INVERSE FUNCTION THEOREMS AND JACOBIANS OVER METRIC SPACES

LUCA GRANIERI

ABSTRACT. We present inversion results for Lipschitz maps $f : \Omega \subset \mathbb{R}^N \to (Y, d)$ and stability of inversion for uniformly convergent sequences. These results are based on the Area Formula and on the l.s.c. of metric Jacobians.

Keywords. Calculus of Variations, Geometric Measure Theory, area formula, lower semicontinuity, Jacobian, Inversion theorem.
MSC 2000. 37J50, 49Q20, 49Q15.

1. INTRODUCTION

The question of inverting maps is recurrent and important in many circumstances, both theoretical or applied ones. Also the literature on the subject is very spread (linear and non-linear functional analysis, differential e/o integral equations, mathematical economics, etc.) and reflecting different approaches and points of view.

Our motivation on invertibility issues relies on mathematical variational models of continuum mechanics. In the setting of nonlinear elasticity, in the undeformed state the material body occupies a bounded open set $\Omega \subset \mathbb{R}^N$. Then one usually looks for minimizers $u : \overline{\Omega} \to \mathbb{R}^N$ of the stored energy $I(u) = \int_\Omega W(\nabla u)\, dx$ in an admissible class $\mathcal{K}$ of deformations u. In this framework, the invertibility of deformations $u \in \mathcal{K}$ corresponds to the physical assumption of *impenetrability* of matter. The variational approach to this kind of problems leads naturally to treat stability of invertibility with respect to suitable notions of convergence. The question is then to find conditions under which the limit map of a sequence of invertible maps is itself invertible. This basic and fundamental question has of course many possible answers. In this paper we are interested in the case of maps $f : \Omega \subset \mathbb{R}^N \to f(\Omega)$ where the target $f(\Omega)$ lies possibly in a metric space (Y, d). Our main motivation in considering this question comes from [19] where the notion of *transport plan* as in Optimal Mass Transportation Theory is proposed as a weak notion of material deformation. In such case (see [19] for the details) it results useful to consider an energy functional on maps $f : \Omega \subset \mathbb{R}^N \to (Y, d)$. Precisely, in [19] Y is the set of probability measures endowed with the Wasserstein metric. In this model, invertibility of maps over a metric space was fundamental to obtain lower semicontinuity of the energy functional. It is then natural to consider the applicability of inverse function theorem for maps with metric space targets, as opposed to the usual setting of Euclidean spaces. This necessitates a shift from degree theory methods (which are standard in the Euclidean case) toward coarser, and therefore more robust, estimates based on volume growth.

For maps over $\mathbb{R}^N$ a widely used tool is the Topological Degree Theory. However, for maps over metric spaces no such theory is available. Observe that also the notion of fixed point has no meaning in this setting. This is a main obstruction in dealing with maps over metric spaces. On the other hand, many concepts of Geometric Measure Theory are well established in this framework ([2, 3, 23, 24, 28]). Following the observation of [20] in which some inversion results are obtained by using the area formula and lower semicontinuity of Jacobians, here we investigate inversion results based on Geometric Measure Theory for Lipschitz maps $f : \Omega \to (Y, d)$.

1.1. **Description of the results.** In Section 2 we provide an overview of some approaches to invertibility based on the Area Formula and on l.s.c. of Jacobians. Many results in the literature deal with global invertibility obtained by local invertibility. In the framework of this paper, we observe that in fact considering open maps is enough. The point is that the so called Invariance Domain Theorem, which state that a continuous locally invertible map is an open map, is peculiar of the Euclidean setting and anyway it does not hold in general. Therefore, the notion of open map seems more appropriate in this metric setting. Outside of the Euclidean setting, i.e. without the benefit of the Invariance Domain Theorem, this seems the natural way to proceed.

A recurrent notion in literature, especially in the Sobolev setting, is that of *open and discrete* maps (see [30, 33, 32, 25, 26]). We also find that this kind of maps is useful from a metric point of view. Actually, this notion is crucial (see Lemma 3.14 and Theorem 3.15) to provide *easy* proofs of l.s.c. of metric Jacobians. Hence we provide a results of stability for invertible maps also valid over a metric target. Our main result (see Theorem 3.16) establishes that the uniform limit f of a sequence of equi-Lipschitz invertible maps is also invertible provided f is open and discrete.

The l.s.c. results for the metric Jacobians are based on l.s.c. of the Hausdorff measure with respect to Hausdorff convergence of sets. A general l.s.c. result for the metric Jacobians would probably require more subtle studies of l.s.c. with respect to Hausdorff convergence. In Section 4 we discuss invertibility of maps related with the so called *quasi-isometries* introduced by John and also considered in [19].

2. Local toward global invertibility

Let $f : \Omega \to f(\Omega) \subset \mathbb{R}^N$ be a (Lipschitz) continuous locally invertible map. Suppose that the Area Formula

$$\int_V N(y, f, \Omega)\, d\mathcal{H}^N(y) = \int_{f^{-1}(V)} Jf(x)\, dx \tag{1}$$

holds true, where $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure, while $N(y, f, U) := \#(f^{-1}(y) \cap U)$ is the multilplicity function. This is the case of course for f Lipschitz. It is possible to use the Area Formula (1) in different ways in order to obtain global invertibility of f.

Let $x_1 \neq x_2$ be such that $f(x_1) = f(x_2)$. Consider two small disjoint balls $B_1 := B(x_1, r_1)$, $B_2 := B(x_2, r_2)$. By the Invariance Domain Theorem we get that f is an open map. Therefore, the set $V := f(B_1) \cap f(B_2)$ is an open non-empty set. Moreover, it results $N(y, f, \Omega) \geq 2$ for every $y \in V$. By using (1) we obtain

$$2\mathcal{H}^N(V) \leq \int_V N(y, f, \Omega)\ d\mathcal{H}^N(y) = \int_{f^{-1}(V)} Jf(x)\ dx. \tag{2}$$

To get a contradiction we have to estimate the above Jacobian integral in terms of $\mathcal{H}^N(V)$. Actually, there are different possible approaches.

(i) Measure preserving maps: In [19] some inversion properties of measure preserving maps are stated. We say that f is measure preserving if $f_\#\mathcal{L}^N = \mathcal{H}^N$, where the push-forward measure is defined by $f_\#\mathcal{L}^N(V) := \mathcal{L}^N(f^{-1}(V))$. These kind of maps are fundamental in Mass Transportation Theory. Denoting by $M := Lip(f)$ the Lipschitz constant of f, by (2) we estimate

$$2\mathcal{H}^N(V) \leq M^N\mathcal{L}^N(f^{-1}(V)) = M^N f_\#\mathcal{L}^N(V) = M^N\mathcal{H}^N(V).$$

Therefore, if $M < \sqrt[N]{2}$ we get invertibility of f.

