A NOTE ON A PHASE-FIELD MODEL FOR ANISOTROPIC SYSTEMS

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Abstract. We investigate, using the framework of Γ-convergence, a phase-field model proposed in [Torabi *et al*, Proc. R. Soc. A, 2009] for strongly anisotropic systems; in particular, we prove a full Γ-convergence result for an anisotropic Modica-Mortola-type energy.

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1. Introduction

In [13] a new phase-field model for strongly anisotropic crystals and epitaxial growth has been proposed using anisotropic Cahn-Hiliard-type equations regularized by an high-order Willmore term. More precisely, the authors introduced the energy functional

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega \gamma\left(\frac{\nabla u}{|\nabla u|}\right)\left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\right)dx + \frac{\beta}{2\varepsilon}\int_\Omega\left(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\right)^2 dx$$

where Ω is a bounded and open subset of $\mathbb{R}^n$, $u\colon \Omega \to \mathbb{R}$ represents the phase in a multiphase system, $\gamma\colon \{x \in \mathbb{R}^n : |x| = 1\} \to (0, +\infty)$ is a Lipschitz continuous function, W is a double well potential, with wells in 0 and 1, β is a fixed positive constant and ε is the measure of the transition layer thickness. In [13] the authors said that F_ε approaches, as $\varepsilon \to 0$, a limit energy of the form

$$\int_S \gamma(\mathbf{n})\, dS + \frac{\beta}{2}\int_S H_S^2\, dS \tag{1.1}$$

where S is the limit sharp interface created by a sequence u_ε which makes the energy equi-bounded, $\mathbf{n}$ is a unit normal to S which comes from the diffuse normal $\nabla u/|\nabla u|$, and H_S is the mean curvature at S, term that arises from the second order penalization that appears in F_ε. Notice that the limit functional (1.1) is composed by two terms: the first one represents an anisotropic perimeter, which is an interesting object recently investigated in view of other applications (see for instance [7] and [8]), while the second one turns out to be, up to a constant, the well known Willmore functional. A rigorous analysis of F_ε using the framework of Γ-convergence (see [3] and [5] for details) reduces to the well known Modica-Mortola setting (see for instance [9] and [10]) when $\gamma = 1$ and $\beta = 0$, and precisely if $F_\varepsilon(u_\varepsilon)$ is bounded then

u_ε converges, up to subsequences, to some $u \in BV(\Omega;\{0,1\})$ strongly in $L^1(\Omega)$ and the Γ-limit of F_ε is given by $c_0\mathcal{H}^{n-1}(J_u)$ where c_0 is a suitable positive constant depending only on W and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. First of all, in this paper we will investigate the behavior of the first part of F_ε; this could be interesting from a mathematical point of view even without the second order penalization. We are able to prove a complete Γ-convergence result, and roughly speaking it turns out that

$$\frac{1}{\varepsilon}\int_\Omega \gamma\left(\frac{\nabla u}{|\nabla u|}\right)\left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\right)dx \xrightarrow{\Gamma} \int_S \bar\gamma(\mathbf{n})\,dS$$

where $\bar\gamma$ is the convexification of the positively one-homogeneous extension of γ. The situation is much more complicated if we try to compute the Γ-limit of the full energy: in this case, if $\gamma = \beta = 1$ it turns out that if $F_\varepsilon(u_\varepsilon)$ is bounded then u_ε converges, up to subsequences, to some $u \in BV(\Omega;\{0,1\})$ strongly in $L^1(\Omega)$ and the Γ-limit of F_ε restricted to sufficiently smooth sets $E \subseteq \mathbb{R}^n$ (boundary of class C^2 for instance is enough) is given by

$$c_0\mathcal{H}^{n-1}(\partial E\cap\Omega) + c_0\int_{\partial E\cap\Omega} H^2_{\partial E\cap\Omega}\,d\mathcal{H}^{n-1}.$$

More precisely, the Γ-$\limsup$ inequality always holds true (see [4]) while the Γ-$\liminf$ estimate can be proved only for dimensions 2 and 3 (see [11] and [12]). In the last section of this paper we try to prove a Γ-convergence result for the energy F_ε when $\gamma \neq 1$, but unfortunately we get only partial results: we are only able to prove, and actually this will be almost straightforward, that the functional

$$\int_S \gamma(\mathbf{n})\,dS + \frac{\beta}{2}\int_S H_S^2\,dS$$

is the Γ-limit of F_ε only in dimension 2 and 3, on sufficiently smooth interfaces, and also under a suitable convexity assumption on γ.

