OPTIMAL PRICING POLICIES FOR PUBLIC TRANSPORTATION NETWORKS

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Abstract. In this paper we study the problem of finding an optimal pricing policy for the use of the public transportation network in a given populated area. The transportation network, modeled by a Borel set $\Sigma \subset \mathbb{R}^n$ of finite length, the densities of the population and of the services (or workplaces), modeled by the respective finite Borel measures φ_0 and φ_1, as well as the effective cost $A(t)$ for a citizen to cover a distance t without the use of the transportation network, are assumed to be given. The pricing policy to be found is then a cost $B(t)$ to cover a distance t with the use of the transportation network (i.e. the "price of the ticket for a distance t"), and has to provide an equilibrium between the needs of the population (hence minimizing the total cost of transportation of the population to the services/workplaces) and that of the owner of the transportation network (hence maximizing the total income of the latter). We present a model for such a choice and discuss the existence as well as some qualitative properties of the resulting optimal pricing policies.

Contents

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1. Introduction

Suppose that a set $\Sigma \subset \Omega$ represent a public transportation network (say, a railway) in a city modeled by a convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. If a path $\theta \subset \Omega$, connecting $x \in \Omega$ to $y \in \Omega$ is interpreted as the itinerary of a passenger moving from x to y, then $\mathcal{H}^1(\theta \setminus \Sigma)$ and $\mathcal{H}^1(\theta \cap \Sigma)$, where $\mathcal{H}^1$ stands for the one-dimensional Hausdorff measure, can be naturally interpreted as the "effective lengths" of the part of the itinerary which the passenger covers, respectively, outside of and along the public transportation network. Let a function A: $\mathbb{R}^+ \to \bar{\mathbb{R}}^+$ represent the effective cost (in terms of time, money and/or fatigue) for a single citizen of travelling without using the public transportation network (i.e. "by own means"), while B: $\mathbb{R}^+ \to \bar{\mathbb{R}}^+$ represent the analogous cost of travelling with the use of such a network (i.e. "by train"). The former clearly characterizes the pattern of behaviour of the population which we consider to be known, while the latter is a pricing policy on the use of the transportation network imposed by its owner. Under this assumption each single citizen moving along a path θ will choose the most convenient distance $L(B,\theta) \leq \mathcal{H}^1(\theta \cap \Sigma)$ to cover with the use of the transportation network (i.e. "by train"), and thus will cover $\mathcal{H}^1(\theta) - L(B,\theta)$ without using the transportation network (i.e. "by own means"). Then the number

$$\delta(B,\theta) := A(\mathcal{H}^1(\theta) - L(B,\theta)) + B(L(B,\theta)),$$

represents the total cost for a citizen of travelling along θ. Suppose the densities of the residents φ_0 and of the services (or workplaces) φ_1 are given. The residents obviously travel to reach the services or workplaces and are free to use or not to use the transportation network. The choice of the way of transportation of the residents can be naturally described by the Monge-Kantorovich optimal transportation model (see e.g. [1, 4, 5, 6] for the detailed presentation of the respective theory).

The problem we study in the present paper is as follows. A company owning the public transportation network Σ would naturally be interested in using it in the most effective way, that is, to earn the maximum profit. The only tool at its disposal, which can be used to accomplish this task, is the pricing policy for the use of the network, i.e. the function B. However, the population reacts on each particular pricing policy B by choosing its own pattern of behaviour, namely, the itineraries and maybe even the destinations of everyday movements. In particular, to maximize the profit, the owner of the public transportation network is tempted to increase B. But one can expect that the higher is the latter, the less people is interested in using the transportation network. For instance, if B is "extremely high" (of course depending on the other factors, such as for instance the cost function A and the size of Ω), nobody will use the public transportation network, and hence such a "greedy" policy would make the company fail. A more modest pricing policy is expected to be therefore optimal. The paper is concerned with the existence and the qualitative properties of such an optimal pricing policy.

2. Notation and preliminaries

In this paper the ambient space $\Omega \subset \mathbb{R}^n$, $n \geq 2$, representing the given city is assumed to be a convex body (i.e. a bounded closed convex set wit nonempty interior) equipped with the Euclidean distance. The convexity assumption on Ω is mainly introduced to simplify the presentation: instead, more general domains Ω can be considered equipped with the geodesic distance relative to Ω, which in case of nonconvex Ω does not coincide with the Euclidean distance. The public transportation network in Ω is assumed to be represented by a Borel set $\Sigma \subset \Omega$ of finite length, i.e. $\mathcal{H}^1(\Sigma) < \infty$.

We call two Lipschitz-continuous paths $\hat{\theta}_1, \hat{\theta}_2$: $[0,1] \to \Omega$ equivalent, if there is a continuous surjective nondecreasing function (called usually "reparameterization") ϕ: $[0,1] \to [0,1]$ such that $\hat{\theta}_1(t) = \hat{\theta}_2(\phi(t))$ for all $t \in [0,1]$. Let then Θ stand for the set of equivalence classes of Lipschitz-continuous paths. In this way each $\theta \in \Theta$ can be clearly identified with some directed rectifiable curve. In the sequel we will frequently slightly abuse the language, identifying the elements of Θ (i.e. directed rectifiable curves) with their parameterizations (i.e. Lipschitz-continuous paths parameterizing such curves), when it cannot lead to a confusion. We consider the set Θ to be equipped with the distance

(2.1)
$$d_\Theta(\theta_1, \theta_2) := \inf\left\{\max_{t\in[0,1]} |\hat{\theta}_1(t) - \hat{\theta}_2(t)| \,:\, \hat{\theta}_i \text{ parameterization of } \theta_i, i = 1,2\right\},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. It is easy to see that $\theta_\nu \to \theta$ in Θ implies the Hausdorff convergence of the respective traces, though the converse is clearly not true.

If $\{\theta_1, \theta_2\} \subset \Theta$ are such that $\theta_1(1) = \theta_2(0)$, then we define $\theta_1 \circ \theta_2 \in \Theta$ the directed rectifiable curve possessing a parameterization

$$\theta_1 \circ \theta_2(t) := \begin{cases} \theta_1(2t), & \text{if } t \in [0,1/2], \\ \theta_2(2t-1), & \text{if } t \in (1/2,1]. \end{cases}$$

It is important to remark that every subset of curves in Θ having uniformly bounded length is compact. In fact, let a sequence $\{\theta_\nu\} \subset \Theta$ satisfy $\mathcal{H}^1(\theta_\nu) \le l$ for some $l \ge 0$ and for all $\nu \in \mathbb{N}$. Then each θ_ν admits a parameterization with Lipschitz constant not exceeding l, and hence by Ascoli-Arzelà theorem $\theta_\nu \to \theta$ in Θ for some $\theta \in \Theta$ and for a subsequence of ν (not relabeled).

For a locally compact metric space X we denote by $\mathcal{M}^+(X)$ the space of positve Radon measures over X equipped with the $*$-weak topology. If $\varphi \in \mathcal{M}^+(X)$, and $\mathscr{B}(X)$ stands for the Borel σ-algebra of X, then by $\mathscr{B}_\varphi(X)$ we denote the completion of the latter with respect ot φ (in other words, $\mathscr{B}_\varphi(X)$ is generated by $\mathscr{B}(X) \cup \varphi^{-1}(\{0\})$, i.e. by Borel sets and φ-nullsets). Then a map f: $X \to Y$ is called φ-measurable, if it is measurable with respect to $\mathscr{B}_\varphi(X)$. If f: $X \to Y$ is a φ-measurable (in particular, Borel measurable) map between locally compact metric spaces X and Y, then we denote by $f_\#\varphi$ the *push-forward* of the measure φ, i.e. the measure defined by

$$(f_\#\varphi)(B) := \varphi(f^{-1}(B))$$

for every Borel $B \subset Y$.

We recall the basic facts about Γ-convergence. For a sequence of functionals f_ν: $X \to \bar{\mathbb{R}}$, $\nu \in \mathbb{N}$, defined over a metric space X, we set

$$\begin{aligned} (\Gamma^- \liminf\nolimits_\nu f_\nu)(x) &:= \inf\{\liminf\nolimits_\nu f_\nu(x_\nu) \,:\, x_\nu \to x\}, \\ (\Gamma^- \limsup\nolimits_\nu f_\nu)(x) &:= \inf\{\limsup\nolimits_\nu f_\nu(x_\nu) \,:\, x_\nu \to x\}, \end{aligned}$$

and we say that this sequence Γ^- converge to a functional f: $X \to \bar{\mathbb{R}}$ as $\nu \to \infty$, written $f := \Gamma^- \lim_\nu f_\nu$, if $f = \Gamma^- \liminf_\nu f_\nu = \Gamma^- \limsup_\nu f_\nu$. If $f_\nu = g$ for all $\nu \in \mathbb{N}$, then the functional $g^- := \Gamma^- \lim_\nu f_\nu$ is called the *relaxation* (or l.s.c.

envelope) of the functional $g\colon X \to \bar{\mathbb{R}}$. Clearly, $\inf_X g = \inf_X g^-$. One also observes that Γ^--convergence is stable with respect to the passage to relaxations, namely,

$$\Gamma^- \lim_\nu f_\nu = \Gamma^- \lim_\nu f_\nu^- .$$

We also define

$$\Gamma^+ \limsup_\nu f_\nu := -\Gamma^- \liminf_\nu (-f_\nu), \qquad \Gamma^+ \liminf_\nu f_\nu := -\Gamma^- \limsup_\nu (-f_\nu),$$

and hence $f := \Gamma^+ \lim_\nu f_\nu$, if $f = \Gamma^+ \liminf_\nu f_\nu = \Gamma^+ \limsup_\nu f_\nu$. It is easy to show that a Γ^--limit (resp. Γ^+-limit) of an arbitrary sequence of functionals is a l.s.c. (resp. u.s.c.) function.

3. Problem setting

3.1. **Behaviour of the population.** In our model the population is characterized by two nonnegative Borel measures φ_0 and φ_1 over Ω, the former modeling the density of the population, the latter the density of the destinations of their everyday movements (say, the density of services or of workplaces), and by the function A: $\mathbb{R}^+ \to \bar{\mathbb{R}}^+$ which is interpreted as the cost per unit mass of moving without the use of public transportation network (i.e. by own means). We request that the total masses of φ_0 and φ_1 be equal, namely,

$$\varphi_0(\Omega) = \varphi_1(\Omega).$$

In the sequel we will always assume that the function A is nondecreasing and continuous. From the applicative point of view, it might be useful to think that $A(0) = 0$ (i.e. the "cost of not moving anywhere" is zero), although technically we will not need such an assumption for the validity of all the results proven in the sequel.

In this section we propose two different models for the choice of the optimal pricing policy B, and consider some of their equivalent formulations. As for the requirements on B, it seems quite natural to request that B be nondecreasing (i.e. the price of a short distance trip should not exceed that of a long distance one) and that $B(0) = 0$ (i.e the owner of the transportation network may only charge effective movements along the network and hence may not charge just "people staying at home"). Below we develop what we think a reasonable model of behaviour of both population and the owner of the transportation network under only the above assumptions on A and B. We will see that in this model there is no way to distinguish between the pricing policy B and its l.s.c. envelope B^- and therefore later on we will be able to assume without loss of generality that B be l.s.c.

3.1.1. *Individual behaviour.* We first consider the question of how each single citizen chooses the distance $L(B, \theta)$ to cover with the use of the transportation network once his itinerary $\theta \in \Theta$ is chosen and supposing that the pricing policy B is a nondecreasing function. Clearly, with $\theta \in \Theta$ fixed, once one chooses to cover the distance l with the use of the transportation network, one pays the cost

$$g(l) := A(\mathcal{H}^1(\theta) - l) + B(l).$$

Therefore, it is reasonable to choose for $L(B, \theta)$ a minimizer of the latter function over $l \in [0, \mathcal{H}^1(\theta \cap \Sigma)]$. Note that the such a minimizer might not exist when B is not l.s.c. However, the l.s.c. envelope g^- of the function g given by the formula

$$g^-(l) = A(\mathcal{H}^1(\theta) - l) + B^-(l),$$

where B^- stands for the l.s.c. envelope of B, attains it minimum value over $[0, \mathcal{H}^1(\theta \cap \Sigma)]$, which coincides with the infimum of g over the same interval. Note that if l minimizes g^- but not g, it means that in l the function B has a jump and $B(l) > \lim_{\xi \to l^-} B(\xi)$. In other words, in this case for a citizen it is convenient to cover along the transportation network any distance $\xi < l$ arbitrarily close to l, but not exactly l, while the cost of transportation along θ will be arbitrarily close to $g^-(l)$. It is natural then to define $L(B, \theta)$ to be one of the minimizers of g^- over $[0, \mathcal{H}^1(\theta \cap \Sigma)]$. It might happen however that a minimizer of g^- is not unique, i.e. covering different distances with the use of the public transportation network along the same path gives an identical cost to the individual. We make therefore the following assumption: every person chooses to cover the minimum possible distance by his own means (or, equivalently, the maximum possible distance with the use of the public transportation network) as long as the total cost for him is the same. Translated in mathematical terms, this leads to the following definition: each person travelling along θ is covering on the transportation network the distance

$$L(B, \theta) := \max\{l \in \operatorname{Argmin}_{\xi \le \mathcal{H}^1(\theta \cap \Sigma)} A(\mathcal{H}^1(\theta) - \xi) + B^-(\xi)\}. \tag{3.1}$$

It is immediate to note that the maximum in the above definition is actually attained and hence the cost of transportation along θ is given by

$$\begin{aligned} \delta(B, \theta) &:= \min_{l \in [0, \mathcal{H}^1(\theta \cap \Sigma)]} (A(\mathcal{H}^1(\theta) - l) + B^-(l)) \\ &= A(\mathcal{H}^1(\theta) - L(B, \theta)) + B^-(L(B, \theta)). \end{aligned} \tag{3.2}$$

It is worth noting that $L(B, \cdot)$ is Borel measurable, and $\delta(B, \cdot)$ is l.s.c. as claimed in Lemma 3.1 below.

