A fractional isoperimetric problem in the Wiener space

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Abstract

We introduce a notion of fractional perimeter in an abstract Wiener space, and we show that halfspaces are the only volume-constrained minimisers.

1 Introduction

The purpose of this paper is introducing a notion of fractional perimeter in an abstract Wiener space, following the approach developed in the seminal work [4], and studying the symmetry properties of minimisers for this functional. More precisely, our main result is to prove that halfspaces are the unique isoperimetric sets for the fractional perimeter, as it happens for the usual perimeter (see [6], [13], [1, Remark 4.7]). Owing to the well-known relation between the isoperimetric problem and the Allen-Cahn energy [14] (see also [12] for an extension of the result to Wiener spaces, and [15] for a nonlocal version in finite dimensions), we also prove the one-dimensional symmetry of minimisers of the corresponding nonlocal Allen-Cahn energy (see Theorem 3.6). We now state the main result of this paper (see Section 2 for the precise definitions). In the whole paper $s \in (0,1)$.

Theorem 1.1. *For any $s \in (0,1)$ and $m \in (0,1)$ there exists a set $E_m \subset X$ which solves the isoperimetric problem*

$$\min\Big\{P_{\gamma,s}(E) : \ E \subset X, \ \gamma(E) = m\Big\}. \tag{1.1}$$

Moreover, the set E_m is necessarily a half-space, i.e., $E = \{\hat{h} < c\}$ for some $h \in H$ and $c \in \mathbb{R}$.

The proof of Theorem 1.1 is based on the extension technique introduced in [5]. Indeed, the fractional perimeter, and more generally the fractional Sobolev seminorm defined in (2.6), can be obtained via the minimisation of a Dirichlet energy after adding an extra variable that lies on a half-line endowed with a degenerate measure. As a consequence, the isoperimetric problem can be tackled by studying this minimisation problem. To this aim, we split the Dirichlet functional in two contributions J_1 and J_2 in a natural way and show that both are decreasing under Ehrhard symmetrisation defined in (3.1). These results are proved in Lemmas 3.1 (which seems to be new even in the finite dimensional case) and 3.3, respectively. In the first proof we adapt a technique in

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[3], in the second we use cylindrical approximations to extend the result from the finite dimensional to the infinite dimensional setting, see Lemma 3.2. These symmetrisation results we believe to be interesting on their own.

Partial symmetrisations in product spaces are also used in [11], with the aim of studying isoperimetric problems with respect to product measures. Also in this case, it is shown that the advantage of symmetrising with respect to a set of variables is not affected by the others.

2 Notation and preliminary definitions

We collect here the definitions and the preliminaries results needed in the sequel. The first two subsections are devoted to the structure of the Wiener space; for all this results we refer to the book [2]. In the third subsection we introduce the fractional perimeters and Sobolev seminorms and use the extension technique in [5, 16] to show some further preliminary results.

2.1 The Wiener space

An abstract Wiener space is a triple (X, γ, H) where X is a separable Banach space, endowed with the norm $\|\cdot\|_X$, γ is a nondegenerate centred Gaussian measure, and H is the Cameron-Martin space associated with the measure γ, that is, H is a separable Hilbert space densely embedded in X, endowed with the inner product $[\cdot,\cdot]_H$ and with the norm $|\cdot|_H$. The requirement that γ is a centred Gaussian measure means that for any $x^* \in X^*$, the measure $x^*_{\#}\gamma$ is a centred Gaussian measure on the real line $\mathbb{R}$, that is, the Fourier transform of γ is given by

$$\hat{\gamma}(x^*) = \int_X e^{-i\langle x, x^*\rangle}\, d\gamma(x) = \exp\left(-\frac{\langle Qx^*, x^*\rangle}{2}\right), \qquad \forall x^* \in X^*;$$

here the operator $Q \in \mathcal{L}(X^*, X)$ is the covariance operator and it is uniquely determined by the formula

$$\langle Qx^*, y^*\rangle = \int_X \langle x, x^*\rangle\langle x, y^*\rangle d\gamma(x), \qquad \forall x^*, y^* \in X^*.$$

The nondegeneracy of γ implies that Q is positive definite: the boundedness of Q follows by Fernique's Theorem (see [2, Theorem 2.8.5]), asserting that there exists a positive number $\beta > 0$ such that

$$\int_X e^{\beta\|x\|_X^2} d\gamma(x) < +\infty.$$

This implies also that the maps $x \mapsto \langle x, x^*\rangle$ belong to $L^p_\gamma(X)$ for any $x^* \in X^*$ and $p \in [1, +\infty)$, where $L^p_\gamma(X)$ denotes the space of all γ-measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X |f(x)|^p d\gamma(x) < +\infty.$$

