# CHARACTERIZATION BY ASYMPTOTIC MEAN FORMULAS OF $q-$HARMONIC FUNCTIONS IN CARNOT GROUPS

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Abstract. Aim of this paper is to extend the work [9] to the Carnot group setting. More precisely, we prove that in every Carnot group a function is $q-$harmonic (here $1 < q < \infty$), if and only if it satisfies a particular asymptotic mean value formula.

## 1. Introduction

It is well known that every continuous function $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is harmonic in the open set $\Omega$ if and only if for every $y \in \Omega$, and for every Euclidean ball $B_E(y,\epsilon) \Subset \Omega$, centered in $y$ with radius $\epsilon$, $u$ satisfies the mean value formula,

$$u(y) = \fint_{B_E(y,\epsilon)} u. \tag{1}$$

This result is still true if instead of the exact mean value formula (1) the following asymptotic mean value formula is satisfied for every $y \in \Omega$:

$$u(y) = \fint_{B_E(y,\epsilon)} u + o(\epsilon^2) \quad \text{as } \epsilon \to 0^+, \tag{2}$$

where $\fint_{B_E(y,\epsilon)} u := \frac{1}{|B_E(y,\epsilon)|}\int_{B_E(y,\epsilon)} u$ and $|B_E(y,\epsilon)|$ denotes the Lebesgue measure of $B_E(y,\epsilon)$.

Starting from the seminal papers [5, 16, 2, 25, 27] many steps have been done in order to extend (1) and (2) to a more general setting. We recall for instance [10, 11, 24] and we refer the interested reader to [23] for a survey on this topic.

Further extensions of formulas (1) and (2) to solutions to linear, possibly degenerate, elliptic and parabolic equations can be found in [7, 12, 26], see also [3, 4, 14] and the references therein.

Recently, a first extension of (2) in a nonlinear framework has been proved in [20]. In that paper it has been proved that every $q-$harmonic function i a viscosity sense, with $1 < q \leq \infty$, satisfies a suitable approximated mean value formula and viceversa.

More precisely, $u$ is a viscosity solution to

$$\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u) = 0 \tag{3}$$

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in a domain $\Omega \subset \mathbb{R}^n$ if and only if for every $x \in \Omega$,

$$u(x) = \frac{\alpha}{2}\Big(\max_{B_E(x,\epsilon)} u + \min_{B_E(x,\epsilon)} u\Big) + \beta \fint_{B_E(x,\epsilon)} u + o(\epsilon^2) \tag{4}$$

as $\epsilon \to 0$ in a weak sense (see [15, 20, 21] for the precise definition) and $\alpha+\beta = 1$, $\alpha/\beta = (q-2)/(n+2)$. In [9], the first named author, Liu and Manfredi extended this result to the Heisenberg group under the assumption that $1 < q < \infty$. We also recall [19], where a similar result has been proved for viscosity solutions for $1 < q \leq \infty$ considering a different type of mean value formula.

We point out that the results obtained in [19] are indeed equivalent (at least for $1 < q < \infty$) to those contained in [9]. However, the approach in [9] and the generalization to Carnot groups described in the present paper, seem to be more flexible for future applications. For instance, in stochastic game theory in particular the *tug of war games* and, possibly, in some geometric aspects of the flow by mean curvature in the Carnot groups. See, e.g., [8] where the $1-$Laplace-Kohn operator appears.

Aim of this paper is to prove that the asymptotic representation formula obtained in [9] holds in the whole class of stratified Carnot groups. In particular, since $\mathbb{R}^n$ is a Carnot group of step 1 then our result contains the Euclidean characterization, for $1 < q < +\infty$, proved in [20] and also the one proved in [9].

In the sequel, we denote by $B(P, r)$ the metric ball, centered at $P$ with radius $r$, obtained in the specific geometry associated with the particular Carnot group considered and by $Q$ the homogeneous dimension of the Carnot group $\mathbb{G}$. Further details about these quantities will be given in Section 2.

Our main result is the following.

**Theorem 1.1.** *Let $\mathbb{G}$ be a Carnot group. Let $1 < q < \infty$ and $u$ be a continuous function defined in a domain $\Omega \subseteq \mathbb{G}$. The asymptotic expansion*

$$u(P) = \frac{\alpha}{2}\left(\min_{B(P,\epsilon)} u + \max_{B(P,\epsilon)} u\right) + \beta \fint_{B(P,\epsilon)} u(x) + o(\epsilon^2), \tag{5}$$

*holds as $\epsilon \to 0^+$ for every $P \in \Omega$ in the viscosity sense if and only if*

$$\Delta_{q,\mathbb{G}} u = 0$$

*in $\Omega$ in the viscosity sense, where*

$$\alpha := \frac{2(q-2)C}{2(q-2)C+1}, \quad \beta := \frac{1}{2(q-2)C+1},$$

*and*

$$C := \frac{1}{2h_1 \mid B_1(0) \mid} \int_{B_1(0)} \|x^{(1)}\|^2 dx^{(1)} dx^{(2)} dx^{(3)} \cdots dx^{(k)}. \tag{6}$$

*Here $h_1$ denotes the dimension of the first layer of the Lie algebra of $\mathbb{G}$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^{h_1}$.*

In order to clarify the statement, we anticipate few definitions from Section 2 below and we refer the reader to that section for the details. Let $\mathbb{G}$ be a Carnot group and let $X_1, \dots, X_{h_1}$ be a basis of the first layer of the Lie algebra of $\mathbb{G}$. For $q \in (1, +\infty)$ the subelliptic $q-$Laplace operator is

$$\Delta_{q,\mathbb{G}} u := \mathrm{div}_{\mathbb{G}}(|\nabla_{V_1} u|^{q-2} \nabla_{V_1} u), \tag{7}$$

where $\nabla_{V_1} u := \sum_{j=1}^{h_1} X_j u X_j$, $|\nabla_{V_1} u| = \sqrt{\sum_{j=1}^{h_1} (X_j u)^2}$ and for $U = (U_1, \dots, U_{h_1})$, $\mathrm{div}_{\mathbb{G}}\, U := \sum_{j=1}^{h_1} X_j U_j$. Notice that, for $q = 2$, $\Delta_{2,\mathbb{G}} u = \Delta_{\mathbb{G}} u$ is the so called Kohn-Laplace operator. We introduce now the notion of viscosity solution for this type of operators. Observe that if $u$ is smooth then, by standard calculations,

$$\Delta_{q,\mathbb{G}} u = |\nabla_{V_1} u|^{q-2} \left((q-2)\Delta_{\infty,\mathbb{G}} u + \Delta_{\mathbb{G}} u\right),$$

where

$$\Delta_{\infty,\mathbb{G}} u = \left\langle D^{2,*}_{V_1} u \,\frac{\nabla_{V_1} u}{|\nabla_{V_1} u|}, \frac{\nabla_{V_1} u}{|\nabla_{V_1} u|} \right\rangle$$

and

$$D^{2,*}_{V_1} u := \left( \frac{(X_i X_j + X_j X_i) u}{2} \right)_{1 \le i,j \le h_1} \tag{8}$$

is the so called symmetrized horizontal Hessian of $u$.

**Definition 1.2.** Fix a value of $q \in (1, \infty)$ and consider the subelliptic $q$-Laplace equation

$$-\,\mathrm{div}_{\mathbb{G}}(|\nabla_{V_1} u|^{q-2} \nabla_{V_1} u) = 0. \tag{9}$$

(i) A lower semi-continuous function $u$ is a viscosity supersolution of (9) if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum at $P_0 \in \Omega$, and $\nabla_{V_1} \phi(P_0) \neq 0$ we have
$$-(q-2)\Delta_{\infty,\mathbb{G}} \phi(P_0) - \Delta_{\mathbb{G}} \phi(P_0) \ge 0.$$

(ii) A lower semi-continuous function $u$ is a viscosity subsolution of (9) if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict maximum in $P_0 \in \Omega$, and $\nabla_{V_1} \phi(P_0) \neq 0$, we have
$$-(q-2)\Delta_{\infty,\mathbb{G}} \phi(P_0) - \Delta_{\mathbb{G}} \phi(P_0) \le 0.$$

(iii) A continuous function $u$ is a viscosity solution of of (9) if it is both a viscosity supersolution and a viscosity subsolution.

As shown in [17, 18] for the Euclidean case and in [1] for the subelliptic case, it suffices to consider smooth functions whose horizontal gradient does not vanish. In addition, in [1], it is shown that the notions of viscosity and weak solutions agree for the equation $-\Delta_{q,\mathbb{G}} u = 0$.

Next we state carefully what we mean when we say that the asymptotic expansion (5) holds in the viscosity sense. We start recalling the following classical definition:

**Definition 1.3.** Let $h$ be a real valued function defined in a neighborhood of zero. We say that

$$h(x) \le o(x^2) \text{ as } x \to 0^+$$

if any of the three equivalent conditions is satisfied:

a) $\displaystyle\limsup_{x \to 0^+} \frac{h(x)}{x^2} \le 0,$

b) there exists a non-negative function $g(x) \geq 0$ such that

$$h(x) + g(x) = o(x^2) \text{ as } x \to 0^+,$$

or

c) $\displaystyle\lim_{x \to 0^+} \frac{h^+(x)}{x^2} \leq 0,$

A similar definition is given for

$$h(x) \geq o(x^2) \text{ as } x \to 0^+.$$

by reversing the inequalities in a) and c), requiring that $g(x) \leq 0$ in b) and replacing $h^+$ by $h^-$ in c)[1]
Let $f$ and $g$ be two real valued functions defined in a neighborhood of $x_0 \in \mathbb{R}$. We say that $f$ and $g$ are asymptotic functions for $x \to x_0$ if there exists a function $h$ defined in a neighborhood $V_{x_0}$of $x_0$ such that:

(i) $f(x) = g(x)h(x)\ \forall x \in V_{x_0} \setminus \{x_0\}$.
(ii) $\lim_{x \to x_0} h(x) = 1$.

If $f$ and $g$ are asymptotic as $x \to x_0$ we simply write $f \sim g$ as $x \to x_0$.

**Definition 1.4.** A continuous function defined in a neighborhood of a point $P \in \mathbb{G}$ satisfies

$$u(P) = \frac{\alpha}{2}\left(\min_{B(P,\epsilon)} u + \max_{B(P,\epsilon)} u\right) + \beta \fint_{B(P,\epsilon)} u + o(\epsilon^2), \tag{10}$$

as $\epsilon \to 0^+$ in viscosity sense, if the following conditions hold:

(i) for every continuous function $\phi$ defined in a neighborhood of a point $P$ such that $u - \phi$ has a strict minimum at $P$ with $u(P) = \phi(P)$ we have

$$-\phi(P) + \frac{\alpha}{2}\left(\min_{B(P,\epsilon)} \phi + \max_{B(P,\epsilon)} \phi\right) + \beta \fint_{B(P,\epsilon)} \phi \leq o(\epsilon^2),$$

as $\epsilon \to 0^+$.

