

A NONLINEAR LIOUVILLE THEOREM FOR FRACTIONAL EQUATIONS IN THE HEISENBERG GROUP

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ABSTRACT. We establish a Liouville-type theorem for a subcritical nonlinear problem, involving a fractional power of the sub-Laplacian in the Heisenberg group. To prove our result we will use the local realization of fractional CR covariant operators, which can be constructed as the Dirichlet-to-Neumann operator of a degenerate elliptic equation in the spirit of Caffarelli and Silvestre [9], as established in [16]. The main tools in our proof are the CR inversion and the moving plane method, applied to the solution of the lifted problem in the half-space $\mathbb{H}^n \times \mathbb{R}^+$.

1. INTRODUCTION AND MAIN RESULTS

In this paper we establish a Liouville-type result for the following fractional nonlinear problem in the Heisenberg group:

$$\mathcal{P}_{\frac{1}{2}}u = u^p \quad \text{in } \mathbb{H}^n. \quad (1.1)$$

Here $\mathcal{P}_{\frac{1}{2}}$ denotes a CR covariant operator of order $1/2$ in \mathbb{H}^n , whose principal symbols agree with the pure fractional power $1/2$ of the Heisenberg Laplacian $-\Delta_{\mathbb{H}}$. In [16] Frank, Gonzalez, Monticelli and one of the authors, study CR covariant operators of fractional orders on orientable and strictly pseudoconvex CR manifolds. In particular, they focus on the construction of such operators as the Dirichlet-to-Neumann map associated to a degenerate elliptic equation in the spirit of Caffarelli and Silvestre [9].

In this context, the Heisenberg group \mathbb{H}^n plays the same role as \mathbb{R}^n in conformal geometry, in the sense that, as shown by Folland and Stein in [14], \mathbb{H}^n approximates the pseudohermitian structure of a general orientable and strictly pseudoconvex CR manifolds (see also [26]).

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Given a Kähler-Einstein manifold \mathcal{X} , CR covariant operators of fractional order γ are pseudodifferential operators whose principal symbol agrees with the pure fractional powers of the CR sub-Laplacian on the boundary $\mathcal{M} = \partial\mathcal{X}$. They can be defined using scattering theory, as done in [12, 22, 21, 20]. In the particular case of the Heisenberg group, they are the intertwining operators on the CR sphere, which can be calculated using representation theory techniques (see [8]).

In [16], in order to construct fractional CR covariant operators in the specific case of the Heisenberg group, \mathbb{H}^n is identified with the boundary of the Siegel domain in \mathbb{R}^{2n+2} (see Section 2 for the precise definition) and it is crucial to use its underlying complex hyperbolic structure.

Another possible approach in the construction of fractional powers of the sub-Laplacian consists in using purely functional analytic tools as done by Ferrari and Franchi [15] who proved an extension result for fractional operators defined by using the spectral resolution of the sub-Laplacian in general Carnot groups, see also [13, 28]. The operators considered in [15] are different in nature from the ones in [16], they correspond to the pure fractional powers of the sub-Laplacian and do not enjoy the CR covariance property.

Since it will be of utmost importance in the sequel, we recall here the extension result proven in [16].

Theorem 1.1 (see Theorem 1.1 in [16]). *Let $\gamma \in (0, 1)$, $a = 1 - 2\gamma$. For each $u \in C^\infty(\mathbb{H}^n)$, there exists a unique solution $\mathcal{E}_\gamma u = U$ for the extension problem*

$$\begin{cases} \frac{\partial^2 U}{\partial \lambda^2} + \frac{a}{\lambda} \frac{\partial U}{\partial \lambda} + \lambda^2 \frac{\partial^2 U}{\partial t^2} + \frac{1}{2} \Delta_{\mathbb{H}} U = 0 & \text{in } \widehat{\mathbb{H}}_+^n := \mathbb{H}^n \times \mathbb{R}^+, \\ U = u & \text{on } \partial \widehat{\mathbb{H}}_+^n = \mathbb{H}^n \times \{\lambda = 0\}. \end{cases} \quad (1.2)$$

Moreover,

$$\mathcal{P}_\gamma u = -c_\gamma \lim_{\lambda \rightarrow 0} \lambda^a \frac{\partial U}{\partial \lambda},$$

where c_γ is a constant depending only on γ which precise value is given by

$$c_\gamma = \frac{\Gamma(\gamma)}{\gamma \Gamma(-\gamma)} \cdot 2^{2\gamma-1}.$$

In (1.2) $\Delta_{\mathbb{H}}$ denotes the sublaplacian in the Heisenberg group, which precise definition is given in Section 2 below. Observe that, differently from the extension result established in [15], here we have the additional term $\lambda^2 \frac{\partial^2 U}{\partial t^2}$ which appears when one considers CR fractional

sub-Laplacian. When $a = 1/2$ the equation in (1.2) satisfied by U becomes:

$$\frac{\partial^2 U}{\partial \lambda^2} + \lambda^2 \frac{\partial^2 U}{\partial t^2} + \frac{1}{2} \Delta_{\mathbb{H}} U = 0, \quad (1.3)$$

and we have

$$\mathcal{P}_{\frac{1}{2}} u = -c_{\frac{1}{2}} \lim_{\lambda \rightarrow 0} \frac{\partial U}{\partial \lambda}.$$

Replacing λ in (1.3) by $\sqrt{2}\lambda$, we will consider the operator

$$\mathcal{L} = \Delta_{\mathbb{H}} + \frac{\partial^2}{\partial \lambda^2} + 4\lambda^2 \frac{\partial^2}{\partial t^2}. \quad (1.4)$$

Our Liouville-type theorem is the analogue, for the fractional operator $\mathcal{P}_{\frac{1}{2}}$, of a result by Birindelli and Prajapat [4], for the sublaplacian $\Delta_{\mathbb{H}}$. In [4], the authors establish a nonexistence result for a class of positive solution of the equation

$$-\Delta_{\mathbb{H}} u = u^p, \quad (1.5)$$

for p subcritical (i.e. $0 < p < \frac{Q+2}{Q-2}$, where $Q = 2n + 2$ denotes the homogeneous dimension of \mathbb{H}^n). The technique they used is based on the moving plane method (which goes back to Alexandrov [1] and Serrin [27]), adapted to the Heisenberg group setting. This method requires two basic tools: the maximum principle and invariance under reflection with respect to a hyperplane. Since the operator $-\Delta_{\mathbb{H}}$ is not invariant under the usual reflection with respect to hyperplanes, Birindelli and Prajapat needed to introduce a new reflection, called H -reflection, under which $-\Delta_{\mathbb{H}}$ is invariant. Since it will be important in the sequel, we recall here the definition of H -reflection.

Definition 1.2. For any $\xi = (x, y, t) \in \mathbb{H}^n$, we consider the plane $T_{\mu} := \{\xi \in \mathbb{H}^n : t = \mu\}$. We define

$$\xi_{\mu} := (y, x, 2\mu - t),$$

to be the H -reflection of ξ with respect to the plane T_{μ} .

Due to the use of this reflection, the proof of the non existence result in [4] requires the solution u of (1.5) to be cylindrical, that is, $u(x, y, t) = u(r_0, t)$ must depend only on r_0 and t where $r_0 = (|x|^2 + |y|^2)^{\frac{1}{2}}$.

We can now state our main result, which is the analogue for the operator $\mathcal{P}_{\frac{1}{2}}$ of the Liouville result contained in [4].

Theorem 1.3. *Let $0 < p < \frac{Q+1}{Q-1}$, where $Q = 2n+2$ is the homogeneous dimension of \mathbb{H}^n . Then there exists no cylindrical solution $u \in C^2(\mathbb{H}^n)$ of*

$$\begin{cases} \mathcal{P}_{\frac{1}{2}} u = u^p & \text{in } \mathbb{H}^n, \\ u > 0 & \text{in } \mathbb{H}^n. \end{cases} \quad (1.6)$$

Using the local formulation (1.2) established in [16], the above theorem will follow as a corollary of the following Liouville-type result for a nonlinear Neumann problem in the half-space $\mathbb{H}^n \times \mathbb{R}^+$.

Theorem 1.4. *Let $0 < p < \frac{Q+1}{Q-1}$ and $U \in C^2(\mathbb{H}^n \times \mathbb{R}^+) \cap C^1(\overline{\mathbb{H}^n \times \mathbb{R}^+})$ be a nonnegative solution of*

$$\begin{cases} \frac{\partial^2 U}{\partial \lambda^2} + 4\lambda^2 \frac{\partial^2 U}{\partial t^2} + \Delta_{\mathbb{H}} U = 0 & \text{in } \mathbb{H}^n \times \mathbb{R}^+, \\ -\frac{\partial U}{\partial \lambda} = U^p & \text{on } \mathbb{H}^n \times \{\lambda = 0\}. \end{cases} \quad (1.7)$$

Suppose that $U(x, y, t, \lambda) = U(r_0, t, \lambda)$ depends only on r_0, t, λ , where $r_0 = (|x|^2 + |y|^2)^{\frac{1}{2}}$. Then $U \equiv 0$.

In the Euclidean case, classical nonexistence results for subcritical nonlinear problems in the all space \mathbb{R}^n are contained in two works by Gidas and Spruck [19] and by Chen and Li [10]. Analogue results for nonlinear Neumann problems in the half-space \mathbb{R}_+^n were established in [24, 25], using the methods of moving planes and moving spheres.

