WEIGHTED SOBOLEV SPACES ON METRIC MEASURE SPACES

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Abstract. We investigate weighted Sobolev spaces on metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$. Denoting by ρ the weight function, we compare the space $W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$ (which always concides with the closure $H^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$ of Lipschitz functions) with the weighted Sobolev spaces $W^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m})$ and $H^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m})$ defined as in the Euclidean theory of weighted Sobolev spaces. Under mild assumptions on the metric measure structure and on the weight we show that $W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) = H^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m})$. We also adapt the results in [24] and in the recent paper [28] to the metric measure setting, considering appropriate conditions on ρ that ensure the equality $W^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m}) = H^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m})$.

1. Introduction

The theory of Sobolev spaces $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ with $p \in (1, \infty)$ on metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$ has by now reached a mature stage, after the seminal papers [11], [25], the more recent developments in [5] and the monographs [9], [20]. In this context, it is natural to investigate to what extent the Sobolev space is sensitive to the reference measure $\mathfrak{m}$. It is clear that the measure $\mathfrak{m}$ is involved, since we impose $L^p(\mathfrak{m})$ summability of the weak gradient, but things are more subtle. Indeed, the measure $\mathfrak{m}$ is also involved in the definition of $(p, \mathfrak{m})$-modulus $\mathrm{Mod}_{p,\mathfrak{m}}$ (Definition 2.2) which, in turn, plays a role in the axiomatization in [25]: by definition, $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ if there exist a representative $\widetilde{f} \in L^p(\mathfrak{m})$ of f and $g \in L^p(\mathfrak{m})$ such that

$$|\widetilde{f}(\gamma(1)) - \widetilde{f}(\gamma(0))| \leq \int_0^1 g(\gamma(t))|\dot{\gamma}(t)|\mathrm{d}t$$

along $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. absolutely continuous curve $\gamma : [0, 1] \to X$. If such a function g exists, then there is one with minimal $L^p(\mathfrak{m})$ norm which is called the minimal gradient.

The definition adopted in [11], instead, is equivalent but based on the approximation in $L^p(\mathfrak{m})$ with functions having an upper gradient in $L^p(\mathfrak{m})$. More recently, in [4, 5] it has been proved that $W^{1,p}(X, \mathrm{d}, \mathfrak{m}) = H^{1,p}(X, \mathrm{d}, \mathfrak{m})$, where the latter space is defined as the collection of all $L^p(\mathfrak{m})$ functions for which there exist $f_n \in \mathrm{Lip}(X) \cap L^p(\mathfrak{m})$ with $\int_X |f_n - f|^p \mathrm{d}\mathfrak{m} \to 0$ and

$$\sup_{n\in\mathbb{N}} \int_X |\nabla f_n|^p \mathrm{d}\mathfrak{m} < \infty$$

(here and in the sequel $|\nabla g|$ denotes the local Lipschitz constant of g). In this case, following [11], the minimal gradient is defined by considering all functions larger than weak $L^p(\mathfrak{m})$ limits of local Lipschitz constants of sequences of Lipschitz functions converging to f in $L^p(\mathfrak{m})$, and considering the function with smallest $L^p(\mathfrak{m})$ norm. This general "$H = W$" result does not depend on structural assumptions on the metric measure structure: (X, d)

complete and separable and $\mathfrak{m}$ a locally finite Borel measure are sufficient for the validity of this identification theorem. In this introduction we shall denote by $|\nabla f|_w$ the minimal gradient arising from both unweighted W and H definitions, not emphasizing its potential dependence on p (see [14]).

Given a Borel weight function $\rho : X \to [0, \infty]$, in this paper we compare spaces $H = W$ relative to the metric measure structure $(X, \mathrm{d}, \rho\mathfrak{m})$ with the weighted spaces built as in the Euclidean theory (namely $X = \mathbb{R}^n$, d =Euclidean distance, $\mathfrak{m} = \mathscr{L}^n$, the Lebesgue measure in $\mathbb{R}^n$). The first weighted space is

$$W^{1,p}_\rho := \left\{ f \in W^{1,1}(X, \mathrm{d}, \mathfrak{m}) : \ |f| + |\nabla f|_w \in L^p(\rho\mathfrak{m}) \right\} \tag{1.1}$$

endowed with the norm

$$\|f\|_\rho := \left(\int_X |f|^p \rho \mathrm{d}\mathfrak{m} + \int_X |\nabla f|^p_w \rho \mathrm{d}\mathfrak{m} \right)^{1/p},$$

where $|\nabla f|_w$ is the minimal $1-$weak gradient of f with respect to the unweighted space $(X, \mathrm{d}, \mathfrak{m})$. Minimal regularity requirements (which provide respectively local finiteness of $\rho\mathfrak{m}$ and a basic embedding in $W^{1,1}$) are that $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ and $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$. The second space we will consider is the subspace $H^{1,p}_\rho$ of $W^{1,p}_\rho$ defined by

$$H^{1,p}_\rho := \overline{\mathrm{Lip}(X) \cap W^{1,p}_\rho}^{\|\cdot\|_\rho}. \tag{1.2}$$

Even when the metric measure structure is Euclidean, it is well known that $H^{1,p}_\rho$ can be strictly included in $W^{1,p}_\rho$, see Section 5 for a more detailed discussion and examples. This gap suggests a discrepancy between the weighted spaces $H = W$ of the metric theory, obtained by considering $\rho\mathfrak{m}$ as reference measure, and the spaces $W^{1,p}_\rho$, $H^{1,p}_\rho$.

Our first main result states that if $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$, $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$, $(X, \mathrm{d}, \mathfrak{m})$ is doubling and supports a 1-Poincaré inequality for Lipschitz functions, then

$$W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) = H^{1,p}_\rho \tag{1.3}$$

and the two spaces are isometric. Hence, the smaller of the two weighted Sobolev spaces can be naturally identified with the Sobolev space of the metric theory, with weighted reference measure $\rho\mathfrak{m}$.

In light of the equality $H = W$ and the Euclidean counterexamples to $H^{1,p}_\rho = W^{1,p}_\rho$, it is natural to expect that stronger integrability properties of ρ are needed to establish the equality $W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) = W^{1,p}_\rho$, namely

$$W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) = \left\{ f \in W^{1,1}(X, \mathrm{d}, \mathfrak{m}) : \ |f| + |\nabla f|_w \in L^p(\rho\mathfrak{m}) \right\}. \tag{1.4}$$

Notice that the inclusion $\subset$ readily follows by (1.3). Our second main result shows that (1.4) holds provided $(X, \mathrm{d}, \mathfrak{m})$ is doubling, supports a 1-Poincaré inequality for Lipschitz functions and ρ satisfies the asymptotic condition

$$\liminf_{n\to\infty} \frac{1}{n^p} \left(\int_X \rho^n \mathrm{d}\mathfrak{m} \right)^{1/n} \left(\int_X \rho^{-n} \mathrm{d}\mathfrak{m} \right)^{1/n} < \infty. \tag{1.5}$$

This condition appeared first in the Euclidean context in [28], dealing with $H^{1,2}_\rho = W^{1,2}_\rho$, see also the recent extension [27] to any power $p > 1$ and even to variable exponents.

As we illustrate below, the proof in [28] is sufficiently robust to be adapted, with minor variants, to a nonsmooth context.

In view of the characterization in [25], we believe that (1.4) is conceptually interesting. Indeed, functions in the left hand side of (1.4) are absolutely continuous (modulo the choice of an appropriate representative) along $\mathrm{Mod}_{p,\rho\mathfrak{m}}$-a.e. curve, while functions in the right hand side are absolutely continuous along $\mathrm{Mod}_{1,\mathfrak{m}}$-a.e. curve. On the other hand, even with $p = 1$, it seems very difficult to connect the two notions of negligibility if ρ and ρ^{-1} are unbounded. As a matter of fact our proof is very indirect and it would be nice to find a more direct explaination of the validity of (1.4).

We conclude the introduction by describing the structure of the paper. In Section 2 we recall aspects of the theory of Sobolev spaces on metric measure spaces; we detail approximation results and the notion of measurable differentiable structure from [11].

In Section 3 we introduce the weighted Sobolev spaces $W^{1,p}_\rho$ and $H^{1,p}_\rho$, showing first completeness of $W^{1,p}_\rho$ under the assumption $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$ and then reflexivity, under the additional assumption that $(X, \mathrm{d}, \mathfrak{m})$ is doubling and satisfies a 1-Poincaré inequality. The proof of reflexivity is particularly tricky and it passes, as in [11] and [2], through the construction of an equivalent uniformly convex norm. This involves a Lusin type approximation by Lipschitz functions. Notice this is not necessarily an approximation in the norm of $W^{1,p}_\rho$, since we know that additional assumptions on ρ are needed to get density of Lipschitz functions, namely the equality $W^{1,p}_\rho = H^{1,p}_\rho$. Then, using reflexivity and the $H = W$ theorem of the metric theory, we prove (1.3) in Theorem 3.6.

Section 4 is devoted to the proof of (1.4), obtained in Theorem 4.1 under the assumption (1.5). Here we follow closely [28], with some minor adaptations due to the lack of differentiability of $f \mapsto \int |\nabla f|^p_w \mathrm{d}\mathfrak{m}$ (potentially even for $p = 2$).

In Section 5 we recall an example from [13], showing that $H^{1,p}_\rho$ can be strictly included in $W^{1,p}_\rho$, and we explore some variants of our results. In particular we relax the 1-Poincaré assumption to a p-Poincaré assumption, modifying consequently the definitions of $W^{1,p}_\rho$ and $H^{1,p}_\rho$. Our main results still work, under the p-Poincaré assumption, for these spaces and we prove that the new definitions coincide with (1.1) and (1.2) assuming the validity of the 1-Poincaré inequality. Finally, we discuss the notion of Muckenhoupt weight and the invariance of our assumptions on $(X, \mathrm{d}, \mathfrak{m})$ under the replacement of $\mathfrak{m}$ by $\eta\mathfrak{m}$, with η a Muckenhoupt weight.

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2. Sobolev Spaces

Throughout this paper we will denote by (X, d) a complete separable metric space and by $\mathfrak{m}$ a locally finite (i.e. finite on bounded sets) Borel regular measure on X. In metric

spaces Lipschitz functions play the role of smooth functions. We recall that a function $f\colon X \to \mathbb{R}$ is called Lipschitz if there exists $L \geq 0$ such that $|f(x) - f(y)| \leq L\mathrm{d}(x,y)$ for all x, $y \in X$; we denote the smallest such constant L by $\mathrm{Lip}(f)$ and denote the set of Lipschitz functions on X by $\mathrm{Lip}(X)$.

For a Lipschitz function f, a natural candidate for the modulus of gradient is given by the slope $|\nabla f|\colon X \to \mathbb{R}$, defined by

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathrm{d}(y,x)}.$$

Definition 2.1 (Absolute continuity)**.** *Let $J \subset \mathbb{R}$ be a closed interval and consider a curve $\gamma : J \to X$. We say that γ is absolutely continuous if*

$$\mathrm{d}(\gamma(t), \gamma(s)) \leq \int_s^t g(r)\mathrm{d}r \quad \forall s,t \in J,\ s < t \tag{2.1}$$

for some $g \in L^1(J)$.

It is well known (see [3, Proposition 4.4] for the proof) that every absolutely continuous curve γ admits a minimal g satisfying (2.1), called metric speed, denoted by $|\dot\gamma(t)|$ and given for a.e. $t \in J$ by

$$|\dot\gamma(t)| = \lim_{s \to t} \frac{\mathrm{d}(\gamma(s), \gamma(t))}{|s-t|}.$$

We will denote by $C([0,1];X)$ the space of continuous curves from $[0,1]$ to (X,d) endowed with the sup norm and by $AC([0,1],X)$ the subset of absolutely continuous curves. Using the metric derivative we can easily define curvilinear integrals, namely

$$\int_\gamma g := \int_0^1 g(\gamma(t))|\dot\gamma(t)|\mathrm{d}t$$

for all $g : X \to [0,\infty]$ Borel and $\gamma \in AC([0,1];X)$.

