A HARNACK'S INEQUALITY FOR MIXED TYPE EVOLUTION EQUATIONS

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Abstract. We define a homogeneous parabolic De Giorgi classes of order 2 which suits a mixed type class of evolution equations whose simplest example is $\mu(x)\frac{\partial u}{\partial t} - \Delta u = 0$ where μ can be positive, null and negative. For functions belonging to this class we prove local boundedness and show a Harnack inequality which, as by-products, gives Hölder-continuity, in particular in the interface I where μ change sign, and a maximum principle.

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1. Introduction

The purpose of this paper is to study problems related to equations of mixed type whose simplest example may be

$$\mu(x)\frac{\partial u}{\partial t} - \Delta u = 0 \qquad \text{in } \Omega \times (0,T) \tag{1}$$

where μ is a function changing sign and possibly taking also the value zero in some region of positive measure, Ω an open subset of $\mathbf{R}^n$ and $T > 0$. This means that the equation can be forward parabolic in a subregion $\Omega_+ \times (0,T)$, backward parabolic in another subregion $\Omega_- \times (0,T)$ and also a family of elliptic equations depending on the parameter t in a third subregion $\Omega_0 \times (0,T)$ of $\Omega \times (0,T)$. For the existence of solutions to such equations we refer to [19] and the forthcoming paper [18]. In these papers coefficient μ is considered depending also on time, but here we confine to μ depending only on the spatial variable.
Precisely we give a Harnack type inequality (see Theorem 7.1 and Theorem 7.2) for a wide class of functions belonging to a proper De Giorgi class. By this result, on one side we give a generalized Harnack inequality which includes the classical ones for elliptic equations and for parabolic equations, on the other we study regularity and maximum principles of solutions of equations like (1); in particular we get some local Hölder continuity on the interfaces where μ change sign (see the examples at the end of the paper).

Just to avoid to confine to consider equations with $\mu : \Omega \to \{-1, 0, 1\}$ and, on the contrary, to consider also, for instance, μ continuous, one is forced to consider weighted spaces. For this

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reason we consider a more general De Giorgi class suitably defined to contain quasi-minima (see Section 4) for the equation

$$\mu \frac{\partial u}{\partial t} - \mathrm{div}(\lambda D u) = 0 \qquad \text{in } \Omega \times (0,T) \tag{2}$$

with μ and λ functions in $L^1_{\mathrm{loc}}(\Omega)$, $\lambda > 0$ while μ is valued in $\mathbf{R}$. Indeed the De Giorgi class we consider contains also solutions of more general equations, like

$$\mu(x) \frac{\partial u}{\partial t} - \mathrm{div}\, A(x,t,u,Du) = B(x,t,u,Du) \tag{3}$$

with

$$\begin{aligned} &\big(A(x,t,u,Du), Du\big) \geqslant \lambda(x)|Du|^2 , \\ &|A(x,t,u,Du)| \leqslant \lambda(x)|Du| , \\ &|B(x,t,u,Du)| \leqslant \lambda(x)|Du| . \end{aligned} \tag{4}$$

To give our main result we follow [5] and [10], but we want to stress that the De Giorgi class we consider is different from the one considered in those papers, also when $\mu \equiv 1$ and that not only because of the more complicate nature of the equations we consider (the reason lies in Lemma 4.6).

Since our class contains parabolic quasi-minima we want recall that quasi-minima or quasi-minimizers (briefly Q-minima) were introduced by Giaquinta and Giusti in [12], where they prove local Hölder continuity extending the result due to De Giorgi for the solution of elliptic equations, while Harnack inequality for quasi-minima was proved by DiBenedetto and Trudinger in [6]. In the parabolic setting the definition of quasi-minima is due to Wieser (see [25]), who proves Hölder continuity for a suitable parabolic De Giorgi class.

Going back to degenerate elliptic and parabolic equations, where by "degenerate" we mean where some weights are involved like in (2), we precise that we consider μ and λ such that

$$|\mu|_\lambda := \begin{cases} |\mu| & \text{if } \mu \neq 0, \\ \lambda & \text{if } \mu = 0 \end{cases} \qquad \text{and} \quad \lambda \qquad \text{are Muckenhoupt weights,}$$

a class of weights we introduce in Section 2. Precisely $|\mu|_\lambda \in A_\infty$ and $\lambda \in A_2$. Moreover we assume a condition relating $|\mu|_\lambda$ and λ, assumption (H.2), which is the existence of two constants $q > 2$ and $K > 0$ such that ($x \in \mathbf{R}^n$, $\rho > r > 0$)

$$\left(\frac{|B_r(x)|}{|B_\rho(x)|} \right)^{1/n} \left(\frac{|\mu|_\lambda(B_r(x))}{|\mu|_\lambda(B_\rho(x))} \right)^{1/q} \left(\frac{\lambda(B_r(x))}{\lambda(B_\rho(x))} \right)^{-1/2} \leqslant K . \tag{5}$$

We stress that we are forced to introduce the weight $|\mu|_\lambda$ extending $|\mu|$ because the weight $|\mu|$ could be zero in some region with positive measure and in that case the measure associated to $|\mu|$, even if non-negative, would not be *doubling*. We recall that $\omega \in L^1_{\mathrm{loc}}(\Omega)$, $\omega : \Omega \to [0, +\infty]$, satisfies a doubling condition if there is a positive constant c such that

$$\omega(B_{2r}(x_0)) \leqslant c\, \omega(B_r(x_0))$$

for every $x_0 \in \Omega$ and $r > 0$ such that $B_{2r}(x_0) \subset \Omega$ (and where $\omega(A)$ denotes $\int_A \omega(x) dx$). Assumption we need for the weights $|\mu|_\lambda$ and λ are summarized in (H.1), (H.2), (H.3), (H.4) in Section 4. In particular (H.4) gives also a condition about the geometry of the interface separating the regions $\Omega_+ = \{\mu > 0\}, \Omega_0 = \{\mu = 0\}, \Omega_- = \{\mu < 0\}$, condition which turns out to be sufficient to get the Harnack inequality. We do not know if this is sharp and are not able to give a counterexample to this condition.

Harnack's inequality for parabolic equations was first proved separately by Hadamard and Pini in 1954 just for the heat equations, then Moser, Aronson, Serrin, Trudinger gave some generalizations of this result. But among the many papers studying Harnack's inequality and regularity of partial differential equations, both parabolic and elliptic, we confine to mention some papers regarding degenerate cases similar to the one we are considering, referring also to the references contained in them for the more classical results.
First we recall [7] where for the first time, at least for our knowledge, a Muckenhoupt condition on λ, and precisely $\lambda \in A_2$, was consider to study regularity of the solutions of equations like

$$\mathrm{div}(\lambda\, Du) = 0$$

or more generally $\mathrm{div}(a \cdot Du) = 0$ where a satisfies

$$\lambda(x)|\xi|^2 \leqslant \big(a(x)\cdot\xi, \xi\big) \leqslant L\,\lambda(x)|\xi|^2$$

In this regard we also recall [23] and [24], where some sommability conditions, and not some local conditions, on the weight were requested.
As regards the parabolic case, we recall that equations like (2) are considered in [2], where $\mu \equiv 1$ is considered, and in [4], where $\mu = \lambda$ is considered. In both these papers a condition A_2 on the weight λ is considered, when $\mu \equiv 1$ to show that L^∞ bounds and Harnack inequality are impossible, in the second paper where $\mu = \lambda$ to show L^∞ bounds and a Harnack inequality. To get the Harnack inequality with $\mu \equiv 1$ a stronger request has to be made, i.e. λ has to belong to $A_{1+2/n}$ which is a proper subclass of A_2 (see [3]).
A more recent paper we mention about linear elliptic equation with principal part in divergence form [17], where the matrix a defining the principal part satisfies

$$\lambda_1(x)|\xi|^2 \leqslant \big(a(x,t)\cdot\xi, \xi\big) \leqslant \lambda_2(x)|\xi|^2\,, \tag{6}$$

but satisfying (5) with λ_1 in the place of λ and λ_2 in the place of $|\mu|_\lambda$; this implies the Sobolev-Poincaré inequality

$$\left[\frac{1}{\nu(B_\rho)}\int_{B_\rho}|u(x)|^q\lambda_2(x)dx\right]^{1/q} \leqslant C\,\rho\left[\frac{1}{\omega(B_\rho)}\int_{B_\rho}|Du(x)|^2\lambda_1(x)dx\right]^{1/2}$$

for every Lipschith function with either support contained in B_ρ or with null mean value. About parabolic equations with some μ in front of ∂_t we also mention [16], where an equation with $\mu = \lambda$ is considered, [8], where the author considers $\mu\,\partial_t u - \mathrm{div}\big(a(x,t)Du\big) = 0$ with a satisfying (6), and [14] where λ_1 and λ_2 are depending also on time. Finally we quote the recent paper [22], where the technique used is the one developed by DiBenedetto, Gianazza and Vespri in [5] and [10] and the result is analogous to that in [3], but it concerns monotone operators with $(p-1)$-growth and the condition about λ is $A_{1+p/n}$.

Coming back to our result, we want to stress that our condition (H.2) on the pair $(|\mu|_\lambda, \lambda)$ simply reduces to require $\lambda \in A_2$ when $\mu \equiv \lambda$, while is sharp to get, among the Muckenhoupt weights, $\lambda \in A_{1+2/n}$ when $\mu \equiv 1$ (for this see Remark 2.7, point $\mathcal{D}$, and Remark 2.8), so our result cover the result obtained in [4] and [3].

Before concluding the introduction we want to stress some difficulties and some interesting thing regarding the main results (Theorem 7.1 and Theorem 7.2). A first comment is the following: given a ball $B_\rho(x_o) \subset \Omega$ and once defined $B_\rho^+(x_o) := B_\rho(x_o) \cap \{\mu > 0\}$, $B_\rho^-(x_o) := B_\rho(x_o) \cap \{\mu <$

$0\}$, $B^0_\rho(x_o) := B_\rho(x_o) \cap \{\mu = 0\}$, we (in particular) show there is a positive constant c such that for every u in a proper class

$$u(x_o,t_o) \leqslant c \inf_{B_\rho(x_o)} \begin{cases} u\left(x, t_o + \vartheta\, \rho^2 \frac{|\mu|_\lambda(B_\rho(x_o))}{\lambda(B_\rho(x_o))}\right) & \text{if } x \in B^+_\rho(x_o) \\ u\left(x, t_o - \vartheta\, \rho^2 \frac{|\mu|_\lambda(B_\rho(x_o))}{\lambda(B_\rho(x_o))}\right) & \text{if } x \in B^-_\rho(x_o) \\ u(x,t_o) & \text{if } x \in B^0_\rho(x_o). \end{cases}$$

Notice that the temporal interval where $\mu \neq 0$ is proportional to

$$\rho^2\, \frac{|\mu|_\lambda(B_\rho(x_o))}{\lambda(B_\rho(x_o))}$$

where (μ_+ and μ_- the positive and negative parts of μ)

$$|\mu|_\lambda(B_\rho(x_o)) = \mu_+(B^+_\rho(x_o)) + \mu_-(B^-_\rho(x_o)) + \lambda(B^0_\rho(x_o))\,;$$

what we want to stress is then that the natural temporal delay, for instance where $\mu > 0$, depends also on the measure of the regions where $\mu < 0$ and $\mu = 0$.
This causes a difficulty in proving our result, in particular Theorem 7.1, because the natural cylinders are alike

$$B_\rho(x) \times \left(t, t + \rho^2\, \frac{|\mu|_\lambda(B_\rho(x))}{\lambda(B_\rho(x))}\right)$$

and so (in general) it not true that

$$B_r(x) \times \left(t, t + r^2\, \frac{|\mu|_\lambda(B_r(x))}{\lambda(B_r(x))}\right) \subset B_R(x) \times \left(t, t + R^2\, \frac{|\mu|_\lambda(B_R(x))}{\lambda(B_R(x))}\right), \quad \text{with} \quad 0 < r < R\,.$$

Other natural difficulties are due to the equation, which can change its nature around an interface, and so every result used by DiBenedetto, Gianazza and Vespri is to be suitably modified and adapted.

The paper is organized as follows: in Section 2 we introduce the class of Muchenhoupt weights and prove some results needed in the following; in particular a simple, but fundamental, extension of a classical lemma will be needed (see Lemma 2.19). Section 3 is devoted to a brief comment about mixed type equations, needed to explain a requirement we make in the De Giorgi class. In Section 4 we introduce a degenerate mixed type evolution equation, the Q-minima for that equation, assumptions about weights involved in that equation and the De Giorgi class suited to that equations which, as already mentioned, turns out to be different from the one introduced in [10] or in [25] also when $\mu \equiv 1$; we also show that Q-minima (and then a large class of solutions) are contained in the De Giorgi class we define. In the following three sections we prove local boundedness, the fundamental step so-called *expansion of positivity* (see Section 6) and a Harnack type inequality stated in Theorem 7.1 and Theorem 7.2. Finally, we give some natural consequences of the inequality we obtain and, due to the particular nature of the equation, some examples in the hope to help comprehension.

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2. Preliminaries on weights

In this section we remind and introduce some definitions and results about A_p weights needed in the following. For most of the results we refer to [9].
By $B_\rho(x_0)$ we will denote the open ball $\{x \in \mathbf{R}^n \,|\, |x - x_0| < \rho\}$, and sometimes we will simply write B_ρ or B if it is not there is no need to specify further. With the word *weight* we will mean a function η such that

$$\eta \text{ weight if:} \quad \eta > 0 \text{ a.e. in } \mathbf{R}^n \quad \text{and} \quad \eta \in L^1_{\rm loc}(\mathbf{R}^n).$$

Given a weight η and a function $u \in L^p(\Omega, \eta)$ with Ω open set of $\mathbf{R}^n$ we will write

$$\eta(B) := \int_B \eta \, dx \,, \qquad \fint_B |u|^p \eta \, dx := \frac{1}{\eta(B)} \int_B |u|^p \eta \, dx \,.$$

Definition 2.1. *Let $p > 1$, $K > 0$ be constants, ω a weight. We say that ω belongs to the class $A_p(K)$ if*

$$\left(\fint_B \omega \, dx \right)^{1/p} \left(\fint_B \omega^{-1/(p-1)} dx \right)^{(p-1)/p} \leqslant K \quad \text{for every ball } B \subset \mathbf{R}^n \tag{7}$$

We say that ω belongs to the class $A_\infty(K, \varsigma)$ if

$$\frac{\omega(S)}{\omega(B)} \leqslant K \left(\frac{|S|}{|B|} \right)^{\varsigma} . \tag{8}$$

We denote by $A_p = \bigcup_{K \geqslant 1} A_p(K)$. It turns out (see, e.g., [9]) that $A_\infty = \bigcup_{p>1} A_p$.
Given a positive weight η, a class $A_p(K;\eta)$ and all the previous classes may be defined in a analogous way simply replacing the measure dx with $\eta \, dx$.
More generally a pair (ν, ω) of weights belong to $A^\alpha_{p,q}(B_0, K)$, $\alpha \in [0, n)$, B_0 ball (possibly $\mathbf{R}^n$) if

$$|B|^{\alpha/n} \left(\fint_B \nu \, dx \right)^{1/q} \left(\fint_B \omega^{-1/(p-1)} dx \right)^{(p-1)/p} \leqslant K \quad \text{for every ball } B \subset B_0 \,. \tag{9}$$

We simply write $A^\alpha_{p,q}(K)$ if $B_0 = \mathbf{R}^n$. For $\alpha = 0$ we get the classical Muckenhoupt class of pairs (for more details we refer to [9]); for $\alpha = 0$, $q = p$, $\nu = \omega$, $B_0 = \mathbf{R}^n$ we get the class A_p.

We remind some properties of A_p weights (the same properties hold for $A_p(\eta)$ weights), for which we refer to [9]. A_p weights verify the *doubling property* which is the following: for a fixed $t > 1$ there exists a constant $c_d > 1$ which we denote by $c_d(\omega)$, such that

$$\int_{tB} \omega \, dx \leqslant c_d(\omega) \int_B \omega \, dx \tag{10}$$

for every ball B of $\mathbf{R}^n$, where by tB we mean the ball concentric to B and whose radius is t times the lenght of the side of B. If $\omega \in A_p(K)$ one has that for every $t > 0$ the constant c_d depends (only) on t, n, p, K.
Moreover $\omega \in A_p(K)$ satisfies the following *reverse Hölder's inequality*: there is $\delta = \delta(n, p, K) > 0$ and a constant $c_{rh} = c_{rh}(p, K) \geqslant 1$ such that

$$\begin{aligned} \left(\fint_B \omega^{1+\delta} dx \right)^{1/(1+\delta)} &\leqslant c_{rh} \left(\fint_B \omega \, dx \right) , \\ \left(\fint_B \omega^{-\frac{1}{p-1}(1+\delta)} dx \right)^{1/(1+\delta)} &\leqslant c_{rh} \left(\fint_B \omega^{-\frac{1}{p-1}} dx \right), \end{aligned} \tag{11}$$

for every ball B. A consequence of the definition of A_p weights and of (11) are the two following inequalities. If $\omega \in A_p(K)$ then, called ς the quantity $\delta/(1+\delta)$, one has

$$\left(\frac{|S|}{|B|}\right)^p \leqslant K\,\frac{\omega(S)}{\omega(B)}\,, \qquad \frac{\omega(S)}{\omega(B)} \leqslant c_{rh}\left(\frac{|S|}{|B|}\right)^{\varsigma} \tag{12}$$

for every measurable $S \subset B$, for every B ball of $\mathbf{R}^n$.

REMARK 2.2. - Another interesting property of A_p weights is the following.
If $\omega \in A_p(K)$ then there is $p' < p$, $p' = p'(n,p,K)$, and $K' = K'(n,p,K)$ such that $\omega \in A_{p'}(K')$.
To prove this fact take $\omega \in A_p(K)$, δ, c_{rh} considered in (11), choose $\bar p$ in such a way that

$$\frac{1}{\bar p - 1} = \frac{1}{p-1}(1+\delta)$$

(precisely $\bar p = (p+\delta)(1+\delta)^{-1} < p$) and using (11) we get

$$\fint_B \omega\,dx \left(\fint_B \omega^{-\frac{1}{p'-1}}dx\right)^{p'-1} \leqslant \fint_B \omega\,dx \left(\fint_B \omega^{-\frac{1}{p-1}(1+\delta)}dx\right)^{\frac{p-1}{1+\delta}} \leqslant$$
$$\leqslant c_{rh}^{p-1}\fint_B \omega\,dx \left(\fint_B \omega^{-\frac{1}{p-1}}dx\right)^{p-1} \leqslant c_{rh}^{p-1}K^p\,.$$

for every $p' \in [\bar p, p]$.

REMARK 2.3. - Suppose to have $\nu, \omega \in A_\infty$, i.e. there are $s_1, s_2, K_1, K_2 > 1$ such that $\omega \in A_{s_1}(K_1)$ and $\nu \in A_{s_2}(K_2)$.
Then the weight $\omega/\nu \in A_\infty(\nu)$, i.e. there is $r > 1$ such that $\omega/\nu \in A_r(c;\nu)$ or

$$\frac{\int_B \omega\,dx}{\int_B \nu\,dx}\left(\frac{\int_B \left(\frac{\omega}{\nu}\right)^{-1/(r-1)}\nu\,dx}{\int_B \nu\,dx}\right)^{r-1} \leqslant c \qquad \text{for every } B \text{ ball in } \mathbf{R}^n. \tag{13}$$

Indeed multiplying and dividing by $|B|^r$ we get that the above inequality is equivalent to

$$\frac{1}{|B|^r}\int_B \omega dx \left(\int_B \omega^{-1/(r-1)}\nu^{r/(r-1)}dx\right)^{r-1} \leqslant c\,\frac{1}{|B|^r}\left(\int_B \nu\,dx\right)^r.$$

Now by Hölder's inequality, if $a^{-1}+b^{-1} = 1$, $a, b > 1$, we get that

$$\frac{1}{|B|^r}\int_B \omega\,dx \left(\int_B \omega^{-1/(r-1)}\nu^{r/(r-1)}dx\right)^{r-1} \leqslant$$
$$\leqslant \fint_B \omega\,dx \left(\fint_B \omega^{-a/(r-1)}dx\right)^{(r-1)/a}\left(\fint_B \nu^{rb/(r-1)}dx\right)^{(r-1)/b}.$$

Since a and r are arbitrary we can choose $1+(r-1)/a = s_1$, so that $\omega \in A_{1+(r-1)/a}(K)$ and consequently

$$\fint_B \omega\,dx \left(\fint_B \omega^{-a/(r-1)}dx\right)^{(r-1)/a} \leqslant K_1\,.$$

Moreover if $\nu \in A_\infty$ by the higher summability property of A_∞ weights, there is $\delta = \delta(s_2, n, K_2) > 0$ such that (11) holds. Notice that it is possible to choose $a, b, r > 1$ in such a way

$$\frac{1}{a}+\frac{1}{b} = 1\,, \qquad \frac{r-1}{a} = s_1 - 1\,, \qquad \frac{rb}{r-1} = 1+\delta\,.$$

With these choices there is $c_1 = c_1(s_2, n, K_2)$ such that

$$\left(\fint_B \nu^{rb/(r-1)} dx \right)^{(r-1)/b} \leqslant c_1 \left(\fint_B \nu \, dx \right)^r .$$

Then (13) holds with $c = K_1 c_1$, $c = c(s_2, n, K_1, K_2)$, $r = r(s_1, s_2, n, K_1, K_2)$.

We recall now some classical results about weighted inequalities. The following one in particular can be found in [1] and is the weighted version of the standard Sobolev-Poincaré inequality. Given two weights ν, ω in $\mathbf{R}^n$ and p, q with $1 < p < q$ the following condition:

$$\text{there is a constant } K > 0 \text{ such that} \quad \left(\frac{|B_r(\bar{x})|}{|B_\rho(\bar{x})|} \right)^{\alpha/n} \left(\frac{\nu(B_r(\bar{x}))}{\nu(B_\rho(\bar{x}))} \right)^{1/q} \left(\frac{\omega(B_r(\bar{x}))}{\omega(B_\rho(\bar{x}))} \right)^{-1/p} \leqslant K \quad \text{for every pair of concentric balls } B_r \text{ and } B_\rho \text{ with } 0 < r < \rho, \tag{14}$$

with $\alpha = 1$ is essentially necessary and sufficient to have the Sobolev-Poincaré inequality. Below we confine to state only the result we need. For more details we refer to [1].

Definition 2.4. *For a pair of weights ν, ω and $\alpha \in [0, n)$ we will write (this is not a standard notation)*

$$(\nu, \omega) \in B^\alpha_{p,q}(K)$$

if it satisfies (14) for every pair of balls $B_r(\bar{x}), B_\rho(\bar{x})$ with $r < \rho$ and $\bar{x} \in \mathbf{R}^n$.

Theorem 2.5. *Consider p, q such that $1 < p < q$, $\rho > 0$, $x_0 \in \mathbf{R}^n$, two weights ν, ω in $\mathbf{R}^n$ such that $\omega \in A_p(K_1)$, $(\nu, \omega) \in B^1_{p,q}(K_2)$ and ν satisfies (10). Then there is a constant γ_1 depending (only) on $n, p, q, K_1, K_2, c_d(\nu)$ (the doubling constants of the weight ν) such that*

$$\left[\frac{1}{\nu(B_\rho)} \int_{B_\rho} |u(x)|^q \nu(x) dx \right]^{1/q} \leqslant \gamma_1 \, \rho \left[\frac{1}{\omega(B_\rho)} \int_{B_\rho} |Du(x)|^p \omega(x) dx \right]^{1/p} \tag{15}$$

for every u Lipschitz continuous function defined in $B_\rho = B_\rho(x_0)$, with either support contained in $B_\rho(x_0)$ or with null mean value.

Remark 2.6. - Notice that the previous theorem holds also for every $q' \in [1, q]$ in the place of q with the same constant γ_1. Indeed condition (14) holds with the same constant K for every $q' \in [1, q]$.
Moreover, using (12), one gets that in particular Theorem 2.5 holds when $\nu = \omega \in A_p$ with $q = np/(n-1) > p$ (and in fact also with some greater value thanks to Remark 2.2).

Remark 2.7. - Here we want to stress some important facts we will need later.
$\mathcal{A}$ - *If $(\nu, \omega) \in A^1_{p,q}(K, B_0)$ with $1 < p < q$, $\nu \in A_\infty$, then there are $\alpha \in (0, 1)$, $\tilde{q} \in (p, q)$, $\tilde{K} \geqslant K$ such that $(\nu, \omega) \in A^\alpha_{p,\tilde{q}}(\tilde{K}, B_0)$.*
By (12), since $\nu \in A_\infty$, we get the existence of $\varsigma > 0$ such that for every $\delta > 0$

$$\left(\frac{\nu(B_r)}{\nu(B_R)} \right)^\delta \leqslant (c_{rh}(\nu))^\delta \left(\frac{|B_r|}{|B_R|} \right)^{\varsigma \delta} .$$

Now we choose δ and consequently define $\tilde{q}$ in such a way that

$$\frac{1}{q} + \delta < \frac{1}{p} \qquad \text{and} \qquad \frac{1}{\tilde{q}} := \frac{1}{q} + \delta .$$

Now we can fix $\alpha \in (0,1)$ and we do that in such a way that $\varsigma\delta = (1-\alpha)/n$. Then we have for $r < R$

$$\begin{aligned} K\left(\frac{\omega(B_r(\bar x))}{\omega(B_R(\bar x))}\right)^{1/p} &\geqslant \left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{1/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/q} = \\ &= \left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{\alpha/n}\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{(1-\alpha)/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/q} \geqslant \\ &\geqslant \frac{1}{(c_{rh}(\nu))^\delta}\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{\alpha/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/q'} . \end{aligned}$$

$\mathcal{B}$ - *If* $(\nu,\omega) \in A^1_{p,q}(K_2, B_0)$ *with* $1 < p < q$, $\nu \in A_\infty$, $\omega \in A_p(K_1)$, *then there are* $p' \in (1,p)$, $q' \in (p,q)$, $K_2' \geqslant K_2$ *such that* $(\nu,\omega) \in A^1_{p',q'}(K_2', B_0)$.

Consider the values of α, q', K' ($K' \geqslant K_2$) of point $\mathcal{A}$: then we know that $(\nu,\omega) \in A^\alpha_{p,q'}(K', B_0)$. If we consider p' in such a way that

$$\frac{p-p'}{p'} = \frac{1-\alpha}{n}$$

we get, using the assumptions, the fact $\omega \in A_p(K_1)$ and (12), for $r < R$

$$\begin{aligned} K'\left(\frac{\omega(B_r(\bar x))}{\omega(B_R(\bar x))}\right)^{1/p'} &\geqslant \left(\frac{\omega(B_r(\bar x))}{\omega(B_R(\bar x))}\right)^{1/p'-1/p}\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{\alpha/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/q'} \geqslant \\ &\geqslant \left(\frac{1}{K_1}\right)^{\frac{p-p'}{p'p}}\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{\frac{p-p'}{p'}}\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{\alpha/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/q'} = \\ &= \left(\frac{1}{K_1}\right)^{\frac{p-p'}{p'p}}\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{1/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/q'} . \end{aligned}$$

Taking $K_2' = K' K_1^{\frac{p-p'}{p'p}}$ (which depends on $K_1, K_2, c_{rh}(\nu), p, q, n$) we conclude.
Actually one can require simply $\omega \in A_\infty$ and get not only $(\nu,\omega) \in A^1_{p',q'}(K_2', B_0)$, but in fact $(\nu,\omega) \in A^{\alpha'}_{p',q'}(K_2', B_0)$ with $p' \in (1,p)$, $q' \in (p,q)$, $\alpha' \in (\alpha, 1)$.

$\mathcal{C}$ - *If* $(\nu,\omega) \in A^1_{2,q}(K, B_0)$ *with* $q > 2$ *the function* $f(\bar x, r) = r^{2\alpha}\frac{\nu(B_r(\bar x))}{\omega(B_r(\bar x))}$ *satisfies the following inequality: by point* $\mathcal{A}$ *and Remark* 2.6 *we get that there are* $\alpha \in (0,1)$ *and* $\tilde K = \tilde K(K, c_{rh}(\nu))$ *such that*

$$f(\bar x, r) \leqslant \tilde K^2 f(\bar x, R)$$

for every $\bar x \in B_0$ *and* r, R *satisfying* $0 < r < R$.
Indeed by assumptions we derive

$$\left(\frac{|B_r(\bar x)|}{|B_R(\bar x)|}\right)^{\alpha/n}\left(\frac{\nu(B_r(\bar x))}{\nu(B_R(\bar x))}\right)^{1/2}\left(\frac{\omega(B_r(\bar x))}{\omega(B_R(\bar x))}\right)^{-1/2} \leqslant \tilde K .$$

Taking the power 2 we immediately get the thesis.

$\mathcal{D}$ - *Consider* $\nu \equiv 1$. *Then there are* $q > p$ *and* $\hat K$ *depending on* n, p, K *such that*

$$\omega \in \begin{cases} A_{1+p/n}(K) & \text{for } n \geqslant \dfrac{p}{p-1}, \\ A_p(K) & \text{for } n \leqslant \dfrac{p}{p-1}. \end{cases} \qquad \Longrightarrow \qquad (1,\omega) \in B^1_{q,p}(\hat K) .$$

First of all notice that for every n we have indeed $\omega \in A_{1+p/n}(K)$. In particular, by Remark 2.2, there is K' and ε such that $\omega \in A_{1+p/n-\varepsilon}(K')$. Using the first one in (12) with $S = B_r(x)$ and $B = B_\rho(x)$ ($x \in \Omega$ and $\rho > r > 0$) we get

$$\left(\frac{|B_r|}{|B_\rho|}\right)^{1+\frac{p}{n}-\varepsilon} \leqslant K' \, \frac{\omega(B_r)}{\omega(B_\rho)}.$$

What we want to prove, since $\nu \equiv 1$, is

$$\left(\frac{|B_r|}{|B_\rho|}\right)^{\frac{1}{n}+\frac{1}{q}} \leqslant \hat{K} \, \left(\frac{\omega(B_r)}{\omega(B_\rho)}\right)^{1/p}.$$

Taking the power p we get the thesis with $\hat{K} = (K')^p$ and choosing some $q \in \left(p, \frac{p}{1-\varepsilon}\right)$.

Remark 2.8. - In this remark we want to stress that the request $\omega \in A_{1+p/n}$ is optimal among the Muckenhoupt class to get that $(1, \omega) \in B^1_{q,p}$ for some $q > p$.
Indeed consider $\omega(x) = |x|^\beta$ which is A_r if and only if $-n < \beta < (r-1)n$. If we consider $r > 1 + p/n$ then it is possible to choose $\beta > p$ and in this case to get $(1, \omega) \in B^1_{q,p}$ for some $q > p$ we should consider $p < \beta < p + (p/q - 1)n$, but this is clearly impossible.

We state now a slight generalization of a result about Muckenhoupt type weights, (see [15] and [14]).

Theorem 2.9. *Consider $B_\rho = B_\rho(x_0)$ a ball of $\mathbf{R}^n$ whose radius's lenght is ρ, $\omega \in A_2(K_1)$, $(\nu, \omega) \in B^1_{2,q}(K_2)$ for some $q > 2$, $\nu \in A_\infty$, $A \subset B_\rho(x_0)$. Then there is $\sigma_1 \in (1, q)$ (see also the remark below) such that for every Lipschitz continuous function u defined in $B_\rho(x_0)$, with either support contained in $B_\rho(x_0)$ or with null mean value and for every $\kappa \in (1, \sigma_1]$*

$$\frac{1}{\upsilon(B_\rho)} \int_A |u|^{2\kappa} \upsilon \, dx \leqslant \gamma_1^2 \, \rho^2 \left(\frac{1}{\nu(B_\rho)} \int_A |u|^2 \nu \, dx\right)^{\kappa-1} \left(\frac{1}{\omega(B_\rho)} \int_{B_\rho} |Du|^2 \omega \, dx\right)$$

where the inequality holds both with $\upsilon = \nu$ and $\upsilon = \omega$ (and in fact with every weight for which Theorem 2.5 holds).

Remark 2.10. - The assumption $\nu \in A_\infty$ means that there is $s > 1, K_3 \geqslant 1$ such that $\nu \in A_s(K_3)$. Following the proof of Theorem 2.9 (and thanks to Remark 2.3) one can see that the constant κ depends (only) on n, q, s, K_1, K_3.

Proof - Consider $\kappa > 1$ (to be chosen) and consider $h_0, r > 1$ in such a way that

$$(\kappa - 1) + \frac{1}{h_0} + \frac{1}{r} = 1 \, .$$

Writing $|u|^{2\kappa}\upsilon$ as $|u|^{2(\kappa-1)}\nu^{k-1}u^2\upsilon^{1/h_0}\upsilon^{1-1/h_0}\nu^{1-\kappa}$ we get

$$\int_A |u|^{2\kappa} \upsilon \, dx \leqslant \left(\int_A u^2 \nu \, dx\right)^{\kappa-1} \left(\int_{B_\rho} |u|^{2h_0} \upsilon \, dx\right)^{\frac{1}{h_0}} \left(\int_{B_\rho} \upsilon^{(1-1/h_0)r} \nu^{(1-\kappa)r} dx\right)^{\frac{1}{r}} .$$

Now we chose $h_0 = q/2$ in such a way Theorem 2.5 holds both with $\upsilon = \nu$ and $\upsilon = \omega$ on the left hand side of the inequality. For such a h_0 we get (we have not chosen k and r yet)

$$\left(\fint_{B_\rho} |u|^{2h_0} \, \upsilon \, dx\right)^{\frac{1}{2h_0}} \leqslant \gamma_1 \, \rho \, \left(\fint_{B_\rho} |Du|^2 \omega \, dx\right)^{1/2} .$$

Now consider $\upsilon = \omega$. The previous inequality becomes

$$\left(\int_{B_\rho} |u|^{2h_0} \upsilon \, dx \right)^{\frac{1}{h_0}} \leqslant \gamma_1^2 \, \rho^2 \, \frac{1}{\big(\omega(B_\rho)\big)^{1-\frac{1}{h_0}}} \left(\int_{B_\rho} |Du|^2 \omega \, dx \right) .$$

Since $(1 - h_0^{-1})r = r(\kappa - 1) + 1$ we may write

$$\int_{B_\rho} \omega^{(1-1/h_0)r} \nu^{(1-\kappa)r} \, dx = \int_{B_\rho} \left(\frac{\omega}{\nu} \right)^{(\kappa-1)r+1} \nu \, dx \, .$$

Since $\omega/\nu \in A_\infty(\nu)$ (see Remark 2.3) the function ω/ν satisfies a reverse Hölder inequality. Then there are two positive constants δ, c_{rh} such that, for every ball B,

$$\frac{1}{\nu(B)} \int_B \left(\frac{\omega}{\nu} \right)^{1+\delta} \nu \, dx \leqslant c_{rh} \left[\frac{1}{\nu(B)} \int_B \frac{\omega}{\nu} \, \nu \, dx \right]^{1+\delta} = c_{rh} \left[\frac{\omega(B)}{\nu(B)} \right]^{1+\delta}$$

(the constants c_{rh}, δ depend on n, s, K_1, K_3 if $\nu \in A_s(K_3)$). Then we will choose κ, r in such a way that $(\kappa - 1)r = \delta$ and consequently, by what remarked above, we get

$$\int_{B_\rho} \omega^{(1-1/h_0)r} \nu^{(1-\kappa)r} \, dx \leqslant c_{rh} \, \nu(B_\rho) \left[\frac{\omega(B_\rho)}{\nu(B_\rho)} \right]^{1+(\kappa-1)r} \leqslant c_{rh} \, \frac{\big(\omega(B_\rho)\big)^{1+(\kappa-1)r}}{\big(\nu(B_\rho)\big)^{(\kappa-1)r}} \, .$$

Then we get the thesis when $\upsilon = \omega$. If $\upsilon = \nu$ the proof is easier since the quantity $\upsilon^{(1-1/h_0)r} \nu^{(1-\kappa)r}$ reduces to ν. □

We briefly recall the definition (a possible definition, in our case equivalent to the other possible ones) of weighted Sobolev spaces for $\nu \in A_\infty$ and $\omega \in A_p$. Given an open and bounded set $\Omega \subset \mathbf{R}^n$ by $L^p(\Omega, \nu)$ we denote the set of measurable functions $u : \Omega \to \mathbf{R}$ such that $\int_\Omega |u|^p \nu \, dx$ is finite. By $W^{1,p}(\Omega, \nu, \omega)$ we denote the space $\{u \in L^p(\Omega, \nu) \cap W^{1,1}_{\mathrm{loc}}(\Omega) \,|\, D_i u \in L^p(\Omega, \omega)\}$ endowed with the obvious norm; by $W^{1,p}_0(\Omega, \nu, \omega)$ we denote the closure of $C^1_c(\Omega)$ in $W^{1,p}(\Omega, \nu, \omega)$. Indeed we will write $H^1(\Omega, \nu, \omega)$ for $W^{1,2}(\Omega, \nu, \omega)$.

Coming back to the result stated in Theorem 2.9, integrating in time one immediately gets what follows.

Corollary 2.11. *With the same assumptions of Theorem* 2.9*, consider moreover* $s_1, s_2 \in (0, T)$*. Consider a family of open sets* $A(t)$*,* $t \in (s_1, s_2)$ *in such a way* $E = \cup_{t \in (s_1, s_2)} A(t)$ *is a an open subset of* $B_\rho \times (s_1, s_2)$*. For every* $v \in C^0([s_1, s_2]; L^2(B_\rho, \nu)) \cap L^2(s_1, s_2; W^{1,2}_0(B_\rho, \nu, \omega))$ *it holds*

$$\frac{1}{\upsilon(B_\rho)} \iint_E |u|^{2\kappa}(x,t) \upsilon(x) \, dx dt \leqslant \gamma_1^2 \, \rho^2 \left(\frac{1}{\nu(B_\rho)} \right)^{\kappa-1} \cdot$$
$$\cdot \Big(\sup_{s_1 < t < s_2} \int_{A(t)} |u|^2(x,t) \nu(x) \, dx \Big)^{\kappa-1} \frac{1}{\omega(B_\rho)} \int_{s_1}^{s_2} \int_{B_\rho} |Du|^2(x,t) \, \omega(x) \, dx dt$$

where the inequality holds both with $\upsilon = \nu$ *and* $\upsilon = \omega$*.*

Lemma 2.12. *Consider* $B_\rho = B_\rho(x_0)$ *a ball,* $p \in (1, +\infty)$*,* $q \in [1, +\infty)$*,* ν*,* ω *and* $v \in W^{1,p}(B_\rho, \nu, \omega)$ *for which assumptions of Theorem* 2.5 *hold,* $k, l \in \mathbf{R}$ *with* $k < l$*. Consider* *also a subset Z of B_ρ and denote by $\bar\nu$ the function taking value 0 in Z and $\bar\nu \equiv \nu$ in $B_\rho \setminus Z$. Then*

$$(l-k)^q\,\bar\nu(\{v<k\})\,\bar\nu(\{v>l\}) \leqslant 2^q\,\gamma_1^q\,\rho^q\,\bar\nu(B_\rho)\,\nu(B_\rho)\,\omega(B_\rho)^{-\frac{q}{p}}\left(\int_{B_\rho\cap\{k<v<l\}}|Dv|^p\,\omega\,dx\right)^{q/p}.$$

Remark 2.13. - The previous result holds in every open set Ω, provided that Theorem 2.5 holds with Ω in the place of B_ρ.