In the Heisenberg group setting there are several papers concerning nonexistence results for problem (1.5). Garofalo and Lanconelli [17] proved some nonexistence results for positive solutions of (1.5) when p is subcritical, under some integrability conditions on u and ∇u . In [23, 29] similar nonexistence results for positive solutions of (1.5) in the half-space are established for the critical exponent $p = \frac{Q+2}{Q-2}$. In [5], a Liouville-type result for solution of (1.5) is proved without requiring any decay condition on u , but only for $0 < p < \frac{Q}{Q-2}$. As explained before, in [4] Birindelli and Prajapat extends this last result to any $0 < p < \frac{Q+2}{Q-2}$ but only in the class of cylindrical solution. A last more recent result in this context was proven by Xu in [30], who established that there are no positive solution of (1.5) for $0 < p < \frac{Q(Q+2)}{(Q-1)^2}$. This result uses a different technique, based on the vector field method, and improves the results contained in [17] and [5], since it does not require any decay on the solution u and it improves the exponent p . Nevertheless it seems not allow to reach the optimal exponent $\frac{Q+2}{Q-2}$ (observe that $\frac{Q}{Q-2} < \frac{Q(Q+2)}{(Q-1)^2} < \frac{Q+2}{Q-2}$).

In this paper we aim to establish a first Liouville-type result for a CR fractional power of $-\Delta_{\mathbb{H}^n}$; this is, to our knowledge, the first nonexistence result in this fractional setting.

Let us comment now on the basic tools in the proof of our main result. Following [4], in order to get a nonexistence result, we combine the method of moving planes with the CR inversion of the solution u .

The CR inversion was introduced by Jerison and Lee in [26], and it is the analogue of the Kelvin transform, in the Heisenberg group context. In Section 3 we will give the precise definition of CR inversion and we will show which problem is satisfied by the CR inversion of a solution of (1.7).

As said before, the moving plane method is based on several version of the maximum principles. More precisely we will recall the classical Bony's maximum principle and we will prove two versions of the Hopf's Lemma (see Propositions 4.3 and 4.6).

The paper is organized as follows:

- in Section 2 we recall some basic facts on the Heisenberg group and we will introduce the fractional CR operator $\mathcal{P}_{\frac{1}{2}}$;
- in Section 3 we will introduce the CR inversion of a function u and prove a lemma concerning the CR inversion of a solution of our problem (1.7);
- in Section 4 we establish a maximum principle and Hopf's Lemma for our operator, which will be basic tools in the method of moving planes;
- in Section 5 we will prove our main results Theorems 1.3 and 1.4.

2. PRELIMINARY FACTS ON THE HEISENBERG GROUP

In this section we recall some basic notions and properties concerning the Heisenberg group, see also [6] and [28].

We will denote the points in \mathbb{H}^n using the notation $\xi = (x, y, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The Heisenberg group \mathbb{H}^n is the space \mathbb{R}^{2n+1} endowed with the group law \circ defined in the following way:

$$\hat{\xi} \circ \xi := (\hat{x} + x, \hat{y} + y, \hat{t} + t + 2 \sum_{j=1}^n (x_j \hat{y}_j - y_j \hat{x}_j)).$$

The natural dilation of the group is given by $\delta_\ell(\xi) := (\ell x, \ell y, \ell^2 t)$, and it satisfies $\delta_\ell(\hat{\xi} \circ \xi) = \delta_\ell(\hat{\xi}) \circ \delta_\ell(\xi)$.

In \mathbb{H}^n we will consider the gauge norm defined as

$$|\xi|_{\mathbb{H}} := \left[\left(\sum_{i=1}^n (x_i^2 + y_i^2) \right)^2 + t^2 \right]^{\frac{1}{4}},$$

which is homogeneous of degree one with respect to δ_ℓ . Using this norm, one can define the distance between two points in the natural way:

$$d_{\mathbb{H}}(\hat{\xi}, \xi) = |\hat{\xi}^{-1} \circ \xi|_{\mathbb{H}},$$

where $\hat{\xi}^{-1}$ denotes the inverse of $\hat{\xi}$ with respect to the group action. We denote the ball associated to the gauge distance by

$$B_{\mathbb{H}}(\xi_0, R) := \{\xi \in \mathbb{H}^n : d_{\mathbb{H}}(\xi, \xi_0) < R\}.$$

Denoting by $|A|$ the Lebesgue measure of the set A , we have that

$$|B_{\mathbb{H}}(\xi_0, R)| = |B_{\mathbb{H}}(0, R)| = R^Q |B_{\mathbb{H}}(0, 1)|.$$

Here $Q = 2n + 2$ denotes the homogeneous dimension of \mathbb{H}^n .

For every $j = 1, \dots, n$, we denote by X_j , Y_j , and T the following vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

They form a basis of the Lie Algebra of left invariant vector fields. Moreover, an easy computations shows that $[X_k, Y_j] = -4\delta_{kj}T$. The Heisenberg gradient of a function f is given by

$$\nabla_{\mathbb{H}} f = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f).$$

Finally, we define the sublaplacian as

$$\begin{aligned} \Delta_{\mathbb{H}} &:= \sum_{j=1}^n (X_j^2 + Y_j^2) \\ &= \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2}. \end{aligned}$$

It can be written also in the form $\Delta_{\mathbb{H}} = \operatorname{div}(\bar{A}\nabla^T)$, where $\bar{A} = \bar{a}_{kj}$ is the $(2n + 1) \times (2n + 1)$ symmetric matrix given by $\bar{a}_{kj} = \delta_{kj}$ for $k, j = 1, \dots, 2n$, $\bar{a}_{j(2n+1)} = \bar{a}_{(2n+1)j} = 2y_j$ for $j = 1, \dots, n$, $\bar{a}_{j(2n+1)} = \bar{a}_{(2n+1)j} = -2x_j$ for $j = n + 1, \dots, 2n$ and $\bar{a}_{(2n+1)(2n+1)} = 4(|x|^2 + |y|^2)$. It is easy to observe that \bar{A} is positive semidefinite for any $(x, y, t) \in \mathbb{H}^n$. This operator is degenerate elliptic, and it is hypoelliptic since it satisfies the Hörmander condition.

We pass now to describe CR covariant operators of fractional orders in \mathbb{H}^n . For more precise notions of CR geometry and for the construction of CR covariant fractional powers of the sub-Laplacian on more general CR manifolds, we refer to [16] and references therein. Here we just consider the case of the Heisenberg group, since it is the one of interest.

Introducing complex coordinates $\zeta = x + iy \in \mathbb{C}^n$, we can identify the Heisenberg group \mathbb{H}^n with the boundary of the Siegel domain $\Omega_{n+1} \subset \mathbb{C}^{n+1}$, which is given by

$$\Omega_{n+1} := \{(\zeta_1, \dots, \zeta_{n+1}) = (\zeta, \zeta_{n+1}) \in \mathbb{C}^n \times \mathbb{C} \mid q(\zeta, \zeta_{n+1}) > 0\},$$

with

$$q(\zeta, \zeta_{n+1}) = \operatorname{Im} \zeta_{n+1} - \sum_{j=1}^n |\zeta_j|^2,$$

through the map $(\zeta, t) \in \mathbb{H}^n \rightarrow (\zeta, t + i|\zeta|^2) \in \partial\Omega_{n+1}$. It is possible to see that $\mathcal{X} = \Omega_{n+1}$ is a Kähler-Einstein manifold, endowed with a Kähler form ω_+ (whose precise expression can be found in formula (1.4) in [16]), and a corresponding Kähler metric g_+ (see formula (1.6) in [16]). Using this metric, one can see that Ω_{n+1} can be identified with the complex hyperbolic space. The boundary manifold $\mathcal{M} = \partial\Omega_{n+1}$ inherits a natural CR structure from the complex structure of the ambient manifold. Given a CR structure, it is possible to associate to it a contact form θ , that in the specific case of the Heisenberg group, is given by

$$\theta = \left[dt + \frac{1}{2} \sum_{j=1}^n (x_j dy_j - y_j dx_j) \right]. \quad (2.1)$$

This form satisfies $\theta(T) = 1$.

Scattering theory tells us that for $s \in \mathbb{C}$, $\operatorname{Re}(s) > \frac{m}{2}$, and except for a set of exceptional values, given f smooth on \mathcal{M} , the eigenvalue equation

$$-\Delta_{g_+} u - s(m - s)u = 0, \quad \text{in } \mathcal{X}$$

has a solution u with the expansion

$$\begin{cases} u = q^{(m-s)} F + q^s G & \text{for some } F, G \in C^\infty(\overline{\mathcal{X}}), \\ F|_{\mathcal{M}} = f. \end{cases}$$

The scattering operator is defined as

$$S(s) : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$$

by

$$S(s)f := G|_{\mathcal{M}}.$$

We set $s = \frac{m+\gamma}{2}$, for $\gamma \in (0, m) \setminus \mathbb{N}$. The conformal fractional sub-Laplacian on $\mathbb{H}^n = \mathcal{M}$ (associated to the contact form θ) is defined in the following way:

$$\mathcal{P}_\gamma^\theta f = C_\gamma S(s)f, \quad (2.2)$$

for a constant

$$C_\gamma = 2^{2\gamma-1} \frac{\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}.$$

For $\gamma = 1$ and $\gamma = 2$ we have (see [16]):

$$P_1^\theta = -\Delta_{\mathbb{H}} \quad \text{and} \quad P_2^\theta = \Delta_{\mathbb{H}}^2 + T^2.$$

A crucial property of $\mathcal{P}_\gamma^\theta$ is its conformal covariance. Indeed, if we consider a conformal change of the contact form $\hat{\theta} = w^{\frac{2}{n+1-\gamma}}\theta$, then the corresponding fractional operator is given by:

$$\mathcal{P}_\gamma^{\hat{\theta}}(\cdot) = w^{-\frac{n+1+\gamma}{n+1-\gamma}} \mathcal{P}_\gamma^\theta(w \cdot).$$

For further details on CR covariance and on the geometric properties of the operator $\mathcal{P}_\gamma^\theta$ we refer to [16]; here we just emphasize that this covariance property is reflected in the fact that the extension operator \mathcal{L} defined in (1.4) well behaves under *CR* inversion (as we will see later in Section 3), and this will be crucial in the proof of our main result.