Definition 2.2 (Modulus)**.** *Given $p \geq 1$ and $\Gamma \subset AC([0,1],X)$, the p-modulus $\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma)$ is defined by*

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma) := \inf\Big\{ \int_X h^p \mathrm{d}\mathfrak{m} \ :\ \int_\gamma h \geq 1\ \forall \gamma \in \Gamma \Big\},$$

where the infimum is taken over all non-negative Borel functions $h : X \to [0,\infty]$.
We say that Γ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible if $\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma) = 0$.

We can now give the definition of weak gradient and Sobolev space which we will use, see [25].

Definition 2.3 (p-upper gradient)**.** *For $p \geq 1$ we say that a Borel function $g : X \to [0,\infty]$ with $\int_X g^p \mathrm{d}\mathfrak{m} < \infty$ is a p-weak upper gradient of f if there exist a function $\widetilde{f}$ and a $\mathrm{Mod}_{p,m}$-negligible set Γ such that $\widetilde{f} = f$ $\mathfrak{m}$-a.e. in X and*

$$|\widetilde{f}(\gamma_0) - \widetilde{f}(\gamma_1)| \leq \int_\gamma g\ \mathrm{d}s \qquad \text{for all } \gamma \in AC([0,1],X) \setminus \Gamma. \tag{2.2}$$

The following Theorem is classical, see [20, 25] for a proof.

Theorem 2.4. *For every $p \geq 1$ the collection of all p-weak upper gradients of a map $f : X \to \mathbb{R}$ is a closed convex lattice in $L^p(\mathfrak{m})$.*
Moreover, if the collection of all p-weak upper gradients of f is nonempty then it contains a unique element of smallest $L^p(\mathfrak{m})$ norm. We shall denote it by $|\nabla f|_{p,\mathfrak{m}}$.

From now on we denote the 1-weak gradient of f with respect to $\mathfrak{m}$ simply by $|\nabla f|_w$, so $|\nabla f|_w = |\nabla f|_{1,\mathfrak{m}}$. Following [25] we can now define the Sobolev space from which we will define weighted Sobolev spaces on metric measure spaces.

Definition 2.5 (Sobolev space $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$)**.** *For each $p \geq 1$ we define $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ to be the Banach space of ($\mathfrak{m}$-a.e. equivalence classes of) functions $f \in L^p(\mathfrak{m})$ having a p-weak upper gradient, endowed with the norm*

$$\|f\|^p_{W^{1,p}(\mathfrak{m})} := \int_X |f|^p \mathrm{d}\mathfrak{m} + \int_X |\nabla f|^p_{p,\mathfrak{m}} \, \mathrm{d}\mathfrak{m}.$$

It can be proved (see [20, Proposition 5.3.25]) that $|\nabla f|_{p,\mathfrak{m}}$ is local, namely

$$|\nabla f|_{p,\mathfrak{m}} = |\nabla g|_{p,\mathfrak{m}} \qquad \mathfrak{m}\text{-a.e. on } \{f = g\} \tag{2.3}$$

for all f, $g \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$.

Definition 2.5 is by now classical and it goes back to the pioneering work [25], where the author also proved that if $p > 1$ then the space $W^{1,p}(\mathfrak{m})$ coincides with the Sobolev space defined by Cheeger [11] in terms of approximation by pairs (f_n, g_n), with $f_n \to f$ in $L^p(\mathfrak{m})$, g_n an upper gradient of f_n and $\{g_n\}_{n\in\mathbb{N}}$ bounded in $L^p(\mathfrak{m})$. More recently, the first named author, Gigli and Savaré (see [5] for $p = 2$ and [4] for $p > 1$) improved this equivalence result proving existence of an approximation by Lipschitz functions, with slopes (or even asymptotic Lipschitz constants, see [1]) as upper gradients. More precisely, defining

$$\begin{aligned} H^{1,p}(X, \mathrm{d}, \mathfrak{m}) \;=\; & \big\{ f \in L^p(\mathfrak{m}) : \ \exists \ f_n \in \mathrm{Lip}(X) \cap L^p(\mathfrak{m}) \\ & \text{with } f_n \to f \text{ in } L^p(\mathfrak{m}) \text{ and } \sup_n \int_X |\nabla f_n|^p \mathrm{d}\mathfrak{m} < \infty \big\}, \end{aligned} \tag{2.4}$$

the following result holds.

Theorem 2.6. *$W^{1,p}(X, \mathrm{d}, \mathfrak{m}) = H^{1,p}(X, \mathrm{d}, \mathfrak{m})$ for all $p > 1$. In addition, for all functions $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ the following holds:*

$$\textit{there exist } f_n \in \mathrm{Lip}(X) \cap L^p(\mathfrak{m}) \textit{ with } f_n \to f \textit{ and } |\nabla f_n| \to |\nabla f|_{p,\mathfrak{m}} \textit{ in } L^p(\mathfrak{m}). \tag{2.5}$$

On the contrary, the picture for $p = 1$ is far from being complete since at least three definitions are available (see also [2] and the forthcoming paper [6] for a discussion on this subject).

For our analysis of weighted Sobolev spaces we will require that the measure $\mathfrak{m}$ is doubling and that a p-Poincaré inequality holds; we recall these properties now. Doubling metric measure spaces which satisfy a p-Poincaré inequality are often called PI_p spaces and there are many known examples which differ from ordinary Euclidean spaces, see for instance [16, 17, 18].

Definition 2.7 (Doubling)**.** *A locally finite Borel measure $\mathfrak{m}$ on (X, d) is doubling if it gives finite positive measure to balls and there exists a constant $C > 0$ such that*

$$\mathfrak{m}(B(x, 2r)) \leq C\mathfrak{m}(B(x, r)) \quad \forall x \in X, \ r > 0. \tag{2.6}$$

In this case we also say that the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ *is doubling.*

A metric measure space gains additional structure if a Poincaré inequality is imposed; this type of inequality is a control on the local variation of a Lipschitz function using infinitesimal behaviour encoded by the slope.

Definition 2.8 (p-Poincaré)**.** *For* $p \in [1, \infty)$*, we say that a* p*-Poincaré inequality holds for Lipschitz functions if there exist constants* τ, $\Lambda > 0$ *such that for every* $f \in \mathrm{Lip}(X)$ *and for every* $x \in \mathrm{supp}(\mathfrak{m})$*,* $r > 0$ *the following inequality holds:*

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \mathrm{d}\mathfrak{m} \leq \tau r \Big(\fint_{B(x,\Lambda r)} |\nabla f|^p \mathrm{d}\mathfrak{m} \Big)^{1/p}, \tag{2.7}$$

where, here and in the sequel,

$$f_A = \fint_A f \mathrm{d}\mathfrak{m} := \frac{1}{\mathfrak{m}(A)} \int_A f \mathrm{d}\mathfrak{m}.$$

We will say that a constant is structural if it depends only on the doubling constant in (2.6) *and the constants* τ, Λ *in* (2.7)*.*

Notice that, by the Hölder inequality, the PI_p condition becomes weaker as p increases, so PI_1 is the strongest possible assumption. A remarkable result (see [22]) is that PI_p is an open ended condition in $(1, \infty)$, namely PI_p for some $p \in (1, \infty)$ implies PI_q for some exponent $q \in (1, p)$.

Thanks to (2.5), for all $p > 1$ and $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$, under the PI_p assumption it holds

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \mathrm{d}\mathfrak{m} \leq \tau r \Big(\fint_{B(x,\Lambda r)} |\nabla f|^p_{p,\mathfrak{m}} \mathrm{d}\mathfrak{m} \Big)^{1/p} \tag{2.8}$$

for all $x \in \mathrm{supp}(\mathfrak{m})$ and $r > 0$, where τ, Λ are as in (2.7) (see also [21, Theorem 2]). The inequality is still valid with $p = 1$ under the PI_1 assumption. Indeed, the space $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ is contained in the space $BV(X, \mathrm{d}, \mathfrak{m})$ of functions having a "measure" upper gradient considered in [2]. The equivalence result with Miranda's definition of BV (which parallels the ideas provided in [5] for the Sobolev spaces) provided in [2] ensures the existence of sequence $(f_n) \subset \mathrm{Lip}(X) \cap L^1(\mathfrak{m})$ with $f_n \to f$ in $L^1(\mathfrak{m})$ and $|\nabla f_n| \mathfrak{m} \to |Df|$ weakly as measures. Taking the limit, one obtains (2.8) with $|Df|(\overline{B}(x, \Lambda r))$ in the right hand side. In the case when $f \in W^{1,1}(X, \mathrm{d}, \mathfrak{m}) \subset BV(X, \mathrm{d}, \mathfrak{m})$ one can use the inequality $|Df| \leq |\nabla f|_{1,\mathfrak{m}} \mathfrak{m}$ to conclude, see also [6] for a more detailed discussion.

We now recall the relevant properties of the maximal operator.

Definition 2.9 (Maximal operator)**.** *Given a locally integrable Borel function* $f : X \to \mathbb{R}$*, we define the maximal function* $Mf \colon X \to [0, \infty]$ *associated to* f *by*

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f| \mathrm{d}\mathfrak{m}. \tag{2.9}$$

Since $\mathfrak{m}$ is doubling we know [17, Theorem 2.2] that for $q > 1$ the maximal operator is a bounded linear map from $L^q(\mathfrak{m})$ to $L^q(\mathfrak{m})$; more precisely, there exists a constant $C > 0$ depending only on the doubling constant such that

$$\|M(f)\|_{L^q(\mathfrak{m})} \leq \frac{C}{(q-1)^{1/q}} \|f\|_{L^q(\mathfrak{m})}$$

for all $f \in L^q(\mathfrak{m})$. For $q = 1$ the maximal operator is also weakly bounded, namely

$$\sup_{\lambda>0} \lambda \mathfrak{m}\big(\{M(g) > \lambda\}\big) \leq \int_X |g| \mathrm{d}\mathfrak{m}. \tag{2.10}$$

We will also need the asymptotic estimate

$$\lim_{\lambda\to\infty} \lambda \mathfrak{m}\big(\{M(g) > \lambda\}\big) = 0. \tag{2.11}$$

This asymptotic version follows by (2.10), taking the inclusion

$$\{M(g) > 2\lambda\} \subset \{M((|g| - \lambda)^+) > \lambda)\}$$

into account.

Recall that x is a Lebesgue point of a locally integrable function u if

$$\lim_{r\downarrow 0} \fint_{B(x,r)} |u(y) - u(x)| \mathrm{d}\mathfrak{m}(y) = 0.$$

This notion is sensitive to modification of u in $\mathfrak{m}$-negligible sets. With a slight abuse of notation we shall also apply this notion to Sobolev functions, meaning that we have chosen a representative in the equivalence class.

We now state a key approximation property for functions in $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$, valid under the doubling and p-Poincaré assumptions. We give a sketch of proof for the reader's convenience, but these facts are well known, see for instance [11], [25] or the more recent paper [1] where some proofs are revisited.

Proposition 2.10. *Assume that $p \in [1, \infty)$ and that $(X, \mathrm{d}, \mathfrak{m})$ is a PI_p metric measure space. Then, for all $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ there exist $f_n \in \mathrm{Lip}(X) \cap W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ and Borel sets E_n with:*

(i) *$E_n \subset E_{n+1}$ and $\sup_n n^p \mathfrak{m}(X \setminus E_n) < \infty$, so that $\mathfrak{m}(X \setminus \cup_n E_n) = 0$,*
(ii) *$|f_n| \leq n$, $\mathrm{Lip}(f_n) \leq Cn$, $f = f_n$ $\mathfrak{m}$-a.e. in E_n,*
(iii) *$|f_n - f| \to 0$ and $|\nabla(f_n - f)|_{p,\mathfrak{m}} \to 0$ in $L^p(\mathfrak{m})$.*

Furthermore, there is a structural constant c such that for all $f \in \mathrm{Lip}(X) \cap W^{1,p}(X, \mathrm{d}, \mathfrak{m})$,

$$|\nabla f|_{p,\mathfrak{m}} \leq |\nabla f| \leq c|\nabla f|_{p,\mathfrak{m}} \qquad \mathfrak{m}\text{-a.e. in } X. \tag{2.12}$$

Proof. Recall the definition of the maximal operator M w.r.t. $\mathfrak{m}$ from (2.9). By iterating the estimate (2.8) on concentric balls (see for instance [11] or [1, Lemma 8.2]) one can obtain the estimate

$$|f(x) - f(y)| \leq C\mathrm{d}(x, y)(M^{1/p}(|\nabla f|^p_{p,\mathfrak{m}})(x) + M^{1/p}(|\nabla f|^p_{p,\mathfrak{m}})(y)) \tag{2.13}$$

whenever $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ and x, $y \in X$ are Lebesgue points of (a representative of) f. Set now

$$g := \max\{|f|, M^{1/p}(|\nabla f|^p_{p,\mathfrak{m}})\}.$$

We also define

$$E_n := \{x \in X : \ x \text{ is a Lebesgue point of } f \text{ and } g(x) \leq n\}.$$

Notice that since $\{g > n\}$ is contained in $\{|f| > n\} \cup \{M(|\nabla f|^p_{p,\mathfrak{m}}) > n^p\}$, the set $X \setminus E_n$ has finite $\mathfrak{m}$-measure, and more precisely Markov inequality and the weak maximal estimate give that $\sup_n n^p \mathfrak{m}(X \setminus E_n) < \infty$.