Lemma 3.1. *For each nondecreasing function B the function $L(B, \cdot)\colon \Theta \to \mathbb{R}^+$ is Borel measurable and the function $\delta(B, \cdot)\colon \Theta \to \mathbb{R}^+$ is l.s.c.*

Proof. Define

$$\begin{aligned} \alpha(s,t) &:= \max\big\{l \in \operatorname{Argmin}\{A(s-\sigma) + B^-(\sigma) : 0 \le \sigma \le t\}\big\}, \\ \beta(s,t) &:= \min\{A(s-\sigma) + B^-(\sigma) : 0 \le \sigma \le t\}, \end{aligned}$$

so that of course $L(B, \theta) = \alpha(\mathcal{H}^1(\theta), \mathcal{H}^1(\theta \cap \Sigma))$ and $\delta(B, \theta) = \beta\big(\mathcal{H}^1(\theta), \mathcal{H}^1(\theta \cap \Sigma)\big)$. The function β is trivially nondecreasing in s and decreasing in t. On the other hand, the function α is u.s.c. Indeed, take two sequences $s_\nu \to s$ and $t_\nu \to t$ as $\nu \to \infty$ and suppose that $\sigma_\nu := \alpha(s_\nu, t_\nu) \to \bar\sigma$. For $\sigma < \bar\sigma$ one has $\sigma < \sigma_\nu \le t_\nu$ for sufficiently large ν and then by definition $A(s_\nu - \sigma) + B^-(\sigma) \ge A(s_\nu - \sigma_\nu) + B^-(\sigma_\nu)$. Passing to the limit in ν and recalling that A is continuous and B^- l.s.c., one gets $A(s-\sigma) + B^-(\sigma) \ge A(s - \bar\sigma) + B^-(\bar\sigma)$, so that $\alpha(s,t) \ge \bar\sigma$. The thesis now follows because $\theta \mapsto \mathcal{H}^1(\theta)$ is l.s.c. by Golab theorem and $\theta \mapsto \mathcal{H}^1(\theta \cap \Sigma)$ is u.s.c. by Lemma 4.1. □

3.1.2. *Collective behaviour. Transport measures.* Now we would like to study which itineraries are chosen by the people in their everyday movements. These itineraries will be modeled by a special measure $\eta \in \mathcal{M}^+(\Theta)$ which will say, roughly speaking, how many people are choosing each particular path. We will call a measure $\eta \in \mathcal{M}^+(\Theta)$ a *transport measure* (or simply a *transport*), if

$$p_{i\#}\eta = \varphi_i, \qquad i = 0, 1,$$

where the map $p_i\colon \Theta \to \Omega$ is defined by $p_i(\theta) := \theta(i)$, $i = 0, 1$. Define now the functional $C(B)$ over $\mathcal{M}^+(\Theta)$ by the formula

$$C(B)(\eta) := \int_\Theta \delta(B, \theta)\, d\eta(\theta).$$

We will show in Proposition 3.2 that $C(B)$ admits a minimizer η_{opt} over the set of all admissible transports. From purely heuristic point of view, the optimal transport η_{opt} says, roughly speaking, how many people take each particular path θ in their everyday movements. Further on we denote by $E_{opt}(B)$ the set of all optimal transports η_{opt} for a given B, namely,

$$E_{opt}(B) := \operatorname{Argmin}\{C(B)(\eta) \,:\, \eta \text{ admissible transport}\}.$$

3.1.3. *Equivalent formulation. Transport plans.* The problem of minimizing C is in fact just a version of the classical Monge-Kantorovich transport problem. To see this, define the function $d_B\colon \Omega \times \Omega \to \bar{\mathbb{R}}$ by setting

$$d_B(x, y) := \inf\left\{\delta(B, \theta) \,:\, \theta \in \Theta, \theta(0) = x, \theta(1) = y\right\},$$

where $\delta(B, \theta)$ is given by (3.2). The number $d_B(x, y)$ can be naturally interpreted as the cost for a single person of travelling from x to y, once the opportunity of using the public transportation network is offered. Then the everyday movement of the population to their workplaces can be also modeled by a *transport plan* $\gamma \in \mathcal{M}^+(\Omega \times \Omega)$, i.e. a finite Borel measure over $\Omega \times \Omega$ satisfying

$$\pi_{i\#}\gamma = \varphi_i, \qquad i = 0, 1, \tag{3.3}$$

where $\pi_i\colon \Omega \times \Omega \to \Omega$, $i = 0, 1$, stand for the projections on the first and the second factor respectively, namely, $\pi_i(x_0, x_1) := x_i$. The term "transport plan" refers to the fact that in a very heuristic way one can think that $\gamma(x, y)$ stands for the amount of people which moves from point x to point y, and hence represents the "plan of transportation" of the population φ_0 to the destinations φ_1. The population chooses the transport plan so as to minimize over all admissible transport plans (i.e. measures $\gamma \in \mathcal{M}^+(\Omega \times \Omega)$ satisfying (3.3)) the total cost of everyday movement given by the Monge-Kantorovich functional I_B defined by the relationship

$$I_B(\gamma) := \int_{\Omega \times \Omega} d_B(x, y)\, d\gamma(x, y).$$

We will denote by $MK(B)$ the respective infimum of I_B. Namely, we set

$$MK(B) := \inf\left\{I_B(\gamma) \,:\, \gamma \text{ admissible transport plan}\right\}.$$

The latter infimum is attained at some optimal transport plan γ_{opt} since the set of admissible transport plans is obviously compact in $*$-weak topology of measures, while the integrand d_B is l.s.c. (in fact, even continuous), as it will be shown in Proposition 5.3(i). We remark that the optimal transport plan γ_{opt} obviously may depend on B and characterizes, though not completely, the pattern of behaviour of the population once the pricing policy B is known. Roughly speaking γ_{opt} says "who goes where" given the pricing policy B for the use of the public transportation network, though it does not say "along which path". Further on we denote by $\Gamma_{opt}(B)$ the set of all optimal transport plans γ_{opt} for a given B, namely,

$$\Gamma_{opt}(B) := \operatorname{Argmin}\{I_B(\gamma) \,:\, \gamma \text{ admissible transport plan}\}.$$

Define the set of the "optimal paths"

$$\widetilde{\Theta}(B) := \{\theta \in \Theta \,:\, d_B(\theta(0), \theta(1)) = \delta(B, \theta)\}$$

and denote by $Q_B(x,y)$ the set of optimal paths connecting x with y, namely,

$$Q_B(x,y) = \{\theta \in \widetilde{\Theta}(B) \,:\, \theta(0) = x, \theta(1) = y\},$$

so that

$$\widetilde{\Theta}(B) = \bigcup_{(x,y)\in\Omega\times\Omega} Q_B(x,y).$$

A trivial by-product of Proposition 5.3(ii) is the fact that both $\widetilde{\Theta}(B)$ and $Q_B(x,y)$ are closed sets (as for the latter, one can also conclude immediately from Lemma 3.1 that each $Q_B(x,y)$ is closed as a set of minima of a l.s.c. function $\delta(B,\cdot)$ on a closed set).

The assertion below provides a relationship between the Monge-Kantorovich problem of finding an optimal transport plan and the problem of minimizing $C(B)$ over all transports.

Proposition 3.2. *If B is a nondecreasing function one has*

$$MK(B) = \min\{C(B)(\eta) \,:\, \eta \textit{ admissible transport}\}, \tag{3.4}$$

Moreover, if $\eta_{opt} \in E_{opt}(B)$, then $\gamma := (p_0 \times p_1)_{\#}\eta_{opt} \in \Gamma_{opt}(B)$. Vice versa, there is a Borel map $q_B\colon \Omega\times\Omega \to \widetilde{\Theta}(B)$ such that whenever $\gamma_{opt} \in \Gamma_{opt}(B)$, then $\eta := q_{B\#}\gamma_{opt} \in E_{opt}(B)$.

Finally, every $\eta_{opt} \in E_{opt}(B)$ is concentrated over $\widetilde{\Theta}(B)$, i.e. $\eta_{opt}(\Theta\setminus\widetilde{\Theta}(B)) = 0$.

Proof. If η is an admissible transport, then $\gamma := (p_0 \times p_1)_{\#}\eta_{opt}$ is clearly an admissible transport plan. Hence, minding the definition (3.2) of d_B, we have

$$\begin{aligned} C(B)(\eta) &= \int_\Theta \delta(B,\theta)\,d\eta \quad \geq \int_\Theta d_B((p_0\times p_1)(\theta))\,d\eta_{opt} \\ &= \int_{\Omega\times\Omega} d_B(x,y)\,d\gamma \;=\; I_B(\gamma) \;\geq\; MK(B). \end{aligned} \tag{3.5}$$

On the other hand, there is a Borel measurable selector $q_B\colon \Omega\times\Omega \to \widetilde{\Theta}(B)$ of the multivalued map $Q_B\colon \Omega\times\Omega \multimap \widetilde{\Theta}(B)$. To verify this claim, it suffices to observe that the graph of Q_B defined by

$$\begin{aligned} \text{Graph } Q_B &:= \{(x,y,\theta) \,:\, (x,y) \in \Omega\times\Omega, \theta \in Q_B(x,y)\} \\ &= \{(x,y,\theta) \in \Omega\times\Omega\times\Theta \,:\, x = \theta(0), y = \theta(1)\} \cap (\Omega\times\Omega\times\widetilde{\Theta}(B)) \end{aligned}$$

is closed because $\widetilde{\Theta}(B)$ is closed (in view of Proposition 5.3(ii)), while $Q_B(x,y)$ is nonempty for each $(x,y) \in \Omega\times\Omega$, and refer to the Kuratowski-Ryll-Nardzewski measurable selection theorem (theorem III.6 from [3] or theorem 5.2.1 from [7]). Then, given an arbitrary optimal transport plan $\gamma_{opt} \in \Gamma_{opt}(B)$ we observe that $\eta_{opt} := q_{B\#}\gamma_{opt}$ is an admissible transport. Moreover,

$$\begin{aligned} MK(B) &= \int_{\Omega\times\Omega} d_B(x,y)\,d\gamma_{opt} \;=\; \int_{\Omega\times\Omega} \delta(B, q_B(x,y))\,d\gamma_{opt} \\ &= \int_\Theta \delta(B,\theta)\,d\eta_{opt} \;=\; C(B)(\eta_{opt}). \;\geq\; \inf C(B), \end{aligned} \tag{3.6}$$

where the infimum of $C(B)$ is taken over the set of all admissible transports. Together with (3.5) this proves that η_{opt} is an optimal trasnport and hence also the validity of the the first claim of the statement being proven. From (3.5) η optimal trasport, we get that $(p_0 \times p_1)_{\#}\eta_{opt}$ is an optimal transport plan. Finally, since in (3.5) all the inequalities are actually equalities when η is an optimal trasport, we get the validity of the last claim. $\Box$

We now introduce another auxiliary construction to be used in the sequel. Define

$$\lambda := \operatorname{diam}\Omega + \mathcal{H}^1(\Sigma) \text{ and } \Theta' := \{\theta \in \Theta \, : \, \mathcal{H}^1(\theta) \leq \lambda\}, \tag{3.7}$$

so that clearly $\Theta' \subset \Theta$ is compact. We prove now the following very easy estimates.

Lemma 3.3. *For every $(x, y) \in \Omega \times \Omega$ one has*

$$d_B(x, y) \leq A(\operatorname{diam}\Omega).$$

Further, $\theta \in \widetilde{\Theta}(B) \setminus \Theta'$, then $B^-(L(B, \theta)) = 0$.