As explained in the introduction, one of the main result in [16], is the characterization of these fractional operators via the extension problem (1.2). Since throughout this paper we will work on this lifted problem in the extended space, let us introduce some notations in $\widehat{\mathbb{H}}^n = \mathbb{H}^n \times \mathbb{R}^+$.

Since, the contact form θ is fixed (and it is the one defined in (2.1)), for simplicity of notations we will write \mathcal{P}_γ instead of $\mathcal{P}_\gamma^\theta$.

Analogously to \mathbb{H}^n , in $\widehat{\mathbb{H}}^n$ we define the following group law (that for simplicity of notation we still denote by \circ):

for $z = (x_1, \dots, x_n, y_1, \dots, y_n, t, \lambda) \in \widehat{\mathbb{H}}^n$ and $\hat{z} = (\hat{x}_1, \dots, \hat{x}_n, \hat{y}_1, \dots, \hat{y}_n, \hat{t}, \hat{\lambda}) \in \widehat{\mathbb{H}}^n$, we set

$$\hat{z} \circ z := (\hat{x} + x, \hat{y} + y, \hat{t} + t + 2 \sum_{j=1}^n (x_j \hat{y}_j - y_j \hat{x}_j), \hat{\lambda} + \lambda).$$

Moreover we consider the norm given by

$$|z|_{\widehat{\mathbb{H}}^n} := [(|x|^2 + |y|^2 + \lambda^2)^2 + t^2]^{\frac{1}{4}}.$$

Finally we denote the distance $d_{\widehat{\mathbb{H}}}$ between z and \hat{z} , by

$$d_{\widehat{\mathbb{H}}}(z, \hat{z}) := |\hat{z}^{-1} \circ z|_{\widehat{\mathbb{H}}^n}.$$

Observe that when $\lambda = \hat{\lambda} = 0$, that is z and \hat{z} belong to \mathbb{H}^n , $d_{\widehat{\mathbb{H}}}(z, \hat{z}) = d_{\mathbb{H}}(z, \hat{z})$. Moreover, given $\bar{z} \in \widehat{\mathbb{H}}^n$ we set

$$\mathcal{B}(\bar{z}, R) = \{z \in \mathbb{C}^{n+1} \mid d_{\widehat{\mathbb{H}}}(z, \bar{z}) < R\}$$

and for any $z_0 \in \mathbb{H}^n \times \{0\}$ we denote

$$\mathcal{B}^+(z_0, R) = \{z \in \mathbb{H}^n \times \mathbb{R}^+ \mid d_{\widehat{\mathbb{H}}}(z, z_0) < R, \lambda > 0\}.$$

The operator \mathcal{L} (defined in (1.4)), writing explicitly all the terms, becomes

$$\begin{aligned} \mathcal{L} &= \frac{\partial^2}{\partial \lambda^2} + \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) \\ &\quad + 4(\lambda^2 + \sum_{j=1}^n (x_j^2 + y_j^2)) \frac{\partial^2}{\partial t^2}. \end{aligned}$$

Also in this case, we can write $\mathcal{L} = \text{div}(A\nabla^T)$, where now A is the $(2n+2) \times (2n+2)$ symmetric matrix given by $a_{kj} = \delta_{kj}$ if $k, j = 1, \dots, 2n$, $a_{j(2n+1)} = a_{(2n+1)j} = 2y_j$ if $j = 1, \dots, n$, $a_{j(2n+1)} = a_{(2n+1)j} = -2x_j$ if $j = n+1, \dots, 2n$, $a_{(2n+1)(2n+1)} = 4(|x|^2 + |y|^2 + \lambda^2)$, $a_{(2n+2)(2n+2)} = 1$, $a_{j(2n+2)} = a_{(2n+2)j} = 0$ if $j = 1, \dots, 2n+1$.

In the sequel it will be useful to express \mathcal{L} for cylindrical and radial functions.

For a point $z = (x, y, t, \lambda) \in \widehat{\mathbb{H}}^n$, let

$$\begin{aligned} r &= (|x|^2 + |y|^2 + \lambda^2)^{1/2}, \\ \rho &= (r^4 + t^2)^{1/4}. \end{aligned}$$

Suppose that Ψ is a radial function, that is, Ψ depends only on ρ ; then a direct computation gives:

$$\begin{aligned} \frac{\partial \Psi}{\partial x_j} &= \frac{\partial \Psi}{\partial \rho} \frac{\partial \rho}{\partial x_j} = \rho^{-3} r^2 x_j \frac{\partial \Psi}{\partial \rho}, & \frac{\partial \Psi}{\partial y_j} &= \frac{\partial \Psi}{\partial \rho} \frac{\partial \rho}{\partial y_j} = \rho^{-3} r^2 y_j \frac{\partial \Psi}{\partial \rho}, \\ \frac{\partial \Psi}{\partial \lambda} &= \frac{\partial \Psi}{\partial \rho} \frac{\partial \rho}{\partial \lambda} = \rho^{-3} r^2 \lambda \frac{\partial \Psi}{\partial \rho}. \end{aligned}$$

Then we deduce that

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial x_j^2} &= \frac{r^4 x_j^2}{\rho^6} \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{\rho^4 r^2 + 2\rho^4 x_j^2 - 3r^4 x_j^2}{\rho^7} \frac{\partial \Psi}{\partial \rho}, \\ \frac{\partial^2 \Psi}{\partial y_j^2} &= \frac{r^4 y_j^2}{\rho^6} \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{\rho^4 r^2 + 2\rho^4 y_j^2 - 3r^4 y_j^2}{\rho^7} \frac{\partial \Psi}{\partial \rho}, \\ \frac{\partial^2 \Psi}{\partial \lambda^2} &= \frac{r^4 \lambda^2}{\rho^6} \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{\rho^4 r^2 + 2\rho^4 \lambda^2 - 3r^4 \lambda^2}{\rho^7} \frac{\partial \Psi}{\partial \rho}, \\ \frac{\partial^2 \Psi}{\partial t^2} &= \frac{t^2}{4\rho^6} \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{2\rho^4 - 3t^2}{4\rho^7} \frac{\partial \Psi}{\partial \rho}.\end{aligned}$$

Hence, by using that

$$\frac{1}{\rho^6} \left(\sum_{j=1}^n (r^4 x_j^2 + r^4 y_j^2) + r^4 \lambda^2 + r^2 t^2 \right) = \frac{r^2}{\rho^2}$$

and

$$\begin{aligned}& \frac{1}{\rho^7} \left[(2n+1)\rho^4 r^2 + (2\rho^4 - 3r^4 + (2\rho^4 - 3t^2)) \left(\sum_{j=1}^n (x_j^2 + y_j^2) + \lambda^2 \right) \right] \\ &= \frac{1}{\rho^7} \left[(2n+1)\rho^4 r^2 + 2\rho^4 r^2 - 3r^6 + r^2(2\rho^4 - 3t^2) \right] = \frac{Qr^2}{\rho^3},\end{aligned}$$

we conclude that

$$\mathcal{L}\Psi(\rho) = \frac{r^2}{\rho^2} \left(\frac{d^2 \Psi(\rho)}{d\rho^2} + \frac{Q}{\rho} \frac{d\Psi(\rho)}{d\rho} \right). \quad (2.3)$$

In a similar way, we deduce that for a cylindrical symmetric function $\phi = \phi(r, t)$,

$$\mathcal{L}\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{Q-2}{r} \frac{\partial \phi}{\partial r} + 4r^2 \frac{\partial^2 \phi}{\partial t^2}. \quad (2.4)$$

Using the radial form (2.3) for \mathcal{L} , an easy computation yields the following lemma.

Lemma 2.1. *Let $\psi(\rho) = \frac{1}{\rho^{Q-1}} = \frac{1}{\rho^{2n+1}}$ for $\rho \neq 0$. Then we have that*

$$\begin{cases} \mathcal{L}\psi(\rho) = 0 & \text{in } \mathbb{H}^n \times \mathbb{R}^+ \setminus \{0\}, \\ -\frac{\partial}{\partial \lambda} \psi(\rho) = 0 & \text{on } \mathbb{H}^n \times \{\lambda = 0\} \setminus \{0\}. \end{cases} \quad (2.5)$$

3. CR INVERSION

Following [4] and [26], we define the CR inversion in the half-space $\widehat{\mathbb{H}}_+^n = \mathbb{H}^n \times \mathbb{R}^+$.

For any $(x, y, t, \lambda) \in \widehat{\mathbb{H}}_+^n$, let as before $r = (|x|^2 + |y|^2 + \lambda^2)^{\frac{1}{2}}$ and $\rho = (r^4 + t^2)^{\frac{1}{4}}$. We set

$$\tilde{x}_i = \frac{x_i t + y_i r^2}{\rho^4}, \quad \tilde{y}_i = \frac{y_i t - x_i r^2}{\rho^4}, \quad \tilde{t} = -\frac{t}{\rho^4}, \quad \tilde{\lambda} = \frac{\lambda}{\rho^2}.$$

The CR inversion of a function U defined on $\widehat{\mathbb{H}}_+^n$, is given by

$$v(x, y, t, \lambda) = \frac{1}{\rho^{Q-1}} U(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}).$$

The following lemma shows which equation is satisfied by the CR inversion of a solution of problem (1.7).