2. Some preliminary considerations

We now move to a more precise setting, and we look better the first part of the energy, that is

$$\frac{1}{\varepsilon}\int_\Omega \gamma\left(\frac{\nabla u}{|\nabla u|}\right)\left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\right)dx \tag{2.1}$$

where the anisotropy function $\gamma\colon S^{n-1} \to (0,+\infty)$ is Lipschitz continuous with Lipschitz constant $L > 0$, and $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. The first obvious remark we can do is that the expression (2.1) does not make sense if $\nabla u = 0$; in order to avoid this problem, and to have an expression which is always well defined, we simply extend γ by its positively one-homogeneous extension $\gamma_0\colon \mathbb{R}^n \to [0,+\infty)$ given by

$$\gamma_0(\xi) := \begin{cases} \gamma\left(\dfrac{\xi}{|\xi|}\right)|\xi| & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}$$

and, instead of $\gamma(\nabla u/|\nabla u|)$ we take

$$\gamma_0\Big(\frac{\nabla u}{\sqrt{r^2+|\nabla u|^2}}\Big), \quad \text{for } r \text{ small.}$$

Notice that when $r \sim 0$ and $\nabla u \neq 0$ we get, since γ_0 is positively one-homogeneous,

$$\gamma_0\Big(\frac{\nabla u}{\sqrt{r^2+|\nabla u|^2}}\Big) = \gamma_0\Big(\frac{\nabla u}{|\nabla u|}\Big)\frac{|\nabla u|}{\sqrt{r^2+|\nabla u|^2}} \sim \gamma_0\Big(\frac{\nabla u}{|\nabla u|}\Big) = \gamma\Big(\frac{\nabla u}{|\nabla u|}\Big).$$

Therefore, the first idea is to replace the functional in (2.1) by the new one

$$\frac{1}{\varepsilon}\int_\Omega \gamma_0\Big(\frac{\nabla u}{\sqrt{r_\varepsilon^2+|\nabla u|^2}}\Big)\Big(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\Big)\,dx \tag{2.2}$$

where now $r_\varepsilon \to 0$ as $\varepsilon \to 0$; we will more precise later about the speed of convergence of r_ε to 0. In any case, the Γ-limit of (2.2) should be finite only on $BV(\Omega;\{0,1\})$, as in the classical Modica-Mortola case where simply $\gamma = 1$, and we expect that it looks like an anisotropic perimeter of the form

$$\int_{J_u} \phi(\nu_u)\,d\mathcal{H}^{n-1}$$

for some anisotropy ϕ related to γ. But, notice that if $u_\varepsilon = c$ for some c constant we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \gamma_0\Big(\frac{\nabla u_\varepsilon}{\sqrt{r^2+|\nabla u_\varepsilon|^2}}\Big)\Big(\frac{\varepsilon^2}{2}|\nabla u_\varepsilon|^2 + W(u_\varepsilon)\Big)\,dx = 0$$

which says that the Γ-limit of (2.2) is 0 on constant functions. We therefore need to prevent $\gamma_0 = 0$ without changing γ_0 on S^{n-1}. In order to do this, let $B_R(0)$ be the ball centered in 0 with radius $R > 0$, take a cut-off function $\varphi\colon \mathbb{R}^n \to [0,1]$ with $\varphi = 1$ on $\mathbb{R}^n \setminus B_{1/2}(0)$, with $\varphi = 0$ on $B_{1/4}(0)$, and with $|\nabla\varphi| \le 4$ everywhere, and let $\tilde\gamma := \varphi\gamma_0 + (1-\varphi)\frac{m}{4}$, where $m := \min\gamma$. Notice that $\tilde\gamma = \gamma_0$ on $\partial B_1(0) = S^{n-1}$. Moreover, since for any $\xi \in \overline{B_1(0)}$ with $|\xi| \ge 1/4$ we have $\gamma_0(\xi) \ge m/4$, it holds $\tilde\gamma \ge m/4$, which is what we wanted.

Proposition 2.1. *The function $\tilde\gamma$ is Lipschitz continuous.*

Proof. In order to prove that $\tilde\gamma$ is Lipschitz continuous it is sufficient to prove that γ_0 is Lipschitz continuous. The lipschitzianity of γ_0 easily descends from the elementary inequality

$$\sqrt{|\xi_1|}\sqrt{|\xi_2|}\left|\frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|}\right| \le |\xi_1 - \xi_2|, \quad \xi_1,\xi_2 \neq 0 \tag{2.3}$$

which can be proved as follows: if $\alpha := \sqrt{|\xi_1|/|\xi_2|}$ then

$$\frac{|\xi_1-\xi_2|}{\sqrt{|\xi_1|}\sqrt{|\xi_2|}} = \left|\alpha\frac{\xi_1}{|\xi_1|} - \frac{1}{\alpha}\frac{\xi_2}{|\xi_2|}\right| = \sqrt{\alpha^2 + \frac{1}{\alpha^2} - 2\frac{\xi_1}{|\xi_1|}\cdot\frac{\xi_2}{|\xi_2|}} \ge \sqrt{2 - 2\frac{\xi_1}{|\xi_1|}\cdot\frac{\xi_2}{|\xi_2|}} = \left|\frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|}\right|.$$