Using (2.13) and the definition of g we obtain

$$|f(x)-f(y)| \le Cn\mathrm{d}(x,y), \qquad |f(x)| \le n$$

for all x, $y \in E_n$. By the McShane lemma (see for instance [17] for the simple proof) we can extend $f|_{E_n}$ to a Lipschitz function f_n on X preserving the Lipschitz constant and the sup estimate, namely $\mathrm{Lip}(f_n) \le Cn$ and $|f_n| \le n$. We claim that $|\nabla f_n|_{p,\mathfrak{m}} \le Cn$ $\mathfrak{m}$-a.e. in X. Indeed, since $|\nabla f_n|_{p,\mathfrak{m}} \le |\nabla f_n|$ $\mathfrak{m}$-a.e in X, we get

$$|\nabla f_n|_{p,\mathfrak{m}} \le |\nabla f_n| \le \mathrm{Lip}(f_n) \le Cn \qquad \mathfrak{m}\text{-a.e. in } X.$$

Furthermore, by the locality of the weak gradient we obtain $|\nabla(f_n - f)|_{p,\mathfrak{m}} = 0$ $\mathfrak{m}$-a.e. on the set E_n.

Using these facts and (2.11) it is straightforward to check, by dominated convergence, that $f_n \to f$ in $L^p(\mathfrak{m})$ and that $|\nabla(f_n - f)|_{p,\mathfrak{m}} \to 0$ in $L^p(\mathfrak{m})$.

The proof of (2.12) relies on a localized version of (2.13), namely

$$|f(x)-f(y)| \le C\mathrm{d}(x,y)(M^{1/p}_{2\Lambda r}(|\nabla f|^p_{p,\mathfrak{m}})(x) + M^{1/p}_{2\Lambda r}(|\nabla f|^p_{p,\mathfrak{m}})(y)) \tag{2.14}$$

for all Lebesgue points x, $y \in X$ of f with $\mathrm{d}(x,y) < r$, where M_s is the maximal operator on scale s (i.e. the supremum in (2.9) is restricted to balls with radius smaller than s). The idea of the proof is to differentiate at Lebesgue points x of $|\nabla f|^p_{p,\mathfrak{m}}$, letting eventually $r \downarrow 0$ and using the fact that $M_r(g) \downarrow |g|$ at Lebesgue points of g as $r \downarrow 0$, see [11] or [1, Proposition 47] for details. □

With a similar proof, using the boundedness of the maximal operator, one can prove the following proposition (see [5], [14] for counterexamples showing that the PI_q assumption can not be removed).

Proposition 2.11. *Let $(X,\mathrm{d},\mathfrak{m})$ be a PI_q metric measure space and $p > q$. If it holds that $f \in W^{1,q}(X,\mathrm{d},\mathfrak{m})$ and both f and $|\nabla f|_{q,\mathfrak{m}}$ belong to $L^p(\mathfrak{m})$, then $f \in W^{1,p}(X,\mathrm{d},\mathfrak{m})$.*

In the sequel we will use the fact that PI_p spaces for some $p \ge 1$ admit a differentiable structure; we conclude this section by recalling some aspects of Cheeger's remarkable theory [11, 23], which provides a differentiable structure that will play a role in the reflexivity of the weighted Sobolev spaces.

Definition 2.12. *A measurable differentiable structure on a metric measure space $(X,\mathrm{d},\mathfrak{m})$ is a countable collection of pairs $\{(U_\alpha,\varphi_\alpha)\}$, called local charts, that satisfy the following conditions:*

(i) *Each U_α is a measurable subset of X with positive measure, and $\mathfrak{m}(X\setminus\bigcup_\alpha U_\alpha) = 0$.*
(ii) *Each φ_α is a Lipschitz map from X to $\mathbb{R}^{N(\alpha)}$ for some integer $N(\alpha) \ge 1$, and moreover $N := \sup_\alpha N(\alpha) < \infty$.*
(iii) *For every $f \in \mathrm{Lip}(X)$ and for every α there exists an $\mathfrak{m}$-measurable function $\mathrm{d}^\alpha f : U_\alpha \to \mathbb{R}^{N(\alpha)}$ such that*

$$\limsup_{y\to x} \frac{|f(y)-f(x)-\mathrm{d}^\alpha f(x)\cdot(\varphi_\alpha(y)-\varphi_\alpha(x))|}{\mathrm{d}(x,y)} = 0 \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \in X$$

and $\mathrm{d}^\alpha f$ is unique up to $\mathfrak{m}$-negligible sets.

The following theorem is proved in [11], here we state it in the form needed in this paper.

Theorem 2.13 (Existence of a measurable differentiable structure)**.** *If $(X, \mathrm{d}, \mathfrak{m})$ is a PI_p metric measure space for some $p \geq 1$, then X admits a measurable differentiable structure and the integer N in Definition 2.12(ii) depends only on the structural constants. Moreover, for all α and $\mathfrak{m}$-a.e. $x \in U_\alpha$, there is a Hilbertian norm $\|\cdot\|_x$ on $\mathbb{R}^{N(\alpha)}$ such that $x \mapsto \|d^\alpha f(x)\|_x$ is $\mathfrak{m}$-measurable in U_α and*

$$\|d^\alpha f(x)\|_x \leq |\nabla f|(x) \leq M\|d^\alpha f(x)\|_x \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in U_\alpha, \text{ for all } f \in \mathrm{Lip}(X), \tag{2.15}$$

where $M > 0$ is a constant independent of α.

3. Weighted Sobolev Spaces

In this section we will define weighted Sobolev spaces and prove that, under natural integrability assumptions on the weight, we obtain a reflexive Banach space. Recall the 1-weak upper gradient of f with respect to $\mathfrak{m}$ is denoted by $|\nabla f|_w$, so $|\nabla f|_w = |\nabla f|_{1,\mathfrak{m}}$.

Definition 3.1 (Weighted space $W^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m})$)**.** *Let $p > 1$ and let $\rho : X \to [0, \infty]$ be a Borel function satisfying $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$; we define the weighted Sobolev space $W^{1,p}_\rho(X, d, \mathfrak{m})$ by*

$$W^{1,p}_\rho(\mathfrak{m}) := \Big\{ f \in W^{1,1}(X, \mathrm{d}, \mathfrak{m}) : \int_X |f|^p \ \rho \mathrm{d}\mathfrak{m} + \int_X |\nabla f|^p_w \ \rho \mathrm{d}\mathfrak{m} < \infty \Big\}. \tag{3.1}$$

We endow $W^{1,p}_\rho(X, \mathrm{d}, \mathfrak{m})$, shortened to $W^{1,p}_\rho$, with the norm:

$$\|f\|^p_\rho := \int_X |f|^p \ \rho \mathrm{d}\mathfrak{m} + \int_X |\nabla f|^p_w \ \rho \mathrm{d}\mathfrak{m}.$$

Remark 3.2. The fact that $\|\cdot\|_\rho$ is a norm is a consequence of the observation $\rho > 0$ $\mathfrak{m}$-a.e. in X and of the following elementary properties

$$\begin{aligned} |\nabla(f+g)|_w &\leq |\nabla f|_w + |\nabla g|_w && \mathfrak{m}\text{-a.e. in } X, \\ |\nabla(\lambda f)|_w &= |\lambda||\nabla f|_w && \mathfrak{m}\text{-a.e. in } X, \ \forall \lambda \in \mathbb{R}. \end{aligned}$$

Note that, using Hölder's inequality, it follows that

$$\|f\|_{W^{1,1}} \leq 2 \left(\int \rho^{-\frac{1}{p-1}} \mathrm{d}\mathfrak{m} \right)^{\frac{p-1}{p}} \|f\|_\rho. \tag{3.2}$$

That is, $W^{1,p}_\rho(\mathfrak{m})$ embeds continuously into $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ provided $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$; a similar calculation shows that the definition of $W^{1,p}_\rho(\mathfrak{m})$ would be unchanged if we replace $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ by $W^{1,1}_{\mathrm{loc}}(X, \mathrm{d}, \mathfrak{m})$ (namely the space of functions whose 1-weak upper gradient is integrable on bounded sets) in (3.1).

We use the embedding of $W^{1,p}_\rho(\mathfrak{m})$ to prove completeness of the weighted Sobolev space, building on the completeness of $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$.

Proposition 3.3 (Completeness of $W^{1,p}_\rho$)**.** *For every $p > 1$, the weighted Sobolev space $(W^{1,p}_\rho, \|\cdot\|_\rho)$ is a Banach space whenever $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$.*

Proof. Suppose that $(f_i)_{i=1}^\infty$ is a Cauchy sequence in $W^{1,p}_\rho$, and let $\omega_i \downarrow 0$ be such that $\|f_n - f_m\|^p_\rho \le \omega_i$ whenever n, $m \ge i$. From (3.2) and the completeness of $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ we obtain that there exists $f \in W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ such that $f_n \to f$ in $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ and hence, up to a subsequence, we can also assume that $f_n \to f$ pointwise $\mathfrak{m}-$a.e. Since $L^p(\rho\mathfrak{m})$ is a Banach space and $L^p(\rho\mathfrak{m})$ convergence implies the existence of subsequences convergent $\rho\mathfrak{m}-$a.e. in X, we deduce f is also the limit of f_n in $L^p(\rho\mathfrak{m})$. As $|\nabla(f_n - f)|_w \to 0$ in $L^1(\mathfrak{m})$ implies $|\nabla(f_n - f_m)|_w \to |\nabla(f - f_m)|_w$ in $L^1(\mathfrak{m})$, we can pass to the limit in $\||\nabla(f_n - f_m)|_w\|^p_{L^p(\rho\mathfrak{m})} \le \omega_i$ to obtain

$$\int_X |\nabla(f - f_m)|^p_w \rho \mathrm{d}\mathfrak{m} \le \omega_i \qquad \forall m \ge i.$$

This proves that $\int_X |\nabla f|^p_w \rho \mathrm{d}\mathfrak{m} < \infty$, so that $f \in W^{1,p}_\rho$, and that $f_m \to f$ in $W^{1,p}_\rho$. □

Theorem 3.4 (Reflexivity of $W^{1,p}_\rho$)**.** *Suppose* $(X, \mathrm{d}, \mathfrak{m})$ *is a* PI_1 *metric measure space,* $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ *and* $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$*. Then* $(W^{1,p}_\rho, \|\cdot\|_\rho)$ *is reflexive for all* $p > 1$*.*

Proof. By assumption $(X, \mathrm{d}, \mathfrak{m})$ is a PI_1 space; therefore, by Theorem 2.13, it admits a differentiable structure consisting of charts $(U_\alpha, \varphi_\alpha)$, where $U_\alpha \subset X$ is measurable and $\varphi_\alpha \colon X \to \mathbb{R}^{N(\alpha)}$ are Lipschitz with $N(\alpha) \le N$, with respect to which Lipschitz functions are differentiable. Further, for all $f \in \mathrm{Lip}(X)$,

$$\|\mathrm{d}^\alpha f(x)\|_x \le |\nabla f|(x) \le M\|\mathrm{d}^\alpha f(x)\|_x \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in U_\alpha \tag{3.3}$$

where $\|\cdot\|_x$ is an inner product norm on $\mathbb{R}^{N(\alpha)}$ for $x \in U_\alpha$ and M is a positive constant depending only on N.

