Global Properties of Transition Probabilities of Singular Diffusions*

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Abstract

We prove global Sobolev regularity and pointwise upper bounds for transition densities associated with second order differential operators in $\mathbf{R}^N$ with unbounded drift. As an application, we obtain sufficient conditions implying the differentiability of the associated transition semigroup on the space of bounded and continuous functions on $\mathbf{R}^N$.

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Contents

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1 Introduction

Given a second order elliptic partial differential operator with real coefficients

$$A=\sum_{i,j=1}^{N}D_i\left(a_{ij}D_j\right)+\sum_{i=1}^{N}F_iD_i=A_0+F\cdot D, \tag{1.1}$$

where $A_0=\sum_{i,j=1}^{N}D_i\left(a_{ij}D_j\right)$, we consider the parabolic problem

$$\begin{cases} u_t(x,t)=Au(x,t), & x\in\mathbf{R}^N,\ t>0,\\ u(x,0)=f(x), & x\in\mathbf{R}^N,\end{cases} \tag{1.2}$$

where $f\in C_b(\mathbf{R}^N)$.

We assume the following conditions on the coefficients of A which will be kept in the whole paper without further mentioning.

(H) $a_{ij}=a_{ji},\ F_i:\mathbf{R}^N\to\mathbf{R}$, with $a_{ij}\in C^{1+\alpha}(\mathbf{R}^N)$, $F_i\in C^{\alpha}_{\rm loc}(\mathbf{R}^N)$ for some $0<\alpha<1$ and

$$\lambda|\xi|^2\le\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2$$

for every $x,\xi\in\mathbf{R}^N$ and suitable $0<\lambda\le\Lambda$.

Notice that the drift $F=(F_1,\dots,F_N)$ is not assumed to be bounded in $\mathbf{R}^N$.

Problem (1.2) has always a bounded solution but, in general, there is no uniqueness. However, if f is nonnegative, it is not difficult to show that (1.2) has a minimal solution u among all non negative solutions. Taking such a solution u, one constructs a semigroup of positive contractions $T(\cdot)$ on $C_b(\mathbf{R}^N)$ such that

$$u(x,t)=T(t)f(x),\quad t>0,\ x\in\mathbf{R}^N,$$

solves (1.2). Furthermore, the semigroup can be represented in the form

$$T(t)f(x)=\int_{\mathbf{R}^N}p(x,y,t)f(y)\,dy,\quad t>0,\ x\in\mathbf{R}^N,$$

for $f\in C_b(\mathbf{R}^N)$. Here p is a positive function and for almost every $y\in\mathbf{R}^N$, it belongs to $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N\times(0,\infty))$ as a function of (x,t) and solves the equation $\partial_t p=Ap$, $t>0$. We refer to Section 2 and [21] for a review of these results as well as for conditions ensuring uniqueness for (1.2).

Now, we fix $x\in\mathbf{R}^N$ and consider p as a function of (y,t). Then p satisfies

$$\partial_t p=A^*_y p,\quad t>0, \tag{1.3}$$

where A^*_y denotes the adjoint operator of A, which acts on the variable y. The great amount of work devoted to these equations (see e.g. [1] – [7], [12] – [14], [19], [20] and the references there) witnesses the interest towards global properties of solutions. Beside the effort to extend as far as possible the classical results on uniformly elliptic and parabolic equations, solution measures are important in stochastics, being stationary distributions in the elliptic case and transition probabilities in the parabolic one.

For global boundedness and Sobolev regularity, as well as Harnack inequalities and pointwise estimates in the elliptic case, we refer to [19] and [4]. Pointwise bounds on kernels of Schrödinger operators, which can be treated with methods similar to those of the present paper, are proved in [20].

The aim of this paper is to study global regularity properties and pointwise bounds of the transition density p as a function of $(y,t)\in\mathbf{R}^N\times(a,T)$ for $0<a<T$.

We prove that $p(x,\cdot,\cdot)$ belongs to $W^{1,0}_k(\mathbf{R}^N\times(a,T))$ provided that

$$\int_{a_0}^{T}\int_{\mathbf{R}^N}|F(y)|^k p(x,y,t)\,dy\,dt<\infty,\quad\forall k>1$$

for fixed $x\in\mathbf{R}^N$ and $0<a_0<a$. This generalises in some sense Theorem 4.1 in [3]. Assuming that certain Lyapunov functions (exponentials or powers) are integrable with respect to $p(x,y,t)dy$ for $(x,t)\in\mathbf{R}^N\times(a,T)$, pointwise upper bounds for p are obtained. If in addition $F\in W^1_{\infty,\rm loc}(\mathbf{R}^N,\mathbf{R}^N)$ and

$|F|^k p$, $|\mathrm{div}\, F|^{k/2} p \in L^1(\mathbf{R}^N \times (a_0, T))$ with $k > 2(N+2)$, then $p \in W^{2,1}_k(\mathbf{R}^N \times (a,T))$ and we get uniform upper bounds on $|D_y p|$. This is the case if F and $\mathrm{div}\, F$ satify some growth conditions of exponential or power type. Analogously, in the case where F and its derivatives up to the second order satisfy growth conditions of exponential type, upper bounds are also obtained for $|D_{yy} p|$ and $|\partial_t p|$. Notice also that, in some situations, the semigroup $(T(t)_{t\geq 0})$ is compact on $C_b(\mathbf{R}^N)$, and hence there is no semigroup in any space $L^p(\mathbf{R}^N)$ (see [22, Remark 4.3]) and $C_0(\mathbf{R}^N)$ is not $T(t)$-invariant, hence $p(x,y,t) \not\to 0$ as $|x| \to \infty$. This means that there is no hope to obtain any decay of p with respect to x.

Finally, if the inward component of the drift term F is of power type, then all upper bounds obtained before are independent of $x \in \mathbf{R}^N$ and as a consequence we deduce that the transition semigroup $T(\cdot)$ is differentiable on $C_b(\mathbf{R}^N)$ for $t > 0$.

Problem (1.3) (even with time-dependent and less regular coefficients) has been considered in [6], [7], where the initial datum is a L^1-function μ. The Authors prove regularity and pointwise estimates for the solution with respect to the space variables under suitable conditions on μ. Lower bounds are obtained in [7] from Harnack inequality. Moreover, a version of our Theorem 5.1 is proved in [6, Theorem 2.1] assuming that the function μ has finite entropy, see also [7, Corollary 3.5]. Our estimates are obtained directly for the fundamental solution (i.e., when μ is the Dirac measure) and have an explicit behaviour with respect to the time variable. Bounds for any initial datum μ can be obtained from those of the fundamental solution after integration, but they explode as $t \to 0$, whereas those in [6], which exploit some smoothness of μ, do not. We refer the reader also to [24], where other bounds on the fundamental solutions are proved, in particular situations, using Lyapunov functions which depend also on the time variable

Notation. $B_R(x)$ denotes the open ball of $\mathbf{R}^N$ of radius R and centre x. If $x = 0$ we simply write B_R. For $0 \leq a < b$, we write $Q(a,b)$ for $\mathbf{R}^N \times (a,b)$ and Q_T for $Q(0,T)$. We write $C = C(a_1, \ldots, a_n)$ to point out that the constant C depends on the quantities $a_1, \ldots, a_n$. To simplify the notation, we understand the dependence on the dimension N and on quantities determined by the matrix (a_{ij}) such as the ellipticity constant or the modulus of continuity of its entries.

If $u : \mathbf{R}^N \times J \to \mathbf{R}$, where $J \subset [0, \infty[$ is an interval, we use the following notation:

$$\partial_t u = \frac{\partial u}{\partial t}, \; D_i u = \frac{\partial u}{\partial x_i}, \; D_{ij} u = D_i D_j u$$
$$Du = (D_1 u, \ldots, D_N u), \; D^2 u = (D_{ij} u)$$

and

$$|Du|^2 = \sum_{j=1}^N |D_j u|^2, \qquad |D^2 u|^2 = \sum_{i,j=1}^N |D_{ij} u|^2.$$

Let us come to notation for function spaces. $C^j_b(\mathbf{R}^N)$ is the space of j times differentiable functions in $\mathbf{R}^N$, with bounded derivatives up to the order j. $C^\infty_c(\mathbf{R}^N)$ is the space of test functions. $C^\alpha(\mathbf{R}^N)$ denotes the space of all bounded and α-Hölder continuous functions on $\mathbf{R}^N$. We also introduce the space

$$C^{2,1}_c(Q(a,b)) = \{\phi \in C^{2,1}(\overline{Q(a,b)}) : \mathrm{supp}\phi \subset B_R \times [a,b] \text{ for some } R > 0\}.$$

Notice that we are not requiring that $u \in C^{2,1}_c(Q(a,b))$ vanishes at $t = a$, $t = b$.

For $1 \leq k \leq \infty$, $j \in \mathbf{N}$, $W^j_k(\mathbf{R}^N)$ denotes the classical Sobolev space of all L^k–functions having weak derivatives in $L^k(\mathbf{R}^N)$ up to the order j. Its usual norm is denoted by $\|\cdot\|_{j,k}$ and by $\|\cdot\|_k$ when $j = 0$.

Let us now define some spaces of functions of two variables following basically the notation of [15]). $C_0(Q(a,b))$ is the Banach space of continuous functions u defined in $Q(a,b)$ such that $\lim_{|x|\to\infty} u(x,t) = 0$ uniformly with respect to $t \in [a,b]$. $C^{2,1}(Q(a,b))$ is the space of all bounded functions u such that $\partial_t u$, Du and $D_{ij} u$ are bounded and continuous in $Q(a,b)$. For $0 < \alpha \leq 1$ we denote by $C^{2+\alpha, 1+\alpha/2}(Q(a,b))$ the space of all bounded function u such that $\partial_t u$, Du and $D_{ij} u$ are bounded and α-Hölder continuous in $Q(a,b)$ with respect to the parabolic distance $d((x,t),(y,s)) := |x-y| + |t-s|^{\frac{1}{2}}$. Local Hölder spaces are defined, as usual, requiring that the Hölder condition holds in every compact subset.

We shall also use parabolic Sobolev spaces. We denote by $W^{2,1}_k(Q(a,b))$ the space of functions $u \in L^k(Q(a,b))$ having weak space derivatives $D^\alpha_x u \in L^k(Q(a,b))$ for $|\alpha| \leq 2$ and weak time derivative $\partial_t u \in L^k(Q(a,b))$ equipped with the norm

$$\|u\|_{W^{2,1}_k(Q(a,b))} := \|u\|_{L^k(Q(a,b))} + \|\partial_t u\|_{L^k(Q(a,b))} + \sum_{1\leq|\alpha|\leq 2} \|D^\alpha u\|_{L^k(Q(a,b))}.$$

$\mathcal{H}^{k,1}(Q_T)$ denotes the space of all functions $u \in W^{1,0}_k(Q_T)$ with $\partial_t u \in (W^{1,0}_{k'}(Q_T))'$, the dual space of $W^{1,0}_{k'}(Q_T)$, endowed with the norm

$$\|u\|_{\mathcal{H}^{k,1}(Q_T)} := \|\partial_t u\|_{(W^{1,0}_{k'}(Q_T))'} + \|u\|_{W^{1,0}_k(Q_T)},$$

where $\frac{1}{k} + \frac{1}{k'} = 1$. Finally, for $k > 2$, $\mathcal{V}^k(Q_T)$ is the space of all functions $u \in W^{1,0}_k(Q_T)$ such that there exists $C > 0$ for which

$$\left|\int_{Q_T} u\partial_t\phi \, dx\, dt\right| \le C\left(\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} + \|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)}\right)$$

for every $\phi \in C^{2,1}_c(Q(a,b))$. Notice that $k/(k-1) = k'$, $k/(k-2) = (k/2)'$. $\mathcal{V}^k(Q_T)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{V}^k(Q_T)} = \|u\|_{W^{1,0}_k(Q_T)} + \|\partial_t u\|_{k/2,k;Q_T},$$

where $\|\partial_t u\|_{k/2,k;Q_T}$ is the best constant C such that the above estimate holds.

In the whole paper the transition density p will be considered as a function of (y,t) for arbitrary but fixed $x \in \mathbf{R}^N$. The writing $\|p\|$ therefore stands for any norm of p as function of (y,t), for a fixed x.

2 Local regularity and integrability of transition densities

As a first step, we construct a semigroup in $C_b(\mathbf{R}^N)$ generated by a suitable realisation of A. Since the domain will not be dense in $C_b(\mathbf{R}^N)$, we cannot use the Hille-Yosida theorem. Instead we follow a classical approximation method based on Schauder's estimates. We only sketch the procedure since it is presented in detail in [21].

Let us fix a ball B_ϱ of centre 0 and radius ϱ. Since A is uniformly elliptic on this ball, the operator A, endowed with the domain

$$D(A) = \Big\{u \in \bigcap_{p\ge 1} W^2_p(B_\varrho) : Au \in C(\overline{B}_\varrho),\ u|_{\partial B_\varrho} = 0\Big\},$$

generates a semigroup $(T_\varrho(t))_{t\ge 0}$ on $C_b(B_\varrho)$, see e.g. [17, Section 3.1.5]. As a consequence, for every $f \in C_b(\mathbf{R}^N)$ there exists a unique function $u_\varrho = T_\varrho f$ satisfying

$$\begin{cases} \partial_t u_\varrho = Au_\varrho, & x \in B_\varrho,\ t > 0,\\ u_\varrho(x,t) = 0, & x \in \partial B_\varrho,\ t > 0,\\ u_\varrho(x,0) = f(x), & x \in \overline{B}_\varrho. \end{cases}$$

The maximum principle yields $\|u_\varrho\|_\infty \le \|f\|_\infty$ and $u_{\varrho_1}(x,t) \le u_{\varrho_2}(x,t)$ if $x \in B_\varrho$ and $\varrho < \varrho_1 < \varrho_2$, provided that $f \ge 0$. Defining

$$T(t)f(x) = \lim_{\varrho\to\infty} u_\varrho(x,t)$$

one constructs a semigroup of positive contractions in $C_b(\mathbf{R}^N)$, named "the minimal semigroup associated with A", which satisfies the following properties.