Lemma 3.1. *Suppose that $U \in C^2(\widehat{\mathbb{H}}_+^n) \cap C(\overline{\widehat{\mathbb{H}}_+^n})$ is a solution of (1.7). Then the CR inversion v of U satisfies*

$$\begin{cases} \mathcal{L}v = 0 & \text{in } \widehat{\mathbb{H}}_+^n \setminus \{0\}, \\ -\frac{\partial v}{\partial \lambda} = \rho^{p(Q-1)-(Q+1)} v^p & \text{on } \mathbb{H}^n \times \{\lambda = 0\} \setminus \{0\}. \end{cases} \quad (3.1)$$

Proof. Since it is just a long computation, we will give the details of the proof for cylindrical solutions $U(r, t)$, but the statement holds true for any solution.

It is clear that

$$\tilde{r}^2 = |\tilde{x}|^2 + |\tilde{y}|^2 + \tilde{\lambda}^2 = \frac{t^2(|x|^2 + |y|^2 + \lambda^2)}{\rho^8} + \frac{r^4(|x|^2 + |y|^2 + \lambda^2)}{\rho^8} = \frac{r^2}{\rho^4},$$

and

$$\tilde{\rho} = (|\tilde{r}|^4 + \tilde{t}^2)^{\frac{1}{2}} = \frac{1}{\rho}.$$

In the same way:

$$\tilde{r}_0^2 = |\tilde{x}|^2 + |\tilde{y}|^2 = \frac{r_0^2}{\rho^4}.$$

Therefore, if U depends only on (r, t) (respectively on (r_0, t, λ)), then so does also v . The following relations are useful

$$\begin{aligned} \frac{\partial \tilde{r}}{\partial r} &= \frac{t^2 - r^4}{\rho^6}, \quad \frac{\partial \tilde{r}}{\partial t} = \frac{-rt}{\rho^6}, \\ \frac{\partial \tilde{t}}{\partial r} &= \frac{4tr^3}{\rho^8}, \quad \frac{\partial \tilde{t}}{\partial t} = \frac{2t^2 - t^2 - r^4}{\rho^8} = \frac{t^2 - r^4}{\rho^8}. \end{aligned}$$

Now we have

$$\frac{\partial v}{\partial r} = \frac{(1-Q)r^3}{\rho^{Q+3}}U + \frac{1}{\rho^{Q-1}}\left[\frac{t^2-r^4}{\rho^6}\frac{\partial U}{\partial \tilde{r}} + \frac{4tr^3}{\rho^8}\frac{\partial U}{\partial \tilde{t}}\right]$$

and

$$\begin{aligned} \frac{\partial^2 v}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{(1-Q)r^3}{\rho^{Q+3}}\right)U + \frac{2(1-Q)r^3}{\rho^{Q+3}}\left[\left(\frac{t^2-r^4}{\rho^6}\right)\frac{\partial U}{\partial \tilde{r}} + \left(\frac{4r^3t}{\rho^8}\right)\frac{\partial U}{\partial \tilde{t}}\right] \\ &+ \frac{1}{\rho^{Q-1}}\left[\frac{\partial}{\partial r}\left(\frac{t^2-r^4}{\rho^6}\right)\frac{\partial U}{\partial \tilde{r}} + \frac{\partial}{\partial r}\left(\frac{4r^3t}{\rho^8}\right)\frac{\partial U}{\partial \tilde{t}}\right] \\ &+ \frac{1}{\rho^{Q-1}}\left[2\left(\frac{t^2-r^4}{\rho^6}\right)^2\frac{\partial^2 U}{\partial \tilde{r}^2} + \left(\frac{t^2-r^4}{\rho^6}\right)\left(\frac{4r^3t}{\rho^8}\right)\frac{\partial^2 U}{\partial \tilde{r}\partial \tilde{t}} + \left(\frac{4r^3t}{\rho^8}\right)^2\frac{\partial^2 U}{\partial \tilde{t}^2}\right]. \end{aligned}$$

The derivatives with respect to t are given by

$$\frac{\partial v}{\partial t} = \frac{(1-Q)t}{2\rho^{Q+3}}U + \frac{1}{\rho^{Q-1}}\left[\frac{-rt}{\rho^6}\frac{\partial U}{\partial \tilde{r}} + \frac{t^2-r^4}{\rho^8}\frac{\partial U}{\partial \tilde{t}}\right]$$

and

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{(1-Q)t}{2\rho^{Q+3}}\right)U + \frac{2(1-Q)t}{2\rho^{Q+3}}\left[\left(\frac{-rt}{\rho^6}\right)\frac{\partial U}{\partial \tilde{r}} + \left(\frac{t^2-r^4}{\rho^8}\right)\frac{\partial U}{\partial \tilde{t}}\right] \\ &+ \frac{1}{\rho^{Q-1}}\left[\frac{\partial}{\partial t}\left(\frac{-rt}{\rho^6}\right)\frac{\partial U}{\partial \tilde{r}} + \frac{\partial}{\partial t}\left(\frac{t^2-r^4}{\rho^8}\right)\frac{\partial U}{\partial \tilde{t}}\right] \\ &+ \frac{1}{\rho^{Q-1}}\left[\left(\frac{-rt}{\rho^6}\right)^2\frac{\partial^2 U}{\partial \tilde{r}^2} + \left(\frac{-rt}{\rho^6}\right)\left(\frac{t^2-r^4}{\rho^8}\right)\frac{\partial^2 U}{\partial \tilde{r}\partial \tilde{t}} + \left(\frac{t^2-r^4}{\rho^8}\right)^2\frac{\partial^2 U}{\partial \tilde{t}^2}\right] \\ &+ \frac{1}{\rho^{Q-2}}\left[\frac{\partial}{\partial t}2\left(\frac{-rt}{\rho^6}\right)\frac{\partial U}{\partial \tilde{r}} + \frac{\partial}{\partial t}\left(\frac{t^2-r^4}{\rho^6}\right)\frac{\partial U}{\partial \tilde{t}}\right]. \end{aligned}$$

Let us denote

$$\mathcal{L}U = a_1\frac{\partial^2 U}{\partial \tilde{r}^2} + a_2\frac{\partial^2 U}{\partial \tilde{r}\partial \tilde{t}} + a_3\frac{\partial^2 U}{\partial \tilde{t}^2} + b_1\frac{\partial U}{\partial \tilde{r}} + b_2\frac{\partial U}{\partial \tilde{t}} + cU.$$

Then

$$c = 0,$$

$$a_1 = \frac{1}{\rho^{Q+11}}[(t^2-r^4)^2 + 4r^2(rt)^2] = \frac{1}{\rho^{Q+3}},$$

$$a_2 = \frac{1}{\rho^{Q+13}}[2(t^2-r^4)(4r^3t) - 8r^2(rt)(t^2-r^4)] = 0,$$

$$a_3 = \frac{1}{\rho^{Q+15}}[16r^6t^2 + 4r^2(t^2-r^4)^2] = \frac{1}{\rho^{Q+3}}\frac{4r^2}{\rho^4}.$$

By using that

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{t^2 - r^4}{\rho^6}\right) &= \frac{-4r^3(t^2 + r^4)}{\rho^{10}} - \frac{6(t^2 - r^4)r^3}{\rho^{10}}, \\ \frac{\partial}{\partial t}\left(\frac{-rt}{\rho^6}\right) &= -\frac{r(t^2 + r^4)}{\rho^{10}} + \frac{3rt^2}{\rho^{10}},\end{aligned}$$

we can see that

$$\begin{aligned}b_1 &= \frac{2(1-Q)r^4(t^2 - r^4)}{r\rho^{Q+9}} + \frac{1}{\rho^{Q-1}}\frac{\partial}{\partial r}\left(\frac{t^2 - r^4}{\rho^6}\right) \\ &\quad + \frac{Q-2}{r}\frac{(t^2 - r^4)(t^2 + r^4)}{\rho^{Q+9}} + \frac{4r^2(1-Q)(-r^2t^2)}{r\rho^{Q+9}} + \frac{4r^2}{\rho^{Q-1}}\frac{\partial}{\partial t}\left(\frac{-rt}{\rho^6}\right) \\ &= \frac{2(1-Q)r^4(t^2 - r^4)}{r\rho^{Q+9}} + \left(\frac{-4r^4(t^2 + r^4)}{r\rho^{Q+9}} - \frac{6(t^2 - r^4)r^4}{r\rho^{Q+9}}\right) \\ &\quad + \frac{Q-2}{r}\frac{(t^2 - r^4)(t^2 + r^4)}{\rho^{Q+9}} + \frac{4(1-Q)(-r^4t^2)}{r\rho^{Q+9}} \\ &\quad - \frac{4r^4(t^2 + r^4)}{r\rho^{Q+9}} + \frac{12r^4t^2}{r\rho^{Q+9}} \\ &= \frac{1}{\rho^{Q+3}}\frac{(Q-2)\rho^2}{r} = \frac{1}{\rho^{Q+3}}\frac{(Q-2)}{\tilde{r}}.\end{aligned}$$

We have that

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{4r^3t}{\rho^8}\right) &= \frac{12r^2t(t^2 + r^4)}{\rho^{12}} - \frac{32r^6t}{\rho^{12}}, \\ \frac{\partial}{\partial t}\left(\frac{t^2 - r^4}{\rho^8}\right) &= \frac{2t(t^2 + r^4)}{\rho^{12}} - \frac{4t(t^2 - r^4)}{\rho^{12}}.\end{aligned}$$