Without loss of generality we assume the sets U_α are disjoint and denote derivatives by $\mathrm{d}f(x)$ instead of $\mathrm{d}^\alpha f(x)$ when $x \in U_\alpha$. We now observe that we can also assume $N(\alpha) = N$ for all α by replacing:

- The inner product norm $\|\cdot\|_x$ on $\mathbb{R}^{N(\alpha)}$ by the semi inner product norm $\|p(\cdot)\|_x$ on $\mathbb{R}^N$, where $p \colon \mathbb{R}^N \to \mathbb{R}^{N(\alpha)}$ is the projection onto the first $N(\alpha)$ coordinates for $x \in U_\alpha$. Here a semi inner product $\langle\cdot,\cdot\rangle$ satisfies the usual properties of an inner product except for positive definiteness - the corresponding semi inner product norm is then given by $\|v\|^2 = \langle v, v\rangle$.
- The derivative $\mathrm{d}f(x) \in \mathbb{R}^{N(\alpha)}$ by $(\mathrm{d}f(x), 0) \in \mathbb{R}^N$ for $x \in U_\alpha$.

After this replacement the map d still satisfies the equivalence (3.3). Clearly d is a linear map from $\mathrm{Lip}(X)$ to the space of ($\mathfrak{m}$-a.e. defined) $\mathbb{R}^N$ valued measurable functions on X. We split the the proof of reflexivity into three steps. Note that the first step is known and follows from results of [15], but we present it explicitly for the sake of readability.

Step 1. *We construct an equivalent norm on* $W^{1,p}_\rho$*.*

We first define a (non linear) map D from $W^{1,p}_\rho$ to non negative ($\mathfrak{m}$-a.e. defined) measurable functions on X, which we denote by $D_x(g)$ instead of $D(g)(x)$, satisfying:

$$\begin{aligned} D_x(\lambda g) &= |\lambda| D_x(g) && \mathfrak{m}\text{-a.e. } x \in X,\ \forall g \in W^{1,p}_\rho,\ \lambda \in \mathbb{R}, \\ D_x(g+h) &\le D_x(g) + D_x(h) && \mathfrak{m}\text{-a.e. } x \in X,\ \forall g, h \in W^{1,p}_\rho, \qquad (3.4) \\ D_x(f) &= \|\mathrm{d}f(x)\|_x && \mathfrak{m}\text{-a.e. } x \in X,\ \forall f \in \mathrm{Lip}(X) \cap W^{1,p}_\rho, \\ D_x(g) \le & |\nabla g|_w(x) \le M D_x(g) && \mathfrak{m}\text{-a.e. } x \in X,\ \forall g \in W^{1,p}_\rho. \qquad (3.5) \end{aligned}$$

Fix $g \in W^{1,p}_\rho \subset W^{1,1}$. By Proposition 2.10, there exists a sequence of Lipschitz functions g_n such that $|\nabla(g_n - g)|_w \to 0$ in $L^1(\mathfrak{m})$ and

$$\sum_{n=1}^{\infty} \mathfrak{m}\{g \neq g_n\} < \infty.$$

Let $A_n := \{g = g_n\}$ and $G_T := \cap_{n \ge T} A_n$; by the Borel-Cantelli lemma it follows that $\cup_T G_T$ has full $\mathfrak{m}$-measure. For n, $m > T$ we have, by Proposition 2.10(i), $|\nabla(g_n - g_m)|_w = 0$ $\mathfrak{m}$-a.e. in G_T. Hence, by inequality (2.12), n, $m > T$ implies $|\nabla(g_n - g_m)| = 0$ $\mathfrak{m}$-a.e. in G_T. We now claim that, for each fixed $x \in G_T$, $\|\mathrm{d}g_n(x)\|_x$ is constant as a function of $n > T$. Indeed, for n, $m > T$, using (3.3),

$$\begin{aligned} |\|dg_n(x)\|_x - \|dg_m(x)\|_x| &\le \|d(g_n - g_m)(x)\|_x \\ &\le |\nabla(g_n - g_m)|(x) \\ &= 0. \end{aligned}$$

We define $D_x(g) := \|\mathrm{d}g_n(x)\|_x$ for $x \in G_T$ and $n > T$. It is easy to show that if we took a different sequence of Lipschitz functions $\widetilde{g}_n$ with $\sum_{n=1}^{\infty} \mathfrak{m}\{g \neq \widetilde{g}_n\} < \infty$ then we obtain the same definition of $D_x(g)$ up to $\mathfrak{m}$-a.e. equivalence.

Using the measurability of the differential map we easily obtain the measurability of $x \mapsto D_x(g)$. Clearly for every $f \in W^{1,p}_\rho$ we have $D_x(f) \ge 0$. For every $f, g \in \mathrm{Lip}(X)$ and $\lambda \in \mathbb{R}$ we know

$$\mathrm{d}(f+g) = \mathrm{d}(f) + \mathrm{d}(g) \text{ and } \mathrm{d}(\lambda f) = \lambda \mathrm{d}(f) \qquad \mathfrak{m}\text{-a.e. in } X.$$

This implies (3.4), since approximations for f and g give rise to approximations for $f+g$ and λf. In order to prove (3.5), we first remark that, by Proposition 2.10(iii) and (3.5), we get

$$\|\mathrm{d}g_n(x)\|_x \le |\nabla g_n|_w \le M \|\mathrm{d}g_n(x)\|_x. \qquad (3.6)$$

Since $|\nabla g_n|_w \to |\nabla g|_w$ in $L^1(\mathfrak{m})$, it follows, up to a subsequence, $|\nabla g_n|_w(x) \to |\nabla g|_w(x)$ for $\mathfrak{m}$-a.e. $x \in X$. Therefore, the conclusion follows by letting $n \to \infty$ in (3.6) and recalling that $\|\mathrm{d}g_n(x)\|_x$ is constant for large n.

Using (3.4) and (3.5) it is easy to see that the following expression defines an equivalent norm on $W^{1,p}_\rho$:

$$\|f\|_{\mathrm{Ch},\rho} := \left(\int_X (|f(x)|^p + (D_x(f))^p) \rho(x) \, \mathrm{d}\mathfrak{m}(x) \right)^{1/p}.$$

Step 2. *Suppose $Q\colon X \to [0,\infty)$ is a $\mathfrak{m}$-measurable function; then the seminorm*

$$f \mapsto \|f\|_{\mathrm{Ch},Q} = \left(\int_X (|f(x)|^p + (D_x(f))^p)Q(x)\,\mathrm{d}\mathfrak{m}(x)\right)^{1/p}$$

is uniformly convex on the intersection $\mathrm{Lip}(X) \cap W^{1,p}_\rho$, with modulus of convexity independent of Q.

Suppose $(Y,\mathcal{F})$ is a measurable space and, for each $y \in Y$, $\mathbb{R}^n$ is equipped with a semi inner product norm $\|\cdot\|_y$ such that $y \mapsto \|v\|_y$ is measurable for any $v \in \mathbb{R}^n$. By polarization, the map $y \mapsto \langle v,w\rangle_y$ is measurable for any v, $w \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle_y$ denotes the induced semi inner product. Representing the semi inner product by a symmetric positive semidefinite matrix $y \mapsto A_y$ in the canonical basis of $\mathbb{R}^n$, it is clear that the entries of A_y are also measurable.

It is well known (see for instance [7]) that for any symmetric positive semidefinite matrix A there exists a unique symmetric matrix $\sqrt{A}$ such that $\sqrt{A}\sqrt{A} = A$; in addition, the map $A \mapsto \sqrt{A}$ is continuous. As the composition of a continuous and a measurable map, the entries of $\sqrt{A_y}$ are measurable. Further, we can write $\langle v,w\rangle_y = (\sqrt{A_y}v)^t(\sqrt{A_y}w)$, which implies $\|v\|_y = |\sqrt{A_y}v|$ for all $v \in \mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Using the discussion above, for each $x \in X$, we choose an $N \times N$ matrix B_x such that $\|v\|_x = |B_x v|$ for all $v \in \mathbb{R}^N$. Let X_1 and X_2 be two disjoint copies of X supporting copies $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{m}$ respectively. We define the following function $\widehat{f}\colon X_1 \cup X_2 \to \mathbb{R}^N$,

$$\widehat{f}(x) := \begin{cases} (Q(x)^{1/p}f(x), 0, \dots, 0) & x \in X_1 \\ Q(x)^{1/p}B_x df(x) & x \in X_2. \end{cases}$$

Clearly $\widehat{f}$ is measurable. By using the equality $D_x(f) = \|\mathrm{d}f(x)\|_x$ $\mathfrak{m}$-a.e. in X it is simple to verify $\|\widehat{f}\|_{L^p(m_1+m_2)} = \|f\|_{\mathrm{Ch},Q}$. Since the transformation $f \mapsto \widehat{f}$ is linear the usual uniform convexity of L^p spaces implies uniform convexity of the norm $\|\cdot\|_{\mathrm{Ch},Q}$ on $\mathrm{Lip}(X) \cap W^{1,p}_\rho$ (with modulus independent of Q).

Step 3. *The norm $\|\cdot\|_{\mathrm{Ch},\rho}$ is uniformly convex on $W^{1,p}_\rho$.*

It is an easy consequence of (3.4), (3.5) and locality for the weak gradient that if $g, h \in W^{1,p}_\rho$ and $g = h$ on a measurable set E then $D_x g = D_x h$ for $\mathfrak{m}$-a.e. $x \in E$. We use this locality property throughout the sequel.

Given $\epsilon \in (0,1)$, let $\delta = \delta(\epsilon) \in (0,1)$ be given by the uniform convexity proved in the previous step. Suppose f, $g \in W^{1,p}_\rho$ satisfy $\|f\|_{\mathrm{Ch},\rho} = \|g\|_{\mathrm{Ch},\rho} = 1$ and $\|f-g\|_{\mathrm{Ch},\rho} \geq \varepsilon$. Using Proposition 2.10, we can find an increasing family of bounded sets E_n such that $\mathfrak{m}(X \setminus \cup_n E_n) = 0$ on which $f|_{E_n}$ and $g|_{E_n}$ are Lipschitz. Set $\rho_n(x) := \rho(x)\chi_{E_n}(x)$, where χ_{E_n} is the characteristic function of E_n. We first extend $f|_{E_n}$ and $g|_{E_n}$ to Lipschitz functions f_n and g_n on X with bounded support. An easy argument using locality of the weak gradient and local integrability of ρ shows that $f_n, g_n \in \mathrm{Lip}(X) \cap W^{1,p}_\rho$.

Next, let $\widetilde{f}_n := f_n/z_n$ and $\widetilde{g}_n := g_n/w_n$ for some scalars z_n and w_n such that

$$\|\widetilde{f}_n\|^p_{\mathrm{Ch},\rho_n} = \int_X (|\widetilde{f}_n(x)|^p + (D_x(\widetilde{f}_n))^p)\rho_n(x)\,\mathrm{d}\mathfrak{m}(x) = 1$$

and

$$\|\widetilde{g}_n\|^p_{\mathrm{Ch},\rho_n} = \int_X (|\widetilde{g}_n(x)|^p + (D_x(\widetilde{g}_n))^p)\rho_n(x)\,\mathrm{d}\mathfrak{m}(x) = 1.$$

Since

$$z_n^p = \int_X (|f_n(x)|^p + (D_x(f_n))^p)\rho_n(x)\,\mathrm{d}\mathfrak{m}(x) = \int_{E_n} (|f(x)|^p + (D_x(f))^p)\rho(x)\,\mathrm{d}\mathfrak{m}(x)$$

and $\|f\|_{\mathrm{Ch},\rho} = 1$, the monotone convergence theorem yields $z_n \uparrow 1$; similarly we obtain $w_n \uparrow 1$. In our choice of f and g we assumed that

$$\|f-g\|^p_{\mathrm{Ch},\rho} = \int_X (|f(x)-g(x)|^p + (D_x(f-g))^p)\rho(x)\,\mathrm{d}\mathfrak{m}(x) > \varepsilon^p.$$

By Fatou's lemma, using locality of D_x to justify pointwise convergence of the integrand, it follows that for n sufficiently large,

$$\|\widetilde{f}_n - \widetilde{g}_n\|_{\mathrm{Ch},\rho_n} = \int_X (|\widetilde{f}_n(x) - \widetilde{g}_n(x)|^p + (D_x(\widetilde{f}_n - \widetilde{g}_n))^p)\rho_n(x)\,\mathrm{d}\mathfrak{m}(x) > \varepsilon^p.$$

