Global properties of transition kernels associated with second order elliptic operators

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Dedicated to Prof. H. Amann on the occasion of his 70^{th} *birthday*

Abstract. We study global regularity properties of transitions kernels associated with second order differential operators in $\mathbf{R}^N$ with unbounded drift and potential terms. Under suitable conditions, we prove Sobolev regularity of transition kernels and pointwise upper bounds. As an application, we obtain sufficient conditions implying the differentiability of the associated semigroup on the space of bounded and continuous functions on $\mathbf{R}^N$.

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1. Introduction

Given a second order elliptic partial differential operator with real coefficients

$$A=\sum_{i,j=1}^{N} D_i\left(a_{ij}D_j\right)+\sum_{i=1}^{N}F_iD_i-V=A_0+F\cdot D-V, \tag{1.1}$$

where $A_0=\sum_{i,j=1}^{N} D_i\left(a_{ij}D_j\right)$, we consider the parabolic problem

$$\begin{cases} u_t(x,t)=Au(x,t), & x\in\mathbf{R}^N,\ t>0,\\ u(x,t)=f(x), & x\in\mathbf{R}^N,\end{cases} \tag{1.2}$$

where $f\in C_b(\mathbf{R}^N)$.

We assume the following conditions on the coefficients of A which will be kept in the whole paper without further mentioning.

(H) $a_{ij} = a_{ji}$, $F_i : \mathbf{R}^N \to \mathbf{R}$, $V : \mathbf{R}^N \to [0, +\infty)$, with $a_{ij} \in C^{1+\alpha}(\mathbf{R}^N)$, V, $F_i \in C^{\alpha}_{\rm loc}(\mathbf{R}^N)$ for some $0 < \alpha < 1$ and

$$\lambda|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for every $x, \xi \in \mathbf{R}^N$ and suitable $0 < \lambda \le \Lambda$.

Notice that neither the drift $F = (F_1, \dots, F_N)$ or the potential V are assumed to be bounded in $\mathbf{R}^N$.

Problem (1.2) has always a bounded solution but, in general, there is no uniqueness. However, if f is nonnegative, it is not difficult to show that (1.2) has a minimal solution u among all non negative solutions. Taking such a solution u one constructs a semigroup of positive contractions $T(\cdot)$ on $C_b(\mathbf{R}^N)$ such that

$$u(x,t) = T(t)f(x), \quad t > 0,\, x \in \mathbf{R}^N$$

solves (1.2). Furthermore, the semigroup can be represented in the form

$$T(t)f(x) = \int_{\mathbf{R}^N} p(x,y,t)f(y)\,dy, \quad t > 0,\, x \in \mathbf{R}^N,$$

for $f \in C_b(\mathbf{R}^N)$. Here p is a positive function and for almost every $y \in \mathbf{R}^N$, it belongs to $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ as a function of (x,t) and solves the equation $\partial_t p = Ap$, $t > 0$. We refer to Section 2, [8, Chapter 1] and [11] (in the case $V = 0$) for a review of these results as well as for conditions ensuring uniqueness for (1.2).

Now, we fix $x \in \mathbf{R}^N$ and consider p as a function of (y,t). Then p satisfies

$$\partial_t p = A^* p, \quad t > 0, \tag{1.3}$$

in the following sense (see [10, Lemma 2.1]): Let $0 \le t_1 < t_2$ and $\varphi \in C^{2,1}(Q(t_1,t_2))$ (see below for the notation) be such that $\varphi(\cdot,t)$ has compact support for every $t \in [t_1,t_2]$. Then

$$\begin{aligned}\int_{Q(t_1,t_2)} &(\partial_t\varphi(y,t) + A\varphi(y,t))\,p(x,y,t)\,dy\,dt \\ &= \int_{\mathbf{R}^N} (p(x,y,t_2)\varphi(y,t_2) - p(x,y,t_1)\varphi(y,t_1))\,dy.\end{aligned} \tag{1.4}$$

The aim of this paper is to study global regularity properties of the kernel p as a function of $(y,t) \in \mathbf{R}^N \times (a,T)$ for $0 < a < T$.

We prove that $p(x,\cdot,\cdot)$ belongs to $W^{1,0}_k(\mathbf{R}^N \times (a,T))$ (see below for the notation) provided that

$$\int_{a_0}^T \int_{\mathbf{R}^N} \left(V(y)^k + |F(y)|^k\right) p(x,y,t)\,dy\,dt < \infty, \quad \forall k > 1$$

for fixed $x \in \mathbf{R}^N$ and $0 < a_0 < a$. This generalizes [10, Corollary 3.1 and Lemma 3.1] and in some sense Theorem 4.1 in [2]. Assuming that certain Lyapunov functions (exponentials or powers) are integrable with respect to $p(x,y,t)\,dy$ for $(x,t) \in \mathbf{R}^N \times (a,T)$, pointwise upper bounds for p are obtained. If in addition $V \in W^{1,\infty}_{\rm loc}(\mathbf{R}^N)$, $F \in W^{1,\infty}_{\rm loc}(\mathbf{R}^N, \mathbf{R}^N)$ such that DV, DF are dominated by some exponential functions, then $p \in W^{2,1}_k(\mathbf{R}^N \times (a,T))$ for all $k > 1$. As a consequence, we obtain also upper bounds for $|D_y p|$. In the case where F and V and their corresponding derivatives up to the second order satisfy growth conditions of exponential type, upper bounds are also obtained for $|D_{yy}p|$ and $|\partial_t p|$. As a consequence, we deduce that the semigroup $T(\cdot)$ is differentiable on $C_b(\mathbf{R}^N)$ for $t > 0$.

In the case where $V = 0$, regularity and pointwise estimates for p can be found in [10], [14] and for the solution of (1.3) with a L^1-function as the initial datum we refer to [3], [4].

Other bounds for the transition densities p are obtained in [1], using time dependent Lyapunov functions techniques.

Notation. $B_R(x)$ denotes the open ball of $\mathbf{R}^N$ of radius R and center x. If $x = 0$ we simply write B_R. For $0 \le a < b$, we use $Q(a,b)$ for $\mathbf{R}^N \times (a,b)$ and Q_T for $Q(0,T)$ (here the intervals can be either open or closed). We write $C = C(a_1, \dots, a_n)$ to point out that the constant C depends on the quantities $a_1, \dots, a_n$. To simplify the notation, we understand the dependence on the dimension N and on quantities determined by the matrix (a_{ij}) as the ellipticity constant or the modulus of continuity of the coefficients.