Proof - Denoted by A the set $\{x\in B_\rho\setminus Z\,|\,v(x)<k\}$ and suppose $\bar\nu(A)>0$, otherwise there is nothing to prove. Following the proof of Theorem 3.16 in [13] we have that for every u which takes the value zero in A

$$\int_{B_\rho}|u-u_{B_\rho}|^q\bar\nu\,dx = \int_{B_\rho\setminus A}|u-u_{B_\rho}|^q\bar\nu\,dx + \int_A|u_{B_\rho}|^q\bar\nu\,dx \geqslant |u_{B_\rho}|^q\int_A\bar\nu\,dx\,, \tag{16}$$

where $u_{B_\rho}=|B_\rho|^{-1}\int_{B_\rho}u(x)\,dx$. Consider the function

$$u := \begin{cases}\min\{v,l\}-k & \text{if } v>k\\ 0 & \text{if } v\leqslant k\,.\end{cases}$$

and estimate, first from below

$$\int_{B_\rho}|u|^q\bar\nu\,dx = \int_{\{v>l\}}(l-k)^q\bar\nu\,dx + \int_{\{k<v<l\}}(v-k)^q\bar\nu\,dx \geqslant (l-k)^q\int_{\{v>l\}}\bar\nu\,dx\,,$$

and then, using (16), from above

$$\begin{aligned}\left(\int_{B_\rho}|u|^q\bar\nu\,dx\right)^{\frac1q} &\leqslant \left(\int_{B_\rho}\big[|u-u_{B_\rho}|+|u_{B_\rho}|\big]^q\bar\nu\,dx\right)^{\frac1q}\\ &\leqslant \left(\int_{B_\rho}|u-u_{B_\rho}|^q\bar\nu\,dx\right)^{\frac1q} + \left(|u_{B_\rho}|^q\int_{B_\rho}\bar\nu\,dx\right)^{\frac1q}\\ &\leqslant 2\left(\frac{\bar\nu(B_\rho)}{\bar\nu(A)}\int_{B_\rho}|u-u_{B_\rho}|^q\bar\nu\,dx\right)^{\frac1q}.\end{aligned}$$

Now, if $q>p$ we can apply Theorem 2.5; if $q\leqslant p$, notice that $(\nu((B_\rho))^{-1}\int_{B_\rho}|u-u_{B_\rho}|^q\bar\nu\,dx)^{1/q}\leqslant(\nu((B_\rho))^{-1}\int_{B_\rho}|u-u_{B_\rho}|^{q'}\bar\nu\,dx)^{1/q'}$ for $q'>q$. Then, by Theorem 2.5 used if necessary with $q'>p$, we finally get

$$(l-k)^q\int_{\{v>l\}}\bar\nu\,dx \leqslant 2^q\,\gamma_1^q\,\rho^q\,\frac{\bar\nu(B_\rho)}{\bar\nu(A)}\,\frac{\nu(B_\rho)}{\omega(B_\rho)^{q/p}}\left(\int_{\{k<v<l\}}|Dv|^p\omega\,dx\right)^{q/p}. \qquad \square$$

Lemma 2.14. *Consider $x_0\in\Omega$ and $\rho>0$ such that $B_{2\rho}(x_0)\subset\Omega$, $\sigma\in(0,\rho)$, $\omega\in A_2(K_1)$, $(\nu,\omega)\in B^1_{2,q}(K_2)$, $\nu\in A_\infty$, $q>2$, $\alpha,\beta>0$. Consider $\mathcal{B}$ an open and non-empty subset of $B_\rho(x_0)$*

such that $\mathcal{B}^\sigma = \{x \in \Omega \,|\, \mathrm{dist}(x, \mathcal{B}) < \sigma\}$ *is a subset of* $B_\rho(x_0)$. *Then, for every* $\varepsilon, \delta \in (0,1)$ *there exists* $\eta \in (0,1)$ *such that for every* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \nu, \omega)$ *satisfying*

$$\int_{\mathcal{B}^\sigma} |Du|^2 \, \omega \, dx \leqslant \beta \, \frac{\omega(B_\rho(x_0))}{\rho^2},$$

and

$$\nu(\{u > 1\} \cap \mathcal{B}) \geqslant \alpha \, \nu(B_\rho(x_0)),$$

there exists $x^* \in \mathcal{B}$ *with* $B_{\eta\rho}(x^*) \subset \mathcal{B}$ *such that*

$$\nu(\{u > \varepsilon\} \cap B_{\eta\rho}(x^*)) > (1 - \delta) \, \nu(B_{\eta\rho}(x^*)).$$

Proof - For any positive η satisfying $\eta\rho < \sigma/2$, we can consider a finite disjoint family of balls $(B_{\eta\rho}(x_i))_{i \in I}$ with the property that

$$\bigcup_{i \in I} B_{\eta\rho}(x_i) \subset \mathcal{B} \subset \bigcup_{i \in I} B_{2\eta\rho}(x_i) \subset \mathcal{B}^\sigma \, .$$

Again for simplicity, we denote by B_i the ball $B_{\eta\rho}(x_i)$ and by B_{ii} the ball $B_{2\eta\rho}(x_i)$. We denote by I^+ and I^- the sets

$$I^+ = \{i \in I : \nu(\{u > 1\} \cap B_{ii}) > \frac{\alpha}{2 \, c_d(\nu)} \, \nu(B_{ii})\},$$

$$I^- = \{i \in I : \nu(\{u > 1\} \cap B_{ii}) \leqslant \frac{\alpha}{2 \, c_d(\nu)} \, \nu(B_{ii})\}$$

where $c_d(\nu)$ is the doubling constant of the weight ν, which, from now on, we will simply denote by c_d. By assumption we then get

$$\begin{aligned}
\alpha \, \nu(B_\rho(x_0)) \leqslant \nu(\{u > 1\} \cap \mathcal{B}) &\leqslant \sum_{i \in I^+} \nu(\{u > 1\} \cap B_{ii}) + \frac{\alpha}{2 \, c_d} \sum_{i \in I^-} \nu(B_{ii}) \leqslant \\
&\leqslant \sum_{i \in I^+} \nu(\{u > 1\} \cap B_{ii}) + \frac{\alpha}{2} \sum_{i \in I^-} \nu(B_i) \leqslant \\
&\leqslant \sum_{i \in I^+} \nu(\{u > 1\} \cap B_{ii}) + \frac{\alpha}{2} \, \nu(\mathcal{B}) \leqslant \\
&\leqslant \sum_{i \in I^+} \nu(\{u > 1\} \cap B_{ii}) + \frac{\alpha}{2} \, \nu(B_\rho(x_0)) \, .
\end{aligned}$$

By this we get that

$$\frac{\alpha}{2} \, \nu(B_\rho(x_0)) \leqslant \sum_{i \in I^+} \nu(\{u > 1\} \cap B_{ii}) \, . \tag{17}$$

Now fix $\varepsilon, \delta \in (0,1)$ and assume by contradiction that

$$\nu(\{u > \varepsilon\} \cap B_i) \leqslant (1 - \delta) \, \nu(B_i), \qquad \text{for every } i \in I := I^+ \cup I^- \, . \tag{18}$$

This clearly would imply in particular that

$$\frac{\nu(\{u \leqslant \varepsilon\} \cap B_{ii})}{\nu(B_{ii})} \geqslant \frac{\delta}{c_d} =: \delta' \qquad \text{for every } i \in I^+ \, .$$

By this last inequality, Lemma 2.12 with $p = q = 2$, $k = \varepsilon$ and $l = 1$, $\bar{\nu} = \nu$ we would obtain that

$$\begin{aligned}\delta'\,\nu(\{u > 1\} \cap B_{ii}) &\leqslant \frac{\nu(\{u \leqslant \varepsilon\} \cap B_{ii})}{\nu(B_{ii})}\,\nu(\{u > 1\} \cap B_{ii}) \leqslant \\ &\leqslant \frac{4\gamma_1^2}{(1-\varepsilon)^2}\,(\eta\rho)^2\,\frac{\nu(B_{ii})}{\omega(B_{ii})}\int_{\{\varepsilon<u<1\}\cap B_{ii}} |Du|^2\,\omega\,dx\,. \qquad (19)\end{aligned}$$

By Remark 2.7, point $\mathcal{A}$, we get the existence of $a \in (0,1)$, K_2', such that (see also Remark 2.6)

$$\left(\frac{|B_{2\eta\rho}(x_i)|}{|B_{2\rho}(x_i)|}\right)^{a/n}\left(\frac{\nu(B_{2\eta\rho}(x_i))}{\nu(B_{2\rho}(x_i))}\right)^{1/2}\left(\frac{\omega(B_{2\eta\rho}(x_i))}{\omega(B_{2\rho}(x_i))}\right)^{-1/2} \leqslant K_2'\,,$$

i.e.

$$\eta^{2a}\,\frac{\nu(B_{2\eta\rho}(x_i))}{\omega(B_{2\eta\rho}(x_i))} \leqslant (K_2')^2\,\frac{\nu(B_{2\rho}(x_i))}{\omega(B_{2\rho}(x_i))}\,.$$

Notice that

$$\nu(B_{2\rho}(x_i)) \leqslant c_d(\nu)\,\nu(B_\rho(x_i)) \leqslant c_d(\nu)\,\nu(B_{2\rho}(x_0)) \leqslant (c_d(\nu))^2\,\nu(B_\rho(x_0))$$
$$\omega(B_{2\rho}(x_0)) \leqslant \omega(B_{4\rho}(x_i)) \leqslant (c_d(\omega))^2\omega(B_\rho(x_i)) \leqslant (c_d(\omega))^2\omega(B_{2\rho}(x_i))$$

by which we get

$$\frac{\nu(B_{2\rho}(x_i))}{\omega(B_{2\rho}(x_i))} \leqslant \frac{(c_d(\nu))^2}{(c_d(\omega))^2}\,\frac{\nu(B_\rho(x_0))}{\omega(B_\rho(x_0))}\,.$$

Summing up on I^+, from (17) and (19) we get

$$\begin{aligned}\frac{\alpha}{2}\,\delta'\,\nu(B_\rho(x_0)) &\leqslant \sum_{i\in I^+}\frac{4\gamma_1^2}{(1-\varepsilon)^2}\,(\eta\rho)^2\,\frac{\nu(B_{ii})}{\omega(B_{ii})}\int_{\{\varepsilon<u<1\}\cap B_{ii}} |Du|^2\,\omega\,dx \leqslant \\ &\leqslant \frac{4\gamma_1^2}{(1-\varepsilon)^2}\,\eta^{2(1-a)}\rho^2(K_2')^2\frac{(c_d(\nu))^2}{(c_d(\omega))^2}\,\frac{\nu(B_\rho(x_0))}{\omega(B_\rho(x_0))}\sum_{i\in I^+}\int_{\{\varepsilon<u<1\}\cap B_{ii}} |Du|^2\,\omega\,dx \leqslant \\ &\leqslant \frac{4\gamma_1^2}{(1-\varepsilon)^2}\,\eta^{2(1-a)}(K_2')^2\frac{(c_d(\nu))^2}{(c_d(\omega))^2}\,\beta\,\nu(B_\rho(x_0))\,.\end{aligned}$$

The conclusion follows by taking the limit $\eta \to 0$. $\square$

Here we state three results, which are corollaries rispectively of Theorem 2.5, Lemma 2.12, Lemma 2.14.

Corollary 2.15. *In the same assumptions of Theorem* 2.5 *suppose moreover* $a, b \in \mathbf{R}$, $a < b$. *Then*

$$\left[\frac{1}{\nu(B_\rho)}\int_a^b\!\!\int_{B_\rho} |u(x,t)|^p\nu(x)dxdt\right]^{1/p} \leqslant \gamma_1\,\rho\left[\frac{1}{\omega(B_\rho)}\int_a^b\!\!\int_{B_\rho} |Du(x,t)|^p\omega(x)dxdt\right]^{1/p} \qquad (20)$$

for every u Lipschitz continuous function in $B_\rho(x_0) \times (a,b)$ such that for every $t \in (a,b)$ $u(\cdot,t)$ has either support contained in $B_\rho(x_0)$ or null mean value (*with respect to the variable x*).

Proof - It is sufficient first to observe that $(\nu((B_\rho))^{-1}\int_{B_\rho}|u-u_{B_\rho}|^p\nu\,dx)^{1/p} \leqslant (\nu((B_\rho))^{-1}\int_{B_\rho}|u-u_{B_\rho}|^{q'}\nu\,dx)^{1/q'}$ for $q > p$, then to take the power p and integrate in time. $\square$

Corollary 2.16. *Consider $B_\rho = B_\rho(x_0)$ a ball, $a, b \in \mathbf{R}$, $a < b$, $p \in (1, +\infty)$, ν, ω and $v \in L^p(a, b; W^{1,p}(B_\rho, \nu, \omega))$ for which assumptions of Theorem 2.5 hold, $k, l \in \mathbf{R}$ with $k < l$. Consider also a subset Z of B_ρ and denote by $\bar\nu$ the function taking value 0 in Z and $\bar\nu \equiv \nu$ in $B_\rho \setminus Z$. Then*

$$(l-k)^p \, \bar\nu \otimes \mathcal{L}^1 \left(\{v < k\}\right) \bar\nu \otimes \mathcal{L}^1 \left(\{v > l\}\right) \leqslant$$
$$\leqslant 2^p \, \gamma_1^p \, \rho^p \, \bar\nu \otimes \mathcal{L}^1 \left(B_\rho \times (a,b)\right) \frac{\nu(B_\rho)}{\omega(B_\rho)} \iint_{(B_\rho \times (a,b)) \cap \{k < v < l\}} |Dv|^p \, \omega \, dxdt \, .$$

Proof - One can follow the proof of Lemma 2.12 integrating in space and time and finally applying Corollary 2.15. □

Corollary 2.17. *Consider $x_0 \in \Omega$ and $\rho > 0$ such that $B_{2\rho}(x_0) \subset \Omega$, $a, b \in \mathbf{R}$, $a < b$, $\sigma \in (0, \rho)$, $\omega \in A_2(K_1)$, $(\nu, \omega) \in B^1_{2,q}(K_2)$, $\nu \in A_\infty$, $q > 2$, $\alpha, \beta > 0$. Consider $\mathcal{B}$ an open and non-empty subset of $B_\rho(x_0)$ such that also $\mathcal{B}^\sigma = \{x \in \Omega \,|\, \mathrm{dist}(x, \mathcal{B}) < \sigma\}$ is a subset of $B_\rho(x_0)$, $a, b \in \mathbf{R}$, $a < b$. Then, for every $\varepsilon, \delta \in (0,1)$ there exists $\eta \in (0,1)$ such that for every $u \in L^2(a, b; W^{1,2}_{\mathrm{loc}}(\Omega, \nu, \omega))$ satisfying*

$$\int_a^b \!\! \int_{\mathcal{B}^\sigma} |Du|^2 \, \omega \, dxdt \leqslant \beta \, (b-a) \, \frac{\omega(B_\rho(x_0))}{\rho^2}$$

and

$$\nu \otimes \mathcal{L}^1 \big(\{u > 1\} \cap (\mathcal{B} \times (a,b))\big) \geqslant \alpha \, (b-a) \, \nu(B_\rho(x_0)),$$

there exists $x^* \in \mathcal{B}$ with $B_{\eta\rho}(x^*) \subset \mathcal{B}$ such that

$$\nu \otimes \mathcal{L}^1 \big(\{u > \varepsilon\} \cap (B_{\eta\rho}(x^*) \times (a,b))\big) > (1-\delta) \, (b-a) \, \nu(B_{\eta\rho}(x^*)).$$

Proof - One can repeat the proof of Lemma 2.14 using a family of disjoint cylinders $(B_{\eta\rho}(x_i) \times (a,b))_{i \in I}$ with the property that

$$\bigcup_{i \in I} B_{\eta\rho}(x_i) \subset \mathcal{B} \subset \bigcup_{i \in I} B_{2\eta\rho}(x_i) \subset \mathcal{B}^\sigma \, ,$$

taking the measure $\nu \otimes \mathcal{L}^1$ instead of ν and finally using Corollary 2.16 to conclude. □

We conclude stating a standard lemma (see, for instance, Lemma 7.1 in [13]) and one of its possible generalizations which will be needed later.

Lemma 2.18. *Let $(y_h)_h$ be a sequence of positive real numbers such that*

$$y_{h+1} \leqslant c \, b^h \, y_h^{1+\alpha}$$

with $c, \alpha > 0$, $b > 1$. If $y_0 \leqslant c^{-1/\alpha} b^{-1/\alpha^2}$ then

$$\lim_{h \to +\infty} y_h = 0 \, .$$

Lemma 2.19. *Let $(y_h)_h$ and $(\epsilon_h)_h$ two sequences of non-negative real numbers such that*

$$y_{h+1} \leqslant c \, b^h \, (y_h + \epsilon_h) \, y_h^\alpha \, , \qquad y_{h+1} \leqslant y_h \, , \qquad \lim_{h \to +\infty} \epsilon_h = 0 \, , \tag{21}$$

$c, \alpha > 0$, $b > 1$. If $y_0 < c^{-1/\alpha} b^{-1/\alpha^2}$ then

$$\lim_{h \to +\infty} y_h = 0 \, .$$

Proof - If $\epsilon_h = 0$ for every h we reduce to Lemma 2.18. Otherwise, say $\bar{y}$ the limit $\lim_h y_h$ which exists by the monotonicity of $(y_h)_h$ and suppose that

$$y_0 < c^{-1/\alpha} b^{-1/\alpha^2} .$$

Now, by contradiction, assume that

$$\bar{y} > 0 .$$

By assumptions we have that for each $\varepsilon > 0$ there is $\bar{h} = \bar{h}(\varepsilon)$ such that

$$\epsilon_h \leqslant \varepsilon \qquad \text{for every } h \geqslant \bar{h} . \tag{22}$$

Now for each $\delta > 0$ we choose ε such that $\varepsilon < \delta \, \bar{y}$ so that we get $\delta \, y_h \geqslant \varepsilon$ for every h. In particular for $h \geqslant \bar{h}$ we get

$$\begin{aligned} y_{h+1} &\leqslant c\, b^h \,(y_h + \epsilon_h)\, y_h^\alpha \leqslant \\ &\leqslant c\, b^h \,(y_h + \varepsilon)\, y_h^\alpha \leqslant \\ &\leqslant c\, b^h \,(y_h + \delta\, y_h)\, y_h^\alpha = \\ &= (1+\delta)\, c\, b^h \, y_h^{1+\alpha} . \end{aligned}$$

Using the lemma above we have that if $y_{\bar{h}} \leqslant (1+\delta)^{-1/\alpha} c^{-1/\alpha} b^{-1/\alpha^2}$ than $\lim_h y_h = \bar{y} = 0$, where $\bar{h}$ depends on ε which depends on the choice of δ. By the monotonicity of $(y_h)_h$ if $y_0 \leqslant (1+\delta)^{-1/\alpha} c^{-1/\alpha} b^{-1/\alpha^2}$ the condition on $y_{\bar{h}}$ is garanteed whatever the value of $\bar{h}$. Since $y_0 < c^{-1/\alpha} b^{-1/\alpha^2}$ there is $\delta > 0$ such that $y_0 \leqslant (1+\delta)^{-1/\alpha} c^{-1/\alpha} b^{-1/\alpha^2}$ and so we would derive that $\bar{y} = 0$, which contradicts the assumption $\bar{y} > 0$. □

3. Preliminaries about mixed type equations

This brief section is devoted to a remark about equations of mixed type, like for example

$$\mu(x) \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,Du)) = 0, \tag{23}$$

where a is a Caratheodory function such that

$$\begin{aligned} & a(x,t,0) = 0 , \\ & (a(x,t,\xi) - a(x,t,\eta), \xi - \eta) \geqslant \lambda(x) |\xi - \eta|^2 , \\ & |a(x,t,\xi) - a(x,t,\eta)| \leqslant L\, \lambda(x) |\xi - \eta| , \end{aligned} \tag{24}$$

for every $\xi, \eta \in \mathbf{R}^n$, where L is a positive constant and $\mu = \mu(x), \lambda = \lambda(x)$ are functions, λ positive, while μ may change sign (and also be zero in some positive measure regions).
Before talking about mixed type equations we want to recall that a weighted Sobolev space $H^1(\Omega, |\mu|, \lambda)$ endowed with the norm

$$\|u\|^2 := \int_\Omega u^2 |\mu| dx + \int_\Omega |Du|^2 \lambda dx$$

can be defined even if the function $|\mu|$ takes the value zero in a subset whose measure is positive (we refer to [20] for the definition and the completeness of this space). If we denote the space

$L^2(0,T;H^1_0(\Omega,|\mu|,\lambda))$ by $\mathcal{V}$ and the space $\{u \in \mathcal{V} \,|\, \mu u' \in \mathcal{V}'\}$ by $\mathcal{W}$ (u' denotes the derivative of u, $\mathcal{V}'$ the dual space of $\mathcal{V}$) one has that a solution of (23) belong to $\mathcal{W}$ and (see [19])

$$u \in \mathcal{W} \quad \Longrightarrow \quad t \mapsto \int_\Omega u^2(x,t)\mu(x)dx \quad \text{is continuous in } [0,T]\,,$$

and

$$\int_\Omega u^2(x,t)|\mu|(x)dx \quad \text{is finite for every } t \in [0,T]\,. \tag{25}$$

On the other hand, for u solution of (23), the function

$$t \mapsto \int_\Omega u^2(x,t)\lambda(x)dx \qquad \text{is not necessarily } L^\infty_{\text{loc}}(0,T) \tag{26}$$

even if it is finite for almost every t since $\displaystyle\int_0^T\int_\Omega u^2(x,t)\lambda(x)dx$ is finite. In the next section we will define a De Giorgi type class of functions requiring

$$t \mapsto \int_A u^2(x,t)|\mu|_\lambda(x)dx \quad \text{belongs to } L^\infty_{\text{loc}}(0,T) \quad \text{for every } A \subset\subset \Omega\,. \tag{27}$$

This is something more of the natural requirement (25) and this a priori is not guaranteed by the equation in a general situation, but in many cases it is true, as we mention below. This condition will be needed only if there is a region in which the equation reduces to a family of elliptic equations, i.e. if there is an open set in which $\mu = 0$.
More in general, using a corollary of Theorem 2.1 in [21] one can prove that, if u is the solution of the problem

$$\begin{cases} \mu\dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t)\cdot Du) = 0 & \text{in } \Omega\times(0,T) \\ u = \phi & \text{in } \partial\Omega\times(0,T) \\ u(x,0) = \varphi(x) & \text{in } \{x \in \Omega \,|\, \mu(x) > 0\} \\ u(x,0) = \psi(x) & \text{in } \{x \in \Omega \,|\, \mu(x) < 0\} \end{cases} \tag{28}$$

for some $\phi \in \mathcal{W}$, $\varphi, \psi \in L^2(\Omega)$, if

$$\phi_t \in \mathcal{W} \quad \text{and} \quad a \text{ is regular in time}$$

(we refer to [21] for the precise requirement about regularity of a) we derive that the function $w = \eta(u-\phi) \in H^1(0,T;H^1(\Omega,|\mu|,\lambda))$, and then in particular

$$u \in C^0((0,T);H^1(\Omega,|\mu|,\lambda)) \tag{29}$$

and as a by-product one gets that u satisfies (27) since $H^1(\Omega,|\mu|,\lambda) \subset L^2(\Omega,\lambda)$.

Analogous considerations hold for Neumann boundary conditions.

We observe that in general a solution of a family of elliptic equation will be not regular in time (if, e.g., a is not regular in time) as we will show with an example in the last section.

4. De Giorgi classes and Q-minima

From this section on we will focus our attention on a class of functions which contains the solutions of some forward-backward evolution equations, also possibly a family of elliptic equations, whose simplest example is the following (λ is positive, but μ is valued in $\mathbf{R}$)

$$\mu \frac{\partial u}{\partial t} - \operatorname{div}(\lambda D u) = 0 \qquad \text{in } \Omega \times (0,T)\,, \tag{30}$$

but one can think to (23) or to (3). The connection of the class we are going to define and this equation will be clarified below. We will show that solutions of such a homogeneous equation, of equation (23) and also of a wider class of homogeneous equations are *quasi-minimizers* (from now on we will call them more simply, and according to the original definition, Q-minima, see Definition 4.3) for equation (30), and Q-minima are contained in the De Giorgi class we are going to define.

Assumptions about μ and λ - Given μ and λ defined in $\mathbf{R}^n$, λ positive almost everywhere, while μ may be positive, null and negative, we define

$$\mu_\lambda := \begin{cases} \mu & \text{if } \mu \neq 0, \\ \lambda & \text{if } \mu = 0. \end{cases}$$

Once considered Ω on open subset of $\mathbf{R}^n$ and $T > 0$ we require μ and λ to satisfy what follows: there is $q > 2$ such that

$$\begin{aligned} &(\text{H.1}) \text{ - } \lambda \in A_2(K_1)\,, \\ &(\text{H.2}) \text{ - } (|\mu|_\lambda, \lambda) \in B^1_{2,q}(K_2)\,, \\ &(\text{H.3}) \text{ - } |\mu|_\lambda \in A_\infty(K_3, \varsigma)\,. \end{aligned}$$

This conditions (see Theorem 2.5) garantees the validity of the Sobolev-Poincaré type inequality

$$\Big[\frac{1}{|\mu|_\lambda(B_\rho)} \int_{B_\rho} |u(x)|^q |\mu|_\lambda(x) dx\Big]^{1/q} \leqslant \gamma_1\, \rho \,\Big[\frac{1}{\lambda(B_\rho)} \int_{B_\rho} |Du(x)|^2 \lambda(x) dx\Big]^{1/2}$$

and of all the results which follows (in particular Theorem 2.9 and Corollary 2.11).
The condition (H.2) (see Remark 2.7, point $\mathscr{A}$) garantees the existence of $\alpha \in (0,1)$, $\tilde{K}_2 > K_2$ depending on K_2 and $c_{rh}(|\mu|_\lambda)$ and $\tilde{q} \in (2,q)$ such that, thanks also to Remark 2.6,

$$(\text{H.2})' \text{ - } (|\mu|_\lambda, \lambda) \in B^\alpha_{2,\tilde{q}}(\tilde{K}_2) \subset B^\alpha_{2,2}(\tilde{K}_2)\,.$$

We will suppose that the sets

$$\Omega_+ := \{x \in \Omega \,|\, \mu(x) > 0\}, \quad \Omega_- := \{x \in \Omega \,|\, \mu(x) < 0\} \quad \text{and} \quad \Omega_0 := \Omega \setminus \big(\Omega_+ \cup \Omega_-\big)$$

are the union of a finite number of open and connected subsets of Ω. This means, for instance, that μ cannot change sign in a Cantor type set with positive measure.
Beyond to μ_+ and μ_-, which will denote respectively the positive and negative part of μ, we define

$$\lambda_+ := \begin{cases} \lambda & \text{in } \Omega_+ \\ 0 & \text{in } \Omega \setminus \Omega_+ \end{cases}, \quad \lambda_- := \begin{cases} \lambda & \text{in } \Omega_- \\ 0 & \text{in } \Omega \setminus \Omega_- \end{cases}, \quad \lambda_0 := \begin{cases} \lambda & \text{in } \Omega_0 \\ 0 & \text{in } \Omega \setminus \Omega_0 \end{cases}. \tag{31}$$

In this way notice that

$$|\mu|_\lambda = |\mu| + \lambda_0 = \mu_+ + \mu_- + \lambda_0.$$

Notice that hypotheses (H.1) and (H.3) (see (10)) implies that λ and $|\mu|_\lambda$ are doubling, i.e. there is a constant $\mathfrak{q}$ such that

$$|\mu|_\lambda\big(B_{2\rho}(x)\big) \leqslant \mathfrak{q}\,|\mu|_\lambda\big(B_\rho(x)\big), \\ \lambda\big(B_{2\rho}(x)\big) \leqslant \mathfrak{q}\,\lambda\big(B_\rho(x)\big) \tag{32}$$

for every $x \in \Omega$ and $\rho > 0$ for which $B_{2\rho}(x) \subset \Omega$.
Moreover by (12), once denoted by $c_{rh}(\lambda)$ the constant satisfying (11) with the weight λ and $\varsigma(\lambda)$ the constant appearing in (12) with $\omega = \lambda$ and $c_{rh}(|\mu|_\lambda)$ and $\varsigma(|\mu|_\lambda)$ the analogous with $\omega = |\mu|_\lambda$, we get that

$$\frac{\lambda(S)}{\lambda(Q)} \leqslant \kappa \left(\frac{|\mu|_\lambda(S)}{|\mu|_\lambda(Q)}\right)^{\tau}, \qquad \frac{|\mu|_\lambda(S)}{|\mu|_\lambda(Q)} \leqslant \kappa \left(\frac{\lambda(S)}{\lambda(Q)}\right)^{\tau} \tag{33}$$

where $\tau = \min\{\varsigma(\lambda)/r, \varsigma(|\mu|_\lambda)/2\}$ and $\kappa = \max\{c_{rh}(\lambda)\, K_3^{\varsigma(\lambda)/r}, c_{rh}(|\mu|_\lambda)\, K_1^{\varsigma(|\mu|_\lambda)/2}\}$.
Once defined I, the set of "interfaces" as follows:

$$I := \big(\partial\Omega_+ \cap \partial\Omega_0\big) \cup \big(\partial\Omega_+ \cap \partial\Omega_-\big) \cup \big(\partial\Omega_0 \cap \partial\Omega_-\big)$$

we moreover will assume the following additional assumptions where, for simplicity, we assume the first holds with the the same constant $\mathfrak{q}$ as before:

$$\text{(H.4) - } \left| \begin{array}{ll} \mu_+\big(B_{2\rho}(x)\big) \leqslant \mathfrak{q}\,\mu_+\big(B_\rho(x)\big) & \text{for every } x \in \overline{\Omega}_+, \\ \mu_-\big(B_{2\rho}(y)\big) \leqslant \mathfrak{q}\,\mu_-\big(B_\rho(y)\big) & \text{for every } x \in \overline{\Omega}_-, \\ \lambda_0\big(B_{2\rho}(z)\big) \leqslant \mathfrak{q}\,\lambda_0\big(B_\rho(z)\big) & \text{for every } x \in \overline{\Omega}_0, \end{array} \right.$$

$$\text{(H.5) - } I \text{ is a such that } \lim_{\varepsilon \to 0^+} |I^\varepsilon| = 0,$$

where (H.4) holds for every $\rho > 0$ for which $B_{2\rho}(x) \subset \Omega$ and I^ε is the open ε-neighbourhood of I and is defined in (34).

Some comments about (H.4) and (H.5) are in order. First notice that since $|\mu|_\lambda$ satisfies (32), at least one of the three requirements in (H.4) holds for every $x \in \Omega$.
Notice moreover that assumption (H.4) is deeply connected to a geometric requirement about the set I of interfaces, indeed (H.4) has to hold in particular for points belonging to I. Finally, about the set I, notice that (H.5) is weaker than the requirement that I is a $\mathcal{H}^{n-1}$-rectifiable set because I could be also not rectifiable. For all these comments we refer to the last section, in which some examples are shown.

Some notations - By $u_+(y)$ the function $\max\{u(y), 0\}$ and by $u_-(y)$ $\max\{-u(y), 0\}$. We will write u_+^2 or u_-^2 to denote

$$u_+^2(y) := (u_+(y))^2, \qquad u_-^2(y) := (u_-(y))^2.$$

Given $A \subset \Omega$ we will denote, for a given $\varepsilon > 0$,

$$A^\varepsilon := \big\{x \in \Omega \,\big|\, \mathrm{dist}(x, A) < \varepsilon\big\}, \qquad A_\varepsilon := \big\{x \in \Omega \,\big|\, \mathrm{dist}(x, A^c) < \varepsilon\big\}, \\ \text{while for } \varepsilon = 0 \quad A^\varepsilon = A_\varepsilon := A. \tag{34}$$

Fix, beyond x_0, $t_0 \in (0,T)$. For a given $\varepsilon > 0$ and a ball $B_\rho(x_0)$ we define the sets

$$
\begin{aligned}
I_{\rho,\varepsilon}(x_0) &:= (I \cap B_\rho(x_0))^\varepsilon\,, \qquad B^0_\rho(x_0) := B_\rho(x_0) \cap \Omega_0 \\
B^+_\rho(x_0) &:= B_\rho(x_0) \cap \Omega_+\,, \qquad B^-_\rho(x_0) := B_\rho(x_0) \cap \Omega_-\,,
\end{aligned}
$$

$$
I^+_\rho(x_0) := I \cap \overline{B^+_\rho}(x_0)\,, \qquad I^-_\rho(x_0) := I \cap \overline{B^-_\rho}(x_0)\,, \qquad I^0_\rho(x_0) := I \cap \overline{B^0_\rho}(x_0)\,,
$$

$$
\begin{aligned}
I^+_{\rho,\varepsilon}(x_0) &:= (I^+_\rho(x_0))^\varepsilon \cap B^+_\rho(x_0)\,, \qquad I^{\rho,\varepsilon}_+(x_0) := (I^+_\rho(x_0))^\varepsilon \setminus I^+_{\rho,\varepsilon}(x_0)\,, \\
I^-_{\rho,\varepsilon}(x_0) &:= (I^-_\rho(x_0))^\varepsilon \cap B^-_\rho(x_0)\,, \qquad I^{\rho,\varepsilon}_-(x_0) := (I^-_\rho(x_0))^\varepsilon \setminus I^-_{\rho,\varepsilon}(x_0)\,, \\
I^0_{\rho,\varepsilon}(x_0) &:= (I^0_\rho(x_0))^\varepsilon \cap B^0_\rho(x_0)\,, \qquad I^{\rho,\varepsilon}_0(x_0) := (I^0_\rho(x_0))^\varepsilon \setminus I^0_{\rho,\varepsilon}(x_0)\,.
\end{aligned}
$$

We define the following functions

$$
h(x_0,\rho) := \frac{|\mu|_\lambda\,(B_\rho(x_0))}{\lambda\,(B_\rho(x_0))}\,, \qquad f(x_0,\rho) := h(x_0,\rho)\rho^2\,. \tag{35}
$$

These functions depend a priori on x_0, but just for simplicity we will not specify this dependence writing only $h(\rho)$ and $f(\rho)$ if not strictly necessary.
Notice that the function h satisfies, if $\mu \neq 0$ almost everywhere, the following inequalities

$$
h(x_0,\rho) \leqslant \mathfrak{q}\, h(x_0,2\rho)\,, \qquad h(x_0,2\rho) \leqslant \mathfrak{q}\, h(x_0,\rho)\,. \tag{36}
$$

Other sets we define are the following: fix $x_0 \in \Omega$ and $t_0 \in (0,T)$, $R > 0$, $\beta > 0$ and $s_1, s_2 \in (0,T)$ with $s_1 < t_0 < s_2$ and satisfying

$$
\begin{array}{lll}
i\,) & s_2 - t_0 = t_0 - s_1 = \beta\, h(x_0,R)R^2 & \text{when we consider } B^+_R(x_0) \text{ or } B^-_R(x_0)\,, \\
ii\,) & s_1, s_2 \quad \text{arbitrary} & \text{when we consider } B^0_R(x_0)\,.
\end{array} \tag{37}
$$

Inside the cylinder $B_R(x_0) \times (s_1,s_2)$ for

$$
\theta \in [0,1)
$$

we define

$$
\sigma_\theta := \theta\, \beta\, h(x_0,R)\, R^2\,. \tag{38}
$$

in such a way that $\sigma_\theta \in [0, \beta\, h(x_0,R)R^2)$; then for $\rho \in (0,R)$ and $\varepsilon > 0$ and taking s_1, s_2 as in (37), point $i\,$), we define the sets

$$
\begin{aligned}
& Q^{\beta,>}_R(x_0,t_0) := B_R(x_0) \times (t_0,s_2)\,, \qquad Q^{\beta,<}_R(x_0,t_0) := B_R(x_0) \times (s_1,t_0)\,, \\
& Q^{\beta,+}_R(x_0,t_0) := B^+_R(x_0) \times (t_0,s_2)\,, \qquad Q^{\beta,-}_R(x_0,t_0) := B^-_R(x_0) \times (s_1,t_0)\,, \\
& Q^{\beta,+}_{R;\rho,\theta}(x_0,t_0) := B^+_\rho(x_0) \times (t_0+\sigma_\theta, s_2)\,, \\
& Q^{\beta,-}_{R;\rho,\theta}(x_0,t_0) := B^-_\rho(x_0) \times (s_1, t_0-\sigma_\theta)\,, \\
& Q^{\beta,+,\varepsilon}_{R;\rho,\theta}(x_0,t_0) := \begin{cases} B_{\rho+\varepsilon}(x_0) \times (t_0+\sigma_\theta, s_2) & \text{if } B^+_{\rho+\varepsilon}(x_0) = B_{\rho+\varepsilon}(x_0), \\ \big((B^+_\rho(x_0))^\varepsilon \times (t_0+\sigma_\theta,s_2)\big) \cup \big((I^+_\rho(x_0))^\varepsilon \times (t_0,s_2)\big) & \text{otherwise,} \end{cases} \\
& Q^{\beta,-,\varepsilon}_{R;\rho,\theta}(x_0,t_0) := \begin{cases} B_{\rho+\varepsilon}(x_0) \times (s_1, t_0-\sigma_\theta) & \text{if } B^-_{\rho+\varepsilon}(x_0) = B_{\rho+\varepsilon}(x_0), \\ \big((B^-_\rho(x_0))^\varepsilon \times (s_1,t_0-\sigma_\theta)\big) \cup \big((I^-_\rho(x_0))^\varepsilon \times (s_1,t_0)\big) & \text{otherwise,} \end{cases}
\end{aligned} \tag{39}
$$

and with s_1, s_2 arbitrary (see (37)) we define

$$
\begin{aligned}
& Q^0_{R;\rho;s_1,s_2}(x_0) := B^0_\rho(x_0) \times (s_1,s_2) \quad \text{for } \rho \leqslant R\,, \\
& Q^{0,\varepsilon}_{R;\rho;s_1,s_2}(x_0) := (B^0_\rho(x_0))^\varepsilon \times (s_1,s_2)
\end{aligned} \tag{40}
$$

The first subscript R below Q denotes that $s_2 - t_0$ and $t_0 - s_1$ are proportional to R^2 and that we consider subsets of $B_R \times (0,T)$.

We now introduce the De Giorgi class for equation (23).
In the following definition we will use the measures μ_+ and μ_- rescaled by the factor $h(x_0,R)$. We will make the implicit assumption that the support of these measures (or functions) is the same of μ_+ and μ_-, i.e.

$$\frac{\mu_+}{h(x_0,R)}(x) := \begin{cases} \dfrac{\mu_+}{h(x_0,R)} & \text{if } \mu_+(x) > 0\,, \\ 0 & \text{if } \mu_+(x) = 0\,, \end{cases} \qquad \frac{\mu_-}{h(x_0,R)} := \begin{cases} \dfrac{\mu_-}{h(x_0,R)} & \text{if } \mu_-(x) > 0\,, \\ 0 & \text{if } \mu_-(x) = 0\,. \end{cases}$$

Moreover in the definition which follows we require that $u \in L^\infty_{\mathrm{loc}}((0,T); L^2_{\mathrm{loc}}(\Omega, |\mu|_\lambda))$ even if only the terms

$$\int_{B_\rho} u^2(x,t)\mu_+(x)dx \quad \text{and} \quad \int_{B_\rho} u^2(x,t)\mu_-(x)dx$$

are, a priori, bounded (see Section 3). The fact that also $\displaystyle\int_{B^0_\rho} u^2(x,t)\lambda(x)dx$ is to be finite will be needed, for instance, to prove point *iii*) of Theorem 5.1.

Definition 4.1 (De Giorgi classes)**.** *Consider Ω an open subset of $\mathbf{R}^n$ and $T > 0$ and a point $(x_0,t_0) \in \Omega \times (0,T)$. Consider $R, r, \tilde r > 0$, $r < \tilde r \leqslant R$, $\beta > 0$, $\theta, \tilde\theta$ such that $0 \leqslant \tilde\theta < \theta < 1$, $s_1, s_2, t_0 \in (0,T)$, $s_1 < t_0 < s_2$ satisfying* (37)*. We say that a function*

$$u \in L^2_{\mathrm{loc}}(0,T; H^1_{\mathrm{loc}}(\Omega, |\mu|, \lambda)) \cap L^\infty_{\mathrm{loc}}((0,T); L^2_{\mathrm{loc}}(\Omega, |\mu|_\lambda))$$

belongs to the De Giorgi class $DG_+(\Omega, T, \mu, \lambda, \gamma)$, being γ a positive constant, if for every $\varepsilon \in [0, R - \tilde r]$ and $\theta - \tilde\theta = (\tilde r - r)^2/R^2$ and every $k \in \mathbf{R}$ the following inequalities hold (σ_θ is defined in (38)*):*

i) *for $s_2 = t_0 + \beta\, h(x_0,R)R^2$ and $B_R(x_0) \times [t_0, s_2] \subset \Omega \times (0,T)$*

$$\begin{aligned} \sup_{t \in (t_0 + \sigma_\theta, s_2)} \int_{B^+_{r+\varepsilon}} (u-k)^2_+(x,t)\mu_+(x)dx + \sup_{t \in (t_0, t_0+\sigma_{\tilde\theta})} \int_{I^{r,\varepsilon}_+} (u-k)^2_+(x,t)\mu_-(x)\,dx \\ + \iint_{Q^{\beta,+,\varepsilon}_{R;r,\theta}} |D(u-k)_+|^2\,\lambda\,dxds \leqslant \\ \leqslant \gamma \Bigg[\sup_{t \in (t_0, t_0+\sigma_{\tilde\theta})} \int_{I^+_{r,\tilde r - r+\varepsilon}} (u-k)^2_+(x,t)\mu_+(x)\,dx + \\ + \sup_{t \in (t_0+\sigma_\theta, s_2)} \int_{I^{r,\tilde r - r+\varepsilon}_+} (u-k)^2_+(x,t)\mu_-(x)\,dx + \\ + \frac{1}{(\tilde r - r)^2} \iint_{Q^{\beta,+,\tilde r - r+\varepsilon}_{R;r,\tilde\theta}} (u-k)^2_+ \left(\frac{\mu_+}{\beta\, h(x_0,R)} + \lambda \right) dxdt \Bigg]; \end{aligned} \tag{41}$$

ii) for $s_1 = t_0 - \beta\, h(x_0,R)R^2$ *and* $B_R(x_0)\times[s_1,t_0]\subset\Omega\times(0,T)$

$$\sup_{t\in(s_1,t_0-\sigma_\theta)}\int_{B^-_{r+\varepsilon}}(u-k)^2_+(x,t)\mu_-(x)dx+\sup_{t\in(t_0-\sigma_{\tilde\theta},t_0)}\int_{I^{r,\varepsilon}_-}(u-k)^2_+(x,t)\mu_+(x)\,dx$$
$$+\iint_{Q^{\beta,-,\varepsilon}_{R;r,\theta}}|D(u-k)_+|^2\,\lambda\,dxds\leqslant$$
$$\leqslant\gamma\Bigg[\sup_{t\in(t_0-\sigma_{\tilde\theta},t_0)}\int_{I^-_{r,\tilde r-r+\varepsilon}}(u-k)^2_+(x,t)\mu_-(x)\,dx+ \tag{42}$$
$$+\sup_{t\in(s_1,t_0-\sigma_\theta)}\int_{I^{r,\tilde r-r+\varepsilon}_-}(u-k)^2_+(x,t)\mu_+(x)\,dx+$$
$$+\frac{1}{(\tilde r-r)^2}\iint_{Q^{\beta,-,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}}(u-k)^2_+\left(\frac{\mu_-}{\beta\,h(x_0,R)}+\lambda\right)dxdt\Bigg];$$

iii) for s_1 *and* s_2 *arbitrary and* $B_R(x_0)\times[s_1,s_2]\subset\Omega\times(0,T)$

$$\iint_{Q^{0,\varepsilon}_{R;r;s_1,s_2}(x_0)}|D(u-k)_+|^2\lambda\,dxdt\leqslant$$
$$\leqslant\gamma\Bigg[\sup_{t\in(s_1,s_2)}\int_{I^{r,\tilde r-r+\varepsilon}_0}(u-k)^2_+(x,t)\mu_-(x)\,dx+$$
$$+\sup_{t\in(s_1,s_2)}\int_{I^{r,\tilde r-r+\varepsilon}_0}(u-k)^2_+(x,t)\mu_+(x)\,dx+ \tag{43}$$
$$+\frac{1}{(\tilde r-r)^2}\iint_{Q^{0,\tilde r-r+\varepsilon}_{R;r;s_1,s_2}}(u-k)^2_+\,\lambda\,dxdt\Bigg];$$

iv) for every $s_2>t_0$ *such that* $B_R(x_0)\times[t_0,s_2]\subset\Omega\times(0,T)$

$$\sup_{t\in(t_0,s_2)}\int_{B^+_r}(u-k)^2_+(x,t)\mu_+(x)dx\leqslant\int_{B^+_{\tilde r}}(u-k)^2_+(x,t_0)\mu_+(x)dx+$$
$$+\sup_{t\in(t_0,s_2)}\int_{I^{r,\tilde r-r}_+}(u-k)^2_+(x,t)\mu_-(x)\,dx+ \tag{44}$$
$$+\ \gamma\,\frac{1}{(\tilde r-r)^2}\int_{t_0}^{s_2}\!\!\int_{B^+_{\tilde r}\cup I^{r,\tilde r-r}_+}(u-k)^2_+\,\lambda\,dxdt;$$

v) for every $s_1<t_0$ *such that* $B_R(x_0)\times[s_1,t_0]\subset\Omega\times(0,T)$

$$\sup_{t\in(s_1,t_0)}\int_{B^-_r}(u-k)^2_+(x,t)\mu_-(x)dx\leqslant\int_{B^-_{\tilde r}}(u-k)^2_+(x,t_0)\mu_-(x)dx+$$
$$+\sup_{t\in(t_0,s_2)}\int_{I^{r,\tilde r-r}_-}(u-k)^2_+(x,t)\mu_+(x)\,dx+ \tag{45}$$
$$+\ \gamma\,\frac{1}{(\tilde r-r)^2}\int_{s_1}^{t_0}\!\!\int_{B^-_{\tilde r}\cup I^{r,\tilde r-r}_-}(u-k)^2_+\,\lambda\,dxdt\,.$$

We will say that u belongs to $DG_-(\Omega, T, \mu, \lambda, \gamma)$ *if the estimates above holds for* $(u-k)_-$ *in the place of* $(u-k)_+$. *We will say that u belongs to* $DG(\Omega, T, \mu, \lambda, \gamma)$ *if* $u \in DG_+(\Omega, T, \mu, \lambda, \gamma) \cap DG_-(\Omega, T, \mu, \lambda, \gamma)$.