In particular, any element $x^* \in X^*$ can be seen as a map $x^* \in L^2_\gamma(X)$, and we denote by $R^* : X^* \to \mathcal{H}$ the identification map $R^*x^*(x) := \langle x, x^*\rangle$. The space $\mathcal{H}$ given by the closure of R^*X^* in

$L^2_\gamma(X)$ is usually called reproducing kernel. By considering the map $R : \mathcal{H} \to X$ defined through the Bochner integral

$$R\hat{h} := \int_X \hat{h}(x)x \, d\gamma(x),$$

we obtain that R is an injective γ–Radonifying operator, which is Hilbert–Schmidt when X is Hilbert. We also have $Q = RR^* : X^* \to X$. The space $H := R\mathcal{H}$, equipped with the inner product $[\cdot, \cdot]_H$ and norm $|\cdot|_H$ induced by $\mathcal{H}$ via R, is the Cameron-Martin space and is a dense subspace of X. The continuity of R implies that the embedding of H in X is continuous, that is, there exists $c > 0$ such that

$$\|h\|_X \le c|h|_H, \qquad \forall h \in H.$$

We have also that the measure γ is absolutely continuous with respect to translation along Cameron-Martin directions; in fact, for $h \in H$, $h = Qx^*$, the measure $\gamma_h(B) = \gamma(B - h)$ is absolutely continuous with respect to γ with density given by

$$d\gamma_h(x) = \exp\left(\langle x, x^*\rangle - \frac{1}{2}|h|^2_H\right) d\gamma(x).$$

2.2 Cylindrical functions and differential operators

For $j \in \mathbb{N}$ we choose $x^*_j \in X^*$ in such a way that $\hat{h}_j := R^* x^*_j$, or equivalently $h_j := R\hat{h}_j = Qx^*_j$, form an orthonormal basis of H. We order the vectors x^*_j in such a way that the numbers $\lambda_j := \|x^*_j\|^{-2}_{X^*}$ form a non-increasing sequence. Given $m \in \mathbb{N}$, we also let $H_m := \langle h_1, \ldots, h_m\rangle \subseteq H$, and $\Pi_m : X \to H_m$ be the closure of the orthogonal projection from H to H_m

$$\Pi_m(x) := \sum_{j=1}^m \left\langle x, x^*_j \right\rangle h_j \qquad x \in X.$$

The map Π_m induces the decomposition $X \simeq H_m \oplus X^\perp_m$, with $X^\perp_m := \ker(\Pi_m)$, and $\gamma = \gamma_m \otimes \gamma^\perp_m$, with γ_m and $\gamma^\perp_m$ Gaussian measures on H_m and $X^\perp_m$ respectively, having H_m and $H^\perp_m$ as Cameron-Martin spaces. When no confusion is possible we identify H_m with $\mathbb{R}^m$; with this identification the measure $\gamma_m = \Pi_{m\#}\gamma$ is the standard Gaussian measure on $\mathbb{R}^m$. Given $x \in X$, we denote by $\underline{x}_m \in H_m$ the projection $\Pi_m(x)$, and by $\overline{x}_m \in X^\perp_m$ the infinite dimensional component of x, so that $x = \underline{x}_m + \overline{x}_m$. When we identify H_m with $\mathbb{R}^m$ we rather write $x = (\underline{x}_m, \overline{x}_m) \in \mathbb{R}^m \oplus X^\perp_m$.

We say that $u : X \to \mathbb{R}$ is a *cylindrical function* if $u(x) = v(\Pi_m(x))$ for some $m \in \mathbb{N}$ and $v : \mathbb{R}^m \to \mathbb{R}$. We denote by $\mathcal{F}C^k_b(X)$, $k \in \mathbb{N}$, the space of all C^k_b cylindrical functions, that is, functions of the form $v(\Pi_m(x))$ with $v \in C^k_b(\mathbb{R}^m)$, with continuous and bounded derivatives up to the order k. We denote by $\mathcal{F}C^k_b(X, H)$ the space generated by all functions of the form uh, with $u \in \mathcal{F}C^k_b(X)$ and $h \in H$.

Given $u \in L^2(X, \gamma)$, we consider the canonical cylindrical approximation operators $\mathbb{E}_m$ given by

$$\mathbb{E}_m u(x) = \int_{X^\perp_m} u(\Pi_m(x), y) \, d\gamma^\perp_m(y). \tag{2.1}$$