If $u : \mathbf{R}^N \times J \to \mathbf{R}$, where $J \subset [0,\infty[$ is an interval, we use the following notation:

$$\partial_t u = \frac{\partial u}{\partial t},\ D_i u = \frac{\partial u}{\partial x_i},\ D_{ij}u = D_i D_j u$$
$$Du = (D_1 u, \dots, D_N u),\ D^2 u = (D_{ij}u)$$

and

$$|Du|^2 = \sum_{i=1}^N |D_i u|^2, \qquad |D^2 u|^2 = \sum_{i,j=1}^N |D_{ij}u|^2.$$

Let us come to notation for function spaces. $C^j_b(\mathbf{R}^N)$ is the space of j times differentiable functions in $\mathbf{R}^N$, with bounded derivatives up to the order j. $C^\infty_c(\mathbf{R}^N)$ is the space of test functions. $C^\alpha(\mathbf{R}^N)$ denotes the space of all bounded and α-Hölder continuous functions on $\mathbf{R}^N$.

For $1 \le k \le \infty$, $j \in \mathbf{N}$, $W^j_k(\mathbf{R}^N)$ denotes the classical Sobolev space of all L^k–functions having weak derivatives in $L^k(\mathbf{R}^N)$ up to the order j. Its usual norm is denoted by $\|\cdot\|_{j,k}$ and by $\|\cdot\|_k$ when $j = 0$.

Let us now define some spaces of functions of two variables (following basically the notation of [7]). $C_0(Q(a,b))$ is the Banach space of continuous functions u defined in $Q(a,b)$ such that $\lim_{|x|\to\infty} u(x,t) = 0$ uniformly with respect to $t \in [a,b]$. $C^{2,1}(Q(a,b))$ is the space of all bounded functions u such that $\partial_t u$, Du

and $D_{ij}u$ are bounded and continuous in $Q(a,b)$. For $0<\alpha\le 1$ we denote by $C^{2+\alpha,1+\alpha/2}(Q(a,b))$ the space of all bounded functions u such that $\partial_t u$, Du and $D_{ij}u$ are bounded and α-Hölder continuous in $Q(a,b)$ with respect to the parabolic distance $d((x,t),(y,s)) := |x-y| + |t-s|^{\frac{1}{2}}$. Local Hölder spaces are defined, as usual, requiring that the Hölder condition holds in every compact subset.

We shall also use parabolic Sobolev spaces. We denote by $W^{r,s}_k(Q(a,b))$ the space of functions $u\in L^k(Q(a,b))$ having weak space derivatives $D^\alpha_x u\in L^k(Q(a,b))$ for $|\alpha|\le r$ and weak time derivatives $\partial^\beta_t u\in L^k(Q(a,b))$ for $\beta\le s$, equipped with the norm

$$\|u\|_{W^{r,s}_k(Q(a,b))} := \|u\|_{L^k(Q(a,b))} + \sum_{|\alpha|\le r}\|D^\alpha_x u\|_{L^k(Q(a,b))} + \sum_{|\beta|\le s}\|\partial^\beta_t u\|_{L^k(Q(a,b))}.$$

$\mathcal{H}^{k,1}(Q_T)$ denotes the space of all functions $u\in W^{1,0}_k(Q_T)$ with $\partial_t u\in (W^{1,0}_{k'}(Q_T))'$, the dual space of $W^{1,0}_{k'}(Q_T)$, endowed with the norm

$$\|u\|_{\mathcal{H}^{k,1}(Q_T)} := \|\partial_t u\|_{(W^{1,0}_{k'}(Q_T))'} + \|u\|_{W^{1,0}_k(Q_T)}$$

where $\frac{1}{k}+\frac{1}{k'}=1$. Finally, for $k>2$, $\mathcal{V}^k(Q_T)$ is the space of all functions $u\in W^{1,0}_k(Q_T)$ such that there exists $C>0$ for which

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\left(\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} + \|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)}\right)$$

for every $\phi\in C^{2,1}_c(Q(a,b))$. Notice that $\frac{k}{k-1}=k'$, $\frac{k}{k-2}=\left(\frac{k}{2}\right)'$. $\mathcal{V}^k(Q_T)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{V}^k(Q_T)} = \|u\|_{W^{1,0}_k(Q_T)} + \|\partial_t u\|_{\frac{k}{2},k;Q_T},$$

where $\|\partial_t u\|_{\frac{k}{2},k;Q_T}$ is the best constant C such that the above estimate holds.

The space $\mathcal{H}^{k,1}(Q_T)$ was introduced and studied by Krylov [6]. All properties of the spaces $\mathcal{H}^{k,1}(Q_T)$ and $\mathcal{V}^k(Q_T)$ needed here, can be found in [10, Appendix].

In the whole paper the transition density p will be considered as a function of (y,t) for arbitrary but fixed $x\in\mathbf{R}^N$. The writing $\|p\|$ therefore stands for any norm of p as function of (y,t), for a fixed x.

2. Local regularity and integrability of transition densities

As a first step we recall some local regularity results for the kernel p associated with the minimal semigroup

$$T(t)f(x) = \int_{\mathbf{R}^N} p(x,y,t)f(y)\,dy,$$

i.e., the semigroup which defines the minimal bounded positive solutions of equation (1.2) when $f\ge 0$.

Regularity properties of the kernels p with respect to the variables (y,t) are known even under weaker conditions than our hypothesis (H), see [2]. We combine the results of [2] with the Schauder estimates to obtain regularity of p with respect to all the variables (x, y, t). The proof is similar to the one of Proposition 2.1 in [10].

Proposition 2.1. *Under assumption (H) the kernel $p = p(x, y, t)$ is a positive continuous function in $\mathbf{R}^N \times \mathbf{R}^N \times (0, \infty)$ which enjoys the following properties.*

(i) *For every $x \in \mathbf{R}^N$, $1 < s < \infty$, the function $p(x, \cdot, \cdot)$ belongs to $\mathcal{H}^{s,1}_{\rm loc}(\mathbf{R}^N \times (0, \infty))$. In particular $p, D_y p \in L^s_{\rm loc}(\mathbf{R}^N \times (0, \infty))$ and $p(x, \cdot, \cdot)$ is continuous.*

(ii) *For every $y \in \mathbf{R}^N$ the function $p(\cdot, y, \cdot)$ belongs to $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N \times (0, \infty))$ and solves the equation $\partial_t p = Ap$, $t > 0$. Moreover*

$$\sup_{|y|\le R} \|p(\cdot, y, \cdot)\|_{C^{2+\alpha,1+\alpha/2}(B_R\times[\varepsilon,T])} < \infty$$

for every $0 < \varepsilon < T$ and $R > 0$.

(iii) *If, in addition, $F \in C^1(\mathbf{R}^N)$, then $p(x, \cdot, \cdot) \in W^{2,1}_{s,\rm loc}(Q_T)$ for every $x \in \mathbf{R}^N$, $1 < s < \infty$, and satisfies the equation $\partial_t p - A^*_y p = 0$, where*

$$A^* = A_0 - F \cdot D - (V + \operatorname{div} F)$$

is the formal adjoint of A.