Let now $\xi_1, \xi_2 \in \mathbb{R}^n$ with $\xi_i \neq 0$ for $i = 1, 2$ (the case $\xi_1 = 0$ or $\xi_2 = 0$ is trivial). Without loss of generality we can assume $|\xi_1| \leq |\xi_2|$. Applying (2.3) we obtain

$$\begin{aligned}
|\gamma_0(\xi_1) - \gamma_0(\xi_2)| &= \left|\gamma\left(\frac{\xi_1}{|\xi_1|}\right)|\xi_1| - \gamma\left(\frac{\xi_2}{|\xi_2|}\right)|\xi_2|\right| \\
&= \left|\gamma\left(\frac{\xi_1}{|\xi_1|}\right)|\xi_1| - \gamma\left(\frac{\xi_2}{|\xi_2|}\right)|\xi_2 - \xi_1 + \xi_1|\right| \\
&\leq \left|\gamma\left(\frac{\xi_1}{|\xi_1|}\right)|\xi_1| - \gamma\left(\frac{\xi_2}{|\xi_2|}\right)|\xi_1|\right| + \max\gamma\,|\xi_1 - \xi_2| \\
&\leq L|\xi_1|\left|\frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|}\right| + \max\gamma\,|\xi_1 - \xi_2| \\
&\leq L\sqrt{|\xi_1|}\sqrt{|\xi_2|}\left|\frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|}\right| + \max\gamma|\xi_1 - \xi_2| \\
&\leq (L + \max\gamma)|\xi_1 - \xi_2|
\end{aligned}$$

and this concludes the proof. □

We have therefore obtained a Lipschitz continuous function $\tilde{\gamma}\colon \mathbb{R}^n \to (0, +\infty)$, and the idea is to consider the energy functional

$$\frac{1}{\varepsilon}\int_\Omega \tilde{\gamma}\left(\frac{\nabla u}{\sqrt{r^2 + |\nabla u|^2}}\right)\left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\right)dx$$

which will be studied in the next section.

3. Analysis of the anisotropic Modica-Mortola energy

In this section we investigate the behavior of the first part of the energy in a general setting.

3.1. **The anisotropy.** Let $\gamma\colon \mathbb{R}^n \to (0, +\infty)$ be a Lipschitz continuous function with Lipschitz constant $L > 0$ and with $\gamma(\xi) = \gamma(-\xi)$ for each $\xi \in S^{n-1}$. For any $r > 0$ let $\gamma_r\colon \mathbb{R}^n \to (0, +\infty)$ be given by

$$\gamma_r(\xi) := \gamma\left(\frac{\xi}{\sqrt{r^2 + |\xi|^2}}\right).$$

For any $\xi \in \mathbb{R}^n$ let

$$\gamma_0(\xi) := \begin{cases} \gamma\left(\dfrac{\xi}{|\xi|}\right)|\xi| & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

For any $r > 0$ and for any $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ we have

$$\left|\gamma\left(\frac{\xi}{\sqrt{r^2 + |\xi|^2}}\right) - \gamma\left(\frac{\xi}{|\xi|}\right)\right| \leq L\left|\frac{\xi}{\sqrt{r^2 + |\xi|^2}} - \frac{\xi}{|\xi|}\right| \leq \frac{Lr}{2|\xi|}$$

from which, multiplying both sides by $|\xi|$, we get the following useful estimate:

$$|\gamma_r(\xi)|\xi| - \gamma_0(\xi)| \leq Lr/2. \tag{3.1}$$

3.2. **Statement of the convergence result.** Let $W\colon \mathbb{R} \to [0,+\infty)$ be the double-well potential given by $W(t) := t^2(1-t)^2/4$. For each $\varepsilon > 0$ let $r_\varepsilon = O(\varepsilon)$. We define the energy $G_\varepsilon\colon L^1(\Omega) \to [0,+\infty]$ given by

$$G_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega \gamma_{r_\varepsilon}(\nabla u)\left(\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\right)dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

We are able to prove a full Γ-convergence result for G_ε.

Theorem 3.1. *The family G_ε Γ-converges, as $\varepsilon \to 0$, strongly in $L^1(\Omega)$, to*

$$G(u) := \begin{cases} c_0 \displaystyle\int_{J_u} \gamma_0^{**}(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega;\{0,1\}) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where γ_0^{**} is the convex envelope of γ_0, ν_u is a unit normal at J_u and $c_0 := \int_0^1 \sqrt{2W(s)}\,ds$.

3.3. **The lower estimate.** We first recall a relaxation result (see Thm. 1.1 in [6]).

Theorem 3.2. *Assume that $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty)$ is a Borel integrand, $f(x,u,\cdot)$ is convex in $\mathbb{R}^n$, and for all $(x_0,u_0) \in \Omega \times \mathbb{R}$ and $\eta > 0$ there exists $\delta > 0$ such that $f(x_0,u_0,\xi) - f(x,u,\xi) \le \eta(1 + f(x,u,\xi))$ for all $(x,u) \in \Omega \times \mathbb{R}$ with $|x-x_0| + |u-u_0| \le \delta$ and for all $\xi \in \mathbb{R}^n$. Let $u \in BV_{\rm loc}(\Omega)$ and let $u_j \to u$ strongly in $L^1_{\rm loc}(\Omega)$, with $u_j \in W^{1,1}_{\rm loc}(\Omega)$. Then*

$$\int_\Omega f(x,u,\nabla u)\,dx + \int_\Omega f^\infty(x,u,dC(u)) + \int_{J_u}\int_{u^-(x)}^{u^+(x)} f^\infty(x,s,\nu_u)\,ds\,d\mathcal{H}^{n-1} \le \liminf_j \int_\Omega f(x,u_j,\nabla u_j)\,dx$$

where $dC(u)$ is the Cantor part of the distributional derivative of u and the so called recession function of f is given by

$$f^\infty(x,u,\xi) := \limsup_{t\to+\infty} \frac{f(x,u,t\xi)}{t}.$$

In the next Proposition we prove the Γ-liminf inequality for G_ε; for, let us fix a positive infinitesimal sequence (ε_j).