Notice that $\mathbb{E}_m u$ depends only on the first m variables and $\mathbb{E}_m u$ converges to u in $L^p_\gamma(X)$ for all $1 \le p < \infty$. We let

$$\nabla_\gamma u := \sum_{j\in\mathbb{N}} \partial_j u \, h_j \qquad \text{for } u \in \mathcal{F}C^1_b(X)$$

$$\mathrm{div}_\gamma \varphi := \sum_{j\ge 1} \partial^*_j [\varphi, h_j]_H \quad \text{for } \varphi \in \mathcal{F}C^1_b(X,H)$$

$$\Delta_\gamma u := \mathrm{div}_\gamma \nabla_\gamma u \qquad \text{for } u \in \mathcal{F}C^2_b(X)$$

where $\partial_j := \partial_{h_j}$ and $\partial^*_j := \partial_j - \hat{h}_j$ is the adjoint operator of ∂_j. With this notation, the following integration by parts formula holds:

$$\int_X u \, \mathrm{div}_\gamma \varphi \, d\gamma = -\int_X [\nabla_\gamma u, \varphi]_H \, d\gamma \qquad \forall \varphi \in \mathcal{F}C^1_b(X,H). \tag{2.2}$$

In particular, thanks to (2.2), the operator ∇_γ is closable in $L^p_\gamma(X)$, and we denote by $W^{1,p}_\gamma(X)$ the domain of its closure. The Sobolev spaces $W^{k,p}_\gamma(X)$, with $k \in \mathbb{N}$ and $p \in [1,+\infty]$, can be defined analogously, and $\mathcal{F}C^k_b(X)$ is dense in $W^{j,p}_\gamma(X)$, for all $p < +\infty$ and $k, j \in \mathbb{N}$ with $k \ge j$. Given a vector field $\varphi \in L^p_\gamma(X;H)$, $p \in (1,\infty]$, using (2.2) we can define $\mathrm{div}_\gamma \varphi$ in the distributional sense, taking test functions u in $W^{1,q}_\gamma(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that $\mathrm{div}_\gamma \varphi \in L^p_\gamma(X)$ if this linear functional can be extended to all test functions $u \in L^q_\gamma(X)$. This is true in particular if $\varphi \in W^{1,p}_\gamma(X;H)$.

Let $u \in W^{2,2}_\gamma(X)$, $\psi \in \mathcal{F}C^1_b(X)$ and $i, j \in \mathbb{N}$. From (2.2), with $u = \partial_j u$ and $\varphi = \psi h_i$, we get

$$\int_X \partial_j u \, \partial_i \psi \, d\gamma = \int_X -\partial_i(\partial_j u)\, \psi + \partial_j u \, \psi \langle x^*_i, x\rangle d\gamma \tag{2.3}$$

Let now $\varphi \in \mathcal{F}C^1_b(X,H)$. If we apply (2.3) with $\psi = [\varphi, h_j] =: \varphi^j$, we obtain

$$\int_X \partial_j u \, \partial_i \varphi^j \, d\gamma = \int_X -\partial_j(\partial_i u)\, \varphi^j + \partial_j u \, \varphi^j \langle x^*_i, x\rangle d\gamma$$

which, summing up in j, gives

$$\int_X [\nabla_\gamma u, \partial_i \varphi] \, d\gamma = \int_X -[\nabla_\gamma(\partial_i u), \varphi] + [\nabla_\gamma u, \varphi]\langle x^*_i, x\rangle d\gamma$$

for all $\varphi \in \mathcal{F}C^1_b(X,H)$.

The operator $\Delta_\gamma : W^{2,p}_\gamma(X) \to L^p_\gamma(X)$ is usually called the Ornstein-Uhlenbeck operator on X. Notice that, if u is a cylindrical function, that is $u(x) = v(y)$ with $y = \Pi_m(x) \in \mathbb{R}^m$ and $m \in \mathbb{N}$, then

$$\Delta_\gamma u = \sum_{j=1}^m \partial_{jj} u - \langle x^*_j, x\rangle \partial_j u = \Delta v - \langle y, \nabla v\rangle_{\mathbb{R}^m} .$$

2.3 Fractional Sobolev spaces and fractional perimeters

Since the operator $-\Delta_\gamma$ is positive and self-adjoint in $L^2_\gamma(X)$, one can define its fractional powers by means of the standard formula in spectral theory

$$(-\Delta_\gamma)^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\gamma} - \mathrm{Id}\right) \frac{dt}{t^{1+s}},$$

where $s \in (0,1)$ and $e^{t\Delta_\gamma}$ denotes the Ornstein-Uhlenbeck semigroup on X.