(ii) Ciarlet-Necas: In [9] it is considered the condition

$$\int_U Jf(x)\ dx \leq \mathcal{H}^N(f(U)), \tag{3}$$

which of course holds true with equality sign for invertible maps. Under (3), by (2) we estimate

$$2\mathcal{H}^N(V) \leq \mathcal{H}^N(f(f^{-1}V)) \leq \mathcal{H}^N(V),$$

leading to invertibility of f.

By using the area formula (see [2, 23, 24, 28]) we get

$$\mathcal{H}^N(f(U)) \leq \int_Y N(y, f, U)\ d\mathcal{H}^N(y) = \int_U Jf(x)\ dx.$$

Therefore, the opposite inequality

$$\mathcal{H}^N(f(U)) \leq \int_U Jf(x)\ dx$$

is always true also in a metric space setting, whenever the area formula holds true.

(iii) Stability: In [20] it is contained a discussion about stability of invertibility. In particular it is observed that invertibility of a limit map f of a sequence f_n of invertible maps can be obtained by using the lower semicontinuity of Jacobian's integrals under uniform convergence (see [1]). Indeed, fixed $\varepsilon > 0$, by uniform convergence we find $f_n(A) \subset f(A)^\varepsilon$ for large n, where B^ε denotes the ε-neighborhood of the set B. By (2) we obtain

$$2\mathcal{H}^N(V) \leq \int_{f^{-1}(V)} Jf(x)\ dx \leq \liminf_{n\to+\infty} \int_{f^{-1}(V)} Jf_n(x)\ dx =$$

$$\liminf_{n\to+\infty} \mathcal{H}^N(f_n(f^{-1}(V)) \leq \mathcal{H}^N(f(f^{-1}(V))^\varepsilon) \leq \mathcal{H}^N(V^\varepsilon).$$

Letting $\varepsilon \to 0$ we get (see also Lemma 3.11)

$$2\mathcal{H}^N(V) \leq \mathcal{H}^N(\overline{V}) = \mathcal{H}^N(V),$$

where we used that $\partial V \subset \partial f(B_1) \cup \partial f(B_2)$ and that f is open and Lipschitz, so that $\mathcal{H}^n(\partial V) = 0$.

(iv) Quasi-isometries: Let f be an $(m - M)$-quasi-isometry, namely such that

$$0 < m \leq D^- f(x) \leq D^+ f(x) \leq M < +\infty,$$

where

$$D^- f(x_0) := \liminf_{x\to x_0} \frac{d_Y(f(x), f(x_0))}{d_X(x, x_0)}, \quad D^+ f(x_0) := \limsup_{x\to x_0} \frac{d_Y(f(x), f(x_0))}{d_X(x, x_0)}.$$

The *metric derivatives* D^+, D^- are related to inversion properties, see [22, 16, 7] for an account on these maps. The constants m, M, roughly speaking, provides estimates of pointwise Lipschitz behavior of f from below and above respectively. By using (2), and given the volume control condition $m^N \mathcal{L}^N(f^{-1}(V) \leq \mathcal{H}^N(V)$, it is possible to estimate (see Section 4)

$$2\mathcal{H}^N(V) \leq \int_{f^{-1}(V)} Jf(x)\ dx \leq M^N \mathcal{L}^N(f^{-1}(V)) \leq \left(\frac{M}{m}\right)^N \mathcal{H}^N(V).$$

Therefore, if $\frac{M}{m} < \sqrt[N]{2}$ we get invertibility of f.

Observe that in the approaches stated above, local invertibility is not actually needed. The fact that f is an open map is enough to conclude, since the main point relies in finding the open set V such that $\mathcal{H}^N(V) > 0$.

3. Invertibility over metric spaces

We address the question of invertibility for Lipschitz maps

$$f : \Omega \subset \mathbb{R}^N \to f(\Omega) \subset (Y, d),$$

for a metric space Y. We assume that Y is geodesic and the Hausdorff N-dimensional measure over Y is non-trivial, i.e. for every ball $B \subset Y$ it results $0 < \mathcal{H}^N(B) < +\infty$. A metric space (Y, d) is said to be geodesic if for each pair of points $x, y \in Y$ the distance $d(x, y)$ is given by

$$d(x, y) := \min \left\{ length(\gamma) := \int_0^1 |\dot{\gamma}|(t)\ dt\ :\ \gamma \in Lip([0, 1], Y), \gamma(0) = x, \gamma(1) = y \right\}, \tag{4}$$

where $Lip([0,1],Y)$ denotes the set of Lipschitz maps of $[0,1]$ over Y, while $|\dot\gamma|(t)$ denotes the metric derivative defined by

$$\lim_{h\to+\infty}\frac{d(\gamma(t+h),\gamma(t))}{h}$$

which exists for a.e. t due to the Lipschitz condition (see [5] for the basic properties). Any curve γ achieving the minimum value in (4) is called geodesic. Therefore, the metric space Y is geodesic if any distance $d(x,y)$ can be obtained as the length of a shortest path (geodesic) connecting x and y.

Moreover, we shall assume that the balls of Y are somehow *continuous*, namely that every continuous path connecting interior and exterior points of a ball B intersects the boundary ∂B. In terms of parametrisation of a Lipschitz curve $\gamma:[0,1]\to Y$, we assume that if $\gamma(0)\in B, \gamma(t)\notin \bar B$, then there exists $0<s<t$ such that $\gamma(s)\in\partial B$.

3.1. **Open and discrete maps.** The Invariance Domain Theorem is peculiar of $\mathbb{R}^N$ and heavily relies on Topological Degree Theory. However, such result does not hold in general metric spaces. Therefore, to our purpose it seems better to handle directly with open maps. In [27] some metric conditions to obtain open maps are given. In this section we make some remarks needed in the sequel.

Every open map f maps interior point into interior point. Hence, for every open set it results $\partial f(U)\subset f(\partial U)$. On the other hand, maps f such that $\partial f(U)=f(\partial U)$ are not necessarily open. For a metric space X, we say that $f:X\to Y$ is *discrete (or isolated)* if for every $x_0\in X$ there exists $r>0$ such that $f(x)\neq f(x_0)$ for every $x\in\overline{B(x_0,r)}$. If in addition it results $\partial f(B)\subset f(\partial B)$, such balls are sometimes called *normal neighborhoods*. We also say that f is *normal* if $\partial f(B)\subset f(\partial B)$ for every ball B. We have the following

Lemma 3.1. *Let $f:X\to Y$ be a continuous discrete map. If f is normal, i.e. $\partial f(B)\subset f(\partial B)$ for every ball, then f is an open map.*

Proof. Let $y_0=f(x_0)\in f(U)$, for an open set $U\subset X$. Consider a normal neighborhood $B_0:=B(x_0,r)$. We claim that y_0 is an interior point of $f(B_0)$. If not, then $y_0\in\partial f(B_0)\subset f(\partial B_0)$. Therefore $y_0=f(x)$ with $x\in\partial B_0$, contradicting the discreteness of f.