The uniqueness of the bounded solution of (1.2) does not hold in general, but it is ensured by the existence of a *Lyapunov function* (cf. [10, Proposition 2.2]), that is a $C^{2+\alpha}_{\rm loc}$-function $W : \mathbf{R}^N \to [0, \infty)$ such that $\lim_{|x|\to\infty} W(x) = +\infty$ and $AW \le \lambda W$ for some $\lambda > 0$. Lyapunov functions are easily found imposing suitable conditions on the coefficients of A. For instance, $W(x) = |x|^2$ is a Lyapunov function for A provided that $\sum_i a_{ii}(x) + F(x) \cdot x - |x|^2 V(x) \le C|x|^2$ for some $C > 0$. The following result can be proved as in [10, Proposition 2.2].

Proposition 2.2. *Let W be a Lyapunov function for A and let $u, v \in C_b(\mathbf{R}^N \times [0, T]) \cap C^{2,1}(\mathbf{R}^N \times (0, T])$ solve (1.2). Then $u = v$.*

Now we turn our attention to integrability properties of p and show how they can be deduced from the existence of suitable Lyapunov functions. In the proof of Proposition 2.4 below we need to approximate the semigroup $(T(t))_{t\ge0}$ with semigroups generated by uniformly elliptic operators. This is done in the next lemma.

Lemma 2.3. *Assume that A has a Lyapunov function W. Take $\eta \in C^\infty_c(\mathbf{R})$ with $\eta(s) = 1$ for $|s| \le 1$, $\eta(s) = 0$ for $|s| \ge 2$, and define $\eta_n(x) = \eta\left(\left|\frac{x}{n}\right|\right)$, $F_n = \eta_n F$, $V_n := \eta_n V$ and $A_n = A_0 + F_n \cdot D - V_n$. Consider the analytic semigroup $(T_n(t))_{t\ge0}$ generated by A_n in $C_b(\mathbf{R}^N)$. Then, for every $f \in C^{2+\alpha}(\mathbf{R}^N)$ there exists a sequence (n_k) such that $T_{n_k}(\cdot)f(\cdot) \to T(\cdot)f(\cdot)$ in $C^{2,1}(\mathbf{R}^N \times [0, T])$.*

Proof. Let $u_n(x, t) = T_n(t)f(x)$, $u(x, t) = T(t)f(x)$ and fix a radius $\varrho > 0$. If $n > \varrho + 1$ the Schauder estimates for the operator A (see e.g. [5, Theorem 8.1.1]) yield

$$\|u_n\|_{C^{2+\alpha,1+\alpha/2}(B_\varrho\times[0,T])} \le C_\varrho \|f\|_{C^{2+\alpha}(\mathbf{R}^N)}.$$

By a standard diagonal argument we find a subsequence (n_k) such that u_{n_k} converges to a function u in $C^{2,1}(\mathbf{R}^N\times(0,\infty))$. Since $\partial_t u_{n_k}-Au_{n_k}=0$ in $B_\varrho\times[0,T]$ for $n_k>\varrho$ we have $\partial_t u-Au=0$ in $\mathbf{R}^N\times[0,T]$. Moreover, $u(x,0)=f(x)$ and $|u(x,t)|\le\|f\|_\infty$, since this is true for u_n. By Proposition 2.2 we infer that $u(x,t)=T(t)f(x)$. □

The integrability of Lyapunov functions with respect to the measures $p(x,y,t)\,dy$ is given by the following result, which is an extension of [12, Lemma 3.9], where the case $V=0$ is considered.

Proposition 2.4. *A Lyapunov function W is integrable with respect to the measures $p(x,\cdot,t)$. Setting*

$$\zeta(x,t)=\int_{\mathbf{R}^N}p(x,y,t)W(y)\,dy, \tag{2.1}$$

the inequality $\zeta(t,x)\le e^{\lambda t}W(x)$ holds. Moreover, $|AW|$ is integrable with respect to $p(x,\cdot,t)$, $\zeta\in C^{2,1}(\mathbf{R}^N\times(0,\infty))\cap C(\mathbf{R}^N\times[0,\infty))$ and $D_t\zeta(x,t)\le\int_{\mathbf{R}^N}p(x,y,t)AW(y)\,dy$.

Proof. For $\alpha\ge0$, set $W_\alpha:=W\wedge\alpha$ and $\zeta_\alpha(x,t):=\int_{\mathbf{R}^N}p(x,y,t)W_\alpha(y)\,dy$.

Let us consider, for every $0<\varepsilon<1$, $\psi_\varepsilon\in C^\infty(\mathbf{R})$ such that $\psi_\varepsilon(t)=t$ for $t\le\alpha$, ψ_ε constant in $[\alpha+\varepsilon,\infty)$, $\psi'_\varepsilon\ge0$, and $\psi''_\varepsilon\le0$. Since $\psi''_\varepsilon\le0$ one deduces that

$$t\psi'_\varepsilon(t)\le\psi_\varepsilon(t),\quad\forall t\ge0. \tag{2.2}$$

Now we approximate A with $A_n:=A_0+F_n\cdot\nabla-V_n$ and $(T(t))_{t\ge0}$ with $(T_n(t))_{t\ge0}$ as in Lemma 2.3. Denoting by $p_n(x,y,t)$ the kernel of $(T_n(t))_{t\ge0}$, since $\psi_\varepsilon\circ W\in C_b^{2+\alpha}(\mathbf{R}^N)$ we have

$$\partial_tT_n(t)(\psi_\varepsilon\circ W)(x)=\int_{\mathbf{R}^N}p_n(x,y,t)A_n(\psi_\varepsilon\circ W)(y)\,dy.$$

On the other hand, by (2.2), we obtain

$$\begin{aligned}A_n(\psi_\varepsilon\circ W)(x)=&\psi'_\varepsilon(W(x))A_nW(x)+V_n(x)\left[\psi'_\varepsilon(W(x))W(x)-\psi_\varepsilon(W(x))\right]\\&+\psi''_\varepsilon(W(x))\sum_{i,j=1}^N a_{ij}(x)D_iW(x)D_jW(x)\\\le&\psi'_\varepsilon(W(x))A_nW(x).\end{aligned}$$

Thus,

$$\partial_tT_n(t)(\psi_\varepsilon\circ W)(x)\le\int_{\mathbf{R}^N}p_n(x,y,t)\psi'_\varepsilon(W(y))A_nW(y)\,dy$$

and also

$$\partial_tT_n(t)(\psi_\varepsilon\circ W)(x)\le\int_{\mathbf{R}^N}p_n(x,y,t)\psi'_\varepsilon(W(y))AW(y)\,dy$$

if n is sufficiently large since, for fixed ε, the function $\psi'_\varepsilon(W(y))$ has compact support. Letting $n \to \infty$ and using Lemma 2.3 (possibly passing to a subsequence) we deduce

$$\partial_t T(t)(\psi_\varepsilon \circ W)(x) \le \int_{\mathbf{R}^N} p(x,y,t)\psi'_\varepsilon(W(y))AW(y)\,dy. \tag{2.3}$$