REMARK 4.2. - Notice that if $|\mu|(B_R(x_0)) = 0$, that is $B_R(x_0) \subset \Omega_0$, (41), (42) and (43) coincide and reduce to

$$\int_{s_1}^{s_2}\int_{B_r} |D(u-k)_+|^2 \lambda \, dxdt \leqslant \gamma \frac{1}{(\tilde{r}-r)^2} \int_{s_1}^{s_2}\int_{B_r} (u-k)_+^2 \lambda \, dxdt$$

by which we can derive

$$\int_{B_r(x_0)} |D(u-k)_+|^2(x,t) \, \lambda(x) \, dx \leqslant \gamma \frac{1}{(\tilde{r}-r)^2} \int_{B_{\tilde{r}}(x_0)} (u-k)_+^2(x,t) \, \lambda(x) \, dx \tag{46}$$

for *almost* every $t \in [s_1, s_2]$. Since by assumption $u \in L^\infty_{\text{loc}}((0,T); L^2_{\text{loc}}(\Omega, |\mu|_\lambda))$ we get as a by-product that $u \in L^\infty_{\text{loc}}((0,T); H^1_{\text{loc}}(\Omega_0, \lambda, \lambda))$.
In some cases we can derive that (46) can hold for *every* $t \in [s_1, s_2]$ (see the previous section).

The estimates given in Definition 4.1 are also known as *energy estimates* or *Caccioppoli's estimates* and we will often refer to them in this way.

Now denote by $\mathcal{K}(\Omega \times (0,T))$ the set $\{K \subset \Omega \times (0,T) \,|\, K \text{ compact}\}$ and consider the functional

$$E : L^2(0,T; H^1(\Omega)) \times \mathcal{K}(\Omega \times (0,T)) \to \mathbf{R}, \qquad E(w,K) = \frac{1}{2} \iint_K |Dw|^2 \lambda \, dxdt \,.$$

We are going to define a Q-minimum following the definition given in [25] (see also [12] for the elliptic case).

Definition 4.3. *We will call a function* $u : \Omega \times (0,T) \to \mathbf{R}$ *a* Q*-minimum for the equation* (30) *if* $u \in L^2_{\text{loc}}(0,T; H^1_{\text{loc}}(\Omega, \, |\mu|, \lambda)) \cap L^\infty_{\text{loc}}((0,T); L^2_{\text{loc}}(\Omega, |\mu|_\lambda))$ *and there is a constant* $Q \geqslant 1$ *such that*

$$-\iint_{\text{supp}(\phi)} u \frac{\partial \phi}{\partial t} \mu \, dxdt + E(u, \text{supp}(\phi)) \leqslant Q \, E(u-\phi, \text{supp}(\phi)) \tag{47}$$

for every $\phi \in C^1_c(\Omega \times (0,T))$.

REMARK 4.4. - It is easy to verify that if $u \in L^2(0,T; H^1(\Omega, |\mu|, \lambda))$ is a Q-minimum for equation (30) than the map $L\phi := -\iint_{\text{supp}(\phi)} u \frac{\partial\phi}{\partial t} \mu \, dxdt$ with $\phi \in C^1_c(\Omega \times (0,T))$ turns out to be a linear and continuous form in $L^2(0,T; H^1_0(\Omega, |\mu|, \lambda))$, i.e. L belongs to the dual space $L^2(0,T; (H^1(\Omega, |\mu|, \lambda))')$ (the proof can be obtained following the analogous one in [25]).

Solutions are Q-minima - Following the analogous proof in [25] one can verify that u is a solution of (30) if and only if u is a 1-minimum for (30).
A second interesting fact is that a solution of (23) is a Q-minimum for the equation (30). Indeed using (24) it is easy to see that a solution of (23) satisfies (47) with $Q = 2LM$.

Q-minima belong to the class DG - We now want to show that the De Giorgi class defined above contains Q-minima and in particular solutions of (30). In Section 7 we will show a Harnack type inequality, and then Hölder continuity, for functions in the De Giorgi classes, and consequently for Q-minima and solutions of (30). To show this, first of all notice that if u

satisfies (47) for every $\phi \in C^1_c(\Omega \times (0,T))$ then, by density of $C^1_c(\Omega \times (0,T))$ in $\mathcal{W}$, u satisfies (47) also for $\phi \in \mathcal{W}$; then in particular we could choose $\phi = (u-k)_+\zeta^2$ with ζ a Lipschitz continuous and non-negative function such that $\zeta(\cdot,t) \in \mathrm{Lip}_0(B_R(x_0))$, $|\nabla\zeta|, \zeta_t \in L^\infty$, $\zeta_t\mu \geqslant 0$. To show this fact it is sufficient to consider a point $(x_0,t_0) \in \Omega \times (0,T)$, a function $(u-k)_+\zeta^2$ with ζ defined in $[s_1,s_2] \times B_R(x_0)$ with $0 < s_1 < t_0 < s_2 < T$ and $s_2 - t_0 = \beta\, h(x_0,R)R^2$ if $\mu_+(B_R(x_0)) > 0$, $t_0 - s_1 = \beta\, h(x_0,R)R^2$ if $\mu_-(B_R(x_0)) > 0$, while if $B_R(x_0) \subset \Omega_0$ s_1 and s_2 arbitrary; then for arbitrary σ_1, σ_2 satisfying $s_1 \leqslant \sigma_1 < \sigma_2 \leqslant s_2$ choose $\phi_\epsilon = (u-k)_+\zeta^2\tau_\epsilon$ where

$$\tau_\epsilon(t) = \begin{cases} 1 & t \in [\sigma_1,\sigma_2] \\ \epsilon^{-1}(t-\sigma_1+\epsilon) & t \in [\sigma_1-\epsilon,\sigma_1] \\ -\epsilon^{-1}(t-\sigma_2-\epsilon) & t \in [\sigma_2,\sigma_2+\epsilon] \\ 0 & t \notin [\sigma_1-\epsilon,\sigma_2+\epsilon] \end{cases}$$

for a suitable $\epsilon > 0$. Taking such a ϕ_ϵ in (47) and letting ϵ go to zero one gets that

$$\begin{aligned} &\frac{1}{2}\int_{B_R} (u-k)_+^2(x,\sigma_2)\zeta^2(x,\sigma_2)\,\mu(x)\,dx + E(u,K) \leqslant Q\,E(u-\phi,K)+ \\ &\quad +\frac{1}{2}\int_{B_R} (u-k)_+^2(x,\sigma_1)\zeta^2(x,\sigma_1)\,\mu(x)\,dx + \int_{\sigma_1}^{\sigma_2}\!\!\int_{B_R} (u-k)_+^2\zeta\zeta_t\,\mu\,dxdt \end{aligned} \tag{48}$$

where we simply denote B_R instead of $B_R(x_0)$ and K denotes the part of the support of ζ contained in $B_R \times [\sigma_1,\sigma_2]$.

1° - First suppose $\mu_+(B_R(x_0)) > 0$ and show (41) and (44). We proceed as follows: consider $\phi = (u-k)_+\zeta^2$ with ζ a Lipschitz continuous function to be choosen later. Since we have that

$$u - \phi = \begin{cases} u & u \leqslant k \\ (u-k)(1-\zeta^2)+k & u > k\,. \end{cases}$$

and $\mathrm{supp}(\phi) \subset \{u > k\}$ we have that

$$\begin{aligned} E(u-\phi,\mathrm{supp}(\phi)) &= \frac{1}{2}\iint_{\mathrm{supp}(\phi)} \left|D\left[(u-k)_+(1-\zeta^2)\right]\right|^2 \lambda\,dxdt \leqslant \\ &\leqslant \iint_{\mathrm{supp}(\phi)} \left[(1-\zeta^2)^2|D(u-k)_+|^2 + 4(u-k)_+^2\zeta^2|D\zeta|^2\right]\lambda\,dxdt\,. \end{aligned} \tag{49}$$

We first prove (41). We consider $r, \tilde{r} > 0$ with $r < \tilde{r} < R$, $t_0, s_2 \in (0,T)$ with $s_2 - t_0 = \beta\, h(x_0,R)R^2$, $\theta, \tilde{\theta}$ such that $0 \leqslant \tilde{\theta} < \theta < 1$. By assuming in addition that for $\varepsilon \geqslant 0$ (and sufficiently small, say $\varepsilon < R - \tilde{r}$)

$$K := \mathrm{supp}(\zeta) \cap \big(B_R(x_0) \times [s_1,s_2]\big) \subset Q^{\beta,+,\tilde{r}-r+\varepsilon}_{R;r,\tilde{\theta}}(x_0,t_0)$$

and that $|\zeta| \leqslant 1$, on the right hand side we estimate $(1-\zeta^2)^2$ by $1-\zeta^2$ and the second term by $4(u-k)_+^2|D\zeta|^2$. Moreover using the assumption that u is a Q-minimum and since

$E(u,K) = E((u-k)_+, K)$ we get that for every $\tau_1, \tau_2 \in [t_0, s_2]$ with $\tau_1 < \tau_2$

$$\begin{aligned}\int_{B_{\tilde r+\varepsilon}} (u-k)_+^2(x,\tau_2)\zeta^2(x,\tau_2)\mu(x)\,dx - \int_{B_{\tilde r+\varepsilon}} (u-k)_+^2(x,\tau_1)\zeta^2(x,\tau_1)\mu(x)\,dx + \\ + 2Q\int_{\tau_1}^{\tau_2}\int_{B_{\tilde r+\varepsilon}} |D(u-k)_+|^2\zeta^2\lambda\,dxdt \leqslant \\ \leqslant 2\int_{\tau_1}^{\tau_2}\int_{B_{\tilde r+\varepsilon}} (u-k)_+^2\,\zeta\zeta_t\,\mu\,dxdt + 8Q\int_{\tau_1}^{\tau_2}\int_{B_{\tilde r+\varepsilon}} (u-k)_+^2|D\zeta|^2\lambda\,dxdt + \\ + (2Q-1)\iint_{Q^{\beta,+,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}\cap(B_R\times[\tau_1,\tau_2])} |D(u-k)_+|^2\lambda\,dxdt\,.\end{aligned}$$

We then choose a Lipschitz continuous function ζ (see also Figure A below where we show an example where $\mu > 0$ and $\mu < 0$) satisfying also

$$\begin{aligned}&\zeta = 1 \quad \text{in } Q^{\beta,+,\varepsilon}_{R;r,\theta}(x_0,t_0)\,, \qquad \zeta = 0 \quad \text{in } Q^{\beta,>}_{R}(x_0,t_0)\setminus Q^{\beta,+,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}(x_0,t_0)\,, \\ &|D\zeta| \leqslant \frac{1}{\tilde r - r}\,, \qquad \theta - \tilde\theta = \frac{(\tilde r - r)^2}{R^2}\,, \\ &|\zeta_t| \leqslant \frac{1}{\sigma_\theta - \sigma_{\tilde\theta}} = \frac{1}{\beta\, h(x_0,R)(\tilde r - r)^2}\,, \qquad \zeta_t\mu \geqslant 0\,, \qquad \zeta_t\mu_- = 0\,.\end{aligned} \tag{50}$$

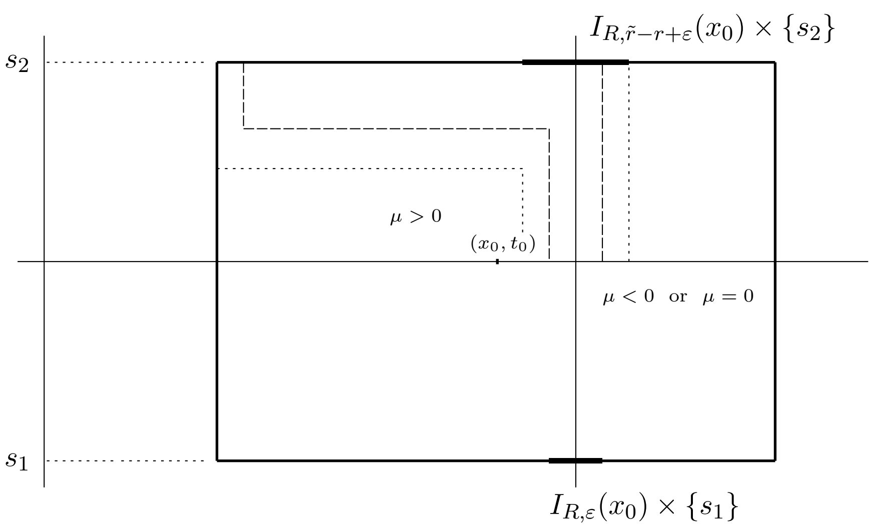


Figure A

Plugging such a ζ into the last inequality and dividing by $2Q$ we get that

$$\frac{1}{2Q}\int_{B^+_{r+\varepsilon}\cup I^{r,\tilde r-r+\varepsilon}_+}(u-k)^2_+(x,\tau_2)\mu(x)\,dx-\frac{1}{2Q}\int_{I^+_{r,\tilde r-r+\varepsilon}\cup I^{r,\varepsilon}_+}(u-k)^2_+(x,\tau_1)\mu(x)\,dx+$$
$$+\iint_{Q^{\beta,+,\varepsilon}_{R;r,\theta}\cap(B_R\times[\tau_1,\tau_2])}|D(u-k)_+|^2\lambda\,dxdt\leqslant$$
$$\leqslant\frac{1}{2Q}\frac{1}{(\tilde r-r)^2}\iint_{Q^{\beta,+,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}\cap(B_R\times[\tau_1,\tau_2])}(u-k)^2_+\left(8Q\lambda+\frac{2}{\beta\,h(x_0,R)}\mu_+\right)dxdt+ \tag{51}$$
$$+\frac{2Q-1}{2Q}\iint_{Q^{\beta,+,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}\cap(B_R\times[\tau_1,\tau_2])}|D(u-k)_+|^2\lambda\,dxdt$$

with

$$\tau_1\in[t_0,t_0+\sigma_{\tilde\theta}(R)]\quad\text{and}\quad\tau_2\in[t_0+\sigma_\theta(R),s_2]\,.$$

Before going on with the proof we state two lemmas, the first result is a slight generalization of Lemma 5.1 in [11] (see also Section 4 in [25]).

Lemma 4.5. *Consider some non-negative functions $f,g_1,g_2:[t_0,s_2]\times(0,R]\times[0,R]\to[0,M]$, $F,G:[t_0,s_2]\times(0,R]\times[0,1)\times[0,R]\to(0,M]$, M positive constant, satisfying*

$$f(\tau_2,\rho,\varepsilon)+g_2(\tau_1,\rho,\varepsilon)+F(\tau_1,\tau_2;\rho,\vartheta,\varepsilon)\leqslant g_1(\tau_1,\rho,\tilde\varepsilon)+g_2(\tau_2,\rho,\tilde\varepsilon)+$$
$$+\frac{1}{(\tilde\varepsilon-\varepsilon)^2}G(\tau_1,\tau_2;\rho,\tilde\vartheta,\tilde\varepsilon)+\delta\,F(\tau_1,\tau_2;\rho,\tilde\vartheta,\tilde\varepsilon) \tag{52}$$

and

$$g_1(\tau_1,\rho,\varepsilon)\leqslant g_1(\tau_1,\tilde\rho,\tilde\varepsilon)\,,\quad g_2(\tau_2,\rho,\varepsilon)\leqslant g_2(\tau_2,\tilde\rho,\tilde\varepsilon)\,,$$
$$F(\tau_1,\tau_2;\rho,\vartheta,\varepsilon)\leqslant F(\tau_1,\tau_2;\tilde\rho,\tilde\vartheta,\tilde\varepsilon)$$

for every $\tau_1,\tau_2\in[t_0,s_2]$, $\tau_1<\tau_2$, for every $\rho\leqslant\tilde\rho,\tilde\vartheta\leqslant\vartheta,\varepsilon\leqslant\tilde\varepsilon$ and $\delta\in(0,1)$. Then there is a constant $c>1$ depending only on δ such that

$$f(\tau_2,\rho,\varepsilon)+g_2(\tau_1,\rho,\varepsilon)+F(\tau_1,\tau_2;\rho,\vartheta,\varepsilon)\leqslant$$
$$\leqslant\frac{1}{1-\delta}\big[g_1(\tau_1,\rho,\tilde\varepsilon)+g_2(\tau_2,\rho,\tilde\varepsilon)\big]+\frac{c}{(\tilde\varepsilon-\varepsilon)^2}G(\tau_1,\tau_2;\rho,\tilde\vartheta,\tilde\varepsilon)\,.$$

Proof - We take the sequence ρ_n defined by (η to be chosen)

$$\vartheta_0=\vartheta\,,\quad\vartheta_{n+1}=\vartheta_n+(1-\eta)(\tilde\vartheta-\vartheta)\eta^n,\qquad\eta\in(0,1)\,,$$
$$\varepsilon_0=\varepsilon\,,\quad\varepsilon_{n+1}=\varepsilon_n+(1-\eta)(\tilde\varepsilon-\varepsilon)\eta^n,\qquad\eta\in(0,1)\,.$$

Notice that

$$\varepsilon_{n+1}-\varepsilon_0=\varepsilon_{n+1}-\varepsilon=(\tilde\varepsilon-\varepsilon)(1-\eta^{n+1})\,,$$
$$\varepsilon_0+\sum_{k=0}^{\infty}(\varepsilon_{n+1}-\varepsilon_n)=\tilde\varepsilon\,,\qquad\vartheta_0+\sum_{k=0}^{\infty}(\vartheta_{n+1}-\vartheta_n)=\tilde\vartheta\,.$$

By (52) we have

$$
\begin{aligned}
& f(\tau_2,\rho,\varepsilon_0) + g_2(\tau_1,\rho,\varepsilon_0) + F(\tau_1,\tau_2;\rho,\vartheta_0,\varepsilon_0) \leqslant \\
& \qquad \leqslant g_1(\tau_1,\rho,\varepsilon_1) + g_2(\tau_2,\rho,\varepsilon_1) + \\
& \qquad\quad + \frac{1}{(\varepsilon_1-\varepsilon_0)^2} G(\tau_1,\tau_2;\rho,\vartheta_1,\varepsilon_1) + \delta\, F(\tau_1,\tau_2;\rho,\vartheta_1,\varepsilon_1) \leqslant \\
& \qquad \leqslant g_1(\tau_1,\rho,\varepsilon_1) + g_2(\tau_2,\rho,\varepsilon_1) + \frac{1}{(\varepsilon_1-\varepsilon_0)^2} G(\tau_1,\tau_2;\rho,\vartheta_1,\varepsilon_1) + \\
& \qquad\quad + \delta \bigg[g_1(\tau_1,\rho,\varepsilon_2) + g_2(\tau_2,\rho,\varepsilon_2) + \\
& \qquad\quad + \frac{1}{(\varepsilon_2-\varepsilon_1)^2} G(\tau_1,\tau_2;\rho,\vartheta_2,\varepsilon_2) + \delta F(\tau_1,\tau_2;\rho,\vartheta_2,\varepsilon_2) \bigg] .
\end{aligned}
$$

By the monotonicity property of the functions we have in fact

$$
\begin{aligned}
& f(\tau_2,\rho,\varepsilon_0) + g_2(\tau_1,\rho,\varepsilon_0) + F(\tau_1,\tau_2;\rho,\vartheta_0,\varepsilon_0) \leqslant \\
& \qquad \leqslant (1+\delta) \Big[g_1(\tau_1,\rho,\varepsilon_2) + g_2(\tau_2,\rho,\varepsilon_2) \Big] + \\
& \qquad\quad + \left(\frac{1}{(\varepsilon_1-\varepsilon_0)^2} + \frac{\delta}{(\varepsilon_2-\varepsilon_1)^2} \right) G(\tau_1,\tau_2;\rho,\vartheta_2,\varepsilon_2) + \\
& \qquad\quad + \delta^2 F(\tau_1,\tau_2;\rho,\vartheta_2,\varepsilon_2) .
\end{aligned}
$$

Iterating N times these inequalities we first get

$$
\begin{aligned}
& f(\tau_2,\rho,\varepsilon_0) + g_2(\tau_1,\rho,\varepsilon_0) + F(\tau_1,\tau_2;\rho,\vartheta_0,\varepsilon_0) \leqslant \ldots \leqslant \\
& \qquad \leqslant \Big[g_1(\tau_1,\rho,\varepsilon_{N+1}) + g_2(\tau_2,\rho,\varepsilon_{N+1}) \Big] \sum_{n=0}^{N} \delta^n + \\
& \qquad\quad + G(\tau_1,\tau_2;\rho,\vartheta_{N+1},\varepsilon_{N+1}) \sum_{n=0}^{N} \frac{\delta^n}{(\varepsilon_{n+1}-\varepsilon_n)^2} + \\
& \qquad\quad + \delta^{N+1} F(\tau_1,\tau_2;\rho,\vartheta_{N+1},\varepsilon_{N+1}) ;
\end{aligned}
$$

then taking the limit as $N \to +\infty$ we finally obtain

$$
\begin{aligned}
& f(\tau_2,\rho,\varepsilon) + g_2(\tau_1,\rho,\varepsilon) + F(\tau_1,\tau_2;\rho,\vartheta,\varepsilon) \leqslant \\
& \qquad \leqslant \frac{1}{1-\delta} \Big[g_1(\tau_1,\rho,\tilde{\varepsilon}) + g_2(\tau_1,\rho,\tilde{\varepsilon}) \Big] + \\
& \qquad\quad + G(\tau_1,\tau_2;\rho,\tilde{\vartheta},\tilde{\varepsilon}) \frac{1}{(\tilde{\varepsilon}-\varepsilon)^2} \frac{1}{(1-\eta)^2} \sum_{n=0}^{\infty} \left(\frac{\delta}{\eta^2} \right)^n .
\end{aligned}
$$

Taking $\eta \in (\sqrt{\delta}, 1)$ we are done. Taking for instance $\eta = \sqrt{(1+\delta)/2}$ one could have $c = (1+\delta)/(1-\delta)$. □

Call

$$
\begin{aligned}
f(\tau,\rho,\varepsilon) &:= \frac{1}{2Q}\int_{B^+_{\rho+\varepsilon}} (u-k)^2_+(x,\tau)\mu_+(x)\,dx\,,\\
g_2(\tau,\rho,\varepsilon) &:= \frac{1}{2Q}\int_{I^{\rho,\varepsilon}_+} (u-k)^2_+(x,\tau)\mu_-(x)\,dx\,,\\
g_1(\tau,\rho,\varepsilon) &:= \frac{1}{2Q}\int_{I^+_{\rho,\varepsilon}} (u-k)^2_+(x,\tau)\mu_+(x)\,dx\,,\\
F(\tau_1,\tau_2;\rho,\vartheta,\varepsilon) &:= \iint_{Q^{\beta,+,\varepsilon}_{R;\rho,\vartheta}\cap(B_R\times[\tau_1,\tau_2])} |D(u-k)_+|^2\lambda\,dxdt\,,\\
G(\tau_1,\tau_2;\rho,\vartheta,\varepsilon) &:= \frac{1}{2Q}\iint_{Q^{\beta,+,\varepsilon}_{R;\rho,\vartheta}\cap(B_R\times[\tau_1,\tau_2])} (u-k)^2_+\left(8Q\lambda+\frac{2}{\beta\, h(x_0,R)}\mu_+\right)dxdt\,,
\end{aligned}
\tag{53}
$$

for $\rho,\vartheta,\varepsilon\geqslant 0$; now we apply the previous lemma in (51) with $\delta=\frac{2Q-1}{2Q}$, $\rho=r$, $\tilde\varepsilon=\tilde r-r+\varepsilon$ and since $(1-\delta)^{-1}=2Q$ we derive the existence of a positive constant c_Q depending only on Q (for instance, as shown at the end of the proof, one could consider $c_Q=4Q-1$) such that

$$
\begin{aligned}
&\frac{1}{2Q}\int_{B^+_{r+\varepsilon}} (u-k)^2_+(x,\tau_2)\mu_+(x)\,dx+\frac{1}{2Q}\int_{I^{r,\varepsilon}_+} (u-k)^2_+(x,\tau_1)\mu_-(x)\,dx+\\
&\qquad+\iint_{Q^{\beta,+,\varepsilon}_{R;r,\theta}\cap(B_R\times[\tau_1,\tau_2])} |D(u-k)_+|^2\lambda\,dxdt\leqslant\\
&\leqslant\int_{I^{r,\tilde r-r+\varepsilon}_+} (u-k)^2_+(x,\tau_2)\mu_-(x)\,dx+\int_{I^+_{r,\tilde r-r+\varepsilon}} (u-k)^2_+(x,\tau_1)\mu_+(x)\,dx+\\
&\quad+\frac{c_Q}{2Q}\frac{1}{(\tilde r-r)^2}\iint_{Q^{\beta,+,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}\cap(B_R\times[\tau_1,\tau_2])} (u-k)^2_+\left(8Q\lambda+\frac{2}{\beta\, h(x_0,R)}\,\mu_+\right)dxdt\,.
\end{aligned}
\tag{54}
$$

Here is the second lemma, a simple but important lemma.

Lemma 4.6. *Consider some non-negative functions* $f,g_1,g_2,g_3:[t_0,s_2]\to[0,M]$, $F,G:[s_1,s_2]\to(0,M]$, *M positive constant, satisfying*

$$
f(\tau_2)+g_3(\tau_1)+\int_{\tau_1}^{\tau_2}F(t)dt\leqslant g_2(\tau_2)+g_1(\tau_1)+\int_{\tau_1}^{\tau_2}G(t)dt
$$

for every $\tau_1<\tau_2$. *Let* θ *and* $\tilde\theta$ *be the values considered in* (50), $\sigma_\theta=\theta\,\beta\,h(x_0,R)R^2$, $\sigma_{\tilde\theta}=\tilde\theta\,\beta\,h(x_0,R)R^2$ *for some positive* β. *Then*

$$
\begin{aligned}
&\sup_{t\in(t_0+\sigma_\theta,s_2)} f(t)+\sup_{t\in(t_0,t_0+\sigma_{\tilde\theta})} g_3(t)+\int_{t_0}^{s_2}F(t)dt\leqslant\\
&\qquad\leqslant\;2\left[\sup_{t\in(t_0+\sigma_\theta,s_2)} g_2(t)+\sup_{t\in(t_0,t_0+\sigma_{\tilde\theta})} g_1(t)+\int_{s_1}^{s_2}G(t)dt\right].
\end{aligned}
$$

Proof - By the assumptions in particular we have

$$
\begin{aligned}
f(\tau_2) + g_3(\tau_1) \leqslant g_2(\tau_2) + g_1(\tau_1) + \int_{\tau_1}^{\tau_2} G(t)dt \, , \\
\int_{\tau_1}^{\tau_2} F(t)dt \leqslant g_2(\tau_2) + g_1(\tau_1) + \int_{\tau_1}^{\tau_2} G(t)dt \, .
\end{aligned}
$$

Taking the supremum in both the inequalities we get

$$
\begin{aligned}
\sup_{\substack{\tau_1 \in (t_0, t_0 + \sigma_{\tilde\theta}) \\ \tau_2 \in (t_0 + \sigma_\theta, s_2)}} \big[f(\tau_2) + g_3(\tau_1) \big] = \sup_{\tau_2 \in (t_0 + \sigma_\theta, s_2)} f(\tau_2) + \sup_{\tau_1 \in (t_0, t_0 + \sigma_{\tilde\theta})} g_3(\tau_1) \leqslant \\
\leqslant \sup_{\substack{\tau_1 \in (t_0, t_0 + \sigma_{\tilde\theta}) \\ \tau_2 \in (t_0 + \sigma_\theta, s_2)}} \left[g_2(\tau_2) + g_1(\tau_1) + \int_{\tau_1}^{\tau_2} G(t)dt \right] \leqslant \\
\leqslant \sup_{\tau_2 \in (t_0 + \sigma_\theta, s_2)} g_2(\tau_2) + \sup_{\tau_1 \in (t_0, t_0 + \sigma_{\tilde\theta})} g_1(\tau_1) + \int_{t_0}^{s_2} G(t)dt
\end{aligned}
$$

and

$$
\int_{t_0}^{s_2} F(t)dt \leqslant \sup_{\tau_2 \in (t_0 + \sigma_\theta, s_2)} g_2(\tau_2) + \sup_{\tau_1 \in (t_0, t_0 + \sigma_{\tilde\theta})} g_1(\tau_1) + \int_{t_0}^{s_2} G(t)dt \, .
$$

Summing the two inequalities we get the thesis. □

Now we multiply by $2Q$ the inequality (54) and apply the previous lemma. We get

$$
\begin{aligned}
\sup_{t \in (t_0 + \sigma_\theta, s_2)} \int_{B^+_{r+\varepsilon}} (u-k)^2_+(x,t) \mu_+(x) \, dx + \sup_{t \in (t_0, t_0 + \sigma_{\tilde\theta})} \int_{I^{r,\varepsilon}_+} (u-k)^2_+(x,t) \mu_-(x) \, dx + \\
+ 2Q \iint_{Q^{\beta,+,\varepsilon}_{R;r,\theta}} |D(u-k)_+|^2 \lambda \, dxdt \leqslant \\
\leqslant 4Q \sup_{t \in (t_0 + \sigma_\theta, s_2)} \int_{I^{r,\tilde r - r + \varepsilon}_+} (u-k)^2_+(x,t) \mu_-(x) \, dx + \\
+ 4Q \sup_{t \in (t_0, t_0 + \sigma_{\tilde\theta})} \int_{I^+_{r,\tilde r - r + \varepsilon}} (u-k)^2_+(x,t) \mu_+(x) \, dx + \\
+ \frac{2 \, c_Q}{(\tilde r - r)^2} \iint_{Q^{\beta,+,\tilde r - r + \varepsilon}_{R;r,\tilde\theta}} (u-k)^2_+ \left(8Q\lambda + \frac{2}{\beta \, h(x_0, R)} \mu_+ \right) dxdt \, .
\end{aligned}
$$

Finally, calling γ the quantity $16\, c_Q\, Q$ (which turns out to be greater than 1) we get (41)

$$\sup_{t\in(t_0+\sigma_\theta,s_2)} \int_{B^+_{r+\varepsilon}} (u-k)^2_+(x,t)\mu_+(x)\,dx + \sup_{t\in(t_0,t_0+\sigma_{\tilde\theta})} \int_{I^{r,\varepsilon}_+} (u-k)^2_+(x,t)\mu_-(x)\,dx + $$

$$+ \iint_{Q^{\beta,+,\varepsilon}_{R;r,\theta}} |D(u-k)_+|^2 \lambda\,dxdt \leqslant$$

$$\leqslant \gamma \Bigg[\sup_{t\in(t_0,t_0+\sigma_{\tilde\theta})} \int_{I^+_{r,\tilde r - r+\varepsilon}} (u-k)^2_+(x,t)\mu_+(x)\,dx +$$

$$+ \sup_{t\in(t_0+\sigma_\theta,s_2)} \int_{I^{r,\tilde r-r+\varepsilon}_+} (u-k)^2_+(x,t)\mu_-(x)\,dx +$$

$$+ \frac{1}{(\tilde r - r)^2} \iint_{Q^{\beta,+,\tilde r-r+\varepsilon}_{R;r,\tilde\theta}} (u-k)^2_+ \left(\lambda + \frac{1}{\beta\, h(x_0,R)}\mu_+\right) dxdt \Bigg].$$

Now we prove (44). We integrate in $B_R(x_0)\times[\tau_1,\tau_2]$ with $[\tau_1,\tau_2]\subset[t_0,s_2]$ for an arbitrary s_2 (we mean that it is not necessary to consider $s_2 = t_0 + \beta\, h(x_0,R)R^2$) and, as done before to obtain (48), we get for every $[\tau_1,\tau_2]\subset[t_0,s_2]$

$$\frac{1}{2}\int_{B_R} (u-k)^2_+(x,\tau_2)\zeta^2(x,\tau_2)\,\mu(x)\,dx + E(u,K) \leqslant Q\,E(u-\phi,K) +$$

$$+ \frac{1}{2}\int_{B_R} (u-k)^2_+(x,\tau_1)\zeta^2(x,\tau_1)\,\mu(x)\,dx + \int_{\tau_1}^{\tau_2}\!\!\int_{B_R} (u-k)^2_+\zeta\zeta_t\,\mu\,dxdt.$$

Now choosing ζ (whose support depends on τ) such that

$$\zeta = 1 \quad \text{in } B^+_r(x_0)\times[t_0,\tau]\,, \qquad \zeta = 0 \quad \text{in } B_R(x_0)\setminus\big(B^+_{\tilde r}(x_0)\cup I^{r,\tilde r-r}_+\big)\times[t_0,\tau]\,,$$

$$\zeta_t \equiv 0, \qquad\qquad |D\zeta| \leqslant \frac{1}{\tilde r - r}\,,$$

using the estimate (49) and the inequality which follows it and taking $\tau_1 = t_0$, we get that for every $\tau\in[t_0,s_2]$

$$\frac{1}{2Q}\int_{B^+_r} (u-k)^2_+(x,\tau)\mu_+(x)\,dx + \int_{t_0}^{\tau}\!\!\int_{B^+_r} |D(u-k)_+|^2\lambda\,dxdt \leqslant$$

$$\leqslant \frac{1}{2Q}\int_{B^+_{\tilde r}} (u-k)^2_+(x,t_0)\mu_+(x)\,dx + \frac{1}{2Q}\int_{I^{r,\tilde r-r}_+} (u-k)^2_+(x,\tau)\mu_-(x)\,dx +$$

$$+ \frac{4}{(\tilde r-r)^2}\int_{t_0}^{\tau}\!\!\int_{B^+_{\tilde r}\cup I^{r,\tilde r-r}_+} (u-k)^2_+\lambda\,dxdt + \frac{2Q-1}{2Q}\int_{t_0}^{\tau}\!\!\int_{B^+_{\tilde r}\cup I^{r,\tilde r-r}_+} |D(u-k)_+|^2\lambda\,dxdt\,.$$

As done to obtain (41), we first use Lemma 4.5 with the analogous functions considered in (53) (notice that with $\varepsilon = 0$ we get $g_2(t_0,r,0) = 0$), then we use Lemma 4.6 to conclude and get (44).

In an analogous way one can prove (42) and (45), provided that $\mu_-(B_R(x_0)) > 0$.

2° - We now drop the assumptions $\mu_+(B_R(x_0)) > 0$ and $\mu_-(B_R(x_0)) > 0$ and prove (43). We recall that in this case we consider $K = B_R(x_0)\times[s_1,s_2]$ with s_1 and s_2 arbitrary (but belonging to $[0,T]$). Now proceeding similarly as before, taking $\phi = (u-k)_+\zeta^2$ with ζ independent of t and satisfying

$$\zeta \equiv 1 \quad \text{in } (B^0_r(x_0))^{\varepsilon}\,, \qquad \zeta \equiv 0 \quad \text{in } B_R(x_0) \setminus (B^0_r(x_0))^{\tilde r - r + \varepsilon}\,,$$

$$0 \leqslant \zeta \leqslant 1\,, \qquad 0 \leqslant |D\zeta| \leqslant \frac{1}{\tilde r - r}\,,$$

from (48), integrating over $(B^0_r)^{\tilde r - r + \varepsilon} \times (\tau_1, \tau_2)$, we derive for every $\tau_1, \tau_2 \in [s_1, s_2]$, $\tau_1 < \tau_2$,

$$\begin{aligned}
\frac{1}{2Q} \int_{I_0^{r,\varepsilon}} (u-k)_+^2(x,\tau_2)\mu_+(x)\,dx + \frac{1}{2Q} \int_{I_0^{r,\varepsilon}} (u-k)_+^2(x,\tau_1)\mu_-(x)\,dx + \\
+ \iint_{Q^{0,\varepsilon}_{R;r;\tau_1,\tau_2}} |D(u-k)_+|^2 \lambda\,dxdt \leqslant \\
\leqslant \frac{1}{2Q} \int_{I_0^{r,\tilde r - r+\varepsilon}} (u-k)_+^2(x,\tau_2)\mu_-(x)\,dx + \frac{1}{2Q} \int_{I_0^{r,\tilde r - r+\varepsilon}} (u-k)_+^2(x,\tau_1)\mu_+(x)\,dx + \\
+ \frac{4}{(\tilde r - r)^2} \iint_{Q^{0,\tilde r - r+\varepsilon}_{R;r;\tau_1,\tau_2}} (u-k)_+^2\,\lambda\,dxdt + \frac{2Q-1}{2Q} \iint_{Q^{0,\tilde r - r+\varepsilon}_{R;r;\tau_1,\tau_2}} |D(u-k)_+|^2 \lambda\,dxdt\,.
\end{aligned}$$

We can apply Lemma 4.5 with $\vartheta = \tilde\vartheta = 0$, $\rho = r$, $\tilde\rho = \tilde r$, $\varepsilon \geqslant 0$, $\tilde\varepsilon = \tilde r - r$, $\delta = (2Q-1)/2Q$ and

$$\begin{aligned}
g_2(\tau,\rho,\epsilon) &:= \frac{1}{2Q} \int_{I_0^{\rho,\epsilon}} (u-k)_+^2(x,\tau)\mu_-(x)\,dx\,, \\
f(\tau,\rho,\epsilon) = g_1(\tau,\rho,\epsilon) &:= \frac{1}{2Q} \int_{I_0^{\rho,\epsilon}} (u-k)_+^2(x,\tau)\mu_+(x)\,dx\,, \\
F(\tau_1,\tau_2;\rho,\vartheta,\epsilon) &:= \iint_{Q^{0,\epsilon}_{R;\rho;\tau_1,\tau_2}} |D(u-k)_+|^2 \lambda\,dxdt\,, \\
G(\tau_1,\tau_2;\rho,\vartheta,\epsilon) &:= 4 \iint_{Q^{0,\epsilon}_{R;\rho;\tau_1,\tau_2}} (u-k)_+^2\,\lambda\,dxdt\,,
\end{aligned}$$

and get the existence of c_Q such that

$$\begin{aligned}
\frac{1}{2Q} \int_{I_0^{r,\varepsilon}} (u-k)_+^2(x,\tau_2)\mu_+(x)\,dx + \frac{1}{2Q} \int_{I_0^{r,\varepsilon}} (u-k)_+^2(x,\tau_1)\mu_-(x)\,dx + \\
+ \iint_{Q^{0,\varepsilon}_{R;r;\tau_1,\tau_2}} |D(u-k)_+|^2 \lambda\,dxdt \leqslant \\
\leqslant \int_{I_0^{r,\tilde r - r+\varepsilon}} (u-k)_+^2(x,\tau_2)\mu_-(x)\,dx + \int_{I_0^{r,\tilde r - r+\varepsilon}} (u-k)_+^2(x,\tau_1)\mu_+(x)\,dx + \\
+ \frac{4c_Q}{(\tilde r - r)^2} \iint_{Q^{0,\tilde r - r+\varepsilon}_{R;r;\tau_1,\tau_2}} (u-k)_+^2\,\lambda\,dxdt\,.
\end{aligned}$$

Taking the supremum for $\tau_1, \tau_2 \in (s_1, s_2)$ we get that u satisfies (43) with $\gamma = 4c_Q$.

5. Local boundedness for functions in DG

In this section we prove that functions belonging to the De Giorgi class are locally bounded in $\Omega \times (0,T)$.

We start proving that a generic function $u \in DG(\Omega, T, \mu, \lambda, \gamma)$ is bounded in $(B_\rho \times (a,b)) \cap (\Omega_+ \times (0,T))$ for some set $B_\rho \times (a,b) \subset\subset \Omega \times (0,T)$.
Fix $x_0 \in \Omega$, $t_0 \in (0,T)$, $R > 0$ and in what follows assume

$$\mu_+(B_R(x_0)) > 0 \, .$$

Then consider $\beta > 0$ and $s_2 \in (0,T)$ with

$$s_2 - t_0 = \beta \, h(x_0, R) R^2 \, , \qquad B_R(x_0) \times (t_0, s_2) \subset \Omega \times (0,T) \, .$$

Consider now $r, \tilde{r}, \hat{r} \in (0, R]$ such that

$$\frac{R}{2} \leqslant r < \tilde{r} < \hat{r} \leqslant R \quad \text{and} \quad \tilde{r} - r = \frac{\hat{r} - \tilde{r}}{2}$$

and $\theta, \tilde{\theta}, \hat{\theta}$ such that

$$0 \leqslant \hat{\theta} < \tilde{\theta} < \theta < 1 \quad \text{and} \quad \tilde{\theta} - \hat{\theta} = \frac{(\hat{r} - \tilde{r})^2}{R^2} \, , \;\; \theta - \tilde{\theta} = \frac{(\tilde{r} - r)^2}{R^2}$$

and define analogously as done in (38) (but here we simplify the notation)

$$\sigma := \theta \, \beta \, h(x_0, R) \, R^2 \, , \qquad \tilde{\sigma} := \tilde{\theta} \, \beta \, h(x_0, R) \, R^2 \, , \qquad \hat{\sigma} := \hat{\theta} \, \beta \, h(x_0, R) \, R^2 \, ,$$

in such a way that

$$0 \leqslant \hat{\sigma} < \tilde{\sigma} < \sigma < s_2 - t_0 \, .$$

Since t_0, x_0 will remain fixed we will often use the following simplified notations: we will write

$$h(\rho), \; B_\rho, \; Q_R^{\beta,+}, \; Q_R^{\beta,>}, Q_{R;\rho,\theta}^{\beta,+,\delta}, \; Q_{R;\rho,\theta}^{\beta,+}$$

instead of respectively

$$h(x_0,\rho), \; B_\rho(x_0), \; Q_R^{\beta,+}(x_0,t_0), \; Q_R^{\beta,>}(x_0,t_0), \; Q_{R;\rho,\theta}^{\beta,+,\delta}(x_0,t_0), \; Q_{R;\rho,\theta}^{\beta,+}(x_0,t_0).$$

In fact, to further simplify the notations, we will suppose that (it is always possible, up to a translation)

$$t_0 = 0 \, .$$

Finally, from now on, we will use this short notations for the following measures

$$M := \mu \otimes \mathcal{L}^1 \, , \qquad \Lambda := \lambda \otimes \mathcal{L}^1 \, , \qquad |M|_\Lambda := |\mu|_\lambda \otimes \mathcal{L}^1 \, ,$$
$$M_+ := \mu_+ \otimes \mathcal{L}^1 \, , \qquad M_- := \mu_- \otimes \mathcal{L}^1 \, ,$$
$$\Lambda_+ := \lambda_+ \otimes \mathcal{L}^1 \, , \qquad \Lambda_- := \lambda_- \otimes \mathcal{L}^1 \, , \qquad \Lambda_0 := \lambda_0 \otimes \mathcal{L}^1$$

where we recall that $\lambda_+, \lambda_-, \lambda_0$ have been defined in (31).