Theorem 2.1 *For $f \in C_b(\mathbf{R}^N)$, let $u(x,t) = T(t)f(x)$, for $t \ge 0$, $x \in \mathbf{R}^N$. Then*

(i) *u belongs to the space $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ and satisfies the equation*

$$\partial_t u(x,t) = \sum_{i,j=1}^N D_i(a_{ij}(x)D_j)u(x,t) + \sum_{i=1}^N F_i(x)D_iu(x,t).$$

Moreover, if $f \in C^2_c(\mathbf{R}^N)$, then $\partial_t u(x,t) = T(t)Af(x)$.

(ii) *$T(t)f(x) \to f(x)$ as $t \to 0$, uniformly on compact sets of $\mathbf{R}^N$.*

(iii) *Let (g_n) be a bounded sequence in $C_b(\mathbf{R}^N)$ and suppose that $g_n(x) \to g(x)$ for every $x \in \mathbf{R}^N$, with $g \in C_b(\mathbf{R}^N)$. Then $T(t)g_n(x) \to T(t)g(x)$ in $C^{2,1}(\mathbf{R}^N \times (0,\infty))$.*

In [21] it is also proved that the semigroup is given by a transition density $p(x, y, t)$, that is

$$T(t)f(x) = \int_{\mathbf{R}^N} p(x, y, t)f(y)\,dy.$$

Local regularity properties of the transition densities with respect to the variables (y, t) are known even under conditions weaker than our hypothesis (H), see [3]. We combine the results of [3] with the Schauder estimates to obtain regularity of p with respect to all the variables (x, y, t).

Proposition 2.2 *Under assumption (H) the kernel $p = p(x, y, t)$ is a positive continuous function in $\mathbf{R}^N \times \mathbf{R}^N \times (0,\infty)$ which enjoys the following properties.*

(i) *For every $x \in \mathbf{R}^N$, $1 < s < \infty$, the function $p(x, \cdot, \cdot)$ belongs to $\mathcal{H}^{s,1}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$. In particular $p, D_y p \in L^s_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ and $p(x, \cdot, \cdot)$ is continuous.*

(ii) *For every $y \in \mathbf{R}^N$ the function $p(\cdot, y, \cdot)$ belongs to $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ and solves the equation $\partial_t p = Ap$, $t > 0$. Moreover*

$$\sup_{|y|\le R} \|p(\cdot, y, \cdot)\|_{C^{2+\alpha,1+\alpha/2}(B_R\times[\varepsilon,T])} < \infty$$

for every $0 < \varepsilon < T$ and $R > 0$.

(iii) *If, in addition, $F \in C^1(\mathbf{R}^N)$, then $p(x, \cdot, \cdot) \in W^{2,1}_{s,\rm loc}(Q_T)$ for every $x \in \mathbf{R}^N$, $1 < s < \infty$, and satisfies the equation $\partial_t p - A^*_y p = 0$, where*

$$A^* = A_0 - F \cdot D - \operatorname{div} F$$

is the formal adjoint of A.

Proof. Assertion (i) is stated in [3, Corollary 3.9].

Let us prove (ii). Since $p(x, \cdot, \cdot)$ is continuous in (y, t) for every fixed x we have $p(x, y, t) < \infty$ for every $t > 0$ and $x, y \in \mathbf{R}^N$. Under this condition, the proof of [21, Theorem 4.4] ensures that $p(\cdot, y, \cdot) \in C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ for every $y \in \mathbf{R}^N$ and that p solves $\partial_t p = Ap$.

Let us fix $y \in \mathbf{R}^N$, $0 < \varepsilon < \tau$ and $t_1 > \tau$. If $|y| \le R$, the parabolic Harnack inequality (see e.g. [16, Chapter VII]) yields

$$\sup_{\varepsilon\le t\le\tau, x\in B_{2R}} p(x, y, t) \le Cp(0, y, t_1) \le C \sup_{|y|\le R} p(0, y, t_1) = M$$

for a suitable $M > 0$. By the interior Schauder estimates (see e.g. [11, Theorem 8.1.1]) we deduce that

$$\begin{aligned}\sup_{|y|\le R} \|p(\cdot, y, \cdot)\|_{C^{2+\alpha,1+\alpha/2}(B_R\times[\varepsilon,\tau])} \le& C\Big(\sup_{|y|\le R} \|\partial_t p(\cdot, y, \cdot) - Ap(\cdot, y, \cdot)\|_{C^{\alpha,\alpha/2}(B_{2R}\times[\varepsilon/2,\tau])} + M\Big)\\ =& CM < \infty.\end{aligned}$$

Finally, we prove that p is continuous in $\mathbf{R}^N \times \mathbf{R}^N \times (0,\infty)$. If $(x_n, y_n, t_n) \to (x_0, y_0, t_0)$ with $t_0 > 0$, then

$$|p(x_n, y_n, t_n) - p(x_0, y_0, t_0)| \le |p(x_n, y_n, t_n) - p(x_0, y_n, t_0)| + |p(x_0, y_n, t_0) - p(x_0, y_0, t_0)|.$$

The last term tends to zero by the continuity of $p(x_0, \cdot, t_0)$ and the first too, as, by the above estimate, $D_x p$ is uniformly bounded in a neighbourhood of (x_0, y_0, t_0).

Assertion (iii) follows from standard local parabolic regularity. □

The minimal semigroup selects one among all bounded solutions of equation (1.2), actually the minimal among all positive solutions, when f is positive. The uniqueness of the bounded solution does not hold, in general but it is ensured by the existence of a *Lyapunov function*, that is of a $C^{2+\alpha}_{\rm loc}$-function $W : \mathbf{R}^N \to [0,\infty)$ such that $\lim_{|x|\to\infty} W(x) = +\infty$ and $AW \le \lambda W$ for some $\lambda > 0$. Lyapunov functions are easily found imposing suitable conditions on the coefficients of A. For instance, $W(x) = |x|^2$ is a Lyapunov function for A provided that $\sum_i a_{ii}(x) + F(x) \cdot x \le C|x|^2$ for some $C > 0$.

Proposition 2.3 *Assume that A has a Lyapunov function W and let $u, v \in C_b(\mathbf{R}^N \times [0, T]) \cap C^{2,1}(\mathbf{R}^N \times (0, T])$ solve (1.2). Then $u = v$.*

Proof. It is sufficient to show that if such a u solves (1.2) with $f \ge 0$, then $u \ge 0$. Define $v_\varepsilon = e^{-\lambda t}u+\varepsilon W$, where $\varepsilon > 0$ and λ is such that $AW \le \lambda W$. Then v_ε has a minimum point $(x_0, t_0) \in \mathbf{R}^N\times[0, T]$. If $v_\varepsilon(x_0, t_0) < 0$, then $t_0 > 0$, since $f \ge 0$, and hence $\partial_t v_\varepsilon(x_0, t_0) \le 0$. Since $Dv_\varepsilon(x_0, t_0) = 0$ and $\sum_{i,j} a_{ij}D_{ij}v_\varepsilon(x_0, t_0) \ge 0$, we have also $(A - \lambda)v_\varepsilon(x_0, t_0) > 0$ and this contradicts the equation $\partial_t v_\varepsilon - (A - \lambda)v_\varepsilon \ge 0$. Therefore $v_\varepsilon \ge 0$ and, letting $\varepsilon \to 0$, $u \ge 0$. □

Now we turn our attention to integrability properties of p and show how they can be deduced from the existence of suitable Lyapunov functions.

The integrability of Lyapunov functions with respect to the measures $p(x,y,t)\,dy$ is given by the following result, which is proved in [22], see also [1].

Proposition 2.4 *A Lyapunov function W is integrable with respect to the measures $p(x,y,t)dy$. Setting*

$$\zeta(x,t) = \int_{\mathbf{R}^N} p(x,y,t)W(y)\,dy, \tag{2.1}$$

the inequality

$$\zeta(x,t) \le e^{\lambda t}W(x)$$

holds. Moreover, $|AW|$ is integrable with respect to $p(x,y,t)dy$, $\zeta \in C^{2,1}(\mathbf{R}^N \times (0,\infty)) \cap C(\mathbf{R}^N \times [0,\infty))$ and

$$\partial_t \zeta(x,t) \le \int_{\mathbf{R}^N} p(x,y,t)AW(y)\,dy.$$

Assuming that AW tends to $-\infty$ faster than $-W$ one obtains, by Proposition 2.4, that the function ζ in (2.1) is bounded with respect to the space variables, see [20, Proposition 2.6]. We repeat here the proof for reader's convenience.

Proposition 2.5 *Assume that the Lyapunov function W satisfies the inequality $AW \le -g(W)$ where $g : [0,\infty) \to \mathbf{R}$ is a convex function such that $\lim_{s\to+\infty} g(s) = +\infty$ and $1/g$ is integrable in a neighbourhood of $+\infty$. Then for every $a > 0$ the function ζ defined in (2.1) is bounded in $\mathbf{R}^N \times [a,\infty)$. Moreover, the semigroup $(T(t))_{t\ge 0}$ is compact in $C_b(\mathbf{R}^N)$.*

Proof. Observe that, since g is convex, then

$$\int_{\mathbf{R}^N} p(x,y,t)g(W(y))\,dy \ge g(\zeta(x,t)).$$

Then, form Proposition 2.4 we deduce

$$\partial_t\zeta(x,t) \le \int_{\mathbf{R}^N} p(x,y,t)AW(y)\,dy \le -\int_{\mathbf{R}^N} p(x,y,t)g(W(y))\,dy \le -g(\zeta(x,t))$$

and therefore $\zeta(x,t) \le z(x,t)$, where z is the solution of the ordinary Cauchy problem

$$\begin{cases} z' = -g(z) \\ z(x,0) = W(x). \end{cases}$$

Let ℓ denote the greatest zero of g. Then $z(x,t) \le \ell$ if $W(x) \le \ell$. On the other hand, if $W(x) > \ell$, then z is decreasing and satisfies

$$t = \int_{z(x,t)}^{W(x)} \frac{ds}{g(s)} \le \int_{z(x,t)}^{\infty} \frac{ds}{g(s)}. \tag{2.2}$$

This inequality easily yields, for every $a > 0$, a constant $C(a)$ such that $z(x,t) \le C(a)$ for every $t \ge a$ and $x \in \mathbf{R}^N$. The compactness of the semigroup is proved in [22, Theorem 3.10]. □

Let us state a condition under which certain exponentials are Lyapunov functions. Propositions 2.6, 2.7 will be used to check the integrability of $|F|^k$ with respect to p.

Proposition 2.6 *Let Λ be the maximum eigenvalue of (a_{ij}) as in (H). Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x) \cdot \frac{x}{|x|} \le -c, \tag{2.3}$$

$0 < c < \infty$, for some $c > 0$, $\beta > 1$. Then $W(x) = \exp\{\delta|x|^\beta\}$ is a Lyapunov function for $\delta < (\beta\Lambda)^{-1}c$. Moreover, if $\beta > 2$, there exist positive constants c_1, c_2 such that

$$\zeta(x,t) \le c_1 \exp\left(c_2 t^{-\beta/(\beta-2)}\right) \tag{2.4}$$

for $x \in \mathbf{R}^N,\ t > 0$.

Proof. Let $W(x) = \exp\{\delta |x|^\beta\}$ and set $G_i = F_i + \sum_j D_j a_{ij}$. We obtain, by a straightforward computation,

$$\begin{aligned} AW(x) =& \delta\beta |x|^{\beta-1} e^{\delta|x|^\beta} \left(\frac{1}{|x|} \sum_i a_{ii}(x) + \frac{\beta-2}{|x|^3} \sum_{i,j} a_{ij}(x) x_i x_j \right. \\ & \left. + \delta\beta |x|^{\beta-3} \sum_{i,j} a_{ij}(x) x_i x_j + G \cdot \frac{x}{|x|} \right) \\ \le & C_1 |x|^{\beta-1} e^{\delta|x|^\beta} \left(1 + (\delta\beta\Lambda - c)|x|^{\beta-1}\right) \\ \le & - C_2 |x|^{2\beta-2} e^{\delta|x|^\beta} \le 0 \end{aligned}$$

for $|x|$ large. This shows that W is a Lyapunov function. Finally, if $\beta > 2$ it follows that $AW \le -g(W)$ with $g(s) = C_3 s (\log s)_+^{2-2/\beta} - C_4$, for suitable $C_3, C_4 > 0$. Then Proposition 2.5 yields the boundedness of $\zeta(\cdot, t)$. To obtain (2.4) we recall that $\zeta \le z$ where z satisfies (2.2). If ℓ denotes the zero of g and $z(x,t) \le 2\ell$ we have simply to choose a suitable c_1. If $z(x,t) \ge 2\ell$, then

$$t \le \int_z^\infty \frac{ds}{g(s)} \le C_5 \int_z^\infty \frac{ds}{s(\log s)^{2-2/\beta}} \le C_6 (\log z)^{2/\beta - 1}$$

and (2.4) follows. □

The right hand side of (2.4) becomes very big as $t \to 0$. In order to have a milder behaviour we investigate when powers are Lyapunov functions.