Next we observe that $\psi_\varepsilon \circ W \le \alpha + 1$, $\psi'_\varepsilon(t) \to \chi_{(-\infty,\alpha]}(t)$, and $\psi_\varepsilon \circ W \to W_\alpha$ pointwise as $\varepsilon \to 0$. From [8, Proposition 2.2.9] we deduce that $T(t)(\psi_\varepsilon \circ W) \to T(t)W_\alpha$ in $C^{2,1}(\mathbf{R}^N \times (0,\infty))$. So, letting $\varepsilon \to 0$ in (2.3) and using dominated convergence in the right hand side (all the integrals can be taken on the compact set $\{W \le \alpha + 1\}$, where AW is bounded) we get

$$D_t\zeta_\alpha(x,t) \le \int_{\{W\le\alpha\}} p(x,y,t)AW(y)\,dy. \tag{2.4}$$

To conclude we proceed as in the proof of [12, Lemma 3.9]. From (2.4) we obtain

$$D_t\zeta_\alpha(x,t) \le \lambda\zeta_\alpha(x,t) \tag{2.5}$$

and hence, by Gronwall's lemma, $\zeta_\alpha(x,t) \le e^{\lambda t}W_\alpha(x)$. Letting $\alpha \to \infty$ we obtain $\zeta(x,t) \le e^{\lambda t}W(x)$ and then W is summable with respect to the measure $p(x,\cdot,t)$. The inequality $0 \le \zeta_\alpha \le \zeta$ and the interior Schauder estimates show that the family (ζ_α) is relatively compact in $C^{2,1}(\mathbf{R}^N \times (0,\infty))$. Since $\zeta_\alpha \to \zeta$ pointwise as $\alpha \to +\infty$, it follows that $\zeta \in C^{2,1}(\mathbf{R}^N \times (0,\infty))$. Moreover, the inequality $\zeta_\alpha(x,t) \le \zeta(x,t) \le e^{\lambda t}W(x)$ implies that $\zeta(\cdot,t) \to W(\cdot)$ as $t \to 0^+$, uniformly on compact sets. Set $E = \{x \in \mathbf{R}^N : AW(x) \ge 0\}$. Clearly

$$\int_E p(x,y,t)AW(y)\,dy \le \lambda \int_E p(x,y,t)W(y)\,dy \le \lambda\zeta(x,t) < \infty. \tag{2.6}$$

Moreover, letting $\alpha \to +\infty$ in (2.3), we obtain that

$$D_t\zeta(x,t) \le \liminf_{\alpha\to+\infty} \int_{\{W\le\alpha\}} p(x,y,t)AW(y)\,dy.$$

This fact and (2.6) imply that $|AW|$ is summable with respect to $p(x,\cdot,t)$ and that the above lim inf is a limit, so that the proof is complete. □

Assuming that AW tends to $-\infty$ faster than $-W$ one obtains, by Proposition 2.4, that the function ζ in (2.1) is bounded with respect to the space variables, see [12, Theorem 3.10] for the case $V = 0$.

Proposition 2.5. *Assume that the Lyapunov function W satisfies the inequality $AW \le -g(W)$ where $g : [0,\infty) \to \mathbf{R}$ is a differentiable convex function such that $g(0) \le 0$, $\lim_{s\to+\infty} g(s) = +\infty$ and $1/g$ is integrable in a neighbourhood of $+\infty$. Then for every $a > 0$ the function ζ defined in (2.1) is bounded in $\mathbf{R}^N \times [a,\infty)$. Moreover, the semigroup $(T(t))_{t\ge0}$ is compact in $C_b(\mathbf{R}^N)$.*

Proof. Observe that $g(s) \le sg'(s)$, since g is convex with $g(0) \le 0$. Let us prove that

$$\int_{\mathbf{R}^N} p(x,y,t) g(W(y))\,dy \ge g(\zeta(x,t)). \tag{2.7}$$

For, fix x and t and set $s_0 = \zeta(x,t)$. Then, for all $y \in \mathbf{R}^N$ we have

$$g(W(y)) \ge g(s_0) + g'(s_0)(W(y) - s_0)$$

and therefore, multiplying by $p(x,y,t)$ and integrating

$$\begin{aligned}&\int_{\mathbf{R}^N} p(x,y,t) g(W(y))\,dy \\ &\ge g(s_0) \int_{\mathbf{R}^N} p(x,y,t)\,dy + g'(s_0)s_0\Big(1 - \int_{\mathbf{R}^N} p(x,y,t)\,dy\Big) \ge g(s_0).\end{aligned}$$

From Proposition 2.4 and (2.7) we deduce

$$D_t\zeta(x,t) \le \int_{\mathbf{R}^N} p(x,y,t) AW(y)\,dy \le -\int_{\mathbf{R}^N} p(x,y,t) g(W(y))\,dy \le -g(\zeta(x,t))$$

and therefore $\zeta(x,t) \le z(x,t)$, where z is the solution of the ordinary Cauchy problem

$$\begin{cases} z' = -g(z) \\ z(x,0) = W(x). \end{cases}$$

Let ℓ denote the greatest zero of g. Then $z(x,t) \le \ell$ if $W(x) \le \ell$. On the other hand, if $W(x) > \ell$, then z is decreasing and satisfies

$$t = \int_{z(x,t)}^{W(x)} \frac{ds}{g(s)} \le \int_{z(x,t)}^{\infty} \frac{ds}{g(s)}. \tag{2.8}$$

This inequality easily yields, for every $a > 0$, a constant $C(a)$ such that $z(x,t) \le C(a)$ for every $t \ge a$ and $x \in \mathbf{R}^N$.

The compactness of $(T(t))_{t\ge 0}$ in $C_b(\mathbf{R}^N)$ can be proved as in [12, Theorem 3.10]. □

Remark 2.6. If $\int_{\mathbf{R}^N} p(x,y,t)\,dy = 1$ (as is the case if $V = 0$) then (2.7) follows from Jensen's inequality and the condition $g(0) \le 0$ is not needed.

Let us state a condition under which certain exponentials or polynomials are Lyapunov functions. Using the same procedure as for the case $V = 0$ (see [10, Proposition 2.5 and 2.6]) we obtain the following results.

Proposition 2.7. *Let Λ be the maximum eigenvalue of (a_{ij}) as in (H). Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} \left(F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta\beta |x|^{\beta-1}} \right) < -c, \tag{2.9}$$

$0 < c < \infty$, *for some* $c, \delta > 0, \beta > 1$ *such that* $\delta < (\beta\Lambda)^{-1}c$. *Then* $W(x) = \exp\{\delta|x|^\beta\}$ *is a Lyapunov function. Moreover, if* $\beta > 2$, *there exist positive constants* c_1, c_2 *such that*

$$\zeta(x,t) \le c_1 \exp\left(c_2 t^{-\beta/(\beta-2)}\right) \tag{2.10}$$

for $x \in \mathbf{R}^N,\ t > 0$.