Proof. Let $[x, y] \in \Theta$ stand for the line segment with endpoints $x \in \Omega$ and $y \in \Omega$. One has therefore

$$d_B(x, y) \leq \delta(B, [x, y]) \leq A(\mathcal{H}^1([x, y])) = A(|x - y|) \leq A(\operatorname{diam}\Omega),$$

since $|x - y| \leq \operatorname{diam}\Omega$ and A is nondecreasing. Hence, the first claim is proven.

For $\theta \in \widetilde{\Theta}(B)$ we just proved

$$\delta(B, \theta) = A(\mathcal{H}^1(\theta) - L(B, \theta)) + B^-(L(B, \theta)) \leq A(\operatorname{diam}\Omega).$$

If, moreover, $\mathcal{H}^1(\theta) \geq \lambda := \operatorname{diam}\Omega + \mathcal{H}^1(\Sigma)$, then minding $L(B, \theta) \leq \mathcal{H}^1(\theta \cap \Sigma) \leq \mathcal{H}^1(\Sigma)$, we get

$$A(\mathcal{H}^1(\theta) - L(B, \theta)) \geq A(\lambda - \mathcal{H}^1(\Sigma)) = A(\operatorname{diam}\Omega),$$

and hence $B^-(L(B, \theta)) = 0$ concluding the proof. $\Box$

Remark. If either A is strictly increasing, or unbounded (i.e. $A(l) \to \infty$ as $l \to +\infty$), then it easily follows from Lemma 3.3 that all the paths in $\widetilde{\Theta}(B)$ must have uniformly bounded length (independent of B). In fact, for all $\theta \in \widetilde{\Theta}(B)$ one has $\delta(B, \theta) = A(\mathcal{H}^1(\theta) - L(B, \theta)) + B^-(L(B, \theta)) \leq A(\operatorname{diam}\Omega)$, hence

$$A(\mathcal{H}^1(\theta) - \mathcal{H}^1(\Sigma)) \leq A(\mathcal{H}^1(\theta) - L(B, \theta)) \leq A(\operatorname{diam}\Omega),$$

which implies that $\mathcal{H}^1(\theta) \leq l$ for some $l > 0$ independent of both θ and B.

3.2. **Income of the owner of the transportation network.** We have to calculate now how much each single person pays to the owner of the public transportation network.

Consider first the case when one knows only the transport plan γ. We assume then that people moving from x to y choose an optimal path $\theta \in \widetilde{\Theta}(B)$ connecting those two points. One needs then to calculate how much of this path is actually covered with the use of the public transportation network. It can happen however, that there are many optimal paths connecting the same couple of points. We assume then in line with the assumption made when defining L that every person chooses the maxmum possible distance to cover with the use of the transportation network as long as the total cost for him is the same, and hence covers the distance

$$\Lambda(B, x, y) := \sup_{\theta \in Q_B(x,y)} L(B, \theta) \tag{3.8}$$

along the transportation network. Under this assumption the income of the owner of the transportation network becomes

$$\hat{G}(B,\gamma) := \int_{\Omega\times\Omega} B^-(\Lambda(B,x,y))\,d\gamma(x,y).$$

Note that we integrate $B^-(\Lambda)$ rather than $B(\Lambda)$ minding that, if B is not l.s.c., for people it might be more convenient to use the transportation network for the distance just slightly lower than Λ.

Since $\Lambda(B,\cdot,\cdot)$ is a Borel function as we show below, then the above integral is well defined.

Lemma 3.4. *For each nondecreasing function B the function $\Lambda(B,\cdot,\cdot)\colon \Omega\times\Omega \to \mathbb{R}^+$ is Borel measurable.*

Proof. We have

$$\Lambda(B,x,y) = \sup_{k\in\mathbb{N}} \Lambda^k(B,x,y), \text{ where } \Lambda^k(B,x,y) := \sup_{\theta\in Q_B(x,y),\mathcal{H}^1(\theta)\le k} L(B,\theta).$$

Let $(x_\nu,y_\nu)\to(x,y)$ in $\Omega\times\Omega$ as $\nu\to\infty$, and let $\theta_\nu\in Q_B(x_\nu,y_\nu)$ be such that

$$L(B,\theta_\nu)\ge \Lambda^k(B,x_\nu,y_\nu)-1/\nu \text{ and } \mathcal{H}^1(\theta_\nu)\le k$$

for all $\nu\in\mathbb{N}$. Then up to a subsequence (not relabeled) $\theta_\nu\to\theta$ as $\nu\to\infty$ and $\theta\in Q_B(x,y)$ according to Proposition 5.3(ii). Therefore, minding that $L(B,\cdot)$ is u.s.c. on $\widetilde{\Theta}(B)$ as a consequence of Lemma 5.7, we get

$$\Lambda^k(B,x,y)\ge L(B,\theta)\ge \limsup_\nu L(B,\theta_\nu)\ge \limsup_\nu \Lambda^k(B,x_\nu,y_\nu).$$

This means that each $\Lambda^k(B,\cdot,\cdot)$ is u.s.c., which suffices to conclude the proof. □

Remark. If either A is strictly increasing, or unbounded, then by the remark following Lemma 3.3 one has that all paths in $\widetilde{\Theta}(B)$ have uniformly bounded length. Hence, using the notation of the above proof, $\Lambda=\Lambda^k$ for some $k\in\mathbb{N}$ sufficiently large, and hence $\Lambda(B,\cdot,\cdot)$ is u.s.c. Since in this case each $Q_B(x,y)$ is compact, then in fact the supremum in the definition (3.8) of Λ is attained (i.e. is a maximum).

Now consider the more particular case when the movement of the population is described by some optimal transport $\eta\in E_{opt}(B)$. Then the income of the owner of the transportation network can be expressed as

$$\hat{F}(B,\eta) := \int_\Theta B^-(L(B,\theta))\,d\eta(\theta)$$

(again, we integrate $B^-(L)$ rather than $B(L)$, because if B is not l.s.c., for people it might be more convenient to use the transportation network for a distance just slightly lower than L).

3.3. **How to choose an optimal pricing policy.** Below we assume the public transportation network Σ and the function A (the cost of moving "by own means") to be given, and will be interested in the optimal choice of B. We note that all the functionals introduced actually depend on B^- rather than on B. Thus, further on we will define the class $\mathcal{B}$ of admissible pricing policies $B\colon \mathbb{R}^+\to\mathbb{R}^+$ which are the nondecreasing l.s.c. functions satisfying $B(0)=0$.

The choice of the optimality criterium for B leads to two different models.

3.3.1. *Short term optimality.* The first and simplest model suggests to choose a B as a maximizer of $\hat{G}(\cdot,\gamma)$ once the transport plan γ describing the everyday movement of the population is known. Namely, the problem reads as follows.

Problem 1. *Given a transport plan γ, find a $B \in \mathcal{B}$ maximizing $\hat{G}(\cdot,\gamma)$ over $\mathcal{B}$.*

Such a model refers to the situation when the pattern of behaviour of the population in terms of "who goes where" is independent on the choice of the pricing policy B and therefore is simply initially fixed. It reasonably suits the company which is planning a short term profit: within a short period of time the movements of the people are unlikely to change with the change of the pricing policy for the use of the transportation network. We remark that in such a case, though the sources and desinations of movement of the population (given by γ) are known and independent of the pricing policy B, the actual itineraries (described by a transport η) might depend on B.

The statement below provides an equivalent formulation of Problem 1.

Proposition 3.5. *One has*

$$\begin{aligned}&\sup\{\hat{G}(B,\gamma)\,:\,B\in\mathcal{B}\} = \\ &\qquad \sup\left\{\hat{F}(B,\eta)\,:\,B\in\mathcal{B}, \eta\in \operatorname{Argmin}\{C(B)(\eta')\,:\,(p_0\times p_1)_{\#}\eta'=\gamma\}\right\}.\end{aligned}$$

Moreover, if one of the above suprema is attained in $B \in \mathcal{B}$, then so is also the other one.

Proof. Let $\eta \in \operatorname{Argmin}\{C(B)(\eta')\,:\,(p_0\times p_1)_{\#}\eta'=\gamma\}$. Then η is concentrated on $\widetilde{\Theta}(B)$. In fact, otherwise, minding that $\delta(B,\theta) > d_B((p_0\times p_1)(\theta))$ over $\Theta\setminus\widetilde{\Theta}(B)$, we get $C(\eta) > I_B(\gamma)$ by (3.5). But by (3.6) we have $I_B(\gamma) = C(q_{B\#}\gamma)$, where q_B is defined as in Proposition 3.2. Since by construction $(p_0\times p_1)_{\#}(q_{B\#}\gamma)=\gamma$, this contradicts the minimality of η.

Since η is η is concentrated in $\tilde{\Theta}(B)$, and minding $(p_0\times p_1)_{\#}\eta=\gamma$, one has

$$\begin{aligned}\hat{F}(B,\eta) &= \int_\Theta B(L(B,\theta))\,d\eta(\theta)\\ &\le \int_\Theta B(\Lambda(B,\theta(0),\theta(1)))\,d\eta(\theta)\\ &= \int_{\Omega\times\Omega} B(\Lambda(B,x,y))\,d\gamma(x,y) \quad = \quad \hat{G}(B,\gamma).\end{aligned}\tag{3.9}$$

On the other hand, define $\widehat{\Theta}(B)\subset\widetilde{\Theta}(B)$ and $\widehat{Q}_B(x,y)\subset Q_B(x,y)$ as follows:

$$\begin{aligned}\widehat{\Theta}(B) &:= \{\theta\in\widetilde{\Theta}(B)\,:\,L(B,\theta)=\Lambda(B,\theta(0),\theta(1))\},\\ \widehat{Q}_B(x,y) &:= \{\theta\in\widehat{\Theta}(B)\,:\,\theta(0)=x,\theta(1)=y\}.\end{aligned}$$

We observe that $\widehat{\Theta}(B)$ is the set where the u.s.c. function $\theta\in\widetilde{\Theta}(B)\mapsto L(B,\theta)$ equals the Borel function $\theta\in\widetilde{\Theta}(B)\mapsto\Lambda(B,\theta(0),\theta(1))$. Hence, $\widehat{\Theta}(B)$ is a Borel set. Then

$$\begin{aligned}\text{Graph}\ \widehat{Q}_B &:= \left\{(x,y,\theta)\,:\,(x,y)\in\Omega\times\Omega,\theta\in\widehat{Q}_B(x,y)\right\}\\ &= \{(x,y,\theta)\,:\,\theta(0)=x,\theta(1)=y\}\cap(\Omega\times\Omega\times\widehat{\Theta}(B))\end{aligned}$$

is also a Borel set since the set $\{(x,y,\theta)\,:\,\theta(0)=x,\theta(1)=y\}$ is closed. By the von Neumann-Aumann measurable selection theorem (theorems III.222 and III.23

of [3], or, equivalently, corollary 5.5.8 from [7]) one has

$$\Delta := \{(x,y) \in \Omega \times \Omega \,:\, \widehat{Q}_B(x,y) \neq \emptyset\} \in \mathscr{B}_\gamma,$$

and, moreover, there is a γ-measurable (but not necessarily Borel measurable) selection $\hat{q}_{B,\gamma}\colon \Delta \to \widehat{\Theta}(B)$.

We show now that for every couple $(x,y) \notin \Delta$ one has

$$B(\Lambda(B,x,y)) = 0. \tag{3.10}$$

In fact, if $\{\theta_\nu\} \subset Q_B(x,y)$ is a maximizing sequence for $L(B,\cdot)$ (i.e. $L(B,\theta_\nu) \uparrow \Lambda(B,x,y)$ as $\nu \to \infty$), then necessarily $\mathcal{H}^1(\theta_\nu) \geq \lambda$ for all sufficiently large ν: otherwise, up to a subsequence, one would have $\theta_\nu \to \theta$ as $\nu \to \infty$, and due to the fact that $L(B,\cdot)$ is u.s.c. on $\widetilde{\Theta}(B)$ one would have $L(B,\theta) = \Lambda(B,x,y)$ contrary to the assumtpion $(x,y) \notin \Delta$. By Lemma 3.3 for all sufficiently large ν, we have $B(L(B,\theta_\nu)) = 0$. Minding that B is l.s.c. and nondecreasing, we get therefore (3.10).

Consider now a $\hat{\theta} \in \Theta$ such that $\mathcal{H}^1(\hat{\theta}) > \lambda$. For every $(x,y) \in \Omega \times \Omega \setminus \Delta$ we set

$$\hat{q}_{B,\gamma}(x,y) := q_B(x,y),$$

where $q_B\colon \Omega \times \Omega \to \widetilde{\Theta}(B)$ is defined in Proposition 3.2. Clearly, for such (x,y) according to (3.10) one has

$$B(L(B,\hat{q}_{B,\gamma}(x,y))) \geq 0 = B(\Lambda(B,x,y)).$$

Hence, due to the definition of $\hat{q}_{B,\gamma}$, the above relationship is satisfied for all $(x,y) \in \Omega \times \Omega$.