For non local PDEs involving the fractional laplacian it is by now classical to use the so-called Caffarelli-Silvestre extension (see [5]). Here we use a general formulation of it, due to Stinga and Torrea [16], which can be easily adapted to our infinite dimensional setting. More precisely, a consequence of their main result is the following:

Theorem 2.1. *Let $u \in \mathrm{dom}((-\Delta_\gamma)^s)$. A solution of the extension problem*

$$\begin{cases} \Delta_\gamma v + \dfrac{1-2s}{y}\partial_y v + \partial_y^2 v = 0 & \text{on } X \times (0,+\infty) \\[2ex] v(x,0) = u & \text{on } X, \end{cases} \tag{2.4}$$

is given by

$$v(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta_\gamma}((-\Delta_\gamma)^s u)(x) e^{-y^2/4t} \frac{dt}{t^{1-s}}$$

and furthermore, one has in $L^2_\gamma(X)$

$$-\lim_{y\to 0^+} y^{1-2s}\partial_y v(x,y) = \frac{2s\Gamma(-s)}{4^s\Gamma(s)}(-\Delta_\gamma)^s u(x). \tag{2.5}$$

After defining the fractional laplacian, let us introduce the fractional Sobolev space

$$H^s_\gamma(X) = \left\{ u \in L^2_\gamma(X) : \ [u]_{H^s_\gamma} < \infty \right\}$$

where

$$[u]^2_{H^s_\gamma} = \inf\left\{ \int_{X\times\mathbb{R}^+} \left(|\nabla_\gamma v|_H^2 + |\partial_y v|^2\right) y^{1-2s} d\gamma(x) dy : \ v \in H^1_{\mathrm{loc}}(X\times\mathbb{R}^+), \ v(\cdot,0) = u(\cdot) \right\}. \tag{2.6}$$

The space H^s_γ is endowed with the Hilbert norm

$$\|u\|^2_{H^s_\gamma} = \|u\|^2_{L^2_\gamma} + [u]^2_{H^s_\gamma}.$$

Remark 2.2. Let us define the space

$$H^1(X\times\mathbb{R}^+, \gamma\otimes y^{1-2s}dy) = \left\{ v \in H^1_{\mathrm{loc}}(X\times\mathbb{R}^+) : \int_{X\times\mathbb{R}^+} \left(|v|^2 + |\nabla_\gamma v|_H^2 + |\partial_y v|^2\right) y^{1-2s} d\gamma(x)dy < \infty \right\}.$$

A function $u \in L^2_\gamma(X)$ belongs to H^s_γ if and only if there is $v_u \in H^1(X \times \mathbb{R}^+, \gamma \otimes y^{1-2s}dy)$ such that the infimum in (2.6) is attained by v_u. We may therefore define the inner product

$$\langle u, w\rangle_{\dot H^s_\gamma} = \int_{X\times\mathbb{R}^+} \left([\nabla_\gamma v_u, \nabla_\gamma v_w]_H + \partial_y v_u\, \partial_y v_w\right) y^{1-2s} d\gamma(x) dy, \qquad u, w \in H^s_\gamma(X).$$

We relate in the next lemma the fractional laplacian with the spaces described above.

Lemma 2.3. *for every $u,w \in H^s_\gamma$ with $u \in \mathrm{dom}((-\Delta_\gamma)^s)$ the following equality holds:*

$$\langle u, w\rangle_{\dot H^s_\gamma} = c_s \int_X (-\Delta_\gamma)^s u\, w \; d\gamma,$$

where c_s is the constant in (2.5).

Proof. For $u,w \in H^s_\gamma(X)$, let v_u be as above. It easily follows from the minimality and elliptic regularity that v_u is a solution of problem (2.4). Indeed, let us consider the test function $\varphi(x)\psi(y)$ with $\varphi \in \mathcal{F}C^\infty_b(X)$ and $\psi \in C^\infty_c(\mathbb{R})$; we have

$$\begin{aligned}
\langle u, \varphi\psi\rangle_{\dot H^s_\gamma} &= \int_{X\times\mathbb{R}^+} \Big[[\nabla_\gamma v_u, \nabla_\gamma \varphi(x)]_H \psi(y) + \varphi(x)\partial_y v_u \psi'(y)\Big] y^{1-2s} d\gamma(x)dy \\
&= \int_{X\times\mathbb{R}^+} \big(-\Delta_\gamma v_u - \partial_y^2 v_u - \frac{1-2s}{y}\partial_y v_u\big)\varphi(x)\psi(y) y^{1-2s} d\gamma(x)dy \\
&\quad - \int_X \lim_{y\to 0+} (y^{1-2s}\partial_y v_u(x,y)\psi(y))\varphi(x)\, d\gamma(x).
\end{aligned}$$

Since $v_u(\cdot,0) = u(\cdot)$, from (2.5) and the density of the test functions in H^s_γ we obtain the thesis. □

We are now ready to define the fractional perimeter of a set in X.

Definition 2.4. For every measurable set $E \subset X$ ans $0 < s < 1$ we define the fractional s-perimeter by setting

$$P_{\gamma,s}(E) = \frac{1}{2}[\chi_E]^2_{H^{s/2}_\gamma}$$

according to (2.6), i.e.,

$$P_{\gamma,s}(E) = \frac{1}{2}\inf\Big\{\int_{X\times\mathbb{R}^+} \big(|\nabla_\gamma v|^2_H + |\partial_y v|^2\big)\, y^{1-s} d\gamma(x)dy : \; v \in H^1_{\mathrm{loc}}(X\times\mathbb{R}^+),\; v(\cdot,0) = \chi_E(\cdot)\Big\}.$$

We say that E has finite s-perimeter in X if $P_{\gamma,s}(E) < \infty$.