Proposition 2.8. *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{|x|}{2\alpha}V(x)\right) < 0, \tag{2.11}$$

for some $\alpha > 0,\ \beta > 2$. *Then* $W(x) = \left(1+|x|^2\right)^\alpha$ *is a Lyapunov function and there exists a positive constant* c *such that*

$$\zeta(x,t) \le c t^{-(2\alpha)/(\beta-2)} \tag{2.12}$$

for $x \in \mathbf{R}^N,\ 0 < t \le 1$.

Remark 2.9. Proposition 2.7 will be used to check the integrability of $|F|^k$ and V^k with respect to p, assuming that $|F|, V$ grow at infinity not faster than $\exp\{|x|^\gamma\}$ for some $\gamma < \beta$.

3. Uniform and pointwise bounds on transition densities

In this section we fix $T > 0$ and consider p as a function of $(y,t) \in \mathbf{R}^N \times (0,T)$ for arbitrary, but fixed, $x \in \mathbf{R}^N$. Further, fix $0 < a_0 < a < b < b_0 \le T$ and assume for definiteness $b_0 - b \ge a - a_0$. Setting

$$\Gamma(k,x,a_0,b_0) := \left(\int_{Q(a_0,b_0)} (1+|F(y)|^k + V(y)^k)p(x,y,t)\,dy\,dt\right)^{\frac{1}{k}}, \tag{3.1}$$

the proofs of Proposition 3.1, Lemma 3.1 and Proposition 3.2 in [10] remain valid for the case $V \ne 0$. So, we obtain that

$$p \in \mathcal{H}^{s,1}(Q(a,b)) \quad \text{for all } s \in (1,k),$$

provided that $\Gamma(k,x,a_0,b_0) < \infty$ for some $k > N+2$. Hence, by the embedding theorem for $\mathcal{H}^{s,1}$, $s > N+2$, (see [10, Theorem 7.1]), we have

Theorem 3.1. *If* $\Gamma(k,x,a_0,b_0) < \infty$ *for some* $k > N+2$, *then* p *belongs to* $L^\infty(Q(a,b))$.

To obtain uniform and pointwise bounds on p we introduce the functions

$$\Gamma_1(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} (1+|F(y)|^k)p(x,y,t)\,dy\,dt\right)^{\frac{1}{k}}, \tag{3.2}$$

$$\Gamma_2(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} V^{\frac{k}{2}}(y)p(x,y,t)\,dy\,dt\right)^{\frac{2}{k}}. \tag{3.3}$$

Clearly $\Gamma_1(k,x,a_0,b_0)+\Gamma_2(k,x,a_0,b_0)\le C\Gamma(k,x,a_0,b_0)$. The following result shows that only the assumption $\Gamma_1(k,x,a_0,b_0),\Gamma_2(k,x,a_0,b_0)<\infty$ for some $k>N+2$ is needed to obtain the boundedness of p.

Theorem 3.2. *If $\Gamma_1(k,x,a_0,b_0),\Gamma_2(k,x,a_0,b_0)<\infty$ for some $k>N+2$ then*

$$\|p\|_{L^\infty(Q(a,b))} \le C\left(\Gamma_1^k(k,x,a_0,b_0)+\Gamma_2^{\frac{k}{2}}(k,x,a_0,b_0)+\frac{b_0-a_0}{(a-a_0)^{\frac{k}{2}}}\right). \tag{3.4}$$

Proof. **Step 1.** Assume first that $\Gamma(k,x,a_0,b_0)<\infty$ so that $p\in L^\infty(Q(a,b))$ for every $a_0<a<b<b_0$ by Theorem 3.1 and consider $q=\eta^{\frac{k}{2}}p\in L^\infty(Q_T)$ where η is a smooth function with compact support in (a_0,b_0) such that $0\le\eta\le 1$, $\eta(t)=1$ for $a\le t\le b$. Clearly $q\in L^\infty(Q_T)$.

Let $\varphi\in C^{2,1}(Q_T)$ be such that $\varphi(\cdot,t)$ has compact support for every t. From (1.4) we obtain

$$\left|\int_{Q_T} q(\partial_t\varphi + A_0\varphi)\,dy\,dt\right| = \left|\int_{Q_T}(qF\cdot D\varphi - Vq\varphi+\frac{k}{2}p\varphi\eta^{\frac{k-2}{2}}\partial_t\eta)\,dy\,dt\right|.$$

Next we note that

$$\|p\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}(b_0-a_0)^{\frac{2}{k}}$$

and that

$$\|Fq\|_{L^k(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Gamma_1(k,x,a_0,b_0) \quad \|Vq\|_{L^{\frac{k}{2}}(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\Gamma_2(k,x,a_0,b_0).$$

Since also

$$\|q\|_{L^k(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}(b_0-a_0)^{\frac{1}{k}}, \quad \|q\|_{L^{\frac{k}{2}}(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}(b_0-a_0)^{\frac{2}{k}},$$

Theorem 7.3 in [10] now implies that

$$\|q\|_{L^\infty(Q_T)} \le C\Big(\|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Gamma_1(k,x,a_0,b_0) + \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\Big(\Gamma_2(k,x,a_0,b_0)+\frac{(b_0-a_0)^{\frac{2}{k}}}{a-a_0}\Big)\Big)$$

and hence, after a simple calculation,

$$\|q\|_{L^\infty(Q_T)} \le C\left(\Gamma_1^k(k,x,a_0,b_0)+\Gamma_2^{\frac{k}{2}}(k,x,a_0,b_0)+\frac{b_0-a_0}{(a-a_0)^{\frac{k}{2}}}\right)$$

and (3.4) follows.