(ii) for every continuous function $\phi$ defined in a neighborhood of a point $P$ such that $u - \phi$ has a strict maximum at $P$ with $u(P) = \phi(P)$ then

$$-\phi(P) + \frac{\alpha}{2}\left(\min_{B(P,\epsilon)} \phi + \max_{B(P,\epsilon)} \phi\right) + \beta \fint_{B(P,\epsilon)} \phi \geq o(\epsilon^2).$$

as $\epsilon \to 0^+$.

As in [9, 20] a key role in the proof of Theorem 1.1 is played by the following Lemmas:

**Lemma 1.5.** *Let $u$ be a smooth function. If $\nabla_{V_1} u(0) \neq 0$, then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ there exist points $P_{\epsilon,m}, P_{\epsilon,M} \in \partial B(0, \epsilon)$ such that*

$$\max_{B(0,\epsilon)} u = u(P_{\epsilon,M})$$

[1]As usual, we denote by $h^+(x) := \max\{h(x), 0\}$ and $h^-(x) := -\min\{h(x), 0\}$.

*and*

$$\min_{B(0,\epsilon)} u = u(P_{m,\epsilon}).$$

**Lemma 1.6.** *For small $\epsilon > 0$, consider points $P_{M,\epsilon}$ and $P_{m,\epsilon}$ in $\partial B(0,\epsilon)$ such that*

$$\max_{B(0,\epsilon)} u = u(P_{M,\epsilon}) \text{ and } \min_{B(0,\epsilon)} u = u(P_{m,\epsilon}).$$

*Whenever $\nabla_{V_1} u(0) \neq 0$ we have*

$$\lim_{\epsilon\to 0} \frac{x^{(1)}_{M,\epsilon}}{\epsilon} = \frac{\nabla_{V_1} u(0)}{| \nabla_{V_1} u(0) |}$$

*and*

$$\lim_{\epsilon\to 0} \frac{x^{(1)}_{m,\epsilon}}{\epsilon} = -\frac{\nabla_{V_1} u(0)}{| \nabla_{V_1} u(0) |},$$

*here $P_{M,\epsilon} = (x^{(1)}_{M,\epsilon}, \cdots, x^{(k)}_{M,\epsilon})$, $P_{m,\epsilon} = (x^{(1)}_{m,\epsilon}, \cdots, x^{(k)}_{m,\epsilon}) \in \mathbb{G}$.*

Both in [20] and [9] Lemma 1.6 follows by rather simple and explicit computations. In our case, due to the quite complicate geometry underlying a general Carnot group, it seems to us that the computations made in [20, 9] cannot be easily extended to this general framework. Hence a new proof based on an asymptotic analysis is required. Indeed, with our approach we do not need to compute explicitly the exact values of the points where the maximum and the minimum of $u$ are realized on the sphere of size $\epsilon$.

Therefore, in our opinion, the main novelty of the present paper is an alternative proof of Lemma 1.6, which allows us to generalize the result proved in [9] to every Carnot group using only their intrinsic homogeneous properties.

We conclude this introduction summarizing the structure of the paper. In Section 2 we introduce Carnot groups and we recall some results about this subject. In Section 3 we prove the key Lemmas 1.5 and 1.6. In Section 4 we introduce approximated mean value formulas for sublaplacians in Carnot groups and in Section 5 we give the proof of Theorem 1.1. Finally, in the Appendix 6 we provide, just to emphasize the main ideas used to prove Theorem 1.1, a complementary proof of Lemma 1.6 in the case of the Engel group.

## 2. Carnot groups

In what follows we briefly recall some standard facts on Carnot groups, see [4, 6, 13, 22] for a more careful discussion.

**Definition 2.1.** A finite dimensional Lie algebra $\mathfrak{g}$ is said to be stratified of step $k \in \mathbb{N}$ if there exist subspaces $V_1, \dots, V_k$ of $\mathfrak{g}$ with linear dimension $v_k := \dim V_k$ such that:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k;$$
$$[V_1, V_i] = V_{i+1} \quad i = 1, \dots, k-1; \quad [V_1, V_k] = \{0\}.$$

A connected and simply connected Lie group $\mathbb{G}$ is a Carnot group if its Lie algebra $\mathfrak{g}$ is finite dimensional and stratified. We also denote by $h_0 := 0$, $h_i := \sum_{j=1}^{i} v_j$ and $m := h_k$

Using the exponential map every Carnot group $\mathbb{G}$ of step $k$ is isomorphic as a Lie group to $(\mathbb{R}^m, \cdot)$ where $\cdot$ is the group operation obtained projecting on $\mathbb{G}$ the Baker-Campbell-Hausdorff formula. For every $\lambda > 0$ and for every $x \in \mathbb{G}$ we denote by $\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}$ and $\tau_x : \mathbb{G} \longrightarrow \mathbb{G}$ the maps defined respectively by:

$$\delta_\lambda(x) = \delta_\lambda(x_1, \ldots, x_m) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_k} x_m) \tag{11}$$

$$\tau_y(x) := y \cdot x, \tag{12}$$

where $\sigma_i \in \mathbb{N}$ is called the homogeneity of the variable $x_i$ in $\mathbb{G}$ and it is defined by

$$\sigma_j := i \quad \text{whenever} \quad h_{i-1} < j \le h_i.$$

We endow $\mathbb{G}$ with a norm and a quasi-distance defining

$$|x|_{\mathbb{G}} := |(x^{(1)}, \ldots, x^{(k)})|_{\mathbb{G}} := \Big( \sum_{j=1}^{k} \|x^{(j)}\|^{\frac{2k!}{j}} \Big)^{\frac{1}{2k!}} \tag{13}$$

$$\mathrm{d}(x, y) := |y^{-1} \cdot x|_{\mathbb{G}}, \tag{14}$$

here $x^{(j)} := (x_{h_{j-1}+1}, \ldots, x_{h_j})$ and $\|x^{(j)}\|$ denotes the standard Euclidean norm in $\mathbb{R}^{h_j - h_{j-1}}$. We define the gauge ball centered at $x \in \mathbb{G}$ of radius $R > 0$ by

$$B(x, R) := \{y \in \mathbb{G} \mid |y^{-1} \cdot x|_{\mathbb{G}} < R\}.$$

The following Proposition is proved in [4].

**Proposition 2.2.** *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group. Then the Lebesgue measure on $\mathbb{R}^m$ is invariant with respect to the left and the right translations on $\mathbb{G}$. Precisely, if we denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^m$, we then have*

$$|x \cdot E| = |E| = |E \cdot x| \quad \forall x \in \mathbb{G}.$$

*Moreover,*

$$|\delta_\lambda(E)| = \lambda^Q |E| \quad \forall \lambda > 0,$$

*where*

$$Q := \sum_{j=1}^{m} \sigma_j. \tag{15}$$

A basis $X = (X_1, \ldots, X_m)$ of $\mathfrak{g}$ is called Jacobian basis if $X_j = J(e_j)$ where $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$ and $J : \mathbb{R}^m \longrightarrow \mathfrak{g}$ is defined by

$$J(\eta)(x) := \mathcal{J}_{\tau_x}(0) \cdot \eta$$

here $\mathcal{J}_{\tau_x}$ denotes the Jacobian matrix of $\tau_x$.

The following Proposition is classical, see [4, Corollary 1.3.19] for a proof.

**Proposition 2.3.** *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group of step $k \in \mathbb{N}$. Then the Jacobian basis $X_1, \ldots, X_m$ have polynomial coefficients and if $h_{l-1} < j \le h_l$, $1 \le l \le k$,*

$$X_j(x) = \partial_j + \sum_{i > h_l}^{m} a_i^{(j)}(x) \partial_i$$

*where $a_i^{(j)}(\delta_\lambda(x)) = \lambda^{\sigma_i - \sigma_j} a_i^{(j)}(x)$ and if $h_{l-1} < i \le h_l$ then $a_i^{(j)}(x) = a_i^{(j)}(x_1, \ldots, x_{h_{l-1}})$.*

Let $X = (X_1, \dots, X_m)$ be a Jacobian basis of $\mathbb{G} = (\mathbb{R}^m, \cdot)$, for any function $f \in C^1(\mathbb{R}^m)$ we define the horizontal gradient by

$$\nabla_{V_1} f := \sum_{i=1}^{h_1} (X_i f) X_i$$

and

$$\nabla_{V_j} f := \sum_{h_{j-1} < i \leq h_j} (X_i f) X_i. \tag{16}$$

Moreover, we define the horizontal Laplacian of $f : \mathbb{G} \longrightarrow \mathbb{R}$ and we denote it by $\Delta_{\mathbb{G}} f$ the following function

$$\Delta_{\mathbb{G}} f := \sum_{i=1}^{h_1} X_i X_i f. \tag{17}$$

*Example* 2.4. The usual Euclidean space $(\mathbb{R}^n, +)$ is trivially a Carnot group of step 1, $|\cdot|_{\mathbb{R}^n}$ is the classical Euclidean norm and $\Delta_{\mathbb{R}^n}$ is the Laplace operator.

*Example* 2.5. If $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^n$ (see [6, 4] for the definition) then it is well known that a Jacobian basis of $\mathfrak{g}$ is $X_j = \partial_j + 2x_{j+n}\partial_{2n+1}$ and $X_{j+n} = \partial_{j+n} - 2x_j \partial_{2n+1}$, for $j = 1, \dots, n$ and $X_{2n+1} = \partial_{2n+1}$. Moreover,

$$|(x_1, \dots, x_{2n+1})|_{\mathbb{H}^n} = \Big( (\sum_{i=1}^{2n} x_i)^2 + x_{2n+1}^2 \Big)^{\frac{1}{4}}.$$

*Example* 2.6. We denote the Engel group $\mathbb{E} \equiv \mathbb{R}^4$. Its Lie algebra is $\mathfrak{e} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = \operatorname{span}\{X_1, X_2\}$, $V_2 = \operatorname{span}\{X_3\}$ and $V_3 = \operatorname{span}\{X_4\}$ and the only nonzero commutation relations are

$$[X_1, X_2] = X_3 \quad , \quad [X_1, X_3] = [X_2, X_3] = X_4.$$

Using exponential coordinates for each $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{E}$ we get

$$\begin{aligned} x \cdot y = &\Big( x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1), \\ &, x_4 + y_4 + \frac{1}{2}\Big( (x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2) \Big) + \frac{1}{12}(x_1 - y_1 + x_2 - y_2)(x_1 y_2 - x_2 y_1) \Big) \end{aligned} \tag{18}$$

and

$$\begin{aligned} X_1(x_1, x_2, x_3, x_4) &= \partial_1 - \frac{x_2}{2}\partial_3 - \Big( \frac{x_3}{2} + \frac{x_2}{12}(x_1 + x_2) \Big) \partial_4 \\ X_2(x_1, x_2, x_3, x_4) &= \partial_2 + \frac{x_1}{2}\partial_3 - \Big( \frac{x_3}{2} - \frac{x_1}{12}(x_1 + x_2) \Big) \partial_4 \\ X_3(x_1, x_2, x_3, x_4) &= \partial_3 + \frac{1}{2}(x_1 + x_2)\partial_4 \\ X_4(x_1, x_2, x_3, x_4) &= \partial_4. \end{aligned}$$

Moreover,

$$|x|_{\mathbb{E}} = |(x_1, x_2, x_3, x_4)|_{\mathbb{E}} = \Big( (x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 \Big)^{\frac{1}{12}}$$

and

$$\Delta_{\mathbb{E}} = X_1 X_1 + X_2 X_2.$$

We conclude this section recalling the Taylor formula (of degree two) in general Carnot groups; see [4, Proposition 20.3.11] for the proof.