Hence, since $\widetilde{f}_n, \widetilde{g}_n \in \mathrm{Lip}(X) \cap W^{1,p}_\rho$, the uniform convexity of the norm $\|\cdot\|_{\mathrm{Ch},\rho_n}$ on $\mathrm{Lip}(X) \cap W^{1,p}_\rho$, proved in Step 2, gives

$$\left\|\frac{\widetilde{f}_n + \widetilde{g}_n}{2}\right\|^p_{\mathrm{Ch},\rho_n} = \int_X \left(\left|\frac{\widetilde{f}_n(x) + \widetilde{g}_n(x)}{2}\right|^p + \left(D_x\left(\frac{\widetilde{f}_n + \widetilde{g}_n}{2}\right)\right)^p\right)\rho_n(x)\mathrm{d}\mathfrak{m}(x) \le (1-\delta)^p.$$

By using locality of D and the definitions of f and g we obtain

$$\int_{E_n} \left(\left|\frac{f(x)/z_n + g(x)/w_n}{2}\right|^p + \left(D_x\left(\frac{f/z_n + g/w_n}{2}\right)\right)^p\right)\rho(x)\mathrm{d}\mathfrak{m}(x) \le (1-\delta)^p.$$

By letting $n \to \infty$ and using the dominated convergence theorem we obtain the inequality $\|(f+g)/2\|_{\mathrm{Ch},\rho} \le (1-\delta)$. Hence the norm $\|\cdot\|_{\mathrm{Ch},\rho}$ is uniformly convex on $W^{1,p}_\rho$ and so the Banach space $W^{1,p}_\rho$ is reflexive. □

As previously stated (see (5.2)) another natural definition of weighted Sobolev space is provided by the closure of Lipschitz functions with respect to the weighted norm.

Definition 3.5 (Weighted space $H^{1,p}_\rho(X,\mathrm{d},\mathfrak{m})$)**.** *Let $p > 1$ and let $\rho : X \to [0,\infty]$ be a Borel function satisfying $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$. We define $H^{1,p}_\rho(X,d,\mathfrak{m})$ as the closure of Lipschitz functions in $W^{1,p}_\rho$, namely*

$$H^{1,p}_\rho(X,d,\mathfrak{m}) := \overline{\mathrm{Lip}(X) \cap W^{1,p}_\rho}^{\|\cdot\|_\rho}.$$

As for the W space, we will adopt the shorter notation $H^{1,p}_\rho$.

We can now use the general identification Theorem 2.6 and the reflexivity of $W^{1,p}_\rho$ to prove our first main result, namely that $H^{1,p}_\rho$ coincides with the metric Sobolev space $W^{1,p}(X,\mathrm{d},\rho\mathfrak{m})$ under the PI_1 assumption on $(X,\mathrm{d},\mathfrak{m})$ and no extra assumption on the weight, besides local integrability of ρ and global integrability of $\rho^{1/(1-p)}$.

Theorem 3.6. *Suppose* $(X, \mathrm{d}, \mathfrak{m})$ *is a* PI_1 *metric measure space. Let* $p > 1$ *and suppose* $\rho : X \to [0, \infty]$ *is a Borel function satisfying* $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ *and* $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$*. Then*

$$W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) = H^{1,p}_\rho.$$

Proof. Let $f \in W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$. By Theorem 2.6 and the inequality $|\nabla g|_w \leq |\nabla g|$ for g Lipschitz, we can approximate f in $L^p(\rho\mathfrak{m})$ by functions $f_n \in \mathrm{Lip}(X) \cap L^p(\rho\mathfrak{m})$ with $\|f_n\|_\rho$ uniformly bounded and even $|\nabla f_n| \to |\nabla f|_{p,\rho\mathfrak{m}}$ in $L^p(\rho\mathfrak{m})$. By reflexivity we have that f_n weakly converge to f in $W^{1,p}_\rho$, and since $H^{1,p}_\rho$ by definition is a closed subspace it follows that $f \in H^{1,p}_\rho$ as well. In addition, the weak lower semicontinuity of the norm gives

$$\int_X (|f|^p + |\nabla f|^p_w)\rho \mathrm{d}\mathfrak{m} \leq \int_X (|f|^p + |\nabla f|^p_{p,\rho\mathfrak{m}})\rho \mathrm{d}\mathfrak{m}. \tag{3.7}$$

Conversely, let $f \in H^{1,p}_\rho$ and let $f_n \in \mathrm{Lip}(X) \cap W^{1,p}_\rho$ be convergent to f in $W^{1,p}_\rho$ norm. Using (2.12) of Proposition 2.10 we obtain that

$$\lim_{n,\, m\to\infty} \int_X |\nabla(f_n - f_m)|^p \rho \mathrm{d}\mathfrak{m} = 0.$$

It follows that $\int |\nabla f_n|^p \mathrm{d}\rho\mathfrak{m}$ is uniformly bounded, therefore one obtains $f \in H^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$ and therefore, by Theorem 2.6, $f \in W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$. □

Remark 3.7 ($W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$ and $H^{1,p}_\rho$ are isometric)**.** The second part of the proof of Theorem 3.6 can be improved if we use the finer information (see Theorem 12.5.1 in [20], while [11] covered only the case $p > 1$) that

$$|\nabla f| = |\nabla f|_w \qquad \mathfrak{m}\text{-a.e. in } X \tag{3.8}$$

for all $f \in \mathrm{Lip}(X) \cap W^{1,1}(X, \mathrm{d}, \mathfrak{m})$, under the PI_1 assumption (recall $|\nabla f|_w$ stands for $|\nabla f|_{1,\mathfrak{m}}$). Indeed, using (3.8) one can get

$$\int_X |\nabla f|^p_{p,\rho\mathfrak{m}} \rho \mathrm{d}\mathfrak{m} \leq \int_X |\nabla f|^p_w \rho \mathrm{d}\mathfrak{m} \tag{3.9}$$

which, combined with (3.7), gives that the spaces are isometric.

4. Identification of Weighted Sobolev Spaces

In this section we prove that for $p > 1$, under certain assumptions on the space $(X, \mathrm{d}, \mathfrak{m})$ and on the weight ρ, the weighted Sobolev spaces $W^{1,p}_\rho$ and $H^{1,p}_\rho$ coincide. Unless otherwise stated, all integrals are with respect to $\mathfrak{m}$. Our second main result is the following.

Theorem 4.1 (Identification of weighted and metric Sobolev spaces)**.** *Suppose* $(X, \mathrm{d}, \mathfrak{m})$ *is a* PI_1 *metric measure space. Let* $p > 1$ *and* $\rho\colon X \to [0, \infty]$ *be such that* $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$*,* $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$ *and*

$$L_\rho := \liminf_{n\to\infty} \frac{1}{n^p} \left(\int_X \rho^n \mathrm{d}\mathfrak{m}\right)^{1/n} \left(\int_X \rho^{-n} \mathrm{d}\mathfrak{m}\right)^{1/n} < \infty.$$

Then $W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) = W^{1,p}_\rho$ *and, in particular,* $H^{1,p}_\rho = W^{1,p}_\rho$*.*

Suppose that the hypotheses of Theorem 4.1 hold. Since $L_\rho < \infty$, there exists $N \in \mathbb{N}$ such that $\|\rho\|_{L^N(\mathfrak{m})}\|\rho^{-1}\|_{L^N(\mathfrak{m})} < \infty$. The assumptions $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ and $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$ imply that ρ is not identically 0 or ∞. Hence $\|\rho\|_{L^N(\mathfrak{m})} < \infty$ and $\|\rho^{-1}\|_{L^N(\mathfrak{m})} < \infty$. These imply $\mathfrak{m}\{\rho \geq 1\}$ and $\mathfrak{m}\{\rho \leq 1\}$ respectively; hence $\mathfrak{m}(X) < \infty$. Further, the integrability $\rho \in L^N(\mathfrak{m})$ (for some N) and $\mathfrak{m}(X) < \infty$ imply $\rho \in L^1(\mathfrak{m})$. We use these facts freely in what follows.

We already know that we can identify $H^{1,p}_\rho$ with $W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m})$, thanks to Theorem 3.6. Hence, we argue by contradiction, assuming that $H^{1,p}_\rho \subsetneq W^{1,p}_\rho$, and we derive a contradiction. Since $H^{1,p}_\rho$ is a closed subspace of the reflexive space $W^{1,p}_\rho$ (Theorem 3.4) there exists an element $u \in W^{1,p}_\rho \setminus H^{1,p}_\rho$ such that $\|u+v\|_\rho \geq \|u\|_\rho$ for any $v \in H^{1,p}_\rho$ (it suffices, given $z \in W^{1,p}_\rho \setminus H^{1,p}_\rho$, to minimize $\|z-h\|_\rho$ as h runs in $H^{1,p}_\rho$ and then define $u = z - h$, where h is a minimizer. Now, suppose v is of the form $v = -tw$ for $t \in (0,1)$, with $w \in H^{1,p}_\rho$ and $w = u$ on a Borel set $E \subset X$. Then we obtain,

$$\int_E (|u|^p + |\nabla u|^p_w)\rho \mathrm{d}\mathfrak{m} + \int_{X\setminus E} (|u|^p + |\nabla u|^p_w)\rho \mathrm{d}\mathfrak{m}$$
$$\leq (1-t)^p \int_E (|u|^p + |\nabla u|^p_w)\rho \mathrm{d}\mathfrak{m} + \int_{X\setminus E} (|u-tw|^p + |\nabla(u-tw)|^p_w)\rho \mathrm{d}\mathfrak{m}$$

and hence

$$\frac{(1-(1-t)^p)}{t} \int_E (|u|^p + |\nabla u|^p_w)\rho \mathrm{d}\mathfrak{m}$$
$$\leq \int_{X\setminus E} \frac{(|u| + t|w|)^p - |u|^p + (|\nabla u|_w + t|\nabla w|_w)^p - |\nabla u|^p_w}{t} \rho \mathrm{d}\mathfrak{m}.$$

By letting $t \to 0$ and using the dominated convergence theorem we obtain

$$\int_E (|u|^p + |\nabla u|^p_w)\rho \mathrm{d}\mathfrak{m} \leq \int_{X\setminus E} (|u|^{p-1}|w| + |\nabla u|^{p-1}_w |\nabla w|_w)\rho \mathrm{d}\mathfrak{m}. \tag{4.1}$$

To apply (4.1) we need to use $u \in W^{1,p}_\rho$ to construct an appropriate test function in $H^{1,p}_\rho$. To do this, as in the proof of Proposition 2.10, we use a maximal operator estimate to obtain Lipschitz bounds on the restriction of u to a smaller subset and then extend this restriction to a Lipschitz map on X.

Now let $u \in W^{1,p}_\rho \setminus H^{1,p}_\rho$ such that $\|u+v\|_\rho \geq \|u\|_\rho$ for any $v \in H^{1,p}_\rho$ and set

$$g := \max\{|u|, M(|\nabla u|_w)\},$$

where M is the maximal operator w.r.t. $\mathfrak{m}$, defined in (2.9). We also define the set

$$F_\lambda := \{x \in X : \ x \text{ is a Lebesgue point of } u \text{ and } g(x) \leq \lambda\}.$$

Arguing as in the proof of Proposition 2.10 we obtain a Lipschitz function u_λ with $|u_\lambda| \leq \lambda$, $\mathrm{Lip}(u_\lambda) \leq C\lambda$, and equal to u $\mathfrak{m}$-a.e. in F_λ. Since $\mathfrak{m}(X) < \infty$ and $\rho \in L^1(\mathfrak{m})$, we deduce $u_\lambda \in H^{1,p}_\rho$.

