Interior gradient regularity for BV minimizers of singular variational problems

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Abstract

We consider a class of vectorial integrals with linear growth, where, as a key feature, some degenerate/singular behavior is allowed. For generalized minimizers of these integrals in BV, we establish interior gradient regularity and — as a consequence — uniqueness up to constants. In particular, these results apply, for $1 < p < 2$, to the singular model integrals

$$\int_\Omega (1 + |\nabla w(x)|^p)^{\frac{1}{p}} \,\mathrm{d}x\,.$$

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Contents

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1 Introduction

In this paper we are concerned with variational integrals of the form

$$F[w] \coloneqq \int_\Omega f(\nabla w(x))\,\mathrm{d}x \qquad \text{for } w\colon \Omega \to \mathbb{R}^N\,. \tag{1.1}$$

Here, the dimensions $n, N \in \mathbb{N}$ (with $n \geq 2$) and a bounded, open, connected[1] set Ω in $\mathbb{R}^n$ with a Lipschitz boundary[2] are permanently and arbitrarily fixed, and $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is a given convex integrand with linear growth. Prescribing boundary values by means of a given function $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, we study the minimization problem for F in the Dirichlet class

$$\mathcal{D} \coloneqq u_0 + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N).$$

This problem has a natural generalized formulation — which will be explained in Section 2 and which we adopt in the following — in the space $\mathrm{BV}(\Omega, \mathbb{R}^N)$ of functions of bounded variation, and we are mainly interested in regularity and uniqueness results for the generalized minimizers.

Before addressing the case of general integrands f, we want to draw the attention to the singular model integrals

$$\mathrm{M}_p[w] \coloneqq \int_\Omega (1 + |\nabla w(x)|^p)^{\frac{1}{p}}\,\mathrm{d}x \qquad \text{for } w\colon \Omega \to \mathbb{R}^N \tag{1.2}$$

with $1 < p < 2$ (where $|\nabla w(x)|$ denotes the Hilbert-Schmidt norm of the gradient matrix $\nabla w(x) \in \mathbb{R}^{Nn}$). In this situation, our main result guarantees $\mathrm{C}^{1,\alpha}$ regularity and uniqueness up to additive constants for the generalized minimizers:

Theorem 1.1 ($\mathrm{C}^{1,\alpha}$ regularity and uniqueness for singular model problems)**.** *For $1 < p < 2$ consider two generalized minimizers $u, v \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ of M_p in a Dirichlet class $\mathcal{D}$. Then u and v are of class $\mathrm{C}^{1,\alpha}$ in the interior of Ω with some positive Hölder exponent α, which depends only on n, N, and p. Moreover, for some constant vector $y \in \mathbb{R}^N$ there holds $u = v + y$ almost everywhere on Ω.*

A formal deduction of Theorem 1.1 from the more general Theorem 1.3 is implemented at the end of Section 5.

We stress that the second derivatives of the Lagrangian m_p in (1.2), given by

$$\mathrm{m}_p(z) \coloneqq (1 + |z|^p)^{\frac{1}{p}} \qquad \text{for } z \in \mathbb{R}^{Nn}\,,$$

blow up for $z \to 0$. The main novelty of Theorem 1.1 and the present paper lies in the treatment of such singular structures in a BV-setting. For non-singular integrals in BV, in contrast, closely related statements are already available in the literature; we refer specifically to the results of [10, 7, 24], which we eventually discuss in a more general context. We also mention that, in the two-dimensional, scalar case $n = 2$, $N = 1$, a closely related model equation motivated by nonlinear elasticity has recently been considered in [11]: under strong assumptions on Ω and u_0, it has been shown that this non-singular equation has even a strong solution, which realizes the boundary values in the $\mathrm{W}^{1,1}$ sense.

We further emphasize that by a classical example of Santi [25] (see also [21, Example 15.12] and [4, Proposition 3.11]) full uniqueness of minimizers generally fails in the BV-context; thus, one cannot hope to infer $u = v$ in the situation of Theorem 1.1. Nevertheless, one can go slightly beyond the uniqueness assertion made above (and likewise in the more general Theorem 1.3 below): indeed, once uniqueness

[1]The connectedness assumption is not mandatory and is only made in order to simplify the formulation of the uniqueness statements below. Evidently, if this assumption is removed, our theorems still apply on each connected component of Ω.

[2]With a few technical modifications, generalized minimizers in $\mathrm{BV}(\Omega, \mathbb{R}^N)$ still make sense for less regular and possibly unbounded domains Ω, and our results continue to hold in this more general setting. However, in order to avoid some technicalities, which have already been extensively discussed in [5, 29], we prefer to impose the above-stated stronger hypotheses on Ω.

up to constants is established, it follows from [4, Theorem 1.16] that the generalized minimizers in $\mathcal{D}$ form not only an N-parameter, but even a 1-parameter family. This aspect has already been discussed at length in [4] and will not be reconsidered in the sequel.

At this stage let us rather contrast Theorem 1.1 with the known results for the model integrals M_p: In the scalar case $N = 1$, the area functional M_2 is of course covered by classical theory (see again [21]). Still assuming $N = 1$, Tausch [33] discussed an alternative approach, which is based on the global gradient estimates of Serrin [30] and Trudinger [35] and which yields the existence of $\mathrm{C}^{1,\alpha}$ minimizers of M_p — for arbitrary $1 < p < \infty$, but under extra assumptions on the domain and the data. In the following account, however, we are mainly interested in the vectorial case with arbitrary $N \in \mathbb{N}$, which seems to require different methods: in this generality, everywhere regularity results for the functional M_2 were obtained in [7, 9, 4], and specifically in [4] we have proved $\mathrm{L}\log\mathrm{L}$ gradient estimates and uniqueness up to constants for the generalized minimizers. Moreover, *almost-everywhere* $\mathrm{C}^{1,\alpha}$ regularity of generalized minimizers of M_p has been established, first by Anzellotti & Giaquinta [3] for $p = 2$ and eventually in [28] for arbitrary $1 < p < \infty$. In the case $1 < p < 2$ this picture is finally completed by our Theorem 1.1, while it remains an open question whether or not the analogous assertions are generally true for $2 \le p < \infty$.

Now we return to functionals of the type (1.1), where the convex integrand f satisfies the linear growth condition

$$L_f := \limsup_{|z|\to\infty} \frac{f(z)}{|z|} < \infty\,. \tag{H1}$$

In the first part of our paper, we will establish local Lipschitz regularity for a minimizer u of F, when further assumptions on f are imposed only in a neighborhood of ∞ in $\mathbb{R}^{Nn}$. For integrands of superlinear growth, the sufficiency of such asymptotic conditions was studied in detail (see for instance [12, 14, 16, 26, 27] and the references given there) and is generally very plausible. However, a suitable result for our purposes is not yet available in the literature and will be established here:

Theorem 1.2 ($\mathrm{W}^{1,\infty}_{\mathrm{loc}}$ regularity for one minimizer of an asymptotically μ-elliptic problem)**.** *Suppose that f is convex with* (H1)*. Moreover, assume that for some radius R we have*

$$\begin{gathered} f \in \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^{Nn} \setminus \overline{\mathrm{B}_R})\,, \qquad f(z) = g(|z|) \text{ for } |z| \ge R\,, \\ \frac{\gamma|\xi|^2}{|z|^\mu} \le \nabla^2 f(z)(\xi,\xi) \le \frac{\Gamma|\xi|^2}{|z|} \qquad \text{for } \xi \in \mathbb{R}^{Nn} \text{ and a.e. } z \in \mathbb{R}^{Nn} \setminus \mathrm{B}_R \end{gathered} \tag{H2}$$

with a function $g\colon [R,\infty) \to \mathbb{R}$, with positive constants γ and Γ, and with an exponent

$$1 < \mu < 3\,.$$

Then, for every $u_0 \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$, there exists a generalized minimizer u of F in the Dirichlet class $\mathcal{D} = u_0 + \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)$ such that we have $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and

$$|\nabla u(x_0)| \le C\left(1 + \operatorname{dist}(x_0,\partial\Omega)^{-n}\int_\Omega |\nabla u_0|\,\mathrm{d}x\right)^{1+\frac{3(\mu-1)}{2(3-\mu)}} \qquad \text{for a.e. } x_0 \in \Omega\,, \tag{1.3}$$

where C depends only on n, L_f, R, μ, γ, and Γ.