**Lemma 2.7.** *Let $\Omega \subset \mathbb{G}$ an open neighborhood of* $0$ *and let $u \in C^\infty(\Omega)$. Then,*

(19)
$$u(P) = u(0) + \langle \nabla_{V_1} u(0), x^{(1)} \rangle_{\mathbb{R}^{h_1}} + \langle \nabla_{V_2} u(0), x^{(2)} \rangle_{\mathbb{R}^{h_2-h_1}} + \frac{1}{2}\left( \langle D^{2*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} + o(\|P\|^2) \right)$$

*for every $P = (x^{(1)}, x^{(2)}, \dots, x^{(k)}) \in \Omega$. Here $D^{2*}_{V_1} u(0)$ is the symmetrized horizontal Hessian matrix of $u$ defined in* (8).

## 3. Proofs of Lemmas 1.5 and 1.6

In this section we prove Lemmas 1.6 and 3.1, which are the main ingredients for the proof of Theorem 1.1.

***Proof of Lemma 1.5:*** The proof of this result is exactly the same as the one proposed in [9, Lemma 3.3], we recall it only for the sake of completeness. Let us consider the case of the maximum, the case of the minimum being analogous. Let us proceed by contradiction. Assume that a sequence of positive numbers $\{\epsilon_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ and a sequence of points $\{P_j\}_{j\in\mathbb{N}} \subset B(0,\epsilon_j)$ such that $\epsilon_j \to 0$, as $j \to +\infty$ and

$$\max_{\overline{B(0,\epsilon_j)}} u = u(P_j).$$

Then for every $j \in \mathbb{N}$, we have that $\nabla u(P_j) = 0$ because $P_j$ is in the interior of $B(0,\epsilon_j)$. Hence we get a contradiction with the fact that by continuity of $\nabla u$ gives $\nabla u(0) = 0$, which implies $\nabla_{V_1} u(0) = 0$. □

***Proof of Lemma 1.6:*** We consider the case of the maximum by using the method of Lagrange multipliers, the case of minimum can be treated in the same way. There exists $\lambda_\epsilon \in \mathbb{R}$ such that

(20)
$$\begin{cases} (\nabla_{h_1} u(P_{M,\epsilon}), \nabla_{h_2-h_1} u(P_{M,\epsilon}), \cdots, \nabla_{h_k-h_{k-1}} u(P_{M,\epsilon})) = \lambda_\epsilon \nabla \sum_{j=1}^k \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}} \\ \sum_{j=1}^k \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}} = \epsilon^{2k!} \end{cases}$$

where $\nabla_{h_{j+1}-h_j} u$ denotes the Euclidean gradient of $u$ with respect to the variables with homogeneity $\sigma_j$. Thus we get

(21)
$$\begin{cases} \nabla_{h_1} u(P_{M,\epsilon}) = 2k!\lambda_\epsilon \|x_\epsilon^{(1)}\|^{2k!-1} \frac{x_\epsilon^{(1)}}{\|x_\epsilon^{(1)}\|} \\ \nabla_{h_2-h_1} u(P_{M,\epsilon}) = \lambda_\epsilon k! \|x_\epsilon^{(2)}\|^{k!-1} \frac{x_\epsilon^{(2)}}{\|x_\epsilon^{(2)}\|} \\ \vdots = \vdots \\ \nabla_{h_k-h_{k-1}} u(P_{M,\epsilon}) = 2(k-1)!\lambda_\epsilon \|x_\epsilon^{(k)}\|^{2(k-1)!-1} \frac{x_\epsilon^{(k)}}{\|x_\epsilon^{(k)}\|} \\ \|x_\epsilon^{(1)}\|^{2k!} = \epsilon^{2k!} - \sum_{j=2}^k \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}}. \end{cases}$$

Hence, by Proposition 2.3, for every $l_1 = 1, \dots, h_1$

(22)
$$X_{l_1} u(P_{M,\epsilon}) = 2k!\lambda_\epsilon \|x_\epsilon^{(1)}\|^{2k!-2} x_\epsilon^{(1,l)} + \lambda_\epsilon \sum_{j=2}^k \sum_{h_{j-1}<i\le h_j} a_i^{(l_1)}(P_{M,\epsilon}) \frac{2k!}{j} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)}$$

Analogously, for the vectors of the second layer $X_{l_2}$ ( $h_1 < l_2 \le h_2$) we get

$$X_{l_2} u(P_{M,\epsilon}) = \lambda_\epsilon k! \|x_\epsilon^{(2)}\|^{k!-2} x_\epsilon^{(2,l)} + \lambda_\epsilon \sum_{j=3}^{k} \sum_{h_{j-1}<i\le h_j} a_i^{(l_2)}(P_{M,\epsilon}) \frac{2k!}{j} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)} \tag{23}$$

and in general for the vector fields of the $p-$th layer $X_{h_{p-1}+1}, \dots, X_{h_p}$, $h_{p-1} < l_p \le h_p$ for $p < k$, we get:

$$X_{l_p} u(P_{M,\epsilon}) = \lambda_\epsilon \frac{2k!}{p} \|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2} x_\epsilon^{(p,l)} + \lambda_\epsilon \sum_{j=p+1}^{k} \sum_{h_{j-1}<i\le h_j} a_i^{(l_p)}(P_{M,\epsilon}) \frac{2k!}{j} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)} \tag{24}$$

finally, for $h_{k-1} < l_k \le h_k$, $X_{h_{k-1}+1} u(P_{M,\epsilon}), \dots, X_{h_k} u(P_{M,\epsilon})$ can be written as follows:

$$X_{l_k} u(P_{M,\epsilon}) = 2\lambda_\epsilon (k-1)! \|x_\epsilon^{(k)}\|^{2(k-1)!-2} x_\epsilon^{(k,l_k)}. \tag{25}$$

We split the proof in two parts. In the first we assume that $|\nabla_{V_k} u(0)| \ne 0$, in the second we treat the case $|\nabla_{V_k} u(0)| = 0$. Since in the sequel we often consider cases of functions like $f \sim g$ as $\epsilon \to 0$, we always write $f \sim g$ sometimes omitting to recall that $\epsilon \to 0$ whenever this choice does not create ambiguity.

Case $|\nabla_{V_k} u(0)| \ne 0$.
Using (25), squaring and summing we get

$$| \lambda_\epsilon | = \frac{\|\nabla_{V_k} u(P_{M,\epsilon})\|}{2(k-1)! \|x_\epsilon^{(k)}\|^{2(k-1)!-1}}. \tag{26}$$

Therefore, for every $1 \le p \le k-1$ and $h_{p-1} < l_p \le h_p$

$$X_{l_p} u(P_{M,\epsilon}) = \pm \frac{\|\nabla_{V_k} u(P_{M,\epsilon})\|}{2(k-1)! \|x_\epsilon^{(k)}\|^{2(k-1)!-1}} \Big( \frac{2k!}{p} \|x_\epsilon^{(p)}\|^{2(k!/p)-2} x_\epsilon^{(p,l_p)} + \sum_{j=p+1}^{k} I_j^{l_p}(P_{M,\epsilon}) \Big) \tag{27}$$

where

$$I_j^{l_p}(P_{M,\epsilon}) := \frac{2k!}{j} \sum_{h_{j-1}<i\le h_j} a_{i,l_p}^j (P_{M,\epsilon}) \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)}. \tag{28}$$

We claim that for every $1 \le p \le k-1$ and $h_{p-1} < l_p \le h_p$ the following formula holds

$$|X_{l_p} u(0)| = \lim_{\epsilon \to 0^+} \frac{k}{p} |\nabla_{V_k} u(P_{M,\epsilon})| \frac{\|x_\epsilon^{(p)}\|^{2(k!/p)-2} x_\epsilon^{(p,l_p)}}{\|x_\epsilon^{(k)}\|^{2(k-1)!-1}}. \tag{29}$$

In order to emphasize the main ideas we prefer to postpone proof of the claim to a separate Lemma, see Lemma 3.1 below.
Moreover adding , squaring (29), and using $|\nabla_{V_k} u(0)| \ne 0$, we get

$$\lim_{\epsilon \to 0^+} \frac{\|x_\epsilon^{(p)}\|^{4\frac{k!}{p}-2}}{\|x_\epsilon^{(k)}\|^{4(k-1)!-2}} = \frac{p^2}{k^2} \frac{|\nabla_{V_p} u(0)|^2}{|\nabla_{V_k} u(0)|^2}. \tag{30}$$

Let us denote $\rho_{\epsilon,p} = \|x_\epsilon^{(p)}\|$. Hence from (30) it follows

$$\lim_{\epsilon\to 0^+} \frac{\rho_{\epsilon,p}^{4\frac{k!}{p}-2}}{\rho_{\epsilon,k}^{4(k-1)!-2}} = \frac{p^2}{k^2}\frac{|\nabla_{V_p}u(0)|^2}{|\nabla_{V_k}u(0)|^2}. \tag{31}$$

Moreover, since $P_{M,\epsilon}\in\partial B(0,\epsilon)$ from (20) we also get

$$\sum_{j=1}^{k-1}\rho_{\epsilon,j}^{\frac{2k!}{j}} + \rho_{\epsilon,k}^{2(k-1)!} = \epsilon^{2k!}. \tag{32}$$

Notice that, if $|\nabla_{V_p}u(0)|\neq 0$, then the limits in (31) can be written also as

$$\rho_{\epsilon,p} \sim \left(\frac{p^2}{k^2}\frac{|\nabla_{V_p}u(0)|^2}{|\nabla_{V_k}u(0)|^2}\right)^{\frac{p}{4k!-2p}} \rho_{\epsilon,k}^{p\frac{4(k-1)!-2}{4k!-2p}}. \tag{33}$$

Hence, if we substitute (33) in (32) we get

$$\sum_{j=1}^{k-1}\left(\frac{j^2}{k^2}\frac{|\nabla_{V_j}u(0)|^2}{|\nabla_{V_k}u(0)|^2}\right)^{\frac{2k!}{(4k!-2j)}} \rho_{\epsilon,k}^{2k!\frac{4(k-1)!-2}{(4k!-2j)}} + \rho_{\epsilon,k}^{2(k-1)!} \sim \epsilon^{2k!}. \tag{34}$$

In particular, this implies that, for $\epsilon\to 0^+$,

$$\left(\frac{1}{k^2}\frac{|\nabla_{V_1}u(0)|^2}{|\nabla_{V_k}u(0)|^2}\right)^{\frac{2k!}{(4k!-2)}} \rho_{\epsilon,k}^{2k!\frac{4(k-1)!-2}{(4k!-2)}} \sim \left(\frac{1}{k^2}\frac{|\nabla_{V_1}u(0)|^2}{|\nabla_{V_k}u(0)|^2}\right)^{\frac{k!}{(2k!-1)}} \rho_{\epsilon,k}^{2k!\frac{2(k-1)!-1}{(2k!-1)}} \sim \epsilon^{2k!}. \tag{35}$$