Now we apply (4.1) with $w = u_\lambda$ and $E = F_\lambda$ to obtain

$$\int_{F_\lambda} (|u|^p + |\nabla u|_w^p)\rho \mathrm{d}\mathfrak{m} \le C\lambda \int_{X\setminus F_\lambda} (|u|^{p-1} + |\nabla u|_w^{p-1})\rho \mathrm{d}\mathfrak{m}. \tag{4.2}$$

Next, we prove the following estimate:

Proposition 4.2. *Let $f\colon (0,\infty) \to (0,\infty)$ be a continuously differentiable and strictly decreasing function such that $f(\lambda) \to 0$ as $\lambda \to \infty$ and $-\int_0^t \lambda f'(\lambda)\mathrm{d}\lambda < \infty$ for all $t > 0$. Then, under assumption* (4.2)*, there exists a constant $C > 0$ such that*

$$\int_X f(g)(|u|^p + |\nabla u|_w^p)\rho \mathrm{d}\mathfrak{m} \le C \int_X \Phi(g)(|u|^{p-1} + |\nabla u|_w^{p-1})\rho \mathrm{d}\mathfrak{m}$$

with $\Phi(t) = -\int_0^t \lambda f'(\lambda)\mathrm{d}\lambda$.

Proof. First of all we notice that the statement makes sense, since $g > 0$ $\mathfrak{m}$-a.e. on the set where $|u|^p + |\nabla u|_w^p$ is positive (therefore, understanding the integrand as being null on $\{g = 0\}$). As in [28, Lemma 1], we can apply Cavalieri's formula $\int \psi \mathrm{d}\mu = \int_0^\infty \mu(\{\psi > t\})\mathrm{d}t$ with ψ Borel nonnegative and μ a finite Borel measure. For $k \in L^1(\mathfrak{m})$, choosing $\mu = k\mathfrak{m}$ and $\psi = f(g)$, $\psi = \Phi(g)$ and using the change of variable $\lambda = f^{-1}(t)$ yields

$$\int_X f(g)k = -\int_0^\infty f'(\lambda) \int_{F_\lambda} k \mathrm{d}\lambda, \tag{4.3}$$

$$\int_X \Phi(g)k = \int_0^\infty \Phi'(\lambda) \int_{X\setminus F_\lambda} k \mathrm{d}\lambda \tag{4.4}$$

for all $k \in L^1(\mathfrak{m})$ nonnegative, and eventually for any nonnegative Borel k. Now observe that multiplying inequality (4.2) by $-f'(\lambda)$ (recall that $-f'(\lambda) \ge 0$ in $(0,\infty)$) and integrating from 0 to ∞ we get

$$\begin{aligned} -\int_0^\infty f'(\lambda) \int_{F_\lambda} (|u|^p + |\nabla u|_w^p)\rho \mathrm{d}\lambda \\ \le -C \int_0^\infty \lambda f'(\lambda) \int_{X\setminus F_\lambda} (|u|^{p-1} + |\nabla u|_w^{p-1})\rho \mathrm{d}\lambda \\ = C \int_0^\infty \Phi'(\lambda) \int_{X\setminus F_\lambda} (|u|^{p-1} + |\nabla u|_w^{p-1})\rho \mathrm{d}\lambda. \end{aligned} \tag{4.5}$$

By applying (4.3) to (4.5) with $k = (|u|^p + |\nabla u|_w^p)\rho$ we get

$$\int_X f(g)(|u|^p + |\nabla u|_w^p)\rho \le C \int_0^\infty \Phi'(\lambda) \int_{X\setminus F_\lambda} (|u|^{p-1} + |\nabla u|_w^{p-1})\rho \mathrm{d}\lambda. \tag{4.6}$$

Further, we choose $k = (|u|^{p-1} + |\nabla u|_w^{p-1})\rho$ and apply inequality (4.4) to (4.6) to obtain the thesis. □

We modify the estimate from Proposition 4.2 by using Hölder's inequality for the measure $\rho\mathfrak{m}$; for any non negative Borel function G on X we have,

$$\int_X \Phi(g)G^{p-1}\rho \le \left(\int_X G^p\rho\right)^{1/p'} \left(\int_X \Phi(g)^p\rho\right)^{1/p}$$

where $1/p + 1/p' = 1$. By applying Proposition 4.2 then using the above inequality with $G = |u|$ and $G = |\nabla u|_w$ we deduce

$$\begin{aligned}\int_X f(g)(|u|^p + |\nabla u|_w^p)\rho \\ &\le C\left(\left(\int_X |u|^p\rho\right)^{1/p'} + \left(\int_X |\nabla u|_w^p\rho\right)^{1/p'}\right)\left(\int_X \Phi(g)^p\rho\right)^{1/p} \\ &\le C\left(\int_X (|u|^p + |\nabla u|_w^p)\rho\right)^{1/p'}\left(\int_X \Phi(g)^p\rho\right)^{1/p} \\ &= C\|u\|_\rho^{p/p'}\left(\int_X \Phi(g)^p\rho\right)^{1/p}.\end{aligned} \tag{4.7}$$

Now we fix $p^* \in (1,p)$ and choose $\varepsilon > 0$ such that $\varepsilon < \min\left\{\frac{1}{2(p-1)}, \frac{1}{p-1}\left(1 - \frac{p^*}{p}\right)\right\}$. Therefore, $(p-1)\varepsilon < 1/2$ and $p_\varepsilon := p(1 + (1-p)\varepsilon) > p^*$. Our next goal is to prove the inequality

$$\int_X g^{(1-p)\varepsilon}(|u|^p + |\nabla u|_w^p)\rho \le C\varepsilon\|u\|_\rho^{p/p'}\left(\int_X g^{p_\varepsilon}\rho\right)^{1/p}. \tag{4.8}$$

In order to prove (4.8), let $f(\lambda) := \lambda^{(1-p)\varepsilon}$, so that

$$\Phi(\lambda) := \frac{(p-1)\varepsilon}{1-(p-1)\varepsilon}\lambda^{1-(p-1)\varepsilon}.$$

Now we observe that

$$\|\Phi \circ g\|_{L^p(\rho\mathfrak{m})}^p = \frac{(p-1)^p\varepsilon^p}{(1-(p-1)\varepsilon)^p}\int_X g^{p(1-(p-1)\varepsilon)}\rho \le 2^p(p-1)^p\varepsilon^p\int_X g^{p(1-(p-1)\varepsilon)}\rho,$$

by our choice of ε. Hence, by applying (4.7) with our choice of f and Φ we obtain (4.8).

Now, recalling that $g = \max\{|u|, M(|\nabla u|_w)\}$ and using the triangle inequality, we estimate

$$\|g\|_{L^{p_\varepsilon}(\rho\mathfrak{m})}^{p_\varepsilon/p} \le (\|u\|_{L^{p_\varepsilon}(\rho\mathfrak{m})} + \|M(|\nabla u|_w)\|_{L^{p_\varepsilon}(\rho\mathfrak{m})})^{p_\varepsilon/p}. \tag{4.9}$$

We will use Hölder's inequality and boundedness of the maximal operator to bound the right hand side in terms of $\|u\|_\rho$ and ρ. Notice that the constants C appearing in the estimates below are independent of ϵ, since we are going to apply the maximal estimates with exponent $p_\varepsilon r$ and $p_\varepsilon r > p^*$, by our choice of ε. We handle $\|u\|_{L^{p_\varepsilon}(\rho\mathfrak{m})}$ and $\|M(|\nabla u|_w)\|_{L^{p_\varepsilon}(\rho\mathfrak{m})}$ separately but with a similar argument; we apply Hölder's inequality twice with the following exponents:

$$r = r_\varepsilon := \frac{2+(1-p)\varepsilon}{2+2(1-p)\varepsilon}, \qquad s = s_\varepsilon := \frac{2}{2+(1-p)\varepsilon}.$$

It is easy to see that r, $s > 1$ and that the conjugate Hölder exponents r', s' are respectively given by

$$r' = \frac{2+(1-p)\varepsilon}{(p-1)\varepsilon}, \qquad s' = \frac{2}{(p-1)\varepsilon}.$$

Furthermore, these exponents satisfy the equations $p_\varepsilon rs = p$ and $s' = r's$. Now, let us derive the inequalities for $M(|\nabla u|_w)$; the case of $|u|$ is similar but easier, since we don't need to use boundedness of the maximal operator.

$$\begin{aligned}\int_X M(|\nabla u|_w)^{p_\varepsilon}\rho &\le \left(\int_X M(|\nabla u|_w)^{p_\varepsilon r}\right)^{1/r}\left(\int_X \rho^{r'}\right)^{1/r'} \\ &\le C\left(\int_X |\nabla u|_w^{p_\varepsilon r}\rho^{1/s}\rho^{-1/s}\right)^{1/r}\left(\int_X \rho^{r'}\right)^{1/r'} \\ &\le C\left(\int_X |\nabla u|_w^{p_\varepsilon rs}\rho\right)^{1/(rs)}\left(\int_X \rho^{r'}\right)^{1/r'}\left(\int_X \rho^{-s'/s}\right)^{1/(rs')} \\ &\le C\|u\|_\rho^{p/(rs)}\left(\int_X \rho^{r'}\right)^{1/r'}\left(\int_X \rho^{-r'}\right)^{1/(rs')}.\end{aligned} \tag{4.10}$$

Similarly we obtain

$$\int |u|^{p_\varepsilon}\rho \le C\|u\|_\rho^{p/(rs)}\left(\int_X \rho^{r'}\right)^{1/r'}\left(\int_X \rho^{-r'}\right)^{1/rs'}.$$

Hence we have

$$(\|u\|_{L^{p_\varepsilon}(\rho\mathfrak{m})} + \|M(|\nabla u|_w)\|_{L^{p_\varepsilon}(\rho\mathfrak{m})})^{p_\varepsilon/p} \le C\|u\|_\rho^{1/(rs)}\left(\int_X \rho^{r'}\right)^{1/(pr')}\left(\int_X \rho^{-r'}\right)^{1/(prs')}. \tag{4.11}$$

By combining our estimates (4.8), (4.9) and (4.11) we obtain

$$\int g^{(1-p)\varepsilon}(|u|^p + |\nabla u|_w^p)\rho \le C\varepsilon\|u\|_\rho^{\frac{p}{p'}+\frac{1}{rs}}\left(\int_X \rho^{r'}\right)^{1/pr'}\left(\int_X \rho^{-r'}\right)^{1/prs'}. \tag{4.12}$$

As $\varepsilon \to 0$ Fatou's lemma gives

$$\int_X (|u|^p + |\nabla u|_w^p)\rho \le \liminf_{\varepsilon\downarrow 0}\int_X g^{(1-p)\varepsilon}(|u|^p + |\nabla u|_w^p)\rho. \tag{4.13}$$

Therefore in order to estimate $\int_X (|u|^p + |\nabla u|_w^p)\rho$ from above we can estimate the right hand side of (4.12) as $\varepsilon \downarrow 0$ (notice that r_ε, $s_\varepsilon \downarrow 1$, while $p_\varepsilon \uparrow p$) in the following lemma.

Lemma 4.3. *The following inequality holds:*

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^p\left(\int_X \rho^{r'}\right)^{1/r'}\left(\int_X \rho^{-r'}\right)^{1/(rs')} \le \frac{2^p}{(p-1)^p}L_\rho \tag{4.14}$$

where L_ρ is defined as in Theorem 4.1.

Proof. Setting $\varepsilon = 2/[(n+1)(p-1)]$ gives $n = r'$ and $rs' = n(n+1)/(n-1)$. Hence the left hand side of (4.14) is bounded from above by

$$\liminf_{n\to\infty}\frac{2^p}{(p-1)^p(n+1)^p}\left(\int_X \rho^n\right)^{\frac{1}{n}}\left(\int_X \rho^{-n}\right)^{\frac{n-1}{n(n+1)}}.$$

Since $\mathfrak{m}(X) < \infty$ and $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$, it follows that $\|\rho^{-1}\|_{L^n(\mathfrak{m})} \to \|\rho^{-1}\|_{L^\infty(\mathfrak{m})}$ as $n \to \infty$. Hence if $0 < \|\rho^{-1}\|_{L^\infty(\mathfrak{m})} < \infty$ then

$$\begin{aligned}\liminf_{n\to\infty} &\frac{2^p}{(p-1)^p(n+1)^p}\|\rho\|_{L^n(\mathfrak{m})}\|\rho^{-1}\|_{L^n(\mathfrak{m})}^{\frac{n-1}{n+1}}\\ &= \liminf_{n\to\infty} \frac{2^p}{(p-1)^p(n+1)^p}\|\rho\|_{L^n(\mathfrak{m})}\|\rho^{-1}\|_{L^n(\mathfrak{m})}\\ &= \frac{2^p}{(p-1)^p}L_\rho\end{aligned}$$

and the thesis follows. If $\|\rho^{-1}\|_{L^\infty(\mathfrak{m})} = \infty$ then $\|\rho^{-1}\|_{L^n(\mathfrak{m})} > 1$ for sufficiently large n and we can use the trivial inequality

$$\|\rho^{-1}\|_{L^n(\mathfrak{m})}^{\frac{n-1}{n+1}} < \|\rho^{-1}\|_{L^n(\mathfrak{m})}$$

for every n to obtain the same bound. The case $\|\rho^{-1}\|_{L^\infty(\mathfrak{m})} = 0$ is impossible since $\rho \in L^1(\mathfrak{m})$. □

Using (4.12), (4.13), (4.14) and the fact that $p/p' = p-1$ we get

$$\|u\|_\rho^p \le CL_\rho^{1/p}\|u\|_\rho^p$$

for some structural constant C. Since $u \neq 0$, we obtain

$$1 \le CL_\rho. \tag{4.15}$$

As in [28], the strategy is now to use the fact that C is independent of ρ to derive a contradiction. Following the notation of Theorem 4.1 we write

$$L_\eta = \liminf_{n\to\infty} \frac{1}{n^p}\|\eta\|_{L^n(\mathfrak{m})}\|\eta^{-1}\|_{L^n(\mathfrak{m})}$$

for any Borel $\eta\colon X \to [0,\infty]$.