The proof of Theorem 1.2 will be carried out in Sections 3 and 4, while for the moment we only compare our statement with related results in the literature.

We first emphasize that our assumption (H2) does not bound the ratio between the largest and the smallest eigenvalue of $\nabla^2 f(z)$ for a.e. $z \in \mathbb{R}^N \setminus \mathrm{B}_{2R}$ but that this ellipticity ratio[3] may (and actually must[4]) blow up for $|z| \to \infty$, at most at the rate of $|z|^{\mu-1}$. In this sense, Theorem 1.2 deals with

[3]The related quantities $\frac{f(z)}{\nabla^2 f(z)(z,z)}$ and $\frac{|\nabla^2 f(z)|\,|z|^2}{\nabla^2 f(z)(z,z)}$ may also blow up at the rate of $|z|^{\mu-1}$.

[4]If (H2) holds with $\mu = 1$ (this corresponds to a uniformly bounded ellipticity ratio), then integration of the lower bound in (H2) shows $f(z) \ge \frac{1}{2}\gamma|z|\log|z|$ for $|z| \gg 1$, so that f necessarily violates (H1). Therefore, integrands which satisfy (H1) and (H2) with $\mu = 1$ do not exist.

a type of non-uniform ellipticity, which has already been investigated by Fuchs & Mingione [17] and Bildhauer & Fuchs [10] in terms of their μ-ellipticity condition and by Marcellini & Papi [24] in a general growth setting. While the results of [10, 24] apply for $1<\mu<1{+}2/n$, the improved bound $1<\mu<3$, which appears also in our statement, has first been reached by Bildhauer [7] under an extra L^∞ assumption on u_0. We remark that the upper bound $\mu<3$ has some optimality property, and we refer to [8, 4] for a further discussion. In contrast to [10, 7] we here impose (H2) only asymptotically near ∞ and not globally on $\mathbb{R}^{Nn}$, and in contrast to [24] we do not assume a priori that there exists a minimizer of class $\mathrm{W}^{1,1}$.

Turning to the estimate (1.3), we find it worth remarking that the exponent on the right-hand side tends to 1 for $\mu\searrow 1$ and to ∞ for $\mu\nearrow 3$. In this sense, (1.3) behaves almost homogeneously near the limit case $\mu=1$ of uniform ellipticity, but it only yields a rough control for $\mu\lesssim 3$.

The choice $f=\mathrm{m}_p$ satisfies (H2) with the optimal exponent $\mu=1+p$, and thus the model integrals M_p of Theorem 1.1 are included in Theorem 1.2. Even though our main motivation stems from these concrete singular cases, we would like to stress that the asymptotic hypotheses do in fact allow for other locally irregular — for instance degenerate — behavior of the integrand as well. In these regards, Theorem 1.2 resembles the asymptotic regularity result of Cupini & Guidorzi & Mascolo [12], and to some extent it can be seen as an adaptation of [12] from the superlinear to the linear growth case. We remark that some refinements of Theorem 1.2 in the spirit of [12] (x-dependent integrands, convexity only near ∞, no upper bound on $\nabla^2 f$ in case $\mu<2$) may be achievable. However, we have not made an effort to deal with these issues and to establish the most general statement here.

Finally, we point out that Theorem 1.2 establishes the existence of only *one* Lipschitz minimizer, and that we do not know whether or not Lipschitz regularity holds, under the same hypotheses, for *every* generalized minimizer. However, under stronger assumptions on the integrand f (which are once more satisfied for the model integrals), the result of Theorem 1.2 can be improved a posteriori: $\mathrm{C}^{1,\alpha}$ regularity of the Lipschitz minimizer follows from known results [36, 34, 19, 1, 23] for the superlinear growth case, and then uniqueness up to additive constants (and thus regularity for every minimizer) can be concluded by duality methods (in fact, [5, Corollary 2.4] suffices for our purposes, but should also be compared to previous statements of [6, 10]). This line of argument — which will be discussed formally in Section 5 — yields the following statement.

Theorem 1.3 (C^1 regularity and uniqueness for degenerate/singular problems in BV)**.** *Suppose that f is convex with* (H1)*. Moreover, assume that we have*

$$\begin{aligned} &f\in\mathrm{C}^2(\mathbb{R}^{Nn}\setminus\{0\})\cap\mathrm{C}^1(\mathbb{R}^{Nn})\,, \qquad f(z)=g(|z|)\ \text{for } z\in\mathbb{R}^{Nn}\,,\\ &\frac{\gamma|\xi|^2}{|z|^{2-q}+|z|^{\mu}}\le\nabla^2 f(z)(\xi,\xi)\le\frac{\Gamma|\xi|^2}{|z|^{2-q}+|z|}\qquad \text{for } \xi\in\mathbb{R}^{Nn}\text{ and } z\in\mathbb{R}^{Nn}\setminus\{0\}\\ &|\nabla^2 f(\widetilde z)-\nabla^2 f(z)|\le\Psi(|z|+|\widetilde z|)\mathrm{S}_{q,\beta}(|z|,|\widetilde z|)|\widetilde z-z|^\beta\qquad \text{for } z,\widetilde z\in\mathbb{R}^{Nn}\setminus\{0\} \end{aligned}\tag{H3}$$

with positive constants γ, Γ, with exponents

$$1<q<\infty\,,\qquad 1<\mu<3\,,\qquad \text{and}\qquad 0<\beta\le 1\,,$$

with some function $g\colon[0,\infty)\to\mathbb{R}$ and a non-decreasing function $\Psi\colon(0,\infty)\to[0,\infty)$, and with the scaling factor $\mathrm{S}_{q,\beta}(a,b):=(a{+}b)^{q-2-\beta}$ for $q>2$, $\mathrm{S}_{2,\beta}(a,b):=1$, and $\mathrm{S}_{q,\beta}(a,b):=a^{q-2}b^{q-2}(a{+}b)^{2-q-\beta}$ for $q<2$. If, in this situation, $u\in\mathrm{BV}(\Omega,\mathbb{R}^N)$ is a generalized minimizer of F, then for every subdomain $\Omega'\Subset\Omega$ there is some $\alpha>0$ such that u is of class $\mathrm{C}^{1,\alpha}$ on Ω'. Moreover, whenever $u,v\in\mathrm{BV}(\Omega,\mathbb{R}^N)$ are generalized minimizers of F in a Dirichlet class $\mathcal{D}$, then we have $u=v+y$ almost everywhere on Ω for some $y\in\mathbb{R}^N$.

Let us comment on the set of assumptions in (H3). For $|z|\gg 1$, the assumptions are basically the same as in (H2) above, but, for $|z|\ll 1$, we additionally impose the condition that $\nabla^2 f(z)$ is bounded from above and below by $|z|^{q-2}$. Clearly, this extra condition means that $\nabla^2 f$ has a singularity at 0 in the case $q<2$, while we deal with a degenerate ellipticity for $q>2$. For technical reasons, we also need the last requirement in (H3), a local Hölder continuity condition for $\nabla^2 f$ with a scaling factor $\mathrm{S}_{q,\beta}$, which reflects the singular/degenerate behavior.