Therefore, since $|\nabla_{V_1}u(0)|\neq 0$, we conclude

$$\rho_{\epsilon,k} \sim \left(k^2\frac{|\nabla_{V_k}u(0)|^2}{|\nabla_{V_1}u(0)|^2}\right)^{\frac{1}{4(k-1)!-2}} \epsilon^{\frac{(2k!-1)}{2(k-1)!-1}}. \tag{36}$$

Moreover, keeping in mind (33) we get for $p = 1,\dots,k-1$

$$\rho_{\epsilon,p} \sim \left(\frac{p^2}{k^2}\frac{|\nabla_{V_p}u(0)|^2}{|\nabla_{V_k}u(0)|^2}\right)^{\frac{p}{4k!-2p}} \rho_{\epsilon,k}^{p\frac{2(k-1)!-1}{2k!-p}} \sim p^{\frac{p}{2k!-p}}\epsilon^{\frac{p(2k!-1)}{2k!-p}}. \tag{37}$$

As a consequence, applying (29) with $p=1$, we conclude that

$$\frac{|X_{l_1}u(0)|}{|\nabla_{V_k}u(0)|} \sim k\frac{\rho_{\epsilon,1}^{2k!-2}x_\epsilon^{(1,l_1)}}{\rho_{\epsilon,k}^{2(k-1)!-1}} \sim \frac{x_\epsilon^{(1,l_1)}}{\epsilon}, \quad l_1 = 1,\cdots,h_1. \tag{38}$$

Case $|\nabla_{V_k}u(0)| = 0$.

Assume that $\nabla_{V_k}u(P_{M,\epsilon})\to 0$ as $\epsilon\to 0^+$. Then, recalling (25)

$$X_{l_k}u(P_{M,\epsilon}) \sim 2\lambda_\epsilon(k-1)!\|x_\epsilon^{(k)}\|^{k!-2}x_\epsilon^{(k,l_k)}. \tag{39}$$

Hence, by (22), for every $l_1 = 1\dots,h_1$ we get

$$X_{l_1}u(P_{M,\epsilon}) \sim 2k!\lambda_\epsilon\|x_\epsilon^{(1)}\|^{2k!-2}x_\epsilon^{(1,l)} + \lambda_\epsilon\sum_{j=2}^{k-1}\sum_{h_{j-1}<i\leq h_j} a_i^{(l_1)}(P_{M,\epsilon})\frac{2k!}{j}\|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2}x_\epsilon^{(j,i)} \tag{40}$$

Analogously for the vectors of the second layer $X_{l_2}$, $h_1 < l_2 \leq h_2$ we get

$$X_{l_2} u(P_{M,\epsilon}) \sim \lambda_\epsilon k! \|x_\epsilon^{(2)}\|^{k!-2} x_\epsilon^{(2,l)} + \lambda_\epsilon \sum_{j=3}^{k-1} \sum_{h_{j-1} < i \leq h_j} a_i^{(l_2)}(P_{M,\epsilon}) \frac{2k!}{j} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)}$$

and in general for the vector fields of the $p-$th layer $X_{h_{p-1}+1}, \ldots, X_{h_p}$, $h_{p-1} < l_p \leq h_p$ for $p \leq k-1$, we get:

$$X_{l_p} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{p} \|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2} x_\epsilon^{(p,l)} + \lambda_\epsilon \sum_{j=p+1}^{k-1} \sum_{h_{j-1} < i \leq h_j} a_i^{(l_p)}(P_{M,\epsilon}) \frac{2k!}{j} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)}$$

In particular for $p = k-1$ and using (39)

$$X_{l_{k-1}} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{k-1} \|x_\epsilon^{(k-1)}\|^{\frac{2k!}{k-1}-2} \sim 2(k-1)! \lambda_\epsilon \|x_\epsilon^{(k-1)}\|^{2(k-1)!-2}.$$

As a consequence, substituting in the previous $k-2$ equations we deduce that $k-2$ equation becomes

$$X_{l_{k-2}} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{k-2} \|x_\epsilon^{(k-2)}\|^{\frac{2k!}{k-2}-2} x_\epsilon^{(k-2,l)}$$

and for $1 \leq p < k-2$ we obtain

$$X_{l_p} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{p} \|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2} x_\epsilon^{(p,l)} + \lambda_\epsilon \sum_{j=p+1}^{k-2} \sum_{h_{j-1} < i \leq h_j} a_i^{(l_p)}(P_{M,\epsilon}) \frac{2k!}{j} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}-2} x_\epsilon^{(j,i)}.$$

Arguing by induction we conclude that for all $1 \leq p \leq k$ and for every $h_{p-1} \leq l_p < h_p$

$$X_{l_p} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{p} \|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2} x_\epsilon^{(p,l)}. \tag{41}$$

We recall that in our hypotheses the gradient of the first layer does not vanish at 0. Then without restrictions we can assume that $X_{l_1} u(0) \neq 0$ and

$$\lambda_\epsilon \sim \frac{X_{l_1} u(0)}{2k! \|x_\epsilon^{(1)}\|^{2k!-2} x_\epsilon^{(1,1)}}. \tag{42}$$

Thus, using (42) in (41) we get

$$X_{l_p} u(P_{M,\epsilon}) \sim \frac{X_{l_1} u(0)}{2k! \|x_\epsilon^{(1)}\|^{2k!-2} x_\epsilon^{(1,1)}} \frac{2k!}{p} \|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2} x_\epsilon^{(p,l)} \sim \frac{\|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2}}{p \|x_\epsilon^{(1)}\|^{2k!-2} x_\epsilon^{(1,1)}} x_\epsilon^{(p,l)} X_{l_1} u(0). \tag{43}$$

In particular we can put in evidence the values of $x_\epsilon^{(p,l_p)}$ obtaining

$$x_\epsilon^{(p,l_p)} \sim \frac{p \|x_\epsilon^{(1)}\|^{2k!-2}}{\|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2}} \frac{X_{l_p} u(P_{M,\epsilon})}{X_{l_1} u(0)} x_\epsilon^{(1,1)}.$$

Squaring and adding we get

$$\|x_\epsilon^{(p)}\| \sim \frac{p \|x_\epsilon^{(1)}\|^{2k!-2}}{\|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2}} \frac{|\nabla_{V_p} u(P_{M,\epsilon})|}{|X_{l_1} u(0)|} |x_\epsilon^{(1,1)}|,$$

which implies

(44)
$$\|x_\epsilon^{(p)}\| \sim \left( p\|x_\epsilon^{(1)}\|^{2k!-2} \frac{|\nabla_{V_p} u(P_{M,\epsilon})|}{|X_{l_1} u(0)|} |x_\epsilon^{(1,1)}| \right)^{\frac{p}{2k!-p}}.$$

Hence, recalling that $P_{M,\epsilon} \in \partial B(0,\epsilon)$ we deduce that

$$\epsilon^{2k!} = \sum_{j=1}^{k} \|x_\epsilon^{(j)}\|^{\frac{2k!}{j}} \sim \|x_\epsilon^{(1)}\|^{2k!} + \sum_{j=2}^{k} \left( p\|x_\epsilon^{(1)}\|^{2k!-2} \frac{|\nabla_{V_j} u(P_{M,\epsilon})|}{|X_{l_1} u(0)|} |x_\epsilon^{(1,1)}| \right)^{\frac{2k!}{2k!-j}}.$$

In particular, as $\epsilon \to 0$, we get

$$\|x_\epsilon^{(1)}\| \sim \epsilon.$$

Inserting the last estimate in (43) we get

(45)
$$X_{l_p} u(P_{M,\epsilon}) \sim \frac{\|x_\epsilon^{(p)}\|^{\frac{2k!}{p}-2}}{p\epsilon^{2k!-2} x_\epsilon^{(1,1)}} x_\epsilon^{(p,l)} X_1 u(0).$$

We consider the following cases for the vectors of the first layer. Either there exist some vectors, say without restrictions only the second one $X_2 u(0)$, such that $X_2 u(0) = 0$ and the others that do not vanish, or all the remaining vectors after $X_1 u(0)$ do not vanish in 0, i.e. $X_2 u(0) \neq 0, \cdots, X_{h_1} u(0) \neq 0$. In the first case it follows, from (45), that

(46)
$$X_2 u(P_{M,\epsilon}) \sim \frac{\|x_\epsilon^{(1)}\|^{2k!-2}}{\epsilon^{2k!-2} x_\epsilon^{(1,1)}} x_\epsilon^{(1,2)} X_1 u(0) \sim X_1 u(0) \frac{x_\epsilon^{(1,2)}}{x_\epsilon^{(1,1)}},$$

that is

(47)
$$x_\epsilon^{(1,1)} X_2 u(P_{M,\epsilon}) \sim X_1 u(0) x_\epsilon^{(1,2)},$$

that implies

$$x_\epsilon^{(1,2)} = o(x_\epsilon^{(1,1)})$$

for $\epsilon \to 0$, and for $l_1 = 3, \ldots, h_1$

(48)
$$x_\epsilon^{(1,1)} X_{l_1} u(0) \sim X_1 u(0) x_\epsilon^{(1,l_1)}.$$

Then recalling that

$$\|x_\epsilon^{(1)}\| \sim \epsilon,$$

from (47) and (48), we deduce squaring and summing

(49)
$$\epsilon^2 \sim (x_\epsilon^{(1,1)})^2 + (x_\epsilon^{(1,2)})^2 + \sum_{l_1=3}^{h_1} \frac{X_{l_1} u(0)^2}{X_1 u(0)^2} (x_\epsilon^{(1,1)})^2 \sim (x_\epsilon^{(1,1)})^2 + \frac{X_2 u(P_{M,\epsilon})^2}{X_1 u(0)^2} (x_\epsilon^{(1,1)})^2 + \sum_{l_1=3}^{h_1} \frac{X_{l_1} u(0)^2}{X_1 u(0)^2} (x_\epsilon^{(1,1)})^2$$
$$\sim (x_\epsilon^{(1,1)})^2 \frac{\sum_{m=1, m\neq 2}^{h_1} X_m u(0)^2 + X_2 u(P_{M,\epsilon})^2}{X_1 u(0)^2} \sim (x_\epsilon^{(1,1)})^2 \frac{|\nabla_{V_1} u(0)|^2}{X_1 u(0)^2}.$$

Hence

(50)
$$\epsilon \sim |x_\epsilon^{(1,1)}| \frac{|\nabla_{V_1} u(0)|}{|X_1 u(0)|},$$

that is

$$\lim_{\epsilon\to 0}\frac{|x_\epsilon^{(1,1)}|}{\epsilon}=\frac{|X_1u(0)|}{|\nabla_{V_1}u(0)|}.$$