Let us show that a form of the coarea formula holds in this framework as well (see [17]).

Proposition 2.5. *Setting for $u \in L^1_\gamma(X)$*

$$V_s(u) = \int_{\mathbb{R}} P_{\gamma,s}(\{u > t\})\, dt,$$

V_s is convex and lower semicontinuous on $L^1_\gamma(X)$. Moreover, if $u_n = \mathbb{E}_n[u]$ are the canonical cylindrical approximation of u then $V_s(u) \le V_s(u_n)$.

Proof. The convexity of V_s has been proved in [8, Proposition 3.4], while the lower semicontinuity easily follows from the lower semicontinuity of perimeters. The last inequality follows immediately from Jensen's inequality. □

3 The fractional isoperimetric problem

In order to discuss the isoperimetric properties of half-spaces, following [9] we introduce a suitable notion of symmetrisation. For $h \in H$ with $|h|_H = 1$, we consider the projection $x' = \pi_h x = x - \hat{h}(x)h$ and write $x = x' + th$ with $t \in \mathbb{R}$. Therefore, for fixed $h \in H$ and for any $I \subset \mathbb{R}$ we set

$$I^* = (-\infty, \phi^{-1}(\gamma_1(I))), \qquad \text{where} \quad \phi(t) = \int_{-\infty}^{t} e^{-\tau^2/2} d\tau. \tag{3.1}$$

In the same vein, for every measurable function $u : X \to \mathbb{R}$ we define the symmetrised function

$$u_h^*(x' + th) = \sup\Big\{c \in \mathbb{R} : \ t \in \{u(x', \cdot) > c\}^*\Big\}. \tag{3.2}$$

Since symmetrisation preserves characteristic functions, we may define the set E_h^* through the equality

$$\chi_{E_h^*} = (\chi_E)_h^*.$$

The proof of Theorem 1.1 relies on the following lemma.

Lemma 3.1. *Let $v \in H^1(X \times \mathbb{R}^+, \gamma \otimes y^{1-2s}dy)$ and let $h \in X^*$ with $|h|_H = 1$. Let v_h^* be as in* (3.2) *and let*

$$J_1(v) := \int_{X \times \mathbb{R}^+} |\partial_y v|^2 \, y^{1-2s} d\gamma(x) dy.$$

Then we have the inequality $J_1(v_h^*) \le J_1(v)$.

Proof. The proof follows that of Theorem 1 in [3] with minor modifications, we repeat it for the reader's convenience. There are some differences: Brock's result is in finite dimensions, the underlying measure is the Lebesgue one and he uses the Steiner symmetrisation, whereas we work in $X \times \mathbb{R}^+$ with the product measure $\gamma \otimes y^{1-2s}dy$ and we are concerned with the Ehrhard symmetrisation. On the other hand, the functionals considered by Brock are much more general than ours. In order to simplify the notation, suppose that $h = h_1$, write as before $x = x' + th_1$ split $X = H_1 \oplus X_1^\perp$ and decompose the gaussian measure as $\gamma = \gamma_1 \otimes \gamma_1^\perp$. Since for every $v \in H^1(X \times \mathbb{R}^+, \gamma \otimes y^{1-2s}dy)$ we have

$$J_1(v) = \int_{X_1^\perp} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}^+} |\partial_y v(x', t, y)|^2 y^{1-2s} d\gamma_1(t) dy \Big) d\gamma_1^\perp(x'),$$

we may limit ourselves to the inner double integral, for fixed x'. Moreover, following the reduction explained in [3], we may deal only with the dense class of *nice functions*, i.e., piecewise affine functions $v : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ such that for every $c > \inf v$ the equation $v(t, y) = c$ has for every $y \in \mathbb{R}^+$ a finite (even) number of solutions $t_1, \ldots, t_{2m}$. Once the result is proved for nice functions, the general case follows as in [3]. For v nice, set $\Omega = \{v > 0\}$ and decompose the vertical set $\{(y, z) \in \mathbb{R}^+ \times \mathbb{R}^+ : \ \exists\, (t, y) \in \Omega \text{ such that } v(t, y) = z\}$ into N disjoint domains G_j such that for any $(y, z) \in G_j$ the equation $v(t, y) = z$ has exactly $2m$ (with m depending on j) solutions