Proposition 2.7 *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x) \cdot \frac{x}{|x|} < 0, \tag{2.5}$$

for some $\beta > 2$. Then $W(x) = \left(1 + |x|^2\right)^\alpha$ is a Lyapunov function for every $\alpha > 0$ and there exists a positive constant c such that

$$\zeta(x,t) \le c t^{-(2\alpha)/(\beta-2)} \tag{2.6}$$

for $x \in \mathbf{R}^N$, $0 < t \le 1$.

Proof. We have, with the notation of Proposition 2.6,

$$\begin{aligned} AW(x) =& \left(1 + |x|^2\right)^\alpha \left(\frac{2\alpha}{1+|x|^2} \sum_i a_{ii}(x) + \frac{4\alpha(\alpha-1)}{(1+|x|^2)^2} \sum_{i,j} a_{ij}(x) x_i x_j + \frac{2\alpha}{1+|x|^2} G \cdot x \right) \\ \le & - C_1 \left(1+|x|^2\right)^{\alpha + (\beta-2)/2} = -C_1 W^\gamma \end{aligned}$$

for $|x|$ large and with $\gamma = 1 + (\beta-2)/(2\alpha) > 1$. This shows $AW \le -g(W)$ with $g(s) = C_2 s^\gamma - C_3$ for suitable $C_2, C_3 > 0$. Proceeding as in the proof of (2.4) one shows (2.6), the only difference being that the function $t^{-(2\alpha)/(\beta-2)}$ goes to 0 as $t \to +\infty$, and then the estimate is not true, in general, for all $t > 0$. □

Remark 2.8 Conditions (2.3), (2.5) are assumptions on the radial component of F. Of course, changing $x/|x|$ to $(x - x_0)/|x - x_0|$ leads to new conditions that, though not equivalent to (2.3), (2.5), yield similar conclusions.

Finally we clarify in which sense the identity $p_t = A_y^* p$ is satisfied.

Lemma 2.9 *Let $0 \le a < b$ and $\varphi \in C_c^{2,1}(Q(a,b))$. Then*

$$\begin{aligned} & \int_{Q(a,b)} \left(\partial_t \varphi(y,t) + A\varphi(y,t)\right) p(x,y,t)\, dy\, dt \\ = & \int_{\mathbf{R}^N} \left(p(x,y,b)\varphi(y,b) - p(x,y,a)\varphi(y,a)\right)\, dy. \end{aligned} \tag{2.7}$$

Proof. If $\psi \in C_c^2(\mathbf{R}^N)$, then $\partial_t T(t)\psi = T(t)A\psi$, see Theorem 2.1(i).

If $\varphi \in C_c^{2,1}(Q(a,b))$, then $\partial_t \left(T(t)\varphi(\cdot,t)\right) = T(t)\partial_t\varphi(\cdot,t) + T(t)A\varphi(\cdot,t)$. Integrating this identity over $[a,b]$ and writing $T(t)$ in terms of the kernel p, we obtain (2.7). □

3 Sobolev regularity: Preliminary estimates

In this section we fix $T > 0$ and consider p as a function of $(y,t) \in \mathbf{R}^N \times (0,T)$ for arbitrary, but fixed, $x \in \mathbf{R}^N$. Further, fix $0 < a_0 < a < b < b_0 \le T$ and assume for definiteness $b_0 - b \ge a - a_0$. Setting

$$\Gamma(k,x,a_0,b_0) := \left(\int_{Q(a_0,b_0)} |F(y)|^k p(x,y,t)\,dy\,dt\right)^{\frac{1}{k}}, \tag{3.1}$$

we show global regularity results for p with respect to the variables (y,t) assuming $\Gamma(k,x,a_0,b_0) < \infty$ for suitable $k \ge 1$. Observe that if $\Gamma(k,x,a_0,b_0) < \infty$ then $\Gamma(h,x,a_0,b_0) < \infty$ for all $h \le k$. We also recall that this assumption can be verified, in many concrete cases, using Propositions 2.6, 2.7.

In the following proposition we show that $p \in L^r(Q(a_0,b_0))$ for small values of $r > 1$.

Proposition 3.1 *If $\Gamma(1,x,a_0,b_0) < \infty$, then $p \in L^r(Q(a_0,b_0))$ for all $r \in [1, \frac{N+2}{N+1})$ and*

$$\|p\|_{L^r(Q(a_0,b_0))} \le C\,(1 + \Gamma(1,x,a_0,b_0))$$

for some constant $C > 0$.

Proof. For every $\varphi \in C_c^{2,1}(Q_T)$ such that $\varphi(\cdot,T) = 0$, by (2.7), we obtain, with A_0 as in (1.1),

$$\begin{aligned}\int_{Q(a_0,b_0)} p(\partial_t\varphi + A_0\varphi)\,dy\,dt = &-\int_{Q(a_0,b_0)} pF\cdot D\varphi\,dy\,dt\\ &+\int_{\mathbf{R}^N} (p(x,y,b_0)\varphi(y,b_0) - p(x,y,a_0)\varphi(y,a_0))\,dy.\end{aligned}$$

Since $\int_{\mathbf{R}^N} p(x,y,t)\,dy \le 1$ for all $t \ge 0$, $x \in \mathbf{R}^N$, it follows that

$$\begin{aligned}\left|\int_{Q(a_0,b_0)} p(\partial_t\varphi + A_0\varphi)\,dy\,dt\right| \le &\Gamma(1,x,a_0,b_0)\|\varphi\|_{W^{1,0}_\infty(Q(a_0,b_0))} + 2\|\varphi\|_\infty\\ \le &\Big(2 + \Gamma(1,x,a_0,b_0)\Big)\|\varphi\|_{W^{1,0}_\infty(Q(a_0,b_0))}.\end{aligned} \tag{3.2}$$

Fix $\psi \in C_c^\infty(Q(a_0,b_0))$ and consider the parabolic problem

$$\begin{cases} \partial_t\varphi + A_0\varphi = \psi & \text{in } Q_T,\\ \varphi(y,T) = 0, & y \in \mathbf{R}^N.\end{cases} \tag{3.3}$$

The Schauder theory, see [11, Chapter 9], provides a solution $\varphi \in C^{2+\alpha,1+\alpha/2}(Q_T)$. Fixing $r_1' > N+2$, by [15, Theorem IV.9.1] we see that φ belongs to $W^{2,1}_{r_1'}(Q_T)$ and satisfies the estimate

$$\|\varphi\|_{W^{2,1}_{r_1'}(Q_T)} \le C\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}. \tag{3.4}$$

Since $r_1' > N+2$, from the Sobolev embedding theorems (cf. [15, Lemma II.3.3]) and (3.4) it follows that

$$\|\varphi\|_{W^{1,0}_\infty(Q(a_0,b_0))} \le \|\varphi\|_{W^{1,0}_\infty(Q_T)} \le C\|\varphi\|_{W^{2,1}_{r_1'}(Q_T)} \le C\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}.$$

Note that the solution φ of (3.3) cannot be inserted directly in (3.2), since it does not have compact support with respect to the space variables. To overcome this problem we fix a smooth function $\theta \in C_c^\infty(\mathbf{R}^N)$ such that $\theta(y) = 1$ for $|y| \le 1$ and write (3.2) for $\varphi_n(y,t) = \theta(y/n)\varphi(y,t)$. Letting $n \to \infty$ and using dominated convergence we see that (3.2) holds also for such a φ. Therefore

$$\left|\int_{Q(a_0,b_0)} p\psi\,dy\,dt\right| \le C\Big(1 + \Gamma(1,x,a_0,b_0)\Big)\|\psi\|_{L^{r_1'}(Q(a_0,b_0))}$$

and hence $p \in L^{r_1}(Q(a_0,b_0))$, where $\frac{1}{r_1} + \frac{1}{r_1'} = 1$. Since $r_1' > N+2$ is chosen arbitrarily, $p \in L^r(Q(a_0,b_0))$ for all $r \in \Big[1, \frac{N+2}{N+1}\Big)$, and

$$\|p\|_{L^r(Q(a_0,b_0))} \le C\Big(1 + \Gamma(1,x,a_0,b_0)\Big). \tag{3.5}$$

□

Lemma 3.2 *If $\Gamma(k,x,a_0,b_0) < \infty$ for $k > 1$ and $p \in L^r(Q(a_0,b_0))$ for some $1 < r \le \infty$, then $p \in \mathcal{H}^{s,1}(Q(a,b))$ for $s := \frac{rk}{r+k-1}$ if $r < \infty$, $s = k$ if $r = \infty$.*

Proof. In the proof we denote by c for a generic constant depending on k, x, a_0, b_0.

Let η be a smooth function such that $0 \le \eta \le 1$, $\eta(t) = 1$ for $a \le t \le b$ and $\eta(t) = 0$ for $t \le a_0$ and $t \ge b_0$. Consider $\varphi \in C_c^{2,1}(Q_T)$. Plugging $\eta\varphi$ in place of φ in (2.7) and setting $q := \eta p$, we obtain

$$\int_{Q_T} q(\partial_t\varphi + A_1\varphi)\,dy\,dt - \int_{Q_T} (qG\cdot D\varphi + p\varphi\partial_t\eta)\,dy\,dt, \tag{3.6}$$

where $A_1 = \sum_{i,j} a_{ij}D_{ij}$ and $G_i = F_i + D_i(\sum_{j=1}^N a_{ij})$.

By Hölder's inequality we have

$$\begin{aligned}\int_{Q(a_0,b_0)} |F|^s p^s\,dy\,dt &= \int_{Q(a_0,b_0)} |F|^s p^{\frac{s}{k}} p^{s(1-\frac{1}{k})}\,dy\,dt \\ &\le \left(\int_{Q(a_0,b_0)} |F|^k p\,dy\,dt\right)^{\frac{s}{k}} \left(\int_{Q(a_0,b_0)} p^{\frac{s(k-1)}{k-s}}\,dy\,dt\right)^{1-\frac{s}{k}} \\ &= \left(\int_{Q(a_0,b_0)} |F|^k p\,dy\,dt\right)^{\frac{s}{k}} \left(\int_{Q(a_0,b_0)} p^r\,dy\,dt\right)^{1-\frac{s}{k}} \\ &\le \Gamma(k,x,a_0,b_0)^s \left(\int_{Q(a_0,b_0)} p^r\,dy\,dt\right)^{1-\frac{s}{k}},\end{aligned}$$

whence

$$\|Gp\|_{L^s(Q(a_0,b_0))} \le c\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}}.$$

This yields

$$\left|\int_{Q_T} q(\partial_t\varphi + A_1\varphi)\,dy\,dt\right| \le c\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}} \|\varphi\|_{W^{1,0}_{s'}(Q_T)},$$

where $\frac{1}{s} + \frac{1}{s'} = 1$. Replacing φ by its difference quotients with respect to the variable y

$$\tau_{-h}\varphi(y,t) := |h|^{-1}(\varphi(y-he_j,t) - \varphi(y,t)), \quad (y,t) \in Q_T,\ 0 \ne h \in \mathbf{R},$$

and since $a_{ij} \in C_b^1(\mathbf{R}^N)$, we obtain

$$\left|\int_{Q_T} \tau_h q(\partial_t\varphi + A_1\varphi)\,dy\,dt\right| \le c\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}} \|\varphi\|_{W^{2,1}_{s'}(Q_T)}. \tag{3.7}$$

As in the proof of Proposition 3.1 we approximate φ in $W^{2,1}_{s'}(Q_T)$ with a sequence of functions $\varphi_n \in C_c^{1,2}(Q_T)$. Since $q \in L^s(Q_T)$, writing (3.7) for φ_n and letting $n \to \infty$ we see that (3.7) holds for φ.

Since $s = (s-1)s' < r$, and then $|\tau_h q|^{s-2}\tau_h q \in L^{s'}(Q_T)$. Using [15, Theorem 9.2.3] we choose now $\varphi \in W^{2,1}_{s'}(Q_T)$ such that

$$\begin{cases} \partial_t\varphi + A_1\varphi = |\tau_h q|^{s-2}\tau_h q, & \text{in } Q_T, \\ \varphi(y,T) = 0, & y \in \mathbf{R}^N \end{cases}$$

and

$$\|\varphi\|_{W^{2,1}_{s'}(Q_T)} \le C\||\tau_h q|^{s-1}\|_{L^{s'}(Q_T)}.$$

Therefore we get

$$\int_{Q_T} |\tau_h q|^s\,dy\,dt \le c\|p\|_{L^r(Q(a_0,b_0))}^{\frac{k-1}{k}} \|\tau_h q\|_{L^s(Q_T)}^{s-1},$$

hence

$$\|Dq\|_{L^s(Q_T)} \le c\|p\|_{L^r(Q_T)}^{\frac{k-1}{k}}$$

and $Dq \in L^s(Q_T)$, $q \in W^{1,0}_s(Q_T)$.