Then we see that the following term vanishes, that is,

$$\begin{aligned}b_2 &= \frac{8(1-Q)r^6t}{\rho^{Q+11}} + \frac{1}{\rho^{Q-1}}\frac{\partial}{\partial r}\left(\frac{4r^3t}{\rho^8}\right) + \frac{4(Q-2)r^3t(t^2 + r^4)}{r\rho^{Q+11}} \\ &\quad + \frac{4(1-Q)r^2t(t^2 - r^4)}{\rho^{Q+11}} + \frac{4r^2}{\rho^{Q-1}}\frac{\partial}{\partial t}\left(\frac{t^2 - r^4}{\rho^8}\right) \\ &= \frac{8(1-Q)r^6t}{\rho^{Q+11}} + \left(\frac{12r^2t(t^2 + r^4)}{\rho^{Q+11}} - \frac{32r^6t}{\rho^{Q+11}}\right) + \frac{4(Q-2)r^2t(t^2 + r^4)}{\rho^{Q+11}} \\ &\quad + \frac{4(1-Q)r^2t(t^2 - r^4)}{\rho^{Q+11}} + \left(\frac{8r^2t(t^2 + r^4)}{\rho^{Q+11}} - \frac{16r^2t(t^2 - r^4)}{\rho^{Q+11}}\right) = 0.\end{aligned}$$

This shows that $\mathcal{L}v(x, y, t, \lambda) = \frac{1}{\rho^{(Q+3)}}\mathcal{L}U(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}) = 0$. Finally, the Neumann data becomes:

$$\begin{aligned} -\lim_{\lambda \rightarrow 0} \frac{\partial v}{\partial \lambda}(x, y, t, \lambda) &= -\lim_{\tilde{\lambda} \rightarrow 0} \frac{1}{\rho^{Q-1}} \rho^{-2} \frac{\partial U}{\partial \tilde{\lambda}}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\lambda}) \\ &= \frac{1}{\rho^{(Q+1)}} U^p(\tilde{x}, \tilde{y}, \tilde{t}, 0) = \rho^{p(Q-1)-(Q+1)} v(x, y, t, 0). \end{aligned}$$

□

4. MAXIMUM PRINCIPLE AND HOPF'S LEMMA

Basic tools in the method of moving planes are the maximum principle and Hopf's Lemma. We start by recalling the classical maximum principle for Hörmander-type operators due to Bony [7].

Proposition 4.1. ([7]) *Let \mathcal{V} be a bounded domain in $\widehat{\mathbb{H}}^n$, let Z be a smooth vector field on \mathcal{V} and let a be a smooth nonnegative function. Assume that $U \in C^2(\mathcal{V}) \cap C^1(\overline{\mathcal{V}})$ is a solution of*

$$\begin{cases} -\mathcal{L}U + Z(z)U + a(z)U \geq 0 & \text{in } \mathcal{V}, \\ U \geq 0 & \text{on } \partial\mathcal{V}. \end{cases} \quad (4.1)$$

Then $U \geq 0$ in \mathcal{V} .

We prove now two Hopf's Lemmas. The first one is Hopf's Lemma for the operator \mathcal{L} in a subset \mathcal{V} of $\widehat{\mathbb{H}}^n$. We first define the interior ball condition in this setting.

Definition 4.2. *Let $\mathcal{V} \subset \widehat{\mathbb{H}}^n$. We say that \mathcal{V} satisfies the interior $d_{\widehat{\mathbb{H}}}$ -ball condition at $P \in \partial\mathcal{V}$ if there exist a constant $R > 0$ and a point $z_0 \in \mathcal{V}$, such that the ball $\mathcal{B}(z_0, R) \subset \mathcal{V}$ and $P \in \partial\mathcal{B}(z_0, R)$, where $\mathcal{B}(z_0, R) = \{z \in \widehat{\mathbb{H}}^n \mid d_{\widehat{\mathbb{H}}}(z, z_0) < R\}$.*

Lemma 4.3. *Let $\mathcal{V} \subset \widehat{\mathbb{H}}^n$ satisfy the interior $d_{\widehat{\mathbb{H}}}$ -ball condition at the point $P_0 \in \partial\mathcal{V}$ and let $U \in C^2(\mathcal{V}) \cap C^1(\overline{\mathcal{V}})$, be a solution of*

$$-\mathcal{L}U \geq c_1(z)U \text{ in } \mathcal{V} \quad (4.2)$$

with $c_1 \in L^\infty(\mathcal{V})$. Suppose that $U(z) > U(P_0) = 0$ for every $z \in \mathcal{V}$.

Then

$$\lim_{s \rightarrow 0} \frac{U(P_0) - U(P_0 - s\nu)}{s} < 0.$$

where ν is the outer normal to $\partial\mathcal{V}$ in P_0 .

Remark 4.4. *We observe that if the function c_1 in Lemma 4.3 is identically zero, then we can drop the assumption $U(P_0) = 0$.*

Proof. By assumption, there exist a point $z_0 = (\hat{x}_1, \dots, \hat{x}_n, \hat{y}_1, \dots, \hat{y}_n, \hat{t}, \hat{\lambda})$ and a radius $R > 0$ such that the ball $\mathcal{B}(z_0, R) \subset \mathcal{V}$ and $P_0 \in \partial\mathcal{B}(z_0, R)$.

We consider the function

$$\psi = Ue^{-K(x_1 - \hat{x}_1)^2}, \text{ for } K > 0. \quad (4.3)$$

An easy computation yields

$$\frac{\partial^2 \psi}{\partial x_1^2} = e^{-K(x_1 - \hat{x}_1)^2} [4K^2(x_1 - \hat{x}_1)^2 U - 2KU - 4K(x_1 - \hat{x}_1) \frac{\partial U}{\partial x_1} + \frac{\partial^2 U}{\partial x_1^2}],$$

and

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x_j^2} &= e^{-K(x_1 - \hat{x}_1)^2} \frac{\partial^2 U}{\partial x_j^2}, & \frac{\partial^2 \psi}{\partial y_k^2} &= e^{-K(x_1 - \hat{x}_1)^2} \frac{\partial^2 U}{\partial y_k^2}, \\ \frac{\partial^2 \psi}{\partial \lambda^2} &= e^{-K(x_1 - \hat{x}_1)^2} \frac{\partial^2 U}{\partial \lambda^2}, \end{aligned}$$

where $j = 2, \dots, n, k = 1, \dots, n$.

Moreover,

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x_1 \partial t} &= e^{-K(x_1 - \hat{x}_1)^2} [-2K(x_1 - \hat{x}_1) \frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x_1 \partial t}], \\ \frac{\partial^2 \psi}{\partial x_j \partial t} &= e^{-K(x_1 - \hat{x}_1)^2} \frac{\partial^2 U}{\partial x_j \partial t}. \end{aligned}$$

Therefore, we have

$$\mathcal{L}\psi + 4K(x_1 - \hat{x}_1)X_1\psi = e^{-K(x_1 - \hat{x}_1)^2} [-4K^2(x_1 - \hat{x}_1)^2 U + \mathcal{L}U - 2KU]$$

and hence for K sufficiently large, we deduce

$$-\mathcal{L}\psi - 4K(x_1 - \hat{x}_1)X_1\psi \geq 0.$$

We introduce now the function $\phi = e^{-\alpha R^2} - e^{-\alpha \rho^2}$, where $\rho = d_{\widehat{\mathbb{H}}}(z, z_0)$, and $0 < \rho < R$. Since ϕ depends only on the distance from z_0 , and \mathcal{L} and X_1 are invariant with respect to the group action in $\widehat{\mathbb{H}}^n$, we can use formula (2.3) where now $\rho = d_{\widehat{\mathbb{H}}}(z_0^{-1} \circ z, 0)$, and the factor $\frac{r^2}{\rho^2}$ is replaced by the function $G(z_0^{-1} \circ z)$, where

$$G(x_1, \dots, x_n, y_1, \dots, y_n, t, \lambda) := \frac{\sum_{j=1}^n (x_j^2 + y_j^2) + \lambda^2}{[(\sum_{j=1}^n (x_j^2 + y_j^2) + \lambda^2)^2 + t^2]^{\frac{1}{2}}}. \quad (4.4)$$

Choosing α sufficiently large, we have

$$\begin{aligned} & -\mathcal{L}\phi - 4K(x_1 - \hat{x}_1)X_1\phi \\ &= [G(z_0^{-1} \circ z)(4\alpha^2 \rho^2 - 2(Q+1)\alpha) \\ & \quad - 8K\alpha(x_1 - \hat{x}_1)\rho X_1\rho] e^{-\alpha \rho^2} \geq 0, \end{aligned}$$

Let $\mathcal{A} := \mathcal{B}(z_0, R) \setminus \mathcal{B}(z_0, R_1)$ for $0 < R_1 < R$. For ε small enough

$$\psi(z) + \varepsilon\phi(z) \geq 0 \quad \text{in } \partial\mathcal{A} := \partial\mathcal{B}(z_0, R) \cup \partial\mathcal{B}(z_0, R_1).$$

Then, by Proposition 4.1 we obtain that $\psi(z) + \varepsilon\phi(z) \geq 0$ in \mathcal{A} .

Therefore, using that $\psi(P_0) = \phi(P_0) = 0$, we deduce that for s small

$$\psi(P_0) - \psi(P_0 - s\nu) + \varepsilon(\phi(P_0) - \phi(P_0 - s\nu)) \leq 0.$$

Using that ϕ is strictly increasing in ρ , we deduce

$$\lim_{s \rightarrow 0} \frac{\psi(P_0) - \psi(P_0 - s\nu)}{s} < 0,$$

which, in turn, implies that

$$\lim_{s \rightarrow 0} \frac{U(P_0 - s\nu) - U(P_0)}{s} < 0.$$

□

For any $\Omega \subset \mathbb{H}^n$, we denote by \mathcal{C} the infinite cylinder

$$\mathcal{C} = \Omega \times (0, +\infty).$$

Before proving our second Hopf's Lemma, let us recall the notion of interior ball condition in the Heisenberg group.

