-periodic and}$$
$$\#\{i \in ([0,n) \cap \mathbb{Z})^d : u_i = 1\} = l_n\}.$$

We assume that there exists $p \in [0,1]$ such that

$$\lim_{n \to +\infty} n^{-d} l_n = p \qquad (5.22)$$

Then, consider the functionals $E_n^{l_n} : L^1(\mathbb{T}^d) \to [0, +\infty]$ defined as

$$E_n^{l_n}(u) = \begin{cases} \displaystyle\sum_{|j-i|=1} n^{1-d}|u_j - u_i| + \gamma_0 \sum_{\xi \in \mathbb{Z}^d} \sum_{i \in nQ \cap \mathbb{Z}^d} n^{-2d}G\big(\frac{i}{n}, \frac{(i+|\xi|)}{n}\big)u_{i+\xi}u_i \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathcal{P}C_n^{l_n}(\mathbb{T}^d) \\ \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \qquad (5.23)$$

Then, by following the same steps of the proof of Theorem 4.11 and taking into account Theorem 7.1 and Remark 7.2, it can be proved the following result.

Theorem 5.8 *Let (l_n) satisfy (5.22). Then, the functionals $E_n^{l_n}$, defined in (5.23), Γ-converge with respect to the $L^1(\mathbb{T}^d)$ strong topology to the functional $E^p : L^1(\mathbb{T}^d) \to [0, +\infty]$ defined by*

$$E^p(u) := \begin{cases} \displaystyle\int_{S_u} |\nu_u|_1 \, d\mathcal{H}^{d-1}(S_u) + \gamma_0 \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} G(x,y)u(x)u(y) \, dx \, dy \\ \qquad\qquad\qquad \text{if } u \in BV(\mathbb{T}^d; \{\pm1\}) \text{ and } |\{u=1\}| = p \\ \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases} \qquad (5.24)$$

The previous result allows to deduce the existence of 'lamellar-type' local minimizers for the discrete energies $E_n^{p_n}$: according to a definition introduce in [1], we say that a function $u \in \mathcal{P}C_n^{l_n}(\mathbb{T}^d)$ is an isolated L^1-local minimizer for the functional $E_n^{l_n}$ if there exists $\delta > 0$ such that

$$E_n^{l_n}(u) \leq E_n^{l_n}(v), \text{ for all } v \in \mathcal{P}C_n^{l_n}(\mathbb{T}^d) \text{ with } 0 < \min_{\tau \in \frac{1}{n}\mathbb{Z}^d} \|u - v(\cdot + \tau\|_{L^1(\mathbb{T}^d)} \leq \delta.$$

An analogous definition of isolated local minimizers can be given in the continuum for the functional $E^p(u)$.

In [1], Proposition 5.6 it was proved the following result: given $p \in (0,1)$, set

$$L_k := T^{d-1} \times \cup_{l=1}^k \left[\frac{l-1}{k}, \frac{l-1}{k} + \frac{p}{k}\right], \quad k \in \mathbb{N} \qquad (5.25)$$

and denote by $\mathcal{L}_{p,k}$ the collection of all sets which may be obtained from L_k by translations and relabeling of coordinates. Then, for any $\gamma_0 > 0$ and $L \in \mathcal{L}_{p,k}$ the function $u_L := \chi_L - \chi_{L^c}$ is an isolated L^1-local minimizer for the functional $\mathcal{E}^p : L^1(\mathbb{T}^d) \to [0, +\infty]$ defined by

$$
\mathcal{E}^p(u) := \begin{cases} \mathcal{H}^{d-1}(S_u) + \gamma_0 \displaystyle\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} G(x,y) u(x) u(y) \, dx \, dy \\ \qquad\qquad\qquad \text{if } u \in BV(\mathbb{T}^d; \{\pm 1\}) \text{ and } |\{u = 1\}| = p \quad (5.26) \\[2ex] +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}
$$

for k large enough. Since $\mathcal{H}^{d-1}(S_u) \leq \int_{S_u} |\nu_u|_1 \, d\mathcal{H}^{d-1}$ and the equality holds for u_L, we get that u_L is also an isolated L^1-local minimizer for the functional E^p. Hence, as a straightforward consequence of Theorem 2.3 and Theorem 5.8, we deduce the following result.

Theorem 5.9 (Local minimality of lamellae) *There exist $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $k > k_0$ and $L \in \mathcal{L}_{p,k}$ there exists $n_0 \in \mathbb{N}$ and a family $\{u_n\}_{n > n_0}$ of isolated local minimizers of $E_n^{l_n}$ such that $u_n \to u_L$ strongly $L^1(\mathbb{T}^d)$, as $n \to +\infty$, where $u_L := \chi_L - \chi_{L^c}$.*

6 General criterion for a ferromagnetic behavior

In this section we consider energies of type (3.1) in the homogeneous case $c_{i,\xi}^\varepsilon = c_\xi$ for all i and ξ, but without constraints on the sign of such constants. Our main goal is to provide a general criterion which ensures that such energies still have a ferromagnetic behavior, that is the ground states are still the uniform states $u \equiv 1$ and $u \equiv -1$ and the continuum limit of the scaled energies is an interfacial energy of the form (4.4). We restrict our analysis to the case in which only short range interactions are taken into account, that is there exists $R > 0$ such that $c_\xi = 0$ if $|\xi| > R$. Hence, we consider energies of the form

$$
F_\varepsilon(u) = - \sum_{\xi \in \mathbb{Z}^d \cap B(0,R)} \sum_{i, i+\xi \in \Omega_\varepsilon} c_\xi \varepsilon^d u_i u_{i+\xi}. \qquad (6.1)
$$

on $\mathcal{P}C_\varepsilon(\Omega)$ and equal to $+\infty$ otherwise in $L^1(\Omega)$. In order to introduce and better explain the ferromagnetic criterion we are going to define we first consider two one-dimensional examples.