Proposition 3.3. *Let $u \in L^1(\Omega)$ and let (u_j) be a sequence in $H^1(\Omega)$ with $\sup_j G_{\varepsilon_j}(u_j) < +\infty$. Then, up to subsequence, not relabeled, we have $u_j \to u$ strongly in $L^1(\Omega)$ for some $u \in BV(\Omega;\{0,1\})$. Moreover, for any $u \in BV(\Omega;\{0,1\})$ and for any sequence (u_j) in $H^1(\Omega)$ with $u_j \to u$ strongly in $L^1(\Omega)$, we have*

$$\liminf_j G_{\varepsilon_j}(u_j) \ge G(u). \tag{3.2}$$

Proof. First of all, notice that

$$\sup_j G_{\varepsilon_j}(u_j) \geq \frac{\min\gamma}{\varepsilon_j}\int_\Omega \left(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2 + W(u_j)\right) dx.$$

Hence, the compactness of the sequence (u_j) descends from the well known compactness property of the Modica-Mortola functional (see for instance [9] and [10]). Therefore, in order to prove (3.2) we can assume $G_{\varepsilon_j}(u_j) \leq M$ for some $M > 0$ and $u \in BV(\Omega; \{0,1\})$. First, notice that by Hölder inequality

$$\int_\Omega \sqrt{2W(u_j)}\, dx \leq \sqrt{\frac{2|\Omega| M\varepsilon_j}{\min\gamma}}$$

and in particular $\int_\Omega \sqrt{2W(u_j)}\, dx$ is bounded. Combining Young inequality with (3.1) we obtain, since trivially $\gamma_0 \geq \gamma_0^{**}$,

$$G_{\varepsilon_j}(u_j) \geq \int_\Omega \sqrt{2W(u_j)}\gamma_{r_{\varepsilon_j}}(\nabla u_j)|\nabla u_j|\, dx \geq \int_\Omega \sqrt{W(u_j)}\gamma_0^{**}(\nabla u_j)\, dx - \frac{r_{\varepsilon_j}L}{2}\int_\Omega \sqrt{2W(u_j)}\, dx.$$

Since $r_{\varepsilon_j} \to 0$ we get

$$\liminf_j G_{\varepsilon_j}(u_j) \geq \liminf_j \int_\Omega \sqrt{W(u_j)}\gamma_0^{**}(\nabla u_j)\, dx.$$

We are now in position to apply Theorem 3.2 with choice $f(x,u,\xi) := \sqrt{2W(u)}\gamma_0^{**}(\xi)$, from which we immediately obtain, since γ_0^{**} remains positively one-homogeneous and since the very definition of c_0,

$$\liminf_j \int_\Omega \sqrt{2W(u_j)}\gamma_0^{**}(\nabla u_j)\, dx \geq c_0 \int_{J_u} \gamma_0^{**}(\nu_u)\, d\mathcal{H}^{n-1}$$

which concludes the proof of (3.2). □

3.4. **The upper estimate.** We first recall some well known results in order to construct the recovery sequence; for more details see [4]. We notice that in order to use the same construction also for the higher order part of the full energy functional we directly treat the case of C^2-boundaries, even if for the first order energy we actually need less regularity. Let $E \subset \mathbb{R}^n$ be a bounded and open set with $\partial E \cap \Omega \in C^2$ and let d_E be the signed distance from $\partial E \cap \Omega$, that is

$$d_E(x) := \begin{cases} d(x, \partial E) & \text{if } x \in E \\ -d(x,\partial E) & \text{if } x \notin E. \end{cases}$$

Then, in a small tubular neighborhood of ∂E it holds $d_E \in C^2$ and $|\nabla d_E| = 1$. Moreover, if we let $E_t := \{x \in \Omega : d_E(x) = t\}$, for $t \in \mathbb{R}$, it turns out that

$$\lim_{t\to 0} \mathcal{H}^{n-1}(E_t) = \mathcal{H}^{n-1}(\partial E \cap \Omega) \tag{3.3}$$

and

$$\Delta d_E = -|H_{E_t}|, \quad \text{on } E_t. \tag{3.4}$$

Let now $\varphi\colon \mathbb{R} \to (0,1)$ be the unique solution of the problem

$$\begin{cases} \varphi'(t) = \sqrt{2W(\varphi(t))} \\ \lim_{t\to-\infty} \varphi(t) = 0, \ \lim_{t\to+\infty} \varphi(t) = 1. \end{cases} \tag{3.5}$$

For any $\varepsilon > 0$ let

$$\Phi_\varepsilon(t) := \begin{cases} \varphi(t) & \text{if } 0 \le t < |\log\varepsilon| \\ p_\varepsilon(t) & \text{if } |\log\varepsilon| \le t \le 2|\log\varepsilon| \\ 1 & \text{if } t > 2|\log\varepsilon| \\ -\Phi_\varepsilon(-t)+1 & \text{if } t < 0 \end{cases}$$

where p_ε is a third-degree polynomial in such a way

$$\Phi_\varepsilon \ \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{\pm|\log\varepsilon|, \pm 2|\log\varepsilon|\}).$$