Definition 4.5. *Let $\Omega \subset \mathbb{H}^n$. We say that Ω satisfies the interior Heisenberg ball condition at $\xi \in \partial\Omega$ if there exist a constant $R > 0$ and a point $\xi_0 \in \Omega$, such that the Heisenberg ball $B_{\mathbb{H}}(\xi_0, R) \subset \Omega$ and $\xi \in \partial B_{\mathbb{H}}(\xi_0, R)$.*

Lemma 4.6. *Let $\Omega \subset \mathbb{H}^n$ satisfy the interior Heisenberg ball condition at the point $P \in \partial\Omega$ and let $U \in C^2(\mathcal{C}) \cap C^1(\bar{\mathcal{C}})$, be a nonnegative solution of*

$$\begin{cases} -\mathcal{L}U \geq c_1(z)U & \text{in } \mathcal{C}, \\ -\partial_\lambda U \geq c_2(\xi)U & \text{on } \Omega, \end{cases} \quad (4.5)$$

with $c_1, c_2 \in L^\infty(\mathcal{C})$ and c_1 is nonnegative. Suppose that $U((P, 0)) = 0$ and U is not identically null.

Then

$$\partial_\nu U(P, 0) < 0, \quad (4.6)$$

where ν is the outer normal to $\partial\Omega$ in P .

Proof. We follow the proof of Lemma 2.4 in [11].

By the strong maximum principle and by Lemma 4.3, we have that

$$U > 0 \quad \text{on } \mathcal{C} \cup \Omega. \quad (4.7)$$

Indeed, the strong maximum principle ensures that $U > 0$ in \mathcal{C} , moreover U cannot vanish at a point in Ω , otherwise at this point the Neumann condition would be violated by Lemma 4.3.

We start by proving the lemma in the case $c_1(z) = c_2(\xi) \equiv 0$.

Since Ω satisfies the interior Heisenberg ball condition at P , there exist $z_0 \in \Omega \times \{0\}$ and $R > 0$, such that the ball $\mathcal{B}^+(z_0, R)$ is contained in the cylinder \mathcal{C} and $(\overline{\partial\mathcal{C}} \cap \{\lambda > 0\}) \cap \partial\mathcal{B}^+(z_0, R) = \{(P, 0)\}$. We consider the set

$$\mathcal{A} = \left(\mathcal{B}^+(z_0, R) \setminus \overline{\mathcal{B}^+(z_0, R/2)} \right) \cap \{\lambda > 0\}.$$

We observe that $\{(P, 0)\} = \partial\mathcal{A} \cap \overline{\partial\mathcal{C}} \cap \{\lambda > 0\}$.

For $z \in \mathcal{A}$ we consider the function $\eta(z) = e^{-\alpha\rho^2} - e^{-\alpha R^2}$, where $\rho = d_{\mathbb{H}}(z, z_0)$. Writing \mathcal{L} in radial coordinates as in (2.3), we have that

$$\mathcal{L}\eta(\rho) = G(z_0^{-1} \circ z) (4\alpha^2\rho^2 - 2(Q+1)\alpha) e^{-\alpha\rho^2},$$

where G is defined as in (4.4).

Therefore, for α sufficiently large, we have that

$$-\mathcal{L}\eta \leq 0. \tag{4.8}$$

By (4.7) we deduce that $U > 0$ on $\partial\mathcal{B}^+(z_0, R/2) \cap \{\lambda \geq 0\}$. Hence, we may choose $\varepsilon > 0$ such that

$$U - \varepsilon\eta \geq 0 \quad \text{on } \partial\mathcal{B}^+(z_0, R/2) \cap \{\lambda \geq 0\}.$$

Claim: $U - \varepsilon\eta \geq 0$ in \mathcal{A} .

Indeed, using (4.8), we deduce that $-\mathcal{L}(U - \varepsilon\eta) \geq 0$ in \mathcal{A} . Hence, by the maximum principle, we have that the minimum of $U - \varepsilon\eta$ is attained only on $\partial\mathcal{A}$ (unless $U - \varepsilon\eta$ is constant). Now, on one side we have that

$$U - \varepsilon\eta \geq 0 \quad \text{on } \partial\mathcal{A} \cap \{\lambda > 0\}.$$

On the other side, since $\partial_\lambda\eta = 0$ on $\{\lambda = 0\}$, we deduce that $-\partial_\lambda(U - \varepsilon\eta) \geq 0$ on $\partial\mathcal{A} \cap \{\lambda = 0\}$.

Thus, using Lemma 4.3, we conclude that the minimum of $U - \varepsilon\eta$ cannot be achieved on $\left(\mathcal{B}^+(z_0, R) \setminus \overline{\mathcal{B}^+(z_0, R/2)} \right) \cap \{\lambda = 0\}$. This reaches the claim.

Finally, since $(U - \varepsilon\eta)((P, 0)) = 0$, we deduce that $\partial_\nu(U - \varepsilon\eta)((P, 0)) \leq 0$, which in turn implies that $\partial_\nu U((P, 0)) < 0$ using that $\partial_\nu\eta((P, 0)) < 0$. This concludes the case $c_1(z) = c_2(\xi) \equiv 0$.

In the general case, we introduce the function $v = e^{-\beta\lambda}U$ and we compute

$$\begin{aligned} -\mathcal{L}v &= -e^{-\beta\lambda}\mathcal{L}U + 2\beta e^{-\beta\lambda}U_\lambda - \beta^2v \\ &\geq c_1(z) - \beta^2v + 2\beta e^{-\beta\lambda}(\beta e^{\beta\lambda}v + e^{\beta\lambda}v_\lambda) \\ &= c_1(z) + \beta^2v + 2\beta v_\lambda. \end{aligned}$$

Therefore, we have

$$-\mathcal{L}v - 2\beta v_\lambda \geq c_1(z) + \beta^2v \geq 0.$$

Moreover, for β large enough

$$-\partial_\lambda v \geq (\beta + c_2(z))v \geq 0.$$

We can apply the first part of the proof to the function v , noting that the same argument works when the operator \mathcal{L} is replaced by $\mathcal{L} + 2\beta\partial_\lambda$. \square

5. PROOF OF THEOREMS 1.3 AND 1.4

In this last section we give the proof of our Liouville-type result.

We consider a solution $U = U(|(x, y)|, t, \lambda)$ of

$$\begin{cases} \mathcal{L}U = 0 & \text{in } \widehat{\mathbb{H}}_+^n := \mathbb{H}^n \times \mathbb{R}^+, \\ -\partial_\lambda U = U^p & \text{on } \mathbb{H}^n = \partial\widehat{\mathbb{H}}_+^n, \\ U > 0 & \text{in } \widehat{\mathbb{H}}_+^n. \end{cases} \quad (5.1)$$

First of all, we perform the CR inversion of U . For $z = (x, y, t, \lambda) \in \widehat{\mathbb{H}}^n$, let

$$w(z) = \frac{1}{\rho^{Q-1}}U(\tilde{z}),$$

where $\tilde{z} = \frac{1}{\rho^4}(xt + yr^2, yt - xr^2, -t, \lambda\rho^2)$, $r = (\sum_{j=1}^n(x_j^2 + y_j^2) + \lambda^2)^{\frac{1}{2}}$ and $\rho(z) = (r^4 + t^2)^{\frac{1}{4}} = d_{\widehat{\mathbb{H}}}(z, 0)$. We have seen in Lemma 3.1 that w satisfies

$$\begin{cases} \mathcal{L}w = 0 & \text{in } \widehat{\mathbb{H}}_+^n \setminus \{0\}, \\ -\partial_\lambda w = \rho^{p(Q-1)-(Q+1)}w^p & \text{on } \mathbb{H}^n \times \{\lambda = 0\} \setminus \{0\}. \end{cases} \quad (5.2)$$

Observe that the function w could be singular at the origin and it satisfies $\lim_{\rho \rightarrow \infty} \rho^{Q-1}w(z) = U(0)$, hence

$$0 < w(z) \leq \frac{\tilde{C}}{\rho(z)^{Q-1}}, \quad \text{for } \rho(z) \geq 1, \quad (5.3)$$

for some positive constant \tilde{C} . We start now applying the moving plane method. We will move a hyperplane orthogonal to the t -direction and use the H -reflection. More precisely, for any $\mu \leq 0$, let

$T_\mu = \{z \in \widehat{\mathbb{H}}_+^n \mid t = \mu\}$, and $\Sigma_\mu = \{z \in \widehat{\mathbb{H}}_+^n \mid t < \mu\}$. For $z \in \Sigma_\mu$, we define $z_\mu = (y, x, 2\mu - t, \lambda)$. To avoid the singular point, we consider

$$\tilde{\Sigma}_\mu = \Sigma_\mu \setminus \{e_\mu\},$$

where $e_\mu = (0, 0, 2\mu, 0)$ is the reflection of the origin. We recall that, as shown in the proof of Lemma 3.1, if U depends only on (r_0, t, λ) , then so does w .