Step 2. Let us now consider the general case. Fix a smooth function $\theta \in C_c^\infty(\mathbf{R})$ such that $\theta(s) = 1$ for $|s| \leq 1$, $\theta(s) = 0$ for $|s| \geq 2$ and define $\theta_n(x) = \theta\left(\frac{|x|}{n}\right)$, $V_n = V\theta_n$. We consider the minimal semigroup $(U_n(t))_{t\geq 0}$ generated in $C_b(\mathbf{R}^N)$ by the operator $A_n = A_0 + F\cdot D - V_n$. Since $V_n \leq V$ the procedure for constructing the minimal semigroup recalled in Section 2 and the maximum principle yield $U_n(t)f \leq T(t)f$ for every $f \in C_b(\mathbf{R}^N)$. If p_n denotes the kernel of U_n the above inequality is equivalent to $p_n(x,y,t) \leq p_(x,y,t)$. To show that p_n converges pointwise to p we consider the analytic semigroup $(T_n(t))_{t\geq 0}$ generated by A on $C_b(B_n)$, under Dirichlet boundary conditions (B_n is the ball of centre 0 and radius n). Since $V_n = V$ in B_n, the maximum principle gives $T_n(t)f \leq U_n(t)f \leq T(t)f$ in B_n for every $f \in C_b(\mathbf{R}^N)$, $f \geq 0$. Then $r_n(x,y,t) \leq p_n(x,y,t) \leq p(x,y,t)$ for $x, y \in B_\rho$ with $\rho < n$, where r_n is the kernel of T_n in B_n. Letting $n \to \infty$ we see that $p_n \to p$ pointwise, since this is true for r_n, see [11, Theorem 4.4].
The proof now easily follows by approximation from Step 1. Let $\Gamma_i^n(k,x,a_0,b_0)$ be the functions defined in (3.1), (3.2), (3.3) relative to p_n. Since $p_n \leq p$ and $V_n \leq V$, it follows that $\Gamma_i^n(k,x,a_0,b_0) \leq \Gamma_i(k,x,a_0,b_0)$. Moreover, $\Gamma^n(k,x,a_0,b_0) < \infty$ for every n, since V_n is bounded. Then we obtain from Step 1

$$\|p_n\|_{L^\infty(Q(a,b))} \leq C\left(\Gamma_1^k(k,x,a_0,b_0) + \Gamma_2^{\frac{k}{2}}(k,x,a_0,b_0) + \frac{b_0 - a_0}{(a-a_0)^{\frac{k}{2}}}\right)$$

and the statement follows letting $n \to \infty$. □

Now we apply similar techniques to obtain pointwise bounds.

We consider the following assumption depending on the weight function ω which, in our examples, will be a polynomial or an exponential.

(H1) W_1, W_2 are Lyapunov functions for A, $W_1 \leq W_2$ and there exists $1 \leq \omega \in C^2(\mathbf{R}^N)$ such that

(i) $\omega \leq cW_1, \quad |D\omega| \leq c\omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}, \quad |D^2\omega| \leq c\omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$
(ii) $\omega V^{\frac{k}{2}} \leq cW_2$ and $\omega |F|^k \leq cW_2$
for some $k > N+2$ and a constant $c > 0$.

We denote by ζ_1, ζ_2 the functions defined by (2.1) and associated with W_1, W_2, respectively.

By Proposition 2.4 we know that (H1) implies $\Gamma_i(k,x,a_0,b_0) < \infty$ for $i = 1,2$. In particular, since $k > N+2$, Theorem 3.2 shows that $p(x,\cdot,\cdot) \in L^\infty(Q(a,b))$ for every $x \in \mathbf{R}^N$.

The use of different Lyapunov functions allows us to obtain more precise estimates in the theorem below and its corollaries.

The proof of the following result is similar to the one of [10, Theorem 4.1]. For reader's convenience we give the details of the proof.

Theorem 3.3. *Assume (H1). Then, there exists a constant $C>0$ such that*

$$0<\omega(y)p(x,y,t)\le C\left(\int_{a_0}^{b_0}\zeta_2(x,t)\,dt+\frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0}\zeta_1(x,t)\,dt\right) \tag{3.5}$$

for all x, $y\in\mathbf{R}^N$, $a\le t\le b$.

Proof. **Step 1.** Assume first that ω is bounded. As in the proof of Theorem 3.2 we choose a smooth function $\eta(t)$ such that $\eta(t)=1$ for $a\le t\le b$ and $\eta(t)=0$ for $t\le a_0$ and $t\ge b_0$, $0\le\eta'\le\frac{2}{a-a_0}$. We consider $\psi\in C^{2,1}(Q_T)$ such that $\psi(\cdot,T)=0$ and such that $\psi(\cdot,t)$ has compact support for all t. Setting $q=\eta^{\frac{k}{2}}p$ and taking $\varphi(y,t)=\eta^{\frac{k}{2}}(t)\omega(y)\psi(y,t)$, from (1.4) we obtain

$$\begin{aligned}\int_{Q_T}\omega q\left(-\partial_t\psi-A_0\psi\right)dydt=\int_{Q_T}\Big[q\Big(\psi A_0\omega+2\sum_{i,j=1}^N a_{ij}D_i\omega D_j\psi+&\\ \omega F\cdot D\psi+\psi F\cdot D\omega-V\omega\psi\Big)+\frac{k}{2}p\omega\psi\eta^{\frac{k-2}{2}}\partial_t\eta\Big]\,dy\,dt.&\end{aligned} \tag{3.6}$$

Since ω is bounded, then $\omega q\in L^1(Q_T)\cap L^\infty(Q_T)$, by Theorem 3.2 and then [10, Theorem 7.3] yields

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)}\le& C\Big(\|\omega q\|_{L^k(Q_T)}+\|\omega q\|_{L^{\frac{k}{2}}(Q_T)}+\|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)}+\|qD\omega\|_{L^k(Q_T)}\\ &+\|\omega qF\|_{L^k(Q_T)}+\|qFD\omega\|_{L^{\frac{k}{2}}(Q_T)}+\|qV\omega\|_{L^{\frac{k}{2}}(Q_T)}\\ &+\frac{1}{a-a_0}\|p\omega\eta^{\frac{k-2}{2}}|_{L^{\frac{k}{2}}(Q_T)}\Big).\end{aligned}$$

Next observe that

$$\|\omega q\|_{L^k(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\|\omega q\|_{L^1(Q_T)}^{\frac{1}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{1}{k}},$$

$$\|\omega q\|_{L^{\frac{k}{2}}(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\|\omega q\|_{L^1(Q_T)}^{\frac{2}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{2}{k}},$$

and that, by (H1)(ii),

$$\|\omega qF\|_{L^k(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\|\omega qF^k\|_{L^1(Q_T)}^{\frac{1}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\zeta_2\,dt\right)^{\frac{1}{k}},$$

$$\|\omega qV\|_{L^{\frac{k}{2}}(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\|\omega qV^{\frac{k}{2}}\|_{L^1(Q_T)}^{\frac{2}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_2\,dt\right)^{\frac{2}{k}}.$$

Moreover, as in the proof of Theorem 3.2 one has

$$\|\omega p\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\|\omega p\|_{L^1(Q(a_0,b_0))}^{\frac{2}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{2}{k}}.$$