For any $t \in \mathbb{R}$ let $\varphi_\varepsilon(t) := \Phi_\varepsilon(t/\varepsilon)$. It is straightforward to verify that

$$\varphi_\varepsilon \to \chi_{(0,+\infty)}, \quad \text{as } \varepsilon \to 0 \text{ in } L^1(\mathbb{R}), \tag{3.6}$$

and

$$||\varphi_\varepsilon'||_{L^\infty(\varepsilon|\log\varepsilon|, 2\varepsilon|\log\varepsilon|)} = o(\varepsilon^2). \tag{3.7}$$

Let now

$$u_\varepsilon(x) := \varphi_\varepsilon(d_E(x)), \quad \forall x \in \Omega. \tag{3.8}$$

Since the regularity of d_E we obtain $u_\varepsilon \in H^2(\Omega)$ for ε sufficiently small. Moreover, notice that $\chi_E(x) = \chi_{(0,+\infty)}(d_E(x))$ for each $x \in \Omega$; then, by the coarea formula, since $|\nabla d_E| = 1$ near $\partial E \cap \Omega$,

$$\begin{aligned} \int_\Omega |u_\varepsilon(x) - \chi_E(x)|\,dx &= \int_\Omega |\varphi_\varepsilon(d_E(x)) - \chi_{(0,+\infty)}(d(x))|\,dx \\ &= \int_\Omega |\varphi_\varepsilon(d_E(x)) - \chi_{(0,+\infty)}(d_E(x))|\,|\nabla d_E(x)|\,dx \\ &= \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(E_t)|\varphi_\varepsilon(t) - \chi_{(0,+\infty)}(t)|\,dt. \end{aligned}$$

Using now (3.3) and (3.6) we get, from the previous computation,

$$u_\varepsilon \to \chi_E, \quad \text{as } \varepsilon \to 0 \text{ in } L^1(\Omega). \tag{3.9}$$

We are ready to start with the proof of the estimates from above. For, fix a positive infinitesimal sequence (ε_j).

Proposition 3.4. *Let $E \subset \mathbb{R}^n$ be a bounded and open set with $\partial E \cap \Omega \in C^2$ and let $u_j := u_{\varepsilon_j}$, where u_{ε_j} is given by* (3.8)*. Then*

$$\limsup_j G_{\varepsilon_j}(u_j) \le c_0 \int_{\partial E \cap \Omega} \gamma(\nu_E)\,d\mathcal{H}^{n-1}. \tag{3.10}$$

Proof. For any $j \in \mathbb{N}$ we let

$$A_j := \{x \in \Omega : |d_E(x)| < \varepsilon_j |\log \varepsilon_j|\}, \quad B_j := \{x \in \Omega : \varepsilon_j |\log \varepsilon_j| < |d_E(x)| < 2\varepsilon_j |\log \varepsilon_j|\}.$$

Since

$$\nabla u_j(x) = \frac{1}{\varepsilon_j}\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\nabla d_E(x), \quad \text{a.e. } x \in A_j \tag{3.11}$$

taking into account (3.5) we easily deduce that, for j sufficiently large,

$$\frac{\varepsilon_j^2}{2}|\nabla u_j(x)|^2 + W(u_j(x)) = \varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W(u_j(x))}, \quad \text{a.e. } x \in A_j. \tag{3.12}$$

We first claim that

$$\limsup_j \frac{1}{\varepsilon_j}\int_{A_j}\gamma_{\varepsilon_j}(\nabla u_j)\left(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2 + W(u_j)\right)dx \le c_0\int_{\partial E\cap\Omega}\gamma(\nu_E)\,d\mathcal{H}^{n-1}. \tag{3.13}$$

Using (3.1), (3.12), and the positive one-homogeneity of γ_0, we get

$$\begin{aligned}
&\frac{1}{\varepsilon_j}\int_{A_j}\gamma_{\varepsilon_j}(\nabla u_j(x))\left(\frac{\varepsilon_j^2}{2}|\nabla u_j(x)|^2 + W(u_j(x))\right)dx\\
&\quad= \frac{1}{\varepsilon_j}\int_{A_j}\gamma_{\varepsilon_j}\left(\frac{1}{\varepsilon_j}\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\nabla d_E(x)\right)\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W(u_j(x))}\,dx\\
&\quad\le \frac{1}{\varepsilon_j}\int_{A_j}\gamma(\nabla d_E(x))\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W(u_j(x))}\,dx + \frac{Lr_{\varepsilon_j}}{2\varepsilon_j}\int_{A_j}\sqrt{2W(u_j(x))}\,dx.
\end{aligned}$$

Since $W(u_j)$ is bounded, $|A_j| \to 0$ and $r_{\varepsilon_j} = O(\varepsilon_j)$, we have

$$\lim_j \frac{r_{\varepsilon_j}}{\varepsilon_j}\int_{A_j}\sqrt{2W(u_j(x))}\,dx = 0.$$