□

Lemma 3.2. *Let $f:\overline{\Omega}\to Y$ be a continuous injective map. If $f(\Omega)$ is an open set then $f:\Omega\to Y$ is open.*

Proof. We may suppose Ω bounded. Given an open set $U\subset\Omega$, let $y_0=f(x_0)\in f(U)$, with $x_0\in B\subset\subset U$. Suppose that y_0 is a boundary point of $f(B)$. Then we find a sequence $y_n\to y_0$ such that $y_n\notin f(B)$. We claim that there exists $\delta>0$ such that $f(\overline{B})^\delta\subset f(\Omega)$.

Indeed, for $z\in f(\overline{B})$, since $f(\Omega)$ is open, we find $\delta_z>0$ such that $B(z,\delta_z)\subset f(\Omega)$. By compactness $f(\overline{B})\subset\bigcup_{i=1}^h B(z_i,\frac{\delta_i}{2})$, where $\delta_i:=\delta_{z_i}$. Consider $\delta=\min\{\frac{\delta_i}{2}\ :\ i=1,\cdots,h\}$. For $z\in f(\overline{B})^\delta$ we find $x\in\overline{B}$ such that $d(z,f(x))<\delta$ and an index i such that $d(z_i,f(x))<\frac{\delta_i}{2}$. Hence

$$d(z,z_i)\le d(z,f(x))+d(f(x),z_i)<\delta+\frac{\delta_i}{2}<\delta_i\Rightarrow z\in f(\Omega).$$

Therefore, for large n, we have $y_n = f(x_n)$ with $x_n \notin B$. Considering a subsequence if necessary, passing to the limit we find $y_0 = f(x)$ with $x \notin B$ contradicting the injectivity of f. $\square$

Lemma 3.3. *Let $f : \overline{\Omega} \subset \mathbb{R}^N \to Y$ be a discrete map such that $f(\Omega)$ is open. Let f_n be a sequence of continuous, injective normal maps such that $f_n \to f$ uniformly. Then $f : \Omega \to Y$ is open.*

Proof. Given an open set $U \subset \Omega$, let $y_0 = f(x_0) \in f(U)$. Consider $B_0 := B(x_0, r) \subset\subset U$ such that $f(x) \neq f(x_0)$ for every $x \in \overline{B_0}$. Let us prove that y_0 is an interior point of $f(B_0)$.

If not, then $y_0 \in \partial f(B_0)$. Therefore, we find a sequence $y_n \to y_0$ with $y_n \notin f(B_0)$. Since $f(\Omega)$ is open, as in the proof of the above Lemma 3.2, we get $y_n = f(x_n)$ with $x_n \notin B_0$. By invertibility of f_n it results $z_n := f_n(x_n) \notin f_n(B_0)$. By uniform convergence we have $z_n \to y_0$. On the other hand, $y'_n := f_n(x_0) \in f_n(B_0)$ also converges to y_0. Claim: we may assume that $z_n \in \partial f_n(B_0)$.

Indeed, if not consider a point y''_n realizing

$$\min_{z \in f_n(\overline{B}_0)} d(z_n, z).$$

If such points y''_n are not interior points, by the continuity assumption on the balls of Y, we would find points of $f_n(B_0)$ with lower distance from z_n. Therefore we have $y''_n \in \partial f_n(B_0) \subset f_n(\partial B_0)$. Since

$$d(y''_n, y_0) \leq d(y''_n, z_n) + d(z_n, y_0) \leq d(y'_n, z_n) + d(z_n, y_0) \to 0,$$

we also have $y''_n \to y_0$.

Since $y''_n = f_n(x'_n)$ with $x'_n \in \partial B_0$, passing to a subsequence if necessary, we find $y_0 = f(x)$ with $x \in \partial B_0$, contradicting the discreteness of f.

$\square$

Lemma 3.4. *Let $f : X \to Y$ be a map such that $D^- f(x) \geq m > 0$. Then f is a discrete map.*

Proof. Suppose by contradiction that $f^{-1}(y)$ contains an accumulation point x_0. Therefore, we get $x_n \to x_0$, with $x_n \in f^{-1}(y)$ and $x_n \neq x_0$. Since $D^- f(x_0) \geq m$, there exists a radius $r > 0$ such that $d_X(x, x_0) \leq \frac{1}{m} d_Y(f(x), f(x_0))$ for every $x \in B(x_0, r)$. For sufficiently large n, since $f(x_n) = f(x_0)$ we obtain the contradiction $x_n = x_0$.

$\square$

The volume estimate considered in Section 2 are devoted to get $N(y, f, \Omega) = 1$ a.e. which is then compared with the condition of open map for f. We get the following

Lemma 3.5. *Let $f : \Omega \to Y$ be an M-Lipschitz open map. If $N(y, f, \Omega) = 1$ $\mathcal{H}^N$-a.e. then f is invertible. Moreover, if the extension of f to $\overline{\Omega}$ is injective on the boundary, then f is globally invertible on $\overline{\Omega}$.*

Proof. Let $x_1 \neq x_2$ be two points in Ω such that $f(x_1) = f(x_2)$. Consider two small disjoint balls $B_1 := B(x_1, r_1)$, $B_2 := B(x_2, r_2)$. Since f is an open map, $V := f(B_1) \cap f(B_2)$ is an open non-empty set. Moreover, it results $N(y,f,\Omega)\ge 2$ for every $y\in V$, which is a contradiction since $\mathcal{H}^N$ is non-trivial.

Suppose by contradiction to have $x_1,x_2\in\overline{\Omega}$ such that $f(x_1)=f(x_2)=y_0$. We may assume $x_1\in\Omega, x_2\in\partial\Omega$. Let B_1 be a small ball around x_1. Since f is Lipschitz and open we find a small radius $r>0$ such that $f(\overline{B(x_2,r)}\cap\overline{\Omega})\subset B(y_0,Mr)\subset f(B_1)$, with $B_1\cap B(x_2,r)=\emptyset$. Again, it follows that $V:=f(B(x_2,r)\cap\Omega)\subset f(B_1)$ is a non-empty open set such that $N(y,f,\Omega)\ge 2$ for every $y\in V$.