Example 6.1 (NN and NNN interactions) *Let $\Omega = (0,1)$ and let F_ε the energy accounting for nearest and next-to-nearest neighbours interactions defined by*

$$
F_\varepsilon(u) := -c_1 \sum_{i,(i+1) \in \mathbb{Z} \cap (0, \varepsilon^{-1})} \varepsilon u_i u_{i+1} - c_2 \sum_{i,(i+2) \in \mathbb{Z} \cap (0, \varepsilon^{-1})} \varepsilon u_i u_{i+2}.
$$

Note that, given $\lambda \in \mathbb{R}$, we may rewrite F_ε as

$$F_\varepsilon(u) = \sum_{i \in \mathbb{Z} \cap (0, \varepsilon^{-1} - 2)} \varepsilon(-c_1(\lambda u_i u_{i+1} + (1-\lambda) u_{i+1} u_{i+2}) - c_2 u_i u_{i+2}) + O(\varepsilon), \quad (6.2)$$

where the additional infinitesimal term is due to the energetic contribution of the interactions near 0 and 1. Set, for $v : \{0, 1, 2\} \to \{-1, 1\}$,

$$F^2_{cell}(v) := -c_1(\lambda v_0 v_1 + (1-\lambda) v_1 v_2) - c_2 v_0 v_2.$$

Then a condition which guarantees that the uniform states -1 and 1 are the only two minimizers of F_ε up to lower order terms is the following

$$F^2_{cell}(v) > F^2_{cell}(\pm 1) = -c_1 - c_2 \quad \forall\, v : \{0, 1, 2\} \to \{-1, 1\}, \ v \not\equiv \pm 1. \qquad (6.3)$$

Indeed, if (6.3) is satisfied, formula (6.2) immediately infers that

$$F_\varepsilon(\pm 1) = \min F_\varepsilon + O(\varepsilon).$$

It is easy to show that (6.3) is satisfied with $\lambda = \frac{1}{2}$ if and only if $c_1, c_2 \in A := \{c_1 > 0, \ 2c_2 > -c_1\}$. Moreover, easy computations show that

1) *if $c_1, c_2 \in B := \{c_1 < 0, \ 2c_2 > c_1\}$ we have two ground states given by the alternating states $(-1)^i$ and $-(-1)^i$;*

2) *if $c_1, c_2 \in C := \{c_1 < 0, \ 2c_2 < c_1\} \cup \{c_1 > 0, \ 2c_2 < -c_1\}$ we have four ground states given by $(-1)^{\left[\frac{i+k}{2}\right]}$, $k = 0, 1, 2, 3$;*

3) *if $c_1, c_2 \in \overline{A} \cap \overline{B} = \{c_1 = 0, \ c_2 > 0\}$ we have four ground states given by the uniform states 1 and -1 and the alternating states given by $(-1)^i$ and $-(-1)^i$; if $c_1, c_2 \in \overline{A} \cap \overline{C} = \{c_1 > 0, \ 2c_2 = -c_1\}$ we have six ground states given by the uniform states 1 and -1 and the states $(-1)^{\left[\frac{i+k}{2}\right]}$, $k = 0, 1, 2, 3$; if $c_1, c_2 \in \overline{B} \cap \overline{C} = \{c_1 < 0, \ 2c_2 = c_1\}$ we have six ground states given by the alternating states $(-1)^i$ and $-(-1)^i$ and the states $(-1)^{\left[\frac{i+k}{2}\right]}$, $k = 0, 1, 2, 3$.*

Example 6.2 (NN, NNN and NNNN interactions) *Let $\Omega = (0, 1)$ and let F_ε the energy accounting for nearest, next-to-nearest and next-to-next-to-nearest neighbours interactions defined by*

$$F_\varepsilon(u) \quad := \quad -c_1 \sum_{i, (i+1) \in \mathbb{Z} \cap (0, \varepsilon^{-1})} \varepsilon u_i u_{i+1} - c_2 \sum_{i, (i+2) \in \mathbb{Z} \cap (0, \varepsilon^{-1})} \varepsilon u_i u_{i+2}$$
$$-c_3 \sum_{i, (i+3) \in \mathbb{Z} \cap (0, \varepsilon^{-1})} \varepsilon u_i u_{i+3}.$$

Adopting the same argument of the previous example, given $\lambda, \alpha, \beta \in \mathbb{R}$, we may rewrite F_ε as

$$F_\varepsilon(u) \quad = \quad \sum_{i \in \mathbb{Z} \cap (0, \varepsilon^{-1} - 3)} \varepsilon(-c_1(\alpha u_i u_{i+1} + \beta u_{i+1} u_{i+2} + (1 - \alpha - \beta) u_{i+2} u_{i+3})$$

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$$-c_2(\lambda u_i u_{i+2} + (1-\lambda)u_{i+1}u_{i+3}) - c_3 u_i u_{i+3}) + O(\varepsilon). \tag{6.4}$$