The exemplary choice $f = \mathrm{m}_p$ with $1 < p < 2$ satisfies (H3) for $q = p$ and $\mu = 1+p$, and thus the singular model cases of Theorem 1.1 are included in Theorem 1.3. However, also in the degenerate case $q \geq 2$ and in fact for arbitrary $q, \mu \in (1, \infty)$ one can construct integrands f which satisfy (H3). Without entering into a detailed discussion of such examples, let us mention the following ones: if f has the form

$$f(z) = \big(1 + h_{q/p}(|z|)^p\big)^{\frac{1}{p}} ,$$

where $1 < p \leq q < \infty$ are arbitrary and the auxiliary convex function $h_{q/p} \colon [0, \infty) \to [0, \infty)$ is of class C^2 on $(0, \infty)$ with $h_{q/p}(t) = t^{q/p}$ for $t \ll 1$ and $h_{q/p}(t) = t$ for $t \gg 1$, then (H3) holds with the given q and $\mu = 1+p$. Thus, for these specific integrands, Theorem 1.3 is applicable whenever $p < 2$ holds.

To conclude the introductory exposition we mention that the overall strategy of our proofs is quite close to [10, 7] and is widely inspired by these papers. Nevertheless, our reasoning differs from the one of [10, 7] in a number of adaptations and technical refinements, for instance in the precise form of the regularization procedure and of the $\mathrm{W}^{1,\infty}$ estimates. The changes are partially caused by the singular structure of the model integrands m_p or the asymptotic formulation of (H2), and all in all we believe that they deserve the detailed treatment which we provide in the present paper.

2 Preliminaries

Some notation. The open ball in $\mathbb{R}^m$ with center x_0 and radius R is abbreviated by $\mathrm{B}_R(x_0)$, and for $x_0 = 0$ we write $\mathrm{B}_R := \mathrm{B}_R(0)$. Furthermore, $\mathcal{L}^m$ stands for the Lebesgue measure and $\mathcal{H}^\ell$ for the ℓ-dimensional Hausdorff measure on $\mathbb{R}^m$. For an arbitrary set $S \subset \mathbb{R}^m$ we further denote by ∂S its topological boundary and by $\overline{S}$ its closure. For a measurable set S with $0 < \mathcal{L}^m(S) < \infty$ and an integrable function defined on S, we indicate by $w_S := \fint_S w \,\mathrm{d}x := \mathcal{L}^m(S)^{-1} \int_A w \,\mathrm{d}x$ the mean value of w over S and by $w_+ := \max\{w, 0\}$ the positive part of w (with the convention $w_+^p := (w_+)^p$). We further write I_m for the $(m{\times}m)$ identity matrix.

Generalized minimizers in BV. Under a Dirichlet class $\mathcal{D}$ we understand a subset of $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ of the form $\mathcal{D} = u_0 + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N)$ with some function $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. In order to minimize F in a Dirichlet class, one commonly extends F to all of $\mathrm{BV}(\Omega, \mathbb{R}^N)$ by semicontinuity (see [20, 15] for motivation and discussion). For convex integrands $f \colon \mathbb{R}^{Nn} \to \mathbb{R}$ with (H1) this extension has the integral representation

$$\mathcal{F}^{\mathcal{D}}[w] := \int_\Omega f(\nabla w)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mathrm{D}^s w}{\mathrm{d}|\mathrm{D}^s w|}\Big)\,\mathrm{d}|\mathrm{D}^s w| + \int_{\partial\Omega} f^\infty((u_0 - w) \otimes \nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}$$

for $w \in \mathrm{BV}(\Omega, \mathbb{R}^N)$. Here, $\mathrm{D}w = \nabla w \cdot \mathcal{L}^n + \mathrm{D}^s w$ denotes the decomposition of the $\mathbb{R}^{Nn}$-valued gradient measure $\mathrm{D}w$ into its absolutely continuous and its singular part with respect to $\mathcal{L}^n$, ν_Ω stands for the outward unit normal vector to $\partial\Omega$, and w and u_0 in the boundary integral are meant as traces. Moreover, the function f^∞ is the recession function of f defined via

$$f^\infty(z) := \lim_{s\to\infty} \frac{f(sz)}{s} \qquad \text{for } z \in \mathbb{R}^{Nn} .$$

We stress that, for a convex integrand f with (H1), its recession function f^∞ is well-defined, real-valued, and 1-homogeneous.

We now introduce the notions of a generalized minimizer and a minimizing sequence for F in a Dirichlet class $\mathcal{D}$.

Definition 2.1 (generalized minimizer)**.** *Suppose that f is convex with* (H1)*. A function $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ is called a generalized minimizer of F in $\mathcal{D}$ if there holds*

$$\mathcal{F}^{\mathcal{D}}[u] \leq \mathcal{F}^{\mathcal{D}}[w] \qquad \text{for all } w \in \mathrm{BV}(\Omega, \mathbb{R}^N)\,.$$

Definition 2.2 (minimizing sequence)**.** *Suppose that f is convex with* (H1)*. A sequence $(u_k)_{k\in\mathbb{N}}$ in a Dirichlet class $\mathcal{D}$ is called a minimizing sequence for F in $\mathcal{D}$ if there holds*

$$\lim_{k\to\infty} F[u_k] = \inf_{\mathcal{D}} F\,.$$

Since $\mathcal{F}^{\mathcal{D}}$ is the extension of F by semicontinuity, generalized minimizers are characterized as limits of minimizing sequences, compare for instance [20], [6, Theorem A.3] and [4, Theorem 1.8].

Theorem 2.3 (characterization of generalized minimizers)**.** *Suppose that f is convex with*

$$0 < \liminf_{|z|\to\infty} \frac{f(z)}{|z|} \leq \limsup_{|z|\to\infty} \frac{f(z)}{|z|} < \infty\,.$$

Then $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ is a generalized minimizer of F in $\mathcal{D}$ if and only if there exists a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ for F in $\mathcal{D}$ such that u_k converges to u in $\mathrm{L}^1(\Omega, \mathbb{R}^N)$. Moreover, one has

$$\inf_{\mathrm{BV}(\Omega,\mathbb{R}^N)} \mathcal{F}^{\mathcal{D}} = \inf_{\mathcal{D}} F\,.$$

Estimates for convex integrands. Next, we collect some basic properties of a convex integrand f which satisfies the assumptions (H1) and (H2).

Lemma 2.4. *Consider a convex function $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$. Then the following statements are true*:

(i) *if f satisfies* (H1)*, then f is globally Lipschitz continuous with Lipschitz constant L_f. In particular, we have $|\nabla f(z)| \leq L_f$ for every point $z \in \mathbb{R}^{Nn}$ of differentiability of f;*

(ii) *if $f \in \mathrm{C}^1(\mathbb{R}^{Nn})$ satisfies* (H1) *and* (H2)*, then there hold*

$$\begin{aligned} \zeta^{\mathrm{T}}\nabla f(z)\cdot\zeta^{\mathrm{T}}z &\geq -C|\zeta|^2\,,\\ \nabla f(z)\cdot z &\geq \nu|z| - C\,,\\ f(z) - f(0) &\geq \nu|z| - C\,,\\ f(z) - f(0) - \nabla f(0)\cdot z &\geq (\nu|z| - C)_+ \end{aligned}$$

for all $z, \zeta \in \mathbb{R}^{Nn}$, with a positive constant $\nu = \nu(R,\mu,\gamma)$ and with $C = C(L_f, R, \mu, \gamma)$.