□

3.2. **Sequences of invertible maps.**

Definition 3.6. *A surjective map $f:X\to Y$ is said to be a covering map if for every $y\in Y$ there exists a neighborhood V_y such that $f^{-1}(V_y)$ is a disjoint union of open sets each of which is mapped by f on V_y homeomorphically.*

For the basic properties of covering maps we refer for instance to [14]. We mention the following

Lemma 3.7. *Let $f:X\to Y$ be a covering map. We have the following*

1. *For every open set $V\subset Y$, $f:f^{-1}(V)\to V$ is a covering map*
2. *If Y is connected and locally connected, and C is a connected component of X, then $f:C\to Y$ is a covering map.*
3. *(Path lifting) Let $\gamma:[0,1]\to Y$ be a continuous path in Y. Let $x\in X$ such that $f(x)=\gamma(0)$. Then there exists a unique continuous path $\tilde{\gamma}:[0,1]\to X$ such that $\tilde{\gamma}(0)=x$ and $f\circ\tilde{\gamma}=\gamma$ in $[0,1]$.*
4. *If X is connected and Y is simply connected, then f is a global homeomorphism.*

Lemma 3.8. *Let X be a geodesic metric space satisfying the continuity condition on balls. Let X be locally compact and locally pathwise connected. Let Y be locally pathwise connected and locally simply connected. Let $f_n:X\to Y$ be local homeomorphisms and $f:X\to Y$ be an open and discrete map. If $f_n\to f$ (locally) uniformly, then for every $x_0\in X$ there exists a neighborhood $x_0\in U$ such that every f_n is an homeomorphism over U.*

Proof. Let $x_0\in X$ be fixed. By uniform convergence, we may assume that $f_n\to f$ in $\overline{B(x_0,s)}$ with $\overline{B(x_0,s)}\subset X$ compact. Since f is discrete we may assume that $f(x)\ne f(x_0)$ for every $x\in\overline{B(x_0,s)}$. As in [25, Proposition 7], we look for a common neighborhood of x_0 in which the functions f_n are all simultaneously invertible.

Setting $y_0=f(x_0), y_n=f_n(x_0)$, let U_1 be the connected component of x_0 contained in $f^{-1}(B(y_0,r))$.

Claim: It is possible to choose r sufficiently small so that U_1 is a normal neighborhood of x_0 and $f(U_1)=B(y_0,r)$.

Indeed, let $0<r<r_0:=\min\{d(y_0,y)\ :\ y\in f(\partial B(x_0,s))\}$. Let us show that $U_1\subset B(x_0,s)$.

For if, let $x\in U_1\setminus B(x_0,s)$. By continuity of balls we would find $x'\in\partial B(x_0,s)\cap U_1$ which would imply $r_0\le d(y_0,f(x'))<r$.

It remains to check that U_1 is normal and $f(U_1)=B(y_0,r)$.

Let $y = f(x)$ with $x \in \partial U_1$. Since U_1 is a connected component it results $x \notin f^{-1}(B(y_0, r))$. Hence $y \notin B(y_0, r)$. We get $y \in f(\overline{U_1}) \setminus f(U_1) = \overline{f(U_1)} \setminus f(U_1) = \partial f(U_1)$. Therefore U_1 is normal. Moreover, since$f(\partial U_1) \cap B(y_0, r) = \emptyset$ we have $f(U_1) = B(y_0, r) \cap f(\overline{U_1})$. Then, $f(U_1)$ is closed and open in $B(y_0, r)$. Hence $f(U_1) = B(y_0, r)$.

By uniform convergence, we may assume that $d_Y(f_n(x), f(x)) < \frac{r}{3}$ for every $x \in \overline{B(x_0, s)}$. Since $d_Y(y_n, y_0) < \frac{r}{3}$, there exists a connected component Q_n of x_0 contained in $f_n^{-1}(B(y_0, \frac{2r}{3})) \cap U_1$. We claim that $\overline{Q_n} \subset U_1$.

Indeed, if $x_j \to x$ with $x_j \in Q_n$ we have

$$d_Y(f(x), y_0) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(x_j)) + d_Y(f_n(x_j), y_0) <$$

$$d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(x_j)) + \frac{2}{3}r.$$

Letting $j \to +\infty$ we get

$$d_Y(f(x), y_0) \leq d_Y(f(x), f_n(x)) + \frac{2}{3}r < r.$$

Since U_1 is a connected component it results $x \in U_1$.

Setting $B_0 := B(y_0, \frac{2r}{3})$, consider the local homeomorphisms $f_n : Q_n \to B_0$. Let us check that f_n is a proper map, i.e. $f_n^{-1}(K)$ is compact in Q_n for every compact set K of B_0.

Indeed, consider a sequence $x_h \in f_n^{-1}(K)$. By compactness, by passing to subsequences, we may assume that $x_h \to x$ and $f_n(x) \in K \subset B_0$. If $x \notin Q_n$, since $\overline{Q_n} \subset U_1$ it results $x \in \partial Q_n \cap U_1$. Hence, there exists a small connected neighborhood $x \in U_x \subset U_1$ such that $f_n(U_x) \subset B_0$. Therefore, $U_x \subset f_n^{-1}(B_0) \cap U_1$ contradicting the fact that Q_n is a connected component.

Since we may assume that B_0 is connected, it turns out that (see for instance [8, Th. 1]) f_n is onto. Moreover, by [8, Lemma 3] f_n is a covering map.

Moreover, since we may assume that B_0 is simply connected, we find (see also [8, Th. 2]) that each f_n maps homeomorphically Q_n onto B_0.

As in the previous part of the proof, we find a connected normal neighborhood $x_0 \in U_2 \subset B(x_0, s)$ such that $f(U_2) = B(y_0, \frac{r}{3})$. For every $x \in U := U_1 \cap U_2$ we have

$$d_Y(f_n(x), y_0) \leq d_Y(f_n(x), f(x)) + d_Y(f(x), y_0) < \frac{r}{3} + \frac{r}{3} = \frac{2r}{3}.$$

Therefore, $U \subset f_n^{-1}(B_0) \cap U_1$ which implies that $U \subset Q_n$. Therefore, the restriction of f_n to U is injective for every $n \geq 1$. □

Lemma 3.9. *Let $f_n : X \to Y$ be a sequence of invertible maps such that $f_n \to f$ uniformly. We have the following*

- **(i):** *if f_n^{-1} are equi-Lipschitz, then f is invertible.*
- **(ii):** *if f_n^{-1} uniformly converge to a continuous map g, then f is invertible.*

In the same assumptions of Lemma 3.8, if f_n^{-1} are locally equi-Lipschitz, or f_n^{-1} is locally uniformly convergent, then f is locally invertible.