Moreover, recalling (48), and (50) we get for $l_1=3,\dots,h_1$,

(51) $$\epsilon\frac{|X_1u(0)|}{|\nabla_{V_1}u(0)|}X_{l_1}u(0)\sim x_\epsilon^{(1,1)}X_{l_1}u(0)\sim X_1u(0)x_\epsilon^{(1,l_1)}$$

and in particular, up to the sign,

(52) $$\epsilon\frac{X_{l_1}u(0)}{|\nabla_{V_1}u(0)|}\sim x_\epsilon^{(1,l_1)},$$

that is the thesis. Eventually, recalling (47), and (50) we obtain

(53) $$\epsilon\sim|x_\epsilon^{(1,1)}|\frac{|\nabla_{V_1}u(0)|}{|X_1u(0)|},$$

(54) $$\epsilon\frac{|X_1u(0)|}{|\nabla_{V_1}u(0)|}X_2u(P_{M,\epsilon})\sim X_1u(0)x_\epsilon^{(1,2)},$$

that is

$$\lim_{\epsilon\to 0}\frac{x_\epsilon^{(1,2)}}{\epsilon}=0=\frac{X_2u(0)}{|\nabla_{V_1}u(0)|}.$$

If the second case occurs, then for $\epsilon\to 0$, and for $l_1=2,\dots,h_1$ we get

(55) $$x_\epsilon^{(1,1)}X_{l_1}u(0)\sim X_1u(0)x_\epsilon^{(1,l_1)}.$$

Thus

(56) $$x_\epsilon^{(1,l_1)}\sim\frac{X_{l_1}u(0)}{X_1u(0)}x_\epsilon^{(1,1)},$$

squaring and summing we get

$$\epsilon^2\sim\|x^{(1)}\|^2\sim(x_\epsilon^{(1,1)})^2\left(1+\sum_{j=2}^{h_1}\frac{X_{l_1}u(0)^2}{X_1u(0)^2}\right)\sim(x_\epsilon^{(1,1)})^2\frac{|\nabla_{V_1}u(0)|^2}{X_1u(0)^2},$$

that is

$$x_\epsilon^{(1,1)}\sim\epsilon\frac{X_1u(0)}{|\nabla_{V_1}u(0)|}$$

and from (56) we get

$$x_\epsilon^{(1,l_1)}\sim\epsilon\frac{X_{l_1}u(0)}{|\nabla_{V_1}u(0)|}.$$

We just need to justify the sign of the limit. Using (19) we get

$$u(P_{\epsilon,M})=u(0)+\left\langle\nabla_{V_1}u(0),x_\epsilon^{(1)}\right\rangle+\left\langle\nabla_{V_2}u(0),x_\epsilon^{(2)}\right\rangle+\frac{1}{2}\left\langle D^{2*}_{V_1}u(0)x_\epsilon^{(1)},x_\epsilon^{(1)}\right\rangle+o(\epsilon^2)$$

and, dividing by $\epsilon>0$

$$0\le\frac{u(P_{M,\epsilon})-u(0)}{\epsilon}=\left\langle\nabla_{V_1}u(0),\frac{x_\epsilon^{(1)}}{\epsilon}\right\rangle+\left\langle\nabla_{V_2}u(0),\frac{x_\epsilon^{(2)}}{\epsilon}\right\rangle+\frac{1}{2}\left\langle D^{2*}_{V_1}u(0)\frac{x_\epsilon^{(1)}}{\epsilon},x_\epsilon^{(1)}\right\rangle+o(\epsilon)$$

letting $\epsilon \to 0^+$ and making use of (44), we conclude

$$\lim_{\epsilon\to 0^+} \frac{x_\epsilon^{(1)}}{\epsilon} = \frac{\nabla_{V_1} u(0)}{|\nabla_{V_1} u(0)|}.$$

□

We complete the previous proof by proving the claim that is contained in the following result.

**Lemma 3.1.** *Let $u \in C^\infty(\Omega)$ with $|\nabla_{V_k} u(0)| \neq 0$. For every $1 \leq p \leq k-1$ and $h_{p-1} < l_p \leq h_p$ it holds*

$$\lim_{\epsilon\to 0^+} \frac{\|x_\epsilon^{(p)}\|^{2(k!/p)-2} x_\epsilon^{(p,l_p)}}{\|x_\epsilon^{(k)}\|^{2(k-1)!-1}} = \frac{p}{k} \frac{X_{l_p} u(0)}{|\nabla_{V_k} u(0)|}. \tag{57}$$

*Proof.* We start observing that, by (25)

$$X_{l_{k-1}} u(P_{M,\epsilon}) = \lambda_\epsilon \frac{2k!}{(k-1)} \|x_\epsilon^{(k-1)}\|^{\frac{2k!}{k-1}} x_\epsilon^{(k-1,l_{k-1})} + \frac{2}{k-1} \sum_{h_{k-1}<i\leq h_k} a_i^{(l_{k-1})}(P_{M,\epsilon}) X_{l_k} u(P_{M,\epsilon}) \tag{58}$$

letting $\epsilon \to 0^+$ and recalling that for every $i$ and $l_{k-1}$, $a_i^{(l_{k-1})}(0) = 0$ we get that for every $h_{k-2} < l_{k-1} \leq h_{k-1}$

$$X_{l_{k-1}} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{(k-1)} \|x_\epsilon^{(k-1)}\|^{\frac{2k!}{k-1}} x_\epsilon^{(k-1,l_{k-1})}. \tag{59}$$

Taking into account (59) and proceeding as before, we get that for every $h_{k-3} < l_{k-2} \leq h_{k-2}$

$$X_{l_{k-2}} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{(k-2)} \|x_\epsilon^{(k-2)}\|^{\frac{2k!}{k-2}} x_\epsilon^{(k-2,l_{k-2})}. \tag{60}$$

A simple induction argument ensures that for every $1 \leq p \leq k-1$ and $h_{p-1} < l_p \leq h_p$ we have

$$X_{l_p} u(P_{M,\epsilon}) \sim \lambda_\epsilon \frac{2k!}{p} \|x_\epsilon^{(k-2)}\|^{\frac{2k!}{p}} x_\epsilon^{(p,l_p)}, \tag{61}$$

the thesis follows recalling that by (25) and $|\nabla_{V_k} u(0)| \neq 0$

$$|\lambda_\epsilon| = \frac{|\nabla_{V_k} u(P_{M,\epsilon})|}{2(k-1)!\|x_\epsilon^{(k)}\|^{2(k-1)!-1}} \tag{62}$$

□

*Remark* 3.2. Notice that when $\mathbb{G}$ is the Heisenberg group, Lemma 1.6 provides an alternative proof of Lemma 3.2 in [9].

*Remark* 3.3. In order to better understand the ideas behind the proofs of Lemma 1.6 and 3.1, we suggest to the interested reader to have a look to the Appendix, where we prove the results in a concrete case, namely in the Engel group.

## 4. Approximated mean value formulas for sublaplacians in Carnot groups

**Lemma 4.1.** *Let $u$ be a $C^2$ function in a homogenous Carnot group of step $k$. Then,*

$$\fint_{B_\epsilon(P_0)} u(P)dP = u(P_0) + C\Delta_{\mathbb{G}}u(P_0)\epsilon^2 + o(\epsilon^2) \quad \text{as } \epsilon \to 0^+, \tag{63}$$

*where*

$$C := \frac{1}{2h_1 \mid B_1(0) \mid} \int_{B_1(0)} \|x^{(1)}\|^2 dx^{(1)} dx^{(2)} dx^{(3)} \cdots dx^{(k)}. \tag{64}$$

*and $h_1$ is as in Definition 2.1.*

*Proof.* Without loss of generality we can assume $P_0 = 0$. Using (19) we get

$$\begin{aligned}
&\fint_{B_\epsilon(0)} u(P)dP = \fint_{B_\epsilon(0)} (u(0) + \langle \nabla_{V_1} u(0), x^{(1)} \rangle_{\mathbb{R}^{h_1}})dP \\
&+ \fint_{B_\epsilon(0)} \langle \nabla_{V_2} u(0), x^{(2)} \rangle_{\mathbb{R}^{h_2-h_1}} dP + \fint_{B_\epsilon(0)} \frac{1}{2} \left( \langle D^{2*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} + o(\|P\|^2) \right) dP \\
&= u(0) + \fint_{B_\epsilon(0)} \frac{1}{2} \left( \langle D^{2*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} + o(\|P\|^2) \right) dP \\
&= u(0) + \frac{1}{2} \fint_{B_\epsilon(0)} \left( \langle D^{2*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} \right) dP + o(\epsilon^2).
\end{aligned} \tag{65}$$

Of course, in the cancellation of the first order term a key role is played by the symmetry of the gauge ball.
We claim that for every $\epsilon > 0$,

$$\frac{1}{2} \fint_{B_\epsilon(0)} \left( \langle D^{2*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} \right) dP = C(Q)\Delta_{\mathbb{G}} u(0)\epsilon^2, \tag{66}$$

where $C(Q)$ is as in (64). Indeed, denoted by $S(\epsilon, x^{(2)}, \dots, x^{(k)}) := \{x^{(1)} \mid \|x^{(1)}\|^{2k!} \leq \epsilon^{2k!} - \sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}}\}$ we get

$$\begin{aligned}
&\frac{1}{2}\fint_{B_\epsilon(0)} \left(\langle D^{2*}_{V_1}u(0)x^{(1)}, x^{(1)}\rangle_{\mathbb{R}^{h_1}}\right) dP \\
&= \frac{1}{2 \mid B_\epsilon(0) \mid}\int_{\{\sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}} \leq \epsilon^{2k!}\}} \left(\int_{S(\epsilon, x^{(2)}, \dots, x^{(k)})} \langle D^{2*}_{V_1}u(0)x^{(1)}, x^{(1)}\rangle_{\mathbb{R}^{h_1}} dx^{(1)}\right) dx^{(2)}dx^{(3)}\cdots dx^{(k)} \\
&= \sum_{i,j=1}^{h_1} \frac{X_iX_ju(0) + X_jX_iu(0)}{4 \mid B_\epsilon(0) \mid}\int_{\{\sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}} \leq \epsilon^{2k!}\}} \left(\int_{S(\epsilon, x^{(2)}, \dots, x^{(k)})} x_i^{(1)}x_j^{(1)} dx^{(1)}\right) dx^{(2)}dx^{(3)}\cdots dx^{(k)} \\
&= \sum_{i=1}^{h_1} \frac{X_iX_iu(0)}{2 \mid B_\epsilon(0) \mid}\int_{\{\sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}} \leq \epsilon^{2k!}\}} \left(\int_{S(\epsilon, x^{(2)}, \dots, x^{(k)})} (x_i^{(1)})^2 dx^{(1)}\right) dx^{(2)}dx^{(3)}\cdots dx^{(k)} \\
&= \frac{\Delta_{\mathbb{G}}u(0)}{2h_1 \mid B_\epsilon(0) \mid}\int_{\{\sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}} \leq \epsilon^{2k!}\}} \left(\int_{S(\epsilon, x^{(2)}, \dots, x^{(k)})} \|x_1^{(1)}\|^2 dx^{(1)}\right) dx^{(2)}dx^{(3)}\cdots dx^{(k)}.
\end{aligned}$$