Minding that $\hat{q}_{B,\gamma}$ is clearly γ-measurable, we define $\eta_\gamma := (\hat{q}_{B,\gamma})_\# \gamma$, getting then

$$\begin{aligned} \hat{F}(B,\eta_\gamma) &= \int_\Theta B(L(B,\theta))\, d\eta_\gamma(\theta) \\ &= \int_{\Omega\times\Omega} B(L(B,\hat{q}_{B,\gamma}(x,y)))\, d\gamma(x,y) \\ &\geq \int_{\Omega\times\Omega} B(\Lambda(B,x,y))\, d\gamma(x,y) \qquad = \quad \hat{G}(B,\gamma). \end{aligned}$$

Together with (3.9) this concludes the proof. □

3.3.2. *Long term optimality.* Another and more complex problem refers to a quite different situation when the pattern of behaviour of the population (i.e. the transport plan γ) is a priori unknown and is chosen by the population so as to minimize its own travel costs. In other words, γ is chosen among optimal transport plans for the given pricing policy B, i.e. $\gamma \in \Gamma_{opt}(B)$, or, equivalently, the transport η is chosen among optimal transports for B, i.e. $\eta \in E_{opt}(B)$. To state formally the respective problem, we define

$$F(B,\eta) := \begin{cases} \hat{F}(B,\eta), & \text{if } \eta \in E_{opt}(B), \\ -\infty, & \text{otherwise.} \end{cases}$$

The problem reads now as follows.

Problem 2. *Find a $B \in \mathcal{B}$ such that for some transport η the pair (B,η) maximizes F among all functions $B \in \mathcal{B}$ and admissible transports.*

This problem suits well a company looking for a long term profit. In fact, in a long term prospective, the directions of the population movements are adjusted to the pricing policy on the use of the transportation network.

The latter problem admits clearly an equivalent formulation. Namely, set

$$G(B,\gamma) := \begin{cases} \hat{G}(B,\gamma), & \text{if } \gamma \in \Gamma_{opt}(B), \\ -\infty, & \text{otherwise.} \end{cases}$$

Problem 2 is then equivalent to that of finding a B: $\mathbb{R}^+ \to \bar{\mathbb{R}}^+$ maximizing $G(B,\gamma)$ among all functions $B \in \mathcal{B}$, where γ varies among all admissible transport plans, as stated in the following assertion.

Proposition 3.6. *One has*

$$\sup F = \sup G.$$

Moreover, if $(B',\eta') \in \operatorname{Argmax} F$, *then setting* $\gamma' := (p_0 \times p_1)_\# \eta'$, *one has* $(B',\gamma') \in \operatorname{Argmax} G$. *Vice versa, if* $(B',\gamma') \in \operatorname{Argmax} G$, *then there is an admissible transport* η', *such that* $(B',\eta') \in \operatorname{Argmax} F$.

Proof. With the use of Proposition 3.5, we get from (3.9) that whenever $\eta \in E_{opt}(B)$, we have $\hat{F}(B,\eta) \leq \hat{G}(B,(p_0 \times p_1)_\# \eta)$ and hence, minding that $(p_0 \times p_1)_\# \eta) \in \Gamma_{opt}(B)$ by Proposition 3.2, we get

$$\hat{F}(B,\eta) \leq \sup\{\hat{G}(B,\gamma)\ :\ \gamma \in \Gamma_{opt}(B)\}. \tag{3.11}$$

On the other hand, for a transport plan γ, the proof of Proposition 3.5 provides the existence of an admissible transport η_γ such that

$$\hat{F}(B,\eta_\gamma) \geq \hat{G}(B,\gamma). \tag{3.12}$$

It is immediate to verify by construction of η_γ that $\eta_\gamma \in E_{opt}(B)$ whenever $\gamma \in \Gamma_{opt}(B)$, because the map $\hat{q}_{B,\gamma}$ is a selector of the multivalued map Q_B. Now, (3.11) and (3.12) together show the statement. ☐

3.4. **Game theoretic interpretation.** Problem 2 admits also an obvious game theoretic interpretation. Namely, consider a game between two players: the "population" P (users of the transportation network) and the "service provider" S (the owner of the network). The set of strategies of the former is the set of admissible transport plans γ (equivalently, the set of all admissible transports η), while the set of strategies of the latter is $\mathcal{B}$. The payoff function of the population K_P is given by the formula

$$K_P(B,\gamma) := -I_B(\gamma)$$

or, equivalently,

$$K_P(B,\eta) := -C(B)(\eta)$$

while the payoff function of the service provider K_S is given by the formula

$$K_S(B,\gamma) := \int_{\Omega\times\Omega} B(\Lambda(B,x,y))\, d\gamma(x,y),$$

or, equivalently,

$$K_S(B,\eta) := \int_\Theta B(L(B,\theta))\, d\eta(\theta).$$

It is immediate to observe that when (B,γ) is a maximizing pair of the functional G (equivalently, (B,η) is a maximizing pair of the functional F), so that B solves Problem 2, then (B,γ) (equivalently, (B,η)) is a Nash equilibrium point of the above game, moreover, it is such an equilibrium point which is "most convenient" for the owner of the transportation network.

4. Auxiliary lemmata

In this section we collect a couple of auxiliary statements to be used in the sequel. We start with the following simple assertion.

Lemma 4.1. *Let $\mu \in \mathcal{M}^+(\Omega)$ be a finite Borel measure, and for the sequence of compact sets $\{K_\nu\}$, $K_\nu \subset \Omega$ one has $K_\nu \to K$ in the sense of Hausdorff for some $K \subset \Omega$. Then $\mu(K) \geq \limsup_\nu \mu(K_\nu)$. In particular, if $\Sigma \subset \Omega$ is a Borel set satisfying $\mathcal{H}^1(\Sigma) < \infty$, then putting $\mu := \mathcal{H}^1 \llcorner \Sigma$, one obtains*

$$\mathcal{H}^1(K \cap \Sigma) \geq \limsup_\nu \mathcal{H}^1(K_\nu \cap \Sigma)$$

Proof. For a $\varepsilon > 0$ let $\bar{U}_\varepsilon$ stand for the closed ε-neighborhood of K. One has then $K_\nu \subset U_\varepsilon$ for all sufficiently large ν, and hence

$$\limsup \mu(K_\nu) \leq \mu(\bar{U}_\varepsilon).$$

Passing to a limit in the above relationship as $\varepsilon \to 0^+$ and minding that in this case $\mu(\bar{U}_\varepsilon) \to \mu(K)$, one gets the desired result. □

We will also need the following general property of Γ-convergence, which is provided in a slightly weaker form by theorem 7.2 from [2].

Proposition 4.2. *Let f_ν, $f \colon X \to \bar{\mathbb{R}}$ be functionals defined over a metric space X, and suppose that $\inf_X f_\nu = \inf_{X'} f_\nu$ for all $\nu \in \mathbb{N}$ and for some compact $X' \subset X$. Then the following assertions hold.*

(i) *If $f(x) \leq \liminf_\nu f_\nu(x_\nu)$ whenever $\{x_\nu\} \subset X'$, $x_\nu \to x$ in X' as $\nu \to \infty$, then*

$$\inf_X f \leq \liminf_\nu \inf_X f_\nu.$$

(ii) *If, moreover, for every $x \in X'$ there is a sequence $\{\tilde{x}_\nu\} \subset X$ such that $\limsup_\nu f_\nu(\tilde{x}_\nu) \leq f(x)$ (in particular, this holds when $\Gamma^- \limsup_\nu f_\nu \leq f$), then $\inf_X f_\nu \to \inf_X f$ as $\nu \to \infty$ and whenever $x_\nu \in X'$ are such that*

$$f_\nu(x_\nu) \leq \inf_X f_\nu + \varepsilon_\nu, \tag{4.1}$$

where $\varepsilon_\nu \to 0$ as $\nu \to \infty$ (in particular, if $x_\nu \in \operatorname{Argmin} f_\nu$), and $x_\nu \to x$ for an $x \in X$, then $x \in \operatorname{Argmin} f$, i.e. $f(x) = \inf_X f$.

Remark. If instead of the assumption in claim (i) one has the stronger property $f \leq \Gamma^- \liminf_\nu f_\nu$, then the proof shows that in fact (ii) is valid for every convergent sequence $\{x_\nu\} \subset X$ (i.e. not just for sequences in X') satisfying (4.1).

5. Existence of solutions

The aim of this section is to show that the statements of the problems 2 and 1 make sense and that both problems admit solutions. Namely, we prove the following result.

Theorem 5.1. *There is a function $B \in \mathcal{B}$ which solves Problem 2 (resp. Problem 1 with given transport plan γ).*

To prove the above theorem, we need several auxiliary statements. Our main tool will be the following Γ-convergence result.

Proposition 5.2. *Assume that for a sequence of nondecreasing functions $\{B_\nu\}$ one has $\Gamma^- \lim_\nu B_\nu = B$. Then the following assertions hold.*

(i) *$\delta(B,\theta) \leq \liminf_\nu \delta(B_\nu,\theta_\nu)$ whenever $\theta_\nu \to \theta$ in Θ;*

(ii) *Let $\Theta' \subset \Theta$ be an arbitrary subset of Θ of paths having uniformly bounded length. Then, for every $\theta \in \Theta'$ and for a subsequence of ν (not relabeled) one has*

$$\delta(B_\nu,\theta) \leq \delta(B,\theta) + \varepsilon_\nu$$

for some sequence $\varepsilon_\nu \to 0$ independent of θ.

In particular,

$$\Gamma^- \lim_\nu \delta(B_\nu,\cdot) = \delta(B,\cdot). \tag{5.1}$$

Furthermore, one has that

(iii) *for every $\theta \in \Theta$, when $x_\nu \to \theta(0)$ and $y_\nu \to \theta(1)$ in Ω, there is a sequence $\{\hat{\theta}_\nu\} \subset \Theta$, such that $\hat{\theta}_\nu(0) = x_\nu$, $\hat{\theta}_\nu(1) = y_\nu$, $\hat{\theta}_\nu \to \theta$ and $\delta(B,\theta) \geq \limsup_\nu \delta(B_\nu,\hat{\theta}_\nu)$.*

Proof. The relationship (5.1) follows immediately from (i) and (ii), since (ii) implies $\delta(B,\theta) \geq \limsup_\nu \delta(B_\nu,\tilde{\theta}_\nu)$.

To prove (i), assume $\theta_\nu \to \theta$ in Θ. Consider any subsequence of θ_ν (not relabeled) such that the sequence $l_\nu := L(B_\nu,\theta_\nu)$ is convergent, and let $l := \lim_\nu l_\nu$. Since $l_\nu \leq \mathcal{H}^1(\theta_\nu \cap \Sigma)$, we get

$$0 \leq l \leq \mathcal{H}^1(\theta \cap \Sigma)$$

in view of Lemma 4.1. On the other hand, minding that A is continuous and nondecreasing, $B(l) \leq \liminf_\nu B_\nu(l_\nu)$ (by Γ^--convergence assumption) and $\mathcal{H}^1(\theta) \leq \liminf_\nu \mathcal{H}^1(\theta_\nu)$ (by Golab theorem), we obtain

$$\begin{array}{rcl} \delta(B,\theta) & \leq & A(\mathcal{H}^1(\theta) - l) + B(l) \\ & \leq & A(\liminf_\nu(\mathcal{H}^1(\theta_\nu) - l_\nu)) + \liminf_\nu B_\nu(l_\nu) \\ & \leq & \liminf_\nu(A(\mathcal{H}^1(\theta_\nu) - l_\nu) + B_\nu(l_\nu)) \\ & = & \liminf_\nu \delta(B_\nu,\theta_\nu). \end{array}$$

To prove (ii), choose an $\varepsilon > 0$ and cover $[0,\mathcal{H}^1(\Sigma)]$ with a finite number of disjoint intervals of the form $I_i := (\bar{l}_i - \delta_i, \bar{l}_i]$, with $\delta_i \leq \varepsilon$ (possibly zero) sufficiently small such that $B(l) \geq B(\bar{l}_i) - \varepsilon/2$ for all $l \in I_i$. Such a cover exists since B is nondecreasing and l.s.c., while the interval $[0,\mathcal{H}^1(\Sigma)]$ is compact. By definition of Γ^--convergence, for each $\bar{l}_i$ there is a sequence $l_{i,\nu} \to \bar{l}_i$ such that $B_\nu(l_{i,\nu}) \to B(\bar{l}_i)$, so that $B_\nu(l_{i,\nu}) \leq B(\bar{l}_i) + \varepsilon/2$ for all sufficiently large ν, and hence

$$B_\nu(l_{i,\nu}) \leq B(l) + \varepsilon$$

for all $l \in I_i$. By the finiteness of the intervals I_i, there is a unique $\nu = \nu(\varepsilon)$ such that the above relationship is true for all i. Define the function $l'_\nu\colon [0,\mathcal{H}^1(\Sigma)] \to [0,\mathcal{H}^1(\Sigma)]$ associating to any $l \in I_i$ the number $l_{i,\nu} \wedge l$. One has then $B_\nu(l'_\nu) \leq B(l) + \varepsilon$ and $0 \leq l - l'_\nu \leq \bar{l}_i - l_{i,\nu}$. Thus, possibly up to passing to a (not relabeled) subsequence of ν, we get a sequence of functions $l'_\nu\colon [0,\mathcal{H}^1(\Sigma)] \to [0,\mathcal{H}^1(\Sigma)]$ such that

$$B_\nu(l'_\nu) \leq B(l) + 1/\nu, \qquad l'_\nu \leq l, \qquad l - l'_\nu \leq 1/\nu. \tag{5.2}$$

Now, for any $\theta \in \Theta'$ we write $l_\nu(\theta) := l'_\nu(L(B,\theta))$, and since $l'_\nu \le L(B,\theta) \le \mathcal{H}^1(\theta \cap \Sigma)$ for every ν, from (5.2) we get

$$A(\mathcal{H}^1(\theta) - l_\nu) + B_\nu(l_\nu) \le A(\mathcal{H}^1(\theta) - l_\nu) + B(L(B,\theta)) + 1/\nu, \tag{5.3}$$

where $l_\nu := l_\nu(\theta)$.