Now we treat the time derivative. Using the above estimates we deduce

$$\begin{aligned}\left|\int_{Q_T} q\partial_t\varphi\,dy\,dt\right| \le& \left|\int_{Q_T} qA_0\varphi\,dy\,dt\right| + c\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\varphi\|_{W^{1,0}_{s'}(Q_T)}\\ =&\left|\int_{Q_T}\sum_{i,j=1}^N a_{ij}D_i\varphi D_j q\,dy\,dt\right| + c\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\varphi\|_{W^{1,0}_{s'}(Q_T)}\\ \le& c\|Dq\|_{L^s(Q_T)}\|\varphi\|_{W^{1,0}_{s'}(Q_T)} + c\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\varphi\|_{W^{1,0}_{s'}(Q_T)}\\ \le& c\|p\|^{\frac{k-1}{k}}_{L^r(Q(a_0,b_0))}\|\varphi\|_{W^{1,0}_{s'}(Q_T)}\end{aligned}$$

and the statement follows. □

Proposition 3.3 *If $\Gamma(k,x,a_0,b_0)<\infty$ for some $1<k\le N+2$, then $p\in L^r(Q(a,b))$ for all $r\in[1,\frac{N+2}{N+2-k})$ and $p\in\mathcal{H}^{s,1}(Q(a,b))$ for all $s\in(1,\frac{N+2}{N+3-k})$.*

Proof. Let us see how the arguments in the proof of Lemma 3.2 can be iterated. Let $r_1<(N+2)/(N+1)$, so that Proposition 3.1 applies, and fix a parameter m (to be chosen later) depending upon k and r. Set $a_n=a_0+n(a-a_0)/m$, $b_n=b_0-n(b_0-b)/m$ for $n=1,\dots,m$. Suppose that $p\in L^{r_n}(Q(a_n,b_n))$ and take $s_n:=\frac{kr_n}{k+r_n-1}$. Then, $1<s_n<r_n$, $s_n<k$ and $r_n=\frac{s_n(k-1)}{k-s_n}$.

We consider again $q=\eta p$ with $\eta(t)=1$ for $a_{n+1}\le t\le b_{n+1}$ and $\eta(t)=0$ for $t\le a_n$, $t\ge b_n$. As in the proof of Lemma 3.2 we get

$$\left|\int_{Q_T} q\partial_t\varphi\,dy\,dt\right| \le c\|p\|^{\frac{k-1}{k}}_{L^{r_n}(Q(a_n,b_n))}\|\varphi\|_{W^{1,0}_{s_n'}(Q_T)},$$

where c denotes a constant depending on k,x,a_0,b_0. Therefore, $p\in\mathcal{H}^{s_n,1}(Q(a_{n+1},b_{n+1}))$ and, by Theorem A.3, we obtain that $p\in L^{r_{n+1}}(Q(a_{n+1},b_{n+1}))$ where

$$\frac{1}{r_{n+1}}=\frac{1}{s_n}-\frac{1}{N+2}=\frac{1}{r_n}\left(1-\frac{1}{k}\right)+\frac{1}{k}-\frac{1}{N+2}.$$

Since $\frac{1}{r_1}>\frac{N+1}{N+2}$, it follows that

$$\frac{1}{r_2}-\frac{1}{r_1}<-\frac{1}{k}\left(1-\frac{1}{N+2}\right)+\frac{1}{k}-\frac{1}{N+2}=\frac{1}{N+2}\left(\frac{1}{k}-1\right)<0.$$

Hence, by induction, $\left(\frac{1}{r_n}\right)$ is a positive and decreasing sequence which converges to $\frac{N+2-k}{N+2}$. Therefore, for any $r<\frac{N+2}{N+2-k}$, after finitely many, say m, iterations we get $r_n>r$ and $p\in L^r(Q(a,b))$. The second half of the statement now follows from Lemma 3.2. □

Corollary 3.4 *If $\Gamma(k,x,a_0,b_0)<\infty$ for some $k>N+2$, then p belongs to $L^\infty(Q(a,b))$.*

Proof. We know from Proposition 3.3 that $p\in L^r(Q(a,b))$ for all $r\in[1,\infty)$. Hence, by Lemma 3.2, $p\in\mathcal{H}^{s,1}(Q(a,b))$ for all $s\in(1,k)$. Choosing $N+2<s<k$ it follows from Theorem A.3 that $p\in L^\infty(Q(a,b))$. □

A closer look at the above proof shows that p is globally Hölder continuous in (y,t).

Proposition 3.5 *Assume that $\Gamma(k,x,a_0,b_0)<\infty$ for some $k>N+2$. Then, p belongs to $C^\nu([a,b],C_b^\theta(\mathbf{R}^N))$ for some ν, $\theta>0$.*

Proof. Since $k>N+2$, we can choose $\alpha>0$ such that $\frac{1}{k}<\alpha<\frac{1}{2}$ and $k(1-2\alpha)>N$. So, applying the embedding theorem in [13, Corollary 7.5] for the space $\mathcal{H}^{k,1}(Q_T)$ (with $q=p=k$, $\gamma=1$ and $\beta=2\alpha$) we obtain

$$\|p(t)-p(\tau)\|_{W^{1-2\alpha,k}(\mathbf{R}^N)}\le C|t-\tau|^{\alpha-\frac{1}{k}}\|p\|_{\mathcal{H}^{k,1}(Q(a,b))}$$

for $a\le\tau<t\le b$, where the constant $C>0$ is independent of τ,t. Thus, p belongs to the space $C^{\alpha-\frac{1}{k}}([a,b],W^{1-2\alpha,k}(\mathbf{R}^N))$. Since $k(1-2\alpha)>N$, it follows from the Sobolev embedding theorem that

$$p\in C^{\alpha-\frac{1}{k}}([a,b],C_b^\theta(\mathbf{R}^N)),\qquad\text{for some }\theta>0.$$

□

4 Uniform and pointwise bounds on transition densities

We consider the following assumption depending on the weight function ω which, in our examples, will be a power or the exponential of a power.

(H1) W_1, W_2 are Lyapunov functions for A, $W_1 \le W_2$ and there exists $1 \le \omega \in C^2(\mathbf{R}^N)$ such that for some $c > 0$ and $k > N+2$

(i) $\omega \le cW_1, \quad |D\omega| \le c\omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}, \quad |D^2\omega| \le c\omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$;

(ii) $\omega |F|^k \le cW_2$.

We denote by ζ_1, ζ_2 the functions defined by (2.1) and associated with W_1, W_2, respectively.

We use different Lyapunov functions to obtain more precise estimates in the theorem below and its corollaries.

Theorem 4.1 *Assume (H1). Then, there exists a constant $C > 0$ such that*

$$0 < \omega(y)p(x,y,t) \le C\left(\int_{a_0}^{b_0} \zeta_2(x,t)\,dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1(x,t)\,dt\right) \tag{4.1}$$

for all $x,\, y \in \mathbf{R}^N, a \le t \le b$.

Proof. **Step 1.** Assume first that ω is bounded. Since $\Gamma(k,x,a_0,b_0) < \infty$ then $p \in L^\infty(Q(a,b))$ for every $a_0 < a < b < b_0$, by Corollary 3.4. We choose a smooth function $\eta(t)$ such that $\eta(t) = 1$ for $a \le t \le b$ and $\eta(t) = 0$ for $t \le a_0$ and $t \ge b_0$, $|\eta'| \le \frac{2}{a-a_0}$. We consider $\psi \in C_c^{2,1}(Q_T)$ such that $\psi(\cdot, T) = 0$. Setting $q = \eta^{\frac{k}{2}} p$ and taking $\varphi(y,t) = \eta^{\frac{k}{2}}\omega(y)\psi(y,t)$ in (2.7) we obtain

$$\begin{aligned}\int_{Q_T} \omega q\left(-\partial_t\psi - A_0\psi\right)\,dy\,dt = \int_{Q_T} \Big[q\Big(\psi A_0\omega + 2\sum_{i,j=1}^N a_{ij}D_i\omega D_j\psi + \omega F\cdot D\psi + \psi F\cdot D\omega\Big) \\ + \frac{k}{2}p\omega\psi\eta^{\frac{k-2}{2}}\partial_t\eta\Big]\,dy\,dt.\end{aligned} \tag{4.2}$$

Since $\omega q \in L^1(Q_T) \cap L^\infty(Q_T)$, Theorem A.5 yields

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} \le & C\Big(\|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)} + \|qD\omega\|_{L^k(Q_T)} + \|\omega qF\|_{L^k(Q_T)} \\ & + \|qF\cdot D\omega\|_{L^{\frac{k}{2}}(Q_T)} + \frac{1}{a-a_0}\|p\omega\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)}\Big).\end{aligned}$$

Next observe that, by (H1)(ii),

$$\|\omega qF\|_{L^k(Q_T)} \le \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \|\omega qF^k\|_{L^1(Q_T)}^{\frac{1}{k}} \le c\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\zeta_2\,dt\right)^{\frac{1}{k}},$$

and that

$$\|\omega p\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)} \le \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \|\omega p\|_{L^1(Q(a_0,b_0))}^{\frac{2}{k}} \le c\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{2}{k}}.$$

Next we combine (H1)(i) and (H1)(ii) to estimate the remaining terms

$$\|qFD\omega\|_{L^{\frac{k}{2}}(Q_T)} \le \left(\int_{Q_T} q^{\frac{k}{2}}\omega^{\frac{k-2}{2}}W_2\,dy\,dt\right)^{\frac{2}{k}} \le c\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_2\,dt\right)^{\frac{2}{k}}$$

and, similarly,

$$\|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)} \le c\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{2}{k}}$$

$$\|qD\omega\|_{L^k(Q_T)} \le c\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{1}{k}}.$$

Collecting similar terms and recalling that $W_1 \le W_2$ we obtain

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} \le & C\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \left(\int_{a_0}^{b_0} \zeta_2 \, dt\right)^{\frac{1}{k}} \\ & + C\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left(\left(\int_{a_0}^{b_0} \zeta_2 \, dt\right)^{\frac{2}{k}} + \frac{1}{a-a_0}\left(\int_{a_0}^{b_0} \zeta_1 \, dt\right)^{\frac{2}{k}}\right).\end{aligned}$$

Hence, after simple computations,

$$\|\omega q\|_{L^\infty(Q_T)} \le C\left(\int_{a_0}^{b_0} \zeta_2 \, dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1 \, dt\right),$$

and (4.1) follows.
Step 2. If ω is not bounded, we consider $\omega_\varepsilon = \omega/(1+\varepsilon\omega)$. A straightforward computation shows that ω_ε satisfies (H1) with a constant c independent of ε. Therefore, from Step 1 we obtain

$$0 < \omega_\varepsilon(y)p(x,y,t) \le C\left(\int_{a_0}^{b_0} \zeta_2(x,t) \, dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1(x,t) \, dt\right) \tag{4.3}$$

with c independent of ε and, letting $\varepsilon \to 0$ the statement is proved. □

Theorem 4.1 can be applied with $\omega = W_1 = 1$ yielding uniform bounds on p, for fixed x.

Corollary 4.2 *Take $\omega = W_1 = 1$ in (H1)(i) and assume that (H1)(ii) holds. Then*

$$\|p\|_{L^\infty(Q(a,b))} \le C\left(\int_{a_0}^{b_0} \zeta_2(x,t) \, dt + \frac{b_0 - a_0}{(a-a_0)^{\frac{k}{2}}}\right).$$

Let us now see some special cases.

Corollary 4.3 *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x)\cdot\frac{x}{|x|} \le -c, \qquad 0 < c < \infty \tag{4.4}$$

for some $c > 0, \beta > 2$, and that $|F(x)| \le c_1 e^{c_2|x|^{\beta-\varepsilon}}$ for some $\varepsilon, c_1, c_2 > 0$. Then, if $\gamma < (\beta\Lambda)^{-1}c$, where Λ is the maximum eigenvalue of (a_{ij}), the inequality

$$0 < p(x,y,t) \le c_3 \exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right) \exp\left(-\gamma|y|^\beta\right)$$

holds for $x, y \in \mathbf{R}^N$, $0 < t \le T$ and suitable $c_3, c_4 > 0$.

Proof. We take $\omega(y) = e^{\gamma|y|^\beta}$, $W_1(y) = W_2(y) = e^{\delta|y|^\beta}$ for some $\gamma < \delta < (\beta\Lambda)^{-1}c$ and use Theorem 4.1 with $a = t$ and $a - a_0 = b_0 - b = b - a = (1/2)t$. The thesis then follows using Proposition 2.6. □

Example 4.4 Let us specialise the above corollary to the case of the operators

$$A = \Delta - |x|^r \frac{x}{|x|}\cdot D$$

with $r > 1$. Then Corollary 4.3 applies with $\beta = r+1$ and any $\gamma < 1/(r+1)$. Therefore

$$0 < p(x,y,t) \le c_1 \exp\left(c_2 t^{-\frac{r+1}{r-1}}\right) \exp\left(-\gamma|y|^{r+1}\right)$$

for all $0 < t \le T$, $x, y \in \mathbf{R}^N$.

Under conditions similar to those of Corollary 4.3, the estimate of p can be improved with respect to the time variable, loosing the exponential decay in y.