Let now

$$\begin{aligned} w_\mu(z) &= w_\mu(|(x, y)|, t, \lambda) := w(|(x, y)|, 2\mu - t, \lambda) \\ &= w(y, x, 2\mu - t, \lambda) = w(z_\mu), \end{aligned}$$

and

$$W_\mu(z) := w_\mu(z) - w(z) = w(z_\mu) - w(z), \quad z \in \Sigma_\mu.$$

By using the invariance of the operator under the CR transform as in Lemma 3.1 and the fact that $\rho(z_\mu) \leq \rho(z)$, we deduce that

$$\begin{cases} \mathcal{L}W_\mu = 0 & \text{in } \widehat{\mathbb{H}}_+^n \setminus \{0, e_\mu\}, \\ -\partial_\lambda W_\mu \geq c(z, \mu)W_\mu & \text{on } \mathbb{H}^n \setminus \{0, e_\mu\}, \end{cases}$$

where $c(z, \mu) = \frac{p\Psi_\mu^{p-1}}{\rho^{(Q+1)-p(Q-1)}}$ and $\Psi_\mu(z)$ is between $w(z)$ and $w_\mu(z)$. By the definition of w_μ and w , we have that $c(z, \mu) \approx C/\rho^2$ at infinity.

We now define the function:

$$h_0 = \rho(z + \beta_2 e_{2n+2})^{-\beta_1},$$

where $e_{2n+2} = (0, \dots, 0, 1)$, $\beta_2 > 0$ and

$$0 < \beta_1 < Q - 1, \quad p\tilde{C}^{p-1} < \beta_1\beta_2, \quad (5.4)$$

where \tilde{C} is the constant in (5.3).

We consider

$$\widetilde{W}_\mu = W_\mu/h_0 \quad \text{in } \Sigma_\mu.$$

The following lemma will let us start moving the hyperplane T_μ from $-\infty$.

Proposition 5.1. *Assume that $w \in C^2(\widehat{\mathbb{H}}_+^n) \cap C^1(\overline{\widehat{\mathbb{H}}_+^n} \setminus \{0\})$ satisfies (5.2). Then (i) For $\mu < 0$ with $|\mu|$ large enough, if $\inf_{\Sigma_\mu} \widetilde{W}_\mu < 0$, then the infimum is attained at some point $z_0 \in \overline{\Sigma_\mu} \setminus \{e_\mu\}$. (ii) For any μ_0 , there exists an $R_1 > 0$ such that whenever $\inf_{\Sigma_\mu} \widetilde{W}_\mu$ is attained at $z_0 \in \overline{\Sigma_\mu} \setminus \{e_\mu\}$ with $\widetilde{W}_\mu(z_0) < 0$ and $\mu \leq \mu_0$, then $\rho(z_0) = d_{\widehat{\mathbb{H}}^n}(z_0, 0) \leq R_1$.*

Proof. We follow the proof of Proposition 6.3 in [11].

We first observe that by the maximum principle and Hopf's Lemma (see Lemma 4.3)

$$\min\{w(z) : z \in \partial\mathcal{B}^+(0, 1) \cap \{\lambda > 0\}\} = d > 0.$$

We define for $z \in \mathcal{A}_\varepsilon := \overline{\mathcal{B}^+(0, 1)} \setminus \mathcal{B}^+(0, \varepsilon)$ the function

$$\phi_\varepsilon(z) = d \frac{\rho(z)^{1-Q} - \varepsilon^{1-Q}}{1 - \varepsilon^{Q-1}}.$$

By Lemma 2.1 ϕ_ε is harmonic in \mathcal{A}_ε and satisfies $\frac{\partial \phi_\varepsilon}{\partial \lambda} = 0$ on $\mathcal{A}_\varepsilon \cap \{\lambda = 0\}$.

By the maximum principle and Hopf's Lemma we have that $w(z) \geq \phi_\varepsilon(z)$ for $z \in \mathcal{A}_\varepsilon$. Since for every z we have $\lim_{\varepsilon \rightarrow 0} \phi_\varepsilon(z) = d\rho(z)^{1-Q} \geq d$, we deduce that

$$w(z) \geq d \quad \text{on } \overline{\mathcal{B}^+(0, 1)} \setminus \{0\}.$$

Since $w(z) \rightarrow 0$ as $\rho(z) \rightarrow \infty$, it follows that for μ large in absolute value, we have $w \leq d$ in $\mathcal{B}^+(z_\mu, 1)$. For such μ we have clearly $W_\mu \geq 0$ on $\mathcal{B}^+(z_\mu, 1)$, and therefore

$$\widetilde{W}_\mu \geq 0 \quad \text{on } \mathcal{B}^+(z_\mu, 1). \quad (5.5)$$

It follows that, for $\mu < 0$ large in absolute value,

$$\inf_{\Sigma_\mu} \widetilde{W}_\mu < 0 \quad \text{implies} \quad \inf_{\Sigma_\mu} \widetilde{W}_\mu = \inf_{\Sigma_\mu \setminus \mathcal{B}^+(z_\mu, 1)} \widetilde{W}_\mu.$$

We conclude the proof of i) observing that $\widetilde{W}_\mu(z) \rightarrow 0$ as $\rho(z) \rightarrow \infty$.

To prove (ii), suppose that z_0 is a minimum point of \widetilde{W}_μ such that $\widetilde{W}_\mu(z_0) < 0$. We want to show that $\rho(z_0)$ cannot be too large.

By the definition of \widetilde{W}_μ , a direct calculation gives

$$\begin{cases} -\mathcal{L}\widetilde{W}_\mu = \frac{\mathcal{L}(h_0)}{h_0}\widetilde{W}_\mu + 2\left(\frac{\nabla_{\mathbb{H}} h_0}{h_0} \cdot \nabla_{\mathbb{H}} \widetilde{W}_\mu + \frac{\partial_\lambda h_0}{h_0} \partial_\lambda \widetilde{W}_\mu + 4\lambda^2 \frac{\partial_t h_0}{h_0} \partial_t \widetilde{W}_\mu\right) & \text{in } \Sigma_\mu \setminus \{e_\mu\}, \\ -\partial_\lambda \widetilde{W}_\mu \geq \left(c(z, \mu) + \frac{\partial_\lambda h_0}{h_0}\right) \widetilde{W}_\mu & \text{on } \partial\Sigma_\mu \cap \{\lambda = 0\} \setminus \{z_\mu\}, \end{cases}$$

where $c(z, \mu) = p\rho^{p(Q-1)-(Q+1)}\Psi_\mu^{p-1}$ and $\Psi_\mu(z)$ is between $w_\mu(z)$ and $w(z)$. Since $\frac{\mathcal{L}(h_0)}{h_0} = \beta_1(\beta_1 - (Q-1))\rho(z + \beta_2 e_{2n+2})^{-2} < 0$, using the maximum principle we deduce that z_0 does not belong to the interior of Σ_μ .

Assume now that $z_0 \in \partial\Sigma_\mu \cap \{\lambda = 0\} \setminus \{z_\mu\}$. As above we conclude that $\Psi_\mu(z_0) \leq w(z_0) \leq \frac{\widetilde{C}}{\rho(z_0)^{q-1}}$, and hence that $c(z_0, \mu) \leq \frac{p\widetilde{C}^{p-1}}{\rho(z_0)^2}$. Since $\frac{\partial_\lambda h_0}{h_0} = -\beta_1\beta_2\rho(z + \beta_2 e_{2n+2})^{-2}$, using assumptions (5.4), we would deduce that $\partial_\lambda \widetilde{W}_\mu(z_0) < 0$ if $\rho(z_0)$ were large enough. This is inconsistent with the fact z_0 is a minimum and concludes the proof of ii). \square

Proof of Theorem 1.4. By Proposition 5.1, we deduce that for μ negative and large in absolute value we have that $\widetilde{W}_\mu \geq 0$ and hence $W_\mu \geq 0$ in Σ_μ . Let us define $\mu_0 = \sup\{\mu < 0 \mid W_\sigma \geq 0 \text{ on } \Sigma_\sigma \setminus e_\sigma \text{ for all } \sigma < \mu\}$. We only need to prove that $\mu_0 = 0$. Suppose that $\mu_0 \neq 0$ by contradiction. By continuity, $W_{\mu_0} \geq 0$ in Σ_{μ_0} . By the maximum principle (and Hopf's Lemma), we deduce that $W_{\mu_0} \equiv 0$ in Σ_{μ_0} or

$$W_{\mu_0} > 0 \quad \text{on} \quad \Sigma_{\mu_0} \cup (\partial\Sigma_{\mu_0} \cap \{\lambda = 0\} \cap \{t < \mu_0\}) \setminus \{e_{\mu_0}\}. \quad (5.6)$$

If $W_{\mu_0} \equiv 0$, then w would be even in the t variable with respect to $t = \mu_0 < 0$ and this would contradict the Neumann condition satisfied by w (we remind that $\rho(z_{\mu_0}) < \rho(z)$ since we are assuming $\mu_0 < 0$), hence $W_{\mu_0} \equiv 0$ is impossible and therefore (5.6) holds.

By the definition of μ_0 there exists $\mu_k \rightarrow \mu_0$, $\mu_0 < \mu_k < 0$ such that $\inf_{\Sigma_{\mu_k}} W_{\mu_k} < 0$.

We observe that for some positive b_1 :

$$\min \left\{ W_{\mu_0}(z) \mid z \in \partial\mathcal{B}^+(e_{\mu_0}, |\mu_0|/2) \cap \widehat{\mathbb{H}}_+^n \right\} = b_1.$$

From this fact, using a similar argument to the one of point i) in Proposition 5.1, we deduce that

$$W_{\mu_0} \geq b_1 \quad \text{in} \quad \overline{\mathcal{B}^+(e_{\mu_0}, |\mu_0|/2)} \setminus \{e_{\mu_0}\}.$$

Therefore, we have that

$$\liminf_{k \rightarrow \infty} \left\{ W_{\mu_k}(z) \mid z \in \mathcal{B}^+(e_{\mu_k}, |\mu_0|/2) \setminus \{e_{\mu_k}\} \right\} \geq b_1.$$

Using this bound and the fact that $W_{\mu_k}(z) \rightarrow 0$ as $\rho(z) \rightarrow \infty$, we deduce that for k large enough, the negative infimum of W_{μ_k} is attained at some point $z_k \in \overline{\Sigma_{\mu_k}} \setminus \mathcal{B}^+(e_{\mu_k}, |\mu_0|/2)$.