Proof. To prove statement (i), we initially observe that f is locally Lipschitz continuous by [13, Theorem 2.31] and thus almost-everywhere differentiable by Rademacher's theorem, see [2, Theorem 2.14]. We then fix an arbitrary point $z_0 \in \mathbb{R}^{Nn}$ of differentiability of f and estimate by convexity of f

$$L_f = \limsup_{|z|\to\infty} \frac{f(z)}{|z|} \geq \limsup_{|z|\to\infty} \frac{f(z_0) + \nabla f(z_0)\cdot(z - z_0)}{|z|} \geq |\nabla f(z_0)|\,.$$

This implies all claims in (i).

In order to verify (ii), we first observe

$$g(|2s-1||z|) = f(sz + (1-s)(-z)) \leq sf(z) + (1-s)f(-z) = g(|z|)$$

for all $s \in [0,1]$ and $z \in \mathbb{R}^{Nn}$ with $|2s-1||z| > R$, by convexity and radial structure of f. Hence, $g \in \mathrm{C}^1([R,\infty))$ is non-decreasing, and from $\nabla f(z) = g'(|z|)z/|z|$ we conclude $\zeta^{\mathrm{T}}\nabla f(z)\cdot\zeta^{\mathrm{T}}z \geq 0$

and $\nabla f(z) \cdot z \geq 0$ for $|z| \geq R$ and all $\zeta \in \mathbb{R}^N$. From these inequalities we will now deduce the assertions in (ii). Indeed, the first inequality is immediate when we also take into account the bound $|\nabla f| \leq L_f$ from (i). For the verification of the remaining claims, we employ (H2) to compute[5], for every $z \in \mathbb{R}^{Nn} \setminus \mathrm{B}_{2R}$,

$$\begin{aligned} \nabla f(z) \cdot z &\geq \big(\nabla f(z) - \nabla f(Rz/|z|)\big) \cdot z \\ &= \int_R^{|z|} \nabla^2 f\big(tz/|z|\big)(z/|z|, z)\,\mathrm{d}t \geq \gamma|z| \int_R^{2R} t^{-\mu}\,\mathrm{d}t =: \nu|z|\,. \end{aligned} \tag{2.1}$$

In combination with (i), this gives the second assertion in (ii) with $\nu = \gamma R^{1-\mu}(1-2^{1-\mu})/(\mu-1)$ and $C = 2R(L_f+\nu)$. The third claim in (ii) is obtained from (i) and inequality (2.1) as follows: for every $z \in \mathbb{R}^{Nn}$, we have

$$\begin{aligned} f(z) - f(0) &= \int_0^{\min\{|z|,2R\}} \nabla f\big(tz/|z|\big) \cdot z/|z|\,\mathrm{d}t + \int_{\min\{|z|,2R\}}^{|z|} \nabla f\big(tz/|z|\big) \cdot z/|z|\,\mathrm{d}t \\ &\geq -2RL_f + \nu(|z| - 2R)\,. \end{aligned}$$

Finally, we observe that the auxiliary function $f_*\colon \mathbb{R}^{Nn} \to \mathbb{R}$, defined by $f_*(z) \coloneqq f(z) - f(0) - \nabla f(0)\cdot z$, is non-negative and convex with $f_*(0) = 0$, (H1) (with $L_{f_*} \leq 2L_f$), and (H2). Hence, the last claim in (ii) follows by applying the third one with f_* in place of f, and the proof of the proposition is complete. □

An iteration lemma. We restate [31, Lemme 5.1], which is employed later to prove $\mathrm{W}^{1,\infty}$ regularity of generalized minimizers.

Lemma 2.5. *Consider a function $\varphi\colon [\ell_0, \infty) \times (0, r_0] \to [0, \infty)$ and assume that $\ell \mapsto \varphi(\ell, \rho)$ is non-increasing for fixed ρ, and that $\rho \mapsto \varphi(\ell, \rho)$ is non-decreasing for fixed ℓ. Then*

$$\varphi(m, \rho) \leq K(m - \ell)^{-\alpha_1}(r - \rho)^{-\alpha_2}\big[\varphi(\ell, r)\big]^{1+\delta} \qquad \text{for all } m > \ell \geq \ell_0 \text{ and } \rho < r \leq r_0\,, \tag{2.2}$$

with some positive constants K, α_1, α_2, and δ, implies

$$\varphi(\ell_0 + d, r_0/2) = 0$$

with d given by

$$d^{\alpha_1} = 2^{\alpha_2 + \frac{(1+\delta)(\alpha_1+\alpha_2)}{\delta}} K r_0^{-\alpha_2}[\varphi(\ell_0, r_0)]^{\delta}\,.$$

Proof. For convenience of the reader, we provide the details of the proof. Following [32, Proof of Lemme 5.1 (with $\sigma = 1/2$)] we proceed by iteration. For this purpose we define two sequences $(\ell_i)_{i\in\mathbb{N}_0}$ and $(\rho_i)_{i\in\mathbb{N}_0}$ via

$$\begin{aligned} \ell_i &\coloneqq \ell_0 + d(1 - 2^{-i})\,, \\ \rho_i &\coloneqq 2^{-1}r_0 + 2^{-i-1}r_0\,. \end{aligned}$$

We observe that the sequence $(\ell_i)_{i\in\mathbb{N}_0}$ is increasing with limit ℓ_0+d, whereas the sequence $(\rho_i)_{i\in\mathbb{N}_0}$ is decreasing with limit $r_0/2$. Furthermore, we notice that differences of two subsequent members are given by

$$\ell_i - \ell_{i-1} = 2^{-i}d \qquad \text{and} \qquad \rho_{i-1} - \rho_i = 2^{-i-1}r_0\,.$$

Applying formula (2.2) with $m = \ell_i$, $\ell = \ell_{i-1}$ and $\rho = \rho_i$, $r = \rho_{i-1}$ for arbitrary $i \in \mathbb{N}$, we obtain

$$\varphi(\ell_i, \rho_i) \leq 2^{\alpha_2 + i(\alpha_1+\alpha_2)} K d^{-\alpha_1} r_0^{-\alpha_2}\big[\varphi(\ell_{i-1}, \rho_{i-1})\big]^{1+\delta}\,.$$

[5] As we just assume $\mathrm{W}^{2,\infty}$ regularity of f, its second derivatives need not exist everywhere. However, the radial structure of f implies that $\nabla^2 f(tz/|z|)$ exists and fulfills the estimates in (H2) at least for a.e. $t \in (R, |z|)$.

In the next step we prove by induction the estimate

$$\varphi(\ell_i,\rho_i) \le 2^{-i\frac{\alpha_1+\alpha_2}{\delta}}\varphi(\ell_0,\rho_0) \tag{2.3}$$

for all $i \in \mathbb{N}_0$. Indeed, this inequality is trivially valid for $i = 0$, and for the induction step with $i \in \mathbb{N}$, we employ the choice of d to calculate

$$\begin{aligned}\varphi(\ell_i,\rho_i) &\le 2^{\alpha_2+i(\alpha_1+\alpha_2)}Kd^{-\alpha_1}r_0^{-\alpha_2}\big[\varphi(\ell_{i-1},\rho_{i-1})\big]^{1+\delta}\\ &\le 2^{\alpha_2+i(\alpha_1+\alpha_2)}2^{-(i-1)\frac{(1+\delta)(\alpha_1+\alpha_2)}{\delta}}Kd^{-\alpha_1}r_0^{-\alpha_2}\big[\varphi(\ell_0,\rho_0)\big]^{\delta}\varphi(\ell_0,\rho_0)\\ &= 2^{i(\alpha_1+\alpha_2)-i\frac{(1+\delta)(\alpha_1+\alpha_2)}{\delta}}\varphi(\ell_0,\rho_0) = 2^{-i\frac{\alpha_1+\alpha_2}{\delta}}\varphi(\ell_0,\rho_0)\,.\end{aligned}$$