Proof. Suppose by contradiction that $f(x_1) = f(x_2)$ with $x_1 \neq x_2$. Fixed $\varepsilon > 0$, by uniform convergence and the equi-Lipschitz condition we get

$$d_X(x_1, x_2) = d_X(f_n^{-1}(f_n(x_1)), f_n^{-1}(f_n(x_2))) \leq H d_Y(f_n(x_1), f_n(x_2)) \leq$$

$$H\left(d_Y(f_n(x_1), f(x_1)) + d_Y(f(x_1), f(x_2)) + d_Y(f(x_2), f_n(x_2))\right) \leq 2H\varepsilon,$$

where H is a common Lipschitz constant for f_n^{-1}. By the arbitrariness of ε we get the contradiction $x_1 = x_2$.

If the sequence f_n^{-1} is uniformly convergent, denote by $g_n := f_n^{-1}$. Hence $g_n \to g$ uniformly. We derive the contradiction

$$x_1 = \lim_{n\to+\infty} g_n(f_n(x_1)) = g(f(x_1)) = g(f(x_2)) = \lim_{n\to+\infty} g_n(f_n(x_2)) = x_2.$$

For the local statement, fixed $x_0 \in X$, following the notation of the proof of Lemma 3.8, by reducing the radius r if necessary, we may assume that conditions (i) or (ii) are satisfied on B_0. Then, we may use the same above arguments on the common neighborhood U. □

In the sequel we will provide some more conditions under which the limit of invertible maps is still an invertible map.

For $N = 1$ the quantity $D^- f(t)$ coincide a.e. with the metric derivative $|f'|(t)$ (see [5]). Observe that the length of a Lipschitz curve coincides with the total variation (see [5, Th. 4.1.6]). Moreover, the total variation is a lower semicontinuous functional as the supremum of a family of continuous functionals (see the proof of [5, Th. 4.3.2]).

By using the metric area formula and the l.s.c. of the total variation, using the same reasonings of point (iii) of Section 2, we can state the following

Theorem 3.10. *Let $f_n, f : I \to (Y, d)$ be Lipschitz maps. Suppose that $f_n \to f$ uniformly and f_n be invertible. If f is open then f is invertible.*

Proof. Following the reasonings of point (iii) of Section 2, we get

$$2\mathcal{H}^1(V) \leq \int_V N(y, f, I)\, d\mathcal{H}^1(y) = \int_{f^{-1}(V)} |f'|(t)\, dt \leq \liminf_{n\to+\infty} \int_{f^{-1}(V)} |f_n'|(t)\, dt$$

$$= \liminf_{n\to+\infty} \mathcal{H}^1(f_n(f^{-1}(V))) \leq \mathcal{H}^1(V).$$

□

The above computation is related to the lower semicontinuity of the Mass of metric currents. Actually, every Lipschitz map $f : \Omega \to Y$ induces in a canonical way a 1-dimensional metric current (see [2]). See also [18] for an explicit computation. For $N = 1$ the mass is given just by $\int_I |f'|(t)\, dt$.

In higher dimensions the question is more complicated since in the computation of the mass a volume factor λ appears.

Precisely, given a norm $\|\cdot\|$ on $\mathbb{R}^N$, the volume factor $\lambda_{\|\cdot\|}$ (see [2]) is defined as

$$\lambda_{\|\cdot\|} = \frac{2^N}{\omega_N} \sup \left\{ \frac{\mathcal{L}^N(B_1^{\|\cdot\|})}{\mathcal{L}^N(P)} \ : \ B_1^{\|\cdot\|} \subset P, P \text{ parallelepiped} \right\},$$

where as usual ω_N denotes the volume of the unit sphere in $\mathbb{R}^N$. It is possible to show

$$N^{-\frac{1}{N}} \leq \lambda_{\|\cdot\|} \leq \frac{2^N}{\omega_N}. \tag{5}$$

Introducing the metric differential (see [24])

$$md_f(x)(v) = \lim_{t \to 0} \frac{d(f(x+tv), f(x))}{t}$$

which is defined a.e. and it results a seminorm on $\mathbb{R}^N$ for a.e. $x \in \Omega$, the total mass correspondent to the (invertible) Lipschitz map f (see [3]) is given by

$$\int_\Omega \lambda_{md_f(x)} J(md_f(x)) \ dx, \tag{6}$$

where, for a seminorm s on $\mathbb{R}^N$, $J(s) = \frac{\omega_N}{\mathcal{H}^N(\{x \,:\, s(x) \leq 1\})} = \frac{\omega_N}{\mathcal{H}^N(B_1^s)}$ is the metric Jacobian. The metric Jacobian is also related to the metric area formula ([2, 23, 28, 24]) and for a.e. $x \in \Omega$ it can be also expressed by the volume derivative

$$J(md_f(x)) = \lim_{r \to 0} \frac{\mathcal{H}^N(f(B(x,r)))}{\omega_N r^N}. \tag{7}$$

Another way to introduce the metric Jacobian is by considering the so called *pullback measure* (see [28]). Anyway, for Lipschitz maps all these notions coincide a.e.

3.3. **Lower semicontinuity of metric Jacobians.** Given $A, B \subset Y$ we define the Hausdorff distance

$$d_H(A,B) = \inf_{\varepsilon > 0} \left\{ A \subset B^\varepsilon, B \subset A^\varepsilon \right\}.$$

We say that $A_n \xrightarrow{H}$ iff $d_H(A_n, A) \to 0$. Observe that d_H is actually a semi-distance, while it is a distance on closed sets. We have the following semicontinuity properties

Lemma 3.11. *We have the following*

(Upper semicontinuity): *If $A_n \xrightarrow{H} A$, with A bounded, then*

$$\limsup_{n \to +\infty} \mathcal{H}^N(A_n) \leq \mathcal{H}^N(\overline{A}). \tag{8}$$

(Lower semicontinuity): *Let $K \subset Y$ be closed and bounded. If $K \setminus A_n \xrightarrow{H} K \setminus A$, then*

$$\mathcal{H}^N(K \cap \overset{\circ}{A}) \leq \liminf_{n \to +\infty} \mathcal{H}^N(K \cap A_n). \tag{9}$$

Proof. Let $\varepsilon > 0$. For large n we have $A_n \subset A^\varepsilon$. Hence it results $\limsup_{n\to+\infty} \mathcal{H}^N(A_n) \leq \mathcal{H}^N(A^\varepsilon)$. Since $\mathcal{H}^N$is non-trivial, by continuity properties of the Hausdorff measure we have

$$\lim_{\varepsilon\to 0} \mathcal{H}^N(A^\varepsilon) = \mathcal{H}^N\left(\bigcap_{\varepsilon>0} A^\varepsilon\right) \leq \mathcal{H}^N(\overline{A}).$$

By (8) we have

$$\limsup_{n\to+\infty} \mathcal{H}^N(K \setminus A_n) \leq \mathcal{H}^N(\overline{K \setminus A}) \leq \mathcal{H}^N(K\setminus \overset{\circ}{A}).$$