Set, for $v : \{0,1,2,3\} \to \{-1,1\}$,

$$F_{cell}^3(v) := -c_1(\alpha v_0 v_1 + \beta v_1 v_2 + (1-\alpha-\beta)v_2 v_3) - c_2(\lambda v_0 v_2 + (1-\lambda)v_1 v_3) - c_3 v_0 v_3.$$

In this case the uniform states -1 and 1 are the only two minimizers of F_ε up to lower order term if there holds

$$F_{cell}^3(v) > F_{cell}^3(\pm 1) = -c_1 - c_2 - c_3 \quad \forall\, v : \{0,1,2,3\} \to \{-1,1\},\ v \not\equiv \pm 1. \tag{6.5}$$

One can check that (6.5) is satisfied in particular if $c_1, c_2, c_3 \in \{c_1 > -3c_3 > 0,\ 3c_2 > -c_1 - 3c_3\} \cup \{c_1 = 0,\ c_2 > 0,\ c_3 > 0\}$. Note that for other values of c_1, c_2 and c_3 it is in principle possible that neither the uniform states 1 and -1 nor different periodic states are minimizers of F_{cell}^3. In this case one could push further the argument before as follows: given $M \in \mathbb{N}$, $M \geq 3$, three set of constants $\{\alpha_1^k, \ldots, \alpha_{M-k+1}^k\}$, $k = 1,2,3$, satisfying $\sum_{j=1}^{M-k+1} \alpha_j^k = 1$ for any k, one can rewrite F_ε as

$$F_\varepsilon(u) \quad = \quad \sum_{i \in \mathbb{Z} \cap (0, \varepsilon^{-1}-M)} \varepsilon \left(-\sum_{k=1}^{3} c_l \sum_{j=1}^{M-k+1} \alpha_j^k u_{i+j-1} u_{i+j+k-1} \right) + O(\varepsilon)$$

and define, for $v : \{0,1,\ldots,M\} \to \{-1,1\}$,

$$F_{cell}^M(v) := -\sum_{k=1}^{3} c_k \sum_{j=1}^{M-k+1} \alpha_j^k v_{j-1} v_{j+k-1}$$

Then, as before, the uniform states -1 and 1 are the only ground states if there holds

$$F_{cell}^M(v) > F_{cell}^M(\pm 1) = -c_1 - c_2 - c_3 \quad \forall\, v : \{0,1,\ldots,M\} \to \{-1,1\},\ v \not\equiv \pm 1. \tag{6.6}$$

We now generalize the argument exploited in the previous examples. Let F_ε be defined by (6.1). Note that, for any $M \in \mathbb{N}$, $M \geq R$, we may rewrite F_ε as

$$F_\varepsilon(u) = \sum_{\varepsilon i \in \Omega_\varepsilon^M} \varepsilon^d \left(-\sum_{|\xi| \leq R} c_\xi \sum_{j,j+\xi \in C_M} \varphi_M(j,\xi) u^{i+j} u^{i+j+\xi} \right) + O(\varepsilon), \tag{6.7}$$

where $C_M := [0,M]^d \cap \mathbb{Z}^d$, $\Omega_\varepsilon^M := \{i \in \mathbb{Z}^d : \varepsilon i + \varepsilon C_M \subset \Omega\}$ and $\varphi_M : C_M \times \mathbb{Z}^d \to \mathbb{R}$ satisfies for any fixed $\xi \in \mathbb{Z}^d$

$$\sum_{j,j+\xi \in C_M} \varphi_M(j,\xi) = 1.$$

Note that the additional infinitesimal term in (6.7) is due to the energetic contribution of the interactions near the boundary of Ω. Set, for $v : C_M \to \{\pm 1\}$,

$$F_{cell}^M(v) := -\sum_{|\xi| \leq R} c_\xi \sum_{j, j+\xi \in C_M} \varphi_M(j, \xi) v^j v^{j+\xi}. \tag{6.8}$$

Definition 6.3 (ferromagnetic criterion) *We say that the family $\{c_\xi : \xi \in \mathbb{Z}^d \cap B(0, R)\}$ has a* ferromagnetic behavior *if there exists $M \in \mathbb{N}$, $M \geq R$ and $\varphi_M : C_M \times \mathbb{Z}^d \to \mathbb{R}$ such that*

$$\sum_{j, j+\xi \in C_M} \varphi_M(j, \xi) = 1 \quad \forall \, \xi \in \mathbb{Z}^d$$

and

$$F_{cell}^M(v) > F_{cell}^M(\pm 1) = -\sum_{|\xi| \leq R} c_\xi \quad \forall \, v : C_M \to \{\pm 1\}, \; v \not\equiv \pm 1. \tag{6.9}$$

Formula (6.9) is a sufficient condition for the minimality of the uniform states 1 and -1. Indeed, if (6.9) is satisfied, formula (6.7) immediately infers that

$$F_\varepsilon(\pm 1) = \min F_\varepsilon + O(\varepsilon)$$

We may, then, consider the scaled energies

$$E_\varepsilon(u) \quad = \quad \frac{F_\varepsilon(u) - F_\varepsilon(\pm 1)}{\varepsilon}, \tag{6.10}$$

which can be written on $\mathcal{PC}_\varepsilon(\Omega)$ as

$$E_\varepsilon(u) = \sum_{\varepsilon i \in \Omega_\varepsilon^M} \varepsilon^{d-1} \left(\sum_{|\xi| \leq R} c_\xi \sum_{j, j+\xi \in C_M} (1 - \varphi_M(j, \xi)) u^{i+j} u^{i+j+\xi} \right) + O(1)$$