By the monotonicity properties of φ we deduce from (2.3)

$$\varphi(\ell_0+d,r_0/2) \le \varphi(\ell_i,\rho_i) \le 2^{-i\frac{\alpha_1+\alpha_2}{\delta}}\varphi(\ell_0,\rho_0)$$

for every $i \in \mathbb{N}_0$, and the assertion follows from the passage to the limit $i \to \infty$. □

3 Regularization and approximation

Several arguments and computations in this paper cannot be carried out directly for generalized minimizers of F, since their second derivatives need not exist (in the case of Theorem 1.2, even a posteriori they need not exist anywhere). This leads us to employ a regularization and approximation procedure for the minimization problem $\inf_{\mathcal{D}} F$. This procedure consists in the construction of a sequence $(F_k)_{k\in\mathbb{N}}$ of functionals and a sequence $(\mathcal{D}_k)_{k\in\mathbb{N}}$ of Dirichlet classes with the following properties:

(i) regularization: every minimum problem $\inf_{\mathcal{D}_k} F_k$ has a regular, unique minimizer u_k;

(ii) approximation: these problems approximate $\inf_{\mathcal{D}} F$ in the sense of

$$\lim_{k\to\infty}\inf_{\mathcal{D}_k} F_k = \inf_{\mathcal{D}} F\,,$$

and the sequence $(u_k)_{k\in\mathbb{N}}$ (sub)converges to a generalized minimizer of F in $\mathcal{D}$.

In what follows, we start by discussing the construction of the functionals F_k (by mollification of the integrand f and addition of a $\mathrm{W}^{1,2}$ regularization term). Afterwards, we verify the approximation property (ii), first for $\mathrm{W}^{1,2}$ boundary values and then for general ones. We finally provide uniform L^∞ estimates for the minimizers u_k, which rely on standard techniques, but which are essential for the subsequent gradient estimates in Section 4.

Regularization of the integrand. We introduce regularized integrands $f_k\colon \mathbb{R}^{Nn} \to \mathbb{R}$ by mollification of f and addition of a quadratic term (where mollification is mainly needed, since (H2) allows non-existence of $\nabla^2 f$ on B_R). Precisely, for $k \in \mathbb{N}$ we define

$$f_k(z) := \tfrac{1}{2}\gamma_k|z|^2 + (\chi_k * f)(z) \qquad \text{for all } z \in \mathbb{R}^{Nn}\,. \tag{3.1}$$

Here, $(\gamma_k)_{k\in\mathbb{N}}$ is a null sequence in $(0,1]$, $\chi\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is a fixed smooth, rotationally symmetric, non-negative function, supported in the unit ball with $\int_{\mathbb{R}^{Nn}}\chi\,\mathrm{d}\mathcal{L}^{Nn} = 1$, and $\chi_k * f$ denotes the usual[6] $1/k$-mollification of f. The assumptions (H1) and (H2) are preserved in the following sense.

Lemma 3.1. *Suppose that f is convex. Then, for every $k \in \mathbb{N}$, the mollification $\chi_k * f$ is smooth and convex, and there holds $\chi_k * f \ge f$. Moreover,*

[6] Precisely, the scaled kernels χ_k are given by $\chi_k(z) := k^{Nn}\chi(kz)$ and $\chi_k * f$ is the convolution of χ_k with f.

(i) *if f additionally satisfies* (H1)*, then* (H1) *is also valid for the mollification $\chi_k * f$ with $L_{\chi_k * f} = L_f$, and $\chi_k * f$ has Lipschitz constant L_f;*

(ii) *if f additionally satisfies* (H2)*, then* (H2) *is also valid for the mollification $\chi_k * f$ — with the same exponent $\mu \in (0,\infty)$, with $R+2/k$ in place of R, with $(2/3)^\mu\gamma$ and 2Γ in place of γ and Γ, and with a different function g.*

Proof. In view of the regularity of χ, the convolution $\chi_k * f$ is smooth, and the non-negativity, normalization, and rotational symmetry of χ_k ensure $\chi_k * f \geq f$ via Jensen's inequality. Moreover, it is easy to verify that mollification preserves convexity and the growth condition (H1), and the claim about the Lipschitz constant follows from Lemma 2.4 (i).

We now assume that f satisfies (H2). In this case, $\chi_k * f$ is rotationally symmetric outside $\mathrm{B}_{R+1/k}$, and for $|z| \geq R + 2/k \geq 2/k$ we have

$$\frac{\gamma|\xi|^2}{(\frac{3}{2}|z|)^\mu} \leq \frac{\gamma|\xi|^2}{(|z|+k^{-1})^\mu} \leq (\chi_k * \nabla^2 f)(z)(\xi,\xi) \leq \frac{\Gamma|\xi|^2}{|z|-k^{-1}} \leq \frac{\Gamma|\xi|^2}{\frac{1}{2}|z|}\,,$$

where we estimated the integrand in the convolution integral first via (H2) and then by its infimum or supremum on $\mathrm{B}_{1/k}(z)$. As we have $\nabla^2(\chi_k * f) = \chi_k * \nabla^2 f$, this readily yields the claims about (H2). □

We next record some consequences of Lemma 3.1 for the regularized integrands. If f is convex with (H2), we gain upper and lower bounds for $\nabla^2 f_k(z) = \gamma_k \mathrm{I}_{Nn} + \nabla^2(\chi_k * f)(z)$. Indeed, for $z, \xi \in \mathbb{R}^{Nn}$, $k \in \mathbb{N}$, and any constant $\nu \leq (2/3)^\mu \gamma$ we have

$$(\gamma_k + \nu|z|^{-\mu})|\xi|^2 \leq \nabla^2 f_k(z)(\xi,\xi) \leq (\gamma_k + 2\Gamma|z|^{-1})|\xi|^2 \qquad \text{whenever } |z| > R+2\,. \tag{3.2}$$

If f additionally satisfies (H1), then Lemma 2.4 can be applied to the mollified integrands $\chi_k * f$, and taking into account $\nabla f_k(z) = \gamma_k z + \nabla(\chi_k * f)(z)$ we obtain the following bounds:

$$|\nabla f_k(z)| \leq \gamma_k|z| + L_f\,, \tag{3.3}$$

$$\zeta^{\mathrm{T}}\nabla f_k(z) \cdot \zeta^{\mathrm{T}} z \geq -C|\zeta|^2\,, \tag{3.4}$$

$$\nabla f_k(z) \cdot z \geq \gamma_k|z|^2 + \nu|z| - C\,, \tag{3.5}$$

$$f_k(z) - f_k(0) \geq \gamma_k|z|^2 + \nu|z| - C\,, \tag{3.6}$$

$$f_k(z) - f_k(0) - \nabla f_k(0)\cdot z \geq (\nu|z| - C)_+ \tag{3.7}$$

for all $z \in \mathbb{R}^{Nn}$, $\zeta \in \mathbb{R}^N$, and $k \in \mathbb{N}$. Here, the constants $\nu = \nu(R,\mu,\gamma) > 0$ and $C = C(L_f, R, \mu, \gamma)$ in (3.2)–(3.7) can be taken with the same dependencies as in Lemma 2.4 (ii), since the mollifications $\chi_k * f$ satisfy (H2) with $R+2$ instead of R and with the common constants $(2/3)^\mu\gamma$ and 2Γ in the lower and upper bounds. For our purposes, it will be crucial that ν and C are independent of $k \in \mathbb{N}$.