Corollary 4.5 *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x) \cdot \frac{x}{|x|} < 0, \tag{4.5}$$

for some $\beta > 2$. *If* $|F(x)| \le c(1+|x|^2)^{\gamma_1}$ *with* $\gamma_1 \ge \frac{\beta-2}{4}$, *then for every* $\gamma_2 \ge 0$, $k > N+2$, *there exists a constant* $C > 0$ *such that*

$$0 < p(x,y,t) \le \frac{C}{t^\sigma}(1+|y|^2)^{-\gamma_2}$$

for all x, $y \in \mathbf{R}^N, 0 < t \le 1$, *where*

$$\sigma = \frac{2}{\beta-2}\left((k-2)\gamma_1 + \gamma_2\right).$$

Proof. Observe that $W_r(x) = (1+|x|^2)^r$ is a Lyapunov function for every $r > 0$. If $\zeta_r(x,t)$ is the corresponding function defined in (2.1), then Proposition 2.7 yields

$$\zeta_r(x,t) \le c_r t^{\frac{-2r}{\beta-2}}$$

for $x \in \mathbf{R}^N$ and $0 < t \le 1$. We set $a = t$ and $a - a_0 = b_0 - b = b - a = (1/2)t^s$, where $s \ge 1$ will be chosen later, and we apply Theorem 4.1 with $\omega(x) = W_1(x) = \left(1+|x|^2\right)^{\gamma_2}$ and $W_2(x) = \left(1+|x|^2\right)^{k\gamma_1+\gamma_2}$. Thus we obtain

$$p(x,y,t) \le C\left(t^{-\frac{2(k\gamma_1+\gamma_2)}{\beta-2}+s} + t^{-\frac{2\gamma_2}{\beta-2}-s\frac{k}{2}+s}\right)(1+|y|^2)^{-\gamma_2}.$$

Minimising over s we get $s = (4\gamma_1)/(\beta-2)$ and the thesis follows. □

Example 4.6 (i) Choosing $\gamma_1 = \frac{\beta-1}{2}$, $\gamma_2 = 0$ in the above corollary one obtains the following estimate of the norm of $T(t)$ as an operator from $L^1(\mathbf{R}^N)$ to $L^\infty(\mathbf{R}^N)$

$$\|T(t)\|_{L^1(\mathbf{R}^N)\to L^\infty(\mathbf{R}^N)} \le ct^{-(k-2)\frac{\beta-1}{\beta-2}}, \qquad 0 < t \le 1.$$

Observe, finally, that the operator $T(t)$ need not map $L^p(\mathbf{R}^N)$ into itself, for any $p \ge 1$. A simple example of this situation is given by the 1-dimensional operator $D^2 - x^3 D$ (for which $\beta = 4$ is in the estimate above), see [22, Remark 4.3].

(ii) Let us consider again the operators

$$A = \Delta - |x|^r \frac{x}{|x|} \cdot D$$

with $r > 1$. Then Corollary 4.5 applies with $\beta = r+1$ and $\gamma_1 = r/2$ yielding

$$p(x,y,t) \le Ct^{-(k-2)\frac{r}{r-1}-\frac{2\gamma_2}{r-1}}\left(1+|y|^2\right)^{-\gamma_2}.$$

5 Pointwise bounds for the derivatives of transition densities

In this section we derive pointwise estimates on the derivatives of the kernel. The first step consists in showing that $p^{\frac{1}{2}}$ belongs to $W_2^{1,0}(Q(a_1,b_1))$. Observe that estimates in this space are known for invariant measures, that is for the limit, as $t \to \infty$, of the transition kernels $p(x,\cdot,t)$, see [2], [5], [19], [4].

As in Section 4, we fix $0 < a_0 < a < a_1 < b_1 < b < b_0 \le T$ with $b - b_1 \ge a_1 - a$, $a_1 - a \ge a - a_0$.

Theorem 5.1 *Assume that (H1) holds for a certain weight function* ω *such that*

$$\int_{\mathbf{R}^N}\left(\frac{1}{\omega(y)}\right)^{1-\varepsilon} dy < \infty \tag{5.1}$$

for some $\varepsilon \in (0,1)$. *Then the function* $p\log p$ *is integrable in* $\mathbf{R}^N$ *for all* $t \in [a,b]$ *and*

$$\int_{Q(a,b)} \frac{|Dp(x,y,t)|^2}{p(x,y,t)}\,dy\,dt \le \frac{1}{\lambda^2}\int_{Q(a,b)} |F(y)|^2 p(x,y,t)\,dy\,dt - \frac{2}{\lambda}\int_{\mathbf{R}^N}\Big[p(x,y,t)\log p(x,y,t)\Big]_{t=a}^{t=b} dy < \infty.$$

In particular, $p^{\frac{1}{2}}$ *belongs to* $W_2^{1,0}(Q(a,b))$.

Proof. Let us first observe that the functions $p\log^2 p$ and $p\log p$ are integrable in $Q(a,b)$ and in $\mathbf{R}^N$ for all fixed $t\in[a,b]$, respectively, as follows from Theorem 4.1 and (5.1).

Since $p\in W^{1,0}_k(Q(a,b))$ by Lemma 3.2, we get from (2.7)

$$\int_{Q(a,b)} p\partial_t\varphi\,dy\,dt = \int_{Q(a,b)}\left(\sum_{i,j} a_{ij}D_i\varphi D_j p - p\,F\cdot D\varphi\right)dy\,dt + \int_{\mathbf{R}^N}\Big[p(x,y,t)\varphi(t,y)\Big]_{t=a}^{t=b}dy \tag{5.2}$$

for every $\varphi\in C^{2,1}_c(Q(a,b))$. By density, the previous equality holds if φ belongs to $W^{1,1}_2(Q(a,b))$ with compact support in y. Let us take $\xi\in C^\infty_c(\mathbf{R}^N)$ such that $\xi(y)=1$ for $|y|\le 1$ and $\xi(y)=0$ for $|y|\ge 2$, $\xi_n(y)=\xi(\frac{y}{n})$ and note that, by Proposition 2.2, the functions $\xi_n^2\log p(x,\cdot,\cdot)$ belong to $W^{1,1}_2(Q(a,b))$. Plugging $\varphi=\xi_n^2\log p$ in (5.2) and writing $a(\xi,\eta)$ for $\sum_{i,j}a_{ij}\xi_i\eta_j$ we get

$$\begin{aligned}\int_{Q(a,b)}\xi_n^2\partial_t p\,dy\,dt =& \int_{Q(a,b)}\Big(\xi_n^2\frac{a(Dp,Dp)}{p}+2\xi_n\log p\,a(Dp,D\xi_n)\\ &-\xi_n^2F\cdot Dp-2\xi_n p\log p\,F\cdot D\xi_n\Big)dy\,dt\\ &+\int_{\mathbf{R}^N}\Big[p(x,y,t)\xi_n^2(y)\log p(x,y,t)\Big]_{t=a}^{t=b}dy.\end{aligned}$$

That is

$$\int_{Q(a,b)}\xi_n^2\frac{a(Dp,Dp)}{p}\,dy\,dt=-2I_n+J_n+2K_n+\int_{\mathbf{R}^N}\xi_n^2\Big[p-p\log p\Big]_{t=a}^{t=b}dy, \tag{5.3}$$

where

$$\begin{aligned} I_n &= \int_{Q(a,b)}\xi_n\log p\,a(Dp,D\xi_n)\;dy\,dt\\ J_n &= \int_{Q(a,b)}\xi_n^2(F\cdot Dp)\;dy\,dt\\ K_n &= \int_{Q(a,b)}\xi_n p\log p\,F\cdot D\xi_n\;dy\,dt.\end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned}|I_n|\le&\left(\int_{Q(a,b)}\xi_n^2\frac{a(Dp,Dp)}{p}\,dy\,dt\right)^{1/2}\left(\int_{Q(a,b)}p\log^2 p\;a(D\xi_n,D\xi_n)\,dy\,dt\right)^{1/2}\\ \le&\varepsilon\int_{Q(a,b)}\xi_n^2\frac{a(Dp,Dp)}{p}\,dy\,dt+\frac{C}{\varepsilon n^2}\int_{Q(a,b)}p\log^2 p\,dy\,dt.\end{aligned} \tag{5.4}$$

Also

$$\begin{aligned}|J_n|\le&\left(\int_{Q(a,b)}|F|^2p\,dy\,dt\right)^{1/2}\left(\int_{Q(a,b)}\xi_n^2\frac{|Dp|^2}{p}\,dy\,dt\right)^{1/2}\\ \le&\frac{\varepsilon}{\lambda}\int_{Q(a,b)}\xi_n^2\frac{a(Dp,Dp)}{p}\,dy\,dt+\frac{C}{\varepsilon}\int_{Q(a,b)}|F|^2p\,dy\,dt\end{aligned}$$

and

$$|K_n|\le\frac{C}{n}\int_{Q(a,b)}|F|\,p|\log p|\,dy\,dt.$$

Hence (5.3) yields

$$\begin{aligned}\Big(1-\Big(2+\frac{1}{\lambda}\Big)\varepsilon\Big)\int_{Q(a,b)}\xi_n^2\frac{a(Dp,Dp)}{p}\,dy\,dt\le&\frac{C}{\varepsilon n^2}\int_{Q(a,b)}p\log^2p\,dy\,dt\\ &+\frac{C}{\varepsilon}\int_{Q(a,b)}|F|^2p\,dy\,dt+\frac{C}{n}\int_{Q(a,b)}|F|\,p\log p\,dy\,dt+\int_{\mathbf{R}^N}\xi_n^2\Big[p-p\log p\Big]_{t=a}^{t=b}dy.\end{aligned}$$

Letting $n \to \infty$, since the function $p \log^2 p$ is integrable in $Q(a,b)$, it follows that

$$\int_{Q(a,b)} \frac{a(Dp,Dp)}{p}\, dy\, dt < \infty$$

and hence, by the first inequality in (5.4), $I_n \to 0$ as $n \to \infty$. Since also $K_n \to 0$, letting $n \to \infty$ in (5.3) and estimating J_n as above we obtain

$$\begin{aligned}
\lambda \int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt \le & \int_{Q(a,b)} \frac{a(Dp,Dp)}{p}\, dy\, dt \\
\le & \left(\int_{Q(a,b)} |F|^2 p\; dy\, dt\right)^{1/2} \left(\int_{Q(a,b)} \frac{|D_y p|^2}{p}\, dy\, dt\right)^{1/2} \\
& + \int_{\mathbf{R}^N} \Big[p - p\log p\Big]_{t=a}^{t=b} dy \\
\le & \varepsilon \int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt + \frac{1}{4\varepsilon} \int_{Q(a,b)} |F|^2\, p\, dy\, dt \\
& + \int_{\mathbf{R}^N} \Big[-p\log p\Big]_{t=a}^{t=b} dy,
\end{aligned}$$

because $\int_{\mathbf{R}^N} p(x,y,a)dy = \int_{\mathbf{R}^N} p(x,y,b)dy = 1$, see [21, Proposition 5.9], and the statement follows choosing $\varepsilon = \frac{\lambda}{2}$. □

Assuming also that $F \in W^1_{\infty,\mathrm{loc}}(\mathbf{R}^N)$ and

$$\int_{Q(a_0,b_0)} (|F|^k + |\mathrm{div}\, F|^{k/2}) p\, dy\, dt < \infty, \qquad k > 2(N+2), \tag{5.5}$$

we can now prove that Dp is bounded.

Lemma 5.2 *Assume that (H1), (5.1) and (5.5) hold. Then*

$$Dp \in L^s(Q(a_1,b_1))$$

for all $1 \le s \le \infty$.

Proof. From Corollary 3.4 and Lemma 3.2 we know that $Dp \in L^k(Q(a,b))$.
Consider the function $q = \eta p$, where $\eta(t) = 1$, $a_1 \le t \le b_1$, and $\eta(t) = 0$ for $t \le a$, $t \ge b$. Observe that, by Theorem 5.1, $\sqrt{q} \in W_2^{1,0}(Q_T)$. Let us consider $r_1 > 1$ with

$$\frac{1}{r_1} = \left(1 - \frac{2}{k}\right)\frac{1}{k} + \frac{2}{k}.$$

By taking $\alpha = \frac{k}{r_1}$ and $\beta > 1$ such that $\frac{2}{\alpha} + \frac{1}{\beta} = 1$, we deduce, using Hölder's inequality and Theorem 5.1, that

$$\begin{aligned}
\int_{Q_T} |F|^{r_1} |Dq|^{r_1}\, dy\, dt &= \int_{Q_T} |F|^{r_1} q^{\frac{1}{\alpha}} q^{-\frac{1}{\alpha}} |Dq|^{\frac{2}{\alpha}} |Dq|^{r_1 - \frac{2}{\alpha}}\, dy\, dt \\
&\le \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{\alpha}} \left(\int_{Q_T} |F|^{r_1\alpha} q\, dy\, dt\right)^{\frac{1}{\alpha}} \left(\int_{Q_T} |Dq|^{(r_1 - \frac{2}{\alpha})\beta}\, dy\, dt\right)^{\frac{1}{\beta}} \\
&= \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{\alpha}} \left(\int_{Q_T} |F|^k q\, dy\, dt\right)^{\frac{1}{\alpha}} \left(\int_{Q_T} |Dq|^k\, dy\, dt\right)^{\frac{1}{\beta}} < \infty.
\end{aligned}$$

By Proposition 2.2(iii) the function q belongs to $W^{2,1}_{r_1,\mathrm{loc}}(Q_T) \cap L^{r_1}(Q_T)$ and solves the parabolic problem

$$\begin{cases} \partial_t q - A_0 q = -F \cdot Dq - q\,\mathrm{div}\, F + p\partial_t \eta, & \text{in } Q_T, \\ q(y,0) = 0, & y \in \mathbf{R}^N, \end{cases}$$

whose right hand side belongs to $L^{r_1}(Q_T)$ by (5.5) and the previous estimate. By parabolic regularity (see [15, Theorem IV.9.1]), we deduce that $q \in W^{2,1}_{r_1}(Q_T)$.

If $r_1 < N+2$ we use again the Sobolev embedding theorem to deduce that $Dq \in L^{s_1}(Q_T)$ for $\frac{1}{s_1} = \frac{1}{r_1} - \frac{1}{N+2}$.