Now fix a function $u \in DG(\Omega, T, \mu, \lambda, \gamma)$ and define (since β will remain fix we omit it in the definition of the following set)

$$A_R^{+,\delta}(k; \rho, \theta) = \{(x,t) \in Q_{R;\rho,\theta}^{\beta,+,\delta} \, | \, u(x,t) > k\} \, .$$

Consider a function $\zeta \in \mathrm{Lip}(B_{\tilde{r}}(x_0) \times [t_0, s_2])$ such that $\zeta(\cdot, t) \in \mathrm{Lip}_c(B_{\tilde{r}}(x_0))$ for every t such that (notice that $\tilde{r} - \frac{R}{2} = r - \frac{R}{2} + (\tilde{r} - r)$ and $\hat{r} - \frac{R}{2} = \tilde{r} - \frac{R}{2} + (\hat{r} - \tilde{r})$)

$$\zeta \equiv 1 \quad \text{in } Q_{R;\frac{R}{2},\theta}^{\beta,+,r-\frac{R}{2}}(x_0,t_0) \, , \qquad \zeta \equiv 0 \quad \text{in } Q_R^{\beta,>}(x_0,t_0) \setminus Q_{R;\frac{R}{2},\tilde{\theta}}^{\beta,+,\tilde{r}-\frac{R}{2}}(x_0,t_0) \, , \, ,$$

$$0 \leqslant \zeta \leqslant 1 \, , \qquad |D\zeta| \leqslant \frac{1}{\tilde{r} - r} \, , \qquad 0 \leqslant \zeta_t \mu \, , \qquad \zeta_t \mu_- = 0 \, , \qquad |\zeta_t| \leqslant \frac{1}{\beta \, h(x_0, R)} \frac{1}{(\tilde{r} - r)^2} \, .$$

In what follows we will denote by $Q^{\beta,+,\tilde{r}-R/2}_{R;R/2,\tilde{\theta}}(s)$ the set $\{(x,t) \in Q^{\beta,+,\tilde{r}-R/2}_{R;R/2,\tilde{\theta}} \,|\, t = s\}$.
First using Hölder's inequality, then applying Corollary 2.11 to the function $(u-k)_+\zeta$ with $\upsilon = \nu = |\mu|_\lambda$ and $\omega = \lambda$, $E = Q^{\beta,+,\tilde{r}-R/2}_{R;R/2,\tilde{\theta}} \cap \Omega_+$ (we integrate first in $Q^{\beta,+,r-R/2}_{R;R/2,\theta}$, then in

$Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}$, with respect to the measure $\mu_+ dxdt$ which is supported in E), we estimate

$$\frac{1}{|\mu|_\lambda(B_R)}\iint_{Q^{\beta,+,r-R/2}_{R;R/2,\theta}}(u-k)^2_+\mu_+\,dxdt \leqslant \frac{1}{|\mu|_\lambda(B_R)}\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}(u-k)^2_+\zeta^2\mu_+\,dxdt \leqslant$$

$$\leqslant \frac{\big(M_+(A^{+,\tilde r-R/2}_R(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\left[\frac{1}{|\mu|_\lambda(B_R)}\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}(u-k)^{2\kappa}_+\zeta^{2\kappa}\mu_+\,dxdt\right]^{\frac{1}{\kappa}} \leqslant$$

$$\leqslant \frac{\big(M_+(A^{+,\tilde r-R/2}_R(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\,\gamma_1^{2/\kappa}R^{2/\kappa}\left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\frac{1}{(\lambda(B_R))^{1/\kappa}}\cdot$$

$$\cdot\left(\sup_{0<t<s_2}\int_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}(t)}(u-k)^2_+(x,t)\zeta^2(x,t)\mu_+(x)\,dx\right)^{\frac{\kappa-1}{\kappa}}\cdot$$

$$\cdot\left(\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}|D((u-k)_+\zeta)|^2(x,t)\,\lambda(x)\,dxdt\right)^{\frac{1}{\kappa}} \leqslant$$

$$\leqslant \frac{\big(M_+(A^{+,\tilde r-R/2}_R(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\;\gamma_1^{2/\kappa}\frac{R^{2/\kappa}}{(\lambda(B_R))^{1/\kappa}}\left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\cdot$$

$$\cdot\left(\sup_{0<t<s_2}\int_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}(t)}(u-k)^2_+(x,t)\zeta^2(x,t)\mu_+(x)\,dx+\right.$$

$$\left.+\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}|D((u-k)_+\zeta)|^2(x,t)\,\lambda(x)\,dxdt\right) \leqslant$$

$$\leqslant \frac{\big(M_+(A^{+,\tilde r-R/2}_R(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\;\gamma_1^{2/\kappa}\frac{R^{2/\kappa}}{(\lambda(B_R))^{1/\kappa}}\left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\cdot$$

$$\cdot\left(\sup_{0<t<s_2}\int_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}(t)}(u-k)^2_+(x,t)\mu_+(x)\,dx+\right.$$

$$\left.+\,2\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}|D(u-k)_+|^2(x,t)\,\lambda(x)\,dxdt+\frac{2}{(\tilde r-r)^2}\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}(u-k)^2_+(x,t)\,\lambda(x)\,dxdt\right)$$

$$\leqslant \frac{\big(M_+(A^{+,\tilde r-R/2}_R(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\;\gamma_1^{2/\kappa}\frac{R^{2/\kappa}}{(\lambda(B_R))^{1/\kappa}}\left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\cdot$$

$$\cdot\left(\sup_{t\in(\tilde\sigma,s_2)}\int_{B^+_{\tilde r}}(u-k)^2_+(x,t)\mu_+(x)dx+\sup_{t\in(0,\tilde\sigma)}\int_{I^+_{R/2,\tilde r-R/2}}(u-k)^2_+(x,t)\mu_+(x)dx+\right.$$

$$\left.+\,2\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}|D(u-k)_+|^2(x,t)\,\lambda(x)\,dxdt+\frac{8}{(\hat r-\tilde r)^2}\iint_{Q^{\beta,+,\tilde r-R/2}_{R;R/2,\tilde\theta}}(u-k)^2_+(x,t)\,\lambda(x)\,dxdt\right)$$

where in the last inequality we have used the fact that $2(\tilde r - r) = \hat r - \tilde r$.
Now we can continue using the energy estimates (41) (with $\varepsilon = \tilde r - R/2$)

$$\frac{1}{|\mu|_\lambda(B_R)} \iint_{Q^{\beta,+,r-R/2}_{R;R/2,\theta}} (u-k)_+^2 \mu_+ \, dxdt \leqslant \frac{\big(M_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}} \; \gamma_1^{2/\kappa} \frac{R^{2/\kappa}}{(\lambda(B_R))^{1/\kappa}} \left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}} \cdot$$

$$\cdot \Bigg[2\gamma \sup_{t\in(0,\hat\sigma)} \int_{I^+_{R/2,\hat r - R/2}} (u-k)_+^2(x,t)\mu_+(x)\,dx + 2\gamma \sup_{t\in(\tilde\sigma,s_2)} \int_{I_+^{R/2,\hat r - R/2}} (u-k)_+^2(x,t)\mu_-(x)\,dx +$$

$$+ \frac{2\gamma}{(\hat r - \tilde r)^2} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 \left(\frac{\mu_+}{\beta\, h(R)} + \lambda\right) dxds + \sup_{t\in(\hat\sigma,\tilde\sigma)} \int_{I^+_{R/2,\hat r - R/2}} (u-k)_+^2(x,t)\mu_+(x)dx +$$

$$+ \frac{8}{(\hat r - \tilde r)^2} \iint_{Q^{\beta,+,\tilde r - R/2}_{R;\tilde r,\tilde\theta}} (u-k)_+^2(x,t)\,\lambda(x)\,dxdt \Bigg] \leqslant$$

$$\leqslant \frac{\big(M_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}} \; \gamma_1^{2/\kappa} \frac{R^{2/\kappa}}{(\lambda(B_R))^{1/\kappa}} \left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}} \cdot$$

$$\cdot \left[\frac{2\gamma+8}{(\hat r - \tilde r)^2} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 \left(\frac{\mu_+}{\beta\, h(R)} + \lambda\right) dxdt + (2\gamma+1) \sup_{t\in(0,s_2)} \int_{(I^+_{R/2})^{\hat r - R/2}} (u-k)_+^2(x,t)|\mu|(x)dx \right] =$$

$$= \frac{\big(M_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|\mu|_\lambda(B_R))^{\frac{\kappa-1}{\kappa}}} \; \gamma_1^{2/\kappa} \frac{R^{2/\kappa}}{(\lambda(B_R))^{1/\kappa}} \left(\frac{1}{|\mu|_\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}} \frac{2\gamma+8}{(\hat r - \tilde r)^2} \cdot$$

$$\cdot \lambda(B_R) \Bigg[\frac{1}{\beta\,|\mu|_\lambda(B_R)} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 \mu_+ \, dxdt + \frac{1}{\lambda(B_R)} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 \lambda \, dxdt +$$

$$+ \frac{2\gamma+1}{2\gamma+8} (\hat r - \tilde r)^2 \frac{1}{\lambda(B_R)} \sup_{t\in(0,s_2)} \int_{(I^+_{R/2})^{\hat r - R/2}} (u-k)_+^2(x,t)|\mu|(x)dx \Bigg].$$

Now we divide by $s_2 - t_0 = \beta\, h(R) R^2$, estimate $\frac{2\gamma+1}{2\gamma+8}$ by 1 and finally multiply and divide in the right hand side by $(\beta\, h(R)R^2)^{\frac{\kappa-1}{\kappa}}$. We get

$$\frac{1}{|M|_\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,r-R/2}_{R;R/2,\theta}} (u-k)_+^2 \mu_+ \, dxdt \leqslant$$

$$\leqslant \gamma_1^{2/\kappa} R^2 \, \beta^{\frac{\kappa-1}{\kappa}} \; \frac{2\gamma+8}{(\hat r - \tilde r)^2} \; \frac{\big(M_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(|M|_\Lambda(Q_R^{\beta,>}))^{\frac{\kappa-1}{\kappa}}} \; \left(\frac{1}{\beta} + 1\right) \cdot$$

$$\cdot \Bigg[\frac{1}{|M|_\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 \mu_+ \, dxdt + \frac{1}{\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 \lambda_+ \, dxdt +$$

$$+ \frac{1}{\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,\hat r - R/2}_{R;R/2,\hat\theta}} (u-k)_+^2 (\lambda_0 + \lambda_-) \, dxdt +$$

$$+ (\hat r - \tilde r)^2 \frac{1}{\Lambda(Q_R^{\beta,>})} \sup_{t\in(0,s_2)} \int_{(I^+_{R/2})^{\hat r - R/2}} (u-k)_+^2(x,t)|\mu|(x)dx \Bigg]. \tag{55}$$

Notice that

$$\iint_{Q^{\beta,+,\hat r-R/2}_{R;R/2,\hat\theta}} (u-k)_+^2(\lambda_0+\lambda_-)\,dxdt \quad \text{is in fact} \quad \int_0^{s_2}\int_{(I^+_{R/2})^{\hat r-R/2}} (u-k)_+^2(\lambda_0+\lambda_-)\,dxdt.$$

In a similar way one can estimate $\iint_{Q^{\beta,+,r-R/2}_{R;R/2,\theta}}(u-k)_+^2\lambda_+\,dxdt$. The main difference is that we use Corollary 2.11 with $\nu=|\mu|_\lambda$ and $\upsilon=\omega=\lambda$. We get

$$\frac{1}{\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,r-R/2}_{R;R/2,\theta}}(u-k)_+^2\lambda_+\,dxdt \leqslant$$

$$\leqslant \gamma_1^{2/\kappa}R^2\,\frac{1+\beta}{\beta^{\frac{1}{\kappa}}}\,\frac{2\gamma+8}{(\hat r-\tilde r)^2}\,\frac{\big(\Lambda_+(A_R^{+,\tilde r-R/2}(k;R/2,\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(\Lambda(Q_R^{\beta,>}))^{\frac{\kappa-1}{\kappa}}}\,\cdot$$

$$\cdot\Bigg[\frac{1}{|M|_\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,\hat r-R/2}_{R;R/2,\hat\theta}}(u-k)_+^2\mu_+\,dxdt+\frac{1}{\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,\hat r-R/2}_{R;R/2,\hat\theta}}(u-k)_+^2\lambda_+\,dxdt+ \tag{56}$$

$$+\frac{1}{\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,\hat r-R/2}_{R;R/2,\hat\theta}}(u-k)_+^2(\lambda_0+\lambda_-)\,dxdt+$$

$$+(\hat r-\tilde r)^2\,\frac{1}{\Lambda(Q_R^{\beta,>})}\sup_{t\in(0,s_2)}\int_{(I^+_{R/2})^{\hat r-R/2}}(u-k)_+^2(x,t)|\mu|(x)dx\Bigg].$$

Once defined (for $\rho\in[R/2,R]$)

$$\tilde u_{\mu_+}(l;\rho,\vartheta;\varepsilon):=\left(\frac{1}{|M|_\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,\varepsilon}_{R;\rho,\vartheta}}(u-l)_+^2\mu_+\,dxdt\right)^{1/2},$$

$$\tilde u_{\lambda_+}(l;\rho,\vartheta;\varepsilon):=\left(\frac{1}{\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,\varepsilon}_{R;\rho,\vartheta}}(u-l)_+^2\lambda_+\,dxdt\right)^{1/2},$$

$$\big(\tilde u_+(l;\rho,\vartheta,\varepsilon)\big)^2:=\big(\tilde u_{\mu_+}(l;\rho,\vartheta,\varepsilon)\big)^2+\big(\tilde u_{\lambda_+}(l;\rho,\vartheta,\varepsilon)\big)^2,$$

we sum the two inequalities and get

$$\big(\tilde u_+(k;\tfrac{R}{2},\theta;\,r-\tfrac{R}{2})\big)^2\leqslant\frac{C_1}{(\hat r-\tilde r)^2}\left[\frac{\big(M_+(A_R^{+,\tilde r-R/2}(k;\frac{R}{2},\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{|M|_\Lambda(Q_R^{\beta,>})^{\frac{\kappa-1}{\kappa}}}+\right.$$

$$\left.+\frac{\big(\Lambda_+(A_R^{+,\tilde r-R/2}(k;\frac{R}{2},\tilde\theta))\big)^{\frac{\kappa-1}{\kappa}}}{(\Lambda(Q_R^{\beta,>}))^{\frac{\kappa-1}{\kappa}}}\right]\cdot\Big[\big(\tilde u_+(k;\tfrac{R}{2},\hat\theta;\hat r-\tfrac{R}{2})\big)^2+\big(\omega^{\hat r-\tilde r}(u;k;\hat r;\hat\theta)\big)^2\Big]$$

where $C_1=\gamma_1^{2/\kappa}R^2\,\dfrac{1+\beta}{\beta^{\frac{1}{\kappa}}}\,(2\gamma+8)$ and

$$\big(\omega^{\hat r-\tilde r}(u;k;\hat r;\hat\theta)\big)^2:=\frac{1}{\Lambda(Q_R^{\beta,>})}\iint_{Q^{\beta,+,\hat r-R/2}_{R;R/2,\hat\theta}}(u-k)_+^2(\lambda_0+\lambda_-)\,dxdt+$$

$$+(\hat r-\tilde r)^2\,\frac{1}{\Lambda(Q_R^{\beta,>})}\sup_{t\in(0,s_2)}\int_{(I^+_{R/2})^{\hat r-R/2}}(u-k)_+^2(x,t)|\mu|(x)dx\,.$$

Notice that for $h < k$ we have

$$(k-h)^2 M_+(A_R^{+,\tilde r - R/2}(k;\tfrac{R}{2},\tilde\theta)) \leqslant \iint_{A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta)} (u-h)_+^2 \mu_+ \, dxdt \leqslant$$
$$\leqslant \iint_{A_R^{+,\tilde r - R/2}(h;R/2,\tilde\theta)} (u-h)_+^2 \mu_+ \, dxdt \,,$$

that is

$$M_+(A_R^{+,\tilde r - R/2}(k;\tfrac{R}{2},\tilde\theta)) \leqslant \frac{M_+(Q_R^{\beta,>})}{(k-h)^2} \, \big(\tilde u_{\mu_+}(h;\tfrac{R}{2},\tilde\theta;\tilde r - \tfrac{R}{2})\big)^2 .$$

From that (and the analogous estimate for $\Lambda_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))$) we derive

$$\frac{M_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))}{|M|_\Lambda(Q_R^{\beta,>})} \leqslant \frac{M_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))}{M_+(Q_R^{\beta,>})} \leqslant \frac{1}{(k-h)^2} \, \big(\tilde u_{\mu_+}(h;\tfrac{R}{2},\tilde\theta;\tilde r - \tfrac{R}{2})\big)^2 ,$$
$$\frac{\Lambda_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))}{\Lambda(Q_R^{\beta,>})} \leqslant \frac{\Lambda_+(A_R^{+,\tilde r - R/2}(k;R/2,\tilde\theta))}{\Lambda_+(Q_R^{\beta,>})} \leqslant \frac{1}{(k-h)^2} \, \big(\tilde u_{\lambda_+}(h;\tfrac{R}{2},\tilde\theta;\tilde r - \tfrac{R}{2})\big)^2 .$$

Then, applying these inequalities we get

$$\tilde u_+(k;\tfrac{R}{2},\theta;r-\tfrac{R}{2}) \leqslant \frac{C_1^{1/2}}{\hat r - \tilde r} \, \frac{1}{(k-h)^{\frac{\kappa-1}{\kappa}}} \, \tilde u_+(h;\tfrac{R}{2},\tilde\theta;\tilde r - \tfrac{R}{2})^{\frac{\kappa-1}{\kappa}} \, \Big[\tilde u_+(k;\tfrac{R}{2},\hat\theta;\hat r - \tfrac{R}{2}) + \omega^{\hat r - \tilde r}(u;k;\hat r;\hat\theta)\Big] \leqslant$$
$$\leqslant \frac{C_1^{1/2}}{\hat r - \tilde r} \, \frac{1}{(k-h)^{\frac{\kappa-1}{\kappa}}} \, \tilde u_+(h;\tfrac{R}{2},\tilde\theta;\tilde r - \tfrac{R}{2})^{\frac{\kappa-1}{\kappa}} \, \Big[\tilde u_+(h;\tfrac{R}{2},\hat\theta;\hat r - \tfrac{R}{2}) + \omega^{\hat r - \tilde r}(u;h;\hat r;\hat\theta)\Big] \leqslant$$
$$\leqslant \frac{C_1^{1/2}}{\hat r - \tilde r} \, \frac{1}{(k-h)^{\frac{\kappa-1}{\kappa}}} \, \tilde u_+(h;\tfrac{R}{2},\hat\theta;\hat r - \tfrac{R}{2})^{\frac{\kappa-1}{\kappa}} \, \Big[\tilde u_+(h;\tfrac{R}{2},\hat\theta;\hat r - \tfrac{R}{2}) + \omega^{\hat r - \tilde r}(u;h;\hat r;\hat\theta)\Big] . \tag{57}$$

Consider the following choices: for $n \in \mathbf{N}$, $k_0 \in \mathbf{R}$ and a fixed d we define

$$k_n := k_0 + d\left(1 - \frac{1}{2^n}\right) \nearrow k_0 + d \,,$$
$$r_n := \frac{R}{2} + \frac{R}{2^{n+1}} \searrow \frac{R}{2} \,,$$
$$\theta_n := \frac{1}{2}\left(1 - \frac{1}{4^n}\right) \nearrow \frac{1}{2} \,,$$
$$\sigma_n := \theta_n \, \beta \, h(x_0,R) \, R^2 \nearrow \frac{1}{2} \, \beta \, h(x_0,R) \, R^2 \,.$$

Notice that (for these choices)

$$2\,(r_n - r_{n+1}) = r_{n-1} - r_n \,.$$

With this choice of θ_n (and since $\beta \, h(x_0,R)R^2 = s_2 - t_0 = s_2$ since we are supposing $t_0 = 0$) we have that

$$\sigma_n = \theta_n \, \beta \, h(x_0,R) \, R^2 = \theta_n \, s_2 \nearrow \frac{s_2}{2} \,.$$

With this choices we define the sequences

$$u_n^+ := \tilde u_+(k_n;\tfrac{R}{2},\theta_n;r_n - \tfrac{R}{2}) \,, \qquad \omega_n^+ := \omega^{r_n - r_{n+1}}(u;k_n;r_n;\theta_n)$$

and show that with the particular choices just made above the sequence $(u_n)_n$ is infinitesimal. To get that it is sufficient to observe that from (57) and using

$$\begin{array}{llll} r_{n+1} & \text{in the place of } r\,, & \theta_{n+1} & \text{in the place of } \theta\,, \\ r_n & \text{in the place of } \tilde r\,, & \theta_n & \text{in the place of } \tilde\theta\,, \\ r_{n-1} & \text{in the place of } \hat r\,, & \theta_{n-1} & \text{in the place of } \hat\theta\,, \\ k_{n+1} & \text{in the place of } k\,, & k_{n-1} & \text{in the place of } h\,, \end{array}$$

we derive

$$u^+_{n+1} \leqslant C_+\, 2^{n+1} \frac{2^{(n+1)\frac{\kappa-1}{\kappa}}}{(3d)^{\frac{\kappa-1}{\kappa}}} \left(u^+_{n-1} + \omega^+_{n-1}\right) (u^+_{n-1})^{\frac{\kappa-1}{\kappa}}\,, \qquad n \geqslant 1\,, \tag{58}$$

where $C_+ = \sqrt{C_1}/R = \gamma_1^{1/\kappa}\,(1+\beta)^{1/2}\beta^{-\frac{1}{2\kappa}}\,(2\gamma+8)^{1/2}$. Setting

$$\alpha = \frac{\kappa-1}{\kappa}\,, \quad c = C_+\,\frac{4^{1+\alpha}}{3^\alpha d^\alpha}\,, \quad b = 2^{1+\alpha}\,, \quad y_n = u^+_n\,, \quad \epsilon_n = \omega^+_n\,.$$

(58) becomes

$$u^+_{n+1} \leqslant c\, b^{n-1}\left(u^+_{n-1} + \omega^+_{n-1}\right)(u^+_{n-1})^\alpha\,, \qquad n \geqslant 1.$$

In particular we get

$$u^+_{2(n+1)} \leqslant c\, b^{2n}\left(u^+_{2n} + \omega^+_{2n}\right)(u^+_{2n})^\alpha\,, \qquad n \geqslant 0.$$

Now notice that $(u^+_n)_n$ is decreasing. Then, using Lemma 2.19, provided that

$$u^+_0 < \left(C_+\,\frac{4^{1+\alpha}}{3^\alpha d^\alpha}\right)^{-1/\alpha} 2^{-\frac{2}{\alpha}-\frac{2}{\alpha^2}} = 3d\,(C_+)^{-\frac{1}{\alpha}}\;4^{-\frac{2}{\alpha}-\frac{1}{\alpha^2}-1}\,, \tag{59}$$

that is

$$\left(\frac{1}{|M|_\Lambda(Q^{\beta,>}_R)}\iint_{Q^{\beta,+,R/2}_{R;R/2,0}}(u-k_0)^2_+\mu_+\,dxdt + \frac{1}{\Lambda(Q^{\beta,>}_R)}\iint_{Q^{\beta,+,R/2}_{R;R/2,0}}(u-k_0)^2_+\lambda_+\,dxdt\right)^{1/2} < \\ < 3d\,(C_+)^{-\frac{1}{\alpha}}\;4^{-\frac{2}{\alpha}-\frac{1}{\alpha^2}-1}\,,$$

we get that the subsequence $(u_{2n})_n$ is infinitesimal and since $(u_n)_n$ is decreasing we finally derive

$$\lim_{n\to+\infty} u^+_n = \tilde u_+\left(k_0+d;\frac{R}{2},\frac{1}{2}\right) = 0 \tag{60}$$

where

$$\big(\tilde u_+(l;\varrho,\vartheta)\big)^2 := \big(\tilde u_+(l;\varrho,\vartheta;0)\big)^2 = \\ = \frac{1}{|M|_\Lambda(Q^{\beta,>}_R)}\iint_{Q^{\beta,+}_{R;\varrho,\vartheta}}(u-l)^2_+\mu_+\,dxdt + \frac{1}{\Lambda(Q^{\beta,>}_R)}\iint_{Q^{\beta,+}_{R;\varrho,\vartheta}}(u-l)^2_+\lambda_+\,dxdt\,.$$

In a complete analogous way, if $\mu_-(B_R) > 0$ and taking $s_1 = t_0 - \beta\, h(x_0,R)R^2$, one can prove that

$$\begin{aligned} &\iint_{Q^{\beta,-}_{R;R/2,1/2}(x_0,t_0)}(u-k_0-d)^2_+\mu_-\,dxdt = 0\,, \\ &\iint_{Q^{\beta,-}_{R;R/2,1/2}(x_0,t_0)}(u-k_0-d)^2_+\lambda_-\,dxdt = 0\,, \end{aligned} \tag{61}$$

where $Q^{\beta,-}_{R;R/2,1/2}(x_0,t_0) = B^-_R(x_0) \times (t_0 - \beta\, h(x_0,R)R^2, t_0 - \frac{1}{2}\beta\, h(x_0,R)R^2)$, provided that

$$\left(\frac{1}{|M|_\Lambda(Q^{\beta,<}_R)} \iint_{Q^{\beta,-,R/2}_{R;R,0}} (u-k_0)^2_+ \mu_-\, dxdt + \frac{1}{\Lambda(Q^{\beta,<}_R)} \iint_{Q^{\beta,-,R/2}_{R;R,0}} (u-k_0)^2_+ \lambda_-\, dxdt \right)^{1/2} < \\ < 3d\,(C_-)^{-\frac{1}{\alpha}}\; 4^{-\frac{2}{\alpha}-\frac{1}{\alpha^2}-1},$$

where $C_- = C_+ = \gamma_1^{1/\kappa}\, \beta^{1/2}\, (2\gamma+8)^{1/2}$.

The proof regarding the part in which $\mu \equiv 0$ is slightly different and we show it. We define

$$\sigma_1 := t_0 - \frac{R^2}{2}, \quad \sigma_2 := t_0 + \frac{R^2}{2} \qquad \text{so that } \; \sigma_2 - \sigma_1 = R^2.$$

Moreover we suppose that

$$\lambda_0(B_R) > 0,$$

otherwise there is nothing to prove. We consider $r, \tilde r, \hat r \in (R/2, R)$ as before. Consider a function $\zeta \in \mathrm{Lip}_c(B_{\tilde r}(x_0))$ (independent of t!) such that

$$\zeta \equiv 1 \quad \text{in } Q^{0,r-\frac{R}{2}}_{R;\frac{R}{2};\sigma_1,\sigma_2}(x_0), \qquad \zeta \equiv 0 \quad \text{in } \big(B_R(x_0)\times(\sigma_1,\sigma_2)\big) \setminus Q^{0,\tilde r-\frac{R}{2}}_{R;\frac{R}{2};\sigma_1,\sigma_2}(x_0), \\ 0 \leqslant \zeta \leqslant 1, \qquad |D\zeta| \leqslant \frac{1}{\tilde r - r}.$$

We moreover define

$$A^{0,\delta}_R(k;\rho;\sigma_1,\sigma_2) := \{(x,t) \in Q^{0,\delta}_{R;\rho;\sigma_1,\sigma_2}(x_0) \,|\, u(x,t) > k\}.$$

Then we proceed in a way similar to that above and estimate $(\lambda(B_R))^{-1} \iint_{Q^{0,r-R/2}_{R;R/2;\sigma_1,\sigma_2}} (u - k)^2_+ \lambda\, dxdt$ using first Corollary 2.11 with $\nu = \upsilon = \omega = \lambda$. One has (we write $Q^{0,\varepsilon}_{R;\rho;s_1,s_2}$ to mean $Q^{0,\varepsilon}_{R;\rho;s_1,s_2}(x_0)$)

$$\frac{1}{\lambda(B_R)} \iint_{Q^{0,r-R/2}_{R;R/2;\sigma_1,\sigma_2}} (u-k)^2_+ \lambda_0\, dxdt \leqslant \frac{1}{\lambda(B_R)} \iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}} (u-k)^2_+ \zeta^2 \lambda_0\, dxdt \leqslant \\ \leqslant \frac{\big(\Lambda_0(A^{0,\tilde r-R/2}_R(k;R/2;\sigma_1,\sigma_2))\big)^{\frac{\kappa-1}{\kappa}}}{(\lambda(B_R))^{\frac{\kappa-1}{\kappa}}} \left[\frac{1}{\lambda(B_R)} \iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}} (u-k)^{2\kappa}_+ \zeta^{2\kappa} \lambda_0\, dxdt \right]^{\frac{1}{\kappa}} \leqslant \\ \leqslant \frac{\big(\Lambda_0(A^{0,\tilde r-R/2}_R(k;R/2;\sigma_1,\sigma_2))\big)^{\frac{\kappa-1}{\kappa}}}{(\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\, \gamma_1^{2/\kappa}\, R^{2/\kappa} \frac{1}{\lambda(B_R)} \cdot \\ \cdot \left[\sup_{t\in(\sigma_1,\sigma_2)} \int_{(B^0_{R/2})^{\tilde r-R/2}} (u-k)^2_+(x,t)\lambda_0(x)dx \right]^{\frac{\kappa-1}{\kappa}} \left[\iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}} |D\big((u-k)_+\zeta\big)|^2 \lambda\, dxdt \right]^{\frac{1}{\kappa}}.$$

Then using the energy estimate (43) we get

$$\begin{aligned}
&\iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}} |D\big((u-k)_+\zeta\big)|^2\lambda\,dxdt \leqslant \\
&\leqslant 2\iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}} \Big[|D(u-k)_+|^2\zeta^2+|D\zeta|^2(u-k)_+^2\Big]\lambda\,dxdt \leqslant \\
&\leqslant 2\iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}} \Big[|D(u-k)_+|^2+\frac{1}{(\tilde r-r)^2}(u-k)_+^2\Big]\lambda\,dxdt \leqslant \\
&\leqslant 2\gamma\Bigg[\sup_{t\in(\sigma_1,\sigma_2)}\int_{I_0^{R/2,\hat r-R/2}}(u-k)_+^2(x,t)\mu_-(x)\,dx+ \\
&\qquad+\sup_{t\in(\sigma_1,\sigma_2)}\int_{I_0^{R/2,\hat r-R/2}}(u-k)_+^2(x,t)\mu_+(x)\,dx+ \\
&\qquad+\frac{1}{(\hat r-\tilde r)^2}\iint_{Q^{0,\hat r-R/2}_{R;R/2;\sigma_1,\sigma_2}}(u-k)_+^2\,\lambda\,dxdt\Bigg]+ \\
&\qquad\quad+\frac{2}{(\tilde r-r)^2}\iint_{Q^{0,\tilde r-R/2}_{R;R/2;\sigma_1,\sigma_2}}(u-k)_+^2\lambda\,dxdt\,.
\end{aligned}$$

Then we have, dividing by $\sigma_2-\sigma_1$ in both sides,

$$\begin{aligned}
&\frac{1}{(\sigma_2-\sigma_1)\lambda(B_R)}\iint_{Q^{0,r-R/2}_{R;R/2;\sigma_1,\sigma_2}}(u-k)_+^2\lambda_0\,dxdt \leqslant \\
&\leqslant \frac{\big(\Lambda_0(A_R^{0,\tilde r-R/2}(k;R/2;\sigma_1,\sigma_2))\big)^{\frac{\kappa-1}{\kappa}}}{(\sigma_2-\sigma_1)^{\frac{\kappa-1}{\kappa}}(\lambda(B_R))^{\frac{\kappa-1}{\kappa}}}\,\gamma_1^{2/\kappa}\,\frac{R^{2/\kappa}}{(\sigma_2-\sigma_1)^{\frac{1}{\kappa}}}\,\frac{(\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)\lambda(B_R)}\cdot \\
&\quad\cdot\Bigg[\sup_{t\in(\sigma_1,\sigma_2)}\int_{(B^0_{R/2})^{\tilde r-R/2}}(u-k)_+^2(x,t)\lambda_0(x)dx+ \\
&\qquad+2\gamma\sup_{t\in(\sigma_1,\sigma_2)}\int_{I_0^{R/2,\hat r-R/2}}(u-k)_+^2(x,t)\mu_-(x)\,dx+ \\
&\qquad+2\gamma\sup_{t\in(\sigma_1,\sigma_2)}\int_{I_0^{R/2,\hat r-R/2}}(u-k)_+^2(x,t)\mu_+(x)\,dx+ \\
&\qquad+\frac{2\gamma+8}{(\hat r-\tilde r)^2}\iint_{Q^{0,\hat r-R/2}_{R;R/2;\sigma_1,\sigma_2}}(u-k)_+^2(\lambda_++\lambda_-)\,dxdt+ \\
&\qquad+\frac{2\gamma+8}{(\hat r-\tilde r)^2}\iint_{Q^{0,\hat r-R/2}_{R;R/2;\sigma_1,\sigma_2}}(u-k)_+^2\lambda_0\,dxdt\Bigg].
\end{aligned} \tag{62}$$

Now defining

$$\big(\tilde u_0(l;\rho;\varepsilon;\sigma_1,\sigma_2)\big)^2=\frac{1}{(\sigma_2-\sigma_1)\lambda(B_R)}\iint_{Q^{0,\varepsilon}_{R;\rho;\sigma_1,\sigma_2}}(u-l)_+^2\lambda_0\,dxdt$$

for $\varepsilon \in [0, R/2)$,

$$\big(\omega^{\hat r - \tilde r}(u;k;\hat r)\big)^2 := \frac{(\hat r - \tilde r)^2}{(\sigma_2-\sigma_1)\lambda(B_R)} \cdot \Bigg[\sup_{t\in(\sigma_1,\sigma_2)} \int_{(B^0_{R/2})^{\hat r - R/2}} (u-k)^2_+(x,t)\lambda_0(x) dx + \\ + \sup_{t\in(\sigma_1,\sigma_2)} \int_{I_0^{R/2,\hat r - R/2}} (u-k)^2_+(x,t)\mu_-(x)\, dx + \\ + \sup_{t\in(\sigma_1,\sigma_2)} \int_{I_0^{R/2,\hat r - R/2}} (u-k)^2_+(x,t)\mu_+(x)\, dx \Bigg] + \\ + \frac{1}{(\sigma_2-\sigma_1)\lambda(B_R)} \iint_{Q^{0,\hat r - R/2}_{R;R/2;\sigma_1,\sigma_2}} (u-k)^2_+(\lambda_+ + \lambda_-)\, dxdt$$

and for $k > h$

$$\frac{\Lambda_0(A_R^{0,\tilde r - R/2}(k;R/2;\sigma_1,\sigma_2))}{(\sigma_2-\sigma_1)\lambda(B_R)} \leqslant \frac{1}{(k-h)^2}\big(\tilde u_0(h;\tfrac{R}{2};\tilde r - \tfrac{R}{2};\sigma_1,\sigma_2)\big)^2$$

and since $\sigma_2 - \sigma_1 = \beta\, R^2$ we reach

$$\begin{aligned} \tilde u_0(k;\tfrac{R}{2};r-\tfrac{R}{2};\sigma_1,\sigma_2) &\leqslant \frac{\gamma_1^{1/\kappa}\,\beta^{\frac{\kappa-1}{2\kappa}} R}{(k-h)^{\frac{\kappa-1}{\kappa}}} \frac{(2\gamma+8)^{1/2}}{\hat r - \tilde r} \cdot \\ &\cdot \Big[\omega^{\hat r - \tilde r}(u;k;\hat r) + \tilde u_0(k;\tfrac{R}{2};\hat r - \tfrac{R}{2};\sigma_1,\sigma_2)\Big]\big(\tilde u_0(h;\tfrac{R}{2};\tilde r - \tfrac{R}{2};\sigma_1,\sigma_2)\big)^{\frac{\kappa-1}{\kappa}} \leqslant \\ &\leqslant \frac{\gamma_1^{1/\kappa}\,\beta^{\frac{\kappa-1}{2\kappa}} R}{(k-h)^{\frac{\kappa-1}{\kappa}}} \frac{(2\gamma+8)^{1/2}}{\hat r - \tilde r} \cdot \\ &\cdot \Big[\omega^{\hat r - \tilde r}(u;k;\hat r) + \tilde u_0(h;\tfrac{R}{2};\hat r - \tfrac{R}{2};\sigma_1,\sigma_2)\Big]\big(\tilde u_0(h;\tfrac{R}{2};\hat r - \tfrac{R}{2};\sigma_1,\sigma_2)\big)^{\frac{\kappa-1}{\kappa}} . \end{aligned} \tag{63}$$

As done before, consider the following choices for $n \in \mathbf{N}$, $k_0 \in \mathbf{R}$ and a fixed d:

$$k_n := k_0 + d\left(1 - \frac{1}{2^n}\right) \nearrow k_0 + d\,, \qquad r_n := \frac{R}{2} + \frac{R}{2^{n+1}} \searrow \frac{R}{2}\,,$$

and define the sequences

$$u_n^0 := \tilde u_0(k_n;\tfrac{R}{2};r_n - \tfrac{R}{2};\sigma_1,\sigma_2)\,, \qquad \omega_n^0 := \omega^{r_n - r_{n+1}}(u;k_n;r_n)\,.$$

Making the following choices in (63)

r_{n+1}	in the place of r,	r_n	in the place of $\tilde r$,
r_{n-1}	in the place of $\hat r$,		
k_{n+1}	in the place of k,	k_{n-1}	in the place of h,

we get

$$u^0_{n+1} \leqslant \frac{\gamma_1^{1/\kappa}\beta^{\frac{\kappa-1}{2\kappa}}(2\gamma+8)^{1/2}}{(3d)^{\frac{\kappa-1}{\kappa}}} (2^{\frac{2\kappa-1}{\kappa}})^{n-1} \big(u^0_{n-1} + \omega^0_{n-1}\big)\, (u^0_{n-1})^{\frac{\kappa-1}{\kappa}}\,, \qquad n \geqslant 1 \tag{64}$$

and then, similarly as before, we derive that

$$\lim_{n\to+\infty} u_n^0 = \tilde{u}_0\left(k_0 + d; \frac{R}{2}; \sigma_1, \sigma_2\right) := \tilde{u}_0\left(k_0 + d; \frac{R}{2}; 0; \sigma_1, \sigma_2\right) =$$

$$= \left(\frac{1}{(\sigma_2 - \sigma_1)\lambda(B_R)} \int_{\sigma_1}^{\sigma_2}\int_{B^0_{R/2}} (u - k_0 - d)_+^2 \lambda_0 \, dxdt\right)^{1/2} = 0$$

provided that

$$u_0^0 < 3\, d\, \gamma_1^{-\frac{1}{\kappa-1}}\, \beta^{-1/2}(2\gamma + 8)^{-\frac{\kappa}{2(\kappa-1)}}\, 2^{-\frac{6\kappa^2 - 7\kappa + 2}{(\kappa-1)^2}} .$$

Now we continue and conclude this section showing that u is locally bounded in Ω. In Figure B, supposing only $\mu > 0$ and $\mu < 0$, the sets involved in the stimates of points i) and ii).

Theorem 5.1. *Suppose $u \in DG(\Omega, T, \mu, \lambda, \gamma)$ and consider $(x_0, t_0) \in \Omega \times (0, T)$, $\beta > 0$. Then there is a constant c_∞ depending only on $\gamma, \gamma_1, \kappa, \beta$ such that:*

i) for every $B_R(x_0) \times (t_0, t_0 + \beta\, h(x_0, R)R^2) \subset \Omega \times (0, T)$ if $\mu_+(B_R(x_0)) > 0$ we have

$$\operatorname*{ess\,sup}_{Q^{\beta,+}_{R;R/2,1/2}} |u| \leqslant c_\infty \left[\frac{1}{|M|_\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,R/2}_{R;R,0}} u^2 \mu_+ \, dxdt + \frac{1}{\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,R/2}_{R;R,0}} u^2 \lambda_+ \, dxdt\right]^{1/2};$$

ii) for every $B_R(x_0) \times (t_0 - \beta\, h(x_0, R)R^2, t_0) \subset \Omega \times (0, T)$ if $\mu_-(B_R(x_0)) > 0$ we have

$$\operatorname*{ess\,sup}_{Q^{\beta,-}_{R;R/2,1/2}} |u| \leqslant c_\infty \left[\frac{1}{|M|_\Lambda(Q_R^{\beta,<})} \iint_{Q^{\beta,-,R/2}_{R;R,0}} u^2 \mu_- \, dxdt + \frac{1}{\Lambda(Q_R^{\beta,<})} \iint_{Q^{\beta,-,R/2}_{R;R,0}} u^2 \lambda_- \, dxdt\right]^{1/2};$$

iii) for every $B_R(x_0) \times (\sigma_1, \sigma_2) \subset \Omega \times (0, T)$, $\sigma_2 - \sigma_1 = R^2$, if $\lambda_0(B_R(x_0)) > 0$

$$\operatorname*{ess\,sup}_{B^0_{R/2}\times(\sigma_1,\sigma_2)} |u| \leqslant c_\infty \left(\frac{1}{\Lambda(B_R \times (\sigma_1, \sigma_2))} \iint_{Q^{0,R/2}_{R;R;\sigma_1,\sigma_2}} u^2 \lambda_0 \, dxdt\right)^{1/2}.$$

Proof - We prove the first point, being the others very similar. By (60) we derive that

$$\operatorname*{ess\,sup}_{Q^+_{R;R/2,1/2}} u \leqslant k_0 + d$$

and d has to satisfy (59). For example we can choose

$$d = 2\,(C_+)^{\frac{1}{\alpha}}\; 3^{-1} 4^{\frac{2}{\alpha} + \frac{1}{\alpha^2} + 1}\, u_0^+ .$$

By definition of u_0^+, defining the quantity

$$c_\infty := \frac{d}{u_0^+} = \frac{2}{3}\,(C_+)^{\frac{1}{\alpha}}\; 4^{\frac{2}{\alpha} + \frac{1}{\alpha^2} + 1} = \frac{2}{3}\gamma_1^{\frac{1}{\kappa-1}}\, \frac{(1+\beta)^{\frac{\kappa}{2(\kappa-1)}}}{\beta^{\frac{1}{2(\kappa-1)}}}\,(2\gamma + 8)^{\frac{\kappa}{2(\kappa-1)}}\, 4^{\frac{3\kappa^2 - 3\kappa + 1}{(\kappa-1)^2}} ,$$

choosing $k_0 = 0$ and estimating u_+^2 by u^2 we finally get

$$\operatorname*{ess\,sup}_{Q^{\beta,+}_{R;R/2,1/2}} u \leqslant c_\infty \left(\frac{1}{|M|_\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,R/2}_{R;R,0}} u^2 \mu_+ \, dxdt + \frac{1}{\Lambda(Q_R^{\beta,>})} \iint_{Q^{\beta,+,R/2}_{R;R,0}} u^2 \lambda_+ \, dxdt\right)^{1/2}.$$

Since the analogous argument can be applied to $-u$ we have the first claim. The points *ii*) and *iii*) are completely analogous: the only difference is that the constant c_∞ in point *ii*) is the same as in point *i*), in point *iii*) is $3^{-1}\,\gamma_1^{\frac{1}{\kappa-1}}\,\beta^{1/2}(8\gamma+2)^{\frac{\kappa}{2(\kappa-1)}}\,2^{\frac{6\kappa^2-7\kappa+2}{(\kappa-1)^2}-1}$. □

Remark 5.2. - Notice that from points *i*) and *ii*) it is not possible to derive a pointwise (in time) estimate: indeed letting β go to zero the constant c_∞ goes to $+\infty$.
Also in point *iii*) we cannot obtain a pointwise estimate because $\sigma_2-\sigma_1 = \beta R^2$ and the constant c_∞ depends on β.
Nevertheless one could obtain a pointwise estimate if $B_R \subset \Omega_0$ using (46) and Theorem 2.9.