Approximation of minimizers with $\mathbf{W^{1,2}}$ boundary values. At this stage, we temporarily restrict ourselves to the case of $\mathrm{W}^{1,2}$ boundary values u_0, and we consider the regularized functionals F_k, defined for $k \in \mathbb{N}$ by

$$F_k[w] := \int_\Omega f_k(\nabla w)\,\mathrm{d}x \qquad \text{for } w \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)\,.$$

Assuming that f is convex with (H1), the functionals F_k are finite (precisely) on the subclass $u_0 + \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$ of $\mathcal{D}$; therefore, we need not work with approximations of u_0 and $\mathcal{D}$, but can rather take $\mathcal{D}_k = \mathcal{D}$ for all $k \in \mathbb{N}$. Even though f is merely convex, each functional F_k is strictly convex and has a unique minimizer u_k in $\mathcal{D}$. Evidently, u_k belongs to $u_0 + \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$, but is in fact more regular; see Lemma 4.2, for instance. In addition, we will now show that the sequence $(u_k)_{k\in\mathbb{N}}$ is a minimizing sequence for F in $\mathcal{D}$.

Lemma 3.2. *Consider $u_0 \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ and suppose that f is convex with* (H1). *Then for every null sequence $(\gamma_k)_{k\in\mathbb{N}}$ in $(0,1]$, the corresponding sequence $(u_k)_{k\in\mathbb{N}}$ of minimizers of F_k in $\mathcal{D}$ is a minimizing sequence for F in $\mathcal{D}$. Moreover, we have*

$$\lim_{k\to\infty} \inf_{\mathcal{D}} F_k = \inf_{\mathcal{D}} F \,. \tag{3.8}$$

Proof. We recall from Lemma 2.4 (i) that f is globally Lipschitz continuous, with Lipschitz constant L_f. To establish (3.8), we fix $\varepsilon > 0$, we first choose a suitable function $w_\varepsilon \in u_0 + \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$, and then we take k sufficiently large such that

$$F[w_\varepsilon] \le \inf_{\mathcal{D}} F + \varepsilon/2, \qquad \gamma_k \|\nabla w_\varepsilon\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{Nn})} \le \varepsilon/2 \,, \qquad \text{and} \qquad k^{-1} L_f |\Omega| \le \varepsilon/4 \,.$$

Here, the first inequality follows from the facts that $u_0 + \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$ is dense in $\mathcal{D}$ and that F is continuous in $W^{1,1}(\Omega, \mathbb{R}^N)$ (due to the Lipschitz continuity of f). From Lemma 3.1 we get $f \le \chi_k * f \le f_k$, and thus we have

$$\inf_{\mathcal{D}} F \le F[u_k] \le F_k[u_k] = \inf_{\mathcal{D}} F_k \,. \tag{3.9}$$

In addition, the Lipschitz continuity of f implies

$$0 \le (\chi_k * f)(z) - f(z) = \int_{\mathrm{B}_{1/k}} \big[f(z - \widetilde{z}) - f(z)\big] \chi_k(\widetilde{z}) \,\mathrm{d}\widetilde{z} \le k^{-1} L_f \tag{3.10}$$

for every $z \in \mathbb{R}^{Nn}$. With this estimate and the minimality property of u_k we then find

$$\begin{aligned} \inf_{\mathcal{D}} F_k = F_k[u_k] \le F_k[w_\varepsilon] &= \tfrac{1}{2}\gamma_k \int_\Omega |\nabla w_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega (\chi_k * f)(\nabla w_\varepsilon) \,\mathrm{d}x \\ &\le \tfrac{1}{2}\gamma_k \|\nabla w_\varepsilon\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{Nn})} + F[w_\varepsilon] + k^{-1}|\Omega| L_f \le \inf_{\mathcal{D}} F + \varepsilon \,, \end{aligned}$$

by the choices of the function w_ε and the index k. Combining this inequality with (3.9), we have verified (3.8) and the convergence $F[u_k] \to \inf_{\mathcal{D}} F$ as $k \to \infty$. The proof of the lemma is complete. □

Approximation of general boundary values. Next, we return to the case of arbitrary boundary values $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, which we approximate with $\mathrm{W}^{1,2}$ functions. For this purpose, we choose a sequence $(u_{0,k})_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ (for example by mollifying an extension of u_0) such that

$$\|u_{0,k} - u_0\|_{\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)} \to 0 \quad \text{as } k \to \infty \,, \tag{3.11}$$

we introduce the corresponding Dirichlet classes

$$\mathcal{D}_k \coloneqq u_{0,k} + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N) \,,$$

and we claim that, for a Lipschitz-continuous integrand f with Lipschitz constant L_f (if f is convex with (H1), this is always at hand), there holds

$$\lim_{k\to\infty} \inf_{\mathcal{D}_k} F = \inf_{\mathcal{D}} F \,. \tag{3.12}$$

In fact, we have

$$\begin{aligned} F(u_0 + v) &\le F(u_{0,k} + v) + L_f \|u_{0,k} - u_0\|_{\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)} \,, \\ F(u_{0,k} + v) &\le F(u_0 + v) + L_f \|u_{0,k} - u_0\|_{\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)} \end{aligned}$$

for all $v \in \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N)$, and when we minimize these expressions in v and send $k \to \infty$, we obtain (3.12).

Approximation of minimizers with general boundary values. Next, still for arbitrary $u_0 \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$, we combine the preceding approximation procedures, and thus we look at minimizers u_k of the regularized functionals F_k in the approximating Dirichlet classes $\mathcal{D}_k$. As before, if f is convex with (H1), then these minimizers u_k exist, are unique in $\mathcal{D}_k$, and belong to $u_{0,k} + \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$.

In the following lemma we establish — at least for suitable choice of $(\gamma_k)_{k\in\mathbb{N}}$ — an identity similar to the ones in (3.8) and in (3.12).

Lemma 3.3. *Consider $u_0 \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ with approximations $(u_{0,k})_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ such that the convergence* (3.11) *holds, and suppose that f is convex with* (H1)*. Then there exists a null sequence $(\gamma_k)_{k\in\mathbb{N}}$ in $(0,1]$ such that the sequence $(u_k)_{k\in\mathbb{N}}$ of minimizers u_k of F_k in $\mathcal{D}_k$ satisfies $F[u_k] \to \inf_{\mathcal{D}} F$ as $k\to\infty$ and such that we have*

$$\lim_{k\to\infty} \inf_{\mathcal{D}_k} F_k = \inf_{\mathcal{D}} F\,. \tag{3.13}$$

If, in addition, (H2) *is valid for f, then we also get*

$$\sup_{k\in\mathbb{N}} \int_\Omega \big(\gamma_k|u_k|^2 + \gamma_k|\nabla u_k|^2 + |u_k| + |\nabla u_k|\big)\,\mathrm{d}x < \infty\,, \tag{3.14}$$

and a subsequence of $(u_k)_{k\in\mathbb{N}}$ converges in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ to a generalized minimizer u of F in $\mathcal{D}$.

Proof. We first choose a minimizing sequence $(v_k)_{k\in\mathbb{N}}$ for F in $\mathcal{D}$ and then a sequence $(w_k)_{k\in\mathbb{N}}$ in $u_{0,k} + \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$ with $\|v_k - w_k\|_{W^{1,1}(\Omega,\mathbb{R}^N)} \to 0$ as $k\to\infty$ (this is possible due to the density of $\mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$ in $\mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)$ and the convergence (3.11) of the boundary values). In view of the Lipschitz continuity of f this implies $F[w_k] - F[v_k] \to 0$ as $k\to\infty$, and hence we have

$$F[w_k] \to \inf_{\mathcal{D}} F \qquad \text{as } k\to\infty\,.$$

We point out that, in spite of this convergence, $(w_k)_{k\in\mathbb{N}}$ is not a minimizing sequence of F in $\mathcal{D}$ in the sense of Definition 2.2 since the boundary values of w_k do not agree with those of u_0, but only approximate them. We set

$$\gamma_k \coloneqq k^{-1}\big(1+\|w_k\|^2_{\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)}\big)^{-1}\,,$$

and, given $\varepsilon>0$, we choose k sufficiently large such that

$$F[w_k] \le \inf_{\mathcal{D}} F + \varepsilon/2, \qquad L_f\|u_{0,k}-u_0\|_{\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)} \le \varepsilon\,, \qquad \text{and} \qquad k^{-1} + k^{-1}L_f|\Omega| \le \varepsilon/2\,.$$