$t = t_k^j,\ k = 1, \ldots, 2m(j)$. Thus v can be represented in each G_j by the inverse functions $t = t_k^j(y, v)$. In each domain G_j the following identities hold:

$$\partial_t v(t_k^j, y) = \Big(\frac{\partial t_k^j}{\partial v}\Big)^{-1} \begin{cases} > 0 & \text{if } k \text{ is odd} \\ < 0 & \text{if } k \text{ is even} \end{cases}$$
$$\partial_y v(t_k^j, y) = -\frac{\partial t_k^j}{\partial y}\Big(\frac{\partial t_k^j}{\partial v}\Big)^{-1}.$$

Since v is nice, all the derivatives of t_k^j are constant in G_j and therefore the rearranged function v^* is nice, too. Moreover the symmetrisation procedure reduces the solutions of the equation $v^*(t, y) = z$ to only one, i.e., the following

$$T^j = \phi^{-1}\Big(\sum_{k=1}^{2m(j)} (-1)^{k-1}\phi(t_k^j)\Big)$$

(where ϕ is introduced in (3.1)) in each G_j, for every $y \in \mathbb{R}^+$. Differentiating we get

$$\gamma_1(T^j)\frac{\partial T^j}{\partial y} = \sum_{k=1}^{2m(j)} (-1)^{k-1}\gamma(t_k^j)\frac{\partial t_k^j}{\partial y}$$
$$\gamma_1(T^j)\frac{\partial T^j}{\partial z} = \sum_{k=1}^{2m(j)} \gamma(t_k^j)\Big|\frac{\partial t_k^j}{\partial y}\Big|.$$

It follows (with $x' \in X_1^\perp$ fixed)

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} |\partial_y v(x', t, y)|^2 y^{1-2s}\gamma_1(t)dydt = \sum_{j=1}^{N}\int_{G_j}\sum_{k=1}^{m(j)}\Big|\frac{\partial t_k^j}{\partial y}\Big|^2\Big|\frac{\partial t_k^j}{\partial z}\Big|^{-1}\gamma_1(t_k^j)dydz$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} |\partial_y v^*(x', t, y)|^2 y^{1-2s}\gamma_1(t)dydt = \sum_{j=1}^{N}\int_{G_j}\Big|\frac{\partial T^j}{\partial y}\Big|^2\Big|\frac{\partial T^j}{\partial z}\Big|^{-1}\gamma_1(T^j)dydz$$
$$= \sum_{j=1}^{N}\int_{G_j}\frac{\Big|\sum_{k=1}^{2m(j)}(-1)^{k-1}\gamma(t_k^j)\dfrac{\partial t_k^j}{\partial y}\Big|^2}{\Big|\sum_{k=1}^{2m(j)}\gamma(t_k^j)\Big|\frac{\partial t_k^j}{\partial y}\Big|\Big|}dydz.$$

Setting

$$c_k^j = \gamma_1(t_k^j)\frac{\partial t_k^j}{\partial y}, \qquad b_k^j = \gamma_1(t_k^j)\Big|\frac{\partial t_k^j}{\partial z}\Big|,$$

we have the following equivalence:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} |\partial_y v(x', t, y)|^2 y^{1-2s}\gamma_1(t)dydt \geq \int_{\mathbb{R}}\int_{\mathbb{R}^+} |\partial_y v^*(x', t, y)|^2 y^{1-2s}\gamma_1(t)dydt$$

$$\iff \qquad \sum_{k=1}^{2m(j)} \frac{(c_k^j)^2}{b_k^j} \geq \Big(\sum_{k=1}^{2m(j)} (-1)^{k-1} c_k^j\Big)^2 \Big|\sum_{j=1}^{2m(j)} b_k^j\Big|^{-1} \quad \forall\, j = 1, \dots, N.$$

But, the last inequality is nothing but the Cauchy-Schwarz inequality:

$$\Big(\sum_{k=1}^{2m(j)} (-1)^{k-1} c_k\Big)^2 = \Big(\sum_{k=1}^{2m(j)} (-1)^{k-1} \frac{c_k}{\sqrt{b_k}} \sqrt{b_k}\Big)^2 \leq \sum_{k=1}^{2m(j)} \frac{c_k^2}{b_k} \sum_{k=1}^{2m(j)} b_k$$

and the thesis follows. □

Let us show that the $L^2_\gamma(X)$ norm of the gradient is also decreasing under Ehrhard rearrangement.

Lemma 3.2. *Let $u \in H^1_\gamma(X)$, and let $h \in X^*$ with $|h|_H = 1$. Then $u^*_h \in H^1_\gamma(X)$ and*

$$\int_X |\nabla_\gamma u^*_h|^2_H d\gamma \leq \int_X |\nabla_\gamma u|^2_H d\gamma\,. \tag{3.3}$$

Proof. In [10, Th. 3.1] the inequality (3.3) is proven for Lipschitz functions in finite dimensions. We extend it by approximation to Sobolev functions in $H^1_\gamma(X)$.