Moreover,

(67)

$$\begin{aligned}
&\frac{1}{2h_1 \mid B_\epsilon(0) \mid}\int_{\{\sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}} \leq \epsilon^{2k!}\}} \left(\int_{\{\|x^{(1)}\|^{2k!} \leq \epsilon^{2k!} - \sum_{j=2}^k \|x^{(j)}\|^{\frac{2k!}{j}}\}} \|x^{(1)}\|^2 dx^{(1)}\right) dx^{(2)}dx^{(3)}\cdots dx^{(k)} \\
&= \frac{1}{2h_1 \mid B_\epsilon(0) \mid}\int_{B_\epsilon(0)} \|x^{(1)}\|^2 dx^{(1)}dx^{(2)}dx^{(3)}\cdots dx^{(k)} \\
&= \frac{1}{2h_1\epsilon^Q \mid B_1(0) \mid}\epsilon^{Q+2}\int_{B_1(0)} \|x^{(1)}\|^2 dx^{(1)}dx^{(2)}dx^{(3)}\cdots dx^{(k)} \\
&= \frac{\epsilon^2}{2h_1 \mid B_1(0) \mid}\int_{B_1(0)} \|x^{(1)}\|^2 dx^{(1)}dx^{(2)}dx^{(3)}\cdots dx^{(k)} = C(Q)\epsilon^2
\end{aligned}$$

□

*Remark* 4.2. The constant $C(Q) < \frac{1}{2h_1}$. Indeed

$$\begin{aligned}
C(Q) = \frac{1}{2h_1 \mid B_1(0) \mid}\int_{B_1(0)} \|x^{(1)}\|^2 dx^{(1)}dx^{(2)}dx^{(3)}\cdots dx^{(k)} < \\
\frac{1}{2h_1 \mid B_1(0) \mid}\int_{B_1(0)} dx^{(1)}dx^{(2)}dx^{(3)}\cdots dx^{(k)} = \frac{1}{2h_1}.
\end{aligned}$$

This estimate is rough, however it is sufficient to conclude that whenever $p > 1$ it holds

$$2(p-2)C(Q) + 1 \neq 0.$$

## 5. Proof of Theorem 1.1

Let $P \in \Omega$ and $\phi$ be a $C^2$-function defined in a neighborhood of $P$. We denote by $P_{M,\epsilon} \in \overline{B(P,\epsilon)}$ and $P_{m,\epsilon} \in \overline{B(P,\epsilon)}$ the points defined by

$$\phi(P_{M,\epsilon}) = \max_{\overline{B(P,\epsilon)}} \phi, \text{ and } \phi(P_{m,\epsilon}) = \min_{\overline{B(P,\epsilon)}} \phi.$$

Moreover we denote for simplicity $P = (x^{(1)}, y)$ where $y^V := (x^{(2)}, \cdots, x^{(k)})$ hence

$$P_{M,\epsilon} = (x^{(1)}_{M,\epsilon}, y^V_{M,\epsilon}), \quad P_{m,\epsilon} = (x^{(1)}_{m,\epsilon}, y^V_{m,\epsilon}).$$

**Lemma 5.1.** *Let $q \in (1, +\infty)$and $\phi$ be a $C^2$-function in a domain $\Omega \subset \mathbb{G}$. Let $C$, $\alpha$, $\beta$ be as in Theorem 1.1and in order to emphasize the dependence on the homogeneous dimension of the constant $C$, we shall write $C = C(Q)$. Consider the vectors*

$$(h^{(1)}_{M,\epsilon}, l^V_{M,\epsilon}) = \left( \frac{x^{(1)}_{M,\epsilon} - x^{(1)}}{\epsilon}, \frac{y^V_{M,\epsilon} - y^V}{\epsilon} \right)$$

*and*

$$(h^{(1)}_{m,\epsilon}, l^V_{m,\epsilon}) = \left( \frac{x^{(1)}_{m,\epsilon} - x^{(1)}}{\epsilon}, \frac{y^V_{m,\epsilon} - y^V}{\epsilon} \right).$$

*The following expansions hold near every $P \in \Omega$.*
*If $q \geq 2$, then*

$$\beta\, C(Q)\epsilon^2 \Big[ \Delta_{\mathbb{G}}\phi(P) + (q-2)\langle D^{2,*}_{\mathbb{G}}\phi(P) h^{(1)}_{M,\epsilon}, h^{(1)}_{M,\epsilon}\rangle \Big] \geq \beta \fint_{B(P,\epsilon)} \phi(x^{(1)}, \dots x^{(k)}) dx^{(1)} \dots dx^{(k)} + \frac{\alpha}{2} \left( \min_{\overline{B(P,\epsilon)}} \phi + \max_{\overline{B(P,\epsilon)}} \phi \right) - \phi(P) + o(\epsilon^2),$$

*as $\epsilon \to 0^+$ and, if $1 < q < 2$, the same inequality holds replacing $h^{(1)}_{M,\epsilon}$ with $h^{(1)}_{m,\epsilon}$. Moreover, if $q \geq 2$ then*

$$\beta\, C(Q)\epsilon^2 \Big[ \Delta_{\mathbb{G}}\phi(P) + (q-2)\langle D^{2,*}_{\mathbb{G}}\phi(P) h^{(1)}_{m,\epsilon}, h^{(1)}_{m,\epsilon}\rangle \Big] \leq \beta \fint_{B(P,\epsilon)} \phi(x^{(1)}, \dots x^{(k)}) dx^{(1)} \dots dx^{(k)} + \frac{\alpha}{2} \left( \min_{\overline{B(P,\epsilon)}} \phi + \max_{\overline{B(P,\epsilon)}} \phi \right) - \phi(P) + o(\epsilon^2),$$

*$\epsilon \to 0^+$ and, if $1 < q < 2$, the same inequality holds replacing $h^{(1)}_{m,\epsilon}$ with $h^{(1)}_{M,\epsilon}$.*

*Proof.* Just moving $P$ to the origin of $\mathbb{G}$ by a left translation of the group we can assume that $P = 0$. Applying (19) we obtain

$$\phi(P_{M,\epsilon}) = \phi(0) + \langle \nabla_{V_1}\phi(0), x^{(1)}_{M,\epsilon}\rangle_{\mathbb{R}^{h_1}} + \frac{1}{2}\langle D^{2,*}_{V_1}\phi(0) x^{(1)}_{M,\epsilon}, x^{(1)}_{M,\epsilon}\rangle + \langle \nabla_{V_2}\phi(0), x^{(2)}\rangle_{\mathbb{R}^{h_2 - h_1}} + o(\epsilon^2).$$

and

$$\phi(-P_{M,\epsilon}) = \phi(0) - \langle \nabla_{V_1}\phi(0), x^{(1)}_{M,\epsilon}\rangle_{\mathbb{R}^{h_1}} + \frac{1}{2}\langle D^{2,*}_{V_1}\phi(0)x^{(1)}_{M,\epsilon}, x^{(1)}_{M,\epsilon}\rangle$$
$$-\langle \nabla_{V_2}\phi(0), x^{(2)}\rangle_{\mathbb{R}^{h_2-h_1}} + o(\epsilon^2).$$

The proof is by now standard. Indeed, adding the last two inequalities we get

$$\phi(P_{M,\epsilon}) + \phi(-P_{M,\epsilon}) = 2\phi(0) + \langle D^{2,*}_{V_1}\phi(0)x^{(1)}_{M,\epsilon}, x^{(1)}_{M,\epsilon}\rangle_{\mathbb{R}^{h_2-h_1}} + o(\epsilon^2).$$

Using the definition of $P_{M,\epsilon}$ if follows that

$$\begin{aligned}\max_{B(0,\epsilon)}\phi + \min_{B(0,\epsilon)}\phi &\le \phi(P_{M,\epsilon}) + \phi(-P_{M,\epsilon}) \\ &= 2\phi(0) + \langle D^{2*}_{V_1}\phi(0)x^{(1)}_{M,\epsilon}, x^{(1)}_{M,\epsilon}\rangle_{\mathbb{R}^{h_2-h_1}} + o(\epsilon^2),\end{aligned} \tag{68}$$

which implies the inequality

$$\phi(0) + \frac{1}{2}\langle D^{2*}_{V_1}\phi(0)x^{(1)}_{M,\epsilon}, x^{(1)}_{M,\epsilon}\rangle_{\mathbb{R}^{h_2-h_1}} \ge \frac{1}{2}\left(\max_{B(0,\epsilon)}\phi + \min_{B(0,\epsilon)}\phi\right) + o(\epsilon^2). \tag{69}$$

Multiplying this relation by $\alpha$, the expansion in Lemma 4.1 by $\beta$, adding and using the fact that $\alpha+\beta = 1$ we obtain the inequality

$$\begin{aligned}&\phi(0) + C(Q)\,\beta\,\Delta_{\mathbb{G}}\phi(0)\epsilon^2 + \frac{\alpha}{2}\langle D^{2*}_{V_1}\phi(0)x^{(1)}_{M,\epsilon}, x^{(1)}_{M,\epsilon}\rangle_{\mathbb{R}^{h_2-h_1}} \\ &\quad\ge \beta \fint_{B(0,\epsilon)}\phi(x^{(1)}, x^{(2)}, \ldots x^{(k)})dx^{(1)}dx^{(2)}, \ldots dx^{(k)} + \frac{\alpha}{2}\left(\min_{B(0,\epsilon)}\phi + \max_{B(0,\epsilon)}\phi\right) + o(\epsilon^2).\end{aligned}$$

in the case $\alpha > 0$. In particular let $\alpha$ and $\beta$ be such that

$$\frac{\alpha}{2C(Q)\beta} = p - 2,$$

Hence, using the requirement $\alpha + \beta = 1$, we get

$$\frac{1-\beta}{2C(Q)\beta} = q - 2,$$

giving

$$\alpha = \frac{2(q-2)C(Q)}{2(q-2)C(Q)+1} \text{ and } \beta = \frac{1}{2(q-2)C(Q)+1}.$$

Hence

$$\begin{aligned}&\frac{\epsilon^2 C(Q)}{2(q-2)C(Q)+1}\left(\Delta_{\mathbb{G}}\phi(0) + (q-2)\langle D^{2*}_{V_1}\phi(0)\frac{x^{(1)}_{M,\epsilon}}{\epsilon}, \frac{x^{(1)}_{M,\epsilon}}{\epsilon}\rangle\right) + o(1) \\ &\ge \frac{1}{2(q-2)C(Q)+1}\fint_{B(0,\epsilon)}\phi(x^{(1)}, \ldots, x^{(k)})dx^{(1)}\ldots dx^{(k)} + \\ &+ \frac{(q-2)C(Q)}{2(q-2)C(Q)+1}\left[\min_{B(0,\epsilon)}\phi + \max_{B(0,\epsilon)}\phi\right] - \phi(0).\end{aligned}$$