Proposition 4.4. *For all $\delta > 0$ there exists a weight $\widetilde{\rho}\colon X \to [0,\infty)$ satisfying the hypotheses of Theorem 4.1 such that $L_{\widetilde{\rho}} \le \delta$, $W_\rho^{1,p} = W_{\widetilde{\rho}}^{1,p}$ and $H_\rho^{1,p} = H_{\widetilde{\rho}}^{1,p}$.*

Proof. Throughout this proof we simply write bounded or unbounded instead of essentially bounded (i.e. in $L^\infty(\mathfrak{m})$) or essentially unbounded (not in $L^\infty(\mathfrak{m})$). We start by observing that if both ρ and ρ^{-1} are bounded then $L_\rho = 0$, so there is nothing to prove.

If ρ and ρ^{-1} are both unbounded then for every $t > 0$ we define

$$\rho_t(x) := \begin{cases} \dfrac{1}{t}\rho(x) & \text{if } \rho(x) \le 1;\\[2ex] t\rho(x) & \text{if } \rho(x) > 1.\end{cases} \tag{4.16}$$

Clearly ρ_t satisfies the assumptions of Theorem 4.1 since it is bounded by constant multiples of ρ. For the same reason, the W and H weighted Sobolev spaces induced by ρ and ρ_t are the same. Observe that

$$\int_X \rho_t^n \le t^{-n}\mathfrak{m}(X) + t^n\int_X \rho^n, \qquad \int_X \rho_t^{-n} \le t^{-n}\mathfrak{m}(X) + t^n\int_X \rho^{-n}. \tag{4.17}$$

Since ρ is unbounded, we get

$$\lim_{n\to\infty} \frac{1}{\mathfrak{m}(X)}\lambda^{-n}\int_X \rho^n = \infty \qquad \forall \lambda > 1. \tag{4.18}$$

Using (4.18) with $\lambda = t^{-2}$ we get

$$t^{-n}\mathfrak{m}(X) \le t^n \int_X \rho^n \qquad \text{for } n \text{ sufficiently large.}$$

Therefore the first inequality in (4.17) gives

$$\|\rho_t\|_{L^n(\mathfrak{m})} \le 2^{\frac{1}{n}} t \|\rho\|_{L^n(\mathfrak{m})} \qquad \text{for } n \text{ sufficiently large.} \tag{4.19}$$

Arguing in the same way using the unboundedness of ρ^{-1} and the second inequality in (4.17) we obtain

$$\|\rho_t^{-1}\|_{L^n(\mathfrak{m})} \le 2^{\frac{1}{n}} t \|\rho^{-1}\|_{L^n(\mathfrak{m})} \qquad \text{for } n \text{ sufficiently large.} \tag{4.20}$$

Hence, by (4.19) and (4.20) we obtain the thesis choosing $t > 0$ with $2t^2 < C^{-1}$.

Let us now assume that ρ is bounded but ρ^{-1} is unbounded. For every $t > 0$ we define

$$\rho_t(x) := \begin{cases} \frac{1}{t}\rho(x) & \text{if } 0 \le \rho(x) \le t \\ \rho(x) & \text{if } \rho(x) > t. \end{cases} \tag{4.21}$$

As before, ρ_t satisfies the hypotheses of Theorem 4.1. We observe,

$$\begin{aligned} \int_X \rho_t^{-n} &= t^n \int_{\{\rho \le t\}} \rho^{-n} + \int_{\{\rho > t\}} \rho^{-n} \\ &\le t^n \int_X \rho^{-n} + t^{-n}\mathfrak{m}(X). \end{aligned}$$

Since ρ^{-1} is unbounded proceeding as in (4.19) we obtain

$$\|\rho_t^{-1}\|_{L^n(\mathfrak{m})} \le 2^{\frac{1}{n}} t \|\rho^{-1}\|_{L^n(\mathfrak{m})}, \tag{4.22}$$

while

$$\|\rho_t\|_{L^n(\mathfrak{m})} \le \|\rho\|_{L^\infty(\mathfrak{m})} + 1 \tag{4.23}$$

for every n. Because ρ is bounded we have

$$L_\rho = \liminf_{n\to\infty} \frac{\|\rho\|_{L^\infty(\mathfrak{m})}\|\rho^{-1}\|_{L^n(\mathfrak{m})}}{n^p}. \tag{4.24}$$

Putting together (4.22), (4.23) and (4.24) we get

$$\liminf_{n\to\infty} \frac{1}{n^p}\|\rho_t\|_{L^n(\mathfrak{m})}\|\rho_t^{-1}\|_{L^n(\mathfrak{m})} \le 2tL_\rho(1 + \|\rho\|^{-1}_{L^\infty(\mathfrak{m})})$$

and we conclude, again, choosing $t > 0$ sufficiently small. The case where ρ^{-1} is bounded and ρ is unbounded is analogous. □

We can now conclude the proof of Theorem 4.1. Choose $\delta \in (0, C^{-1})$, where $C > 0$ is the constant in (4.15) and apply Proposition 4.4 to find a weight $\widetilde{\rho}$ satisfying the hypotheses of Theorem 4.1 such that $L_{\widetilde{\rho}} \leq \delta$, $W^{1,p}_{\rho} = W^{1,p}_{\widetilde{\rho}}$ and $H^{1,p}_{\rho} = H^{1,p}_{\widetilde{\rho}}$. Then the assumption that $W^{1,p}_{\rho} \setminus H^{1,p}_{\rho} \neq \varnothing$ implies $W^{1,p}_{\widetilde{\rho}} \setminus H^{1,p}_{\widetilde{\rho}} \neq \varnothing$ and we may repeat all of our arguments to obtain an analogue of (4.15) with $\widetilde{\rho}$ in place of ρ; hence,

$$1 \leq C L_{\widetilde{\rho}} \leq C\delta < 1$$

which gives a contradiction.

5. Examples and Extensions

In this section we discuss some examples and generalize our results by considering Muckenhoupt weights or by requiring a weaker Poincaré inequality.

5.1. **An example where $W^{1,p}_{\rho} \supsetneq H^{1,p}_{\rho}$.** Let us consider the standard Euclidean structure $X =$ closed unit ball of $\mathbb{R}^2$, d $=$ Euclidean distance, $\mathfrak{m} = \mathscr{L}^2$. In [13] examples of weights $\rho \in L^1(\mathfrak{m})$ with $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$ and $H^{1,p}_{\rho} \subsetneq W^{1,p}_{\rho}$ are given for any $p > 1$. Here we report only the example with $p = 2$, with a weight ρ in all L^q spaces having also the inverse in all L^q spaces, $1 \leq q < \infty$.

Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and $\epsilon \in (0, \pi/2)$. Set

$$S_\epsilon := \{(x_1, x_2) \in \Omega \mid \tan(\epsilon) < \frac{x_2}{x_1} < \tan(\frac{\pi}{2} - \epsilon)\},$$

$S^+_\epsilon := S_\epsilon \cap \{x_2 > 0\}$ and $S^-_\epsilon := S_\epsilon \cap \{x_2 < 0\}$. Let us consider $\rho : \mathbb{R}^2 \setminus \{0\} \to [0, \infty)$ defined by

$$\rho(x) := \begin{cases} \left(\ln^{-2}\left(\frac{e}{|x|}\right)\right)^{k \arccos\left(\frac{x_1}{|x_1|}\right)} & \text{if } 0 < |x| \leq 1 \\ 1 & \text{if } |x| > 1, \end{cases}$$

where $k : \mathbb{R} \to [-1, 0]$ is a $\pi-$periodic smooth function such that $k'(0) = 0$ and

$$k \equiv -1 \quad \text{in } \left(\epsilon, \frac{\pi}{2} - \epsilon\right), \qquad k \equiv 0 \quad \text{in } \left(\frac{\pi}{2}, \pi\right).$$

It follows that $\lambda \in C^0(\mathbb{R}^2 \setminus \{0\})$ and that

$$\rho, \rho^{-1} \in \bigcap_{q \in [1,\infty)} L^q_{\text{loc}}(\mathbb{R}^2).$$

It is proved by a direct calculation in [13] that the function

$$u(x) := \begin{cases} 1 & \text{if } x_1 > 0,\ x_2 > 0 \\ 0 & \text{if } x_1 < 0,\ x_2 < 0 \\ x_2/|x| & \text{if } x_1 < 0,\ x_2 > 0 \\ x_1/|x| & \text{if } x_1 > 0,\ x_2 < 0 \end{cases}$$

belongs to $W^{1,2}_{\rho} \setminus H^{1,2}_{\rho}$.

5.2. **Muckenhoupt weights and the PI_p condition.** An important class of weights is the one introduced by Muckenhoupt [24] to study the boundedness of the maximal operator in L^p spaces. In Euclidean spaces, for this p-dependent class of weights ρ it is known that $W^{1,p}_\rho = H^{1,p}_\rho$, see for instance [12, 19] and Theorem 5.7 below. Let us recall the definition of Muckenhoupt weight in the context of metric measure spaces.

Definition 5.1 (Muckenhoupt weight)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and let $\rho : X \to [0, \infty]$ be locally integrable. For $p > 1$, we say that ρ is an A_p-weight if*

$$\sup_B \left(\fint_B \rho \mathrm{d}\mathfrak{m} \right) \left(\fint_B \rho^{-1/(p-1)} \mathrm{d}\mathfrak{m} \right)^{p-1} < \infty$$

where the supremum runs among all balls B. We say that ρ is an A_1-weight if there exists a constant C such that

$$\fint_B \rho \mathrm{d}\mathfrak{m} \leq C \operatorname*{ess\,inf}_B u$$

for all balls $B \subset X$. We denote the class of A_p weights on $(X, \mathrm{d}, \mathfrak{m})$ by $A_p(\mathfrak{m})$.

It is immediate to see, using the Hölder inequality, that $\big(\fint_B \rho \mathrm{d}\mathfrak{m} \big) \big(\fint_B \rho^{-1/(p-1)} \mathrm{d}\mathfrak{m} \big)^{p-1}$ is always larger than 1. This easily yields that $(X, \mathrm{d}, \rho\mathfrak{m})$ is doubling whenever $(X, \mathrm{d}, \mathfrak{m})$ is doubling; indeed, for $p > 1$,

$$\fint_{2B} \rho \mathrm{d}\mathfrak{m} \leq C \left(\fint_{2B} \rho^{-1/(p-1)} \mathrm{d}\mathfrak{m} \right)^{1-p} \leq C \left(\fint_B \rho^{-1/(p-1)} \mathrm{d}\mathfrak{m} \right)^{1-p} \leq C \fint_B \rho \mathrm{d}\mathfrak{m}$$

and a similar argument works for $p = 1$. It follows that the maximal operator with respect to $\rho\mathfrak{m}$ is bounded in $L^p(\rho\mathfrak{m})$ for all $p > 1$. A remarkable fact, proved in the Euclidean case by Muckenhoupt [24], with a proof that extends readily also to doubling metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$ (see [26, Theorem 9]), is the fact that even the maximal operator M in (2.9), namely the maximal operator respect to $\mathfrak{m}$, is bounded in $L^p(\rho\mathfrak{m})$ for all $p > 1$, and weakly bounded if $p = 1$.