Therefore,

$$\begin{aligned}
&\limsup_j \frac{1}{\varepsilon_j}\int_{A_j}\gamma_{\varepsilon_j}(\nabla u_j)\left(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2 + W(u_j)\right)dx\\
&\qquad\le \limsup_j \frac{1}{\varepsilon_j}\int_{A_j}\gamma(\nabla d_E(x))\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W(u_j(x))}\,dx.
\end{aligned}$$

By coarea formula we deduce that

$$\begin{aligned}
&\frac{1}{\varepsilon_j}\int_{A_j}\gamma(\nabla d_E(x))\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W(u_j(x))}\,dx\\
&\quad= \frac{1}{\varepsilon_j}\int_{-\varepsilon_j|\log\varepsilon_j|}^{\varepsilon_j|\log\varepsilon_j|}\int_{E_t}\gamma(\nabla d_E(x))\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W\left(\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\right)}\,d\mathcal{H}^{n-1}(x)dt\\
&\quad= \frac{1}{\varepsilon_j}\int_{-\varepsilon_j|\log\varepsilon_j|}^{\varepsilon_j|\log\varepsilon_j|}\varphi'\left(\frac{t}{\varepsilon_j}\right)\sqrt{2W\left(\varphi\left(\frac{t}{\varepsilon_j}\right)\right)}\int_{E_t}\gamma(\nabla d_E(x))\,d\mathcal{H}^{n-1}(x)dt\\
&\quad= \int_{-|\log\varepsilon_j|}^{|\log\varepsilon_j|}\varphi'(s)\sqrt{2W(\varphi(s))}\int_{E_{\varepsilon_j s}}\gamma(\nabla d_E(x))\,d\mathcal{H}^{n-1}(x)ds.
\end{aligned}$$

For each t with $|t|$ small let $F^t \subseteq \mathbb{R}^n$ be such that $\partial F^t \cap \Omega = E_t$. For any $s \in \mathbb{R}$ we easily get $D\chi_{F^{\varepsilon_j s}} \rightharpoonup^* \mathcal{H}^{n-1} \llcorner (\partial E \cap \Omega)$ as $j \to +\infty$; moreover, since (3.3) holds we also get $|D\chi_{F^{\varepsilon_j s}}|(\Omega) \to \mathcal{H}^{n-1}(\partial E \cap \Omega)$ as $j \to +\infty$. For each $x \in E_{\varepsilon_j s}$ we have

$$\frac{dD\chi_{F^{\varepsilon_j s}}}{d|D\chi_{F^{\varepsilon_j s}}|}(x) = \nabla d_E(x)$$

and therefore

$$\int_{E_{\varepsilon_j s}} \gamma(\nabla d_E(x))\, d\mathcal{H}^{n-1}(x) = \int_\Omega \gamma\left(\frac{dD\chi_{F^{\varepsilon_j s}}}{d|D\chi_{F^{\varepsilon_j s}}|}(x)\right) d|D\chi_{F^{\varepsilon_j s}}|(x).$$

Since γ is continuous and bounded and we have convergence of the masses, applying Reshetnyak Continuity Theorem we can say that

$$\lim_{j\to+\infty} \int_\Omega \gamma\left(\frac{dD\chi_{F^{\varepsilon_j s}}}{d|D\chi_{F^{\varepsilon_j s}}|}(x)\right) d|D\chi_{F^{\varepsilon_j s}}|(x) = \int_{\partial E\cap\Omega} \gamma(\nu_E)\, d\mathcal{H}^{n-1}.$$

Finally, observing that

$$\int_{-|\log\varepsilon_j|}^{|\log\varepsilon_j|} \varphi'(s)\sqrt{2W(\varphi(s))} \int_{E_{\varepsilon_j s}} \gamma(\nabla d_E(x))\, d\mathcal{H}^{n-1}(x) ds \\ \leq c \int_{-\infty}^{+\infty} \varphi'(s)\sqrt{2W(\varphi(s))}\mathcal{H}^{n-1}(E_{\varepsilon_j s})\, ds,$$

using the Dominated Convergence Theorem we deduce that

$$\lim_j \frac{1}{\varepsilon_j} \int_{A_j} \gamma(\nabla d_E(x))\varphi'\left(\frac{d_E(x)}{\varepsilon_j}\right)\sqrt{2W(u_j(x))}\, dx \\ = \int_{-\infty}^{+\infty} \varphi'(s)\sqrt{2W(\varphi(s))}\, ds \int_{\partial E\cap\Omega} \gamma(\nu_E)\, d\mathcal{H}^{n-1}.$$

The proof of (3.13) is now complete: it is sufficient to use (3.5) and the very definition of c_0. In order to conclude it remains to prove that

$$\limsup_j \frac{1}{\varepsilon_j} \int_{B_j} \gamma_{\varepsilon_j}(\nabla u_j)\left(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2 + W(u_j)\right) dx = 0 \tag{3.14}$$

since

$$\int_{\Omega\setminus(A_j\cup B_j)} \gamma_{\varepsilon_j}(\nabla u_j)\left(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2 + W(u_j)\right) dx = 0$$

for any j. Using the very definition of u_j and (3.7), we can say that $\gamma_{\varepsilon_j}(\nabla u_j)$ is bounded on B_j, and thus we find that