Next we combine (H1)(i) and (H1)(ii) to estimate the remaining terms

$$\|D\omega qF\|_{L^{\frac{k}{2}}(Q_T)} \leq \left(\int_{Q_T} q^{\frac{k}{2}}\omega^{\frac{k-2}{2}}W_2\right)^{\frac{2}{k}} \leq \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{2}} \left(\int_{a_0}^{b_0} \zeta_2\, dt\right)^{\frac{2}{k}}$$

and, similarly,

$$\|D^2\omega q\|_{L^{\frac{k}{2}}(Q_T)} \leq \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left(\int_{a_0}^{b_0} \zeta_1\, dt\right)^{\frac{2}{k}}$$

$$\|D\omega q\|_{L^k(Q_T)} \leq \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \left(\int_{a_0}^{b_0} \zeta_1\, dt\right)^{\frac{1}{k}}.$$

Collecting similar terms and recalling that $W_1 \leq W_2$ we obtain

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} \leq & C\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \left(\int_{a_0}^{b_0} \zeta_2\, dt\right)^{\frac{1}{k}} \\ & + C\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left(\left(\int_{a_0}^{b_0} \zeta_2\, dt\right)^{\frac{2}{k}} + \frac{1}{a-a_0}\left(\int_{a_0}^{b_0} \zeta_1\, dt\right)^{\frac{2}{k}}\right)\end{aligned}$$

hence, after simple computations,

$$\|\omega q\|_{L^\infty(Q_T)} \leq C\left(\int_{a_0}^{b_0} \zeta_2\, dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1\, dt\right)$$

and (3.5) follows for a bounded ω.

Step 2. If ω is not bounded, we consider $\omega_\varepsilon = \frac{\omega}{1+\varepsilon\omega}$. A straightforward computation shows that ω_ε satisfies $(H1)$ with a constant C independent of ε. Therefore, from Step 1 we obtain

$$0 < \omega_\varepsilon(y)p(x,y,t) \leq C\left(\int_{a_0}^{b_0} \zeta_2(x,t)\, dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1(x,t)\, dt\right), \tag{3.7}$$

with C independent of ε and, letting $\varepsilon \to 0$, the statement is proved. □

Corollary 3.4. *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}}\right) < -c, \qquad 0 < c < \infty \tag{3.8}$$

for some $\delta > 0, c > 0, \beta > 2$ *such that* $\delta < (\beta\Lambda)^{-1}c$, *where* Λ *is the maximum eigenvalue of* (a_{ij}), *and that* $V(x) + |F(x)| \leq c_1 e^{c_2|x|^{\beta-\varepsilon}}$ *for some* $\varepsilon, c_1, c_2 > 0$. *Then*

$$0 < p(x,y,t) \leq c_3 \exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)\exp\left(-\delta|y|^\beta\right)$$

for $x, y \in \mathbf{R}^N$, $0 < t \leq T$, *for suitable* $c_3, c_4 > 0$.

Proof. We take $\omega(y) = e^{\delta|y|^\beta}$, $W_1(y) = W_2(y) = e^{\gamma|y|^\beta}$ for $\delta < \gamma < (\beta\Lambda)^{-1}c$ and use Theorem 3.3 with $a = t$ and $a - a_0 = b_0 - b = b - a = \frac{t}{2}$. The thesis then follows using Proposition 2.7. □

Example. (i) The above corollary applies with any $\gamma < (\beta\Lambda)^{-1}c$ and without any restriction on $V \geq 0$ when

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|}\right) < -c, \qquad 0 < c < \infty,$$

for some $\beta > 2$ and $|F(x)| \leq c_1 e^{c_2|x|^{\beta-\varepsilon}}$ for some $\varepsilon, c_1, c_2 > 0$. This is obvious if $V = 0$ and, in the general case, it follows by observing that the kernel p is pointwise dominated, by the maximum principle, by the corresponding kernel of the operator with $V = 0$.

(ii) Let us consider the Schrödinger operator $\Delta - a^2|x|^s$ with $a > 0, s > 2$. Then Corollary 3.4 applies with $\beta = 1 + \frac{s}{2}$ and any $\delta < \frac{2a}{s+2}$. This yields

$$0 < p(x, y, t) \leq c_3 \exp\left(c_4 t^{-\frac{s+2}{s-2}}\right) \exp\left(-\delta|y|^{\frac{s+2}{2}}\right) := c(t)\phi(y).$$

Using the symmetry of p and the semigroup law (see [9, Example 3.13]), we obtain

$$p(x, y, t) \leq c_3 \exp\left(c_4 t^{-\frac{s+2}{s-2}}\right) \exp\left(-\delta|x|^{\frac{s+2}{2}}\right) \exp\left(-\delta|y|^{\frac{s+2}{2}}\right).$$

This estimate was obtained in [9, Example 3.13].

(iii) Let us generalize the previous situation to the case of the operators

$$A = \Delta - |x|^r \frac{x}{|x|} \cdot D - |x|^s$$

with $r > 1$. We distinguish three cases.

(a) If $s < 2r$, then $\beta = r + 1$ and δ can be any positive number less than $\frac{1}{r+1}$. Therefore

$$0 < p(x, y, t) \leq c_1 \exp\left(c_2 t^{-\frac{r+1}{r-1}}\right) \exp\left(-\delta|y|^{r+1}\right).$$

(b) If $s = 2r$, then $\beta = r + 1$ as before but now δ must be less than $\frac{1+\sqrt{5}}{2(r+1)}$.

(c) If $s > 2r$, then $\beta = 1 + \frac{s}{2}$ and $\delta < \frac{2}{s+2}$. Then we get, as in (ii)

$$0 < p(x, y, t) \leq c_1 \exp\left(c_2 t^{-\frac{s+2}{s-2}}\right) \exp\left(-\delta|y|^{\frac{s+2}{2}}\right) := c(t)\phi(y). \tag{3.9}$$

In this case one can also obtain estimates with respect to x proceeding as in (ii). We consider the formal adjoint $A^* = \Delta + |x|^r \frac{x}{|x|} \cdot D + (N + r - 1)|x|^{r-1} - |x|^s$. the associated minimal semigroup has the kernel $p^*(x, y, t) = p(y, x, t)$ which satisfies (3.9), by the same argument as above. This yields $p(t, x, y) \leq c(t)\phi(x)$ and, proceeding as in (ii),

$$p(x, y, t) \leq c_1 \exp\left(c_2 t^{-\frac{s+2}{s-2}}\right) \exp\left(-\delta|x|^{\frac{s+2}{2}}\right) \exp\left(-\delta|y|^{\frac{s+2}{2}}\right).$$

Under conditions similar to those of Corollary 3.4, the estimate of p can be improved with respect to the time variable, loosing the exponential decay in y.