By the uniform continuity of A on the bounded interval $[0, \max_{\theta\in\Theta'} \mathcal{H}^1(\theta)]$, we know that $A(\mathcal{H}^1(\theta) - l_\nu) \le A(\mathcal{H}^1(\theta) - L(B,\theta)) + \tilde{\varepsilon}_\nu$ with $\tilde{\varepsilon}_\nu \to 0$ as $\nu \to \infty$ and $\tilde{\varepsilon}_\nu$ not depending on $\theta \in \Theta'$. Therefore, (5.3) implies for each $\theta \in \Theta'$ the estimate

$$\begin{array}{rcl} \delta(B,\theta) & = & A(\mathcal{H}^1(\theta) - L(B,\theta)) + B(L(B,\theta)) \\ & \ge & A(\mathcal{H}^1(\theta) - l_\nu(\theta)) + B_\nu(l_\nu) - 1/\nu - \tilde{\varepsilon}_\nu \\ & \ge & \delta(B_\nu,\theta) - \varepsilon_\nu, \end{array} \tag{5.4}$$

where $\varepsilon_\nu := \tilde{\varepsilon}_\nu + 1/\nu$. Therefore, (ii) follows.

At last, to prove (iii), we let $\alpha_\nu \in \Theta$ stand for the segment connecting x_ν with $\theta(0)$ and $\beta_\nu \in \Theta$ for the segment connecting $\theta(1)$ with y_ν respectively. Set now

$$\hat{\theta}_\nu := \alpha_\nu \circ \theta \circ \beta_\nu,$$

so that clearly $\hat{\theta}_\nu$ connects x_ν with y_ν and $\hat{\theta}_\nu \to \theta$ as $\nu \to \infty$. One has $\mathcal{H}^1(\hat{\theta}_\nu \cap \Sigma) \ge \mathcal{H}^1(\theta \cap \Sigma)$, and hence

$$\delta(B_\nu, \hat{\theta}_\nu) \le A(\mathcal{H}^1(\hat{\theta}_\nu) - L(B_\nu,\theta)) + B_\nu(L(B_\nu,\theta))$$

by (3.2). Mind that

$$|\mathcal{H}^1(\theta) - \mathcal{H}^1(\hat{\theta}_\nu)| \le \mathcal{H}^1(\alpha_\nu) + \mathcal{H}^1(\beta_\nu) = |x_\nu - \theta(0)| + |\theta(1) - y_\nu| \to 0$$

as $\nu \to \infty$. This together with the uniform continuity of A on bounded intervals, and (5.4) provides

$$\delta(B_\nu, \hat{\theta}_\nu) \le \delta(B,\theta) + \varepsilon'_\nu$$

for some sequence $\varepsilon'_\nu \to 0$ as $\nu \to \infty$, which concludes the proof. □

We prove yet another convergence statement concerning the distance d_B.

Proposition 5.3. *Assume $x_\nu \to x$, $y_\nu \to y$ in Ω and $\Gamma^- \lim_\nu B_\nu = B$. Then*

(i) *$d_{B_\nu}(x_\nu, y_\nu) \to d_B(x,y)$. In particular, setting $B_\nu := B$, we get that the function d_B is continuous;*

(ii) *if $\theta_\nu \in \widetilde{\Theta}(B_\nu)$ and $\theta_\nu \to \theta$ in Θ, then $\theta \in \widetilde{\Theta}(B)$. In particular, setting $B_\nu := B$, we get that $Q_B(x,y)$ is closed for all $(x,y) \in \Omega \times \Omega$ and also $\widetilde{\Theta}(B)$ is closed.*

Proof. Define $\hat{\delta}(x,y,B,\cdot)\colon \Theta \to \bar{\mathbb{R}}$

$$\hat{\delta}(x,y,B,\theta) := \begin{cases} \delta(B,\theta), & \text{if } \theta(0) = x, \theta(1) = y, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 5.2, (i) and (iii) imply that

$$\hat{\delta}(x,y,B,\cdot) = \Gamma^- \lim_\nu \hat{\delta}(x_\nu, y_\nu, B_\nu, \cdot)$$

whenever $x_\nu \to x$, $y_\nu \to y$ in Ω and $\Gamma^- \lim_\nu B_\nu = B$. According to (5.6) one has

$$d_B(x,y) = \min_{\Theta'} \hat{\delta}(x,y,B,\cdot),$$

where the minimum is attained since $\Theta' \subset \Theta$ is compact, and hence $Q_B(x,y) = \operatorname{Argmin}_{\Theta'} \hat{\delta}(x,y,B,\cdot)$ for every $(x,y) \in \Omega \times \Omega$ and for every nondecreasing B. Thus we get both (i) and (ii) as an immediate consequence of Proposition 4.2(ii) (minding the remark following this proposition). $\square$

Let $E' \subset \mathcal{M}^+(\Theta')$ stand for the set of all transports concentrated on Θ' and observe that it is compact since all the transports have the same total mass $\varphi_0(\Omega)$. Defining $f'\colon \Theta \to \Theta'$ by the relationship

$$f'(\theta) := \begin{cases} \theta, & \theta \in \Theta', \\ [\theta(0), \theta(1)], & \theta \notin \Theta', \end{cases} \tag{5.5}$$

one clearly has for $\theta' := f'(\theta)$ that

$$\delta(B, \theta) \geq \delta(B, \theta') \tag{5.6}$$

for all $\theta \in \Theta$. In fact, if $\theta \notin \Theta'$, then

$$\mathcal{H}^1(\theta) - L(B, \theta) > \lambda - \mathcal{H}^1(\Sigma) \geq \operatorname{diam} \Omega, \tag{5.7}$$

and hence $\delta(B, \theta) \geq A(\operatorname{diam} \Omega)$, while $\delta(B, \theta') \leq A(\mathcal{H}^1([\theta(0), \theta(1)]) \leq A(\operatorname{diam} \Omega)$. We prove now the folowing general statement.

Lemma 5.4. *For every transport η and for each $B \in \mathcal{B}$ there is a transport $\eta' \in E'$ such that*

(i) $(p_0 \times p_1)_\# \eta' = (p_0 \times p_1)_\# \eta$;

(ii) $C(B)(\eta') \leq C(B)(\eta)$. *In particular,*

$$\eta' \in E_{opt}(B) \text{ whenever } \eta \in E_{opt}(B);$$

(iii) *If η is concentrated on $\tilde{\Theta}(B)$ (in particular, when $\eta \in E_{opt}(B)$), then so is η', and*

$$\hat{F}(B, \eta') \geq \hat{F}(B, \eta).$$

REMARKS.

(A) The first claim of (ii) shows in fact that the infimum of $C(B)(\eta)$ over all admissible transports η coincides with that over E'. Moreover, since $\delta(B, \cdot)$ is l.s.c. and E' is compact, then as a by-product we have $E_{opt}(B) \cap E' \neq \emptyset$.

(B) The claim (iii) is applicable both to $\eta \in E_{opt}(B)$ (such transports are concentrated on $\widetilde{\Theta}(B)$ due to the last claim of Proposition 3.2) and to $\eta \in \operatorname{Argmin}\{C(B)(\eta') : (p_0 \times p_1)_\# \eta' = \gamma\}$ with γ a given transport plan (such transports are also concentrated on $\widetilde{\Theta}(B)$ as shown in the proof of Proposition 3.5).

(C) If either A is strictly increasing, or unbounded, then by the remark following Lemma 3.3 one has $\mathcal{H}^1(\theta) \leq l$ with $l > 0$ independent of B for all $\theta \in \widetilde{\Theta}(B)$. Hence, letting $\tilde{E}$ stand for the set of admissible transports concentrated on the set of paths with length not exceeding l, we get that $E_{opt}(B) \subset \tilde{E}$ for all $B \in \mathcal{B}$. Clearly, $\tilde{E}$ is compact in $*$-weak topology of measures.

Proof. Set $\eta' := f'_\# \eta$, where f' is defined by (5.5). Then (i) is immediate, while (ii) follows from

$$C(B)(\eta') = \int_\Theta \delta(B, \theta)\, d\eta' = \int_\Theta \delta(B, f'(\theta))\, d\eta \leq \int_\Theta \delta(B, \theta)\, d\eta = C(B)(\eta),$$

which in its turn is a consequence of (5.6). Finally, observe that if $\theta \in \widetilde{\Theta}(B) \setminus \Theta'$, then $B(L(B,\theta)) = 0$ by Lemma 3.3. This, minding that in view of (5.6) one has $f'(\widetilde{\Theta}(B)) \subset \widetilde{\Theta}(B)$ (which shows the first part of (iii)), implies the estimate

$$\begin{aligned}
\hat{F}(B,\eta') &= \int_{\widetilde{\Theta}(B)} B(L(B,\theta))\, d\eta' &&= \int_{\widetilde{\Theta}(B)} B(L(B,f'(\theta)))\, d\eta \\
&\geq \int_{\Theta'\cap\widetilde{\Theta}(B)} B(L(B,f'(\theta)))\, d\eta &&= \int_{\Theta'\cap\widetilde{\Theta}(B)} B(L(B,\theta))\, d\eta \\
&= \int_{\widetilde{\Theta}(B)} B(L(B,\theta))\, d\eta &&= \hat{F}(B,\eta),
\end{aligned}$$

which concludes the proof of (iii). □

We will need the following auxiliary statement which will be used both in the proof of the main existence result and of Proposition 5.6 in the sequel.

Lemma 5.5. *Assume that for a sequence of nonnegative Borel functions $\{u_\nu\}\colon X \to \mathbb{R}$ defined over a compact metric space X one has $u^- = \Gamma^- \liminf_\nu u_\nu$. Then for any sequence of measures $\{\eta_\nu\} \subset \mathcal{M}^+(X)$ such that $\eta_\nu \rightharpoonup \eta$ $*$-weakly one has*

$$\int_X u^-\, d\eta \leq \liminf_\nu \int_X u_\nu\, d\eta_\nu.$$

Remark. Analogously, if $\Gamma^+ \limsup_\nu u_\nu = u^+$ and u_ν are uniformly bounded from above, then

$$\int_X u^+\, d\eta \geq \limsup_\nu \int_X u_\nu\, d\eta_\nu.$$

Proof. It is easy to verify that

$$\Gamma^- \liminf u_\nu(x) = \sup_{\nu\in\mathbb{N}} \tau_\nu, \text{ where } \tau_\nu := \big(\inf_{m\geq\nu} u_m\big)^-. \tag{5.8}$$

Fix a $j \in \mathbb{N}$ and evaluate

$$\begin{aligned}
\liminf_{\nu\to\infty} \int_X u_\nu\, d\eta_\nu &\geq \liminf_{\nu\to\infty} \int_X \big(\inf_{m\geq j} u_m\big)\, d\eta_\nu \\
&\geq \liminf_{\nu\to\infty} \int_X \big(\inf_{m\geq j} u_m\big)^-\, d\eta_\nu \\
&= \liminf_{\nu\to\infty} \int_X \tau_j\, d\eta_\nu \qquad \geq \int_X \tau_j\, d\eta.
\end{aligned}$$

Since this is true for any $j \in \mathbb{N}$, by the Beppo-Levi monotone convergence theorem and (5.8) the thesis follows. □

At this moment we may claim another Γ-convergence result.