By additivity of $\mathcal{H}^N$, (9) follows. $\square$

Lemma 3.12. *Let $\Omega \subset \mathbb{R}^N$ be bounded and let $f_n, f : \overline{\Omega} \to Y$ be normal Lipschitz maps such that $f_n \to f$ (locally) uniformly. Moreover, suppose that f is discrete. Then, for every open set $U \subset \Omega$ it results*

$$\mathcal{H}^N(f(U)) \leq \liminf_{n\to+\infty} \mathcal{H}^N(f_n(U)). \tag{10}$$

Proof. Observe that by Lemma 3.1 f is open. Fixed $\varepsilon > 0$, we prove that for large n it results

$$f(\overline{\Omega}) \setminus f_n(U) \subset \left(f(\overline{\Omega}) \setminus f(U)\right)^\varepsilon.$$

If not, we find a sequence $y_n \in f(\overline{\Omega})\setminus f_n(U)$ such that $d(y_n, z) \geq \varepsilon$ for every $z \in f(\overline{\Omega})\setminus f(U)$. Passing to a subsequence, we may suppose $y_n \to y \in f(\overline{\Omega})$. It results $y \in f(U)$. Let $y = f(x_0)$ and let $B_0 \subset\subset U$ be a neighborhood of x_0 such that $f(x) \neq f(x_0)$ for every $x \in \bar{B}_0$. Consider a point z_n realizing

$$\min_{z\in f_n(\overline{B}_0)} d(y_n, z).$$

By the continuity property of balls of Y, we find $z_n \in \partial f_n(B_0)$ such that $z_n \to y$. Since f_n is normal, we get $z_n = f_n(x_n)$, with $x_n \in \partial B_0$. By uniform convergence and by passing to a subsequence if necessary, we get $y = f(x)$ with $x \in \partial B_0$, contradicting the discreteness of f.

By (9) we get

$$\mathcal{H}^N(f(U)) = \mathcal{H}^N(f(\overline{\Omega}) \cap f(U)) \leq \liminf_{n\to+\infty} \mathcal{H}^N\left(f_n(U) \cap f(\overline{\Omega})\right) \leq \liminf_{n\to+\infty} \mathcal{H}^N(f_n(U)).$$

$\square$

Remark 3.13. *Actually, in the proof of the above Lemma it suffices that f is normal and $\partial f_n(B) \to \partial f(B)$.*

Lemma 3.14. *Let $f_n, f : \overline{\Omega} \to Y$ be normal Lipschitz maps such that $f_n \to f$ (locally) uniformly. Moreover, suppose that f is locally invertible. Then*

$$\int_\Omega Jf(x)\ dx \leq \liminf_{n\to+\infty} \int_\Omega Jf_n(x)\ dx,$$

where we denote by $Jf(x) := J(mdf(x))$ the metric Jacobian of f.

Proof. Since Ω can be covered by disjoint balls, we may assume with no loss of generality that Ω is bounded. Since f is locally invertible, by a Vitali covering argument, Ω is a disjoint union of balls B_i on which f is invertible. By using the metric area formula, Lemma 3.12 and Fatou's Lemma we evaluate

$$\int_\Omega Jf(x)\ dx = \sum_i \int_{B_i} Jf(x)\ dx = \sum_i \mathcal{H}^N(f(B_i)) \leq \sum_i \liminf_{n\to+\infty} \mathcal{H}^N(f_n(B_i)) \leq$$

$$\leq \liminf_{n\to+\infty} \sum_i \mathcal{H}^N(f_n(B_i)) \leq \liminf_{n\to+\infty} \sum_i \int_{B_i} Jf_n(x)\ dx = \liminf_{n\to+\infty} \int_\Omega Jf_n(x)\ dx.$$

□

Observe that the Jacobian integral is always u.s.c. (for sequences of invertible maps) on closed (or on sets with boundaries of null measure) sets. Indeed, for such a set C, by the Area Formula and (8) we have

$$\limsup_{n\to+\infty} \int_C Jf_n(x)\ dx = \limsup_{n\to+\infty} \mathcal{H}^N(f_n(C)) \leq$$

$$\mathcal{H}^N(\overline{f(C)}) \leq \mathcal{H}^N(f(\overline{C})) = \mathcal{H}^N(f(C)) \leq \int_C Jf(x)\ dx.$$

In the case $Y = \mathbb{R}^N$, the easiest way to obtain l.s.c. of Jacobians is maybe for equi-Lipschitz sequences. In such case, one can use the weak continuity of Jacobians, see for instance [4]. Also in the metric target case, this is true with no invertibility assumptions on f.

Theorem 3.15. *Let $f_n, f : \overline{\Omega} \to Y$ be normal Lipschitz maps such that $f_n \to f$ (locally) uniformly. Moreover, suppose that $Lip(f_n) \leq M$ (or more generally such that the Jacobians are equi-integrable) and f is discrete. Then*

$$\int_\Omega Jf(x)\ dx \leq \liminf_{n\to+\infty} \int_\Omega Jf_n(x)\ dx.$$

Proof. Since f is Lipschitz, following the arguments of [24] let $\mathcal{D}_r$ be the Borel set on which the metric Jacobian exists and it is a norm. It results, up to a null measure set, $Jf(x) = 0$ for $x \notin \mathcal{D}_r$. By the results of [24], we find a decomposition of $\mathcal{D}_r$ on disjoint compact sets $E_i \subset \Omega$ on which f is invertible. Fixed $\varepsilon > 0$, we also find open sets U_i such that $E_i \subset U_i \subset \Omega$ and $\mathcal{L}^N(U_i \setminus E_i) < \frac{\varepsilon}{2^i}$. We now check the l.s.c. of Jacobians.

$$\int_\Omega Jf(x)\ dx = \int_{\mathcal{D}_r} Jf(x)\ dx = \sum_i \int_{E_i} Jf(x)\ dx = \sum_i \mathcal{H}^N(f(E_i)) \leq \sum_i \mathcal{H}^N(f(U_i)) \leq$$

$$\sum_i \liminf_{n\to+\infty} \mathcal{H}^N(f_n(U_i)) \leq \liminf_{n\to+\infty} \sum_i \mathcal{H}^N(f_n(U_i)) \leq \liminf_{n\to+\infty} \sum_i \int_{U_i} Jf_n(x)\ dx \leq$$

$$\liminf_{n\to+\infty} \int_\Omega Jf_n(x)\ dx + M^N\varepsilon.$$

Letting $\varepsilon \to 0$ the thesis follows. □

In the case $Y = \mathbb{R}^N$ and if $f \in \mathcal{C}^1$ no assumptions on f are actually needed. In fact, in such case we may write $\Omega = U \cup B_f$ where B_f is the branch set of f, namely the set of points where f is not locally invertible. By Local Inversion Theorem we have $B_f \subset Z_f$, where $Z_f := \{x \in \Omega \;:\; f \text{ differentiable at } x, Jf(x) = 0\}$. Moreover, since f is locally invertible on the open set U, we show that $f_n : U \to \mathbb{R}^N$ are open maps for large n.