The local boundedness for a function in the class DG is immediatly needed in the following section.

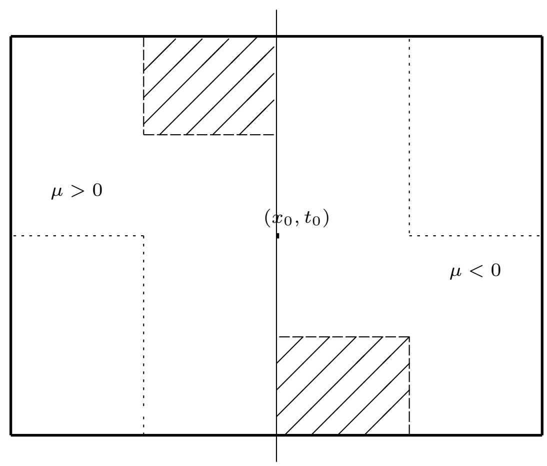


Figure B

6. Expansion of positivity

In this section we will see many preliminary results needed to prove Harnack's inequality.
In what follows we fix the following points and sets: given three points $(x^\diamond, t^\diamond), (x^\circ, t^\circ), (x^\star, t^\star) \in \Omega \times (0,T)$ in such a way that

$$Q_R^{\beta^\diamond,>}(x^\diamond,t^\diamond) = B_R(x^\diamond)\times(t^\diamond, s_2^\diamond) \subset \Omega\times(0,T) \qquad \text{where } s_2^\diamond = t^\diamond + \beta^\diamond\, h(x^\diamond, R)\, R^2\,,$$

$$Q_R^{\beta^\circ,<}(x^\circ,t^\circ) = B_R(x^*)\times(s_1^\circ, t^\circ) \subset \Omega\times(0,T) \qquad \text{where } s_1^\circ = t^* - \beta^\circ\, h(x^*, R)\, R^2\,,$$

$$Q_R^{s_1^\star, s_2^\star}(x^\star,t^\star) := B_R(x^\star)\times(s_1^\star, s_2^\star) \subset \Omega\times(0,T) \qquad \text{where } s_1^\star = t^\star - \frac{\beta^\star}{2}R^2,\ s_2^\star = t^\star + \frac{\beta^\star}{2}R^2\,,$$

with $\beta^\diamond, \beta^\circ, \beta^\star > 0$.
We recall that, thanks to the results of the previous section, a function belonging to the De Giorgi class DG is locally bounded.

Proposition 6.1. *Consider three points* $(x^\diamond, t^\diamond), (x^\circ, t^\circ), (x^\star, t^\star) \in \Omega \times (0,T)$ *and* $\rho \in (0,R)$. *Suppose* $Q_R^{\beta^\diamond, >}(x^\diamond, t^\diamond)$, $Q_R^{\beta^\circ, <}(x^\circ, t^\circ)$, $Q_R^{s_1^\star, s_2^\star}(x^\star, t^\star)$ *are contained in* $\Omega \times (0,T)$. *Then for every choice of* $\theta^\diamond, \theta^\circ \in (0,1)$ *and* $a, \sigma \in (0,1)$ *there are*
$\overline{\nu}^\diamond \in (0,1)$, *depending only on* $\kappa, \gamma_1, \gamma, a, \theta^\diamond, \beta^\diamond$,
$\overline{\nu}^\circ \in (0,1)$, *depending only on* $\kappa, \gamma_1, \gamma, a, \theta^\circ, \beta^\circ$,
$\overline{\nu}^\star \in (0,1)$, *depending only on* $\kappa, \gamma_1, \gamma, a, (R-\rho)/R, \max\{1, 1/\beta^\star\}$,
$\overline{\nu} \in (0,1)$, *depending only on* $\kappa, \gamma_1, \gamma, a, (R-\rho)/R$,
such that for every $u \in DG_+(\Omega, T, \mu, \lambda, \gamma)$ *and fixed* $\overline{m}, \omega$ *satisfying*

i) $\overline{m} \geqslant \sup_{Q_{R;R,0}^{\beta^\diamond,+}(x^\diamond,t^\diamond)} u, \quad \omega \geqslant \operatorname*{osc}_{Q_{R;R,0}^{\beta^\diamond,+}(x^\diamond,t^\diamond)} u$ *we have that if* $\mu_+(B_\rho) > 0$ *and*

$$\frac{M_+(A_0^+)}{|M|_\Lambda(Q_R^{\beta^\diamond, >}(x^\diamond, t^\diamond))} + \frac{\Lambda_+(A_0^+)}{\Lambda(Q_R^{\beta^\diamond, >}(x^\diamond, t^\diamond))} \leqslant \overline{\nu}^\diamond,$$

where $A_0^+ = \{(x,t) \in Q_{R;R,0}^{\beta^\diamond,+}(x^\diamond, t^\diamond) \,|\, u(x,t) > \overline{m} - \sigma\omega\}$, *then*

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } (x,t) \in Q_{R;\rho,\theta^\diamond}^{\beta^\diamond,+}(x^\diamond, t^\diamond)\,;$$

ii) $\overline{m} \geqslant \sup_{Q_{R;R,0}^{\circ,-}(x^\circ,t^\circ)} u, \quad \omega \geqslant \operatorname*{osc}_{Q_{R;R,0}^{\circ,-}(x^\circ,t^\circ)} u$ *we have that if* $\mu_-(B_\rho) > 0$ *and*

$$\frac{M_-(A_0^-)}{|M|_\Lambda(Q_R^{\circ, <}(x^\circ, t^\circ))} + \frac{\Lambda_-(A_0^-)}{\Lambda(Q_R^{\circ, <}(x^\circ, t^\circ))} \leqslant \overline{\nu}^\circ,$$

where $A_0^- = \{(x,t) \in Q_{R;R,0}^{\circ,-}(x^\circ, t^\circ) \,|\, u(x,t) > \overline{m} - \sigma\omega\}$, *then*

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } (x,t) \in Q_{R;\rho,\theta^\circ}^{\circ,-}(x^\circ, t^\circ)\,;$$

iii) $\overline{m} \geqslant \sup_{Q_R^{s_1^\star,s_2^\star}(x^\star,t^\star)} u, \quad \omega \geqslant \operatorname*{osc}_{B_R(x^\star)\times(s_1^\star,s_2^\star)} u$ *we have that if* $\lambda_0(B_\rho) > 0$ *and*

$$\Lambda_0(A_0^0) \leqslant \overline{\nu}^\star\, \Lambda(Q_R^{s_1^\star, s_2^\star}(x^\star, t^\star))$$

where $A_0^0 = \{(x,t) \in Q^0_{R;R,s_1^\star,s_2^\star}(x^\star, t^\star) \,|\, u(x,t) > \overline{m} - \sigma\omega\}$, *then*

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } (x,t) \in Q^0_{R;\rho,s_1^\star,s_2^\star}(x^\star, t^\star)\,;$$

iv) $\overline{m} \geqslant \sup_{B_R(x^\star)} u(\cdot,t), \quad \omega \geqslant \operatorname*{osc}_{B_R(x^\star)} u(\cdot,t)$ *we have that if* $B_R(x^\star) \subset \Omega_0$ *and*

$$\lambda\big(\{x \in B_R(x^\star) \,|\, u(x,t) > \overline{m} - \sigma\omega\}\big) \leqslant \overline{\nu}\ \lambda(B_R(x^\star))$$

then

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } x \in B_\rho(x^\star)$$

for a.e. $t \in (s_1^\star, s_2^\star)$.

Remark 6.2. - The requirement $\mu_+(B_\rho) > 0$ in point *i)* (and analogously $\mu_-(B_\rho) > 0$ in point *ii)* and $\lambda_0(B_\rho) > 0$ in point *iii)* is not technically needed, for the proof it would be sufficient to have $\mu_+(B_R) > 0$. We require it just to give a meaning to the thesis of the theorem.

Proof - We prove only the first claim, being the other similar. Often we will not write the point $(x^\diamond, t^\diamond)$, just to simplify the notation. First of all fix $a, \sigma \in (0,1)$ which will remain fixed for all the proof. Choose $\theta^\diamond \in (0,1)$ and $\rho \in (0,R)$, assume that $\mu_+(B_\rho) > 0$ and consider the following sequences ($h \in \mathbf{N}$)

$$\rho_h = \rho + \varepsilon^h (R - \rho), \qquad \theta_h = \theta^\diamond - \varepsilon^{2h} \theta^\diamond ,$$

where $\varepsilon \in (0,1)$. We require that $(\theta_{h+1} - \theta_h) R^2$ is to be equal to $(\rho_h - \rho_{h+1})^2$ (as required in Definition 4.1 and in the proof that a Q-minimum belongs to the De Giorgi class, see (50)): we derive that $\theta^\diamond$ has to satisfy

$$\theta^\diamond = \frac{1-\varepsilon}{1+\varepsilon} \frac{(R-\rho)^2}{R^2} . \tag{65}$$

Referring to definitions (39) we will consider

$$x_0 = x^\diamond, \qquad t_0 = t^\diamond, \qquad s_2 = s_2^\diamond := t^\diamond + \beta^\diamond \, h(x^\diamond, R) R^2 ,$$

but we will often omit to write them just to simplify the notation. Now we moreover define, for $h \in \mathbf{N}$ and $a, \sigma \in (0,1)$,

$$\begin{gathered}
B_h = B_{\rho_h}(x^\diamond) , \\
\delta_h := \sum_{j=h}^{\infty} (\rho_j - \rho_{j+1}) = \rho_h - \rho = \varepsilon^h (R - \rho) \searrow 0 , \\
Q_h^+ := Q_{R;\rho,\theta_h}^{\beta^\diamond,+,\rho_h - \rho}(x^\diamond, t^\diamond) \\
I_h^+ := (I_\rho^+(x^\diamond))^{\delta_h} , \\
\sigma_h = a\,\sigma + \varepsilon^h (1-a)\,\sigma \searrow a\sigma , \qquad k_h = \overline{m} - \sigma_h \omega \nearrow \overline{m} - a\sigma\omega , \\
A_h^+ = \{(x,t) \in Q_h^+ \,|\, u(x,t) > k_h\} .
\end{gathered} \tag{66}$$

Notice that

$$\begin{gathered}
Q_{h+1}^+ \subset Q_h^+ , \qquad A_{h+1}^+ \subset A_h^+ , \\
\rho_h - \rho_{h+1} = (1-\varepsilon) \varepsilon^h (R - \rho) , \\
\theta_{h+1} \, \beta^\diamond \, h(x^\diamond, R) \, R^2 - \theta_h \, \beta^\diamond \, h(x^\diamond, R) \, R^2 = \theta^\diamond (1 - \varepsilon^2) \, \varepsilon^{2h} \, \beta^\diamond \, h(x^\diamond, R) \, R^2.
\end{gathered}$$

In the next picture we show some possible Q_h^+ marked by dashed lines, while the one marked by longer lines is the limit set (for $h \to +\infty$).

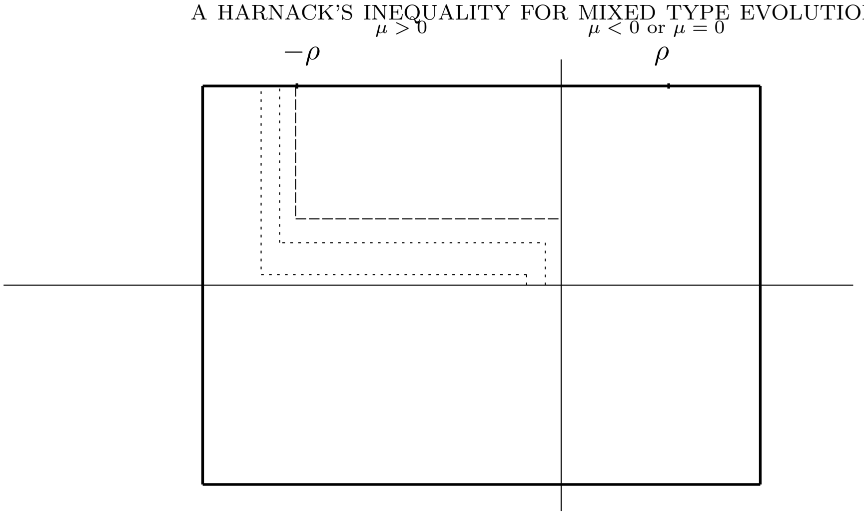


Figure C

First of all notice that since

$$(k_{h+1} - k_h)^2 M_+(A^+_{h+1}) \leqslant \iint_{A^+_{h+1}} (u - k_h)^2_+ \mu_+ \, dxdt \leqslant \iint_{Q^+_{h+1}} (u - k_h)^2_+ \mu_+ \, dxdt$$

and $k_{h+1} - k_h = (1 - a)\, \sigma\, \omega\, \varepsilon^{h+1}$ we can estimate

$$\varepsilon^{2h+2} (1-a)^2 \sigma^2 \omega^2 \, \frac{M_+(A^+_{h+1})}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \leqslant \frac{1}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \iint_{Q^+_{h+1}} (u - k_h)^2_+ \mu_+ \, dxdt \,. \tag{67}$$

Similarly

$$\varepsilon^{2h+2} (1-a)^2 \sigma^2 \omega^2 \, \frac{\Lambda_+(A^+_{h+1})}{\Lambda(Q_R^{\beta^\diamond,>})} \leqslant \frac{1}{\Lambda(Q_R^{\beta^\diamond,>})} \iint_{Q^+_{h+1}} (u - k_h)^2_+ \lambda_+ \, dxdt \,. \tag{68}$$

Then we can argue in a completely analogous way as done to obtain (55) and (56). Taking in (55)

$$\begin{array}{llll} \rho_{h+1} = r \,, & \rho_h = \tilde{r} \,, & \rho_{h-1} = \hat{r} \,, & \rho \text{ in place of } R/2 \,, \\ \theta_{h+1} = \theta \,, & \theta_h = \tilde{\theta} \,, & \theta_{h-1} = \hat{\theta} \,, & k_h = k \,, \end{array} \tag{69}$$

we get (the only difference with (55) is that $2(\rho_h - \rho_{h+1}) \neq \rho_{h-1} - \rho_h$ unless $\varepsilon = 1/2$)

$$\begin{aligned}
\frac{1}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \iint_{Q_{h+1}^+} (u-k_h)_+^2 \mu_+ \, dxdt \leqslant \\
\leqslant \gamma_1^{2/\kappa} R^2 \, \frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}} \, \frac{2\gamma+2}{(\rho_h - \rho_{h+1})^2} \, \cdot \\
\cdot \Bigg[\frac{1}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \iint_{Q_{h-1}^+} (u-k_h)_+^2 \mu_+ \, dxdt + \frac{1}{\Lambda(Q_R^{\beta^\diamond,>})} \iint_{Q_{h-1}^+} (u-k_h)_+^2 \lambda_+ \, dxdt + \\
+ \frac{1}{\Lambda(Q_R^{\beta^\diamond,>})} \iint_{I_{h-1}^+ \times (t^\diamond, s_2^\diamond)} (u-k_h)_+^2 (\lambda_0 + \lambda_-) \, dxdt + \\
+ (\rho_h - \rho_{h+1})^2 \, \frac{1}{\Lambda(Q_R^{\beta^\diamond,>})} \sup_{t \in (t^\diamond, s_2^\diamond)} \int_{I_{h-1}^+} (u-k_h)_+^2 (x,t) |\mu|(x) dx \Bigg] .
\end{aligned}$$

Now since (here we use $\sigma_h \leqslant \sigma$)

$$\begin{aligned}
\iint_{Q_{h-1}^+} (u-k_h)_+^2 \mu_+ \, dxdt \leqslant M_+(A_{h-1}^+) \ \sup_{Q_{h-1}^+} (u-k_h)^2 \leqslant M_+(A_{h-1}^+) (\sigma\omega)^2 \, , \\
\iint_{Q_{h-1}^+} (u-k_h)_+^2 \lambda_+ \, dxdt \leqslant \Lambda_+(A_{h-1}^+) \ \sup_{Q_{h-1}^+} (u-k_h)^2 \leqslant \Lambda_+(A_{h-1}^+) (\sigma\omega)^2 \, ,
\end{aligned}$$

by the above inequality and by (67) we get

$$\begin{aligned}
\frac{M_+(A_{h+1}^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \leqslant \frac{\gamma_1^{2/\kappa} R^2}{\varepsilon^{2h+2}(1-a)^2 \sigma^2 \omega^2} \, \frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}} \, \frac{2\gamma+2}{(1-\varepsilon)^2 \varepsilon^{2h} (R-\rho)^2} \\
\cdot \left(\frac{M_+(A_h^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \right)^{\frac{\kappa-1}{\kappa}} \cdot \Bigg[\frac{M_+(A_{h-1}^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} (\sigma\omega)^2 + \frac{\Lambda_+(A_{h-1}^+)}{\Lambda(Q_R^{\beta^\diamond,>})} (\sigma\omega)^2 + \\
+ \frac{1}{\Lambda(Q_R^{\beta^\diamond,>})} \iint_{I_{h-1}^+ \times (t^\diamond, s_2^\diamond)} (u-k_h)_+^2 (\lambda_0 + \lambda_-) \, dxdt + \\
+ \frac{(R-\rho)^2 (1-\varepsilon)^2 \varepsilon^{2h}}{\Lambda(Q_R^{\beta^\diamond,>})} \sup_{t \in (t^\diamond, s_2^\diamond)} \int_{I_{h-1}^+} (u-k_h)_+^2 (x,t) |\mu|(x) dx \Bigg] .
\end{aligned}$$

Now defining first

$$y_h := y_h^M + y_h^\Lambda \, , \qquad \text{where } y_h^M := \frac{M_+(A_h^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} \quad \text{and} \quad y_h^\Lambda := \frac{\Lambda_+(A_h^+)}{\Lambda(Q_R^{\beta^\diamond,>})} \, ,$$

and, since $(u-k_h)_+^2$ is bounded by $(\sigma\omega)^2$ and estimating

$$\frac{1}{\sigma^2\omega^2\Lambda(Q_R^{\beta^\diamond,>})}\iint_{I_h^+\times(t^\diamond,s_2^\diamond)}(u-k_h)_+^2(\lambda_0+\lambda_-)\,dxdt+$$

$$+\ \frac{(R-\rho)^2(1-\varepsilon)^2\varepsilon^{2(h-1)}}{\sigma^2\omega^2\Lambda(Q_R^{\beta^\diamond,>})}\sup_{t\in(t^\diamond,s_2^\diamond)}\int_{I_h^+}(u-k_h)_+^2(x,t)|\mu|(x)dx\leqslant$$

$$\leqslant\frac{(\Lambda_0+\Lambda_-)(I_h^+\times(t^\diamond,s_2^\diamond))}{\Lambda(Q_R^{\beta^\diamond,>})}+\frac{(R-\rho)^2(1-\varepsilon)^2\varepsilon^{2(h-1)}|\mu|(I_h^+)}{\Lambda(Q_R^{\beta^\diamond,>})}$$

defining also

$$\epsilon_h:=\frac{(\Lambda_0+\Lambda_-)(I_h^+\times(t^\diamond,s_2^\diamond))}{\Lambda(Q_R^{\beta^\diamond,>})}+\frac{(R-\rho)^2(1-\varepsilon)^2\varepsilon^{2(h-1)}|\mu|(I_h^+)}{\Lambda(Q_R^{\beta^\diamond,>})}$$

we first get

$$y_{h+1}^M\leqslant\frac{\gamma_1^{2/\kappa}R^2\,(2\gamma+2)}{(1-a)^2(1-\varepsilon)^2\varepsilon^2(R-\rho)^2}\,\frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}}\,\frac{1}{\varepsilon^{4h}}\,(y_h^M)^{\frac{\kappa-1}{\kappa}}\left[y_{h-1}^M+y_{h-1}^\Lambda+\epsilon_{h-1}\right]\,.$$

Taking (69) in (56) we can argue in a similar way to estimate y_{h+1}^Λ and get

$$y_{h+1}^\Lambda\leqslant\frac{\gamma_1^{2/\kappa}R^2\,(2\gamma+2)}{(1-a)^2(1-\varepsilon)^2\varepsilon^2(R-\rho)^2}\,\frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}}\,\frac{1}{\varepsilon^{4h}}\,(y_h^\Lambda)^{\frac{\kappa-1}{\kappa}}\left[y_{h-1}^M+y_{h-1}^\Lambda+\epsilon_{h-1}\right]\,.$$

Summing the two inequalities and since the sequences $(y_h^M)_h$, $(y_h^\Lambda)_h$ are decreasing we finally get

$$y_{h+1}\leqslant\ \frac{\gamma_1^{2/\kappa}R^2\,(2\gamma+2)}{(1-a)^2(1-\varepsilon)^2\varepsilon^2(R-\rho)^2}\,\frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}}\,\frac{1}{\varepsilon^{4h}}\,y_{h-1}^{\frac{\kappa-1}{\kappa}}\,(y_{h-1}+\epsilon_{h-1})$$

for every $h\geqslant 1$; then, for instance,

$$y_{2(h+1)}\leqslant\ \frac{\gamma_1^{2/\kappa}R^2\,(2\gamma+2)}{(1-a)^2(1-\varepsilon)^2(R-\rho)^2\varepsilon^6}\,\frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}}\,\frac{1}{\varepsilon^{8h}}\,y_{2h}^{\frac{\kappa-1}{\kappa}}\,(y_{2h}+\epsilon_{2h})\,.$$

Using (65) to write $R^2/(R-\rho)^2$ and Lemma 2.19 with

$$c=\frac{\gamma_1^{2/\kappa}\,(2\gamma+2)}{(1-a)^2(1-\varepsilon^2)\,\varepsilon^6\,\theta^*}\,\frac{1+\beta^\diamond}{(\beta^\diamond)^{\frac{1}{\kappa}}}\,,\quad \alpha=\frac{\kappa-1}{\kappa}\,,\quad b=\frac{1}{\varepsilon^8}\,,$$

we derive that the subsequence $(y_{2h})_h$ of even indexes, and in fact the whole sequence $(y_h)_h$ since $(y_h)_h$ is decreasing, is converging to zero provided that

$$\frac{M_+(A_0^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>})}+\frac{\Lambda_+(A_0^+)}{\Lambda(Q_R^{\beta^\diamond,>})}\leqslant\left(\frac{(1-a)^2(1-\varepsilon^2)\,\varepsilon^6\,\theta^\diamond(\beta^\diamond)^{\frac{1}{\kappa}}}{\gamma_1^{2/\kappa}\,(1+\beta^\diamond)\,(2\gamma+2)}\right)^{\frac{\kappa}{\kappa-1}}\varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}}\,.$$

By the definition of A_h we have that

$$Q_0^+=Q_{R;\rho,0}^{\beta^\diamond,+,R-\rho}(x^\diamond,t^\diamond)\qquad\text{and}\qquad A_0^+=\left\{(x,t)\in Q_0^+\,\middle|\,u(x,t)>\overline{m}-\sigma\omega\right\}$$

but we can consider

$$\begin{aligned} A_0^+ &= \big\{ (x,t) \in Q_{R;R,0}^{\beta^\diamond,+}(x^\diamond, t^\diamond) \,\big|\, u(x,t) > \overline{m} - \sigma\omega \big\} = \\ &= \big\{ (x,t) \in B_R^+(x^\diamond) \times (t^\diamond, t^\diamond + h(x^\diamond, R)R^2) \,\big|\, u(x,t) > \overline{m} - \sigma\omega \big\} \end{aligned}$$

since we will consider the measures M_+ and Λ_+ of this set. Then we have derived that

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } (x,t) \in Q_{R;\rho,\theta^\diamond}^{\beta^\diamond,+}(x^\diamond, t^\diamond)$$

provided that

$$\frac{M_+(A_0^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>})} + \frac{\Lambda_+(A_0^+)}{\Lambda(Q_R^{\beta^\diamond,>})} \leqslant \overline{\nu}^\diamond$$

where

$$\overline{\nu}^\diamond = \left(\frac{(1-a)^2 (1-\varepsilon^2)\, \varepsilon^6\, \theta^\diamond\, (\beta^\diamond)^{\frac{1}{\kappa}}}{\gamma_1^{2/\kappa}\, (1+\beta^\diamond)\, (2\gamma + 2)} \right)^{\frac{\kappa}{\kappa-1}} \varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}} .$$

In a complete analogous way: fix a point (x°, t°) such that $\mu_-(B_R(x^\circ)) > 0$. One gets that taking the same values as before for ρ, a, σ and $\theta^\circ \in (0,1)$ there is $\overline{\nu}^\circ > 0$ such that if

$$\frac{M_-(A_0^-)}{|M|_\Lambda(Q_R^{\beta^\circ,<})} + \frac{\Lambda_-(A_0^-)}{\Lambda(Q_R^{\beta^\circ,<})} \leqslant \overline{\nu}^\circ ,$$

where the ball B_R is centred in x° and

$$\begin{aligned} A_0^- &= \Big\{ (x,t) \in Q_{R;\rho,0}^{\beta^\circ,-,R-\rho}(x^\circ, t^\circ) \,|\, u(x,t) > \overline{m} - \sigma\omega \Big\} , \\ \overline{\nu}^\circ &= \left(\frac{(1-a)^2 (1-\varepsilon^2)\, \varepsilon^6\, \theta^\circ\, (\beta^\circ)^{\frac{1}{\kappa}}}{\gamma_1^{2/\kappa}\, (1+\beta^\circ)\, (2\gamma + 2)} \right)^{\frac{\kappa}{\kappa-1}} \varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}} , \end{aligned}$$

then

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } (x,t) \in Q_{R;\rho,\theta^\circ}^{\beta^\circ,-}(x^\circ, t^\circ) .$$

Finally we analyse the part in which $\mu \equiv 0$, which is slightly different. Fix a point $(x^\star, t^\star)$ such that $\lambda_0(B_R(x^\star)) > 0$, consider k_h and σ_h as in (66). Arguing as done to obtain (62) and taking in (62) for $k, r, \tilde{r}, \hat{r}$ the same values as in (69) and for σ_1, σ_2 respectively $s_1^\star$ and $s_2^\star$ we get

$$\begin{aligned}
&\frac{1}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \iint_{Q^0_{h+1}} (u - k_h)_+^2 \lambda_0 \, dxdt \leqslant \\
&\qquad \leqslant \gamma_1^{2/\kappa} (\beta^\star)^{\frac{\kappa-1}{\kappa}} R^2 \, \frac{2\gamma + 2}{(1-\varepsilon)^2 \varepsilon^{2h} (R-\rho)^2} \, \frac{\big(\Lambda_0(A^0_h)\big)^{\frac{\kappa-1}{\kappa}}}{(\Lambda(B_R \times (s_1^\star, s_2^\star)))^{\frac{\kappa-1}{\kappa}}} \, \cdot \\
&\qquad \cdot \Bigg[\frac{1}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \iint_{Q^0_{h-1}} (u - k_h)_+^2 \lambda_0 \, dxdt + \\
&\qquad\quad + \frac{1}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \iint_{I^0_{h-1} \times (s_1^\star, s_2^\star)} (u - k_h)_+^2 (\lambda_+ + \lambda_-) \, dxdt + \\
&\qquad\quad + \frac{(1-\varepsilon)^2 \varepsilon^{2h} (R-\rho)^2}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \sup_{t \in (s_1^\star, s_2^\star)} \int_{(B^0_{\rho_h})^{\rho_h - \rho}} (u - k_h)_+^2 (x,t) \lambda_0(x) dx \\
&\qquad\quad + \frac{(1-\varepsilon)^2 \varepsilon^{2h} (R-\rho)^2}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \sup_{t \in (s_1^\star, s_2^\star)} \int_{I^0_{h-1}} (u - k_h)_+^2 (x,t) \mu_+(x) dx + \\
&\qquad\quad + \frac{(1-\varepsilon)^2 \varepsilon^{2h} (R-\rho)^2}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \sup_{t \in (s_1^\star, s_2^\star)} \int_{I^0_{h-1}} (u - k_h)_+^2 (x,t) \mu_-(x) dx \Bigg]
\end{aligned}$$

where

$$\begin{aligned}
I^0_h &:= (I^0_\rho(x^\star))^{\rho_h - \rho} \setminus I^0_{\rho, \rho_h - \rho}(x^\star) \\
Q^0_h &:= Q^{0, \rho_h - \rho}_{R; \rho, s_1^\star, s_2^\star}(x^\star, t^\star) \, , \\
A^0_h &= \{(x,t) \in Q^0_h \, | \, u(x,t) > k_h\} \, .
\end{aligned}$$

Since, as for (67), we have

$$\varepsilon^{2h+2} (1-a)^2 \sigma^2 \omega^2 \, \frac{\Lambda_0(A^0_{h+1})}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \leqslant \frac{1}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \iint_{Q^0_{h+1}} (u - k_h)_+^2 \lambda_0 \, dxdt \, ,$$

$$\iint_{Q^0_{h-1}} (u - k_h)_+^2 \lambda_0 \, dxdt \leqslant \Lambda_0(A^0_{h-1}) \sup_{Q^0_{h-1}} (u - k_h)^2 \leqslant \Lambda_0(A^0_{h-1}) (\sigma\omega)^2 \, ,$$

we derive

$$y_{h+1} \leqslant \frac{\gamma_1^{2/\kappa} (\beta^\star)^{\frac{\kappa-1}{\kappa}} R^2 \, (2\gamma + 2)}{(1-a)^2 (1-\varepsilon)^2 \varepsilon^2 (R-\rho)^2} \, \frac{1}{\varepsilon^{4h}} \, y_{h-1}^{\frac{\kappa-1}{\kappa}} \, (y_{h-1} + \epsilon_{h-1})$$

where here we have defined

$$\begin{aligned}
y_h &:= \frac{\Lambda_0(A^0_h)}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \\
\epsilon_h &:= \frac{1}{\Lambda(B_R \times (s_1^\star, s_2^\star))} \Bigg[(\Lambda_+ + \Lambda_-)(I^0_{h-1} \times (s_1^\star, s_2^\star)) + \\
&\qquad + (R-\rho)^2 (1-\varepsilon)^2 \varepsilon^{2h} \Big(\Lambda_0((B^0_{\rho_h})^{\rho_h - \rho}) + |\mu|(I^0_{h-1}) \Big) \Bigg].
\end{aligned}$$

Arguing similarly as before we get that y_h tends to zero, that is

$$u(x,t) \leqslant \overline{m} - a\,\sigma\,\omega \qquad \text{for a.e. } (x,t) \in Q^0_{R;\rho,s_1^\star,s_2^\star}(x^\star,t^\star)\,,$$

provided that

$$\frac{\Lambda_0(A_0^0)}{\Lambda(B_R \times (s_1^\star, s_2^\star)} \leqslant \overline{\nu}^\star$$

where

$$\overline{\nu}^\star = \left[\frac{(1-a)^2\,(1-\varepsilon)^2\,\varepsilon^6\,(R-\rho)^2}{\gamma_1^{2/\kappa} R^2\,(2\gamma+2)}\right]^{\frac{\kappa}{\kappa-1}} \frac{1}{\beta^\star}\,\varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}}\,.$$

Notice that ($\gamma_1 > 1$)

$$\left[\frac{(1-a)^2\,(1-\varepsilon)^2\,\varepsilon^6\,(R-\rho)^2}{\gamma_1^{2/\kappa} R^2\,(2\gamma+2)}\right]^{\frac{\kappa}{\kappa-1}} \varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}} \leqslant 1$$

and to garantee $\overline{\nu}^\star \leqslant 1$ for every choice of $\beta^\star$ (say less than 1) we can choose ε in a suitable way. For example taking ε in such a way that $\varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}}/\beta^\star = 1/2$, i.e.

$$\varepsilon = \left(\frac{\beta^\star}{2}\right)^{\frac{(\kappa-1)^2}{8\kappa^2}}$$

we have $\overline{\nu}^\star < 1$ and we get rid of the dependence of $1/\beta^\star$ for $\beta^\star$ small.
For the last point we can proceed as follows: first notice that $B_R := B_R(x^\star) \subset \Omega_0$. With the same k_h and ρ_h as before we consider $B_h := B_{\rho_h}(x^\star)$, define the sequence of test functions

$$\zeta_h : B_R \to [0,1]\,, \qquad \zeta_h(x) = \left\{ \begin{array}{ll} 1 & \text{in } B_{h+1} \\ 0 & \text{in } B_R \setminus B_h \end{array} \right. \qquad |D\zeta_h| \leqslant \frac{1}{\rho_h - \rho_{h+1}}$$

and for almost every $t \in (0,T)$ we define $A_h = \{x \in B_{\rho_h}(x^\star) \,|\, u(x,t) > k_h\}$. Using Theorem 2.5 with 2κ in the place of q (see also Remark 2.6) we have

$$
\begin{aligned}
(1-a)^2\sigma^2\omega^2\varepsilon^{2(h+1)}\frac{\lambda(A_{h+1})}{\lambda(B_R)} &\leqslant \frac{1}{\lambda(B_R)}\int_{B_{h+1}}(u-k_h)_+^2(x,t)\lambda(x)\,dx \leqslant \\
&\leqslant \frac{1}{\lambda(B_R)}\int_{B_h}(u-k_h)_+^2(x,t)\zeta_h^2(x)\lambda(x)\,dx \leqslant \\
&\leqslant \left(\frac{\lambda(A_h)}{\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\left[\frac{1}{\lambda(B_R)}\int_{B_h}(u-k_h)_+^{2\kappa}(x,t)\zeta_h^{2\kappa}(x)\lambda(x)\,dx\right]^{\frac{1}{\kappa}} \leqslant \\
&\leqslant \left(\frac{\lambda(A_h)}{\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\frac{\gamma_1^2\,R^2}{\lambda(B_R)}\int_{B_h}|D\big((u-k_h)_+\zeta_h\big)|^2\lambda\,dx \leqslant \\
&\leqslant \left(\frac{\lambda(A_h)}{\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\frac{2\,\gamma_1^2\,R^2}{\lambda(B_R)}\int_{B_h}\left[|D(u-k_h)_+|^2+\frac{1}{(\rho_h-\rho_{h+1})^2}(u-k_h)_+^2\right]\lambda\,dx \leqslant \\
&\leqslant \left(\frac{\lambda(A_h)}{\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\frac{2\,\gamma_1^2\,R^2}{\lambda(B_R)}\left[\frac{\gamma}{(\rho_{h-1}-\rho_h)^2}\int_{B_{h-1}}(u-k_h)_+^2(x,t)\,\lambda(x)\,dx+\right. \\
&\qquad\qquad\left.+\frac{1}{(\rho_h-\rho_{h+1})^2}\int_{B_h}(u-k_h)_+^2\lambda\,dx\right] \leqslant \\
&\leqslant \left(\frac{\lambda(A_h)}{\lambda(B_R)}\right)^{\frac{\kappa-1}{\kappa}}\frac{2\,\gamma_1^2\,R^2}{\lambda(B_R)}\frac{\gamma+1}{\varepsilon^{2h}(R-\rho)^2(1-\varepsilon)^2}\int_{B_{h-1}}(u-k_h)_+^2(x,t)\,\lambda(x)\,dx \leqslant \\
&\leqslant 2\,\gamma_1^2\,R^2\,\frac{\gamma+1}{\varepsilon^{2h}(R-\rho)^2(1-\varepsilon)^2}\,\sigma^2\omega^2\,\left(\frac{\lambda(A_{h-1})}{\lambda(B_R)}\right)^{1+\frac{\kappa-1}{\kappa}}.
\end{aligned}
$$

We can conclude similarly as before using Lemma 2.18 and provided that

$$
\frac{\lambda(A_0)}{\lambda(B_R)} \leqslant \overline{\nu} = \left[\frac{(1-a)^2\,(1-\varepsilon)^2\,\varepsilon^6\,(R-\rho)^2}{\gamma_1^2R^2\,(2\gamma+2)}\right]^{\frac{\kappa}{\kappa-1}}\;\varepsilon^{\frac{8\kappa^2}{(\kappa-1)^2}}\,. \qquad \square
$$

Proposition 6.3. *Consider three points $(x^\diamond,t^\diamond),(x^\circ,t^\circ),(x^\star,t^\star)\in\Omega\times(0,T)$ and $r\in(0,R)$. Suppose $Q_R^{\beta^\diamond,>}(x^\diamond,t^\diamond)$ $Q_R^{\beta^\circ,<}(x^\circ,t^\circ)$ $Q_R^{s_1^\star,s_2^\star}(x^\star,t^\star)$ are contained in $\Omega\times(0,T)$. Then for every choice of $\theta^\diamond,\theta^\circ\in(0,1)$ and $a,\sigma\in(0,1)$ there are*
$\underline{\nu}^\diamond\in(0,1)$, depending only on $\kappa,\gamma_1,\gamma,a,\ \theta^\diamond,\ \beta^\diamond$,
$\underline{\nu}^\circ\in(0,1)$, depending only on $\kappa,\gamma_1,\gamma,a,\ \theta^\circ,\ \beta^\circ$,
$\underline{\nu}^\star\in(0,1)$, depending only on $\kappa,\gamma_1,\gamma,a,(R-r)/R$, $\max\{1,1/\beta^\star\}$,
$\underline{\nu}\in(0,1)$, depending only on $\kappa,\gamma_1,\gamma,a,(R-r)/R$,
such that for every $u\in DG_-(\Omega,T,\mu,\lambda,\gamma)$ and fixed $\underline{m},\omega$ satisfying

i) $\underline{m}\leqslant\inf_{Q^{\beta^\diamond,+}_{R;R,0}(x^\diamond,t^\diamond)}u$, $\quad\omega\geqslant\operatorname*{osc}_{Q^{\beta^\diamond,+}_{R;R,0}(x^\diamond,t^\diamond)}u$ we have that if $\mu_+(B_r)>0$ and

$$
\frac{M_+(A_0^+)}{|M|_\Lambda(Q_R^{\beta^\diamond,>}(x^\diamond,t^\diamond))}+\frac{\Lambda_+(A_0^+)}{\Lambda(Q_R^{\beta^\diamond,>}(x^\diamond,t^\diamond))}\leqslant\underline{\nu}^\diamond,
$$

where $A_0^+ = \{(x,t) \in Q_{R;R,0}^{\beta^\diamond,+}(x^\diamond,t^\diamond)\,|\, u(x,t) < \underline{m} + \sigma\omega\}$*, then*

$$u(x,t) \geqslant \underline{m} + a\,\sigma\,\omega \qquad \textit{for a.e. } (x,t) \in Q_{R;r,\theta^\diamond}^{\beta^\diamond,+}(x^\diamond,t^\diamond)\,;$$

ii) $\underline{m} \leqslant \inf_{Q_{R;R,0}^{\circ,-}(x^\circ,t^\circ)} u$, $\quad \omega \geqslant \operatorname*{osc}_{Q_{R;R,0}^{\circ,-}(x^\circ,t^\circ)} u$ *we have that if* $\mu_-(B_r) > 0$ *and*

$$\frac{M_-(A_0^-)}{|M|_\Lambda(Q_R^{\circ,<}(x^\circ,t^\circ))} + \frac{\Lambda_-(A_0^-)}{\Lambda(Q_R^{\circ,<}(x^\circ,t^\circ))} \leqslant \underline{\nu}^\circ,$$

where $A_0^- = \{(x,t) \in Q_{R;R,0}^{\circ,-}(x^\circ,t^\circ)\,|\, u(x,t) < \underline{m} + \sigma\omega\}$*, then*

$$u(x,t) \geqslant \underline{m} + a\,\sigma\,\omega \qquad \textit{for a.e. } (x,t) \in Q_{R;r,\theta^\circ}^{\circ,-}(x^\circ,t^\circ)\,;$$

iii) $\underline{m} \leqslant \inf_{Q_R^{s_1^\star,s_2^\star}(x^\star,t^\star)} u$, $\quad \omega \geqslant \operatorname*{osc}_{Q_R^{s_1^\star,s_2^\star}(x^\star,t^\star)} u$ *we have that if* $\lambda_0(B_r) > 0$ *and*

$$\Lambda_0(A_0^0) \leqslant \underline{\nu}^\star\, \Lambda(Q_R^{s_1^\star,s_2^\star}(x^\star,t^\star))$$

where $A_0^0 = \{(x,t) \in Q_{R;R,s_1^\star,s_2^\star}^0(x^\star,t^\star)\,|\, u(x,t) < \underline{m} + \sigma\omega\}$*, then*

$$u(x,t) \geqslant \underline{m} + a\,\sigma\,\omega \qquad \textit{for a.e. } (x,t) \in Q_{R;r,s_1^\star,s_2^\star}^0(x^\star,t^\star)\,;$$

iv) $\underline{m} \leqslant \inf_{B_R(x^\star)} u(\cdot,t)$, $\quad \omega \geqslant \operatorname*{osc}_{B_R(x^\star)} u(\cdot,t)$ *we have that if* $B_R(x^\star) \subset \Omega_0$ *and*

$$\lambda\big(\{x \in B_R(x^\star)\,|\, u(x,t) < \underline{m} + \sigma\omega\}\big) \leqslant \underline{\nu}\ \lambda(B_R(x^\star))$$

then

$$u(x,t) \geqslant \underline{m} + a\,\sigma\,\omega \qquad \textit{for a.e. } x \in B_r(x^\star)$$

for a.e. $t \in (0,T)$.

We now need some results which are preparatory for one fundamental step in view of proving the Harnack's inequality, Lemma 6.7, which is usually referred to as *expansion of positivity.*

We define, for a fixed point $(\bar{y},\bar{s}) \in \Omega \times (0,T)$ and a fixed $h > 0$, the sets

$$\begin{aligned} A_{h,\rho}^+(\bar{y},\bar{s}) &= \{x \in B_\rho^+(\bar{y})\,|\, u(x,\bar{s}) < h\}\,, \\ A_{h,\rho}^-(\bar{y},\bar{s}) &= \{x \in B_\rho^-(\bar{y})\,|\, u(x,\bar{s}) < h\}\,, \\ A_{h,\rho}^0(\bar{y},\bar{s}) &= \{x \in B_\rho^0(\bar{y})\,|\, u(x,\bar{s}) < h\}\,. \end{aligned} \tag{70}$$

Remark 6.4. - Observe that the condition $u(x,\bar{s}) \geqslant h$ for every $x \in B_\rho(\bar{y})$ implies that $A_{h,4\rho}(\bar{y},\bar{s}) \subset B_{4\rho}(\bar{y}) \setminus B_\rho(\bar{y})$ and then in particular, if ω is a doubling weight (c_ω denotes the doubling constant of ω), one has

$$\omega\big(A_{h,4\rho}(x^*,t^*)\big) \leqslant \omega\big(B_{4\rho}(x^*) \setminus B_\rho(x^*)\big) \leqslant \big(1 - c_\omega^{-2}\big)\ \omega\big(B_{4\rho}(x^*)\big)\,.$$

In our situation this holds for $|\mu|_\lambda$, thanks to (32), but also for μ_+, μ_-, λ_0 thanks to the assumtpion (H.4).