Let us set

$$E_{cell}^M(v) := F_{cell}^M(v) - F_{cell}^M(\pm 1) = \sum_{|\xi| \leq R} c_\xi \sum_{j, j+\xi \in C_M} (1 - \varphi_M(j, \xi)) v^j v^{j+\xi}. \tag{6.11}$$

Moreover, with a little abuse of notation, set for $u \in \mathcal{PC}_\varepsilon(\Omega)$ and $i \in \Omega_\varepsilon$

$$E_{cell}^M(u, i + C_M) := E_{cell}^M(u(1/\varepsilon \cdot + i)).$$

Note, then, that we get

$$E_\varepsilon(u) = \sum_{\varepsilon i \in \Omega_\varepsilon^M} \varepsilon^{d-1} E_{cell}^M(u, i + C_M) + O(1). \tag{6.12}$$

In the next proposition we show that the functionals E_ε are equi-coercive with respect to the $L^1(\Omega)$-topology and their possible Γ-limits are finite only on $BV(\Omega; \{-1, \, 1\})$.

Proposition 6.4 *Assume that the family* $\{c_\xi : \ \xi \in \mathbb{Z}^d \cap B(0,R)\}$ *has a ferromagnetic behavior, according to Definition 6.3, and let* E_ε *be defined by (6.10). Let* u_ε *be such that* $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < +\infty$. *Then there exist a subsequence* ε_k *and* $u \in BV(\Omega; \{-1, \ 1\})$ *such that* u_{ε_k} *converge to* u *strongly in* $L^1(\Omega)$.

PROOF. Let $\{\Omega^k\}$ be an increasing sequence of open sets compactly contained in Ω such that $\cup_{k\in\mathbb{N}}\Omega^k = \Omega$. By (6.9) and (6.12), we have that for any $k \in \mathbb{N}$

$$
\begin{aligned}
\mathcal{H}^{d-1}(S(u_\varepsilon) \cap \Omega^k) &\leq C\varepsilon^{d-1}\#\{i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega^k : \ E_{cell}^M(u_\varepsilon, i + C_M) > 0\} \\
&\leq CE_\varepsilon(u_\varepsilon) \leq C
\end{aligned}
$$

By Theorem 2.1 and by a diagonalization argument we get the conclusion. $\qquad\square$

Now we pass to show that the Γ-limit of E_ε is an interfacial energy of the form (4.4). Since it is not our purpose in this paper to investigate boundary layer effects, we assume that $\Omega = \mathbb{T}^d$, $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$. Moreover from now on we will use the notation E_n in place of $E_{\frac{1}{n}}$. In this periodic setting E_n turns out to be defined on all $[0,1)^d$-periodic functions $u : \frac{1}{n}\mathbb{Z}^d \to \{-1,1\}$ as

$$
\begin{aligned}
E_n(u) &= \sum_{\xi \in \mathbb{Z}^d \cap B(0,R)}\sum_{i \in \mathbb{Z}^d \cap nQ} c_\xi n^{1-d}(1 - u_i u_{i+\xi}) \\
&= \sum_{i \in \mathbb{Z}^d \cap nQ} n^{1-d}E_{cell}^M(u, i + C_M), \qquad (6.13)
\end{aligned}
$$

equal to $+\infty$ otherwise in $L^1(\mathbb{T}^d)$, and

$$
E_n(\pm 1) = \min E_n = 0.
$$

Theorem 6.5 *Assume that the family* $\{c_\xi : \ \xi \in \mathbb{Z}^d \cap B(0,R)\}$ *has a ferromagnetic behavior, according to Definition 6.3, and let* E_n *be defined by (6.13). Let* $M > R$ *and* $\varphi : C_M \times \mathbb{Z}^d \to \mathbb{R}$ *such that (6.9) is satisfied. Then the functionals* E_n Γ-*converge with respect to the* $L^1(\mathbb{T}^d)$-*topology to the functional* $E : L^1(\mathbb{T}^d) \to [0,+\infty]$ *defined by*

$$
E(u) = \begin{cases} \displaystyle\int_{S(u)} \varphi(\nu_u)\,d\mathcal{H}^{d-1} & \text{if } u \in BV(\mathbb{T}^d; \{-1,1\}) \\ +\infty & \text{otherwise,} \end{cases} \qquad (6.14)
$$

and $\varphi : S^{d-1} \to [0,+\infty)$ *is the restriction to* S^{d-1} *of a convex and positively homogeneous function of degree one and it is defined by*

$$
\varphi(\nu) := \lim_{T \to +\infty} \frac{1}{T^{d-1}} \min\left\{ \sum_{i \in \mathbb{Z}^d \cap TQ_\nu} E_{cell}^M(u, i + C_M) : \ u \in D(TQ_\nu) \right\}, \qquad (6.15)
$$

where $Q_\nu := Q_\nu(0,1)$ *and*

$$
D(TQ_\nu) := \{u : \mathbb{Z}^d \to \{-1,1\} : \ u(i) = u_{0,\nu}(i) \ \forall \ i \in \mathbb{Z}^d \setminus (T-R)Q_\nu\}. \qquad (6.16)
$$