Indeed, for every $x \in U$, concerning the topological index (see for instance [13, 30] for the basic properties) it results $i(x, f) = \pm 1$. Fix a ball B on which f is invertible. For $x \in B$, by uniform convergence, for large n we have

$$deg(f_n(x), f_n, B) = deg(f(x), f_n, B) = deg(f(x), f, B) = i(x, f) = \pm 1.$$

Let $y = f_n(x) \in f_n(V)$ for an open set $V \subset U$. For $y' \in B(y, r_n)$ in a small ball we get

$$deg(y', f_n, B) = deg(y, f_n, B) = \pm 1.$$

Therefore, there exists $x' \in B$ such that $y' = f_n(x')$. Hence $y' \in B(y, r_n) \subset f_n(B) \subset f_n(V)$.

Hence, Lemma 3.14 applies.

This is also the case of everywhere differentiable maps (see [29]). The same argument also holds for Sobolev maps under topological conditions. Actually, if f is open and discrete, or discrete and sense-preserving, then $B_f \subset Z_f \cup S_f$, where $S_f := \{x \in \Omega \;:\; f \text{ is not differentiable at } x\}$ (see [32]). Analogous results in a metric setting could also related to structure properties of the branch set B_f.

As consequence of the previous results we get the following stability result.

Theorem 3.16. *Let $f_n, f : \Omega \to Y$ (resp. Y be locally pathwise connected and simply connected) be Lipschitz maps such that (resp. locally equi-Lipschitz) $Lip(f_n) \leq M$ (or more generally such that the Jacobians are equi-integrable) and $f_n \to f$ (locally) uniformly. Moreover, suppose that f_n are invertible (resp. local homeomorphisms), $f(\Omega)$ is open (resp. f is open) and f is discrete. Then f is invertible (resp. locally).*

Proof. By Lemma 3.3 f is an open map. Suppose with no loss of generality that Ω is bounded. By using Theorem 3.15 it is sufficient to argue as in Section 2. Actually, suppose that $f(x_1) = f(x_2)$ with $x_1 \neq x_2$. We find small disjoint balls $x_1 \in B_1, x_2 \in B_2$ such that $V = f(B_1) \cap f(B_2)$ is a non-empty open set. We compute

$$2\mathcal{H}^N(V) \leq \int_V N(y, f, \Omega)\; d\mathcal{H}^N(y) = \int_{f^{-1}(V)} Jf(x)\; dx \leq$$

$$\liminf_{n \to +\infty} \int_{f^{-1}(V)} Jf_n(x)\; dx = \liminf_{n \to +\infty} \mathcal{H}^N(f_n(f^{-1}(V))) \leq \mathcal{H}^N(\overline{f(f^{-1}(V))}) \leq \mathcal{H}^N(V),$$

by (8) and since $\mathcal{H}^N(\partial V) = 0$.

For the *local* statement it is sufficient to apply Lemma 3.8. □

Remark 3.17. *It is sufficient to consider f_n injective and normal, or such that $f_n(\Omega)$ is an open set. Of course, in the case $Y \subset \mathbb{R}^n$, all these conditions are redundant by the Invariance Domain Theorem.*

Theorem 3.16 can be compared with the results of [31, 25, 26]. A main point here is that we are not using degree theory.

4. Small contraction maps

Maps satisfying

$$0 < m < D^- f(x) \leq D^+ f(x) < M < +\infty \tag{11}$$

were introduced by John (see for instance [22]). We may obtain some results related to quasi-isometries by using some of the above arguments. Let us mention some preliminary basic facts (see [12, 7, 22]).

Definition 4.1. *A metric space E is said to be C-convex if for every $a, b \in E$ there exists a rectifiable curve γ joining a, b such that* $\text{length}(\gamma) \leq Cd(a, b)$.

It turns out that C-convex spaces are bi-Lipschitz homeomorphic to a length metric space.

Lemma 4.2 (Lemma 2.3 of [12])**.** *Let $M > 0$, $A \subset X$ be a C-convex set and let $f : A \to Y$ be a function such that $D^+ f(x) \leq M$ $\forall x \in A$. Then f is CM-Lipschitz.*

Proposition 4.3. *Let f_n be homeomorphisms such that $D^- f_n \geq m > 0$. If Y is C-convex and $f_n \to f$ uniformly, then f is invertible.*

Proof. By Remark 2.5 and Lemma 3.3 of [19] it follows $D^+ f^{-1}(f(x)) = \frac{1}{D^- f(x)}$. By Lemma 4.2 it follows that the sequence f_n^{-1} is equi-Lipschitz. By using the second part (ii) of Lemma 3.9 the thesis follows. □

Remark 4.4. *By using Lemma 3.8 it is possible to state a correspondent local statement.*

Remark 4.5. *For a.e. $x \in \Omega$ it results $D^- f(x) \geq m \Rightarrow Jf(x) \geq m^N$. Indeed, by the inequality*

$$D^- f(x)|v| \leq mdf_f(x)(v) \leq D^+ f(x)|v|$$

it follows that $Jf \geq m^N$.

In order to get inversion results, we consider the following variant of Ciarlet-Necas condition

$$\text{for open set } U \subset X \ : \ m^N \mathcal{L}^N(U) \leq \mathcal{H}^N(f(U)). \tag{12}$$

By using the area formula, condition (12) is trivially satisfied by invertible maps such that $Jf \geq m^N$.

Theorem 4.6. *Let $f : \Omega \to Y$ be a M-Lipschitz open map satisfying* (12). *If $\frac{M}{m} < \sqrt[N]{2}$, then f is invertible.*

Proof. Following the reasonings of Section 2, by the Metric Area Formula and (12) we obtain the contradiction

$$2(\mathcal{H}^N(V)) \leq \int_{f^{-1}(V)} Jf(x)\, dx \leq M^N \mathcal{L}^N(f^{-1}(V)) \leq \left(\frac{M}{m}\right)^N \mathcal{H}^N(V) < 2\mathcal{H}^N(V). \tag{13}$$

□

This inversion results can be compared with the results of [22, 15, 16, 17]. Actually, by (12) we compute

$$\frac{\mathcal{H}^N(f(B(x,r)))}{\omega_N r^N} \geq m^N.$$

Passing to the limit we obtain $Jf(x) \geq m^N$.