Lemma 6.5. *Given* (x^*, t^*) *such that* $B_{4\rho}(x^*) \subset \Omega$ *then*
i) if $\lambda_0\big(B_{4\rho}(x^*)\big) > \lambda_0\big(B_\rho(x^*)\big) > 0$ *there exists* $\eta \in (0,1)$*, depending only on* $\mathfrak{q}$*, such that for every* $\bar t \in (0,T)$ *we have that, given* $h > 0$ *and* $u \geqslant 0$ *belonging to* $DG(\Omega, T, \mu, \lambda, \gamma)$ *for which the following holds*

$$u(x, \bar t) \geqslant h \qquad \text{a.e. in } B^0_\rho(x^*),$$

then

$$\lambda_0(A^0_{\eta h, 4\rho}(x^*, \bar t)) \leqslant \left(1 - \frac{1}{2}\frac{1}{\mathfrak{q}^2}\right) \lambda_0\big(B^0_{4\rho}(x^*)\big).$$

If $B_{4\rho}(x^*) \times [t^* - \beta\, h(x^*, 4\rho)\, \rho^2, t^* + \beta\, h(x^*, 4\rho)\, \rho^2] \subset \Omega \times (0,T)$ *with* $\beta \in (0, 16]$ *then:*
ii) if $\mu_+\big(B_{4\rho}(x^*)\big) > \mu_+\big(B_\rho(x^*)\big) > 0$ *there exists* $\eta \in (0,1)$*, depending only on* $\gamma, \mathfrak{q}$*, and there exists* $\tilde\beta \in (0, \beta]$*, depending only on* γ *and* β*, such that, given* $h > 0$ *and* $u \geqslant 0$ *belonging to* $DG(\Omega, T, \mu, \lambda, \gamma)$ *for which the following holds*

$$u(x, t^*) \geqslant h \qquad \text{a.e. in } B^+_\rho(x^*),$$

then for every $t \in [t^*, t^* + \tilde\beta\, h(x^*, 4\rho)\, \rho^2]$

$$\mu_+(A^+_{\eta h, 4\rho}(x^*, t)) \leqslant \left(1 - \frac{1}{2}\frac{1}{\mathfrak{q}^2}\right) \mu_+\big(B^+_{4\rho}(x^*)\big);$$

iii) if $\mu_-\big(B_{4\rho}(x^*)\big) > \mu_-\big(B_\rho(x^*)\big) > 0$ *there exists* $\eta \in (0,1)$*, depending only on* $\gamma, \mathfrak{q}$*, and there exists* $\tilde\beta \in (0, \beta]$*, depending only on* γ *and* β*, such that, given* $h > 0$ *and* $u \geqslant 0$ *belonging to* $DG(\Omega, T, \mu, \lambda, \gamma)$ *for which the following holds*

$$u(x, t^*) \geqslant h \qquad \text{a.e. in } B^-_\rho(x^*),$$

then for every $t \in [t^* - \tilde\beta\, h(x^*, 4\rho)\, \rho^2, t^*]$

$$\mu_-(A^-_{\eta h, 4\rho}(x^*, s)) \leqslant \left(1 - \frac{1}{2}\frac{1}{\mathfrak{q}^2}\right) \mu_-\big(B^-_{4\rho}(x^*)\big);$$

iv) there exist $\eta \in (0,1)$*, depending only on* γ *and* $\mathfrak{q}$*, and there exists* $\tilde\beta \in (0, \beta]$*, depending only on* γ *and* β*, such that, given* $h > 0$ *and* $u \geqslant 0$ *belonging to* $DG(\Omega, T, \mu, \lambda, \gamma)$ *for which the following holds*

$$u(x, t^*) \geqslant h \qquad \text{a.e. in } B_\rho(x^*),$$

then

$$|\mu|_\lambda\big(A^+_{\eta h, 4\rho}(x^*, t) \cup A^-_{\eta h, 4\rho}(x^*, s) \cup A^0_{\eta h, 4\rho}(x^*, t^*)\big) \leqslant \left(1 - \frac{1}{2}\frac{1}{\mathfrak{q}^2}\right) |\mu|_\lambda\big(B_{4\rho}(x^*)\big)$$

for every $t \in [t^*, t^* + \tilde\beta\, h(x^*, 4\rho)\, \rho^2]$ *and* $s \in [t^* - \tilde\beta\, h(x^*, 4\rho)\, \rho^2, t^*]$.

Proof - First we prove point *ii)*. Consider $s_1 = t^* - \beta h(x^*, 4\rho)\rho^2$, $s_2 = t^* + \beta h(x^*, 4\rho)\rho^2$. Apply the energy estimate (44) to the function $(u - h)_-$ with $x_0 = x^*$, $t_0 = t^*$, $r = 4\rho(1 - \sigma)$ for an arbitrary $\sigma \in (0,1)$, $R = \tilde r = 4\rho$, $\varepsilon = 0$. With this choice we have $\tilde r - r = 4\rho\sigma$. Then we get

$$\begin{aligned}\sup_{t\in(t^*,s_2)} \int_{B^+_{4\rho(1-\sigma)}(x^*)} & (u-h)^2_-(x,t)\mu_+(x)dx \leqslant \\ & \leqslant \int_{B^+_{4\rho}(x^*)} (u-h)^2_-(x,t^*)\mu_+(x)dx \ + \sup_{t\in(t^*,s_2)} \int_{I_+^{4\rho,4\rho\sigma}} (u-h)^2_-(x,t)\mu_-(x)\,dx + \\ & \quad + \frac{\gamma}{(4\rho\sigma)^2} \int_{t^*}^{s_2}\!\!\int_{B^+_{4\rho}(x^*)\cup I_+^{4\rho,4\rho\sigma}} (u-h)^2_-\,\lambda\,dxds.\end{aligned}$$

Now, in addition to this inequality, we use the two following inequalities: first that in a set $A_{\eta h,r}$ we have that $(u-h)_- \geqslant (1-\eta)h$; moreover, since $u \geqslant 0$, $(u-h)_- \leqslant h$. Then, using also Remark 6.4, we get for every $t \in [t^*, s_2]$

$$\begin{aligned}(1-\eta)^2 h^2 \mu_+\big(A^+_{\eta h,4\rho(1-\sigma)}(x^*,t)\big) & \leqslant \\ & \leqslant \int_{A^+_{\eta h,4\rho(1-\sigma)}(x^*,t)} (u-h)^2_-(x,t)\mu_+(x)dx \leqslant \\ & \leqslant \int_{B^+_{4\rho(1-\sigma)}(x^*)} (u-h)^2_-(x,t)\mu_+(x)dx + \leqslant \\ & \leqslant h^2\,\mu_+\big(B_{4\rho}(x^*)\setminus B_\rho(x^*)\big) + h^2\mu_-(I_+^{4\rho,4\rho\sigma}) + \frac{\gamma h^2}{(4\rho\sigma)^2}\,\Lambda\big((B^+_{4\rho}(x^*)\cup I_+^{4\rho,4\rho\sigma})\times(t^*,s_2)\big).\end{aligned}$$

Using the following decomposition

$$A^+_{\eta h,4\rho}(x^*,t) = A^+_{\eta h,4\rho(1-\sigma)}(x^*,t) \cup \big\{x \in B^+_{4\rho}(x^*)\setminus B^+_{4\rho(1-\sigma)}(x^*)\,\big|\,u(x,t) < \eta h\big\},$$

and then the last estimate we get

$$\begin{aligned}(1-\eta)^2\mu_+\big(A^+_{\eta h,4\rho}(x^*,t)\big) & \leqslant \\ & \leqslant (1-\eta)^2\Big[\mu_+\big(A^+_{\eta h,4\rho(1-\sigma)}(x^*,t)\big) + \mu_+\big(B_{4\rho}(x^*)\setminus B_{4\rho(1-\sigma)}(x^*)\big)\Big] \leqslant \\ & \leqslant \mu_+\big(B_{4\rho}(x^*)\setminus B_\rho(x^*)\big) + \mu_-(I_+^{4\rho,4\rho\sigma}) + \frac{\gamma}{(4\rho\sigma)^2}\,\Lambda\big((B^+_{4\rho}(x^*)\cup I_+^{4\rho,4\rho\sigma})\times(t^*,s_2)\big) + \\ & \quad + (1-\eta)^2\mu_+\big(B_{4\rho}(x^*)\setminus B_{4\rho(1-\sigma)}(x^*)\big).\end{aligned} \tag{71}$$

If the thesis were false we would have that for every $\tilde{\beta} \in (0,\beta]$ and $\eta \in (0,1)$ there would be $\bar t \in [t^*, t^* + \tilde{\beta}\,h(x^*,4\rho)\,\rho^2]$ such that

$$\left(1 - \frac{1}{2}\frac{1}{\mathfrak{q}^2}\right)\,\mu_+\big(B^+_{4\rho}(x^*)\big) < \mu_+(A^+_{\eta h,4\rho}(x^*,\bar t))$$

and then

$$\begin{aligned}(1-\eta)^2\left(1-\frac{1}{2}\frac{1}{\mathfrak{q}^2}\right)\,\mu_+\big(B^+_{4\rho}(x^*)\big) & < \\ & < \mu_+\big(B_{4\rho}(x^*)\setminus B_\rho(x^*)\big) + \mu_-(I_+^{4\rho,4\rho\sigma}) + \frac{\gamma}{(4\rho\sigma)^2}\,\Lambda\big((B^+_{4\rho}(x^*)\cup I_+^{4\rho,4\rho\sigma})\times(t^*,s_2)\big) + \\ & \quad + (1-\eta)^2\mu_+\big(B_{4\rho}(x^*)\setminus B_{4\rho(1-\sigma)}(x^*)\big).\end{aligned}$$

Then taking, for instance, $\beta = \sigma^3$, letting σ and η go to zero we would find the contradiction (and here is needed $\mu_+\big(B_{4\rho}(x^*)\big) > \mu_+\big(B_\rho(x^*)\big) > 0$)

$$\left(1 - \frac{1}{2}\frac{1}{\mathfrak{q}^2}\right) \mu_+\big(B^+_{4\rho}(x^*)\big) \leqslant \mu_+\big(B_{4\rho}(x^*) \setminus B_\rho(x^*)\big)$$

$$\Downarrow$$

$$2\mu_+\big(B_\rho(x^*)\big) \leqslant \mu_+\big(B_\rho(x^*)\big) .$$

In a way analogous to (71) we can derive for every $s \in [s_1, t^*]$

$$\begin{aligned}
&(1-\eta)^2\mu_-\big(A^-_{\eta h,4\rho}(x^*, s)\big) \leqslant \\
&\quad \leqslant (1-\eta)^2\Big[\mu_-\big(A^-_{\eta h,4\rho(1-\sigma)}(x^*, s)\big) + \mu_-\big(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*)\big)\Big] \leqslant \\
&\quad \leqslant \mu_-\big(B_{4\rho}(x^*) \setminus B_\rho(x^*)\big) + \mu_+(I_-^{4\rho,4\rho\sigma}) + \frac{\gamma}{(4\rho\sigma)^2}\,\Lambda\big((B^-_{4\rho}(x^*) \cup I_-^{4\rho,4\rho\sigma}) \times (s_1, t^*)\big) + \\
&\qquad + (1-\eta)^2\mu_-\big(B_{4\rho}(x^*) \setminus B_{4\rho(1-\sigma)}(x^*)\big)
\end{aligned} \tag{72}$$

by which, again by contradiction, we prove point *iii*).

Point *i*) is quite immediate. Since $(u-h)_-(x, \bar t) \geqslant (1-\eta)h$ in $A^0_{\eta h,4\rho}(x^*, \bar t)$ we immediately get

$$\begin{aligned}
&(1-\eta)^2h^2\lambda_0\big(A^0_{\eta h,4\rho(1-\sigma)}(x^*, \bar t)\big) \leqslant \\
&\qquad \leqslant \int_{A^0_{\eta h,4\rho}(x^*,\bar t)} (u-h)^2_-(x, \bar t)\lambda_0(x)dx \leqslant \\
&\qquad \leqslant \int_{B^0_{4\rho}(x^*)} (u-h)^2_-(x, \bar t)\lambda_0(x)dx \leqslant h^2\,\lambda_0\big(B_{4\rho}(x^*) \setminus B_\rho(x^*)\big)
\end{aligned}$$

that is

$$(1-\eta)^2\lambda_0\big(A^0_{\eta h,4\rho(1-\sigma)}(x^*, \bar t)\big) \leqslant \lambda_0\big(B_{4\rho}(x^*) \setminus B_\rho(x^*)\big) \leqslant \left(1 - \frac{1}{\mathfrak{q}^2}\right)\lambda_0\big(B_{4\rho}(x^*)\big)$$

and then η is easily found.

Point *iv*) is obtained simply summing and rearranging the previous inequalities. □

Lemma 6.6. *Consider* $\beta \in (0, 16]$ *and* (x^*, t^*) *such that* $B_{5\rho}(x^*) \times [t^* - \beta\, h(x^*, 4\rho)\,\rho^2, t^* + \beta\, h(x^*, 4\rho)\,\rho^2] \subset \Omega \times (0, T)$*, consider* η *and* $\tilde\beta$ *to be the values determined in Lemma* 6.5*. Consider* κ *and* τ *the constants appearing in* (33)*. Consider* $u \geqslant 0$ *in* $DG(\Omega, T, \mu, \lambda, \gamma)$*,* $h > 0$*.*

i) *If* $\mu_+\big(B_{4\rho}(x^*)\big) > \mu_+\big(B_\rho(x^*)\big) > 0$ *and* $u(\cdot, t^*) \geqslant h$ *a.e. in* $B^+_\rho(x^*)$
then for every $\epsilon > 0$ *there exists* $\eta_1 \in (0, \eta)$*,* η_1 *depending only on* $\gamma_1, \gamma, \mathfrak{q}, \epsilon, \eta, \tilde\beta$ *such that*

$$\begin{aligned}
M_+\Big(\{u < \eta_1 h\} \cap \Big[B^+_{4\rho}(x^*) \times (t^*, t^* + \tilde\beta\,\rho^2 h(x^*, 4\rho))\Big]\Big) &\leqslant \\
\leqslant \epsilon\ |M|_\Lambda\Big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde\beta\,\rho^2 h(x^*, 4\rho))\Big)&, \\
\Lambda_+\Big(\{u < \eta_1 h\} \cap \Big[B^+_{4\rho}(x^*) \times (t^*, t^* + \tilde\beta\,\rho^2 h(x^*, 4\rho))\Big]\Big) &\leqslant \\
\leqslant \kappa\,\epsilon^\tau\ \Lambda\Big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde\beta\,\rho^2 h(x^*, 4\rho))\Big)&;
\end{aligned}$$

ii) if $\mu_-\big(B_{4\rho}(x^*)\big) > \mu_-\big(B_\rho(x^*)\big) > 0$ *and* $u(\cdot,t^*) \geqslant h$ *a.e. in* $B^-_\rho(x^*)$
then for every $\epsilon > 0$ *there exists* $\eta_1 \in (0,\eta)$*,* η_1 *depending only on* $\gamma_1, \gamma, \mathfrak{q}, \epsilon, \eta, \tilde{\beta}$ *such that*

$$\begin{aligned} M_-\Big(\{u < \eta_1 h\} \cap \Big[B^-_{4\rho}(x^*) \times (t^* - \tilde{\beta}\,\rho^2 h(x^*,4\rho), t^*)\Big]\Big) \leqslant \\ \leqslant \epsilon\ |M|_\Lambda \Big(B_{4\rho}(x^*) \times (t^* - \tilde{\beta}\,\rho^2 h(x^*,4\rho), t^*)\Big), \\ \Lambda_-\Big(\{u < \eta_1 h\} \cap \Big[B^-_{4\rho}(x^*) \times (t^* - \tilde{\beta}\,\rho^2 h(x^*,4\rho), t^*)\Big]\Big) \leqslant \\ \leqslant \kappa\,\epsilon^{\tau}\ \Lambda\Big(B_{4\rho}(x^*) \times (t^* - \tilde{\beta}\,\rho^2 h(x^*,4\rho), t^*)\Big); \end{aligned}$$

iii) consider $\beta > 0$ *such that* $B_{5\rho}(x^*) \times [t^* - \beta\, h(x^*,4\rho)\,\rho^2, t^* + \beta\, h(x^*,4\rho)\,\rho^2] \subset \Omega \times (0,T)$*: if* $\lambda_0\big(B_{4\rho}(x^*)\big) > \lambda_0\big(B_\rho(x^*)\big) > 0$ *and* $u \geqslant h$ *a.e. in* $\big(B^0_\rho(x^*) \times (t^* - \beta\,\rho^2 h(x^*,4\rho), t^* + \beta\,\rho^2 h(x^*,4\rho))\big)$ *then for every* $\epsilon > 0$ *there exists* $\eta_1 \in (0,\eta)$*,* η_1 *depending only on* $\gamma_1, \gamma, \mathfrak{q}, \epsilon, \eta, \beta$ *such that*

$$\begin{aligned} \Lambda_0\Big(\{u < \eta_1 h\} \cap \Big[B^0_{4\rho}(x^*) \times (t^* - \beta\,\rho^2 h(x^*,4\rho), t^* + \beta\,\rho^2 h(x^*,4\rho))\Big]\Big) \leqslant \\ \leqslant \epsilon\ \Lambda\Big(B_{4\rho}(x^*) \times (t^* - \beta\,\rho^2 h(x^*,4\rho), t^* + \beta\,\rho^2 h(x^*,4\rho)\Big); \end{aligned}$$

iv) if $B_{5\rho}(x^*) \subset \Omega_0$ *and* $u(\cdot,t) \geqslant h$ *a.e. in* $B_\rho(x^*)$ *then for every* $\epsilon > 0$ *there exists* $\eta_1 \in (0,\eta)$*,* η_1 *depending only on* $\gamma_1, \gamma, \mathfrak{q}, \epsilon, \eta$ *such that for almost every* $t \in [t^* - \beta\, h(x^*,4\rho)\,\rho^2, t^* + \beta\, h(x^*,4\rho)\,\rho^2]$

$$\lambda\Big(\{u < \eta_1 h\} \cap \big(B_{4\rho}(x^*) \times \{t\}\big)\Big) \leqslant \epsilon\ \lambda\Big(B_{4\rho}(x^*)\Big).$$

Proof - We first show point *i*). Consider $\tilde{\beta}$ and η to be the values determined in Lemma 6.5, point *i*). For simplicity, by f we will denote the quantity

$$f(x^*,4\rho) = h(x^*,4\rho)\,\rho^2\,.$$

Now we consider $m \in \mathbf{N}$, $\tau \in [t^*, t^* + \tilde{\beta}\, f(x^*,4\rho)]$ and $\sigma \in [t^* - \tilde{\beta}\, f(x^*,4\rho), t^*]$. First of all notice that for every t^*, for every $\tau \in [t^*, t^* + \tilde{\beta}\, f(x^*,4\rho)]$ and $\sigma \in [t^* - \tilde{\beta}\, f(x^*,4\rho), t^*]$ and every $t \in (t^* - \alpha\,\rho^2 h(x^*,4\rho), t^* + \alpha\,\rho^2 h(x^*,4\rho))$ we derive, using Lemma 6.5 and since for $m \in \mathbf{N}$ it holds $A^+_{\eta h 2^{-m},4\rho}(x^*,\tau) \subset A^+_{\eta h,4\rho}(x^*,\tau)$, $A^-_{\eta h 2^{-m},4\rho}(x^*,\sigma) \subset A^-_{\eta h,4\rho}(x^*,\sigma)$, $A^0_{\eta h 2^{-m},4\rho}(x^*,t^*) \subset A^0_{\eta h,4\rho}(x^*,t^*)$, that if $\mu_+\big(B_\rho(x^*)\big) > 0$, $\mu_-\big(B_\rho(x^*)\big) > 0$, $\lambda_0\big(B_\rho(x^*)\big) > 0$

$$\begin{aligned} &\frac{1}{2\mathfrak{q}^2}\,\mu_+\big(B^+_{4\rho}(x^*)\big) \leqslant \mu_+\big(B^+_{4\rho}(x^*) \setminus A^+_{\eta h,4\rho}(x^*,\tau)\big) \leqslant \mu_+\big(B^+_{4\rho}(x^*) \setminus A^+_{\eta h 2^{-m},4\rho}(x^*,\tau)\big), \\ &\frac{1}{2\mathfrak{q}^2}\,\mu_-\big(B^-_{4\rho}(x^*)\big) \leqslant \mu_-\big(B^-_{4\rho}(x^*) \setminus A^-_{\eta h,4\rho}(x^*,\sigma)\big) \leqslant \mu_-\big(B^-_{4\rho}(x^*) \setminus A^-_{\eta h 2^{-m},4\rho}(x^*,\sigma)\big), \\ &\frac{1}{2\mathfrak{q}^2}\,\lambda_0\big(B^0_{4\rho}(x^*)\big) \leqslant \lambda_0\big(B^0_{4\rho}(x^*) \setminus A^0_{\eta h,4\rho}(x^*,t)\big) \leqslant \lambda_0\big(B^0_{4\rho}(x^*) \setminus A^0_{\eta h 2^{-m},4\rho}(x^*,t)\big). \end{aligned} \tag{73}$$

Again for simplicity, we define (since x^* is fixed we omit it)

$$\begin{aligned} A^+_m(\tau) :=& A^+_{\eta h 2^{-m},4\rho}(x^*,\tau), \qquad & a^+_m :=& \int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)} \mu_+(A^+_m(\tau))\, d\tau, \\ A^-_m(\sigma) :=& A^-_{\eta h 2^{-m},4\rho}(x^*,\sigma), \qquad & a^-_m :=& \int_{t^*-\tilde{\beta}\, f(x^*,4\rho)}^{t^*} \mu_-(A^-_m(\sigma))\, d\sigma, \\ A^0_m(t) :=& A^0_{\eta h 2^{-m},4\rho}(x^*,t), \qquad & a^0_m :=& \int_{t^*-\alpha\, f(x^*,4\rho)}^{t^*+\alpha\, f(x^*,4\rho)} \lambda_0(A^0_m(t))dt, \\ A_m(t) :=& A_{\eta h 2^{-m},4\rho}(x^*,t), \qquad & B_{4\rho} :=& B_{4\rho}(x^*), \end{aligned}$$

$$d^>_m := \int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)} \lambda(A_m(t))dt, \quad d^<_m := \int_{t^*-\tilde{\beta}\, f(x^*,4\rho)}^{t^*} \lambda(A_m(t))dt,$$

$$d_m := \int_{t^*-\alpha\, f(x^*,4\rho)}^{t^*+\alpha\, f(x^*,4\rho)} \lambda(A_m(t))dt.$$

First we prove point i). Now we estimate from above and from below the quantity

$$\mu_+\big(B^+_{4\rho} \setminus A^+_{m-1}(\tau)\big) \int_{B_{4\rho}} \Big(u - \frac{\eta h}{2^m}\Big)_-(x,\tau)\mu_+(x)dx$$

Using also (73) we get that for every $\tau \in [t^*, t^* + \tilde{\beta}\, f(x^*,4\rho)]$

$$\begin{aligned} & \frac{1}{2\mathfrak{q}^2}\, \mu_+\big(B^+_{4\rho}\big) \frac{\eta h}{2^{m+1}} \mu_+\big(A^+_{m+1}(\tau)\big) \leqslant \\ & \leqslant \mu_+\big(B^+_{4\rho} \setminus A^+_{m-1}(\tau)\big) \frac{\eta h}{2^{m+1}} \mu_+\big(A^+_{m+1}(\tau)\big) \leqslant \\ & \leqslant \mu_+\big(B^+_{4\rho} \setminus A^+_{m-1}(\tau)\big) \int_{B_{4\rho}} \Big(u - \frac{\eta h}{2^m}\Big)_-(x,\tau)\mu_+(x)dx = \\ & \leqslant \mu_+\big(B^+_{4\rho} \setminus A^+_{m-1}(\tau)\big) \frac{\eta h}{2^m} \mu_+\big(A^+_m(\tau)\big) \end{aligned}$$

that is we get that for every $\tau \in [t^*, t^* + \tilde{\beta}\, f(x^*,4\rho)]$

$$\frac{1}{2\mathfrak{q}^2}\, \mu_+\big(B^+_{4\rho}\big) \frac{\eta h}{2^{m+1}} \mu_+\big(A^+_{m+1}(\tau)\big) \leqslant \mu_+\big(B^+_{4\rho} \setminus A^+_{m-1}(\tau)\big) \frac{\eta h}{2^m} \mu_+\big(A^+_m(\tau)\big) . \tag{74}$$

Now to estimate the right hand side of (74) we use Lemma 2.12 in the ball $B_{4\rho}(x^*)$ with $k = \eta h/2^m$, $l = \eta h/2^{m-1}$, $q = 1$, $p \in (1,2)$ arbitrary, $\omega = \lambda$, $\nu = |\mu|_\lambda$ ($\bar{\nu} = \mu_+$) we get for every $\tau \in (t^*, t^* + \tilde{\beta} f(x^*,4\rho))$

$$\begin{aligned} & \mu_+\big(B^+_{4\rho} \setminus A^+_{m-1}(\tau)\big) \frac{\eta h}{2^m} \mu_+\big(A^+_m(\tau)\big) \leqslant \\ & \leqslant 8\, \gamma_1\, \rho\, \frac{\mu_+(B^+_{4\rho})\, |\mu|_\lambda(B_{4\rho})}{(\lambda(B_{4\rho}))^{1/p}} \cdot \left(\int_{A_{m-1}(\tau)\setminus A_m(\tau)} |Du|^p(x,\tau)\, \lambda\, dx \right)^{1/p} \end{aligned}$$

By this last inequality and (74) and integrating in time between t^* and $t^* + \tilde{\beta}\, f(x^*, 4\rho)$ we get

$$\begin{aligned}
&\frac{1}{2\mathfrak{q}^2}\,\frac{\eta h}{2^{m+1}} a^+_{m+1} \leqslant \\
&\quad \leqslant 8\,\gamma_1\,\rho\,\frac{|\mu|_\lambda(B_{4\rho})}{(\lambda(B_{4\rho}))^{1/p}}\cdot\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\left(\int_{A_{m-1}(\tau)\setminus A_m(\tau)}|Du|^p(x,\tau)\,\lambda\,dx\right)^{1/p} d\tau \leqslant \\
&\quad \leqslant 8\,\gamma_1\,\rho\,\frac{|\mu|_\lambda(B_{4\rho})}{(\lambda(B_{4\rho}))^{1/p}}\cdot\left(\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{A_{m-1}(\tau)\setminus A_m(\tau)}|Du|^p(x,\tau)\,\lambda\,dxd\tau\right)^{1/p}\big(\tilde{\beta}\, f(x^*,4\rho)\big)^{\frac{p-1}{p}} \leqslant \\
&\quad \leqslant 8\,\gamma_1\,\rho\,\frac{|\mu|_\lambda(B_{4\rho})}{(\lambda(B_{4\rho}))^{1/p}}\big(\tilde{\beta}\, f(x^*,4\rho)\big)^{\frac{p-1}{p}}\cdot\left(\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\big[\lambda(A_{m-1}(\tau))-\lambda(A_m(\tau))\big]\,d\tau\right)^{\frac{2-p}{2p}}\cdot \qquad (75)\\
&\qquad\qquad \cdot\left(\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{B_{4\rho}}\left|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\right|^2(x,\tau)\,\lambda\,dxd\tau\right)^{1/2}.
\end{aligned}$$

Now we want to estimate the term in the right hand side involving the gradient of $u - \frac{\eta h}{2^m}$ and to do this we apply the energy estimates (41), (42), (43) in some suitable subsets of

$$B_{5\rho}(x^*)\times(t^* - \tilde{\beta}\, f(x^*,4\rho), t^* + \tilde{\beta}\, f(x^*,4\rho)).$$

to estimate the quantity $\displaystyle\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{B_{4\rho}}\left|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\right|^2(x,\tau)\,\lambda\,dxd\tau$. First we estimate, taking in (41) $t_0 = t^* - \tilde{\beta} f(x^*,4\rho)$, $s_2 = t^* + \tilde{\beta} f(x^*,4\rho)$, $R = \tilde{r} = 5\rho$, $r = 4\rho$, $\varepsilon = 0$, $\tilde{\theta} = 0$ and $\theta = \tilde{\beta}\,\frac{16}{25}\frac{h(x^*;4\rho)}{h(x^*;5\rho)}$ in (41),

$$\begin{aligned}
&\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{B^+_{4\rho}}\left|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\right|^2(x,\tau)\,\lambda\,dxd\tau \leqslant \\
&\quad\leqslant \gamma\Bigg[\int_{I^+_{5\rho,\rho}}\Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x, t^* - \tilde{\beta} f(x^*,4\rho))\mu_+(x)\,dx + \\
&\qquad + \sup_{t\in(t^*,t^*+\tilde{\beta}\, f(x^*,4\rho))}\int_{I^{5\rho,\rho}_+}\Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,t)\mu_-(x)\,dx + \\
&\qquad + \frac{1}{\rho^2}\iint_{Q^{+,\rho}_{5\rho;5\rho,0}}\Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,\tau)\left(\frac{\mu_+}{h(x^*,5\rho)}+\lambda\right)dxd\tau\Bigg] \leqslant \\
&\quad\leqslant \gamma\Bigg[\left(\frac{\eta h}{2^{m-1}}\right)^2\mu_+(I^+_{5\rho,\rho}) + \left(\frac{\eta h}{2^{m-1}}\right)^2\mu_-(I^{5\rho,\rho}_+) + \\
&\qquad + \left(\frac{\eta h}{2^{m-1}}\right)^2\frac{1}{\rho^2}\left(\frac{M_+}{h(x^*,5\rho)}+\Lambda\right)(Q^{+,\rho}_{5\rho;5\rho,0})\Bigg] \leqslant \\
&\quad\leqslant \gamma\left(\frac{\eta h}{2^{m-1}}\right)^2\frac{1}{\rho^2}\Big[\rho^2\,|\mu|(B_{5\rho}) + 2\,\lambda(B_{5\rho})\,2\,\tilde{\beta}\, f(x^*,4\rho)\Big].
\end{aligned}$$

Then taking in (42) $t_0 = t^* + 2\,\tilde{\beta} f(x^*, 4\rho)$, $s_1 = t^*$, $R = \tilde{r} = 5\rho$, $r = 4\rho$, $\varepsilon = 0$, $\tilde{\theta} = 0$ and $\theta = \tilde{\beta}\,\frac{16}{25}\frac{h(x^*_, 4\rho)}{h(x^*_, 5\rho)}$ in (41),

$$\begin{aligned}
&\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{B^-_{4\rho}} \Big|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\Big|^2(x,\tau)\,\lambda\,dxd\tau \leqslant \\
&\leqslant \gamma\Bigg[\int_{I^-_{5\rho,\rho}} \Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-\big(x, t^*+2\tilde{\beta} f^+(x^*,4\rho)\big)\mu_-(x)\,dx + \\
&\qquad + \sup_{t\in(t^*,\,t^*+\tilde{\beta}\, f(x^*,4\rho))}\int_{I_-^{5\rho,\rho}} \Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,t)\mu_+(x)\,dx + \\
&\qquad\quad + \frac{1}{\rho^2}\iint_{Q^{-,\rho}_{5\rho;5\rho,0}} \Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,\tau)\,\Big(\frac{\mu_-}{h(x^*,5\rho)}+\lambda\Big)\,dxd\tau\Bigg] \leqslant \\
&\leqslant \gamma\Bigg[\Big(\frac{\eta h}{2^{m-1}}\Big)^2\,\mu_-(I^-_{5\rho,\rho}) + \Big(\frac{\eta h}{2^{m-1}}\Big)^2\,\mu_+(I_-^{5\rho,\rho}) + \\
&\qquad\quad + \Big(\frac{\eta h}{2^{m-1}}\Big)^2\,\frac{1}{\rho^2}\,\Big(\frac{M_-}{h(x^*,5\rho)}+\Lambda\Big)\,(Q^{-,\rho}_{5\rho;5\rho,0})\Bigg] \leqslant \\
&\leqslant \gamma\,\Big(\frac{\eta h}{2^{m-1}}\Big)^2\,\frac{1}{\rho^2}\,\Big[\rho^2\,|\mu|(B_{5\rho}) + 4\,\tilde{\beta}\,\lambda(B_{5\rho})\,f(x^*,4\rho)\Big].
\end{aligned}$$

Finally taking in (43) $s_1 = t^*$, $s_2 = t^* + \tilde{\beta} f(x^*, 4\rho)$, $R = \tilde{r} = 5\rho$, $r = 4\rho$, $\varepsilon = 0$ in (41),

$$\begin{aligned}
&\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{B^0_{4\rho}} \Big|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\Big|^2(x,\tau)\,\lambda\,dxd\tau \leqslant \\
&\leqslant \gamma\Bigg[\sup_{t\in(t^*,\,t^*+\tilde{\beta}\, f(x^*,4\rho))}\int_{I_0^{5\rho,\rho}} \Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,t)\mu_-(x)\,dx + \\
&\qquad + \sup_{t\in(t^*,\,t^*+\tilde{\beta}\, f(x^*,4\rho))}\int_{I_0^{5\rho,\rho}} \Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,t)\mu_+(x)\,dx + \\
&\qquad\quad + \frac{1}{\rho^2}\int_{t^*}^{t^*+\tilde{\beta}\, f(x^*,4\rho)}\int_{(B^0_{4\rho})^\rho} \Big(u-\frac{\eta h}{2^{m-1}}\Big)^2_-(x,\tau)\,\lambda\,dxd\tau\Bigg] \leqslant \\
&\leqslant \gamma\Bigg[\Big(\frac{\eta h}{2^{m}}\Big)^2\,|\mu|(I_0^{5\rho,\rho}) + \Big(\frac{\eta h}{2^{m-1}}\Big)^2\,\frac{1}{\rho^2}\,\lambda\big((B^0_{4\rho})^\rho\big)\,\tilde{\beta}\,f(x^*,4\rho)\Bigg] \leqslant \\
&\leqslant \gamma\,\Big(\frac{\eta h}{2^{m-1}}\Big)^2\,\frac{1}{\rho^2}\,\Big[\rho^2\,|\mu|(B_{5\rho}) + \tilde{\beta}\,\lambda\big(B_{5\rho}\big)\,f(x^*,4\rho)\Big].
\end{aligned}$$

Summing up we get

$$
\begin{aligned}
(76)\qquad & \int_{t^*}^{t^*+\tilde{\beta} f(x^*,4\rho)} \int_{B_{4\rho}} \Big| D\Big(u - \frac{\eta h}{2^{m-1}} \Big)_- \Big|^2 (x,\tau) \, \lambda \, dx d\tau \leqslant \\
& \leqslant \gamma \Big(\frac{\eta h}{2^{m-1}} \Big)^2 \frac{1}{\rho^2} \Big[3\rho^2 \, |\mu|(B_{5\rho}) + 9 \, \tilde{\beta} \, f(x^*,4\rho) \, \lambda\big(B_{5\rho}\big) \Big]
\end{aligned}
$$

and so we can conclude, from (75), that

$$
\begin{aligned}
a^+_{m+1} \leqslant 64 \, \gamma_1 \gamma^{1/2} \mathfrak{q}^2 \, \frac{|\mu|_\lambda(B_{4\rho})}{(\lambda(B_{4\rho}))^{1/p}} \big(\tilde{\beta} \, f(x^*,4\rho) \big)^{\frac{p-1}{p}} \cdot \big(d^>_{m-1} - d^>_m \big)^{\frac{2-p}{2p}} \cdot \\
\cdot \Big[3\rho^2 \, |\mu|(B_{5\rho}) + 9 \, \tilde{\beta} \, f^+(x^*,4\rho) \, \lambda\big(B_{5\rho}\big) \Big]^{1/2} .
\end{aligned}
$$

Taking the power $\frac{2p}{2-p}$ and summing between $m = 1$ and $m = m^*$ we have

$$
\begin{aligned}
\sum_{m=1}^{m^*} (a^+_{m+1})^{\frac{2p}{2-p}} \leqslant (64 \, \gamma_1 \gamma^{1/2} \mathfrak{q}^2)^{\frac{2p}{2-p}} \, \frac{(|\mu|_\lambda(B_{4\rho}))^{\frac{2p}{2-p}}}{(\lambda(B_{4\rho}))^{\frac{2}{2-p}}} \big(\tilde{\beta} \, f(x^*,4\rho) \big)^{\frac{2(p-1)}{2-p}} \cdot \\
\cdot \Big[3\rho^2 \, |\mu|(B_{5\rho}) + 9 \, \tilde{\beta} \, f(x^*,4\rho) \, \lambda\big(B_{5\rho}\big) \Big]^{\frac{p}{2-p}} \big(d^>_0 - d^>_{m^*} \big) .
\end{aligned}
$$

Since the sequences $(a^+_m)_{m\in\mathbf{N}}$ and $(d^>_m)_{m\in\mathbf{N}}$ are decreasing we can estimate $\sum_{m=1}^{m^*} (a^+_{m+1})^{\frac{2p}{2-p}}$ from below by $m^* (a^+_{m^*+1})^{\frac{2p}{2-p}}$ and $d^>_0 - d^>_{m^*}$ from above by $d^>_0$ and $d^>_0$ by $\tilde{\beta} \, f(x^*,4\rho) \, \lambda(B_{4\rho})$ and get

$$
\begin{aligned}
(a^+_{m^*+1})^{\frac{2p}{2-p}} & \leqslant \frac{1}{m^*} (64 \, \gamma_1 \gamma^{1/2} \mathfrak{q}^2)^{\frac{2p}{2-p}} \, \frac{(|\mu|_\lambda(B_{4\rho}))^{\frac{2p}{2-p}}}{(\lambda(B_{4\rho}))^{\frac{2}{2-p}}} \big(\tilde{\beta} \, f(x^*,4\rho) \big)^{\frac{2(p-1)}{2-p}} \cdot \\
& \cdot \Big[3\rho^2 \, |\mu|(B_{5\rho}) + 9 \, \tilde{\beta} \, f(x^*,4\rho) \, \lambda\big(B_{5\rho}\big) \Big]^{\frac{p}{2-p}} \tilde{\beta} \, f(x^*,4\rho) \, \lambda(B_{4\rho}) \leqslant \\
& \leqslant \frac{C^{\frac{2p}{2-p}}}{m^*} \big(\tilde{\beta} f(x^*,4\rho) \big)^{\frac{2p}{2-p}} \big(|\mu|_\lambda(B_{4\rho}) \big)^{\frac{2p}{2-p}} = \\
& = \frac{C^{\frac{2p}{2-p}}}{m^*} \big(|M|_\Lambda \big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta} f(x^*,4\rho)) \big) \big)^{\frac{2p}{2-p}} ,
\end{aligned}
$$

where $C = 64 \, \gamma_1 \gamma^{1/2} \mathfrak{q}^{5/2} (3 + 9 \, \tilde{\beta})^{1/2} \tilde{\beta}^{-1/2}$, by which finally

$$
a^+_{m^*+1} \leqslant C \left(\frac{1}{m^*} \right)^{\frac{2-p}{2p}} |M|_\Lambda \big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta} f(x^*,4\rho)) \big) .
$$

Then for every $\epsilon > 0$ one can find m^* such that $C/m^{*\frac{2-p}{2p}} \leqslant \epsilon$. Then we consider

$$
m^* \geqslant \left(\frac{64 \, \gamma_1 \gamma^{1/2} \mathfrak{q}^{5/2} (3 + 9 \, \tilde{\beta})^{1/2}}{\tilde{\beta}^{1/2} \epsilon} \right)^{\frac{2p}{2-p}} \qquad \text{and} \qquad \eta_1 = \frac{\eta}{2^{m^*}}
$$

which depends on $\gamma_1, \gamma, \mathfrak{q}, \epsilon, \tilde{\beta}, \eta$.
Now by (33) we immediately get that

$$\frac{\Lambda_+\Big(\{u<\eta_1 h\}\cap\Big[B^+_{4\rho}(x^*)\times(t^*,t^*+\tilde{\beta}\,\rho^2 h(x^*,4\rho))\Big]\Big)}{\Lambda\Big(B_{4\rho}(x^*)\times(t^*,t^*+\tilde{\beta}\,\rho^2 h(x^*,4\rho))\Big)}\leqslant$$

$$\leqslant \kappa\left(\frac{M_+\Big(\{u<\eta_1 h\}\cap\Big[B^+_{4\rho}(x^*)\times(t^*,t^*+\tilde{\beta}\,\rho^2 h(x^*,4\rho))\Big]\Big)}{|M|_\Lambda\Big(B_{4\rho}(x^*)\times(t^*,t^*+\tilde{\beta}\,\rho^2 h(x^*,4\rho))\Big)}\right)^{\tau}\leqslant \kappa\,\epsilon^{\tau}\,.$$

In a complete analogous way one can prove point ii).

Point iii): the case where $\mu\equiv 0$, as usual, is a bit different. Let us see a sketch of the proof. Integrating between $t^*-\beta\,\rho^2 h(x^*,4\rho)$ and $t^*+\beta\,\rho^2 h(x^*,4\rho)$ and using Lemma 2.12 similarly as before but with $\nu=\lambda$ and $\bar{\nu}=\lambda_0|_{B_r(x^*)}$, we get

$$\frac{1}{2\mathfrak{q}^2}\,\frac{\eta h}{2^{m+1}}a^0_{m+1}\leqslant 8\,\gamma_1\,\rho\,\left(\lambda(B_{4\rho})\right)^{\frac{p-1}{p}}\cdot\left(\int_{t^*-\beta\,f(x^*,4\rho)}^{t^*+\beta\,f(x^*,4\rho)}\big[\lambda(A_{m-1}(t))-\lambda(A_m(t))\big]\,dt\right)^{\frac{2-p}{2p}}\cdot$$

$$\cdot\left(2\beta\,f(x^*,4\rho)\right)^{\frac{p-1}{p}}\left(\int_{t^*-\beta\,f(x^*,4\rho)}^{t^*+\beta\,f(x^*,4\rho)}\int_{B_{4\rho}}\left|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\right|^2(x,t)\,\lambda\,dxdt\right)^{1/2}.$$

Now estimating the part involving the gradient of $u-\frac{\eta h}{2^m}$ similarly as (76) we get

$$\frac{1}{2\mathfrak{q}^2}\,\frac{\eta h}{2^{m+1}}a^0_{m+1}\leqslant 8\,\gamma_1\,\gamma^{\frac12}\,\left(\lambda(B_{4\rho})\right)^{\frac{p-1}{p}}\cdot\left(\int_{t^*-\beta\,f(x^*,4\rho)}^{t^*+\beta\,f(x^*,4\rho)}\big[\lambda(A_{m-1}(t))-\lambda(A_m(t))\big]\,dt\right)^{\frac{2-p}{2p}}\cdot$$

$$\cdot\;\left(2\beta\,f(x^*,4\rho)\right)^{\frac{p-1}{p}}\left(\frac{\eta h}{2^{m-1}}\right)\left[3\rho^2\,|\mu|(B_{5\rho})+18\,\beta\,f(x^*,4\rho)\,\lambda\big(B_{5\rho}\big)\right]^{\frac12}$$

and proceeding as before we reach

$$a^0_{m^*+1}\leqslant C'\left(\frac{1}{m^*}\right)^{\frac{2-p}{2p}}\Lambda\big(B_{4\rho}(x^*)\times(t^*-\beta f(x^*,4\rho),t^*+\beta f(x^*,4\rho))\big)$$

with $C'=64\,\gamma_1\gamma^{1/2}\mathfrak{q}^{5/2}(3+18\,\beta)^{1/2}(2\beta)^{-1/2}$. The conclusion is as before.