Remark 6.6 (bounds) *It can be proved that the function φ defined in (6.15) satisfies the following bounds*

$$\alpha|\nu|_{1,\infty} \leq \varphi(\nu) \leq \beta|\nu|_{1,\infty}, \quad \forall \, \nu \in S^{d-1},$$

where $\alpha := \min\{E_{cell}^M(v) : \, v \not\equiv \pm 1\}$, $\beta := \max\{E_{cell}^M(v) : \, v \not\equiv \pm 1\}$ and $|\nu|_{1,\infty} := \max\{\sum_{k \neq l} |\nu_k| : \, l \in \{1, \ldots N\}\}$. The proof of the lower bound can be obtained by using a slicing argument along the directions $e_{\bar{l}}$, where $\bar{l} \in argmax \, |\nu|_{1,\infty}$, and taking into account that, by (6.9), the transition between the two states 1 and -1 costs a positive energy. The proof of the upper bound can be easily obtained by testing the minimum problem in (6.15) with $u = u_{0,\nu}$ and using again a slicing argument.

The next proposition shows that the definition of φ in (6.15) is well posed. We omit its proof, since it relies on an argument which is standard in homogenization theory and is very similar to that exploited in the construction of the recovery sequence in the proof of Theorem 6.5.

Proposition 6.7 *Let $M > R$ and $\varphi : C_M \times \mathbb{Z}^d \to \mathbb{R}$ such that (6.9) is satisfied and let $D(TQ_\nu)$ be defined by (6.16) for all $\nu \in S^{d-1}$ and $T > 0$. Then the limit*

$$\lim_{T \to +\infty} \frac{1}{T^{d-1}} \min \left\{ \sum_{i \in \mathbb{Z}^d \cap TQ_\nu} E_{cell}^M(u, i + C_M) : \, u \in D(TQ_\nu) \right\}$$

exists for all $\nu \in S^{d-1}$.

Proof of Theorem 6.5. We first prove the $\liminf$ inequality. Let $u_n \to u$ in $L^1(\mathbb{T}^d)$ such that $\liminf_n E_n(u_n) < +\infty$. Up to passing to a subsequence we may assume that $\liminf_n E_n(u_n) = \lim_n E_n(u_n)$. Since u_n is $[0,1)^d$-periodic, then u is $[0,1)^d$-periodic. By Proposition 6.4, we immediately deduce that $u \in BV(\mathbb{T}^d; \{-1,1\})$. We now use a blow-up argument. Let $Q = [0,1)^d$ and set

$$\mu_n := \sum_{i \in \mathbb{Z}^d \cap nQ} n^{1-d} E_{cell}^M(u_n, i + C_M) \delta_{\frac{1}{n}i}.$$

Since $\sup_n \mu_n(Q) = \sup_n E_n(u_n) < +\infty$, we may suppose, up to passing to a further subsequence, that there exists a positive finite measure μ such that $\mu_n \rightharpoonup \mu$. By the Radon-Nikodym Theorem, we may decompose μ into two mutually singular positive measures as

$$\mu = \psi \, \mathcal{H}^{d-1} \lfloor_{S(u)} + \mu_s.$$

The conclusion follows by showing that

$$\psi(x_0) \geq \varphi(\nu_{u(x_0)}) \quad for \, \mathcal{H}^{d-1} \lfloor_{S(u)} \, a.e. \, x_0 \in S(u).$$

For $\mathcal{H}^{d-1} \lfloor_{S(u)} \, a.e. \, x_0 \in S(u)$ it holds:

(i) $\lim_{\rho \to 0} \frac{1}{\rho^d} \int_{x_0 + \rho Q^{\pm}_{\nu_u(x_0)}} |u(x) - \pm 1| \, dx = 0$,

(ii) $\lim_{\rho \to 0} \frac{1}{\rho^{d-1}} \mathcal{H}^{d-1}(S(u) \cap \{x_0 + \rho Q_{\nu_u(x_0)}\}) = 1$,

(iii) $\lim_{\rho \to 0} \frac{\mu(x_0 + \rho Q_{\nu_u(x_0)})}{\mathcal{H}^{d-1}(S(u) \cap \{x_0 + \rho Q_{\nu_u(x_0)}\})} = \psi(x_0)$.

Fix such a $x_0 \in S(u)$ and let (ρ_m) be a sequence of positive numbers converging to zero such that $\mu(x_0 + \rho_m Q_{\nu_u(x_0)}) = 0$. By (ii) and (iii) we get

$$
\begin{aligned}
\psi(x_0) &= \lim_m \frac{\mu(x_0 + \rho_m Q_{\nu_u(x_0)})}{\rho_m^{d-1}} \\
&= \lim_m \frac{1}{\rho_m^{d-1}} \lim_n \sum_{i \in \mathbb{Z}^d \cap n(x_0 + \rho_m Q_{\nu_u(x_0)})} n^{1-d} E_{cell}^M(u_n, i + C_M).
\end{aligned}
$$

Observe that, for any m and n we can find $\rho_{m,n}$ with $\lim_n \rho_{m,n} = \rho_m$ and $x_0^n \in \mathbb{Z}^d$ with $\frac{1}{n} x_0^n \to x_0$, such that

$$
\mathbb{Z}^d \cap (x_0^n + n\rho_{m,n} Q_{\nu_u(x_0)}) = \mathbb{Z}^d \cap n(x_0 + \rho_m Q_{\nu_u(x_0)}).
$$