Proposition 5.6. *Assume that for a sequence of nondecrasing functions $\{B_\nu\}$ one has $\Gamma^- \lim_\nu B_\nu = B$. Then $\min C(B_\nu) \to \min C(B)$, where both minima are taken over the set of all admssible transports. Moreover, if $\eta_\nu \in E_{opt}(B_\nu) \cap E'$ and $\eta_\nu \rightharpoonup \eta$ $*$-weakly in the sense of measures, then $\eta \in E_{opt}(B)$.*

Proof. Applying the first part of Lemma 5.5 with $X := \Theta'$, $u_\nu := \delta(B_\nu, \cdot)$ and $u^- := \delta(B, \cdot)$, minding that $\Gamma^- \lim_\nu u_\nu = u^-$ in view of Proposition 5.2, yields

$$C(B)(\eta') \leq \liminf_\nu C(B_\nu)(\eta'_\nu)$$

whenever $\{\eta'_\nu\} \subset E'$ and $\eta'_\nu \rightharpoonup \eta'$ $*$-weakly in the sense of measures. On the other hand, up to a subsequence of ν (not relabeled) one has by Proposition 5.2(ii)

$$\delta(B_\nu, \theta) \leq \delta(B, \theta) + \varepsilon_\nu$$

for each $\theta \in \Theta'$ and for some sequence $\{\varepsilon_\nu\}$ (independent of θ), $\varepsilon_\nu \to 0$ as $\nu \to \infty$. Hence, integrating the above relationship in arbitrary $\eta \in E'$, we get

$$\limsup_\nu C(B_\nu)(\eta) \leq C(B)(\eta).$$

The validity of the statement being proven follows now directly from Proposition 4.2(ii) minding that according to the remark following Lemma 5.4 the infimum of $C(B)$ over all admissible transports coincides with that over compact set E' of the latter. $\square$

We will also need another convergence result below for the functions $L(B, \cdot)$ and for their compositions with B.

Lemma 5.7. *Assume that for a sequence of nondecreasing functions $\{B_\nu\}$ one has $\Gamma^- \lim_\nu B_\nu = B$ while $\theta_\nu \in \widetilde{\Theta}(B_\nu)$ and $\theta_\nu \to \theta$ in Θ as $\nu \to \infty$. Then*

$$\begin{array}{rcl} \limsup_\nu L(B_\nu, \theta_\nu)) & \leq & L(B, \theta), \\ \limsup_\nu B_\nu^-(L(B_\nu, \theta_\nu)) & \leq & B(L(B, \theta)). \end{array}$$

Remark. Applied with $B_\nu := B$ the above lemma proves in particular that both $L(B, \cdot)$ and its composition with B are u.s.c. over $\widetilde{\Theta}(B)$.

Proof. Withouth loss of generality we assume that $l_\nu := L(B_\nu, \theta_\nu) \to l$. By Proposition 5.3(i) one has

$$d_B(\theta(0), \theta(1)) = \lim_\nu d_{B_\nu}(\theta_\nu(0), \theta_\nu(1)) = \lim_\nu A(\mathcal{H}^1(\theta_\nu) - l_\nu) + B_\nu(l_\nu).$$

Minding that $\mathcal{H}^1(\theta) \leq \liminf_\nu \mathcal{H}^1(\theta_\nu)$ by Golab theorem, the function A is continuous and nondecreasing, while $B(l) \leq \liminf_\nu B_\nu(l_\nu)$, due to the Γ^--convergence assumption, we get from the above relationship the estimate

$$\lim_\nu A(\mathcal{H}^1(\theta_\nu) - l_\nu) + B_\nu(l_\nu) \geq A(\mathcal{H}^1(\theta) - l) + B(l).$$

Since $\mathcal{H}^1(\theta \cap \Sigma) \geq \limsup \mathcal{H}^1(\theta_\nu \cap \Sigma) \geq l$ by Lemma 4.1, we get then by definition of $\delta(B, \cdot)$ that

$$A(\mathcal{H}^1(\theta) - l) + B(l) \geq \delta(B, \theta) \geq d_B(\theta(0), \theta(1)).$$

Combining the above estimates, we have that all the inequalities above are in fact equalities, and in particular

$$A(\mathcal{H}^1(\theta) - l) + B(l) = \delta(B, \theta),$$

which implies $L(B, \theta) \geq l$ by (3.1) hence proving the first part of the statement. As for the second one, the above proven chain of equalities gives $B(l) = \lim_\nu B_\nu(l_\nu)$, so that the thesis follows since $L(B, \theta) \geq l$ and B is nondecreasing. $\square$

Finally, we prove the following truncation result.

Lemma 5.8. *For every $B \in \mathcal{B}$, $c > A(\lambda)$ and $\eta \in E'$ one has*

$$\hat{F}(B \wedge c, \eta) = \hat{F}(B, \eta).$$

Further, $E_{opt}(B) \cap E' = E_{opt}(B \wedge c) \cap E'$.

Proof. For every $\theta \in \Theta'$ and for every $B \in \mathcal{B}$ one has $\delta(B,\theta) \le A(\mathcal{H}^1(\theta)) \le A(\lambda)$, since $\mathcal{H}^1(\theta) \le \lambda$. Hence, $B(L(B,\theta)) \le \delta(B,\theta) \le A(\lambda) < c$. Setting $\bar{B} := B \wedge c$, we have also $\bar{B}(L(\bar{B},\theta)) < c$, which implies $L(B,\theta) = L(\bar{B},\theta)$, and

$$B(L(B,\theta)) = \bar{B}(L(\bar{B},\theta)), \qquad \delta(B,\theta) = \delta(\bar{B},\theta). \tag{5.9}$$

The first equality of (5.9) implies then the first claim of the statement being proven, while the second of the equalities in (5.9) proves the second claim because according to Lemma 5.4(ii) the minima of $C(B)$ and of $C(\bar{B})$ are attained at E'. □

Finally, we may prove Theorem 5.1.

Proof of Theorem 5.1. We provide here the proof only for Problem 2, since the proof for Problem 1 is completely analogous. Let $\{B_\nu, \eta_\nu\}$ be a maximizing sequence for F, and in particular, $\eta_\nu \in E_{opt}(B_\nu)$. By Lemma 5.4 we may suppose without loss of generality that each $\eta_\nu \in E'$. Since E' is compact in $*$-weak topology of measures by remark following Lemma 5.4, then one may assume up to passing to subsequence (not relabeled) that $\eta_\nu \rightharpoonup \eta$ $*$-weakly in the sense of measures.

Let $c > A(\lambda)$. In view of Lemma 5.8 we may assume without loss of generality that $0 \le B_\nu(x) \le c$ for all $x \in \mathbb{R}^+$. We may choose therefore a subsequence of B_ν (not relabeled) such that for some $B\colon \mathbb{R}^+ \to \mathbb{R}$ one has $\Gamma^- \lim_\nu B_\nu = B$. By Proposition 5.6 then $\eta \in E_{opt}(B)$.

Define now for each $B \in \mathcal{B}$ the map $\tilde{B}\colon \Theta \to \bar{\mathbb{R}}$ by $\tilde{B}(\theta) := B(L(B,\theta))$. Note that Lemma 5.7 implies $\Gamma^+ \limsup_\nu \tilde{B}_\nu \le \tilde{B}$. It remains to observe that in view of the remark following the Lemma 5.5 one has

$$\begin{aligned}
\limsup_\nu \int_\Theta B_\nu(L(B_\nu,\theta))\, d\eta_\nu(\theta) &= \limsup_\nu \int_\Theta \tilde{B}_\nu(\theta)\, d\eta_\nu(\theta) \\
&\le \int_\Theta (\Gamma^+ \limsup_\nu \tilde{B}_\nu)(\theta)\, d\eta(\theta), \\
&\le \int_\Theta \tilde{B}(\theta)\, d\eta(\theta) \\
&= \int_\Theta B(L(B,\theta))\, d\eta(\theta)
\end{aligned}$$

and hence B solves Problem 2. □

6. Qualitative properties of optimal policies

The aim of this section is to prove the existence of optimal pricing policies possessing some natural regularity properties. For this purpose we introduce now the following natural notion of a *minimum optimal pricing policy* for Problem 2 (resp. Problem 1).

Definition 6.1. *Let the couple of functions $\{B_1, B_2\} \in \mathcal{B}$ be optimal pricing policies for Problem 2 (resp. Problem 1 with given transport plan γ). We will say that B_1 dominates B_2, if $B_1(u) \ge B_2(u)$ for all $u \in \mathbb{R}^+$.*

A function $\hat{B} \in \mathcal{B}$ is called minimum optimal pricing policy for Problem 2 (resp. Problem 1 with given transport plan γ), if it is an optimal pricing policy for the respective problem which does not dominate any other one.

We will prove now existence of minimum optimal pricing policies.

Theorem 6.2. *There exists a minimum optimal pricing policy for Problem 2 (resp. Problem 1 with given transport plan γ).*

Proof. As in Theorem 5.1, we provide the proof for Problem 2, because the proof for Problem 1 is completely analogous. Let $\mathcal{B}_{opt}$ stand for an arbitrary set of optimal pricing policies for Problem 2 such that for every couple of different policies $\{B_1, B_2\} \in \mathcal{B}_{opt}$ one pricing policy dominates the other. Define $\hat{B}\colon \mathbb{R}^+ \to \mathbb{R}^+$ by the relationship

$$\hat{B}(u) := \inf\{B(u) \,:\, B \in \mathcal{B}_{opt}\}$$

for all $u \in \mathbb{R}^+$. Let $c > A(\lambda)$. According to Lemma 5.8,

$$B(u) = \inf\left\{B(u) \,:\, B \in \bar{\mathcal{B}}_{opt}\right\}, \text{ where } \bar{\mathcal{B}}_{opt} := \{B \wedge c \,:\, B \in \mathcal{B}_{opt}\}$$

for all $u \in \mathbb{R}^+$, while the elements of $\bar{\mathcal{B}}_{opt}$ are still optimal pricing policies. We may therefore obtain a sequence $\{B_\nu\} \subset \bar{\mathcal{B}}_{opt}$ converging to B pointwise, i.e. $B_\nu(u) \to \hat{B}(u)$ for all $u \in \mathbb{R}^+$ as $\nu \to \infty$. Up to a subsequence (not relabeled), we have that $\Gamma^- \lim_\nu B_\nu = \hat{B}^-$. Consider a sequence of transports $\{\eta_\nu\}$ such that $(B_\nu, \eta_\nu) \in \operatorname{Argmax} F$, and hence, in particular, $\eta_\nu \in E_{opt}(B_\nu)$ for all $\nu \in \mathbb{N}$. In view of Lemma 5.4, we may suppose without loss of generality that all $\eta_\nu \in E'$ and since E' is compact in $*$-weak topolgy of measures, then again up to a subsequence (not relabeled), one has $\eta_\nu \rightharpoonup \eta$ in $\mathcal{M}^+(\Theta)$. By Proposition 5.6 one has $\eta \in E_{opt}(\hat{B}^-)$. Now, in a complete analogy with the proof of Theorem 5.1, we observe referring to Lemmata 5.5 and 5.7 that

$$\max F \quad = \quad \limsup_\nu \int_\Theta B_\nu(L(B_\nu, \theta))\, d\eta_\nu(\theta) \quad \leq \quad \int_\Theta \hat{B}^-(L(B, \theta))\, d\eta(\theta)$$

and thus, minding that $\hat{B}^- \in \mathcal{B}$, one has that $\hat{B}^-$ is an optimal pricing policy for Problem 2. Clearly, $\hat{B}^-(u) \leq \hat{B}(u)$ for all $u \in \mathbb{R}^+$ and hence does not dominate any pricing policy from $\mathcal{B}_{opt}$. Due to the arbitrarity of choice of $\mathcal{B}_{opt}$ we conclude referring to the Zorn's lemma the existence of a minimum element with respect to the domination relation in the set of all optimal pricing policies for Problem 2. This minimum element is in fact an optimal pricing policy. $\square$

Remark. We do not know whether in fact the minimum optimal pricing policy is *unique* even for the Problem 1 with fixed transport plan γ. If that were the case, then the minimum optimal pricing policy would be necessarily the pointwise minimum of optimal pricing policies for this problem as one can easily deduce form the above proof. It is worth noting that to prove the uniqueness of minimum optimal pricing policy it would be enough to show that $B_1 \wedge B_2$ is an optimal pricing policy whenever so are both B_1 and B_2.

6.1. **Regularity.** We are able to prove now that every minimum optimal pricing policy for Problem 2 (resp. Problem 1) is always continuous (moreover, its modulus of continuity is determined by that of A).