Corollary 3.5. *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|}-\frac{|x|}{2\alpha}V(x)\right)<0, \tag{3.10}$$

for some $\alpha>0$ *and* $\beta>2$. *If* $|F(x)|+\sqrt{V(x)}\le c(1+|x|^2)^{\gamma_1}$ *and* $\omega(x):=(1+|x|^2)^{\gamma_2}$ *with* $0<k\gamma_1+\gamma_2\le\alpha$, $\gamma_1\ge\frac{\beta-2}{4}$ *and* $k>N+2$, *then there exists a constant* $C>0$ *such that*

$$0<p(x,y,t)\le\frac{C}{t^\sigma}(1+|y|^2)^{-\gamma_2},$$

for all $x,\,y\in\mathbf{R}^N$, $0<t\le 1$ *where*

$$\sigma=\frac{2}{\beta-2}\left((k-2)\gamma_1+\gamma_2\right).$$

Proof. Observe that $W_r(x)=(1+|x|^2)^r$ is a Lyapunov function for every $0<r\le\alpha$. If $\zeta_r(x,t)$ is the corresponding function defined in (2.1), then Proposition 2.8 yields

$$\zeta_r(x,t)\le c_r t^{\frac{-2r}{\beta-2}}$$

for $x\in\mathbf{R}^N$ and $0<t\le 1$. We set $a=t$ and $a-a_0=b_0-b=b-a=\frac{t^s}{2}$ where $s\ge 1$ will be chosen later and we apply Theorem 3.3 with $\omega(x)=W_1(x)=\left(1+|x|^2\right)^{\gamma_2}$ and $W_2(x)=\left(1+|x|^2\right)^{k\gamma_1+\gamma_2}$. Thus we obtain

$$p(x,y,t)\le C\left(t^{-\frac{2(k\gamma_1+\gamma_2)}{\beta-2}+s}+t^{-\frac{2\gamma_2}{\beta-2}-s\frac{k}{2}+s}\right)(1+|y|^2)^{-\gamma_2}.$$

Minimising over s we get $s=\frac{4\gamma_1}{\beta-2}$ and the thesis follows. □

Example. Let us consider again the operators

$$A=\Delta-|x|^r\frac{x}{|x|}\cdot D-|x|^s$$

with $r>1$. Again we distinguish three cases.

(a) If $s+1\le r$, then $\beta=r+1$ and $\gamma_1=\frac{r}{2}$. It is easily seen that (3.10) holds for every $\alpha>0$ and hence

$$p(x,y,t)\le Ct^{-(k-2)\frac{r}{r-1}-\frac{2\gamma_1}{r-1}}\left(1+|y|^2\right)^{-\gamma_2}$$

for every $\gamma_2\ge 0$, $0<t\le 1$, $y\in\mathbf{R}^N$.

(b) If $r<s+1$, then (3.10) holds for $\beta=s+2$ and every $\alpha>0$. So, we have to distinguish two cases.

(i) If $s\le 2r$, then $\gamma_1=\frac{r}{2}$ and

$$p(x,y,t)\le Ct^{-(k-2)\frac{r}{s}-\frac{2\gamma_1}{s}}\left(1+|y|^2\right)^{-\gamma_2},$$

(ii) If $s > 2r$, then $\gamma_1 = \frac{s}{4}$ and

$$p(x,y,t) \le Ct^{-\frac{k-2}{2}-\frac{2\gamma_1}{s}} \left(1+|y|^2\right)^{-\gamma_2},$$

for every $\gamma_2 \ge 0$, $0 < t \le 1$, $y \in \mathbf{R}^N$.

Remark 3.6. The results of this section generalize Theorem 4.1 and its corollaries in [10] and also the results obtained in [9] in the case of exponential decay but not for polynomial decay, where the results in [9] are more precise.

4. Regularity properties

In this section we obtain the differentiability of the transition semigroup $T(\cdot)$ associated with the transition kernels p in $C_b(\mathbf{R}^N)$ in the case where the coefficients F and V are of exponential type.

We assume here that $a_{ij} \in C_b^2(\mathbf{R}^N)$, $V \in C^1(\mathbf{R}^N)$ and $F \in C^2(\mathbf{R}^N)$. All results of this section can be proved exactly by the same arguments as in [10, Section 5 and Section 6].

Theorem 4.1. *Suppose that there exist constants $\beta > 2$, $c > 0$ such that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}}\right) < -c.$$

Assume moreover that

$$V(x) + |DV(x)| + |F(x)| + |DF(x)| + |D^2F(x)| \le c_1 \exp(c_2|x|^{\beta-\varepsilon})$$

for some ε, $c_1, c_2 > 0$. Then the following estimates hold

(i) $0 < p(x,y,t) \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$
(ii) $|D_y p(x,y,t)| \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$
(iii) $|D_y^2 p(x,y,t)| \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$
(iv) $|\partial_t p(x,y,t)| \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$

for suitable c_3, c_4, $\gamma > 0$ and for all $0 < t \le T$ and $x, y \in \mathbf{R}^N$.

Remark 4.2. (a) Assuming only that there exist constants $\beta > 2$, $c > 0$ such that

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}}\right) < -c,$$

and $V(x) + |F(x)| \le C\exp(|x|^\gamma)$ for some $C > 0$ and $\gamma < \beta$, the functions $p\log^2 p$ and $p\log p$ are integrable in $Q(a,b)$ and in $\mathbf{R}^N$ for fixed $t \in [a,b]$

respectively and

$$\int_{Q(a,b)} \frac{|D_y p(x,y,t)|^2}{p(x,y,t)}\, dy\, dt \le \frac{1}{\lambda^2} \int_{Q(a,b)} (|F(y)|^2 + V^2(y))p(x,y,t)\, dy\, dt + \int_{Q(a,b)} p(x,y,t) \log^2 p(x,y,t)\, dy\, dt + \frac{2}{\lambda} \int_{\mathbf{R}^N} \Big[p(x,y,t) - p(x,y,t) \log p(x,y,t)\Big]_{t=a}^{t=b} dy < \infty.$$

In particular, $p^{\frac{1}{2}}$ belongs to $W_2^{1,0}(Q(a,b))$ (see [10, Theorem 5.1]). This implies in particular that $p \in W_k^{2,1}(Q(a,b))$ provided that also DF is of exponential type for some $k > N + 2$ (see [10, Theorem 5.2]).

(b) From Theorem 3.3 and (a) (cf. [10, Theorem 5.3]) one can observe that the assumption $a_{ij} \in C_b^2(\mathbf{R}^N)$ is not needed for (i) and (ii).

As a consequence we obtain the differentiability of $T(\cdot)$ in $C_b(\mathbf{R}^N)$.

Theorem 4.3. *Under the assumptions of Theorem* 4.1*, the transition semigroup* $T(\cdot)$ *is differentiable on* $C_b(\mathbf{R}^N)$ *for* $t > 0$.

Example. Let $a \in \mathbf{R}$. From Theorem 4.3 we deduce that the operator

$$A = \Delta - |x|^r x \cdot D - a^2 |x|^s$$

with $r > 0$ and $s \ge 0$ generates a differentiable semigroup in $C_b(\mathbf{R}^N)$. This result is known for $a = 0$ (see [13, Proposition 4.4]).

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