With these choices we have on the one hand

$$\inf_{\mathcal{D}} F \le F[u_k + u_0 - u_{0,k}] \le F_k[u_k] + L_f\|u_{0,k}-u_0\|_{\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)} \le \inf_{\mathcal{D}_k} F_k + \varepsilon\,,$$

and on the other hand (keeping in mind the estimate (3.10))

$$\begin{aligned}\inf_{\mathcal{D}_k} F_k = F_k[u_k] \le F_k[w_k] &= \tfrac12\gamma_k\int_\Omega |\nabla w_k|^2\,\mathrm{d}x + \int_\Omega (\chi_k * f)(\nabla w_k)\,\mathrm{d}x\\ &\le k^{-1} + F[w_k] + k^{-1}L_f|\Omega| \le \inf_{\mathcal{D}} F + \varepsilon\,.\end{aligned}$$

Thus, we have shown (3.13) and the convergence $F[u_k+u_0-u_{0,k}] \to \inf_{\mathcal{D}} F$. The claim $F[u_k] \to \inf_{\mathcal{D}} F$ then follows from the $\mathrm{W}^{1,1}$ convergence $u_{0,k}\to u_0$ and the Lipschitz continuity of f.

Assuming (H2), we can estimate f_k from below via (3.6) (and the fact that $f_k(0) \ge f(0)$). Hence we infer from (3.13)

$$\sup_{k\in\mathbb{N}} \int_\Omega \big(\gamma_k|\nabla u_k|^2 + |\nabla u_k|\big)\,\mathrm{d}x < \infty\,.$$

This bound implies (3.14), when we also take into account the Poincaré inequalities

$$\int_\Omega \gamma_k|u_k|^2\,\mathrm{d}x \le C\bigg[\int_\Omega \gamma_k|\nabla u_k|^2\,\mathrm{d}x + \gamma_k\|w_k\|^2_{\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)}\bigg]\,,$$
$$\int_\Omega |u_k|\,\mathrm{d}x \le C\bigg[\int_\Omega |\nabla u_k|\,\mathrm{d}x + \|u_{0,k}\|_{\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)}\bigg]\,,$$

combined with the choice of γ_k and the convergence (3.11). By Rellich's theorem, we deduce from (3.14) that a subsequence of $(u_k)_{k\in\mathbb{N}}$ converges in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$. Finally, the fact that the limit is a generalized minimizer of F in $\mathcal{D}$ is a consequence of Theorem 2.3 (note that the coercivity of f can be inferred from Lemma 2.4 (ii), applied to mollifications $\chi_k * f$ if f is not C^1), since $(u_k+u_0-u_{0,k})_{k\in\mathbb{N}}$ is a minimizing sequence for F in $\mathcal{D}$, which has the same L^1-cluster points as $(u_k)_{k\in\mathbb{N}}$. □

L^∞ estimates for the minimizers of the regularized functionals F_k. *For the remainder of this paper we fix* the boundary values $u_0 \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$, approximations $(u_{0,k})_{k\in\mathbb{N}}$ of u_0 in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ with (3.11) and the associated Dirichlet classes $\mathcal{D}_k := u_{0,k} + \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)$, the sequence $(\gamma_k)_{k\in\mathbb{N}}$ from Lemma 3.3 (which in turn determines the functionals F_k), and finally the sequence of minimizers u_k of F_k in $\mathcal{D}_k$.

We record that u_k solves the Euler-Lagrange system of F_k in its weak formulation

$$\int_\Omega \nabla f_k(\nabla u_k)\cdot\nabla\varphi\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)\,. \tag{3.15}$$

In the sequel we will obtain uniform estimates for the u_k from this system. First we establish interior L^∞ bounds by a Moser-type iteration technique.

Lemma 3.4. *Suppose that f is convex with* (H1) *and* (H2)*. Then the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and for all balls $\mathrm{B}_{r_0}(x_0)\subset\Omega$ we have*

$$\sup_{\mathrm{B}_{r_0/2}(x_0)} |u_k| \le C r_0\bigg[\gamma_k r_0^{-2}\fint_{\mathrm{B}_{r_0}(x_0)} |u_k|^2\,\mathrm{d}x + r_0^{-1}\fint_{\mathrm{B}_{r_0}(x_0)} |u_k|\,\mathrm{d}x + 1\bigg]\,,$$

with a constant C, which depends only on n, L_f, R, μ, and γ, but not on $k\in\mathbb{N}$.

The proof of Lemma 3.4 follows mainly the reasoning in [4, Section 4]. In the present setting, however, we have imposed the conditions in (H2) only outside the ball B_R, and thus we repeat the essential arguments in our situation.

Proof. Step 1: A Caccioppoli-type inequality. For arbitrary $s\ge 2$ and $t\ge 1$ we test the Euler-Lagrange system (3.15) with the function

$$\varphi = |u_k|^{t-1}u_k\eta^s\,,$$

where $\eta\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,[0,1])$ is a cut-off function with $M_\eta := \sup_\Omega|\nabla\eta| > 0$. By [12, Theorem 1.1], we have $\varphi\in\mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^N)$, and thus φ is admissible as a test function in (3.15). In order to plug $\nabla\varphi$ into (3.15) we compute

$$\nabla\varphi = s|u_k|^{t-1}u_k\otimes\eta^{s-1}\nabla\eta + |u_k|^{t-1}\nabla u_k\eta^s + (t-1)|u_k|^{t-3}u_k\otimes(u_k^{\mathrm{T}}\nabla u_k)\eta^s =: \Phi_1+\Phi_2+\Phi_3\,,$$

and we observe, taking into account (3.5) and (3.4),

$$\nabla f_k(\nabla u_k)\cdot\Phi_2 \ge \gamma_k|u_k|^{t-1}|\nabla u_k|^2\eta^s + \nu|u_k|^{t-1}|\nabla u_k|\eta^s - C|u_k|^{t-1}\eta^s\,,$$
$$\nabla f_k(\nabla u_k)\cdot\Phi_3 \ge -C(t-1)|u_k|^{t-1}\eta^s\,,$$

where ν denotes the positive constant from (3.5) and where C depends only on L_f, R, μ, and γ. We now use, one by one, the previous estimates, the Euler-Lagrange system (3.15) for $\nabla\varphi = \Phi_1+\Phi_2+\Phi_3$, an explicit estimate for Φ_1, the bound (3.3), and finally Young's inequality to find

$$\begin{aligned}
&\gamma_k \int_\Omega |u_k|^{t-1}|\nabla u_k|^2\eta^s \,\mathrm{d}x + \nu \int_\Omega |u_k|^{t-1}|\nabla u_k|\eta^s \,\mathrm{d}x \\
&\quad\le \int_\Omega \nabla f_k(\nabla u_k)\cdot[\Phi_2+\Phi_3]\,\mathrm{d}x + Ct\int_\Omega |u_k|^{t-1}\eta^s\,\mathrm{d}x \\
&\quad= -\int_\Omega \nabla f_k(\nabla u_k)\cdot\Phi_1\,\mathrm{d}x + Ct\int_\Omega |u_k|^{t-1}\eta^s\,\mathrm{d}x \\
&\quad\le \gamma_k s\int_\Omega |u_k|^t|\nabla u_k|\eta^{s-1}|\nabla\eta|\,\mathrm{d}x + L_f s\int_\Omega |u_k|^t\eta^{s-1}|\nabla\eta|\,\mathrm{d}x + Ct\int_\Omega |u_k|^{t-1}\eta^s\,\mathrm{d}x \\
&\quad\le \frac{\gamma_k}{2}\int_\Omega |u_k|^{t-1}|\nabla u_k|^2\eta^s\,\mathrm{d}x + \gamma_k M_\eta^2 s^2\int_\Omega |u_k|^{t+1}\eta^{s-2}\,\mathrm{d}x \\
&\qquad + CM_\eta(s+t)\int_\Omega |u_k|^t\eta^{s-1}\,\mathrm{d}x + CM_\eta^{1-t}t\int_\Omega \eta^{s+t-1}\,\mathrm{d}x\,.
\end{aligned}\tag{3.16}$$