Denote by ω_A the modulus of continuity of A over $[0, \lambda]$, where $\lambda := \operatorname{diam}\Omega + \mathcal{H}^1(\Sigma)$ as defined by (3.7). Namely, let

$$\omega_A(l) := \sup\{|A(u) - A(v)| \,:\, |u - v| \leq l, 0 \leq u \leq \lambda, 0 \leq v \leq \lambda\}$$

for every $l \leq \lambda$, and set for convenience $\omega_A(l) := \omega_A(\lambda)$ whenever $l > \lambda$. It is easy to observe that $\omega_A\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined in this way is a continuous nondecreasing function.

We are able now to prove the following regularity theorem, which shows that the modulus of continuity of every minimum optimal pricing policy B is dominated by that of A.

Theorem 6.3. *Let B be a minimum optimal pricing policy for Problem 2 (resp. Problem 1 with given transport plan γ). Then*

$$B(u) - B(v) \leq \omega_A(|u-v|)$$

for all $u, v \in \mathbb{R}^+$.

REMARK. In particular, the above theorem shows that every minimum optimal pricing policy B is Lipschitz continuous whenever so is A (and the Lipschitz constant of B does not exceed that of A).

Proof. As usual, we provide the proof for Problem 2, since the proof for Problem 1 is completely analogous. If the claim is false, then there is a $u \in \mathbb{R}^+$ and an $l > 0$ such that $B(u+l) > B(u) + \omega_A(l)$. Let then $\alpha(v) := B(u) + \omega_A(v-u)$ and note that $\alpha\colon [u, +\infty) \to \mathbb{R}^+$ is clearly continuous since so is ω_A. Because B is l.s.c. and α is continuous, then the set $\{v \geq u \,:\, B(v) > \alpha(v)\}$ is open, and since it contains $u+l$ then there is a maximum open interval $(a,b) \ni u+l$ such that

$$B(v) > \alpha(v) \text{ for all } l \in (a,b).$$

Note that

$$B(a) \leq \alpha(a) \tag{6.1}$$

due to the maximality of (a,b). We define a l.s.c. nondecreasing function $z\colon (a,b) \to \mathbb{R}^+$ by the formula

$$z(v) := (\alpha(v) + B(v))/2, \qquad\qquad v \in (a,b).$$

Set now

$$\bar{B}(v) := \begin{cases} B(v), & v \notin (a,b), \\ z(v), & v \in (a,b). \end{cases}$$

Clearly, $\bar{B} \in \mathcal{B}$. In fact, the only nontrivial assertion to verify is that $\bar{B}$ is nondecreasing, i.e. that $\bar{B}(v_1) \leq \bar{B}(v_2)$ whenever $v_1 \leq v_2$. The latter claim is obvious if both $v_1 \in (a,b)$ and $v_2 \in (a,b)$ (or both $v_1 \notin (a,b)$ and $v_2 \notin (a,b)$), since z (resp. B) is nondecreasing. If $v_1 \leq a$, $v_2 \in (a,b)$, then

$$\bar{B}(v_1) = B(v_1) \leq B(a) \leq \alpha(a) \leq \alpha(v_2) \leq z(v_2) = \bar{B}(v_2),$$

the second inequality in the latter chain of estimates being true due to (6.1). Finally, if $v_1 \in (a,b)$, $v_2 \geq b$, then

$$\bar{B}(v_1) = z(v_1) \leq B(v_1) \leq B(b) \leq B(v_2) = \bar{B}(v_2).$$

Observe now that $\bar{B}(v) < B(v)$ for all $v \in (a,b)$ by construction. Hence, by minimality of B, one has that $\bar{B}$ cannot solve Problem 2. This means that whenever η is such that $(B,\eta) \in \operatorname{Argmax} F$, and $\bar{\eta} \in E_{opt}(\bar{B})$, then

$$\hat{F}(\bar{B}, \bar{\eta}) < \hat{F}(B, \eta). \tag{6.2}$$

On the other hand, one has for every $\theta \in \Theta'$ that

$$L(B, \theta) \notin (a,b). \tag{6.3}$$

In fact, if $\mathcal{H}^1(\theta \cap \Sigma) \geq v$ for some $v \in (a,b)$, then

$$\begin{array}{rcl} A(\mathcal{H}^1(\theta) - v) + B(v) & > & A(\mathcal{H}^1(\theta) - v) + B(u) + \omega_A(v-u) \\ & \geq & A(\mathcal{H}^1(\theta) - u) + B(u), \end{array} \tag{6.4}$$

where in the latter relationship we used the fact that for $\theta \in \Theta'$ one has $\mathcal{H}^1(\theta) \leq \lambda$. Clearly, (6.4) implies $L(B, \theta) \neq v$ and hence (6.3). Analogously,

$$L(\bar{B}, \theta) \notin (a, b), \tag{6.5}$$

for $\theta \in \Theta'$ since still

$$\bar{B}(v) = z(v) > B(u) + \omega_A(v-u) = \bar{B}(u) + \omega_A(v-u)$$

whenever $v \in (a, b)$.

By Lemma 5.4 we may suppose without loss of generality that η is concentrated over Θ'. But for every $\theta \in \Theta'$ one has in view of (6.5) and of the definition of $\bar{B}$ that

$$\begin{aligned} \min_{l \in [0, \mathcal{H}^1(\theta \cap \Sigma)]} A(\mathcal{H}^1(\theta) - l) + \bar{B}(l) &= A(\mathcal{H}^1(\theta) - L(\bar{B}, \theta)) + \bar{B}(L(\bar{B}, \theta)) \\ &= A(\mathcal{H}^1(\theta) - L(\bar{B}, \theta)) + B(L(\bar{B}, \theta)), \end{aligned}$$

which implies

$$L(\bar{B}, \theta) = L(B, \theta) \tag{6.6}$$

and also

$$\delta(\bar{B}, \theta) = \delta(B, \theta).$$

The latter relationship implies $\eta \in E_{opt}(\bar{B})$, while from (6.6) together with (6.5) and the definition of $\bar{B}$ one gets

$$\bar{B}(L(\bar{B}, \theta)) = B(L(B, \theta))$$

for η-a.e. $\theta \in \Theta$, and therefore $\hat{F}(\bar{B}, \bar{\eta}) = \hat{F}(B, \eta)$ contradicting (6.2). The latter contradiction concludes the proof. □

6.2. **Natural bounds.** We now prove that the function A (the cost of transportation without the use of the transport network) gives a natural bound to solutions of the problems being studied, i.e. to the optimal pricing policies.

Corollary 6.4. *Suppose that the function A is subadditive, i.e.*

$$A(u+v) \leq A(u) + A(v) \text{ for every } u, v \in \mathbb{R}^+.$$

Then every minimum optimal pricing policy B for Problem 2 (resp. for Problem 1 with given transport plan γ) satisfies $B(u) \leq A(u)$ for all $u \in [0, \mathcal{H}^1(\Sigma)]$.

Proof. Minding that $B(0) = 0$, we get

$$B(u) = B(u) - B(0) \leq \omega_A(u)$$

by Theorem 6.3. Let $v \in [0, \lambda]$ be such that $\omega_A(u) = A(v+u) - A(v)$. Minding the subadditivity of A, we get $\omega_A(u) \leq A(u)$, hence proving $B(u) \leq A(u)$. □

We will show in the example 7.2 that in fact the condition of subadditivity of A is inevitable for the statement of Corollary 6.4 to hold.

7. Some examples

Here we present some simplest examples of the optimal pricing policies solving Problems 2 and 1.

7.1. **Single citizen.** Consider the following sample two-dimensional situation (i.e. $n = 2$). Let $\varphi_1 := \delta_0$ be a Dirac measure concentrated in the origin of the coordinate system, and $\varphi_0 := \delta_S$ be a Dirac measure concentrated in a point $S = (l, h)$. In this case the population is clearly represented by a single citizen living in S, while the service (or working place) is unique and is located in the origin of the coordinate system. We assume the transportation network $\Sigma := [0, l]$ to be the line segment of length l along the x axis (see figure 1). Recall also that we are interested only in l.s.c. pricing policies B.

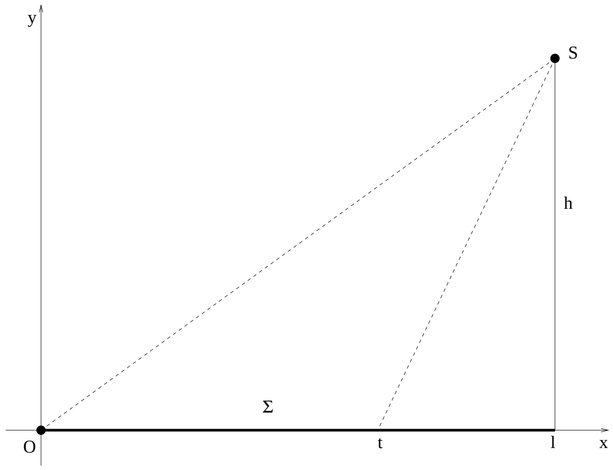

FIGURE 1. Sample transportation network Σ and the choice of the optimal path.

Note that in this sample situation there is only one admissible transport plan $\gamma = \varphi_0 \otimes \varphi_1$, hence every solution to Problem 2 is also a solution to Problem 1, and vice versa.

Clearly, whatever are the cost functions A and B, for the passenger which moves between S and O it is convenient to choose the path θ_t which is a union of two line segments: a line segment connecting S with a point on Σ which has coordinates $(t, 0)$, $t \leq l$, and another one connecting the latter to O. In this case the total cost for the passenger is

$$\delta(B, \theta_t) = A\left((h^2 + (l-t)^2)^{1/2}\right) + B(t), \tag{7.1}$$

and hence

$$d_B(S, 0) = \min_{0 \leq t \leq l} \delta(B, \theta_t) \text{ and } MK(B) = d_B(S, 0).$$

Therefore the scope of the "population" (the users of the transportation network) can be understood as that of simply finding, given a pricing policy B, an optimal $t \in [0, l]$ which minimizes the cost $\delta(B, \theta_t)$ among all $t \in [0, l]$. The optimal transports $\eta_{opt} \in E_{opt}(B)$ will be thus concentrated only on the paths θ_t with t optimal in the above sense. Let t_B be the maximum among all optimal $t \in [0, 1]$. Note that all the paths θ_t with t optimal provide the same cost of movement $\delta(B, \theta_t)$ for the passenger. It is also worth observing that the Dirac measure η_B concentrated on the optimal path θ_{t_B} is the unique transport maximizing $\hat{F}(B,\cdot)$ among all optimal transports.

It is easy to observe therefore that under our assumption the owner of the transportation network earns $B(t_B)$ and hence is interested in maximizing the latter over all nonnegative nondecreasing B satisfying $B(0) = 0$. In other words, Problem 2 reduces to that of maximizing the functional $\tilde{F}$ defined by $\tilde{F}(B) := B(t_B)$ over all $B \in \mathcal{B}$. To solve this problem, we note that

$$d_B(S,0) \leq A(\mathcal{H}^1([OS])) = A\left((h^2+l^2)^{1/2}\right).$$

Hence,

$$\delta(B,\theta_{t_B}) = A\left((h^2+(l-t_B)^2)^{1/2}\right) + B(t_B) \leq A\left((h^2+l^2)^{1/2}\right),$$

which gives a bound to the maximum of $B(t_B)$: in fact,

$$B(t_B) \leq A\Big((h^2+l^2)^{1/2}\Big) - A(h).$$

We claim that there is a pricing policy B such that the latter estimate is an equality. This policy would of course provide the maximum of $\tilde{F}$ (and hence of $\hat{F}$). For this situation, one necessarily has $t_B = l$ and then

$$B(t_B) = B(l) = A\left((h^2+l^2)^{1/2}\right) - A(h). \tag{7.2}$$

In this case

$$MK(B) = d_B(S,0) = \delta(B,\theta_{t_B}) = A(h) + B(l) \tag{7.3}$$

On the other hand, for every $t < t_B = l$ one should have

$$\delta(B,\theta_t) \geq \delta(B,\theta_{t_B}),$$

which implies, in view of (7.1) and (7.3) that

$$B(t) \geq A(h) + B(l) - A\left((h^2+(l-t)^2)^{1/2}\right).$$

Plugging (7.2) in the latter estimate, we get

$$B(t) \geq A\left((h^2+l^2)^{1/2}\right) - A\left((h^2+(l-t)^2)^{1/2}\right), \qquad t < l. \tag{7.4}$$

Summing up, we conclude that a nondecreasing l.s.c. function B satisfying $B(0) = 0$ and verifying simultaneously (7.2) and (7.4) solves Problem 2 in our situation. Since of course (7.2) and (7.4) can be simultaneously satisfied, we infer that the optimal pricing policies are exactly those $B \in \mathcal{B}$ for which both (7.2) and (7.4) hold. The minimum optimal pricing policy is therefore the function $B_{\min}$ given by the formula

$$B_{\min}(t) := A\left((h^2+l^2)^{1/2}\right) - A\left((h^2+(l-t)^2)^{1/2}\right), \qquad t \in [0,l].$$

We consider now some particular cases.