Now, we iterate the above procedure by setting for every $n \in \mathbf{N}$

$$\frac{1}{r_{n+1}} = \left(1 - \frac{2}{k}\right)\frac{1}{s_n} + \frac{2}{k}, \; \frac{1}{s_n} = \frac{1}{r_n} - \frac{1}{N+2} \quad \text{and } s_0 = k.$$

If $r_n < N+2$ for every n, then $0 \le s_n \le s_{n+1}$. Take $s = \lim_{n\to\infty} s_n$. Since $k > 2(N+2)$ one can see that

$$\frac{1}{s} = \left(1 - \frac{2}{k}\right)\frac{1}{s} + \frac{2}{k} - \frac{1}{N+2} < 0.$$

Thus, $r_n > N+2$ for some n and hence $Dq \in L^\infty(Q_T)$, by the Sobolev embedding. Similarly, if $r_n = N+2$ for some n, then $s_n < \infty$ is arbitrary and hence $r_{n+1} > N+2$, taking s_n sufficiently large and using $k > 2(N+2)$. Thus $Dq \in L^\infty(Q_T)$ in all cases.

The statement follows now from Theorem 5.1, since

$$\int_{Q_T} |Dq|\, dy\, dt \le \left(\int_{Q_T} \frac{|Dq|^2}{q}\, dy\, dt\right)^{\frac{1}{2}} \left(\int_{Q_T} q\, dy\, dt\right)^{\frac{1}{2}} < \infty,$$

and the proof is complete. □

We can now refine Lemma 5.2 providing also a quantitative estimate for the $W^{2,1}_{k/2}$-norm of p.

Theorem 5.3 *Assume that (H1), (5.1) and (5.5) hold. Then $p(x,\cdot,\cdot) \in W^{2,1}_{k/2}(Q(a_1,b_1))$. Moreover there is a constant $C > 0$ such that*

$$\begin{aligned}\|p(x,\cdot,\cdot)\|_{W^{2,1}_{\frac{k}{2}}(Q(a_1,b_1))} \le & C\left\{\left(\int_{Q(a,b)} |F|^k p\right)^{\frac{1}{2}} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{2}}\right. \\ & \left. + \|p\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))}\left(\left(\int_{Q(a,b)} |\mathrm{div}\, F|^{\frac{k}{2}} p\right)^{\frac{2}{k}} + \frac{(b-a)^{\frac{2}{k}}}{a_1 - a}\right)\right\}.\end{aligned}$$

Proof. Take η as in Lemma 5.2 such that $|\eta'| \le \frac{2}{a_1 - a}$. Since $Dq \in L^\infty(Q_T)$ by Lemma 5.2, it follows that

$$\begin{aligned}\int_{Q_T} |F|^{\frac{k}{2}} |Dq|^{\frac{k}{2}}\, dy\, dt =& \int_{Q_T} |F|^{\frac{k}{2}} |Dq|^{\frac{k-2}{2}} \frac{|Dq|}{\sqrt{q}} \sqrt{q}\, dy\, dt \\ \le & \|Dq\|^{\frac{k-2}{2}}_{L^\infty(Q_T)} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{2}} \left(\int_{Q_T} |F|^k q\, dy\, dt\right)^{\frac{1}{2}},\end{aligned}$$

since $Dq \in L^\infty(Q_T)$. This gives

$$\||F||Dq|\|_{L^{\frac{k}{2}}(Q_T)} \le \left(\int_{Q(a,b)} |F|^k p\right)^{\frac{1}{k}} \|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{k}}.$$

Let us consider again the parabolic problem satisfied by q

$$\begin{cases} \partial_t q - A_0 q = -F \cdot Dq - q\,\mathrm{div}\, F + p\partial_t \eta, & \text{in } Q_T, \\ q(y,0) = 0, & y \in \mathbf{R}^N. \end{cases}$$

Using (5.5) and the previous computation, the $L^{\frac{k}{2}}$-norm of the right hand side can be estimated through

$$\begin{aligned}&\left(\int_{Q(a,b)} |F|^k p\right)^{\frac{1}{k}} \|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{k}} \\ &+ \|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\left(\int_{Q(a,b)} |\mathrm{div}\, F|^{k/2} p\right)^{\frac{2}{k}} + \frac{(b-a)^{\frac{2}{k}}}{a_1 - a}\right).\end{aligned}$$

Therefore, $q \in W^{2,1}_{\frac{k}{2}}(Q_T)$ and, using the embedding of $W^{1,0}_{\frac{k}{2}}(Q_T)$ in $L^\infty(Q_T)$, we get

$$\begin{aligned}\|q\|_{W^{2,1}_{\frac{k}{2}}(Q_T)} \le & C\Bigg\{\Big(\int_{Q(a,b)} |F|^k p\Big)^{\frac{1}{k}} \|q\|^{\frac{k-2}{k}}_{W^{2,1}_{\frac{k}{2}}(Q_T)} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{k}} \\ & + \|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\Big(\int_{Q(a,b)} |\mathrm{div}\, F|^{k/2} p\Big)^{\frac{2}{k}} + \frac{(b-a)^{\frac{2}{k}}}{a_1-a}\right)\Bigg\} \\ \le & C\Bigg\{\varepsilon \|q\|_{W^{2,1}_{\frac{k}{2}}(Q_T)} + C_\varepsilon \Big(\int_{Q(a,b)} |F|^k p\Big)^{\frac{1}{2}} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{2}} \\ & + \|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\Big(\int_{Q(a,b)} |\mathrm{div}\, F|^{k/2} p\Big)^{\frac{2}{k}} + \frac{(b-a)^{\frac{2}{k}}}{a_1-a}\right)\Bigg\}\end{aligned}$$

and the estimate for $\|q\|_{W^{2,1}_{\frac{k}{2}}(Q_T)}$ follows choosing $C\varepsilon = \frac{1}{2}$. □

The following result is similar to Theorem 4.1, but relies upon Theorem 5.3 rather than Corollary 3.4. In the sequel, we shall use the following assumption.

(H2) $F \in C^2(\mathbf{R}^N, \mathbf{R}^N)$, $W_1 \le W_2$ are Lyapunov functions for A and there exists $1 \le \omega \in C^4(\mathbf{R}^N)$ such that

$$(\omega^k + |D\omega|^k + |D^2\omega|^k + |D^3\omega|^k + |D^4\omega|^k) \le CW_1$$

and

$$\begin{aligned}&(\omega^k + |D\omega|^k + |D^2\omega|^k + |D^3\omega|^k)(1+|F|^k) + (\omega^k + |D\omega|^k + |D^2\omega|^k) \\ &\quad (1 + |D_jF|^k + |\mathrm{div}\, D_jF|^k) \le CW_2, \quad j = 1, \ldots, N,\end{aligned}$$

for some $k > 2(N+2)$ and a constant $C > 0$. Moreover we suppose that (5.1) holds for some $\varepsilon \in (0,1)$.

We still denote by ζ_1, ζ_2 the functions defined by (2.1) and associated with W_1, W_2, respectively.

Remark 5.4 The C^4 requirement on ω is not always necessary. In order to simplify the presentation, we refrain from specifying the minimal regularity needed in each statement.The minimal degree of smoothness will be clear from the context. Notice also that (H2) implies (H1) and (5.5), hence all the estimates depending on (H1) and (5.5) are true under (H2).

Theorem 5.5 *Assume that (H2) holds. Then there is a constant $C > 0$ such that*

$$\begin{aligned}|\omega(y)Dp(x,y,t)| \le & C\Bigg\{\|Dp\|^{\frac{k-2}{k}}_{L^\infty(Q(a_1,b_1))} \left(\int_{Q(a,b)} \frac{|Dp|^2}{p}\, dy\, dt\right)^{\frac{1}{k}} \left(\int_a^b \zeta_2(x,t)\, dt\right)^{\frac{1}{k}} \\ & + \|p\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))} \left(\int_a^b \zeta_2(x,t)\, dt + \frac{1}{(a_1-a)^{k/2}} \int_a^b \zeta_1(x,t)\, dt\right)^{\frac{2}{k}}\Bigg\}.\end{aligned}$$

for all x, $y \in \mathbf{R}^N$, and $a_1 \le t \le b_1$.

Proof. As in the proof of Theorem 5.3, let us take $q = \eta p$. Then we have $\omega(y)q(y,0) = 0$ and

$$\begin{aligned}\partial_t(\omega q) - A_0(\omega q) =& \omega(\partial_t q - A_0 q) - 2a(D\omega, Dq) - qA_0\omega \\ =& -\omega F \cdot Dq - \omega q \mathrm{div}\, F + \omega p \partial_t \eta - 2a(D\omega, Dq) - qA_0\omega.\end{aligned} \tag{5.6}$$

Assumption (H2) easily gives

$$\|\omega q \mathrm{div}\, F\|_{L^{k/2}(Q_T)} + \|qA_0\omega\|_{L^{k/2}(Q_T)} \le C\|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_a^b \zeta_2(x,t)\, dt\right)^{\frac{2}{k}}$$

and

$$\|\omega p \partial_t \eta\|_{L^{k/2}(Q_T)} \le \|p\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))} \frac{C}{a_1-a} \left(\int_a^b \zeta_1(x,t)\, dt\right)^{\frac{2}{k}}.$$

To treat the terms containing Dq we proceed as in Theorem 5.3, getting

$$\int_{Q_T} \omega^{k/2}|F|^{k/2}|Dq|^{k/2}\,dy\,dt = \int_{Q_T} \omega^{k/2}|F|^{k/2}|Dq|^{\frac{k-2}{2}}\frac{|Dq|}{\sqrt{q}}\sqrt{q}\,dy\,dt$$
$$\leq \|Dq\|_{L^\infty(Q_T)}^{\frac{k-2}{2}}\left(\int_{Q_T}\frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{2}}\left(\int_{Q_T}\omega^k|F|^k q\,dy\,dt\right)^{\frac{1}{2}},$$

whence

$$\|\omega|F||Dq|\|_{L^{k/2}(Q_T)} \leq C\|Dq\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{Q_T}\frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{k}}\left(\int_a^b \zeta_2(x,t)\,dt\right)^{\frac{1}{k}}.$$

The term $|D\omega\cdot Dq|$ is estimated in the same way. Then the right hand side of (5.6) belongs to $L^{k/2}(Q_T)$. Hence, $\omega q \in W^{2,1}_{k/2}(Q_T)$ and the following estimate holds

$$\|\omega(\cdot)p(x,\cdot,\cdot)\|_{W^{2,1}_{k/2}(Q(a_1,b_1))} \leq C\Bigg\{\|Dp\|_{L^\infty(Q(a_1,b_1))}^{\frac{k-2}{k}}\left(\int_{Q(a,b)}\frac{|Dp|^2}{p}\,dy\,dt\right)^{\frac{1}{k}}\left(\int_a^b \zeta_2(x,t)\,dt\right)^{\frac{1}{k}}$$
$$+\|p\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}}\left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1-a)^{k/2}}\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}\Bigg\}. \tag{5.7}$$

Since $k > 2(N+2)$, we use Sobolev embedding (see [15, Lemma II.3.3]) to get the same estimate for the L^∞-norm of $D(\omega q)$ in Q_T. Now we use Theorem 4.1 with ω replaced by $\widetilde{\omega} = (1+|D\omega|^2)^{k/2}$, to obtain

$$\begin{aligned}\|qD\omega\|_{L^\infty(Q_T)} &\leq \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\|q|D\omega|^k\|_{L^\infty(Q_T)}^{\frac{1}{k}} \\ &\leq \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\|q\widetilde{\omega}\|_{L^\infty(Q_T)}^{\frac{1}{k}} \\ &\leq C\|p\|_{L^\infty(Q(a,b))}^{\frac{k-1}{k}}\left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1-a)^{k/2}}\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{1}{k}} \\ &\leq C\|p\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}}\left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1-a)^{k/2}}\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}.\end{aligned}$$

Using all the above estimates, one finally gets the result from the inequality

$$\|\omega Dq\|_{L^\infty(Q_T)} \leq \|D(\omega q)\|_{L^\infty(Q_T)} + \|qD\omega\|_{L^\infty(Q_T)}.$$

□

We can prove similar decay for D^2p and $\partial_t p$.

Theorem 5.6 *Assume that (H2) holds for certain weight functions ω and ω_0 such that $\omega|F| \leq \widetilde{c}\omega_0$ for a constant $\widetilde{c} > 0$. If $a_{ij} \in C^2_b(\mathbf{R}^N)$, then there is a constant $C > 0$ such that*

$$|\omega(y)D^2p(x,y,t)| \leq C\left(\|Dp\|_{L^\infty(Q(a_1,b_1))}^{\frac{k-2}{k}}\left(\int_{Q(a,b)}\frac{|Dp|^2}{p}\,dy\,dt\right)^{\frac{1}{k}} + \|p\|_{L^\infty(Q(a,b))}^{\frac{k-2}{k}}\right)$$
$$\left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1-a)^{\frac{k}{2}}}\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}.$$

for all x, $y \in \mathbf{R}^N$, and $a_1 \leq t \leq b_1$.