Finally let us see point iv). If $B_{4\rho}(x^*)\subset\Omega_0$ we have

$$\frac{1}{2\mathfrak{q}^2}\,\frac{\eta h}{2^{m+1}}\lambda\big(A_{m+1}(t)\big)=\frac{1}{2\mathfrak{q}^2}\,\frac{\eta h}{2^{m+1}}\lambda\big(A_{m+1}(t)\big)\leqslant$$

$$\leqslant 8\,\gamma_1\,\rho\,\left(\lambda(B_{4\rho})\right)^{\frac{p-1}{p}}\cdot\left(\int_{A_{m-1}(t)\setminus A_m(t)}|Du|^p(x,t)\,\lambda\,dx\right)^{1/p}d\tau\leqslant$$

$$\leqslant\left(\lambda(A_{m-1}(t))-\lambda(A_m(t))\right)^{\frac{2-p}{2}}\left(\int_{B_{4\rho}}\left|D\Big(u-\frac{\eta h}{2^{m-1}}\Big)_-\right|^2(x,t)\,\lambda(x)\,dx\right)^{1/2}.$$

Since $B_{5\rho}(x^*) \subset \Omega_0$ taking $\tilde{r} = 5\rho$, $r = 4\rho$ and $\varepsilon = 0$ in (46), we get for almost every t that

$$\int_{B_{4\rho}} \Big| D\Big(u - \frac{\eta h}{2^{m-1}}\Big)_- \Big|^2 (x,t)\, \lambda(x)\, dx \leqslant$$

$$\leqslant \gamma \frac{1}{\rho^2} \int_{B_{5\rho}} \Big(u - \frac{\eta h}{2^{m-1}}\Big)^2_- (x,t)\, \lambda(x)\, dx \leqslant \gamma \left(\frac{\eta h}{2^{m-1}}\right)^2 \frac{1}{\rho^2}\, \lambda(B_{5\rho})$$

and then $\lambda\big(A_{m+1}(t)\big) \leqslant 64\, \mathfrak{q}^2\, \gamma_1\, \gamma^{1/2} \left(\lambda(A_{m-1}(t)) - \lambda(A_m(t))\right)^{\frac{2-p}{2}} \left(\lambda(B_{5\rho})\right)^{1/2}$. By that we can conclude similarly as above. □

Now we state a result known as *expansion of positivity.* It will be a fundamental step to prove the Harnack inequality.

Lemma 6.7. *Consider (x^*, t^*) such that $B_{5\rho}(x^*) \times [t^* - 16\, h(x^*, 4\rho)\, \rho^2, t^* + 16\, h(x^*, 4\rho)\, \rho^2] \subset \Omega \times (0, T)$.*
Consider the value $\tilde{\beta}$ determined in Lemma 6.5 and used in in Lemma 6.6. Then for every $\hat{\theta} \in (0,1)$ there is $\lambda > 0$ depending only on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \tilde{\beta}, \hat{\theta}$ such that for every $h > 0$ and $u \geqslant 0$ in $DG(\Omega, T, \mu, \lambda, \gamma)$ points i) and ii) are true:
i) if $\mu_+(B_\rho(x^*)) > 0$ and

$$u(\cdot, t^*) \geqslant h \qquad a.e.\ in\ B_\rho^+(x^*)$$

then

$$u \geqslant \lambda h \qquad a.e.\ in\ B_{2\rho}^+(x^*) \times \big(t^* + \hat{\theta}\, \tilde{\beta}\, h(x^*, 4\rho)\rho^2, t^* + \tilde{\beta}\, h(x^*, 4\rho)\rho^2\big);$$

ii) if $\mu_-(B_\rho^-(x^*)) > 0$ and

$$u(\cdot, t^*) \geqslant h \qquad a.e.\ in\ B_\rho^-(x^*)$$

then

$$u \geqslant \lambda h \qquad a.e.\ in\ B_{2\rho}^-(x^*) \times \big(t^* + \hat{\theta}\, \tilde{\beta}\, h(x^*, 4\rho)\rho^2, t^* + \tilde{\beta}\, h(x^*, 4\rho)\rho^2\big).$$

Moreover for every $\beta > 0$ for which $B_{5\rho}(x^*) \times [t^* - \beta\, h(x^*, 4\rho)\, \rho^2, t^* + \beta\, h(x^*, 4\rho)\, \rho^2] \subset \Omega \times (0, T)$ there is $\lambda > 0$ depending only on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \beta$ such that for every $h > 0$ and $u \geqslant 0$ in $DG(\Omega, T, \mu, \lambda, \gamma)$ point iii) is true:
iii) if $\lambda_0(B_\rho(x^*)) > 0$ and

$$u \geqslant h \qquad a.e.\ in\ B_\rho^0(x^*) \times \big(t^* - \beta\, h(x^*, 4\rho)\rho^2, t^* + \beta\, h(x^*, 4\rho)\rho^2\big)$$

then

$$u \geqslant \lambda h \qquad a.e.\ in\ B_{2\rho}^0(x^*) \times \big(t^* - \beta\, h(x^*, 4\rho)\rho^2, t^* + \beta\, h(x^*, 4\rho)\rho^2\big).$$

If $B_{5\rho}(x^*) \subset \Omega_0$ there is $\lambda > 0$ depending only on $\gamma_1, \gamma, \mathfrak{q}, \kappa$ such that for every $h > 0$ and $u \geqslant 0$ in $DG(\Omega, T, \mu, \lambda, \gamma)$ point iv) is true:
iv) for almost every $t \in (0, T)$ if

$$u(\cdot, t) \geqslant h \qquad a.e.\ in\ B_\rho(x^*)$$

then

$$u(\cdot, t) \geqslant \lambda h \qquad a.e.\ in\ B_{2\rho}(x^*).$$

Proof - The proof is a consequence of Proposition 6.3 and Lemma 6.6. We start from point i): in Proposition 6.3 we consider $\underline{m} = 0$, $R = 4\rho$, $r = 2\rho$, $\beta^\diamond = \tilde{\beta}$ (the value determined in Lemma 6.5 and used in Lemma 6.6 and belonging to $(0, 16]$), $\theta^\diamond$ and $a \in (0,1)$ arbitrary; from Proposition 6.3 we derive the existence of $\underline{\nu}^\diamond \in (0,1)$ such that if, for $c > 0$ an arbitrary constant, the following holds

$$\frac{M_+\Big(\big\{u < c\big\} \cap \big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\rho^2 h(x^*, 4\rho))\big)\Big)}{|M|_\Lambda\Big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\, \rho^2 h(x^*, 4\rho))\Big)} + \\ + \frac{\Lambda_+\Big(\big\{u < c\big\} \cap \big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\rho^2 h(x^*, 4\rho))\big)\Big)}{\Lambda\Big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\, \rho^2 h(x^*, 4\rho))\Big)} \leqslant \underline{\nu}^\diamond$$

then

$$u \geqslant a\, c \qquad \text{in } B^+_{2\rho}(x^*) \times \Big(t^* + \theta^\diamond \tilde{\beta}\, h(x^*, 4\rho)\rho^2, t^* + \tilde{\beta}\, h(x^*, 4\rho)\rho^2\Big)\ .$$

Now we use Lemma 6.6: consider η the value determined in Lemma 6.5 and used in in Lemma 6.6, take $\beta = 16$ and ϵ such that $\epsilon + \kappa\, \epsilon^\tau = \underline{\nu}^\diamond$ and conclude that there is η_1 (depending on $\gamma_1, \gamma, \mathfrak{q}, \tilde{\beta}, \eta, \underline{\nu}^\diamond$ and then on $\gamma_1, \gamma, \mathfrak{q}, \tilde{\beta}, \eta, \kappa, a, \theta^\diamond$, but η depends only on γ and $\mathfrak{q}$) such that

$$\frac{M_+\Big(\big\{u < \eta_1 h\big\} \cap \big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\rho^2 h(x^*, 4\rho))\big)\Big)}{|M|_\Lambda\Big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\, \rho^2 h(x^*, 4\rho))\Big)} + \\ + \frac{\Lambda_+\Big(\big\{u < \eta_1 h\big\} \cap \big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\rho^2 h(x^*, 4\rho))\big)\Big)}{\Lambda\Big(B_{4\rho}(x^*) \times (t^*, t^* + \tilde{\beta}\, \rho^2 h(x^*, 4\rho))\Big)} \leqslant \underline{\nu}^\diamond\ .$$

Then

$$u \geqslant a\, \eta_1 h \qquad \text{in } B^+_{2\rho}(x^*) \times \Big(t^* + \theta^\diamond \tilde{\beta}\, h(x^*, 4\rho)\rho^2, t^* + \tilde{\beta}\, h(x^*, 4\rho)\rho^2\Big)\ .$$

Taking $\hat{\theta} = \theta^\diamond$, $a = 1/2$ for simplicity and $\lambda = \eta_1/2$ we conclude the proof of point i). In the same way one can prove point point ii).

Let us see point point iii). In Proposition 6.3 we consider again $\underline{m} = 0$, $R = 4\rho$, $r = 2\rho$, $\beta^\star = \beta\, h(x^*, 4\rho)/8$ and $a \in (0,1)$ arbitrary. We derive the existence of $\underline{\nu}^\star \in (0,1)$ such that if, for $c > 0$ an arbitrary constant, the following holds

$$\Lambda_0\bigg(\big\{u < c\big\} \cap \Big(B_{4\rho}(x^*) \times \big(t^* - \beta\, h(x^*, 4\rho)\, \rho^2, t^* + \beta\, h(x^*, 4\rho)\, \rho^2\big)\Big)\bigg) \leqslant \\ \leqslant \underline{\nu}^\star\, \Lambda\Big(B_{4\rho}(x^*) \times \big(t^* - \beta\, h(x^*, 4\rho)\, \rho^2, t^* + \beta\, h(x^*, 4\rho)\, \rho^2\big)\Big)\,,$$

then

$$u \geqslant a\, c \qquad \text{in } B^0_{2\rho}(x^*) \times \big(t^* - \beta\, h(x^*, 4\rho)\, \rho^2, t^* + \beta\, h(x^*, 4\rho)\, \rho^2\big)\,.$$

Now in Lemma 6.6 take $\epsilon = \underline{\nu}^\star$ and conclude that there is η_1 (depending on $\gamma_1, \gamma, \mathfrak{q}, \kappa, a, \beta$) such that

$$u \geqslant a\, \eta_1 h \qquad \text{in } B^0_{2\rho}(x^*) \times \big(t^* - \beta\, h(x^*, 4\rho)\, \rho^2, t^* + \beta\, h(x^*, 4\rho)\, \rho^2\big)\,.$$

Taking, e.g, $a = 1/2$ we conclude.

To prove point *iv*) we consider $\underline{m}$, R, r and $a \in (0,1)$ as above and use point *iv*) of Proposition 6.3. Then we get the existence of $\underline{\nu} \in (0,1)$ such that, for $c > 0$, if

$$\lambda\big(\{x \in B_{4\rho}(x^\star) \,|\, u(x,t) < c\}\big) \leqslant \underline{\nu}\; \lambda(B_{4\rho}(x^\star))$$

then $u(x,t) \geqslant a\,c$ for a.e. $x \in B_{2\rho}(x^\star)$. Using Lemma 6.6 we conclude as above. □

7. The Harnack type inequality

The following theorems (Theorem 7.1 and Theorem 7.2) are the main results of the paper.

Theorem 7.1. *Assume* $u \in DG(\Omega, T, \mu, \lambda, \gamma)$, $u \geqslant 0$, $(x_o, t_o) \in \Omega \times (0,T)$ *and fix* $\rho > 0$.

i) *Suppose* $x_o \in \Omega_+ \cup I$. *For every* $\vartheta_+ \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - h(x_o,\rho)\rho^2, t_o + 16\,h(x_o,4\rho)\rho^2 + \vartheta_+ h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_+ > 0$ *depending (only) on* $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{k}, \tau, K_1, K_2, K_3, q, \varsigma, \vartheta_+$ *such that*

$$u(x_o,t_o) \leqslant c_+ \inf_{B^+_\rho(x_o)} u(x, t_o + \vartheta_+\,\rho^2 h(x_o,\rho)).$$

ii) *Suppose* $x_o \in \Omega_- \cup I$. *For every* $\vartheta_- \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - 16\,h(x_o,4\rho)\rho^2 + \vartheta_- h(x_o,\rho)\rho^2, t_o + h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_- > 0$ *depending (only) on* $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{k}, \tau, K_1, K_2, K_3, q, \varsigma, \vartheta_-$ *such that*

$$u(x_o,t_o) \leqslant c_- \inf_{B^-_\rho(x_o)} u(x, t_o - \vartheta_-\,\rho^2 h(x_o,\rho)).$$

iii) *Suppose* $x_o \in \Omega_0 \cup I$. *Suppose* $B_{5\rho}(x_o) \times [t_o - 16\,h(x_o,4\rho)\rho^2, t_o + 16\,h(x_o,4\rho)\rho^2] \subset \Omega \times (0,T)$. *For every* s_1, s_2 *for which* $s_2 - t_o = t_o - s_1 \leqslant 16\,h(x_o,4\rho)\rho^2$, *suppose* $s_2 - t_o = t_o - s_1 = \omega\,h(x_o,4\rho)\rho^2$ *for* $\omega \in (0,16]$, *there is* c_0 *depending (only) on* $K_1, K_2, K_3, q, \varsigma, \kappa, \gamma_1, \gamma, \omega, h(x_o,4\rho), \mathfrak{q}$ *such that*

$$\sup_{B^+_\rho(x_o)\times[s_1,s_2]} u \leqslant c_0 \inf_{B^+_\rho(x_o)\times[s_1,s_2]} u.$$

iv) *Suppose* $B_{5\rho}(x_o) \subset \Omega_0$. *Then there is* c *depending (only) on* $K_1, K_2, K_3, q, \varsigma, \kappa, \gamma_1, \gamma, \mathfrak{q}$ *such that for almost every* $t \in (0,T)$

$$\sup_{B_\rho(x_o)} u(\cdot,t) \leqslant c \inf_{B_\rho(x_o)} u(\cdot,t).$$

Proof - We start by proving the first of the three inequalities under the assumption that $B^+_\rho(x_o) \neq \emptyset$. For some $r_1, r_2 > 0$ and $(\bar{x},\bar{t}) \in B_{5\rho}(x_o) \times [t_o - h(x_o,\rho)\rho^2, t_o + 16\,h(x_o,4\rho)\rho^2 + \vartheta_+ h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ we define the sets

$$Q^{+,<}_{r_1,h(\bar{y},r_2)}(\bar{x},\bar{t}) := \Big(B^+_{r_1}(\bar{x}) \times [\bar{t} - h(\bar{y},r_2)r_1^2, \bar{t}]\Big), \quad Q^{+,>}_{r,h(\bar{y},r_2)}(\bar{x},\bar{t}) := \Big(B^+_r(\bar{x}) \times [\bar{t}, \bar{t} + h(\bar{y},r_2)r_1^2]\Big),$$
$$Q^{<}_{r_1,h(\bar{y},r_2)}(\bar{x},\bar{t}) := \Big(B_{r_1}(\bar{x}) \times [\bar{t} - h(\bar{y},r_2)r_1^2, \bar{t}]\Big), \quad Q^{>}_{r,h(\bar{y},r_2)}(\bar{x},\bar{t}) := \Big(B_r(\bar{x}) \times [\bar{t}, \bar{t} + h(\bar{y},r_2)r_1^2]\Big).$$

We may write $u(x_o,t_o) = b\,\rho^{-\xi}$ for some $b, \xi > 0$ to be fixed later. Define the functions

$$\mathcal{M}(r) = \sup_{Q^{+,<}_{r,h(x_o,\rho)}(x_o,t_o)} u, \qquad \mathcal{N}(r) = b(\rho - r)^{-\xi}, \qquad r \in [0,\rho).$$

Let us denote by $r_o \in [0, \rho)$ the largest solution of $\mathcal{M}(r) = \mathcal{N}(r)$. Define

$$N := \mathcal{N}(r_o) = b(\rho - r_o)^{-\xi}\,.$$

We can find $(y_o, \tau_o) \in Q^{+,<}_{r_o, h(x_o,\rho)}(x_o, t_o)$ such that

$$\frac{3N}{4} < \sup_{Q^{+,<}_{\frac{\rho_o}{4}, h(y_o,\rho_o)}(y_o,\tau_o)} u \leqslant N \tag{77}$$

where $\rho_o \in (0, (\rho - r_o)/2]$. If $\rho_o \leqslant (\rho - r_o)/2$ then $B^+_{\rho_o}(y_o) \subset B_{\frac{\rho+r_o}{2}}(x_o)$. We want the value of ρ_o to be be chosen in such a way that

$$Q^{+,<}_{\rho_o, h(y_o,\rho_o)}(y_o, \tau_o) \subset Q^{+,<}_{\frac{\rho+r_o}{2}, h(x_o,\rho)}(x_o, t_o)$$

and the request $\rho_o \leqslant (\rho - r_o)/2$ may be not sufficient. We also need $\tau_o - h(y_o, \rho_o)\rho_o^2 \geqslant t_o - h(x_o, \rho)(\rho + r_o)^2/4$ and this is guaranteed if

$$h(y_o, \rho_o)\rho_o^2 \leqslant h(x_o, \rho)\left[\frac{(\rho + r_o)^2}{4} - r_o^2\right], \tag{78}$$

which in turn is true, since $r_o^2 < \rho\, r_o$, if

$$h(y_o, \rho_o)\rho_o^2 \leqslant h(x_o, \rho)\,\frac{(\rho - r_o)^2}{4}.$$

so we will choose ρ_o satisfying these two requests. Notice that this last request can be satisfied writing $h(y_o, \rho_o)\,\rho_o^2 = h(y_o, \rho_o)\rho_o^{2\alpha}\rho_o^{2(1-\alpha)}$ because, thanks to Remark 2.7, point $\mathcal{C}$, and (H.2)$'$ we have

$$\begin{aligned} h(y_o, \rho_o)\rho_o^{2\alpha} &\leqslant \tilde{K}_2^2\, h(y_o, 2\rho)(2\rho)^{2\alpha} \leqslant \\ &\leqslant \tilde{K}_2^2\,\frac{|\mu|_\lambda(B_{4\rho}(x_o))}{\lambda(B_{2\rho}(y_o))}(2\rho)^{2\alpha} \leqslant \\ &\leqslant 4^\alpha \tilde{K}_2^2\,\mathfrak{q}^2\, h(x_o, \rho)\rho^{2\alpha} \end{aligned}$$

and then we have

$$h(y_o, \rho_o)\rho_o^2 \leqslant 4^\alpha \tilde{K}_2^2\,\mathfrak{q}^2\, h(x_o, \rho)\rho^{2\alpha}\rho_o^{2(1-\alpha)}.$$

Then (78) holds if in particular

$$4^\alpha \tilde{K}_2^2\,\mathfrak{q}^2\, h(x_o, \rho)\rho^{2\alpha}\rho_o^{2(1-\alpha)} \leqslant h(x_o, \rho)\,\frac{(\rho - r_o)^2}{4}$$

that is

$$\rho_o^{1-\alpha} \leqslant \frac{1}{2^\alpha \tilde{K}_2\,\mathfrak{q}}\,\frac{1}{\rho^\alpha}\,\frac{\rho - r_o}{2} \tag{79}$$

and it is always possible to choose ρ_o small enough such that (79) is satisfied. Therefore ρ_o will be chosen satisfying

$$\rho_o = \min\left\{\frac{\rho - r_o}{2}, \left[\frac{1}{2^\alpha \tilde{K}_2\,\mathfrak{q}}\,\frac{1}{\rho^\alpha}\,\frac{\rho - r_o}{2}\right]^{\frac{1}{1-\alpha}}\right\}. \tag{80}$$

By this choice of ρ_o and by the choice of r_o we have

$$\sup_{Q^{+,<}_{\rho_o,h(y_o,\rho_o)}(y_o,\tau_o)} u \leqslant \sup_{Q^{+,<}_{\frac{\rho+r_o}{2},h(x_o,\rho)}(x_o,t_o)} u < \mathcal{N}\left(\frac{\rho+r_o}{2}\right) = 2^{\xi} N. \tag{81}$$

We now proceed dividing the proof in six steps.

Step 1 - In this step we want to show that there is $\overline{\nu} \in (0,1)$, depending only on $\kappa, \gamma_1, \gamma, \xi, \mathfrak{q}$, such that

$$\begin{aligned} &\frac{M_+\left(\{u > \frac{N}{2}\} \cap Q^{+,<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)}{|M|_\Lambda\left(Q^{<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)} > \overline{\nu}\,, \\ &\frac{\Lambda_+\left(\{u > \frac{N}{2}\} \cap Q^{+,<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)}{\Lambda\left(Q^{<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)} > \overline{\nu} \end{aligned} \tag{82}$$

and that

$$\iint_{Q^{+,<}_{\frac{\rho_o}{2},h(y_o,\rho_o)}(y_o,\tau_o)} |Du|^2 \lambda \, dxdt \leqslant 9\,\gamma\,(2^{\xi}N)^2\, h(y_o,\rho_o)\, \lambda\big(B_{\rho_o}(y_o)\big)\,. \tag{83}$$

To prove (82) first we show that there is $\nu \in (0,1)$ such that

$$\frac{M_+\left(\{u > \frac{N}{2}\} \cap Q^{+,<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)}{|M|_\Lambda\left(Q^{<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)} + \frac{\Lambda_+\left(\{u > \frac{N}{2}\} \cap Q^{+,<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)}{\Lambda\left(Q^{<}_{\rho_o/2,h(y_o,\rho_o)}(y_o,\tau_o)\right)} > \nu\,. \tag{84}$$

Argue by contradiction and suppose that (84) is false. Since

$$\begin{aligned} Q^{+,<}_{\frac{\rho_o}{2},h(y_o,\rho_o)}(y_o,\tau_o) &= \left(B^+_{\frac{\rho_o}{2}}(y_o) \times \left[\tau_o - h(y_o,\tfrac{\rho_o}{2})\tfrac{h(y_o,\rho_o)}{h(y_o,\rho_o/2)}\tfrac{\rho_o^2}{4}, \tau_o\right]\right)\,, \\ Q^{+,<}_{\frac{\rho_o}{4},h(y_o,\rho_o)}(y_o,\tau_o) &= \left(B^+_{\frac{\rho_o}{4}}(y_o) \times \left[\tau_o - h(y_o,\tfrac{\rho_o}{2})\tfrac{h(y_o,\rho_o)}{h(y_o,\rho_o/2)}\tfrac{\rho_o^2}{16}, \tau_o\right]\right)\,, \end{aligned}$$

setting in Proposition 6.1

$$\overline{m} = \omega = 2^{\xi}N, \quad R = \frac{\rho_o}{2}, \quad \rho = \frac{\rho_o}{4}, \quad \sigma = 1 - 2^{-\xi-1}, \quad a = \sigma^{-1}\left(1 - \frac{3}{2^{\xi+2}}\right),$$

$$x^{\diamond} = y_o\,, \qquad t^{\diamond} = \tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{4}\,, \qquad \beta^{\diamond} = \frac{h(y_o,\rho_o)}{h(y_o,\rho_o/2)}\,, \qquad \theta^{\diamond} = \frac{3}{4}\,,$$

we obtain from Proposition 6.1 that

$$u \leqslant \frac{3N}{4} \quad \text{in } Q^{+,<}_{\frac{\rho_o}{4},h(y_o,\rho_o)}(y_o,\tau_o)$$

which contradicts (77). Notice that $\beta^{\diamond} \in [\mathfrak{q}^{-1}, \mathfrak{q}]$. Now by (84) we derive that at least one of the two addends in (84) is greater or equal to $\nu/2$. Now we get (82) by (33) taking

$$\overline{\nu} = \frac{1}{\kappa}\left(\frac{\nu}{2}\right)^{\frac{1}{\alpha}}\,.$$

To prove (83) we use (41). In (41) we choose $x_0 = y_o$, $t_0 = \tau_o - h(y_o,\rho_o)\rho_o^2$, $R = \rho_o$, $\tilde r = \rho_o$, $r = \rho_o/2$, $\varepsilon = 0$, $\beta = 1$, $\theta = \frac{3}{4}$, $\tilde\theta = \frac{1}{2}$, $k = 0$ and since $u \leqslant 2^\xi N$ we get

$$\begin{aligned}
&\iint_{Q^{+,<}_{\frac{\rho_o}{2},\,h(y_o,\rho_o)}(y_o,\tau_o)} |Du|^2\,\lambda\,dxdt \leqslant \\
&\leqslant \gamma\Bigg[(2^\xi N)^2\,\mu_+\left(I^+_{\frac{\rho_o}{2},\frac{\rho_o}{2}}(y_o)\right) + (2^\xi N)^2\,\mu_-\left(I_+^{\frac{\rho_o}{2},\frac{\rho_o}{2}}(y_o)\right) + \\
&\quad + \frac{4}{\rho_o^2}\iint_{\left(B^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\times[\tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{2},\tau_o]\cup\left(I^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\times[\tau_o-h(y_o,\rho_o)\rho_o^2,\tau_o]} u^2\left(\frac{\mu_+}{h(y_o,\rho_o)} + \lambda\right)dxdt\Bigg] \leqslant \\
&\leqslant \gamma\left[(2^\xi N)^2\,\mu_+\left(I^+_{\frac{\rho_o}{2},\frac{\rho_o}{2}}(y_o)\right) + (2^\xi N)^2\,\mu_-\left(I_+^{\frac{\rho_o}{2},\frac{\rho_o}{2}}(y_o)\right)\right] + \\
&\quad + \frac{4\,\gamma}{\rho_o^2}(2^\xi N)^2\left[\rho_o^2\,\mu_+\left(\left(B^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\right) + h(y_o,\rho_o)\,\rho_o^2\,\lambda\left(\left(B^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\right)\right] \leqslant \\
&\leqslant \frac{\gamma}{\rho_o^2}\left[(2^\xi N)^2\,\frac{h(y_o,\rho_o)}{h(y_o,\rho_o)}\,\rho_o^2\,|\mu|\left(\left(B^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\right)\right] + \\
&\quad + \frac{4\,\gamma}{\rho_o^2}(2^\xi N)^2\left[\frac{h(y_o,\rho_o)}{h(y_o,\rho_o)}\,\rho_o^2\,\mu_+\left(\left(B^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\right) + h(y_o,\rho_o)\,\rho_o^2\,\lambda\left(\left(B^+_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}}\right)\right] \leqslant \\
&\leqslant \frac{9\,\gamma}{\rho_o^2}(2^\xi N)^2\,h(y_o,\rho_o)\,\rho_o^2\,\lambda\big(B_{\rho_o}(y_o)\big)\,.
\end{aligned}$$

Step 2 - The goal of this step is to show the existence of $\bar t \in [\tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{4},\tau_o]$ such that

$$\begin{aligned}
&\frac{\mu_+\left(\left\{x\in B^+_{\rho_o/2}(y_o)\,\middle|\,u(x,\bar t) > \frac{N}{2}\right\}\right)}{|\mu|_\lambda\left(B_{\rho_o/2}(y_o)\right)} > \frac{\overline{\nu}}{2}\,, \\
&\frac{\lambda_+\left(\left\{x\in B^+_{\rho_o/2}(y_o)\,\middle|\,u(x,\bar t) > \frac{N}{2}\right\}\right)}{\lambda\left(B_{\rho_o/2}(y_o)\right)} > \frac{\overline{\nu}}{2}\,, \\
&\int_{(B^+_{\frac{\rho_o}{2}}(y_o))^{\frac{\rho_o}{2}}} |Du(x,\bar t)|^2\lambda(x)dx \leqslant \frac{144\gamma}{\overline{\nu}}\,(2^\xi N)^2\,\frac{\lambda(B_{\rho_o}(y_o))}{\rho_o^2}\,.
\end{aligned} \tag{85}$$

To this aim we introduce the following sets (b being a positive number to be fixed later)

$$\begin{aligned}
A^+(t) &= \left\{x\in B^+_{\rho_o/2}(y_o)\,\middle|\,u(x,t) > \frac{N}{2}\right\}\,, \qquad t\in[\tau_o - h(y_o,\rho_o)\tfrac{\rho_o^2}{4},\tau_o] \\
I^+_\mu &= \left\{t\in[\tau_o - h(y_o,\rho_o)\tfrac{\rho_o^2}{4},\tau_o]\,\middle|\,\frac{\mu_+(A^+(t))}{|\mu|_\lambda(B_{\rho_o/2}(y_o))} > \frac{\overline{\nu}}{2}\right\}, \\
J_b &= \left\{t\in[\tau_o - h(y_o,\rho_o)\tfrac{\rho_o^2}{4},\tau_o]\,\middle|\,\int_{(B^+_{\frac{\rho_o}{2}}(y_o))^{\frac{\rho_o}{2}}} |Du(x,t)|^2\lambda(x)dx \leqslant b\,(2^\xi N)^2\,\frac{\lambda(B_{\rho_o}(y_o))}{\rho_o^2}\right\}.
\end{aligned}$$

Using (82) we can write

$$\begin{aligned}
\overline{\nu}\, h(y_o,\rho_o)\frac{\rho_o^2}{4} &< \int_{\tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{4}}^{\tau_o} \frac{\mu_+(A^+(t))}{|\mu|_\lambda\left(B_{\rho_o/2}(y_o)\right)}\, dt = \\
&= \int_{I_\mu^+} \frac{\mu_+(A^+(t))}{|\mu|_\lambda\left(B_{\rho_o/2}(y_o)\right)}\, dt + \int_{[\tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{4},\tau_o]\setminus I^+} \frac{\mu_+(A^+(t))}{|\mu|_\lambda\left(B_{\rho_o/2}(y_o)\right)}\, dt \leqslant \\
&\leqslant |I_\mu^+| + \frac{\overline{\nu}}{2}\, h(y_o,\rho_o)\frac{\rho_o^2}{4}
\end{aligned}$$

by which

$$|I_\mu^+| > \frac{\overline{\nu}}{2}\, h(y_o,\rho_o)\frac{\rho_o^2}{4}\,.$$

Now from one hand we have (83), on the other

$$\int_{[\tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{4},\tau_o]\setminus J_b} \int_{(B^+_{\frac{\rho_o}{2}}(y_o))^{\frac{\rho_o}{2}}} |Du|^2\, \lambda\, dxdt \geqslant b\,(2^\xi N)^2\, \frac{\lambda(B_{\rho_o}(y_o))}{\rho_o^2} \left| \left[\tau_o - h(y_o,\rho_o)\frac{\rho_o^2}{4}, \tau_o\right] \setminus J_b \right|.$$

Then we get

$$\big|J_b\big| \geqslant h(y_o,\rho_o)\frac{\rho_o^2}{4}\left(1 - \frac{36\gamma}{b}\right).$$

Choosing $b > 36\gamma$ this inequality is not trivial. Choosing, e.g., $b = 144\gamma/\overline{\nu}$ one gets

$$|I_\mu^+ \cap J_b| = |I_\mu^+| + |J_b| - |I_\mu^+ \cup J_b| \geqslant \frac{\overline{\nu}}{4}\, h(y_o,\rho_o)\frac{\rho_o^2}{4}\,.$$

Step 3 - Here we show that for every $\bar{\delta} \in (0,1)$ there are $\eta \in (0,1)$ and $y^* \in B^+_{\rho_o/2}(y_o)$, $\eta = \eta(K_1, K_2, q, K_3, \varsigma, \bar{\delta})$, $y^* = y^*(\gamma, 2^\xi N, \overline{\nu}, K_1, K_2, q, K_3, \varsigma, \bar{\delta}) = y^*(\gamma, 2^\xi N, \kappa, \gamma_1, \mathfrak{q}, K_1, K_2, q, K_3, \varsigma, \bar{\delta})$, such that $B_{\eta\frac{\rho_o}{2}}(y^*) \subset B^+_{\frac{\rho_o}{2}}(y_o)$ and such that

$$\mu_+\left(\left\{u(\cdot,\bar{t}) \leqslant \frac{N}{4}\right\} \cap B_{\eta\frac{\rho_o}{2}}(y^*)\right) \leqslant \bar{\delta}\, \mu_+(B_{\eta\frac{\rho_o}{2}}(y^*)). \tag{86}$$

To see that it is sufficient to use the informations of the previous step and to apply Lemma 2.14 to the function $2u/N$ with $\omega = \lambda$, $\nu = |\mu|_\lambda$, $\varepsilon = 1/2$, $\rho = \rho_o$, $x_0 = y_o$, $\mathcal{B} = B^+_{\rho_o/2}(y_o)$, $\sigma = \rho_o/2$, $\alpha = \frac{\overline{\nu}}{2}$, $\beta = \frac{144\gamma}{\overline{\nu}}\,(2^\xi N)^2$ and we get

$$\mu_+\left(\left\{u(\cdot,\bar{t}) > \frac{N}{4}\right\} \cap B_{\eta\frac{\rho_o}{2}}(y^*)\right) > (1-\bar{\delta})\, \mu_+(B_{\eta\frac{\rho_o}{2}}(y^*))$$

which is equivalent to (86). Notice that η depends on $K_1, K_2, q, K_3, \varsigma$, the constants of the weights, $\bar{\delta}$ and not on the value N.

Step 4 - Here we show that an estimate like that of the third step can be established also in a cylinder. Precisely we show that for every $\delta \in (0,1)$ there is $\bar{x} \in B_{\eta\frac{\rho_o}{4}}(y^*)$, $\varepsilon \in (0,1)$ which will depend only on δ and $\mathfrak{q}$, and $s^* = (\varepsilon\,\eta\,\rho_o/4)^2\, h(\bar{x}, \varepsilon\eta\frac{\rho_o}{4})$ such that ($\bar{t}, \bar{\delta}, \eta, \rho_o$ as above)

$$M_+\left(\left\{u \leqslant \frac{N}{8}\right\} \cap \left(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar{x}) \times [\bar{t}, \bar{t}+s^*]\right)\right) \leqslant \delta\, M_+\big(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar{x}) \times [\bar{t}, \bar{t}+s^*]\big)\,. \tag{87}$$

Notice that $\bar{x}$ implicitely depends on y^* and δ and then $\bar{x}$ depends on $\gamma, 2^\xi N, \kappa, \gamma_1, \mathfrak{q}, K_1, K_2, q, K_3, \varsigma, \bar{\delta}, \delta$. To see this we consider $\varepsilon \in (0,1)$ and a disjoint family of balls $\{B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j)\}_{j=1}^m$ such that

$$B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j) \subset B_{\eta\frac{\rho_o}{4}}(y^*)) \quad \text{for every } j = 1, \dots, m, \qquad \text{and}$$
$$B_{\eta\frac{\rho_o}{4}}(y^*) \subset \bigcup_{j=1}^m B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j) \subset B_{\eta\frac{\rho_o}{2}}(y^*)$$

and define

$$s_j^* := (\varepsilon\,\eta\,\rho_o/4)^2\, h\left(x_j, \varepsilon\eta\frac{\rho_o}{4}\right).$$

If necessary one can choose ε small enough so that $\bar{t} + s_j^* < T$. We apply the energy estimate (44) to the function $(u - N/4)_-$ in each of the sets $B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j) \times [\bar{t}, \bar{t} + s_j^*]$. Since $B_{\eta\frac{\rho_o}{2}}(y^*) \subset \Omega_+$ we get

$$\begin{aligned}
&\sup_{t\in[\bar{t},\bar{t}+s_j^*]} \int_{B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j)} \left(u - \frac{N}{4}\right)_-^2 (x,t)\mu_+(x)dx \leqslant \\
&\qquad \leqslant \int_{B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j)} \left(u - \frac{N}{4}\right)_-^2 (x,\bar{t})\mu_+(x)dx + \frac{16\gamma}{\eta^2\rho_o^2} \int_{\bar{t}}^{\bar{t}+s_j^*}\!\!\int_{B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j)} \left(u - \frac{N}{4}\right)_-^2 \lambda\, dxdt
\end{aligned}$$

and summing over j and using (86)

$$\begin{aligned}
&\sum_{j=1}^m \sup_{t\in[\bar{t},\bar{t}+s_j^*]} \int_{B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j)} \left(u - \frac{N}{4}\right)_-^2 (x,t)\mu_+(x)dx \leqslant \\
&\qquad \leqslant \int_{B_{\eta\frac{\rho_o}{2}}(y^*))} \left(u - \frac{N}{4}\right)_-^2 (x,\bar{t})\mu_+(x)dx + \sum_{j=1}^m \frac{16\gamma}{\eta^2\rho_o^2} \int_{\bar{t}}^{\bar{t}+s_j^*}\!\!\int_{B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j)} \left(u - \frac{N}{4}\right)_-^2 \lambda\, dxdt \leqslant \\
&\qquad \leqslant \frac{N^2}{16}\, \mu_+\left(\left\{u(\cdot,\bar{t}) \leqslant \frac{N}{4}\right\} \cap B_{\eta\frac{\rho_o}{2}}(y^*)\right) + \\
&\qquad\qquad + \sum_{j=1}^m \frac{16\gamma}{\eta^2\rho_o^2}\, \frac{N^2}{16}\, \varepsilon^2\, \eta^2\, \frac{\rho_o^2}{16}\, h\left(x_j, \varepsilon\eta\frac{\rho_o}{4}\right)\, \lambda\big(B_{\varepsilon\eta\frac{\rho_o}{2}}(x_j)\big) \leqslant \\
&\qquad \leqslant \frac{N^2}{16}\, \bar{\delta}\, \mu_+\big(B_{\eta\frac{\rho_o}{2}}(y^*)\big) + \sum_{j=1}^m \frac{16\,\gamma}{\eta^2\rho_o^2}\, \frac{N^2}{16}\, \varepsilon^2\eta^2\, \frac{\rho_o^2}{16}\, h\left(x_j, \varepsilon\eta\frac{\rho_o}{4}\right)\, \mathfrak{q}\, \lambda\big(B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j)\big) \leqslant \\
&\qquad \leqslant \mathfrak{q}\, \frac{N^2}{16}\, \bar{\delta}\, |\mu|_\lambda\big(B_{\eta\frac{\rho_o}{4}}(y^*))\big) + \frac{\gamma\, \mathfrak{q}\, N^2\varepsilon^2}{16} \sum_{j=1}^m |\mu|_\lambda\big(B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j)\big) \leqslant \\
&\qquad \leqslant \mathfrak{q}\, \frac{N^2}{16} (\bar{\delta} + \gamma\,\varepsilon^2)\, |\mu|_\lambda\big(B_{\eta\frac{\rho_o}{4}}(y^*)\big)\,.
\end{aligned}$$

On the other side, defining

$$B_j(t) = \left\{x \in B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j) \,\middle|\, u(x,t) \leqslant \frac{N}{8}\right\},$$

we easily get (for $t \in [\bar t, \bar t + s_j^*]$)

$$\int_{B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j)} \Big(u - \frac{N}{4}\Big)_-^2 (x,t)\mu_+(x)dx \geqslant \int_{B_j(t)} \Big(u - \frac{N}{2}\Big)_-^2 (x,t)\mu_+(x)dx \geqslant \frac{N^2}{64} |\mu|_\lambda(B_j(t)) \, .$$

Now putting together these inequalities we get

$$\frac{N^2}{64} \sum_{j=1}^m |M|_\Lambda \left(\left\{ u \leqslant \frac{N}{8} \right\} \cap \big(B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j) \times [\bar t, \bar t + s_j^*] \big) \right) \leqslant$$
$$\leqslant \mathfrak{q} \, \frac{N^2}{16} (\bar\delta + \gamma\,\varepsilon^2) \sum_{j=1}^m |M|_\lambda \Big(B_{\eta\frac{\rho_o}{4}}(y^*) \times [\bar t, \bar t + s_j^*] \Big) \, .$$

Once $\delta \in (0,1)$ is chosen we consider ε and $\bar\delta$ in such a way that

$$4\,\mathfrak{q}\,(\bar\delta + \gamma\,\varepsilon^2) \leqslant \delta$$

and then we get

$$\sum_{j=1}^m |M|_\Lambda \left(\left\{ u \leqslant \frac{N}{8} \right\} \cap \big(B_{\varepsilon\eta\frac{\rho_o}{4}}(x_j) \times [\bar t, \bar t + s_j^*] \big) \right) \leqslant \delta \sum_{j=1}^m |M|_\lambda \Big(B_{\eta\frac{\rho_o}{4}}(y^*) \times [\bar t, \bar t + s_j^*] \Big) \, .$$

Notice that s_j^* depend on ε and consequently on the choice of δ. To find a cylinder, independent of δ, in which the estimate above holds true notice that, whatever the choice of δ, by the last inequality at least one among the x_j's has to satisfy (87). We call $\bar x$ that x_j and $s^* := s_j^*$.

Step 5 - Here we show that

$$(88) \quad u \geqslant \frac{N}{16} \qquad \text{a.e. in } B_{\varepsilon\eta\frac{\rho_o}{8}}(\bar x) \times \left[\bar t + \frac{(\varepsilon\,\eta\,\rho_o)^2}{32}\, h\big(\bar x, \varepsilon\eta\frac{\rho_o}{4}\big), \bar t + \frac{(\varepsilon\,\eta\,\rho_o)^2}{16}\, h\big(\bar x, \varepsilon\eta\frac{\rho_o}{4}\big) \right] \, .$$

First notice that ε depens only on δ and $\mathfrak{q}$. By (87) and (33) we also get

$$(89) \qquad \Lambda \left(\left\{ u \leqslant \frac{N}{8} \right\} \cap \big(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar x) \times [\bar t, \bar t + s^*] \big) \right) \leqslant \kappa\, \delta^{\tau}\, \Lambda\big(B_{\varepsilon\eta\frac{\rho_o}{4}}(y^*) \times [\bar t, \bar t + s^*] \big) \, .$$

Now we want to apply Proposition 6.3, so first notice that, by the choice of $\bar x$ and ρ_o, since $u \geqslant 0$ and by (92) we have, choosing ε even smaller if necessary so that $\bar t + s^* < \tau_o$, that

$$\operatorname*{osc}_{B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar x)\times[\bar t, \bar t + s^*]} \leqslant 2^{\xi} N \, .$$

Then taking in Proposition 6.3, point i), the following values

$$\underline{m} = 0, \qquad \omega = 2^{\xi} N, \qquad r = \varepsilon\eta\frac{\rho_o}{8}, \qquad R = \varepsilon\eta\frac{\rho_o}{4},$$
$$x^\diamond = \bar x, \qquad t^\diamond = \bar t, \qquad \beta^\diamond = 1,$$
$$\sigma = \frac{1}{8}\frac{1}{2^{\xi}}, \qquad a = \frac{1}{2}, \qquad \theta^\diamond = \frac{1}{2}$$

we have the existence of $\underline{\nu}^\diamond$, which in this case depends only on κ, γ_1, γ, such that if

$$\frac{M_+\Big(\big\{ u \leqslant \frac{N}{8} \big\} \cap \big(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar x) \times [\bar t, \bar t + s^*] \big) \Big)}{M_+\big(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar x) \times [\bar t, \bar t + s^*] \big)} + \frac{\Lambda\Big(\big\{ u \leqslant \frac{N}{8} \big\} \cap \big(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar x) \times [\bar t, \bar t + s^*] \big) \Big)}{\Lambda\big(B_{\varepsilon\eta\frac{\rho_o}{4}}(\bar x) \times [\bar t, \bar t + s^*] \big)} \leqslant \underline{\nu}^\diamond$$

then (95) holds. Then, by (87) and (89), it is sufficient to choose δ in the fourth step in such a way that

$$\delta + \kappa\, \delta^{\tau} = \underline{\nu}^{\diamond}$$

to get that (95) holds (so δ depends only on $\kappa, \gamma_1, \gamma, \kappa, \tau$).