Example 7.1. If $A(t) = t$, then a function $B \in \mathcal{B}$ solves Problem 2, if and only if

$$\begin{array}{rcll} B(l) & = & (h^2+l^2)^{1/2} - h, & \\ B(t) & \geq & (h^2+l^2)^{1/2} - (h^2+(l-t)^2)^{1/2}, & \qquad t < l. \end{array}$$

The minimum optimal solution to Problem 2 is given then by the continuous concave function (see figure 2)

$$B_{\min}(t) := (h^2+l^2)^{1/2} - (h^2+(l-t)^2)^{1/2}, \qquad t \leq l.$$

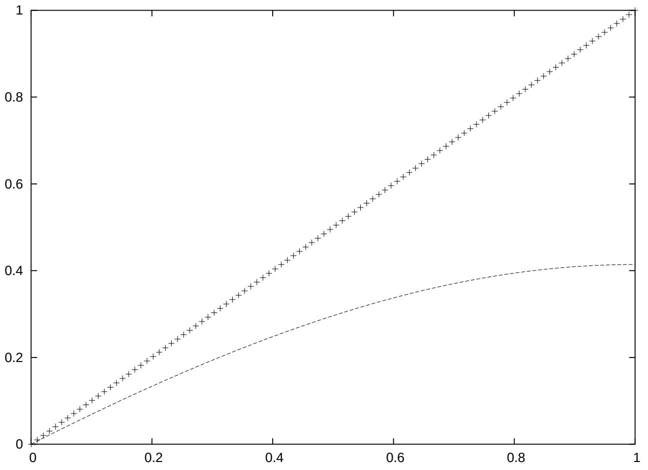

FIGURE 2. Graph of A (dotted line) and of the minimum optimal pricing policy $B_{\min}$ (solid line) for $A(t) := t$ and $h = l = 1$.

Example 7.2. If $A(t) = t^2$, then a function $B \in \mathcal{B}$ solves Problem 2, if and only if

$$\begin{array}{rcll} B(l) & = & l^2, & \\ B(t) & \geq & 2lt - t^2, & \qquad t < l. \end{array}$$

The minimum optimal solution to Problem 2 is given in this case by the continuous concave function (see figure 3)

$$B_{\min}(t) := 2lt - t^2 \qquad\qquad t \leq l.$$

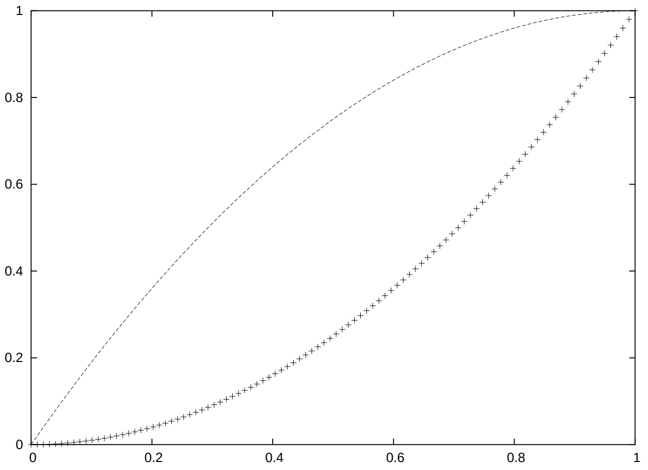

FIGURE 3. Graph of A (dotted line) and of the minimum optimal pricing policy $B_{\min}$ (solid line) for $A(t) := t^2$ and $h = l = 1$.

Note that in this case $B_{\min}(t) > A(t)$ for all $t \in (0, l)$, showing that Corollary 6.4 is sharp.

7.2. **Two citizens.** Let Ω and Σ be as in the above paragraph, while $\varphi_1 := 2\delta_0$ and $\varphi_0 := \delta_{S_1} + \delta_{S_2}$, where $S_i = (l, h_i)$, $i = 1, 2$. This model corresponds to the case of a population represented by two citizens living in S_1 and S_2 but only a single workplace (or service) O. Here, as in the single citizen case, there is only one admissible transport plan $\gamma = \varphi_0 \otimes \varphi_1$, hence every solution to Problem 2 is also a solution to Problem 1, and vice versa.

Clearly, for a passenger moving between S_i and O it is convenient to choose the polygonal path θ_{t_i} consisting of a line segment connecting S_i with a point on Σ which has coordinates $(t_i, 0)$, $t_i \leq l$ (which is the part of the itinerary the passenger moves by own means), and a line segment connecting the latter point to O (which is the part of the itinerary the passenger moves with the help of the transportation network), so that the total cost for the passenger is

$$\delta(B, \theta_{t_i}) = A\left((h_i^2 + (l - t_i)^2)^{1/2}\right) + B(t_i). \tag{7.5}$$

Therefore, each t_i is chosen as the greatest number in $[0, l]$ minimizing $\delta(B, \theta_{t_i})$. On the other hand, let

$$B_i(t) := A\left((h_i^2 + l^2)^{1/2}\right) - A\left((h_i^2 + (l - t)^2)^{1/2}\right), \qquad t \in [0, l], \tag{7.6}$$

and mind that according to (7.4) each B_i is the minimum optimal solution for Problem 2 (and also for Problem 1) with $\varphi_1 = \delta_0$ and $\varphi_0 = \delta_{S_i}$, i.e. for the single citizen case. With this notation one has

$$\delta(B, \theta_{t_i}) = B(t_i) - B_i(t_i) + A\left((h_i^2 + l^2)^{1/2}\right),$$

and since the last term in the righthand side of the above expression is independent of t_i, we get that each t_i is actually chosen as the greatest real number $t \in [0, l]$ minimizing $B(t) - B_i(t)$. This gives $t_i = t_i(B)$. The income F of the owner of the transportation network is then calculated by

$$F(B, t_1(B), t_2(B)) := B(t_1(B)) + B(t_2(B)). \tag{7.7}$$

To simplify the calculations, assume from now on that $A(t) = t$. Let $h_1 \leq h_2$, so that a direct calculation shows $B_1(t) \geq B_2(t)$ and $B_1'(t) \geq B_2'(t)$ for all $t \in [0, l]$. The first important observation is that $t_1 \geq t_2$. In fact, otherwise one would have

$$B(t_1) - B_1(t_1) < B(t_2) - B_1(t_2), \text{ while } B(t_2) - B_2(t_2) \leq B(t_1) - B_2(t_1),$$

and hence

$$\begin{aligned} \int_{t_1}^{t_2} B_1'(\tau)\, d\tau \quad &= \quad B_1(t_2) - B_1(t_1) \quad < \quad B(t_2) - B(t_1) \\ &\leq \quad B_2(t_2) - B_2(t_1) \quad = \quad \int_{t_1}^{t_2} B_2'(\tau)\, d\tau, \end{aligned}$$

which is impossible due to the fact that $B_1'(t) \geq B_2'(t)$ for all $t \in [0, l]$.

Note now that if B is an optimal pricing policy, then $B(t) \geq B_2(t)$ for all $t \in [0, l]$, since otherwise $B \vee B_2$ would give a strictly better pricing policy. On the other hand, one has $B(t) \leq B_2(t)$ for all $t \in [0, t_2]$, since otherwise the citizen living in S_2 would not cover the distance t_2 with the use of the transportation network. Therefore $B(t) = B_2(t)$ for all $t \in [0, t_2]$.

Our next observation is that $t_1 = l$. In fact, if $t_1 < l$ then $B(t) - B_1(t) > B(t_1) - B_1(t_1)$ for each $t_1 < t \leq l$. But then, taking instead of B the new pricing policy $\tilde{B}$ defined by

$$\tilde{B}(t) := \begin{cases} B(t), & \text{if } 0 \leq t \leq t_1, \\ B(t_1) + B_1(t) - B_1(t_1), & \text{if } t_1 < t \leq l, \end{cases}$$

then the effect will be that $t_1(\tilde{B}) = l > t_1(B)$ (i.e. that the citizen living in S_1 move the maximum possible distance l with the help of the transportation network, rather than $t_1(B)$), while in the meantime $t_2(\tilde{B}) = t_2(B)$ (i.e. the citizen living in S_2 will still move the distance $t_2(B)$ with the help of the network). Therefore, the gain of the owner of the transportation network becomes

$$\begin{aligned} F(\tilde{B}, t_1(\tilde{B}), t_2(\tilde{B})) &= \tilde{B}(t_2(B)) + \tilde{B}(l) \\ &= B(t_2(B)) + B(t_1) + B_1(l) - B_1(t_1(B)) \\ &= F(B, t_1(B), t_2(B)) + (B_1(l) - B_1(t_1(B))) \\ &> F(B, t_1(B), t_2(B)). \end{aligned}$$

The latter contradiction with the optimality of B shows then that $t_1 = l$.

We note now that if B is the minimum optimal pricing policy, then

$$B(t) = B(t_2) + B_1(t) - B_1(t_2)$$

for all $t \in [t_2, l]$. In fact, with this choice every path θ_t with $t \in [t_2, l]$ gives the same value of the cost $\delta(B, \theta_t)$ to the citizen living in S_1 (so that $t_1(B) = l$ only because of our assumption that each single person chooses the maximum possible distance to cover with the use of the transportation network as long as the total cost for him is the same).

Summing up, we arrive at a conclusion that the minimum optimal pricing policy in the situation we are studying, is given by

$$B(t) = \begin{cases} B_2(t), & \text{if } 0 \leq t \leq t_2, \\ B_2(t_2) + B_1(t) - B_1(t_2), & \text{if } t_2 < t \leq l. \end{cases} \tag{7.8}$$

According to (7.7), minding that $t_1(B) = l$, the income of the owner of the transportation network becomes then

$$F(B, t_1(B), t_2(B)) = 2B_2(t_2) + B_1(l) - B_1(t_2).$$

Therefore, t_2 has to maximize $2B_2(t) - B_1(t)$ among all $t \in [0, l]$, and hence, by (7.6), it maximizes the function

$$f(t) := \left(h_1^2 + (l-t)^2\right)^{1/2} - 2\left(h_2^2 + (l-t)^2\right)^{1/2}$$

among all $t \in [0, l]$. A straightforward computation ensures then $f'(t) > 0$ whenever $3(l-t)^2 > h_2^2 - 4h_1^2$, and $f'(t) < 0$ in the case of the opposite inequality. One can consider therefore three possible cases.

CASE 1. $4h_1^2 \geq h_2^2$. Then $t_2 = l$ and hence the minimum optimal pricing policy is $B(t) = B_2(t)$ for all $t \in [0, l]$. Both citizens therefore use the transportation network to the full length l.

CASE 2. $h_2^2 - 3l^2 < 4h_1^2 < h_2^2$. Then

$$t_2 = l - \left(\frac{h_2^2 - 4h_1^2}{3}\right)^{1/2}$$

and the minimum optimal pricing policy B is given by (7.8) (see figure 4).

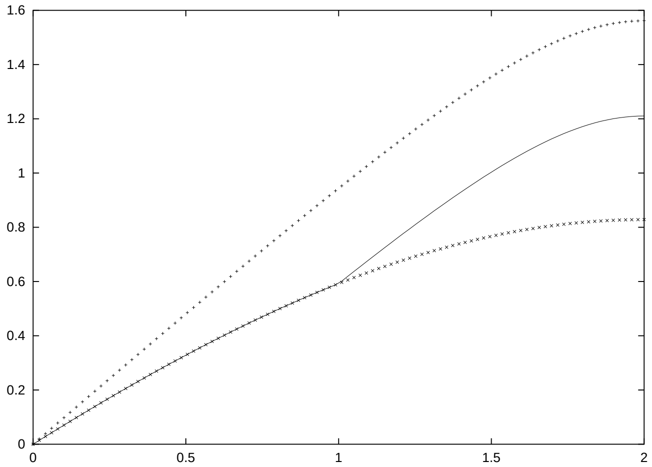

FIGURE 4. Graphs of B_1 (upper dotted line), B_2 (lower dotted line) and B (solid line) for $l = 2$, $h_1 = 1/2$, $h_2 = 2$, in which case $t_2 = 1$.

CASE 3. $4h_1^2 \leq h_2^2 - 3l^2$. Then $t_2 = 0$ (so that the citizen living in S_2 does not use the network) and hence the minimum optimal pricing policy is $B(t) = B_1(t)$ for all $t \in [0, l]$.

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