For $p > 1$ it is well known that an A_p weight ρ on a Euclidean space is p-admissible [19]; this means that the weighted space $(\mathbb{R}^n, |\cdot|, \rho\mathcal{L}^n)$ satisfies PI_p. A generalization in the metric setting is proved in [8]. A converse holds in dimension one but it is an open problem for higher dimensions [10].

Let us now compare the Muckenhoupt condition with the Zhikov one, introduced in [28] and used also in the present paper.

Definition 5.2. *Let $\rho\colon X \to [0, \infty]$ be Borel and let $p > 1$. We say that ρ belongs to the class $Z_p(\mathfrak{m})$ if*

$$\liminf_{n\to\infty} \frac{1}{n^p} \|\rho\|_{L^n(\mathfrak{m})} \|\rho^{-1}\|_{L^n(\mathfrak{m})} < \infty.$$

Even though both the Muckenhoupt and Zhikov conditions lead to the identification of the weighted Sobolev spaces, the following simple examples show that they are not comparable, even in the Euclidean case. One of the reasons is that the class A_p involves a more local condition; for instance there is no reason for $(X, \mathrm{d}, \rho\mathfrak{m})$ to be doubling when $(X, \mathrm{d}, \mathfrak{m})$ is doubling and $\rho \in Z_p(\mathfrak{m})$. As a matter of fact, the Zhikov condition is easier to check. For instance, if both $\exp(t\rho)$ and $\exp(t\rho^{-1})$ belong to $L^1(\mathfrak{m})$ for some $t > 0$, then $\rho \in Z_2(\mathfrak{m})$ (see the simple proof in [28], still valid in the metric measure setting).

Example 5.3. Let X be the unit ball of $\mathbb{R}^2$ and let $\rho\colon X \to \mathbb{R}$ be given by

$$\rho(x) := \begin{cases} \log(1/|x|) & \text{if } x_1x_2 > 0, \\ \dfrac{1}{\log(1/|x|)} & \text{if } x_1x_2 < 0. \end{cases}$$

It is easy to check that $\exp(\rho)$, $\exp(\rho^{-1}) \in L^1(\mathscr{L}^2)$ (see also [28]), which implies $\rho \in Z_2(\mathscr{L}^2)$. Further, one can easily prove that $\rho \notin A_p(\mathscr{L}^2)$ for any $p \geq 1$; indeed, the average of both ρ and ρ^{-1} on balls centred at the origin tends to infinity as the radius of the ball tends to zero.

Example 5.4. Let $X = (0,1)$. Then, by a direct computation, $|x|^\alpha \in A_p(\mathscr{L}^1)$ provided $-1 < \alpha < p-1$. Hence $1/\sqrt{x} \in A_p(\mathscr{L}^1)$ for any $p \geq 1$; however, $1/\sqrt{x} \notin Z_p(\mathscr{L}^1)$ for any $p \geq 1$ since $1/\sqrt{x} \notin L^n(\mathscr{L}^1)$ for any $n \geq 2$.

5.3. **Relaxation of the PI_1 assumption to PI_p.** Suppose $\rho^{-1} \in L^\alpha(\mathfrak{m})$ for some exponent $\alpha \in (1/(p-1), \infty]$ and set $q = p\alpha/(\alpha+1)$ ($q = p$ if $\alpha = \infty$). Then the definition

$$W^{1,p}_{\rho,q} := \left\{ f \in W^{1,q}(X, \mathrm{d}, \mathfrak{m}) : \int_X |f|^p \rho \mathrm{d}\mathfrak{m} + \int_X |\nabla f|^p_{q,\mathfrak{m}} \rho \mathrm{d}\mathfrak{m} < \infty \right\}, \tag{5.1}$$

with the corresponding norm, and the corresponding definition of $H^{1,p}_{\rho,p}$ (namely the closure of $\mathrm{Lip}(X) \cap W^{1,p}_{\rho,q}$ inside $W^{1,p}_{\rho,q}$) are much more natural. Indeed, there is already a natural embedding in $W^{1,q}(X, \mathrm{d}, \mathfrak{m})$ that is missing in the general case, so there is no necessity to invoke the space $W^{1,1}(X, \mathrm{d}, \mathfrak{m})$ and the PI_1 structure (notice that $q = 1$ corresponds precisely to $\alpha = 1/(p-1)$). The embedding provides completeness of $W^{1,p}_{\rho,q}$, via the completeness of $W^{1,q}(X, \mathrm{d}, \mathfrak{m})$, and also the proof of reflexivity can be immediately adapted to the space $W^{1,p}_{\rho,q}$.

Assume now that $\rho^{-1} \in L^\alpha(\mathfrak{m})$ for any $\alpha \in (1/(p-1), \infty)$, as it happens when $L_\rho < \infty$; in this case we can choose the power q in (5.1) as close to p as we wish, and use the fact that PI_p is an open ended condition to choose q in such a way that PI_q still holds. This leads to the following result (which also shows that the space $W^{1,p}_{\rho,q}$ is essentially independent of the exponent q).

Theorem 5.5. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a PI_p metric measure space. Let $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ be a non-negative Borel function satisfying $\rho^{-1} \in L^\alpha(\mathfrak{m})$ for all $\alpha \in (1/(p-1), \infty)$. Then the space $W^{1,p}_{\rho,q}$ in* (5.1) *and its norm do not depend on the choice of $q \in [1,p)$, as soon as PI_q holds, and*

$$H^{1,p}_{\rho,q} = W^{1,p}(X, \mathrm{d}, \rho\mathfrak{m}) \qquad \textit{whenever } \mathsf{PI}_q \textit{ holds.} \tag{5.2}$$

If, in addition, $L_\rho < \infty$, then $H^{1,p}_{\rho,q} = W^{1,p}_{\rho,q}$.

Proof. Recall that (see Theorem 12.5.1 in [20], while [11] covered only the case $q > 1$)

$$|\nabla f| = |\nabla f|_{q,\mathfrak{m}} \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } f \in \mathrm{Lip}(X) \cap W^{1,q}(X, \mathrm{d}, \mathfrak{m})$$

for all $q \in [1, \infty)$, under the PI_q condition. This, combined with the locality of weak gradients and the Lusin approximation with Lipschitz functions gives

$$|\nabla f|_{q,\mathfrak{m}} = |\nabla f|_{q',\mathfrak{m}} \qquad \mathfrak{m}\text{-a.e. in } X$$

whenever $f \in W^{1,q}(X,\mathrm{d},\mathfrak{m}) \cap W^{1,q'}(X,\mathrm{d},\mathfrak{m})$ and both PI_q and $\mathsf{PI}_{q'}$ hold. Then, the independence of $W^{1,p}_{\rho,q}$ with respect to q follows by Proposition 2.11. The identity (5.2) and the last statement can be obtained repeating respectively the proofs of Theorem 3.6 and Theorem 4.1 with this new class of weighted spaces. □

5.4. **Combination of Zhikov and Muckenhoupt weights.** Zhikov [28] proves identification of weighted Sobolev spaces for weights ρ expressable as a product $\rho = \rho_M \rho_Z$ where $\rho_M \in A_p(\mathcal{L}^n)$ and $\rho_Z \in Z_p(\rho_M \mathcal{L}^n)$. The minor adaptations needed to include Muckenhoupt weights work also in the metric setting.

Theorem 5.6. *Suppose $(X,\mathrm{d},\mathfrak{m})$ is a PI_1 metric measure space. Let $p > 1$ and $\rho = \rho_M \rho_Z$ where $\rho_M \in A_p(\mathfrak{m})$ and $\rho_Z \in Z_p(\rho_M \mathfrak{m})$. If $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ and $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$ then $W^{1,p}_\rho(\mathfrak{m}) = H^{1,p}_\rho(\mathfrak{m})$.*

Proof. As remarked in the discussion before Theorem 5.7 we know that, the maximal operator with respect to $\mathfrak{m}$ is bounded in $L^p(\rho_M \mathfrak{m})$ if $p > 1$. To obtain the identification $W^{1,p}_\rho(\mathfrak{m}) = H^{1,p}_\rho(\mathfrak{m})$ we apply exactly the same argument as in the proof of Theorem 4.1 apart from the fact that in (4.10) we apply Hölder's inequality and boundedness of the maximal operator with respect to the measure $\rho_M \mathfrak{m}$; hence (4.10) changes to

$$\begin{aligned}
\int_X M(|\nabla u|_w)^{p_\varepsilon} \rho_Z \rho_M &\le \left(\int_X M(|\nabla u|_w)^{p_\varepsilon r} \rho_M\right)^{1/r} \left(\int_X \rho_Z^{r'} \rho_M\right)^{1/r'} \\
&\le C \left(\int_X |\nabla u|_w^{p_\varepsilon r} \rho_Z^{1/s} \rho_Z^{-1/s} \rho_M\right)^{1/r} \left(\int_X \rho_Z^{r'} \rho_M\right)^{1/r'} \\
&\le C \left(\int_X |\nabla u|_w^{p_\varepsilon rs} \rho_Z \rho_M\right)^{1/(rs)} \left(\int_X \rho_Z^{r'} \rho_M\right)^{1/r'} \left(\int_X \rho_Z^{-s'/s} \rho_M\right)^{1/(rs')} \\
&\le C \|u\|_\rho^{p/(rs)} \left(\int_X \rho_Z^{r'} \rho_M\right)^{1/r'} \left(\int_X \rho_Z^{-r'} \rho_M\right)^{1/(rs')}.
\end{aligned}$$

With this estimate we are able to use the assumption $\rho_Z \in Z_p(\rho_M \mathfrak{m})$ to obtain again the identification $W^{1,p}_\rho(\mathfrak{m}) = H^{1,p}_\rho(\mathfrak{m})$. □

The following result, which is well known in the Euclidean setting, easily follows from Theorem 5.6 by taking $\rho_Z = 1$.

Corollary 5.7. *Suppose $(X,\mathrm{d},\mathfrak{m})$ is a PI_1 space. Let $\rho \in A_p(\mathfrak{m})$ with $\rho \in L^1_{\mathrm{loc}}(\mathfrak{m})$ and $\rho^{-1} \in L^{1/(p-1)}(\mathfrak{m})$. Then we have $W^{1,p}_\rho(\mathfrak{m}) = H^{1,p}_\rho(\mathfrak{m})$.*

A more direct proof of Corollary 5.7 can be obtained using the same approach described in Proposition 2.10. More precisely,

$$\begin{aligned}
\fint_B |f| \mathrm{d}\mathfrak{m} &\le \frac{1}{\mathfrak{m}(B)} \left(\int_B |f|^p \rho \mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \left(\int_B \rho^{-\frac{1}{p-1}} \mathrm{d}\mathfrak{m}\right)^{\frac{p}{p-1}} \\
&\le \left(\fint_B |f|^p \rho \mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \left(\fint_B \rho \mathrm{d}\mathfrak{m} \left(\fint_B \rho^{-\frac{1}{p-1}} \mathrm{d}\mathfrak{m}\right)^{p-1}\right)^{\frac{1}{p}}
\end{aligned}$$

and since $\rho \in A_p(\mathfrak{m})$ we get

$$M_{\mathfrak{m}} f(x) \leq c\,(M_{\rho\mathfrak{m}} f^p(x))^{\frac{1}{p}} \tag{5.3}$$

where $c > 0$ and depends only on ρ. Given $f \in W^{1,p}_\rho \subset W^{1,1}(X, \mathrm{d}, \mathfrak{m})$, by (2.13) and (5.3) we have

$$|f(x) - f(y)| \leq C\mathrm{d}(x,y) \left(\left(M_{\rho\mathfrak{m}} |\nabla f|^p_{1,\mathfrak{m}}(x) \right)^{\frac{1}{p}} + \left(M_{\rho\mathfrak{m}} |\nabla f|^p_{1,\mathfrak{m}}(y) \right)^{\frac{1}{p}} \right)$$

and, proceeding exactly as in Proposition 2.10, we obtain $W^{1,p}_\rho \subset H^{1,p}_\rho$ and therefore $W^{1,p}_\rho = H^{1,p}_\rho$.

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