Then

$$
\begin{aligned}
\psi(x_0) &= \lim_m \lim_n \sum_{i \in \mathbb{Z}^d \cap (x_0^n + n\rho_{m,n} Q_{\nu_u(x_0)})} (n\rho_{m,n})^{1-d} E_{cell}^M(u_n, i + C_M) \\
&= \lim_m \lim_n \sum_{j \in \mathbb{Z}^d \cap (n\rho_{m,n} Q_{\nu_u(x_0)})} (n\rho_{m,n})^{1-d} E_{cell}^M(u_n, x_0^n + j + C_M)
\end{aligned}
$$

Let $v_{m,n} : \frac{1}{n\rho_{m,n}} \mathbb{Z}^d \to \{-1, 1\}$ be defined by

$$
v_{m,n}\left(\frac{1}{n\rho_{m,n}} j\right) = u_n\left(\frac{1}{n}(x_0^n + j)\right) \quad j \in \mathbb{Z}^d
$$

By (i) and since $u_n \to u$ in $L^1(Q)$ we get

$$
\lim_m \lim_n \int_{Q_{\nu_u(x_0)}} |v_{m,n} - u_{0,\nu_u(x_0)}| \, dx = 0.
$$

Hence, by a standard diagonalization argument, we can find a sequence of positive numbers $\lambda_k \to 0$ and a sequence $v_k : \lambda_k \mathbb{Z}^d \to \{-1, 1\}$ such that $v_k \to u_{0,\nu_u(x_0)}$ in $L^1(Q_{\nu_u(x_0)})$ and

$$
\psi(x_0) \geq \lim_k \sum_{j \in \mathbb{Z}^d \cap \lambda_k Q_{\nu_u(x_0)}} \lambda_k^{d-1} E_{cell}^M(v_k, j + C_M)
$$

By Lemma 6.8, there exists $w_k : \lambda_k \mathbb{Z}^d \to \{-1, 1\}$ such that $w_k(\lambda_k j) = u_\nu(\lambda_k j)$ if $j \in \mathbb{Z}^d \setminus \left(\frac{1}{\lambda_k} - R\right) Q_{\nu_{u(x_0)}}$ and

$$\psi(x_0) \geq \liminf_k \sum_{j \in \mathbb{Z}^d \cap \frac{1}{\lambda_k} Q_{\nu_{u(x_0)}}} \lambda_k^{d-1} E_{cell}^M(w_k, j + C_M). \qquad (6.17)$$

Eventually the functions $\hat{w}_k : \mathbb{Z}^d \to \{-1, 1\}$ defined as

$$\hat{w}_k(j) := w_k(\lambda_k j), \quad j \in \mathbb{Z}^d$$

belong to $D\left(\frac{1}{\lambda_k} Q_{\nu_{u(x_0)}}\right)$ and, setting $T_k := \frac{1}{\lambda_k}$, by (4.15), (6.15) and Proposition 6.7, we get

$$\psi(x_0) \geq \liminf_k \ T_k^{1-d} \sum_{j \in \mathbb{Z}^d \cap T_k Q_{\nu_{u(x_0)}}} E_{cell}^M(\hat{w}_k, j + C_M) \geq \varphi(\nu_{u(x_0)}).$$

We now pass to the construction of a recovery sequence. We perform this construction just for functions $u \in BV(\mathbb{T}^d; \{-1, 1\})$ such that $S(u)$ is a polyhedral set, since the set of these functions is dense in $BV(\mathbb{T}^d; \{-1, 1\})$. We further restrict to the case in which $S(u)$ is the restriction to Q of an hyperplane, since this construction is easily generalized to each face of a polyhedral boundary. Fix, then, $\nu \in S^{d-1}$ and let u be such that $S(u) = \Pi_\nu \cap Q$, where Π_ν is an hyperplane orthogonal to ν. Without loss of generality we may assume $\Pi_\nu = \{x \in \mathbb{R}^d : \langle x, \nu \rangle = 0\}$, which corresponds to $u = u_{0,\nu}$ on Q. Given $\delta > 0$, by Proposition 6.7, there exist $T_\delta > 0$ and $u_\delta \in D(T_\delta Q_\nu)$ such that $T_\delta > R\delta^{-1}$, and

$$\frac{1}{T_\delta^{d-1}} \sum_{i \in \mathbb{Z}^d \cap T Q_\nu} E_{cell}^M(u_\delta, i + C_M) \leq \varphi(\nu) + \delta.$$

Let $\{b_1, \ldots, b_d\}$ be an orthonormal base of $\mathbb{R}^d$ such that $b_d = \nu$ and $Q_\nu = \{x \in \mathbb{R}^d : |\langle x, b_l \rangle| < 1/2, \ l = 1, \ldots, d\}$. For any $j \in \bigoplus_{l=1}^{d-1} \mathbb{Z}(T_\delta + 2R)b_l$ set $x_j = [j] = ([j_1], \ldots, [j_d])$. Then let $u_n : \frac{1}{n} \mathbb{Z}^d \cap Q \to \{-1, 1\}$ be such that

$$u_n\left(\frac{i}{n}\right) = u_\delta(i - x_j) \quad for \ i \in \mathbb{Z}^d \cap (T_\delta + 2R)Q_\nu + x_j, \ j \in \bigoplus_{l=1}^{d-1} \mathbb{Z}(T_\delta + 2R)b_l$$

and $u \equiv u_\nu$ otherwise. It can be easily verified that $u_n \to u$ in $L^1(Q)$. Note that, by (6.9), the interactions giving non zero energetic contribution are only those inside the domains of the form $\frac{1}{n}((T_\delta + 2R)Q_\nu + x_j)$ and in a $\frac{R}{n}$-neighborhood of $\partial(\frac{1}{n}((T_\delta + 2R)Q_\nu + x_j)) \cap \Pi_\nu \cap Q$. Summing up all such contributions, we get

$$E''(u) \ \leq \ \limsup_n E_n(u_n) \leq \mathcal{H}^{d-1}(S(u)(T_\delta + 2R)^{1-d} \sum_{i \in \mathbb{Z}^d \cap T Q_\nu} E_{cell}^M(u_\delta, i + C_M)$$