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_{B_j}\gamma_{\varepsilon_j}(\nabla u_j)\Big(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2+W(u_j)\Big)\,dx \\ &\le \frac{c}{\varepsilon_j}\int_{B_j}\Big(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2+W(u_j)\Big)\,dx \\ &= \frac{c}{\varepsilon_j}\int_{B_j}\Big(\frac{1}{2}\Big|\varphi'_{\varepsilon_j}\Big(\frac{d_E(x)}{\varepsilon_j}\Big)\Big|^2+W\Big(\varphi_{\varepsilon_j}\Big(\frac{d_E(x)}{\varepsilon_j}\Big)\Big)\Big)\,dx.\end{aligned}$$

Using again the coarea formula, it follows that

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_{B_j}\Big(\frac{1}{2}\Big|\varphi'_{\varepsilon_j}\Big(\frac{d_E(x)}{\varepsilon_j}\Big)\Big|^2+W\Big(\varphi_{\varepsilon_j}\Big(\frac{d_E(x)}{\varepsilon_j}\Big)\Big)\Big)\,dx \\ &= \int_{|\log\varepsilon_j|}^{2|\log\varepsilon_j|}\Big(\frac{1}{2}|\varphi'_{\varepsilon_j}(s)|^2+W(\varphi_{\varepsilon_j}(s))\Big)\mathcal{H}^{n-1}(E_{\varepsilon_j s})\,ds.\end{aligned}$$

Combining (3.3) with (3.7) we obtain

$$\limsup_j \int_{|\log\varepsilon_j|}^{2|\log\varepsilon_j|}|\varphi'_{\varepsilon_j}(s)|^2\mathcal{H}^{n-1}(E_{\varepsilon_j s})\,ds=0$$

while by the Dominated Convergence Theorem we get

$$\limsup_j \int_{|\log\varepsilon_j|}^{2|\log\varepsilon_j|}W(\varphi_{\varepsilon_j}(s))\mathcal{H}^{n-1}(E_{\varepsilon_j s})\,ds=0$$

since (3.3) and (3.6) hold true and, using the very definition of W,

$$W(\varphi_{\varepsilon_j}(s))\le\frac{|1-\varphi_{\varepsilon_j}(s)|}{2},\quad \forall s\ge 0.$$

Combining all these facts we deduce (3.14). □

We are now in position to complete the estimate from above by means of a relaxation argument. In order to do this, we recall that, by standard Γ-convergence theory, we have to prove that $G''(u)\le G(u)$ everywhere, where G'' is the so called Γ-lim sup given by

$$G''(u):=\inf\{\limsup_j G_{\varepsilon_j}(u_j): u_j\to u \text{ in } L^1(\Omega)\}.$$

We also recall that G'' turns out to be always a lower semicontinuous functional.

Proposition 3.5. *For each $u\in L^1(\Omega)$ we have $G''(u)\le G(u)$.*

Proof. Of course, it is sufficient to consider the case $u\in BV(\Omega;\{0,1\})$. Formula (3.10) says that for any $E\subset\mathbb{R}^n$ bounded, open and with $\partial E\cap\Omega\in C^2$, it holds

$$G''(\chi_E)\le c_0\int_{\partial E\cap\Omega}\gamma(\nu_E)\,d\mathcal{H}^{n-1}.$$

Fix $u \in BV(\Omega;\{0,1\})$; in particular, $u = \chi_E$, where E is a set with finite perimeter in Ω. Let (E_h) be a sequence of bounded, open sets with $\partial E_h \cap \Omega \in C^2$, for any $h \in \mathbb{N}$, such that $\chi_{E_h} \to u$ in $L^1(\Omega)$ and

$$\lim_h |D\chi_{E_h}|(\Omega) = |Du|(\Omega);$$

such a sequence always exists, see for instance Thm. 3.42 in [2]. Let $\mu_h := D\chi_{E_h}$. Using again Reshetniak Continuity Theorem and taking into account that G'' is lower semicontinuous with respect to the L^1-convergence, we get

$$\begin{aligned} G''(u) &\le c_0 \liminf_{h\to+\infty} \int_{\partial E_h \cap \Omega} \gamma(\nu_{E_h})\, d\mathcal{H}^{n-1} = c_0 \liminf_{h\to+\infty} \int_\Omega \gamma\left(\frac{d\mu_h}{d|\mu_h|}\right) d|\mu_h| \\ &= c_0 \int_{\partial E \cap \Omega} \gamma(\nu_E)\, d\mathcal{H}^{n-1} = c_0 \int_{J_u} \gamma(\nu_u)\, d\mathcal{H}^{n-1}. \end{aligned}$$

Therefore, we have that

$$G''(u) \le c_0 \int_{J_u} \gamma(\nu_u)\, d\mathcal{H}^{n-1}$$

holds true for each $u \in BV(\Omega;\{0,1\})$. Let $J\colon L^1(\Omega) \to [0,+\infty]$ be given by

$$J(u) := \begin{cases} \displaystyle\int_{J_u} \gamma(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega;\{0,1\}) \\ +\infty & \text{otherwise.} \end{cases}$$