Observe that our metric approach allow to consider a general open set $\Omega \subset \mathbb{R}^N$, while the results of [22, 15, 16, 17] are restricted to particular geometries of the domain of f. In particular, maps satisfying $Jf \geq m^N$ are considered in [17] in the case of local homeomorphisms $f : B \to \mathbb{R}^N$, defined on a ball $B \subset \mathbb{R}^N$. Therefore, the restriction to balls or to locally invertible maps could be not strictly necessary. Another advantage of the volume condition (12) is the stability with respect to uniform convergence.

Theorem 4.7. *Let $f_n : \Omega \to Y$ be M-Lipschitz maps satisfying* (12)*. Suppose that $f_n \to f$ uniformly. If f is open and $\frac{M}{m} < \sqrt[N]{2}$, then f is invertible.*

Proof. Following the computation in (13) we get

$$2(\mathcal{H}^N(V)) \leq \int_{f^{-1}(V)} Jf(x)\ dx \leq M^N \mathcal{L}^N(f^{-1}(V)).$$

Since $m^N \mathcal{L}^N(f^{-1}(V)) \leq \mathcal{H}^N(f_n(f^{-1}(V)))$, recalling (8), passing to the limit we obtain the contradiction

$$2(\mathcal{H}^N(V)) \leq M^N \mathcal{L}^N(f^{-1}(V)) \leq \left(\frac{M}{m}\right)^N \mathcal{H}^N(\overline{f(f^{-1}(V))}) < 2\mathcal{H}^N(V),$$

since $\mathcal{H}^N(\partial V) = 0$. □

Acknowledgments. *The author wishes to thank Nicola Fusco for interesting discussions and precious comments. This research was supported by the 2008 ERC Advanced Grant Project N. 226234* Analytic Techniques for Geometric and Functional Inequalities.

References

[1] E. Acerbi, G. Buttazzo, N. Fusco, Semicontinuity and Relaxation for Integrals Depending on Vector Valued Functions, J. Math. Pure et Appl. **62** (1983), 371-387.
[2] L. Ambrosio, B. Kircheim, Rectifiable sets in metric and Banach spaces, *Math. Ann.* **318** (3) (2000) 527–555.
[3] L. Ambrosio, B. Kircheim, Metric currents, Acta Mathematica **185** no. 1(2000), 1-80.
[4] L. Ambrosio, N. Fusco, D. Pallara, Function of Bounded Variations and Free Discontinbuity Problems, Oxford University Press, New York, 2000.
[5] L. Ambrosio, P. Tilli, *Topics on Analysis in Metric Spaces*, Oxford University Press, 2004.
[6] Erik Barvínek, Ivan Daler and Jan Franku, *Convergence of sequences of inverse functions*, Archivium Mathematicum, Vol. 27 **3-4** (1991), 201–204.
[7] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol.1, AMS, 2000.
[8] Chung-Wu Ho, A note on proper maps, Proceedings AMS, Vol. 51 **1** (1975), 237-241.
[9] P.G. Ciarlet, J. Necas, Injectivity and self-Contact in nonlinear elasticity, Arch. Rational Mech. Anal. **97** (1987), no. 3, 171-188.
[10] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer 2008.

[11] G. De Marco, G. Gorni, G. Zampieri, Global inversion of functions: an introduction, NODEA **1** (1994), 229-248.

[12] E. Durand-Cartagena, J.A. Jaramillo, Pointwise Lipschitz functions on metric spaces, *J. Math. Anal. Appl.* **363** (2010), 525–548.

[13] I. Fonseca, W. Gangbo, *Degree Theory in Analysis and Applications*, Clarendom Press Oxford, 1995.

[14] W. Fulton, Algebraic Topology, Springer, 1995.

[15] J. Gevirtz, Metric conditions that imply local invertibility, *Communications in Pure and Applied Mathematics* **23** (1969), 243–264.

[16] J. Gevirtz, Injectvity in Banach spaces and the Mazur-Ulam Theorem on Isometries, *Transactions AMS* **274**, no. 1 (1982), 307–318.

[17] J. Gevirtz, A sharp condition for univalence in Euclidean spaces, *Proceedings AMS* **57**, no. 2 (1976), 261–265.

[18] L. Granieri, Metric currents and geometry of Wasserstein spaces, Rend. Semin. Mat. Univ. Padova **124** (2010), 91-125.

[19] L. Granieri, F. Maddalena, A metric approach to elastic reformations, Acta Mathematica Applicandae, published online December 2013.

[20] L. Granieri, G. Mola, Sequences of inverse and implicit functions, in preparation.

[21] C. Gutierrez, C. Biasi, Finite Branched Coverings in a Generalized Inverse Mapping Theorem, *Int. Journal of Math. Analysis* **2** no. 4 (2008), 169–179.

[22] F. John, On Quasi-Isometric Mappings I, *Communications in Pure and Applied Mathematics* **21** (1968), 77–110.

[23] M. B. Karmanova, Area and Coarea Formulas for the mappings of Sobolev classes with values in a metric space, *Siberian Mathematical Journal*, Vol. 48, No. 4 (2007), 621–628. *Proceedings of the American Mathematical Society*, **121**, No. 1 (1994), 113–123.

[24] B. Kirchheim, Rectifiable Metric Spaces: Local Structure and Regularity of the Hausdorff Measure, *Proceedings of the American Mathematical Society*, **121**, No. 1 (1994), 113–123.

[25] L. V. Kovalev, Jani Onninen, On invertibility of Sobolev mappings, *Journal für die reine und angewandte Mathematik*, **656**, (2011), 1–16.

[26] L. V. Kovalev, Jani Onninen, Kai Rajala, Invertibility of Sobolev mappings under minimal hypotheses, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), no. 2, 517–528.

[27] A. Lytchak, Open map Theorem for metric spaces, *St. Petersburg Math. J.* **17** (2006), No. 3, 477–491.

[28] V. Magnani, An Area formula in metric spaces, Colloq. Math. **124** (2011), no. 2, 275-283.

[29] J. S. Raymond, Local inversion for differentiable functions and Darboux property, Mathematika **49** (2002), no. 1-2, 141-158.

[30] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion*, AMS, 1989.

[31] M. Seeotharama Gowda, R. Sznajder, Weak connectedness of inverse images of continuus functions, Mathematics of Operations Reaserch **24** (1999), No. 1, 255-261.

[32] R. Van Der Putten, A note on the local invertibility of Sobolev functions, *Math. Scand.* **83** (1998), 255–264.

[33] O. Martio, S. Rickman and J. Vaisala, *Topological and Metric Properties of Quasiregular Mappings*, Annales Academie Scientiarum Fennicae Mathematica, **488** (1971), 1–31.

Dipartimento di Matematica e Applicazioni, Università Federico II di Napoli, Via Cintia, Monte S. Angelo 80126 Napoli, Italy.

E-mail address: `luca.granieri@unina.it, granieriluca@libero.it.`