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$$+CR\frac{(T_\delta + 2R)^{d-2}}{(T_\delta + 2R)^{d-1}} \leq \mathcal{H}^{d-1}(S(u)(1+\delta)^{d-1}(\varphi(\nu)+\delta) + CR\delta.$$

The conclusion follows by letting δ tend to 0. $\qquad\square$

Lemma 6.8 *Let* $\lambda_k \to 0$ *and let* $v_k : \lambda_k \mathbb{Z}^d \to \{-1,1\}$ *be such that* $v_k \to u_{0,\nu}$ *in* $L^1(Q_\nu)$. *Then there exist* $w_k : \lambda_k \mathbb{Z}^d \to \{-1,1\}$ *such that* $w_k(\lambda_k j) = u_{0,\nu}(\lambda_k j)$ *if* $j \in \mathbb{Z}^d \setminus \left(\frac{1}{\lambda_k} - R\right)Q_\nu$ *and*

$$\liminf_k \sum_{j\in\mathbb{Z}^d \cap \frac{1}{\lambda_k}Q_\nu} \lambda_k^{d-1} E_{cell}^M(w_k, j+C_M) \leq \liminf_k \sum_{j\in\mathbb{Z}^d \cap \frac{1}{\lambda_k}Q_\nu} \lambda_k^{d-1} E_{cell}^M(v_k, j+C_M) \quad (6.18)$$

PROOF. Note that if $\|v_k - u_{0,\nu}\|_{L^1(Q_\nu)} \leq C\lambda_k^2$, then $\#\{j \in \mathbb{Z}^d \cap (Q_\nu \setminus \left(\frac{1}{\lambda_k}-R\right)Q_\nu) : v_k(\lambda_k j) \neq u_{0,\nu}(\lambda_k j)\} \leq C\lambda_k^{2-N}$. In this case it can be easily verified that the thesis holds for

$$w_k := \chi_{(1-R\lambda_k)Q_\nu}v_k + (1 - \chi_{(1-R\lambda_k)Q_\nu})u_{0,\nu}.$$

Otherwise, we may assume that $\liminf_k \frac{1}{\lambda_k^2}\|v_k - u_{0,\nu}\|_{L^1(Q_\nu)} = +\infty$. In this case, set $N_k := \left[\frac{\sqrt{\|v_k - u_{0,\nu}\|_{L^1(Q_\nu)}}}{\lambda_k}\right]$ and, for $l \in \{1, \ldots, N_k\}$, set

$$Q_{\nu,l}^k := \{x \in Q_\nu : \mathrm{dist}(x, Q_\nu^c) \geq Rl\lambda_k\}.$$

Then for any $k \in \{1, \ldots, N_k\}$ consider the family of functions w_k^l defined by

$$w_k^l := \chi_{Q_{\nu,l}^k}v_k + (1 - \chi_{Q_{\nu,l}^k})u_{0,\nu}.$$

Arguing as in the proof of Proposition 4.5, we can find $l_k \in \{1, \ldots, N_k\}$ such that

$$\sum_{j\in\mathbb{Z}^d \cap \frac{1}{\lambda_k}Q_\nu} \lambda_k^{d-1} E_{cell}^M(w_k^{l_k}, j+C_M) \leq \sum_{j\in\mathbb{Z}^d \cap \frac{1}{\lambda_k}Q_\nu} \lambda_k^{d-1} E_{cell}^M(u_k, j+C_M)$$
$$+ \sum_{j\in\mathbb{Z}^d \cap \frac{1}{\lambda_k}(Q_\nu - \lambda_k N_k Q_\nu)} \lambda_k^{d-1} E_{cell}^M(u_{0,\nu}, j+C_M) + o(1).$$

The conclusion follows by choosing $w_k := w_k^{l_k}$ and noticing that

$$\lim_k \sum_{j\in\mathbb{Z}^d \cap \frac{1}{\lambda_k}(Q_\nu - \lambda_k N_k Q_\nu)} \lambda_k^{d-1} E_{cell}^M(u_{0,\nu}, j+C_M) = 0.$$

$\qquad\square$

7 Appendix

In this section we consider sets of finite perimeter in a bounded Lipschitz set Ω with assigned measure and prove a density result with sets having polyhedral boundary in Ω and maintaining the assigned measure.

Theorem 7.1 *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $m \in (0,1)$ be fixed. Then for any set E with $|E| = m|\Omega|$ there exists a sequence E_n such that $\partial E_n \cap \Omega$ is a polyhedral set, $|E_n| = m|\Omega|$, $\lim_{n\to+\infty} |E_n \triangle E| = 0$ and $\lim_{n\to+\infty} \mathrm{Per}(E_n) = \mathrm{Per}(E)$.*

PROOF. Let us fix a set E of finite perimeter and measure equal to $m|\Omega|$. By a standar density result we select a sequence $(F_n)_n$ of sets with polyhedral boundary converging in measure and in perimeter to E. Then the measures of F_n satisfy $F_n = m_n|\Omega|$ with $\lim_{n\to+\infty} m_n = m$ but in general $m_n \neq m$. In the sequel we will slightly modify the sets F_n taking care that all the convergence and structure properties remain valid.