Step 6 - Now, starting from (95), we are in the conditions to apply the expansion of positivity. Before going on we recall the dependence of some parameters that are involved (and that we will need):

$$\begin{aligned} &\eta = \eta(K_1, K_2, q, K_3, \varsigma, \bar{\delta}) = \eta(K_1, K_2, q, K_3, \varsigma, \delta, \tau, \mathfrak{q}) = \eta(K_1, K_2, q, K_3, \varsigma, \kappa, \gamma_1, \gamma, \kappa, \mathfrak{q}), \\ &\varepsilon = \varepsilon(\bar{\delta}) = \varepsilon(\delta, \mathfrak{q}) = \varepsilon(\kappa, \gamma_1, \gamma, \kappa, \tau, \mathfrak{q}), \end{aligned}$$

We call just for simplicity

$$r := \frac{\varepsilon\, \eta\, \rho_o}{8} \qquad \text{and} \qquad \bar{s} := \bar{t} + 4\, h\Big(\bar{x}, \varepsilon\eta \frac{\rho_o}{4}\Big) r^2 .$$

In Lemma 6.7 we consider

$$x^* = \bar{x}, \qquad t^* = \bar{t} + \frac{(\varepsilon\, \eta\, \rho_o)^2}{16}\, h\Big(\bar{x}, \varepsilon\eta \frac{\rho_o}{4}\Big) = \bar{s}, \qquad \rho = r, \qquad h = \frac{N}{16},$$

and get that there is $\tilde{\beta}$ depending on γ and for every $\hat{\theta}$ there is $\lambda > 0$ depending on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \tilde{\beta}, \hat{\theta}$ such that

$$u \geqslant \lambda \frac{N}{16} \qquad \text{a.e. in } B^+_{2r}(\bar{x}) \times \Big[\bar{s} + \hat{\theta}\, \tilde{\beta}\, h(\bar{x}, 4r) r^2, \bar{s} + \tilde{\beta}\, h(\bar{x}, 4r) r^2\Big] .$$

Since this holds for every $t \in [\bar{s} + \hat{\theta}\, \tilde{\beta}\, h(\bar{x}, 4r) r^2, \bar{s} + \tilde{\beta}\, h(\bar{x}, 4r) r^2]$, applying again this lemma we reach

$$u \geqslant \lambda^2 \frac{N}{16} \qquad \text{a.e. in } B^+_{4r}(\bar{x}) \times \Big[\bar{s} + \hat{\theta}\, \tilde{\beta}\, \big(h(\bar{x}, 4r) + 4h(\bar{x}, 8r)\big) r^2, \bar{s} + \tilde{\beta}\big(h(\bar{x}, 4r) + 4h(\bar{x}, 8r)\big) r^2\Big] .$$

Iterating this arguement m times we get

$$u \geqslant \lambda^m \frac{N}{16} \qquad \text{a.e. in } B^+_{2^m r}(\bar{x}) \times \left[\bar{s} + \hat{\theta}\, \tilde{\beta}\, r^2 \sum_{j=1}^m 4^{j-1} h(\bar{x}, 2^{j+2} r), \bar{s} + \tilde{\beta}\, r^2 \sum_{j=1}^m 4^{j-1} h(\bar{x}, 2^{j+2} r)\right] .$$

Now we define the quantities ($m \in \mathbf{N}$ is still to be fixed)

$$\left\{ \begin{aligned} &s_m := \bar{s} + \hat{\theta}\, \tilde{\beta}\, r^2 \sum_{j=1}^m 4^{j-1} h(\bar{x}, 2^{j+2} r) , \\ &t_m := \bar{s} + \tilde{\beta}\, r^2 \sum_{j=1}^m 4^{j-1} h(\bar{x}, 2^{j+2} r) . \end{aligned} \right.$$

Since $\bar{x} \in B_\rho(x_o)$ requiring that $2^m r \geqslant 2\rho$ provides that $B_{2^m r}(\bar{x}) \supset B_\rho(x_o)$ so we require that m is such that

$$2\rho \leqslant 2^m r < 4\rho , \qquad \text{i.e.} \quad 1 + \log_2 \frac{\rho}{r} \leqslant m < 2 + \log_2 \frac{\rho}{r} . \tag{90}$$

What we have still to fix in the times interval is the value of $\hat{\theta}$ and moreover the values of b and ξ. Now notice that for every $x, y \in \Omega$ and $\varrho > 0$ such that $B_{2\varrho}(x) \subset \Omega$ and $B_{2\varrho}(y) \subset \Omega$ and such that $|x - y| < \varrho$ we have

$$|\mu|_\lambda(B_\varrho(x)) \leqslant |\mu|_\lambda(B_{2\varrho}(y)) \leqslant \mathfrak{q}\, \mu|_\lambda(B_\varrho(y))$$
$$\lambda(B_\varrho(y)) \leqslant \lambda(B_{2\varrho}(x)) \leqslant \mathfrak{q}\, \lambda(B_\varrho(x))$$

by which we derive

$$h(x, \varrho) \leqslant \mathfrak{q}^2 h(y, \varrho)\,.$$

Then, using this last estimate, (H.2)$'$ (see also Remark 2.7, point $\mathcal{C}$) and (90) we can estimate

$$\begin{aligned}
\sum_{j=1}^{m} 4^{j-1} r^2 h(\bar{x}, 2^{j+2} r) &\leqslant \mathfrak{q}^2 \sum_{j=0}^{m-1} 4^j r^2 h(x_o, 2^{j+3} r) = \\
&= \frac{\mathfrak{q}^2}{4^3} \sum_{j=0}^{m-1} (4^{j+3} r^2)^{1-\alpha} (4^{j+3} r^2)^\alpha h(x_o, 2^{j+3} r) \leqslant \\
&\leqslant \frac{\mathfrak{q}^2}{4^3} \sum_{j=0}^{m-1} (4^{j+3} r^2)^{1-\alpha} \tilde{K}_2^2 (4^{m+2} r^2)^\alpha h(x_o, 2^{m+2} r) = \\
&= \frac{\mathfrak{q}^2}{4^3} \tilde{K}_2^2 (4^{m+2} r^2)^\alpha h(x_o, 2^{m+2} r) \sum_{j=0}^{m-1} (4^3 r^2)^{1-\alpha} (4^{1-\alpha})^j \leqslant \\
&\leqslant \frac{\mathfrak{q}^2\, \tilde{K}_2^2}{4 - 4^\alpha}\, 4^m r^2 h(x_o, 2^{m+2} r) \leqslant \\
&\leqslant \frac{4\, \mathfrak{q}^6\, \tilde{K}_2^2}{4 - 4^\alpha}\, \rho^2 h(x_o, \rho)
\end{aligned}$$

by which

$$s_m \leqslant \bar{s} + \hat{\theta}\, \tilde{\beta}\, \frac{4\, \mathfrak{q}^6\, \tilde{K}_2^2}{4 - 4^\alpha}\, \rho^2 h(x_o, \rho)\,.$$

Now for a fixed constant $\vartheta_+ \in (0, 1]$ we can choose

$$\hat{\theta} \leqslant \vartheta_+ \, \frac{4 - 4^\alpha}{4}\, \frac{1}{\tilde{\beta}\, \mathfrak{q}^6\, \tilde{K}_2^2}\,,$$

indipendent of m, and, since $\bar{s} < t_o$, we get

$$s_m < t_o + \vartheta_+\, \rho^2 h(x_o, \rho)\,.$$

Notice that once $\hat{\theta}$ is fixed λ depends only on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \tilde{\beta}$. By the choice of m and recalling the definition of N we have

$$u \geqslant \lambda^m\, \frac{b(\rho - r_o)^{-\xi}}{16} \qquad \text{a.e. in } B_\rho^+(x_o) \times [s_m, t_m]\,.$$

By the choice we made of ρ_o in (80) we have that

$$\text{or } \frac{1}{\rho - r_o} = \frac{1}{2\,\rho_o} \qquad \text{either } \frac{1}{\rho - r_o} = \frac{1}{2^{1+\alpha}\, \mathfrak{q}\, \tilde{K}_2}\, \frac{1}{\rho^\alpha}\, \frac{1}{\rho_o^{1-\alpha}}\,.$$

Then, by the definition of r and since $u(x_o, t_o) = b\,\rho^{-\xi}$, in the first case we get

$$u(x,t) \geqslant \frac{(2^\xi \lambda)^m}{16} \frac{b(\varepsilon\,\eta)^\xi}{(2^6\rho)^\xi} = (2^\xi\lambda)^m \frac{(\varepsilon\,\eta)^\xi}{2^{6\xi+4}} u(x_o,t_o) \qquad \text{a.e. in } B_\rho^+(x_o) \times [s_m, t_m] .$$

In the second we get

$$\begin{aligned} u(x,t) &\geqslant \frac{\lambda^m}{16} b \left(\frac{1}{2^{1+\alpha}\, \mathfrak{q}\, \tilde{K}_2} \right)^\xi \frac{1}{\rho^{\alpha\xi}} \left(\frac{\varepsilon\eta}{8}\right)^{(1-\alpha)\xi} \left(\frac{2^m}{4\rho}\right)^{(1-\alpha)\xi} = \\ &= (2^{(1-\alpha)\xi}\lambda)^m\, b\, \frac{(\varepsilon\,\eta)^{(1-\alpha)\xi}}{(2^{6-4\alpha}\mathfrak{q}\,\tilde{K}_2)^\xi} \frac{1}{\rho^\xi} = \\ &= (2^{(1-\alpha)\xi}\lambda)^m \frac{(\varepsilon\,\eta)^{(1-\alpha)\xi}}{(2^{6-4\alpha}\mathfrak{q}\,\tilde{K}_2)^\xi} u(x_o,t_o) \qquad \text{a.e. in } B_\rho^+(x_o)\times[s_m,t_m] . \end{aligned}$$

So we can get rid of the dependence of m choosing now ξ in such a way that

$$\begin{aligned} 2^\xi \lambda = 1 \qquad & \text{in the first case,} \\ 2^{(1-\alpha)\xi}\lambda = 1 \qquad & \text{in the second case.} \end{aligned}$$

Since r depends on ρ_o, which depends on r_o, which depends on ξ, once we have fixed ξ we have also chosen the value of r, and consequently of m. Summing up, we have reached

$$u(x,t) \geqslant c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_\rho^+(x_o) \times [s_m,t_m]$$

with $s_m < t_o + \vartheta_+\, \rho^2 h(x_o,\rho)$, where

$$c_o = \frac{(\varepsilon\,\eta)^\xi}{2^{6\xi+4}} \qquad \text{or} \qquad c_o = \frac{(\varepsilon\,\eta)^{(1-\alpha)\xi}}{(2^{6-4\alpha}\mathfrak{q}\,\tilde{K}_2)^\xi}.$$

By the dependence of η, ε and ξ and since $\tilde{K}_2$ depends only on K_2 we have that

$$c_o \quad \text{depends on} \quad \gamma_1, \gamma, \mathfrak{q}, \kappa, \tilde{\beta}, \alpha, \kappa, \tau, K_1, K_2, K_3, q, \varsigma .$$

Now we are done if $t_m \geqslant t_o + \vartheta_+\, \rho^2 h(x_o,\rho)$ and the constant c_+ is c_o.
If, otherwise, $t_m < t_o + \vartheta_+\, \rho^2 h(x_o,\rho)$ we consider

$$\hat{t} \in [s_m, t_m] \qquad \text{such that} \qquad \hat{t} + \hat{\theta}\,\tilde{\beta}\, h(x_o,4\rho)\rho^2 \leqslant t_o + \vartheta_+ h(x_o,\rho)\rho^2 .$$

By (36) this is true, taking if necessary $\hat{\theta}$ smaller, if

$$\hat{\theta} \leqslant \frac{\vartheta_+}{\mathfrak{q}^2\tilde{\beta}} .$$

Applying again Lemma 6.7 and since $u(x,t) \geqslant c_o\, u(x_o,t_o)$ a.e. in $B_\rho^+(x_o)\times[s_m,t_m]$ (and then also in $B_{\rho/4}^+(x_o)\times[s_m,t_m]$) we get, in particular, that both

$$u(x,t) \geqslant \lambda\, c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_{2\rho}^+(x_o) \times \left[\hat{t} + \hat{\theta}\,\tilde{\beta}\, h(x_o,4\rho)\rho^2, \hat{t} + \tilde{\beta}\, h(x_o,4\rho)\rho^2\right]$$

and

$$u(x,t) \geqslant \lambda\, c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_{\rho/2}^+(x_o) \times \left[\hat{t} + \hat{\theta}\,\tilde{\beta}\, h(x_o,\rho)\rho^2/16, \hat{t} + \tilde{\beta}\, h(x_o,\rho)\rho^2/16\right] ;$$

then in particular

$$u(x,t) \geqslant \lambda\, c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_{\rho}^+(x_o) \times \left[\hat{t} + \hat{\theta}\,\tilde{\beta}\, h(x_o,\rho)\rho^2/16, \hat{t} + \tilde{\beta}\, h(x_o,\rho)\rho^2/16\right] .$$

Repeating this argument for every t in $\left[\hat{t} + \hat{\theta}\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16, \hat{t} + \tilde{\beta}\,h(x_o,\rho)\rho^2/16\right]$ we get

$$u(x,t) \geqslant \lambda^2\, c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_\rho^+(x_o) \times \left[\hat{t} + 2\hat{\theta}\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16, \hat{t} + 2\tilde{\beta}\,h(x_o,\rho)\rho^2/16\right].$$

If necessary, we add the requirement $2\hat{\theta} < 1$ so that $\left[\hat{t} + 2\hat{\theta}\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16, \hat{t} + 2\tilde{\beta}\,h(x_o,\rho)\rho^2/16\right] \cap \left[\hat{t} + \hat{\theta}\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16, \hat{t} + \tilde{\beta}\,h(x_o,\rho)\rho^2/16\right] \neq \emptyset$. Going on, we get

$$u(x,t) \geqslant \lambda^3\, c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_\rho^+(x_o) \times \left[\hat{t} + 3\,\hat{\theta}\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16, \hat{t} + 3\tilde{\beta}\,h(x_o,\rho)\rho^2/16\right]$$

requiring $3\hat{\theta} < 2$, which is free since we already imposed $2\hat{\theta} < 1$. We iterate k times, without additional assumptions about $\hat{\theta}$, till $\hat{t} + k\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16 > t_o + \vartheta\,h(x_o,\rho)\rho^2$ and get

$$u(x,t) \geqslant \lambda^k\, c_o\, u(x_o,t_o) \qquad \text{a.e. in } B_\rho^+(x_o) \times \left[\hat{t} + k\,\hat{\theta}\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16, \hat{t} + k\,\tilde{\beta}\,h(x_o,\rho)\rho^2/16\right].$$

Since $t_o - \hat{t} > h(x_o,\rho)\rho^2$, the inequality

$$\hat{t} + \frac{k\,\tilde{\beta}}{16}\,h(x_o,\rho)\rho^2 > t_o + \vartheta_+ h(x_o,\rho)\rho^2$$

holds if we choose

$$k > \frac{16}{\tilde{\beta}}\,(1 + \vartheta_+)\,.$$

For instance we can choose $[\frac{16}{\tilde{\beta}}\,(1+\vartheta_+)] + 1$, the minimum integer greater that $\frac{16}{\tilde{\beta}}\,(1+\vartheta_+)$ and the constant c_+ is $\lambda^k c_o$, where k depends only on $\tilde{\beta}$ and ϑ_+. Since $\tilde{\beta}$ depends only on γ we conclude that c_+ depends (only) on

$$\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \kappa, \tau, K_1, K_2, K_3, q, \varsigma, \vartheta_+\,.$$

In a complete analogous way one can prove point *ii*).

We see now point *iii*). Since s_1 and s_2 will remain fixed in the following we will use the simplified notations, for some $r > 0$ and $\bar{x} \in \Omega$,

$$Q_r^0(\bar{x}) := B_r^0(\bar{x}) \times [s_1, s_2]\,, \qquad Q_r(\bar{x}) := B_r(\bar{x}) \times [s_1, s_2]\,.$$

Similarly as for point *i*), we may write $u(x_o,t_o) = b\,\rho^{-\xi}$ for some $b, \xi > 0$ to be fixed later. Define the functions

$$\mathcal{M}(r) = \sup_{Q_r^0(x_o)} u, \qquad \mathcal{N}(r) = b(\rho - r)^{-\xi}, \qquad r \in [0,\rho).$$

Let us denote by $r_o \in [0,\rho)$ the largest solution of $\mathcal{M}(r) = \mathcal{N}(r)$. Define

$$N := \mathcal{N}(r_o) = b(\rho - r_o)^{-\xi}\,.$$

We can find $y_o \in B_{r_o}^0(x_o)$ such that

$$\frac{3N}{4} < \sup_{Q^0_{\frac{\rho_o}{4}}(y_o)} u \leqslant N \tag{91}$$

where $\rho_o = (\rho - r_o)/2$, so $B^0_{\rho_o}(y_o) \subset B_{\frac{\rho+r_o}{2}}(x_o)$. By this choice of ρ_o and by the choice of r_o we have

$$\sup_{Q^0_{\rho_o}(y_o)} u \leqslant \sup_{Q^0_{\frac{\rho+r_o}{2}}(x_o)} u < \mathcal{N}\left(\frac{\rho + r_o}{2}\right) = 2^{\xi} N. \tag{92}$$

We now proceed dividing the proof in four steps.

Step 1 - In this step we want to show that there is $\overline{\nu} \in (0,1)$, depending on κ, γ_1, γ, such that

$$\Lambda_0\left(\left\{u > \frac{N}{2}\right\} \cap Q^0_{\rho_o/2}(y_o)\right) > \overline{\nu}\, \Lambda\left(Q_{\rho_o/2}(y_o)\right)$$

and that

$$\iint_{Q^0_{\rho_o/2}(y_o)} |Du|^2 \lambda\, dxdt \leqslant \gamma\, (2^{\xi} N)^2 \left(\frac{2\, K_2^2\, \mathfrak{q}^2}{\omega} + 4\right)(s_2 - s_1)\, \frac{\lambda\big(B_{\rho_o}(y_o)\big)}{\rho_o^2}\,. \tag{93}$$

Arguing by contradiction we immediatly get the first inequality: indeed if that were false, setting in Proposition 6.1, point *iii*),

$$\overline{m} = \omega = 2^{\xi} N, \quad R = \frac{\rho_o}{2}, \quad \rho = \frac{\rho_o}{4}, \quad \sigma = 1 - 2^{-\xi-1}, \quad a = \sigma^{-1}\left(1 - \frac{3}{2^{\xi+2}}\right),$$

$$x^\star = y_o\,, \qquad t^\star = t_o\,, \qquad \beta^\star = \frac{8(s_2 - t_o)}{\rho_o}, \qquad s_1^\star = s_1\,, \qquad s_2^\star = s_2\,,$$

we would get that

$$u \leqslant \frac{3N}{4} \quad \text{in } B^0_{\rho_o/4}(y_o) \times (s_1, s_2)$$

which contradicts (91). To prove (93) we choose in (43) $x_0 = y_o$, $R = \rho_o$, $\tilde{r} = \rho_o$, $r = \rho_o/2$, $\varepsilon = 0$, $k = 0$ and since $u \leqslant 2^{\xi} N$ we get

$$\begin{aligned}
&\iint_{Q^0_{\rho_o/2}(y_o)} |Du|^2 \lambda\, dxdt \leqslant \\
&\quad\leqslant \gamma \left[(2^{\xi} N)^2\, |\mu| \left(I_0^{\frac{\rho_o}{2}, \frac{\rho_o}{2}}(y_o)\right) + \frac{4}{\rho_o^2} \iint_{\left(B^0_{\frac{\rho_o}{2}}(y_o)\right)^{\frac{\rho_o}{2}} \times [s_1, s_2]} u^2 \lambda\, dxdt\right] \leqslant \\
&\quad\leqslant \gamma \left[(2^{\xi} N)^2\, |\mu| \left(I_0^{\frac{\rho_o}{2}, \frac{\rho_o}{2}}(y_o)\right) + (2^{\xi} N)^2\, \frac{4}{\rho_o^2}\, 2\, \omega\, h(x_o, 4\rho)\rho^2\, \lambda\big(B_{\rho_o}(y_o)\big)\right] \leqslant \\
&\quad\leqslant \gamma\, (2^{\xi} N)^2 \left[|\mu|_\lambda\big(B_{\rho_o}(y_o)\big) + \frac{8\, \omega}{\rho_o^2}\, h(x_o, 4\rho)\rho^2\, \lambda\big(B_{\rho_o}(y_o)\big)\right] = \\
&\quad= \gamma\, (2^{\xi} N)^2 \left[h(y_o, \rho_o) + \frac{8\, \omega}{\rho_o^2}\, h(x_o, 4\rho)\rho^2\right] \lambda\big(B_{\rho_o}(y_o)\big) \leqslant \\
&\quad\leqslant \gamma\, (2^{\xi} N)^2 \left[4\, K_2^2\, \mathfrak{q}^2\, h(x_o, 4\rho)\frac{\rho^2}{\rho_o^2} + \frac{8\, \omega}{\rho_o^2}\, h(x_o, 4\rho)\rho^2\right] \lambda\big(B_{\rho_o}(y_o)\big)\,.
\end{aligned}$$

Step 2 - Here we show that for every $\delta \in (0,1)$ there are $\eta \in (0,1)$, which will depend only on

$K_1, K_2, K_3, q, \varsigma, \delta$, and $y^* \in B^0_{\rho_o/2}(y_o)$, which will depend only on $\gamma, 2^{\xi}N, \bar{\nu}, K_1, K_2, K_3, q, \varsigma, \omega, \delta$ (δ will be chosen depending on $\kappa, \gamma_1, \gamma, \omega\, h(x_o, 4\rho)$)), such that $B_{\eta\rho_o/2}(y_o) \subset B^0_{\rho_o/2}(y_o)$ and

$$\Lambda\left(\left\{u \leqslant \frac{N}{4}\right\} \cap \big(Q_{\eta\frac{\rho_o}{2}}(y^*)\big)\right) \leqslant \delta\, \Lambda\big(Q_{\eta\frac{\rho_o}{2}}(y^*)\big)\,. \tag{94}$$

Indeed by *Step 1* and applying Corollary 2.17 to the function $2u/N$ with $\omega = \nu = \lambda$, $\varepsilon = 1/2$, $\rho = \rho_o$, $x_0 = y_o$, $\mathcal{B} = B^0_{\rho_o/2}(y_o)$, $\sigma = \rho_o/2$, $a = s_1$, $b = s_2$, $\alpha = \bar{\nu}$, $\beta = \gamma\,(2^{\xi}N)^2\,\big(2\,K_2^2\,\mathfrak{q}^2\,\omega^{-1} + 4\big)$ we get the existence of $B_{\eta\frac{\rho_o}{2}}(y^*) \subset B^0_{\rho_o/2}(y_o)$ such that

$$\Lambda\left(\left\{u > \frac{N}{4}\right\} \cap \big(Q_{\eta\frac{\rho_o}{2}}(y^*)\big)\right) > (1-\delta)\, \Lambda\big(Q_{\eta\frac{\rho_o}{2}}(y^*)\big)$$

which is equivalent to (87).

Step 3 - Here we show that

$$u \geqslant \frac{N}{8} \qquad\qquad \text{a.e. in } Q_{\eta\frac{\rho_o}{4}}(y^*)\,. \tag{95}$$

Now we want to apply Proposition 6.3 so first notice that, since $u \geqslant 0$ and by (92) we have that

$$\operatorname*{osc}_{Q_{\eta\frac{\rho_o}{2}}(y^*)} \leqslant 2^{\xi}N\,.$$

Then taking in Proposition 6.3, point *iii*), the following values

$$\begin{gathered}
\underline{m} = 0, \qquad \omega = 2^{\xi}N, \qquad r = \eta\frac{\rho_o}{4}, \qquad R = \eta\frac{\rho_o}{2}, \\
x^\star = y^*, \qquad t^\star = t_o, \qquad s_1^\star = s_1 \qquad s_2^\star = s_2, \\
\beta^\star = 8\,\omega\, h(x_o, 4\rho)\,\frac{\rho^2}{\eta^2\rho_o^2}, \qquad \sigma = \frac{1}{4}\frac{1}{2^{\xi}}, \qquad a = \frac{1}{2}
\end{gathered}$$

we have the existence of $\underline{\nu}^\star \in (0,1)$, which in this case depends only on $\kappa, \gamma_1, \gamma, \omega\, h(x_o, 4\rho)$, such that if

$$\Lambda\left(\left\{u \leqslant \frac{N}{4}\right\} \cap \big(Q_{\eta\frac{\rho_o}{2}}(y^*)\big)\right) \leqslant \underline{\nu}^\star\, \Lambda\big(Q_{\eta\frac{\rho_o}{2}}(y^*)\big)\,.$$

then (95) holds. Then it is sufficient to choose $\delta = \underline{\nu}^\star$ (so δ depends only on $\kappa, \gamma_1, \gamma, \omega\, h(x_o, 4\rho)$).

Step 4 - Now we want to apply the expansion of positivity. We call, for simplicity

$$r := \eta\frac{\rho_o}{4}\,.$$

Taking in Lemma 6.7, point *iii*),

$$\rho = r\,, \qquad \beta = \omega$$

we get that

$$u \geqslant \lambda\,\frac{N}{8} \qquad\qquad \text{a.e. in } B^0_{2r}(y^*) \times [s_1, s_2]$$

with λ depending on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \omega$. Now taking in Lemma 6.7, point *iii*),

$$\rho = 2r\,, \qquad \beta = \omega$$

we get that

$$u \geqslant \lambda^2\,\frac{N}{8} \qquad\qquad \text{a.e. in } B^0_{2r}(y^*) \times [s_1, s_2]\,.$$

We iterate this argument m times getting

$$u \geqslant \lambda^m \frac{N}{8} \qquad \text{a.e. in } B^0_{2^m r}(y^*) \times [s_1, s_2]$$

till $B_{2^m r}(y^*) \supset B_\rho(x_o)$ and this is guaranteed if

$$2\rho \leqslant 2^m r < 4\rho \, .$$

As done before, observe that

$$u(x,t) \geqslant \lambda^m \frac{N}{8} = \lambda^m \frac{b\,\eta^\xi}{8^{\xi+1}} \frac{2^{m\xi}}{(2^m r)^\xi} \geqslant \lambda^m \frac{b\,\eta^\xi}{8^{\xi+1}} \frac{2^{m\xi}}{(4\,\rho)^\xi} =$$
$$= (2^\xi \lambda)^m \frac{\eta^\xi}{2^{5\xi+3}} u(x_o, t_o) \, .$$

Then, as before, choosing ξ in such a way $2^\xi \lambda = 1$ we get rid of the dependence of m and then in particular we get

$$u(x,t) \geqslant c_0 \, u(x_o,t_o) \qquad \text{a.e. in } B^0_\rho(x_o) \times [s_1, s_2]$$

where $c_0 = \frac{\eta^\xi}{2^{5\xi+3}}$ depends (only) on $K_1, K_2, K_3, q, \varsigma, \kappa, \gamma_1, \gamma, \omega, h(x_o, 4\rho), \mathfrak{q}$, the constants by which λ and η depend.

Finally the proof of point *iv*) can be obtained similarly to that of point *iii*), using in the order Proposition 6.1, point *iv*), Lemma 2.14, Proposition 6.3, point *iv*), Lemma 6.7, point *iv*). □

The previous theorem has an immediate consequence which we state here below.

Theorem 7.2. *Assume $u \in DG(\Omega, T, \mu, \lambda, \gamma)$, $u \geqslant 0$. Fix $\rho > 0$ and $\vartheta \in (0,1]$ for which $B_{5\rho}(x_o) \times [t_o - 16\,h(x_o,4\rho)\rho^2 - \vartheta h(x_o,\rho)\rho^2, t_o + 16\,h(x_o,4\rho)\rho^2 + \vartheta h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$. Suppose $x_o \in I$. Then there exists $c > 0$ depending on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \kappa, \tau, K_1, K_2, K_3, q, \varsigma, \vartheta_+$ such that*

$$u(x_o,t_o) \leqslant c \inf_{B_\rho(x_o)} \tilde{u}(x)$$

where

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta\, h(x_o,\rho)\rho^2) & \textit{if } x \in B^+_\rho(x_o) \\ u(x, t_o - \vartheta\, h(x_o,\rho)\rho^2) & \textit{if } x \in B^-_\rho(x_o) \end{cases} \qquad \textit{if} \quad x_o \in \partial\Omega_+ \cap \partial\Omega_-,$$

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta\, h(x_o,\rho)\rho^2) & \textit{if } x \in B^+_\rho(x_o) \\ u(x, t_o) & \textit{if } x \in B^0_\rho(x_o) \end{cases} \qquad \textit{if} \quad x_o \in \partial\Omega_+ \cap \partial\Omega_0,$$

$$\tilde{u}(x) = \begin{cases} u(x, t_o - \vartheta\, h(x_o,\rho)\rho^2) & \textit{if } x \in B^-_\rho(x_o) \\ u(x, t_o) & \textit{if } x \in B^0_\rho(x_o) \end{cases} \qquad \textit{if} \quad x_o \in \partial\Omega_- \cap \partial\Omega_0,$$

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta\, h(x_o,\rho)\rho^2) & \textit{if } x \in B^+_\rho(x_o) \\ u(x, t_o - \vartheta\, h(x_o,\rho)\rho^2) & \textit{if } x \in B^-_\rho(x_o) \\ u(x, t_o) & \textit{if } x \in B^0_\rho(x_o). \end{cases} \qquad \textit{if} \quad x_o \in \partial\Omega_+ \cap \partial\Omega_-$$

Proof - By Theorem 7.1 we immediately get the result taking $\vartheta = \vartheta_+ = \vartheta_-$ and $c = \max\{c_+, c_-, c_0\}$. □

One can give many different and equivalent formulations of the classical parabolic Harnack's inequality. We conclude giving only one possible equivalent formulation, which can be proved by standard arguements, to the one given above. Under the assumptions of Theorem 7.2 one has for $u \in DG$, $u \geqslant 0$, and for instance for $x_o \in \partial\Omega_+ \cap \partial\Omega_0 \cap \partial\Omega_-$ (and with obvious generalization in the other cases)

$$\sup_{B_\rho(x_o)} \tilde{u}(x) \leqslant c \inf_{B_\rho(x_o)} u(x, t_o)$$

$$\text{where} \quad \tilde{u}(x) = \begin{cases} u(x, t_o - \vartheta\, h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^+(x_o) \\ u(x, t_o + \vartheta\, h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^-(x_o) \\ u(x, t_o) & \text{if } x \in B_\rho^0(x_o). \end{cases} \tag{96}$$

Some consequences of the Harnack inequality - An important and standard consequence for a function satisfying a Harnack's inequality is Hölder-continuity. By classical computations and assuming (if necessary taking γ bigger)

$$\frac{\gamma}{\gamma - 1} < 2$$

one can get that if $u \in DG(\Omega, T, \mu, \lambda, \gamma)$ then u is locally α-Hölder continuous with respect to x and $\alpha/2$-Hölder continuous with respect to t, where $\alpha = (\log_2 \frac{\gamma}{\gamma-1})$, in $\big(\Omega_+ \cup \Omega_- \cup I\big) \times (0, T)$. As regards Ω_0 we can only get that for every $t \in (0, T)$ $u(\cdot, t)$ is locally α-Hölder continuous in Ω_0. Notice that in the interface I separating Ω_0 and $\Omega_+ \cup \Omega_-$ the function u is regular also with respect to t.

Another consequence is a strong maximum procniple, which one can get, again by standard arguement, using (96). One can derive a "standard" maximum principle from Theorem 7.1, which we do not state, and others from Theorem 7.2.
If, for instance, we suppose $x_o \in \partial\Omega_+ \cap \partial\Omega_0 \cap \partial\Omega_-$ (and again with obvious generalization in the other cases) we could briefly state the maximum principles as follows: suppose $(x_o, t_o) \in \Omega \times (0, T)$ is a maximum point for u in a set

$$\Big(B_\rho^+(x_o) \times (t_o - \vartheta\, h(x_o, \rho)\rho^2, t_o + \vartheta\, h(x_o, \rho)\rho^2)\Big) \cup \Big(B_\rho^0(x_o) \times \{t_o\}\Big) \cup \Big(\cup B_\rho^-(x_o) \times (t_o - \vartheta\, h(x_o, \rho)\rho^2, t_o + \vartheta\, h(x_o, \rho)\rho^2)\Big)$$

for some $\vartheta \in (0, 1]$, then u is constant in the set

$$\Big(B_\rho^+(x_o) \times (t_o - \vartheta\, h(x_o, \rho)\rho^2, t_o]\Big) \cup \Big(B_\rho^0(x_o) \times \{t_o\}\Big) \cup \Big(\cup B_\rho^-(x_o) \times [t_o, t_o + \vartheta\, h(x_o, \rho)\rho^2)\Big).$$

8. Examples

In this section we show some possible examples of μ (and consequently of I) and λ. In all the examples, just for simplicity, we suppose $\Omega \subset \mathbf{R}^2$.

1. In the simplest situation when $\mu \equiv \lambda \equiv 1$ we get the classical case in which the De Giorgi class contains the solutions of

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,Du)) = b(x,t,u,Du)$$

with a, b satisfying

$$\begin{aligned} \big(a(x,t,u,Du), Du\big) &\geqslant \lambda |Du|^p \,, \\ |a(x,t,u,Du)| &\leqslant \Lambda |Du|^{p-1} \,, \\ |b(x,t,u,Du)| &\leqslant \Lambda |Du|^{p-1} \,, \end{aligned}$$

with λ, Λ positive numbers. Obviously if $\mu \equiv -1$ we have the analogous results for backward parabolic equations.

2. If $\mu \equiv 0$ and $\lambda \equiv 1$ we have a family (in the parameter t) of elliptic equations for which one cannot expect regularity in time, neither for "solutions". The same may happen Ω_0 is a proper subset of Ω.
For example, in dimension 1 consider the solutions of

$$\frac{d}{dx}\left(a(x,t)\frac{du}{dx}\right) = 0 \,, \qquad u(0) = 0 \,,\ u(2) = 1 \,,$$

with

$$a(x,t) = \alpha(t) \text{ in } [0,1] \qquad \text{and} \qquad a(x,t) = \beta(t) \text{ in } [1,2]$$

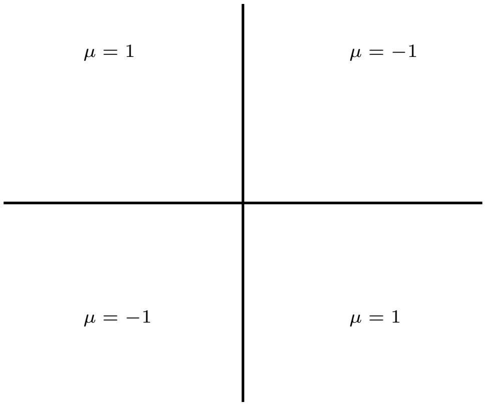
with $\alpha(t) \neq \beta(t)$ for every t and α and β discontinuous. The solutions are clearly discontinuous in time for $x \in (0,2)$.

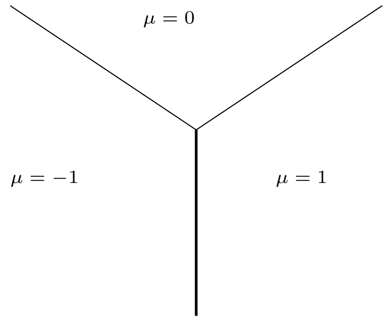
3. If $\mu > 0$ and $\lambda > 0$ we have the Harnack's inequality for doubly weighted equations, like for instance

$$\mu \frac{\partial u}{\partial t} - \operatorname{div}(\lambda Du) = 0 \,. \tag{97}$$

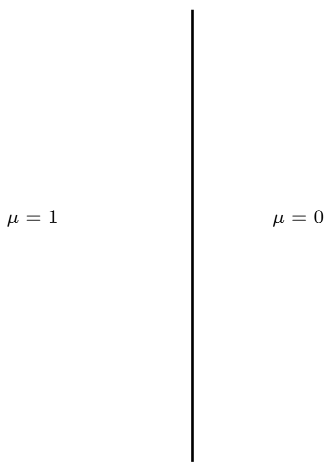
In the particular case $\mu \equiv 1$ we rescue the result contained in [3], while if $\mu \equiv \lambda$ we rescue the result contained in [4].

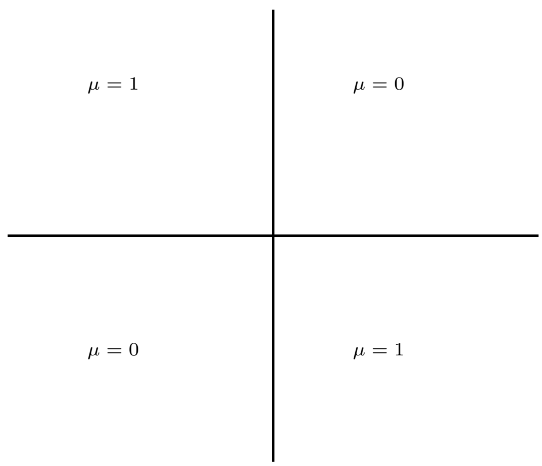
4. Consider now an example where for simplicity $|\mu| \equiv \lambda \equiv 1$ in Ω, but $\mu \not\equiv 1$. Suppose, for instance, that μ changes sign around an interface like that in the first of the two following pictures where I is a cross intersecting in a point x_o. This kind of interface clearly satisfies assumptions (H.4) and (H.5) and then also in a neighbourhood of the points (x_o, t), $t \in (0,T)$, the solution, e.g., of (97) is Hölder-continuous.
Also an interface like that shown in the second of the two following pictures is admitted.





5. Consider $\mu \geqslant 0$ and, for simplicity, suppose that μ takes only the values 1 and 0. in the pictures below there are two simple examples: in the first one the interface is made by just a line, in the second one is made by two intersecting lines. In both cases a function belonging to the De Giorgi class turns out to be Hölder-continuous in $(\Omega_+ \cup I) \times (0, T)$. In particular it is continuous in the interface I both in x and t, even if it could not be continuous in Ω_0 as shown in the second example.



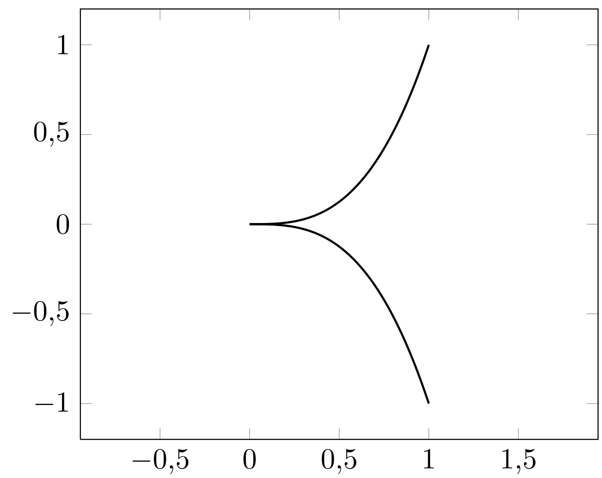


6. Also some cusps like the one in the picture below can be admitted, provided that assumption (H.4) is satisfied. For example, suppose (part of) the interface is that in the picture below and the vertex is the point $(0, 0)$ and suppose $\mu \neq 0$. If μ_+ satisfies (H.4) then we are in the assumptions and the theorems of Section 7 and Section **??** hold.
For instance, suppose $\lambda \equiv 1$ and consider $\mu \equiv -1$ on the left of the curve and $\mu \equiv 1$ on

the other side of the curve which is the union of the graphs of $f(x) = x^n$ and $g(x) = -x^n$ for $x \in [0, L]$, $L > 0$, and $n \in \mathbf{N}$, $n \geqslant 1$. We have that

$$\mu_+\big(B_{2\rho}(0,0)\big) \leqslant \mathfrak{q}\, \mu_+\big(B_\rho((0,0))\big)$$

for some $\mathfrak{q}$ depending on n.
While if, for instance, we consider $f(x) = e^{-1/x}$ and $g(x) = -e^{-1/x}$ the above inequality does not hold any more.



If we consider different μ, i.e. μ which can degenerate to zero, the geometry of the interface can change depending also on how the weights $|\mu|$ and λ degenerate near the interface.

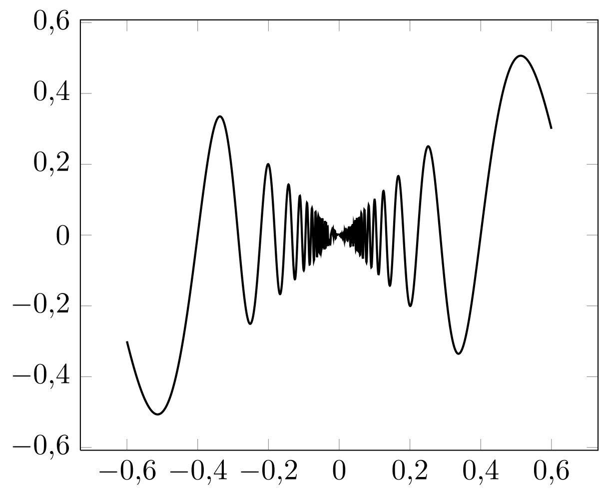
7. The final example is the following: again for simplicity suppose $|\mu| \equiv \lambda \equiv 1$ in $\mathbf{R}^2$ and suppose $\mu \equiv 1$ in the region above the graphic of f, which we will call Ω_+, and $\mu \equiv -1$ in the region below the graphic of f, which we will call Ω_-, where

$$f(y) = y \cos \frac{1}{y} \qquad (f(0) = 0)\,.$$

In spite of the fact that the length of the graphic inside the ball $B := B_1(0,0)$ is infinite, the measure (the 2-dimensional Lebesgue measure $\mathcal{L}^2$) of the ε-neighbourhood of I is of order ε and then going to zero when $\varepsilon \to 0^+$. Moreover, due to the simmetry of the graphic of f we have that

$$\mu_+\big(B_{2\rho}(0,0)\big) = \frac{1}{2}\, \mathcal{L}^2\big(B_{2\rho}(0,0)\big) \leqslant \frac{1}{2}\, c\, \mathcal{L}^2\big(B_\rho(0,0)\big) = \frac{1}{2}\, c\, \mu_+\big(B_\rho(0,0)\big)$$

where c denotes the doubling constant for $\mathcal{L}^2$. Therefore also in this case assumptions (H.4) and (H.5) are satisfied and even if I is not rectifiable can be an admissible interface.



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