We let $u_n \in \mathcal{F}C^1_b(X)$ be the canonical cylindrical approximation of u defined in (2.1). Since $u_n \to u$ in $H^1_\gamma(X)$, we have $(u_n)^*_h \to u^*_h$ in $L^2_\gamma(X)$, so that by the lower semicontinuity of the H^1_γ norm we obtain

$$\int_X |\nabla_\gamma u^*_h|^2_H d\gamma \leq \liminf_{n\to\infty} \int_X |\nabla_\gamma (u_n)^*_h|^2_H d\gamma \leq \liminf_{n\to\infty} \int_X |\nabla_\gamma u_n|^2_H d\gamma = \int_X |\nabla_\gamma u|^2_H d\gamma\,.$$

□

From Lemma 3.2 we immediately get the following result.

Lemma 3.3. *Let $v \in H^1(X \times \mathbb{R}^+, \gamma \otimes y^{1-2s}dy)$ and let $h \in X^*$ with $|h|_H = 1$. Letting v^*_h be as in (3.2) and*

$$J_2(v) = \int_{X\times\mathbb{R}^+} |\nabla_\gamma v|^2_H\, y^{1-2s} d\gamma(x)dy,$$

we have the inequality $J_2(v^*_h) \leq J_2(v)$.

From (2.6), Lemma 3.1 and Lemma 3.3 we immediately get the following result:

Corollary 3.4. *If $u \in H^s_\gamma(X)$ then for every $h \in H$ we have $u^*_h \in H^s_\gamma(X)$ and*

$$[u^*_h]_{H^s_\gamma} \leq [u]_{H^s_\gamma}\,.$$

Given $u \in L^2_\gamma(X)$, let $S_u : \mathbb{R} \to \mathbb{R}$ be the decreasing function defined through its inverse by the equality

$$S_u^{-1}(t) = \phi^{-1}(\gamma(\{u > t\}),$$

for ϕ as in (3.1), so that $\gamma(\{u > t\}) = \gamma(\{S_u > t\})$.

Theorem 3.5. *Let $u \in H^s_\gamma$. Then*

$$[S_u]_{H^s_{\gamma_1}} \leq [u]_{H^s_\gamma}\,, \tag{3.4}$$

with equality if and only if u is one-dimensional, that is, $u(x) = S_u(\hat{h}(x))$ for some $h \in H$ with $|h| = 1$.

Proof. We first show the inequality (3.4). Let (u_n) be the canonical cylindrical approximation of u defined in (2.1), let (h_k) be a sequence dense in $\{h \in H_n : |h|_H = 1\}$ and let $u_{n,k}$ be iteratively defined by $u_{n,0} = u_n$ and $u_{n,k} = (u_{n,k-1})^*_{h_k}$ as in (3.2). Then, $\|u_{n,k}\|_{L^2_\gamma(X)} = \|u_n\|_{L^2_\gamma(X)}$ for every k and by the preceding lemmas, we have that $[u_{n,k}]_{H^s_\gamma} \leq [u_n]_{H^s_\gamma}$, hence (up to a subsequence that we don't relabel) the sequence $(u_{n,k})$ converges to a function $\tilde{u}_n$ in $L^2_\gamma(X)$ with $[\tilde{u}_n]_{H^s_\gamma} \leq [u_n]_{H^s_\gamma}$. Since $\tilde{u}_n$ is symmetric with respect to all the directions in H_n, it can be written as $\tilde{u}_n(x) = S_{u_n}(\hat{h}(x))$ for some $h \in H_n$. From Lemma 3.1 and Lemma 3.3 it follows that

$$[S_{u_n}]_{H^s_{\gamma_1}} = [S_{u_n} \circ \hat{h}]_{H^s_\gamma} \leq [u_n]_{H^s_\gamma} \leq [u]_{H^s_\gamma}.$$

Passing to the limit as $n \to \infty$ and noting that $S_{u_n} \to S_u$ in $L^2_{\gamma_1}(\mathbb{R})$, we get the inequality (3.4).