Since G'' is lower semicontinuous with respect to the L^1-convergence, it holds $G''(u) \le \bar{J}(u)$ where $\bar{J}$ is the L^1-relaxed functional of J. It is well known that $\bar{J}$ takes the form

$$\bar{J}(u) = \begin{cases} \displaystyle\int_{J_u} E\gamma(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega;\{0,1\}) \\ +\infty & \text{otherwise} \end{cases}$$

where $E\gamma\colon S^{n-1} \to [0,+\infty)$ is the BV-elliptic envelope of $\gamma_{|_{S^{n-1}}}$ (see [1] for details), that is the bigger BV-elliptic function less than $\gamma_{|_{S^{n-1}}}$. In order to conclude the proof it is sufficient to show that $E\gamma \le \gamma_0^{**}$. First of all, since $E\gamma \le \gamma$, we also deduce that $(E\gamma)_0 \le \gamma_0$, where we have denoted by $(E\gamma)_0\colon \mathbb{R}^n \to [0,+\infty)$ the positively one-homogeneous extension of $E\gamma$. But since $(E\gamma_0) = E\gamma$ on S^{n-1} the functional

$$u \mapsto \begin{cases} \displaystyle\int_{J_u} (E\gamma)_0(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega;\{0,1\}) \\ +\infty & \text{otherwise} \end{cases}$$

remains lower semicontinuous, and then, taking into account, for instance, Thm. 5.11 in [2], it descends that $(E\gamma)_0$ is convex, so that $(E\gamma)_0 \le \gamma_0^{**}$, which gives $E\gamma \le \gamma_0^{**}$, and this yields the conclusion. □

4. Some remarks about the full energy

In this section we consider the full energy $F_\varepsilon \colon L^1(\Omega) \to [0,+\infty]$ given by $F_\varepsilon := G_\varepsilon + H_\varepsilon$, where

$$H_\varepsilon(u) := \begin{cases} \dfrac{\beta}{2\varepsilon}\displaystyle\int_\Omega \Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big)^2 dx & \text{if } u \in H^2(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

and $\beta > 0$ is a fixed constant. Following Röger and Schätzle (see [11]) it is possibile to prove, by means of varifolds approach, that when $n = 2,3$ and $E \subset \mathbb{R}^n$ is bounded and open with $\partial E \cap \Omega \in C^2$, for any sequence (u_j) in $H^2(\Omega)$ with $u_j \to \chi_E$ strongly in $L^1(\Omega)$ and with

$$\sup_j \frac{1}{\varepsilon_j}\int_\Omega \Big(\frac{\varepsilon_j^2}{2}|\nabla u_j|^2 + W(u_j)\Big)\, dx < +\infty$$

it holds

$$\liminf_j H_{\varepsilon_j}(u_j) \geq \frac{c_0\beta}{2}\int_{\partial E\cap\Omega} H^2_{\partial E\cap\Omega}\, d\mathcal{H}^{n-1}$$

where we remember that $H_{\partial E\cap\Omega}$ denotes the mean curvature of $\partial E \cap \Omega$. Therefore, by the subadditivity of the lim inf operator we deduce that

$$\liminf_j F_{\varepsilon_j}(u_j) \geq c_0 \int_{\partial E\cap\Omega} \gamma_0^{**}(\nu_E)\, d\mathcal{H}^{n-1} + \frac{c_0\beta}{2}\int_{\partial E\cap\Omega} H^2_{\partial E\cap\Omega}\, d\mathcal{H}^{n-1}. \tag{4.1}$$

Concerning the Γ-lim sup estimate we notice first that for the same sequence u_j used for the upper estimate for G_ε we get, always when $n = 2,3$ and $E \subset \mathbb{R}^n$ is bounded and open with $\partial E \cap \Omega \in C^2$,

$$\limsup_j H_{\varepsilon_j}(u_j) \leq \frac{c_0\beta}{2}\int_{\partial E\cap\Omega} H^2_{\partial E\cap\Omega}\, d\mathcal{H}^{n-1}$$

and this fact has been proven in [4]. Combining (3.10) with the superadditivity of the lim sup operator we obtain

$$\limsup_j F_{\varepsilon_j}(u_j) \leq c_0 \int_{\partial E\cap\Omega} \gamma(\nu_E)\, d\mathcal{H}^{n-1} + \frac{c_0\beta}{2}\int_{\partial E\cap\Omega} H^2_{\partial E\cap\Omega}\, d\mathcal{H}^{n-1}. \tag{4.2}$$

Taking into account (4.1) with (4.2) we deduce the following Theorem.

Theorem 4.1. *Assume $n = 2,3$ and γ_0 convex. Then, for each $E \subset \mathbb{R}^n$ bounded, open, with $\partial E \cap \Omega \in C^2$ we have*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(\chi_E) = c_0 \int_{\partial E\cap\Omega} \gamma(\nu_E)\, d\mathcal{H}^{n-1} + \frac{c_0\beta}{2}\int_{\partial E\cap\Omega} H^2_{\partial E\cap\Omega}\, d\mathcal{H}^{n-1}$$

with respect to the strong L^1-convergence.

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