This computation works for $\alpha \geq 0$; that is for every $q \geq 2$.
When $\alpha < 0$ the procedure is the same but the sign of the inequality is reversed, that is

$$\frac{\epsilon^2 C(Q)}{2(q-2)C(Q)+1}\left(\Delta_{\mathbb{G}}\phi(0)+(p-2)\langle D^{2*}_{V_1}\phi(0)\frac{x^{(1)}_{M,\epsilon}}{\epsilon},\frac{x^{(1)}_{M,\epsilon}}{\epsilon}\rangle\right)+o(1)$$
$$\leq\left(\frac{1}{2(q-2)C(n)+1}\fint_{B(0,\epsilon)}\phi+\frac{(q-2)C(n)}{2(q-2)C(n)+1}\left[\min_{B(0,\epsilon)}\phi+\max_{B(0,\epsilon)}\phi\right]-\phi(0)\right).$$

and $1 < q < 2$.
In the same way, using the inequality coming from the minimum we get

$$\phi(0)+\frac{1}{2}\langle D^{2*}_{V_1}\phi(0)x^{(1)}_{m,\epsilon},x^{(1)}_{m,\epsilon}\rangle+o(\epsilon^2)\leq\frac{1}{2}\left(\min_{B(0,\epsilon)}\phi+\max_{B(0,\epsilon)}\phi\right).$$

and

$$\frac{\epsilon^2 C(Q)}{2(q-2)C(Q)+1}\left(\Delta_{\mathbb{G}}\phi(0)+(q-2)\langle D^{2*}_{V_1}\phi(0)x^{(1)}_{m,\epsilon},x^{(1)}_{m,\epsilon}\rangle\right)+o(\epsilon^2)$$
$$\leq\left(\frac{1}{2(q-2)C(Q)+1}\fint_{B(0,\epsilon)}\phi+\frac{(q-2)C(Q)}{2(q-2)C(Q)+1}\left[\min_{B(0,\epsilon)}\phi+\max_{B(0,\epsilon)}\phi\right]-\phi(0)\right),$$

for $q \geq 2$ and

$$\frac{\epsilon^2 C(Q)}{2(q-2)C(Q)+1}\left(\Delta_{\mathbb{G}}\phi(0)+(q-2)\langle D^{2*}_{V_1}\phi(0)\frac{x^{(1)}_{m,\epsilon}}{\epsilon},\frac{x^{(1)}_{m,\epsilon}}{\epsilon}\rangle\right)+o(\epsilon^2)$$
$$\geq\frac{1}{2(q-2)C(Q)+1}\fint_{B(0,\epsilon)}\phi(x^{(1)},\dots,x^{(k)})dx^{(1)}\dots dx^{(k)}+ \tag{70}$$
$$+\frac{(q-2)C(Q)}{2(q-2)C(Q)+1}\left[\min_{B(0,\epsilon)}\phi+\max_{B(0,\epsilon)}\phi\right]-\phi(0),$$

for $1 < q < 2$.

□

*Proof of Theorem 1.1.* Assume that $u$ satisfies the asymptotic expansion in the viscosity sense as in Definition 1.4. Let $\phi$ be a smooth function such that $u-\phi$ has a strict maximum at $P$ and $\nabla_{V_1}\phi(P)\neq 0$. Then it follows, by condition (ii) in Definition 1.4,

$$\frac{1}{2(q-2)C(Q)+1}\fint_{B(P,\epsilon)}\phi+\frac{(q-2)C(Q)}{2(q-2)C(Q)+1}\left[\min_{B(P,\epsilon)}\phi+\max_{B(P,\epsilon)}\phi\right]-\phi(P)\geq 0,$$

and recalling (70) we conclude that

$$\frac{\epsilon^2 C(Q)}{2(q-2)C(Q)+1}\left(\Delta_{\mathbb{G}}\phi(P)+(q-2)\langle D^{2*}_{V_1}\phi(P)\frac{x^{(1)}_{m,\epsilon}}{\epsilon},\frac{x^{(1)}_{m,\epsilon}}{\epsilon}\rangle\right)\geq o(\epsilon^2).$$

Dividing by $\epsilon^2$, using Lemma 1.6 and letting $\epsilon \to 0$ we get

$$\Delta_{\mathbb{G}}\phi(P) + (q-2)\langle D^{2*}_{\mathbb{G}}\phi(P)\frac{\nabla_{V_1}\phi(P)}{|\nabla_{V_1}\phi(P)|}, \frac{\nabla_{V_1}\phi(P)}{|\nabla_{V_1}\phi(P)|}\rangle \geq 0,$$

that is $u$ is a viscosity subsolution of $\Delta_{\mathbb{G},q}u = 0$.

Let us prove the converse implication. Assume that $u$ is a viscosity solution. In particular $u$ is a supersolution so that for every $C^2$ test function $\phi$ such that $u-\phi$ is a strict minimum at the point $P \in \Omega$ with $\nabla_{V_1}\phi(P) \neq 0$ we have

$$-(q-2)\Delta_{\mathbb{G},\infty}\phi(P) - \Delta_{\mathbb{G}}\phi(P) \geq 0.$$

Recalling inequality (70)

$$\begin{aligned} 0 &\geq \frac{\epsilon^2 C(Q)}{2(q-2)C(Q)+1}\left(\Delta_{\mathbb{G}}\phi(P) + (q-2)\langle D^{2*}_{V_1}\phi(P)\frac{x^{(1)}_{m,\epsilon}}{\epsilon}, \frac{x^{(1)}_{m,\epsilon}}{\epsilon}\rangle\right) \\ &\geq \left(\frac{1}{2(q-2)C(Q)+1}\fint_{B(0,\epsilon)}\phi + \frac{(q-2)C(Q)}{2(q-2)C(Q)+1}\left[\min_{B(P,\epsilon)}\phi + \max_{B(P,\epsilon)}\phi\right] - \phi(P)\right) + o(\epsilon^2), \end{aligned}$$

and keeping in mind that

$$\lim_{\epsilon\to 0}\frac{x^{(1)}_{m,\epsilon}}{\epsilon} = -\frac{\nabla_{V_1}\phi(P)}{|\nabla_{V_1}\phi(P)|},$$

we get

$$\frac{1}{2(q-2)C(Q)+1}\fint_{B(P,\epsilon)}\phi + \frac{(q-2)C(Q)}{2(q-2)C(Q)+1}\left[\min_{B(P,\epsilon)}\phi + \max_{B(P,\epsilon)}\phi\right] - \phi(P) + o(\epsilon^2) \leq 0,$$

which is condition (i) in the Definition 1.4. An analogous computation gives the proof of condition (ii).

□

## 6. Appendix

Aim of this Appendix is to prove Lemma 1.6 and 3.1 when $\mathbb{G}$ is the Engel group, see Example 2.6 for the definition.

By direct computations we get:

$$\left\{\begin{array}{rcl} X_1u(P_{M,\epsilon}) &=& \lambda_\epsilon\Big(12x_{1,\epsilon}(x^2_{1,\epsilon}+x^2_{2,\epsilon})^5 - 3x_{2,\epsilon}x^5_{3,\epsilon} - 2\Big(x_{3,\epsilon} + \frac{x_{2,\epsilon}}{6}(x_{1,\epsilon}+x_{2,\epsilon})\Big)x^3_{4,\epsilon}\Big) \\ X_2u(P_{M,\epsilon}) &=& \lambda_\epsilon\Big(12x_{2,\epsilon}(x^2_{1,\epsilon}+x^2_{2,\epsilon})^5 + 3x_{1,\epsilon}x^5_{3,\epsilon} - 2\Big(x_{3,\epsilon} - \frac{x_{1,\epsilon}}{6}(x_{1,\epsilon}+x_{2,\epsilon})\Big)x^3_{4,\epsilon}\Big) \\ X_3u(P_{M,\epsilon}) &=& \lambda_\epsilon\Big(6x^5_{3,\epsilon} + 2(x_{1,\epsilon}+x_{2,\epsilon})x^3_{4,\epsilon}\Big) \\ X_4u(P_{M,\epsilon}) &=& 4\lambda_\epsilon x^3_{4,\epsilon} \\ |P_{M,\epsilon}|^{12}_{\mathbb{E}} &=& \epsilon^{12}. \end{array}\right. \tag{71}$$

Therefore, if $X_4u(P_{M,\epsilon}) \neq 0$

$$\lambda_\epsilon = \frac{X_4u(P_{M,\epsilon})}{4x^3_{4,\epsilon}}$$

and
(72)

$$\begin{cases} X_1u(P_{M,\epsilon}) &= \frac{X_4u(P_{M,\epsilon})}{4x_{4,\epsilon}^3}\Big(12x_{1,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5-3x_{2,\epsilon}x_{3,\epsilon}^5\Big)-\frac{X_4u(P_{M,\epsilon})}{2}\Big(x_{3,\epsilon}+\frac{x_{2,\epsilon}}{6}(x_{1,\epsilon}+x_{2,\epsilon})\Big)\\ X_2u(P_{M,\epsilon}) &= \frac{X_4u(P_{M,\epsilon})}{4x_{4,\epsilon}^3}\Big(12x_{2,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5+3x_{1,\epsilon}x_{3,\epsilon}^5\Big)-\frac{X_4u(P_{M,\epsilon})}{2}\Big(x_{3,\epsilon}-\frac{x_{1,\epsilon}}{6}(x_{1,\epsilon}+x_{2,\epsilon})\Big)\Big)\\ X_3u(P_{M,\epsilon}) &= \frac{3X_4u(P_{M,\epsilon})}{2x_{4,\epsilon}^3}x_{3,\epsilon}^5+\frac{X_4u(P_{M,\epsilon})}{2}(x_{1,\epsilon}+x_{2,\epsilon})\\ X_4u(P_{M,\epsilon}) &= 4\lambda_\epsilon x_{4,\epsilon}^3\\ |P_{M,\epsilon}|_{\mathbb{E}}^{12} &= \epsilon^{12}. \end{cases}$$

Letting $\epsilon\to 0^+$ we get

(73)
$$\begin{aligned} X_1u(0) &= \lim_{\epsilon\to 0^+}\frac{X_4u(P_{M,\epsilon})}{4x_{4,\epsilon}^3}\Big(12x_{1,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5-3x_{2,\epsilon}x_{3,\epsilon}^5\Big)\\ X_2u(0) &= \lim_{\epsilon\to 0^+}\frac{X_4u(P_{M,\epsilon})}{4x_{4,\epsilon}^3}\Big(12x_{2,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5+3x_{1,\epsilon}x_{3,\epsilon}^5\Big)\\ X_3u(0) &= \lim_{\epsilon\to 0^+}\frac{3X_4u(P_{M,\epsilon})}{2x_{4,\epsilon}^3}x_{3,\epsilon}^5. \end{aligned}$$

hence, if $X_4u(0)\neq 0$,

(74)
$$\lim_{\epsilon\to 0^+}\frac{x_{3,\epsilon}^5}{x_{4,\epsilon}^3}=\frac{2X_3u(0)}{3X_4u(0)}.$$