Let $0 < \eta < 1$ be fixed to be chosen later. By the regularity of Ω it is possible to find a positive $r > 0$ and finite set of disjoint cubes of side r such that their union covers Ω up to an error less than η, that is, there exists a (finite) set $\mathcal{I}_r$ such that

$$|\Omega \setminus \sqcup_{x \in \mathcal{I}_r} Q_r(x)| \leq \eta|\Omega|.$$

We now claim that there exist $0 < h < k < 1$ and points $x_h, x_k \in \mathcal{I}_r$ such that

$$|E \cap Q_r(x_h)| > hr^d, \quad |E \cap Q_r(x_k)| < kr^d. \tag{7.1}$$

In fact, assuming by contradiction on k, h that $|E \cap Q_r(x)| \leq hr^d$, $|E \cap Q_r(x)| \geq kr^d$ for any $x \in \mathcal{I}_r$ we get

$$m|\Omega| = |E| \geq \sum_{x \in \mathcal{I}_r} |E \cap Q_r(x)| \geq k \sum_{x \in \mathcal{I}_r} r^d \#(\mathcal{I}_r) \geq k(1-\eta)|\Omega|$$

and

$$m|\Omega| = |E| = |E \setminus \sqcup_{x \in \mathcal{I}_r} Q_r(x)| + |\sqcup_{x \in \mathcal{I}_r} E \cap Q_r(x)|$$
$$\leq |\Omega \cap \sqcup_{x \in \mathcal{I}_r} Q_r(x)| + \sum_{x \in \mathcal{I}_r} |E \cap Q_r(x)| \leq |\Omega| - (1-h)r^d \#(\mathcal{I}_r)$$
$$\leq h|\Omega| + \eta(1-h)|\Omega| = (h(1-\eta) + \eta)|\Omega|.$$

Hence, according to fix at the beginning $\eta < \min\{m, 1-m\}$, we may select h, k such that $0 < h(1-\eta) < m - \eta < m < k(1-\eta)$ and also $k < 1$, and get a contradiction in both the inequalities above.

Thus let x_k, x_h be such that inequalities (7.1) hold for the initial set E. By taking into account the L^1 convergence of F_n to E, for n large enough, a weak version of (7.1) holds for F_n with h, k replaced by $h/2, (k+1)/2$, i.e.

$$|F_n \cap Q_r(x_h)| > \frac{h}{2} r^d, \quad |F_n \cap Q_r(x_k)| < \frac{k+1}{2} r^d. \tag{7.2}$$

In addition we also assume that $|m_n - m| < \min\{(h/4)r^d, (1-k)/4r^d\}$. We will modify F_n in $Q_r(x_k)$ or alternatively in $Q_r(y_k)$, accordingly to the sign of the gap $m_n - m$.

Let us treat first the case in which $m_n < m$. Set $r_n = (4|m_n - m|/(1-k))^{1/d}$ and consider a collection of disjoint cubes $Q_{r_n}(y)$ of radius r_n in $Q_r(x_k)$, intersecting along faces. We have that

$$\sum_{Q_{r_n}(y) \subseteq Q_r(x_k)} |Q_{r_n}(y) \cap F_n| \leq |Q_r(x_k) \cap F_n| < \frac{k+1}{2} r^d,$$

hence, by the mean value theorem, taking into account that the cubes well contained in $Q_r(x_k)$ are $[r/r_n]^d$, we can find $Q_{r_n}(y_k^n)$ such that, refining h, k as above,

$$|F_n \cap Q_{r_n}(y_k^n)| < \frac{k+3}{4} r_n{}^d. \tag{7.3}$$

Note that the function $(0, r_n) \ni t \to |Q_t(y_k^n) \setminus F_n|$ is continuous and, thanks to (7.3), suriective on the set $[0, m - m_n]$, so we may select t_n such that $E_n = F_n \cup Q_{t_n}(y_k^n)$ satisfies the measure constraint. In case $m_n > m$ we will argue analogously subdividing $Q_r(x_h)$ in small cubes with a slight different radius r'_n depending on h but still proportional to $|m_n - m|^{1/d}$ and select z_h^n, t'_n such that $E_n = F_n \setminus Q_{t'_n}(z_h^n)$ has mean volume fraction equal to m. Clearly, in both cases the regularity result of the boundary is maintained as well as the convergence in measure and in perimeter thanks to the fact that the additional perimeter of E_n with respect to F_n is contained in the boundary of a cube of radius t_n or t'_n of order $|m - m_n|^{1/d}$.

$\square$

Remark 7.2 *By the locality of the construction above the same result holds for more general settings, for instance when Ω is substituted by the d-dimensional flat torus. Actually in this last case it is enough to work locally on a single chart of the torus. Moreover, Theorem 7.1 can be extended to a density result with prescribed measure also for functions assuming a finite set of fixed values, as for instance spin functions. Analogously one can replace the convergence of perimeters with suitable surface energies satisfying the hypotheses of Theorem 2.2. Note that in case Ω is the whole space $\mathbb{R}^d$ or a cone for example, the same result can be directly achieved by mean of standard homothety arguments.*

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