By Proposition 5.1 we know that the sequence $\{z_k\}$ is bounded and therefore, after passing to a subsequence, we may assume that $z_k \rightarrow z_0$. By (5.6) we have that $W_{\mu_0}(z_0) = 0$ and $z_0 \in \partial\Sigma_{\mu_0} \cap \{t = \mu_0\}$.

If $z_k \in \Sigma_{\mu_k} \cap \{\lambda > 0\}$ for an infinite number of k , then $\nabla W_{\mu_k}(z_k) = 0$, and therefore, by continuity

$$\nabla W_{\mu_0}(z_0) = 0. \quad (5.7)$$

If $z_0 \in \partial\Sigma_{\mu_0} \cap \widehat{\mathbb{H}}_+^n$, then by Lemma 4.3, we have that $\frac{\partial w}{\partial t}(z_0) < 0$, which gives a contradiction. Analogously, using Lemma 4.6, we get a contradiction if we assume that $z_0 \in \partial\Sigma_{\mu_0} \cap \{\lambda = 0\} \cap \{t = \mu_0\}$.

In the case in which $z_k \in \partial\Sigma_{\mu_k} \cap \{\lambda = 0\} \cap \{t < \mu_k\}$, we still have that the derivatives of W_{μ_k} at z_k in all directions except the λ direction vanish. Passing to the limit and arguing as above, we get a contradiction. Hence we have established that $\mu_0 = 0$. This implies that v is even in t , but since the origin 0 on the t -axis is arbitrary, we

can perform the CR transform with respect to any point and then we conclude that w is constant in the direction t .

This shows that U is actually a solution of the following problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}_+^{2n+1}, \\ -\partial_\lambda U = U^p & \text{on } \mathbb{R}^{2n}. \end{cases} \quad (5.8)$$

Since $\frac{Q+1}{Q-1} = \frac{2n+3}{2n+1} < \frac{2n+1}{2n-1}$, we conclude the proof by using the standard Liouville type theorem for problem (5.8) (see [11, 24]). \square

The following lemma allows us to deduce Theorem 1.3 by Theorem 1.4.

Lemma 5.2. *Let $u \in C^2(\mathbb{H}^n)$ be cylindrically symmetric and positive (respectively nonnegative). Then the corresponding solution U of the extension problem (1.7) is cylindrically symmetric, i.e. $U = U(r_0, t, \lambda)$ with $r_0 = \sqrt{x^2 + y^2}$, U is positive (respectively nonnegative) and moreover $U \in C^2(\widehat{\mathbb{H}}_+^n) \cap C^1(\widehat{\mathbb{H}}_+^n)$.*

Proof. The main tool in the proof of the Lemma relies on the construction of the extension $U = \mathcal{E}_{1/2}u$ by using the Fourier transform in \mathbb{H}^n (we refer for details to Section 5 in [16] and to [2]). Let us recall the Fourier transform of a smooth function $u(\zeta, t)$, $(\zeta, t) \in \mathbb{H}^n$,

$$\mathcal{F}(u)(\mu) = \int_{\mathbb{H}^n} u(\zeta, t) \pi_{\zeta, t}^\mu d\zeta dt, \quad (5.9)$$

where $\pi_{\zeta, t}^\mu$ denotes the irreducible representation

$$\pi_{\zeta, t}^\mu \Psi(\xi) = \begin{cases} \Psi(\xi - \bar{\zeta}) e^{i\mu t + 2\mu(\xi \cdot \zeta - |\zeta|^2/2)}, & \mu > 0, \\ \Psi(\xi + \zeta) e^{i\mu t + 2\mu(\xi \cdot \bar{\zeta} - |\zeta|^2/2)}, & \mu < 0, \end{cases}$$

for a holomorphic function $\Psi(\xi)$, $\xi \in \mathbb{H}^n$ in the Bargmann space with orthogonal basis

$$\Psi_{\alpha, \mu}(\xi) = \frac{(\sqrt{2|\mu|}\xi)^\alpha}{\sqrt{\alpha!}}.$$

The inversion formula is given by

$$u(\zeta, t) = \frac{2^{n-1}}{\pi^{n+1}} \int_{\mathbb{R}} \text{tr } \pi_{\zeta, t}^{*, \mu} \mathcal{F}(u)(\mu) |\mu|^n d\mu \quad (5.10)$$

where $\pi_{\zeta, t}^{*, \mu} = \pi_{(\zeta, t)^{-1}}^\mu$ is the adjoint operator of $\pi_{\zeta, t}^\mu$.

The extension operator $\mathcal{E}_{1/2}$ maps a function u on the Heisenberg group to a function $U = \mathcal{E}_{1/2}u$ on $\mathbb{H}^n \times (0, \infty)$. For every $q > 0$, every

multi-index vector α and $\mu \in \mathbb{R}$, $U(\cdot, q) = \mathcal{E}_{1/2}u(\cdot, q)$ is implicitly given through the Fourier multiplier

$$\widehat{\mathcal{E}_{1/2}u(\cdot, q)}_\alpha(\mu) = \widehat{U}_\alpha(\mu, q) = \phi_\alpha(2|\mu|q) \hat{u}_\alpha(\mu), \quad (5.11)$$

where $\hat{u}_\alpha(\mu) = \mathcal{F}(u)(\mu)\Psi_{\alpha,\mu}$,

$$\phi_\alpha(y) = \frac{\Gamma\left(\frac{\frac{3}{2}+n+2|\alpha|}{2}\right)}{\Gamma(\gamma)} e^{-y/2} y^{\frac{1}{2}} V\left(\frac{\frac{3}{2}+n+2|\alpha|}{2}, \frac{3}{2}, y\right),$$

and $V(a, b, y)$ is the solution of Kummer's equation

$$V(a, b, y) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-\tau y} \tau^{a-1} (1+\tau)^{b-a-1} d\tau.$$

Note that the function $q \mapsto \widehat{\mathcal{E}_{1/2}u(\cdot, q)}_\alpha(\mu)$ solves the equation

$$\left(q\partial_{qq} + \frac{1}{2}\partial_q - \mu^2 q - |\mu|(n+2|\alpha|)\right) \widehat{\mathcal{E}_{1/2}u(\cdot, q)}_\alpha(\mu) = 0,$$

and therefore $U = \mathcal{E}_{1/2}u$ satisfies

$$\left(q\partial_{qq} + \frac{1}{2}\partial_q + q\partial_{tt} + \frac{1}{2}\Delta_b\right) U = 0. \quad (5.12)$$

We observe that the change of variable $q = \frac{\lambda^2}{2}$ transforms equation (5.12) above into the extension (1.2).

For every multi-index vector α and $\mu \in \mathbb{R}$, we have

$$\begin{aligned} U_\alpha(\tilde{\zeta}, \tilde{t}, q) &= \frac{2^{n-1}}{\pi^{n+1}} \int_{\mathbb{R}} \left[\text{tr} \pi_{\tilde{\zeta}, \tilde{t}}^{*, \mu} \phi_\alpha(2|\mu|q) \hat{u}_\alpha(\mu) |\mu|^n \right] d\mu \\ &= \frac{2^{n-1}}{\pi^{n+1}} \int_{\mathbb{R}} \left[\text{tr} \pi_{\tilde{\zeta}, \tilde{t}}^{*, \mu} \phi_\alpha(2|\mu|q) \left(\int_{\mathbb{H}^n} u(\zeta, t) \pi_{\zeta, t}^\mu d\zeta dt \Psi_{\alpha, \mu} \right) |\mu|^n \right] d\mu \\ &= \frac{2^{n-1}}{\pi^{n+1}} \int_{\mathbb{R}} \int_{\mathbb{H}^n} \left[\text{tr} \phi_\alpha(2|\mu|q) \left(u(\zeta, t) \pi_{(\tilde{\zeta}, \tilde{t})^{-1}}^\mu \pi_{\zeta, t}^\mu \Psi_{\alpha, \mu} \right) |\mu|^n \right] d\zeta dt d\mu. \end{aligned}$$

We observe that if u is cylindrically symmetric with respect to ζ , that is, for the rotation $\mathcal{R}_{x,y}$,

$$u(\zeta, t) = u(\mathcal{R}_{x,y}(\zeta), t),$$

then the extension $U_\alpha(\zeta, t, q)$ of $\widehat{\mathcal{E}_{1/2}u(\cdot, q)}_\alpha$ is cylindrically symmetric with respect to $\zeta = (x, y)$ for each q . That is,

$$U(\zeta, t, q) = U(\mathcal{R}_{x,y}(\zeta), t, q).$$

After the change of variables $q = \lambda^2/2$, we deduce that $U(\zeta, t, \lambda)$ is cylindrically symmetric with respect to $\zeta = (x, y)$ for each λ . \square

Proof of Theorem 1.3. By Lemma 5.2, we see that if u is a cylindrical function, that is $u = u(|(x, y)|, t)$, then its extension U satisfying (1.2) is also cylindrical in the all halfspace $\mathbb{H}^n \times \mathbb{R}^+$, in the sense that $U = U(|(x, y)|, t, \lambda)$. Using this fact, the conclusion follows as a corollary of Theorem 1.4. □

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