Assume now that the equality holds in (3.4). Again by Lemma 3.1 and Lemma 3.3, this implies that

$$\int_{X\times\mathbb{R}^+} |\partial_y v_{S_u}|^2 y^{1-2s}\,\gamma(x)\,dy \;=\; \int_{X\times\mathbb{R}^+} |\partial_y v_u|^2 y^{1-2s}\,\gamma(x)\,dy$$

and

$$\int_{\mathbb{R}^+} \|\nabla_\gamma v_{S_u}(\cdot,t)\|^2_{L^2_\gamma(X)} y^{1-2s}\,dy \;=\; \int_{\mathbb{R}^+} \|\nabla_\gamma v_u(\cdot,t)\|^2_{L^2_\gamma(X)} y^{1-2s}\,dy\,,$$

where v_{S_u}, v_u are the corresponding minimisers of the right-hand side of (2.6). Hence, for a.e. $t > 0$ we have

$$\|\nabla_\gamma v_{S_u}(\cdot,t)\|_{L^2_\gamma(X)} = \|\nabla_\gamma v_u(\cdot,t)\|_{L^2_\gamma(X)}\,.$$

Thanks to [12, Prop. 3.12], it follows that v_u is one-dimensional for a.e. $t > 0$, which implies that u is also one-dimensional, and concludes the proof. □

A direct consequence of Theorem 3.5 is the following symmetry result:

Theorem 3.6. *Let $m > 0$ and $F : \mathbb{R} \to \mathbb{R}$ be lower semicontinuous, and assume that the problem*

$$\min\Big\{[w]_{H^s_{\gamma_1}} + \int_{\mathbb{R}} F(w)\,d\gamma_1 :\ \int_{\mathbb{R}} w\,d\gamma_1 = m\Big\} \tag{3.5}$$

admits a minimiser. Then the unique minimisers of the problem

$$\min\Big\{[u]_{H^s_\gamma} + \int_X F(u)\,d\gamma :\ \int_X u\,d\gamma = m\Big\} \tag{3.6}$$

are given by $u(x) = \varphi(\hat{h}(x))$ for some minimiser φ of problem (3.5) and for some $h \in H$.

We can conclude with the proof of Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 follows from Theorem 3.5 with $s/2$ in place of s, by taking $u = \chi_E$ to be the characteristic function of E. □

Acknowledgements

D.P. has been partially supported by the PRIN 2010 MIUR project “Problemi differenziali di evoluzione: approcci deterministici e stocastici e loro interazioni”. Y.S. has been partially supported by the ERC grant ϵ “Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities”, and the ANR project ”HAB”. M.N. and D.P. are members of the italian CNR-GNAMPA.

References

[1] L. Ambrosio, S. Maniglia, M. Miranda, D. Pallara: *BV* functions in abstract Wiener spaces. *J. Funct. Anal.*, 258:785-813, 2010.

[2] V. I. Bogachev: *Gaussian Measures.* American Mathematical Society, 1998.

[3] F. Brock: Weighted Dirichlet-type inequalities for Steiner symmetrization. *Calc. Var. Partial Differential Equations,* 8:15-25, 1999.

[4] L. Caffarelli, J.M. Roquejoffre, O. Savin: Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63:1111-1144, 2010.

[5] L. Caffarelli, L. Silvestre: An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32:1245-1260 ,2007.

[6] E.A. Carlen, C. Kerce: On the cases of equality in Bobkov's inequality and Gaussian rearrangement. *Calc. Var. Partial Differential Equations,* 13:1-18, 2001.

[7] V. Caselles, A. Lunardi, M. Miranda, M. Novaga: Perimeter of sublevel sets in infinite dimensional spaces. *Adv. Calc. Var.*, 5(1):59-76, 2012.

[8] A. Chambolle, A. Giacomini, L. Lussardi: Continuous limits of discrete perimeters. *ESAIM Math. Model Numer. Anal.*, 44:207-230, 2010.

[9] A. Ehrhard: Symétrisation sans l'espace de Gauss. *Math. Scand.* 53:281-301, 1983.

[10] A. Ehrhard: Inégalités isopérimetriques et intégrales de Dirichlet gaussiennes. *Ann. Sci. Ecole Norm. Sup.*, 17: 317-332, 1984.

[11] N. Fusco, F. Maggi, A. Pratelli: On the isoperimetric problem with respect to a mixed Euclidean-Gaussian density. *J. Funct. Anal.*, 260:3678-3717, 2011.

[12] M. Goldman, M. Novaga: Approximation and relaxation of perimeter in the Wiener space. *Ann. Inst. H. Poincaré Anal. Nonlinéaire*, 29:525-544, 2012.

[13] M. Ledoux: A short proof of the Gaussian isoperimetric inequality, in: *High dimensional probability,* (Oberwolfach, 1996). Progr. Probab., Birkhäuser, 43:229-232, 1998.

[14] L. Modica, S. Mortola: Un esempio di Γ-convergenza. *Boll. Unione Mat. Ital.*, 14(5):285-299, 1977.

[15] O. Savin, E. Valdinoci: Γ-convergence for nonlocal phase transitions. *Ann. Inst. Henri-Poincaré (C)*, 29(4):479-500, 2012.

[16] P. R. Stinga, J. L. Torrea: Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations*, 35:2092-2122, 2010.

[17] A. Visintin: Generalized coarea formula and fractal sets. *Japan Journal of Industrial and applied Math.*, 8:175-201, 1991.