Proof. Suppose, for simplicity, that $a_{ij} = \delta_{ij}$. From the proof of Theorem 5.5 we know that the function $v = \omega q$ belongs to $W^{2,1}_{k/2}(Q_T)$ and satisfies $v(y,0) = 0$ and

$$\partial_t v - \Delta v = -\omega F\cdot Dq - \omega q \operatorname{div} F + \omega p\partial_t\eta - 2D\omega\cdot Dq - q\Delta\omega. \tag{5.8}$$

Since $F \in C^2$, by local parabolic regularity $v \in W^{3,1}_{k/2,\mathrm{loc}}(Q_T)$. We can therefore differentiate (5.8) with respect to $y_j \in \mathbf{R}$, $j = 1, \dots, N$, thus obtaining

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) D_j v = &- (D_j\omega)F \cdot Dq - \omega D_j F \cdot Dq - \omega F \cdot DD_j q - q(D_j\omega)\mathrm{div}\, F \\ &- \omega(D_j q)\mathrm{div}\, F - \omega q \mathrm{div}\,(D_j F) + (D_j\omega)p\partial_t\eta + \omega(D_j p)\partial_t\eta \\ &- 2DD_j\omega \cdot Dq - 2D\omega \cdot DD_j q - (D_j q)\Delta\omega - q\Delta D_j\omega. \end{aligned} \tag{5.9}$$

As in the proof of Theorem 5.5 one can see that Assumption (H2) easily implies that

$$\begin{aligned}&\|q\Delta(D_j\omega)\|_{L^{k/2}(Q_T)} + \|\omega q \mathrm{div}\,(D_j F)\|_{L^{k/2}(Q_T)} + \|qD_j\omega \mathrm{div}\, F\|_{L^{k/2}(Q_T)} \\ &\qquad \le C\|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_a^b \zeta_2(x,t)\,dt\right)^{\frac{2}{k}}\end{aligned}$$

and

$$\begin{aligned}&\|(D_j\omega)F \cdot Dq\|_{L^{k/2}(Q_T)} + \|\omega D_j F \cdot Dq\|_{L^{k/2}(Q_T)} + \|\omega \mathrm{div}\, F D_j q\|_{L^{k/2}(Q_T)} \\ &+ \|D_j q \Delta\omega\|_{L^{k/2}(Q_T)} + \|DD_j\omega \cdot Dq\|_{L^{k/2}(Q_T)} \\ &\le C\|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{Q_T} \frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{k}} \left(\int_a^b \zeta_2(x,t)\,dt\right)^{\frac{1}{k}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|(D_j\omega)p\partial_t\eta\|_{L^{k/2}(Q_T)} \le & \frac{C}{a_1 - a}\|p\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))} \left(\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}} \qquad \text{and} \\ \|\omega(D_j p)\partial_t\eta\|_{L^{k/2}(Q_T)} \le & \frac{C}{a_1 - a}\|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{Q_T} \frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{k}} \left(\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{1}{k}}.\end{aligned}$$

To treat the terms containing the second order derivatives of q we use (H2), Theorem 5.5 and (5.7) with ω replaced by ω_0, since $\omega|F| \le \tilde{c}\omega_0$. Hence,

$$\begin{aligned}\|\omega F \cdot DD_j q\|_{L^{k/2}(Q_T)} \le & \tilde{c}\|\omega_0 \cdot DD_j q\|_{L^{k/2}(Q_T)} \\ \le & \tilde{c}\left\{\|q|DD_j\omega_0|\|_{L^{k/2}(Q_T)} + \|D_j\omega_0|Dq|\|_{L^{k/2}(Q_T)}\right. \\ & \left. + \||D\omega_0|D_j q\|_{L^{k/2}(Q_T)} + \|\omega_0 q\|_{W^{2,1}_{k/2}(Q_T)}\right\} \\ \le & C\left\{\|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{Q_T} \frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{k}} \left(\int_a^b \zeta_2(x,t)\,dt\right)^{\frac{1}{k}}\right. \\ & \left. + \|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1 - a)^{k/2}} \int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}\right\}.\end{aligned}$$

Now, applying (5.7) with ω replaced by $(1 + |D\omega|^2)^{1/2}$, the same arguments yield

$$\begin{aligned}\|D\omega \cdot DD_j q\|_{L^{k/2}(Q_T)} \le & C\left\{\|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_{Q_T} \frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{k}} \left(\int_a^b \zeta_2(x,t)\,dt\right)^{\frac{1}{k}}\right. \\ & \left. + \|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)} \left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1 - a)^{k/2}} \int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}\right\}.\end{aligned}$$

Therefore the right hand side of (5.9) is in $L^{k/2}(Q_T)$. Thus, Since $D_j v \in L^{k/2}(Q_T)$ and $D_j v(y,0) = 0$, by the parabolic regularity $D_j v \in W^{2,1}_{k/2}(Q_T)$ and, by Sobolev embedding [15, Lemma II.3.3], $D_{ij} v = D_{ij}(\omega q) \in$

$L^\infty(Q_T)$. Moreover, from the above estimates, we get

$$\|D_{ij}(\omega q)\|_{L^\infty(Q_T)} \le C\left(\|Dq\|^{\frac{k-2}{k}}_{L^\infty(Q_T)}\left(\int_{Q_T}\frac{|Dq|^2}{q}\,dy\,dt\right)^{\frac{1}{k}} + \|q\|^{\frac{k-2}{k}}_{L^\infty(Q_T)}\right)$$
$$\left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1-a)^{k/2}}\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}. \qquad (5.10)$$

Since $\omega D_{ij}q = D_{ij}(\omega q) - qD_{ij}\omega - D_i\omega D_j q - D_j\omega D_i q$, it follows from (H2), Theorem 4.1 with ω replaced by $(1+|D^2\omega|^2)^{1/2}$ and Theorem 5.5 with $(1+|D\omega|^2)^{1/2}$ instead of ω, that $\omega D_{ij}q \in L^\infty(Q_T)$. Finally, the estimate for D^2p follows from Theorem 4.1, Theorem 5.5 and (5.10). □

Theorem 5.7 *Assume that (H2) holds for certain weight functions ω and ω_0 such that $\omega(|F|+|\mathrm{div}\,F|) \le \tilde{c}\omega_0$ for a constant $\tilde{c} > 0$. If $a_{ij} \in C_b^2(\mathbf{R}^N)$, then there is a constant $C > 0$ such that*

$$|\omega(y)\partial_t p(x,y,t)| \le C\left(\|Dp\|^{\frac{k-2}{k}}_{L^\infty(Q(a_1,b_1))}\left(\int_{Q(a,b)}\frac{|Dp|^2}{p}\,dy\,dt\right)^{\frac{1}{k}} + \|p\|^{\frac{k-2}{k}}_{L^\infty(Q(a,b))}\right)$$
$$\left(\int_a^b \zeta_2(x,t)\,dt + \frac{1}{(a_1-a)^{\frac{k}{2}}}\int_a^b \zeta_1(x,t)\,dt\right)^{\frac{2}{k}}.$$

for all x, $y \in \mathbf{R}^N$, and $a_1 \le t \le b_1$.

Proof. As in the proof of Theorem 5.6 we assume, for simplicity, that $a_{ij} = \delta_{ij}$. It follows from Proposition 2.2 that

$$\omega(y)\partial_t p = \omega(y)\Delta p - \omega(y)F\cdot Dp - \omega(y)\mathrm{div}\,Fp.$$

Hence, by assumption we have

$$|\omega(y)\partial_t p(x,y,t)| \le |\omega(y)\Delta p(x,y,t)| + \tilde{c}\omega_0(y)|Dp(x,y,t)| + \tilde{c}\omega_0(y)p(x,y,t).$$

So the estimate for $\partial_t p$ follows now from Theorem 4.1, Theorem 5.5 and Theorem 5.6. □

Remark 5.8 In concrete examples, the weight ω and the Lyapunov functions W_1, W_2 are powers or exponentials of powers. The above results are formulated in a unified way, but the two situations are different. In the exponential case, in fact, slightly simpler statements are possible: typically, one has $\omega(y) = \exp\{\gamma|y|^\beta\}$ and $W_1(y) = W_2(y) = \exp\{\delta|y|^\beta\}$, with $\beta > 0$ and $\delta > \gamma > 0$, so only one Lyapunov function is needed.

6 Some applications

We show that, under the main assumptions of the previous section, the semigroups $T(\cdot)$ associated with the transition kernels p are differentiable in $C_b(\mathbf{R}^N)$. We remark that if the drift F is unbounded, the associated semigroup is rarely analytic in $C_b(\mathbf{R}^N)$, see [23].

Theorem 6.1 *Suppose that $a_{ij} \in C_b^2(\mathbf{R}^N)$, $F \in C^2(\mathbf{R}^N)$ and that there exist constants $c > 0, \beta > 2$ such that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}F(x)\cdot\frac{x}{|x|} \le -c.$$

Assume moreover that $|F(x)| + |DF(x)| + |D^2F(x)| \le c_1\exp(c_2|x|^{\beta-\varepsilon})$ for some $\varepsilon, c_1, c_2 > 0$. Then the inequalities

(i) $0 < p(x,y,t) \le c_3\exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)\exp\left(-\gamma|y|^\beta\right)$

(ii) $|Dp(x,y,t)| \le c_3\exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)\exp\left(-\gamma|y|^\beta\right)$

(iii) $|D^2p(x,y,t)| \le c_3\exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)\exp\left(-\gamma|y|^\beta\right)$

(iv) $|\partial_t p(x,y,t)| \le c_3\exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)\exp\left(-\gamma|y|^\beta\right)$

hold for suitable $c_3, c_4, \gamma > 0$ *and for all* $0 < t \le T$ *and* $x, y \in \mathbf{R}^N$.

Proof. From Proposition 2.6 we deduce that the function $\exp\{\delta|x|^\beta\}$ is a Lyapunov function for a sufficiently small $\delta > 0$. We fix $\omega(y) = \exp\{\gamma|x|^\beta\}$, $\omega_0(y) = \exp\{\gamma_0|x|^\beta\}$, $W_1(y) = W_2(y) = \exp\{\delta|x|^\beta\}$ with $\gamma < \gamma_0$ and $k\gamma_0 < \delta$. With these choices, it is easily seen that assumption (H2) holds both for ω and ω_0 so that all the results of the previous sections apply. Moreover

$$\zeta(x,t) \le c_1 \exp\left(c_2 t^{-\frac{\beta}{\beta-2}}\right)$$

for suitable $c_1, c_2 > 0$ and every $x \in \mathbf{R}^N, t > 0$, where ζ is the function defined in (2.1) and associated with $W_1 = W_2$.

Statement (i) follows from Corollary 4.3. For the proof of the other statements we apply Theorem 5.1 with $a = t, b = 2t$. Estimating the integral of $|F|^2 p$ through ζ and using (i) for that of $p \log p$ we deduce

$$\int_t^{2t} \int_{\mathbf{R}^N} \frac{|Dp(x,y,s)|^2}{p(x,y,s)}\, dy\, ds \le c_3 \exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)$$

for $x \in \mathbf{R}^N, t > 0$ and suitable positive constants c_3, c_4. Inserting this estimate in Theorem 5.3 and using (i) and Sobolev embedding we obtain

$$|Dp(x,y,s)| \le c_3 \exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)$$

for $x, y \in \mathbf{R}^N, t \le s \le 2t$. Finally, (ii), (iii), (iv) follow using these estimates in Theorems 5.5, 5.6, 5.7, respectively. □

Remark 6.2 Observe that the assumption $a_{ij} \in C_b^2(\mathbf{R}^N)$ is not needed for (i) and (ii).

Remark 6.3 Let us point out a variant of Theorem 6.1. We assume that $a_{ij} \in C_b^2(\mathbf{R}^N)$, $F \in C^2(\mathbf{R}^N)$ and that there exist constants $c > 0, \beta > 2$ such that

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x) \cdot \frac{x}{|x|} \le -c.$$

We assume moreover that $|F(x)| + |DF(x)| + |D^2F(x)| \le c_1(1+|x|^2)^{\gamma_1}$ for some $\gamma_1, c_1, c_2 > 0$. Then, for sufficiently large γ_2 the following estimate holds

$$p(x,y,t) + |Dp(x,y,y)| + |D^2 p(x,y,t)| + |\partial_t p(x,y,t)| \le C t^{-\sigma}(1+|y|^2)^{-\gamma_2},$$

for $x, y \in \mathbf{R}^N, 0 < t \le 1$ and with a suitable σ depending on γ_1, γ_2. In fact, the estimate for p is contained in Corollary 4.5 where the dependence of σ on γ_1, γ_2 is explicitly stated. The corresponding bounds for the derivatives of p can be obtained as in Theorem 6.1. We refrain from stating the explicit dependence of σ in the general case since it does not seem to be optimal.

Finally, let us show that the transition semigroup $T(\cdot)$ is differentiable in spaces of continuous functions, under the assumption of Theorem 6.1. We observe that in the case $\beta = 2$ the semigroup need not to be differentiable as the example of the Ornstein-Uhlenbeck operator shows, see [18]. Moreover, even when $\beta > 2$ the semigroup is not, in general, analytic, see [23]. Finally we point out that our methods allow to prove the differentiability of the semigroup without requiring that the drift F is a gradient.

Theorem 6.4 *Under the assumptions of Theorem 6.1, the transition semigroup* $T(\cdot)$ *is differentiable on* $C_b(\mathbf{R}^N)$ *for* $t > 0$.

Proof. Let us fix $0 < a < T$. By Theorem 6.1 we know $|\partial_t p(x,y,t)| \le c_1 \exp(-c_2|y|^\beta)$ for every $a \le t \le T$, $x, y \in \mathbf{R}^N$. Since $p(\cdot, y, \cdot) \in C_{\rm loc}^{1+\alpha/2, 2+\alpha}(\mathbf{R}^N \times (0,\infty))$, for every $f \in C_b(\mathbf{R}^N)$ and $t > 0$ the function

$$T(t)f(\cdot) = \int_{\mathbf{R}^N} p(\cdot, y, t) f(y)\, dy$$

is differentiable with respect to the norm of $C_b(\mathbf{R}^N)$ and

$$\frac{d}{dt} T(t)f(\cdot) = \int_{\mathbf{R}^N} \partial_t p(\cdot, y, t) f(y)\, dy.$$

□

As an example, we obtain that the operator $A = \Delta - x|x|^r \cdot D$, $r > 0$, generates a differentiable semigroup in $C_b(\mathbf{R})$. The same result is proved also in [23, Proposition 4.4], where the proof was based on results on intrinsic ultracontractivity of Schrödinger operator proved in [9] and therefore used the gradient structure of the drift.