Using (74) in (73) we obtain

(75)
$$\lim_{\epsilon\to 0^+}\frac{3x_{1,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5}{x_{4,\epsilon}^3}=\frac{X_1u(0)}{X_4u(0)}$$

(76)
$$\lim_{\epsilon\to 0^+}\frac{3x_{2,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5}{x_{4,\epsilon}^3}=\frac{X_2u(0)}{X_4u(0)}.$$

Squaring, adding and denoting by $\rho_\epsilon := (x_{1,\epsilon}^2+x_{2,\epsilon}^2)^{1/2}$ we get

(77)
$$\lim_{\epsilon\to 0^+}\frac{9\rho_\epsilon^{22}}{x_{4,\epsilon}^6}=\frac{|\nabla_{V_1}u|^2(0)}{(X_4u(0))^2},$$

that in particular it is equivalent to say

(78)
$$\rho_\epsilon\sim\Big(\frac{|\nabla_{V_1}u(0)|}{3|X_4u(0)|}\Big)^{\frac{1}{11}}|x_{4,\epsilon}|^{\frac{3}{11}}.$$

Since $P_{M,\epsilon}\in\partial B(0,\epsilon)$ then from (72) we get

(79)
$$\rho_\epsilon^{12}+x_{3,\epsilon}^6+x_{4,\epsilon}^4=\epsilon^{12}$$

therefore, using (78) in (79) we obtain

(80)
$$\Big(\frac{|\nabla_{V_1}u(0)|}{3|X_4u(0)|}\Big)^{\frac{12}{11}}|x_4|^{\frac{36}{11}}+x_{3,\epsilon}^6+x_{4,\epsilon}^4\sim\epsilon^{12}.$$

Now, using (74) we obtain

$$X_3u(0) \sim \frac{3X_4u(P_{M,\epsilon})}{2x_{4,\epsilon}^3}x_{3,\epsilon}^5$$

that is

$$x_{4,\epsilon}^{\frac{3}{5}}\Big(\frac{2X_3u(0)}{3X_4u(0)}\Big)^{\frac{1}{5}} \sim x_{3,\epsilon}. \tag{81}$$

Hence, using the previous relation in (80) we conclude

$$\Big(\frac{|\nabla_{V_1}u(0)|}{3|X_4u(0)|}\Big)^{\frac{12}{11}}|x_4|^{\frac{36}{11}} + \Big(\frac{2X_3u(0)}{3X_4u(0)}\Big)^{\frac{6}{5}}|x_{4,\epsilon}|^{\frac{18}{5}} + x_{4,\epsilon}^4 \sim \epsilon^{12}.$$

That implies

$$\Big(\frac{|\nabla_{V_1}u(0)|}{3|X_4u(0)|}\Big)^{\frac{12}{11}}|x_{4,\epsilon}|^{\frac{36}{11}} \sim \epsilon^{12}, \tag{82}$$

and hence

$$|x_{4,\epsilon}| \sim \epsilon^{\frac{11}{3}}\Big(\frac{3|X_4u(0)|}{|\nabla_{V_1}u(0)|}\Big)^{\frac{1}{3}}. \tag{83}$$

Hence putting (83) in (81) we get

$$|x_{3,\epsilon}| \sim 2^{\frac{1}{5}}\epsilon^{\frac{11}{5}}\Big(\frac{|X_4u(0)|}{|\nabla_{V_1}u(0)|}\Big)^{\frac{1}{5}}\Big(\frac{X_3u(0)}{X_4u(0)}\Big)^{\frac{1}{5}}$$

and, above all, inserting (82) in (78) we conclude

$$\rho_\epsilon \sim \epsilon.$$

From (75) and (76) and taking into account (83), it follows

$$\lim_{\epsilon\to 0^+}\frac{x_{1,\epsilon}}{\epsilon} = \lim_{\epsilon\to 0^+}\frac{3x_{1,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5}{x_{4,\epsilon}^3} = \pm\frac{X_1u(0)}{|\nabla_{V_1}u(0)|} \tag{84}$$

$$\lim_{\epsilon\to 0^+}\frac{x_{2,\epsilon}}{\epsilon} = \lim_{\epsilon\to 0^+}\frac{3x_{2,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5}{x_{4,\epsilon}^3} = \pm\frac{X_2u(0)}{|\nabla_{V_1}u(0)|}. \tag{85}$$

Suppose now that $X_4u(P_{M,\epsilon}) \to 0$. Then

$$X_3u(P_{M,\epsilon}) \sim 6\lambda_\epsilon x_{3,\epsilon}^5$$

and

$$X_2u(P_{M,\epsilon}) \sim \lambda_\epsilon\Big(12x_{2,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5 + 3x_{1,\epsilon}x_{3,\epsilon}^5\Big) \sim 12\lambda_\epsilon x_{2,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5, \tag{86}$$

analogously

$$X_1u(P_{M,\epsilon}) \sim \lambda_\epsilon\Big(12x_{1,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5 - 3x_{2,\epsilon}x_{3,\epsilon}^5\Big) \sim 12\lambda_\epsilon x_{1,\epsilon}(x_{1,\epsilon}^2+x_{2,\epsilon}^2)^5. \tag{87}$$

Since by hypothesis $\nabla_{V_1} u(0) \neq 0$, we may assume that

$$12\lambda_\epsilon x_{1,\epsilon}(x_{1,\epsilon}^2 + x_{2,\epsilon}^2)^5 \sim X_1 u(0) \neq 0.$$

Hence denoting as usual $\rho_\epsilon^2 = x_{1,\epsilon}^2 + x_{2,\epsilon}^2$ we get that

$$\lambda_\epsilon \sim \frac{X_1 u(0)}{12 x_{1,\epsilon} \rho_\epsilon^{10}}. \tag{88}$$

As a consequence

$$x_{1,\epsilon} X_2 u(P_{M,\epsilon}) \sim x_{2,\epsilon} X_1 u(0), \quad 2x_{1,\epsilon}\rho_\epsilon^{10} X_3 u(P_{M,\epsilon}) \sim X_1 u(0) x_{3,\epsilon}^5,$$
$$3x_{1,\epsilon}\rho_\epsilon^{10} X_4 u(P_{M,\epsilon}) \sim X_1 u(0) x_{4,\epsilon}^3,$$

recalling that

$$\rho_\epsilon^{12} + x_{3,\epsilon}^6 + x_{4,\epsilon}^4 = \epsilon^{12},$$

we deduce that

$$\rho_\epsilon^{12} + \Big|\frac{2x_{1,\epsilon}\rho^{10} X_3 u(P_{M,\epsilon})}{X_1 u(0)}\Big|^{\frac{6}{5}} + \Big|\frac{3x_{1,\epsilon}\rho^{10} X_4 u(P_{M,\epsilon})}{X_1 u(0)}\Big|^{\frac{4}{3}} \sim \epsilon^{12}$$

or equivalently

$$\rho_\epsilon^{12}\left(1 + \Big|\frac{2x_{1,\epsilon} X_3 u(P_{M,\epsilon})}{X_1 u(0)}\Big|^{\frac{6}{5}} + \rho_\epsilon^{\frac{4}{3}}\Big|\frac{3x_{1,\epsilon} X_4 u(P_{M,\epsilon})}{X_1 u(0)}\Big|^{\frac{4}{3}}\right) \sim \epsilon^{12}$$

that when $\epsilon \to 0$ implies $\rho_\epsilon \sim \epsilon$. Hence, by (88)

$$\lambda_\epsilon \sim \frac{X_1 u(0)}{12 x_{1,\epsilon}\epsilon^{10}}.$$

Moreover,

$$x_{1,\epsilon} X_2 u(P_{M,\epsilon}) \sim x_{2,\epsilon} X_1 u(0), \quad 2x_{1,\epsilon}\epsilon^{10} X_3 u(P_{M,\epsilon}) \sim X_1 u(0) x_{3,\epsilon}^5,$$
$$3x_{1,\epsilon}\epsilon^{10} X_4 u(P_{M,\epsilon}) \sim X_1 u(0) x_{4,\epsilon}^3. \tag{89}$$

We have two cases. Either there exists a subsequence such that $X_2 u(P_\epsilon) \to 0$ or $\lim_{\epsilon\to 0^+} X_2 u(P_\epsilon) = X_2 u(0) \neq 0$. In the first case it results that

$$x_{2,\epsilon} = o(x_{1,\epsilon}),$$

hence

$$\rho_\epsilon \sim x_{1,\epsilon} \sim \epsilon,$$
$$\lambda_\epsilon \sim \frac{X_1 u(0)}{12\epsilon^{11}},$$

and

$$2\epsilon^{11} X_3 u(P_{M,\epsilon}) \sim X_1 u(0) x_{3,\epsilon}^5, \quad 3\epsilon^{11} X_4 u(P_{M,\epsilon}) \sim X_1 u(0) x_{4,\epsilon}^3.$$

Recalling (87) and (86) we get

$$X_1 u(P_{M,\epsilon}) \sim X_1 u(0)\frac{x_{1,\epsilon}}{\epsilon}, \quad X_2 u(P_{M,\epsilon}) \sim \frac{x_{2,\epsilon}}{\epsilon} \tag{90}$$

therefore, up to the sign

$$\lim_{\epsilon\to 0} \frac{x_{1,\epsilon}}{\epsilon} = \frac{X_1 u(0)}{|\nabla_{V_1} u(0)|}$$

and

$$\lim_{\epsilon\to 0}\frac{x_{2,\epsilon}}{\epsilon}=X_2u(0)=0.$$

In the second case, namely $\lim_{\epsilon\to 0^+}X_2u(P_\epsilon)=X_2u(0)\neq 0$, we deduce from (89) that

(91)

$$x_{1,\epsilon}X_2u(0)\sim x_{2,\epsilon}X_1u(0),\quad 2x_{1,\epsilon}\epsilon^{10}X_3u(P_{M,\epsilon})\sim X_1u(0)x_{3,\epsilon}^5,\quad 3x_{1,\epsilon}\epsilon^{10}X_4u(P_{M,\epsilon})\sim X_1u(0)x_{4,\epsilon}^3,$$

that is

$$\epsilon^2\sim\rho_\epsilon^2\sim x_{1,\epsilon}^2\Big(1+\frac{X_2u(0)^2}{X_1u(0)^2}\Big)\sim x_{1,\epsilon}^2\frac{X_1u(0)^2+X_2u(0)^2}{X_1u(0)^2}.$$

Hence

$$x_{1,\epsilon}^2\sim\frac{X_1u(0)^2}{X_1u(0)^2+X_2u(0)^2}\epsilon^2$$

and as a consequence

$$x_{2,\epsilon}^2\sim\frac{X_2u(0)^2}{X_1u(0)^2+X_2u(0)^2}\epsilon^2.$$

Hence, up to the sign, we get

$$x_{1,\epsilon}\sim\frac{|\,X_1u(0)\,|}{|\nabla_{V_1}u(0)|}\epsilon,\quad x_{2,\epsilon}\sim\frac{|\,X_2u(0)\,|}{|\nabla_{V_1}u(0)|}\epsilon.$$

In order to justify the sign of the limit we proceed exactly as in Lemma 1.6 and this conclude the proof.

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