A Appendix

In this appendix we present a simple, purely analytical, proof of the embeddings of the spaces $\mathcal{H}^{k,1}(Q_T)$, due to Krylov, see [13]. Krylov proves the above embeddings for the more general case of stochastic parabolic Sobolev spaces. We also prove by the same method an embedding for the spaces $\mathcal{V}^k(Q_T)$ which we used in Section 3. Finally, we prove an estimate for the L^∞ norm of solutions of certain parabolic problems.

We recall that $\mathcal{H}^{k,1}(Q_T)$ consists of all functions $u \in W_k^{1,0}(Q_T)$ with $\partial_t u \in (W_{k'}^{1,0}(Q_T))'$ and that, for $k > 2$, $\mathcal{V}^k(Q_T)$ is the space of all functions $u \in W_k^{1,0}(Q_T)$ such that there exists $C > 0$ for which

$$\left| \int_{Q_T} u \partial_t \phi \, dx \, dt \right| \le C \left(\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} + \|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} \right)$$

for every $\phi \in C_c^{2,1}(Q_T)$. $\|\partial_t u\|_{k/2,k;Q_T}$ denotes the best constant C such that the above estimate holds. Note that if a smooth function belongs to $\mathcal{H}^{k,1}(Q_T)$ or to $\mathcal{V}^k(Q_T)$ the estimate for $\partial_t u$ implies that u vanishes at times $0, T$.

Lemma A.1 *There exist linear, continuous extension operators $E_1 : \mathcal{H}^{k,1}(Q_T) \to \mathcal{H}^{k,1}(\mathbf{R}^{N+1})$ and $E_2 : \mathcal{V}^k(Q_T) \to \mathcal{V}^k(\mathbf{R}^{N+1})$.*

Proof. The proof is easily achieved using standard reflection arguments with respect to the variable t. □

Lemma A.2 *The restrictions of functions in $C_c^\infty(\mathbf{R}^{N+1})$ to Q_T are dense in $\mathcal{H}^{k,1}(Q_T)$ and in $\mathcal{V}^k(Q_T)$.*

Proof. If $u \in \mathcal{H}^{k,1}(Q_T)$ we consider $v = E_1 u \in \mathcal{H}^{k,1}(\mathbf{R}^{N+1})$. By standard arguments involving convolutions and multiplications by cut-off functions, we may approximate v with smooth functions with compact support in the norm of $\mathcal{H}^{k,1}(\mathbf{R}^{N+1})$, hence u. The proof for $\mathcal{V}^k(Q_T)$ is similar. □

Theorem A.3 **(i)** *If $1 < k < N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $\frac{1}{r} = \frac{1}{k} - \frac{1}{N+2}$.*

(ii) *If $k = N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $N+2 \le r < \infty$.*

(iii) *If $k > N+2$, then $\mathcal{H}^{k,1}(Q_T)$ is continuously embedded in $C_0(Q_T)$.*

Proof. By Lemma A.2 it is sufficient to establish the estimate

$$\|u\|_{L^r(Q_T)} \le C \|u\|_{\mathcal{H}^{k,1}(Q_T)}$$

for every $u \in C_c^\infty(\mathbf{R}^{N+1})$, with C independent of u, where r is as in (i), (ii) or $r = \infty$ in case (iii).

We consider the fundamental solution G of the operator $\partial_t - \Delta$ in $\mathbf{R}^{N+1}$. We have

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)N/2} \exp\left(-\frac{1}{4t}|x|^2\right) & t > 0 \\ 0 & t \le 0 \end{cases}$$

Let $u \in C_c^\infty(\mathbf{R}^{N+1})$, $\psi \in C_c^\infty(Q_T)$ and consider $\phi = G * \psi$. The function ϕ belongs to $C^2(\mathbf{R}^{N+1})$ and satisfies $\partial_t \phi - \Delta \phi = \psi$, see e.g. [11, Theorem 8.4.2]. Since ψ has support in $\mathbf{R}^N \times [0,T]$, then $G * \psi = G_T * \psi$, where $G_T = G\chi_{[0,T]}$. By a straightforward computation one sees that $G_T \in L^s(\mathbf{R}^{N+1})$ for $1 \le s < (N+2)/N$ and $DG_T \in L^s(\mathbf{R}^{N+1})$ for $1 \le s < (N+2)(N+1)$. Young's inequality then yields $\|\phi\|_{W_s^{1,0}(Q_T)} \le c_1 \|\psi\|_{L^1(Q_T)}$ for $s < (N+2)/(N+1)$. Since $k > N+2$, $k' < (N+2)/(N+1)$ and we get

$$\left| \int_{Q_T} u\psi \, dx \, dt \right| = \left| \int_{Q_T} u(\partial_t \phi - \Delta\phi) \, dx \, dt \right| = \left| \int_{Q_T} u(\partial_t \phi) + Du \cdot D\phi \, dx \, dt \right|$$
$$\le c_2 \|u\|_{\mathcal{H}^{k,1}(Q_T)} \|\phi\|_{W_{k'}^{1,0}(Q_T)} \le c_3 \|u\|_{\mathcal{H}^{k,1}(Q_T)} \|\psi\|_{L^1(Q_T)}.$$

This proves (iii).

In order to prove (ii) we fix $N+2 < r < \infty$ and choose $1 < s < (N+2)/(N+1)$ such that

$$\frac{1}{k'} = \frac{1}{s} + \frac{1}{r'} - 1.$$

Young's inequality then yields $\|\phi\|_{W_{k'}^{1,0}(Q_T)} \le c_1 \|\psi\|_{L^{r'}(Q_T)}$ hence

$$\left| \int_{Q_T} u\psi \, dx \, dt \right| \le c \|u\|_{\mathcal{H}^{k,1}(Q_T)} \|\psi\|_{L^{r'}(Q_T)}$$

and (ii) is proved.

To prove (i) we use the estimate $\|\phi\|_{W_{r'}^{2,1}(Q_T)} \le c \|\psi\|_{L^{r'}(Q_T)}$, see [15, Theorem 9.2.3] and the embedding $W_{r'}^{2,1}(Q_T) \subset W_{k'}^{1,0}(Q_T)$, see [15, Lemma II.3.3] to conclude as before. □

A closer look at the above proof shows an embedding of the space $\mathcal{V}^k(Q_T)$, used in Section 4.

Theorem A.4 *If $k > N+2$, then $\mathcal{V}^k(Q_T)$ is continuously embedded in $C_0(Q_T)$. Moreover*

$$\|u\|_{L^\infty(Q_T)} \le C \left(\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{k/2,k;Q_T}\right).$$

Proof. As above we may assume that $u \in C_c^\infty(\mathbf{R}^{N+1})$. Choose ϕ, ψ as in the above theorem. Then

$$\begin{aligned}\left|\int_{Q_T} u\psi\, dx\, dt\right| &= \left|\int_{Q_T} u(\partial_t\phi - \Delta\phi)\, dx\, dt\right| = \left|\int_{Q_T} u\partial_t\phi + Du\cdot D\phi\, dx\, dt\right| \\ &\le \left(\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{k/2,k;Q_T}\right)\left(\|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} + \|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\right) \\ &\le C\left(\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{k/2,k;Q_T}\right)\|\psi\|_{L^1(Q_T)}\end{aligned}$$

by the above estimates for ϕ, since $k/(k-1) < (N+2)/(N+1)$ and $k/(k-2) < (N+2)/N$. □

We need the following estimate of the sup norm of solution of parabolic problems.

Theorem A.5 *Let $k > N+2$, $v \in L^k(Q_T)$, $w \in L^{\frac{k}{2}}(Q_T)$ and assume that $u \in L^k(Q_T)$ satisfies*

$$\int_{Q_T} u(\partial_t\phi + A_0\phi)\, dx\, dt = \int_{Q_T} (v\cdot D\phi + w\phi)\, dx\, dt \tag{A.1}$$

for every $\phi \in C_c^{2,1}(Q_T)$. Then $u \in \mathcal{V}^k(Q_T)$ and

$$\|u\|_{L^\infty(Q_T)} \le C_1\|u\|_{\mathcal{V}^k(Q_T)} \le C_2\left(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right)$$

where C_1, C_2 depend on N, T, k and the C_b^1-norm of a_{ij}.

Proof. **Step 1.** First we show that

$$\|u\|_{L^k(Q_T)} \le C\left(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right). \tag{A.2}$$

For $\phi \in W^{2,1}_{k'}(Q_T)$, Sobolev embedding gives

$$\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} \le C\|\phi\|_{W^{2,1}_{k'}(Q_T)} \tag{A.3}$$

since $k > N+2$ and $1 - 1/k - 2/(N+2) < 1 - 2/k < 1 - 1/k$. As a consequence, since $u \in L^k(Q_T)$, by approximation (A.1) holds if ϕ belongs to $W^{2,1}_{k'}(Q_T)$. Let us fix $\psi \in C_c^\infty(Q_T)$. Using [15, Theorem 9.2.3] we choose now $\phi \in W^{2,1}_{k'}(Q_T)$ such that

$$\begin{cases} \partial_t\phi + A_0\phi = \psi, & \text{in } Q_T, \\ \phi(x,T) = 0, & x \in \mathbf{R}^N. \end{cases}$$

We have also

$$\|\phi\|_{W^{2,1}_{k'}(Q_T)} \le C\|\psi\|_{L^{k'}(Q_T)},$$

where C depends on k, T and the coefficients (a_{ij}). Therefore, inserting this ϕ in (A.1) and using (A.3), we obtain

$$\left|\int_{Q_T} u\psi\right| \le C\left(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right)\|\psi\|_{L^{k'}(Q_T)}$$

and (A.2) follows.
Step 2. We have

$$\int_{Q_T} u(\partial_t\phi + A_1\phi)\, dx\, dt = \int_{Q_T} (g\cdot D\phi + w\phi)\, dx\, dt,$$

where $A_1 = \sum_{i,j} a_{ij}D_{ij}$ and $g_i = v_i + uD_i(\sum_{j=1}^N a_{ij})$ and therefore

$$\begin{aligned}\left|\int_{Q_T} u(\partial_t\phi + A_1\phi)\, dx\, dt\right| \le & C\Big[\left(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)}\right)\|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} \\ & + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\Big]\end{aligned}$$

Replacing ϕ by its difference quotients with respect to the variable x we obtain as in Lemma 3.2

$$\begin{aligned}\left|\int_{Q_T} \tau_h u(\partial_t\phi + A_1\phi)\,dx\,dt\right| \le & C\Big[\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)}\big)\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)} \\ & + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|D\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\Big].\end{aligned}$$

By Sobolev embedding

$$\|D\phi\|_{L^s(Q_T)} \le C\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)}$$

if $1/s = 1 - 1/k - 1/(N+2)$. Since $k/(k-1) < k/(k-2) < s$, because $k > N+2$, we can estimate the $L^{k/(k-2)}$-norm of $D\phi$ with its $W^{2,1}_{k/(k-1)}$-norm thus obtaining

$$\left|\int_{Q_T} \tau_h u(\partial_t\varphi + A_1\phi)\,dx\,dt\right| \le C\left(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right)\|\phi\|_{W^{2,1}_{\frac{k}{k-1}}(Q_T)}.$$

We approximate ϕ in $W^{2,1}_{k/(k-1)}(Q_T)$ with a sequence of functions $\varphi_n \in C^{1,2}_c(Q_T)$. Since $u \in L^k(Q_T)$, writing the above inequality for ϕ_n and letting $n \to \infty$ we see that it holds for ϕ.

Proceeding as above we now choose $\phi \in W^{2,1}_{k'}(Q_T)$ such that

$$\begin{cases} \partial_t\phi + A_1\phi = |\tau_h q|^{k-2}\tau_h u, & \text{in } Q_T, \\ \phi(x,T) = 0, & x \in \mathbf{R}^N \end{cases}$$

and

$$\|\phi\|_{W^{2,1}_{k'}(Q_T)} \le C\||\tau_h u|^{k-1}\|_{L^{k'}(Q_T)}.$$

This yields $u \in W^{1,0}_k(Q_T)$ and

$$\|Du\|_{L^k(Q_T)} \le C\left(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right).$$

Now we treat the time derivative. We have

$$\int_{Q_T} u\partial_t\phi\,dx\,dt = \int_{Q_T}\big(\sum_{i,j} a_{ij}D_iuD_j\phi + v\cdot D\phi + w\phi\big)\,dx\,dt$$

and hence, using the above estimates,

$$\begin{aligned}\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le & C\Big[\big(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\big)\|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)} \\ & + \|w\|_{L^{\frac{k}{2}}(Q_T)}\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)}\Big].\end{aligned}$$

Then $u \in \mathcal{V}^k(Q_T)$ and hence Theorem A.4 yields $u \in L^\infty(Q_T)$ and

$$\begin{aligned}\|u\|_{L^\infty(Q_T)} \le C\left(\|Du\|_{L^k(Q_T)} + \|\partial_t u\|_{k/2,k;Q_T}\right) \le & C\left(\|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right) \\ \le & C\left(\|v\|_{L^k(Q_T)} + \|w\|_{L^{\frac{k}{2}}(Q_T)}\right),\end{aligned}$$

having used (A.2) in the last inequality. □

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