Clearly, the first integral on the right-hand side of (3.16) can be absorbed on the left-hand side. Next we relate the left-hand side of (3.16) to the gradients of the functions $|u_k|^{(t+1)/2}\eta^{s/2}$ and $|u_k|^t\eta^s$. Computing these gradients in the first step, and controlling the occurrences of ∇u_k via (3.16) in the second step, we arrive at

$$\begin{aligned}
&\gamma_k\int_\Omega |\nabla(|u_k|^{\frac{t+1}{2}}\eta^{\frac{s}{2}})|^2\,\mathrm{d}x + \nu\int_\Omega |\nabla(|u_k|^t\eta^s)|\,\mathrm{d}x \\
&\quad\le \gamma_k s^2\int_\Omega |u_k|^{t+1}\eta^{s-2}|\nabla\eta|^2\,\mathrm{d}x + \gamma_k(t+1)^2\int_\Omega |u_k|^{t-1}|\nabla u_k|^2\eta^s\,\mathrm{d}x \\
&\qquad + \nu s\int_\Omega |u_k|^t\eta^{s-1}|\nabla\eta|\,\mathrm{d}x + \nu t\int_\Omega |u_k|^{t-1}\eta^s|\nabla u_k|\,\mathrm{d}x \\
&\quad\le C\gamma_k M_\eta^2t^2s^2\int_\Omega |u_k|^{t+1}\eta^{s-2}\,\mathrm{d}x + CM_\eta t^2(s+t)\int_\Omega |u_k|^t\eta^{s-1}\,\mathrm{d}x + CM_\eta^{1-t}t^3\int_\Omega \eta^{s+t-1}\,\mathrm{d}x\,,
\end{aligned}$$

where C still depends only on L_f, R, μ, and γ, and it is in particular independent of the parameters γ_k, s, and t.

Step 2: An inequality of reverse Hölder type. For a ball $\mathrm{B}_{r_0}(x_0)\subset\Omega$ with $r_0\le 1$, we fix $\eta\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ such that $\mathbb{1}_{\mathrm{B}_{r_0/2}(x_0)}\le\eta\le\mathbb{1}_{\mathrm{B}_{r_0}(x_0)}$ and $M_\eta\le 4/r_0$ hold on Ω. Next we observe that Young's inequality with exponents $n/(n-1)$ and n gives

$$\gamma_k r_0^{-1-n}(|u_k|\eta^{2n})^{\frac{n}{n-1}t+1}\eta^{-4n} \le \gamma_k^{\frac{n}{n-1}}r_0^{-\frac{n^2}{n-1}}(|u_k|^{\frac{t+1}{2}}\eta^{nt-n+1})^{\frac{2n}{n-1}} + r_0^{-n}(|u_k|^t\eta^{2nt-2n+2})^{\frac{n}{n-1}}\,.$$

When we employ first the last estimate, then Sobolev's inequality, and finally the resulting estimate of Step 1 with the choice $s=2nt-2n+2\in[2,2nt]$ and $\eta\le1$, we obtain

$$\begin{aligned}
&\left[\gamma_k r_0^{-1-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|\eta^{2n})^{\frac{n}{n-1}t+1}\eta^{-4n}\,\mathrm{d}x + r_0^{-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|\eta^{2n})^{\frac{n}{n-1}t}\eta^{-2n}\,\mathrm{d}x + r_0^{\frac{n}{n-1}t}\right]^{\frac{n-1}{n}} \\
&\le \left[\gamma_k^{\frac{n}{n-1}}r_0^{-\frac{n^2}{n-1}}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|^{\frac{t+1}{2}}\eta^{nt-n+1})^{\frac{2n}{n-1}}\,\mathrm{d}x + 2r_0^{-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|^t\eta^{2nt-2n+2})^{\frac{n}{n-1}}\,\mathrm{d}x + r_0^{\frac{n}{n-1}t}\right]^{\frac{n-1}{n}} \\
&\le C\left[\gamma_k r_0^{1-n}\int_{\mathrm{B}_{r_0}(x_0)}|\nabla(|u_k|^{\frac{t+1}{2}}\eta^{nt-n+1})|^2\,\mathrm{d}x + r_0^{1-n}\int_{\mathrm{B}_{r_0}(x_0)}|\nabla(|u_k|^t\eta^{2nt-2n+2})|\,\mathrm{d}x + r_0^t\right] \\
&\le Ct^4\left[\gamma_k r_0^{-1-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|\eta^{2n})^{t+1}\eta^{-4n}\,\mathrm{d}x + r_0^{-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|\eta^{2n})^t\eta^{-2n}\,\mathrm{d}x + r_0^t\right],
\end{aligned}$$

where C now depends only on n, L_f, R, μ, and γ.

Step 3: Moser iteration and conclusion. In order to iterate the preceding inequality, we introduce the abbreviations

$$t_j := \Big(\frac{n}{n-1}\Big)^j\,, \quad A_j := \big(Ct_j^4\big)^{\frac{1}{t_j}}\,,$$

$$\Psi(j) := \bigg[\gamma_k r_0^{-1-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|\eta^{2n})^{t_j+1}\eta^{-4n}\,\mathrm{d}x + r_0^{-n}\int_{\mathrm{B}_{r_0}(x_0)}(|u_k|\eta^{2n})^{t_j}\eta^{-2n}\,\mathrm{d}x + r_0^{t_j}\bigg]^{\frac{1}{t_j}}\,,$$

with the constant C from the inequality in Step 2. As in [4, Proof of Theorem 1.11] we thus have $\Psi(j{+}1) \le A_j\Psi(j)$, and the iteration of this inequality yields

$$\Psi(m+1) \le \bigg(\prod_{j=0}^m A_j\bigg)\Psi(0)\,.$$

Observing $\prod_{j=0}^\infty A_j < \infty$ we can pass to the limit $m \to \infty$. Recalling the choice of η in Step 2, we find the claimed estimate

$$\sup_{\mathrm{B}_{r_0/2}(x_0)}|u_k| \le C\bigg[\gamma_k r_0^{-1-n}\int_{\mathrm{B}_{r_0}(x_0)}|u_k|^2\,\mathrm{d}x + r_0^{-n}\int_{\mathrm{B}_{r_0}(x_0)}|u_k|\,\mathrm{d}x + r_0\bigg]\,.$$

Using the bound (3.14) from Lemma 3.3, we infer that the u_k are uniformly bounded on $\mathrm{B}_{r_0/2}(x_0)$, and since x_0 is arbitrary in Ω, this means that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. $\square$

Remark 3.5. *The L^∞ estimate of Lemma 3.4 remains valid in the more general form*

$$\sup_{\mathrm{B}_{r_0/2}(x_0)}|u_k-\zeta_k| \le Cr_0\bigg[\gamma_k r_0^{-2}\fint_{\mathrm{B}_{r_0}(x_0)}|u_k-\zeta_k|^2\,\mathrm{d}x + r_0^{-1}\fint_{\mathrm{B}_{r_0}(x_0)}|u_k-\zeta_k|\,\mathrm{d}x + 1\bigg]$$

with an arbitrary sequence $(\zeta_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^N$. This results from the observation that also $u_k-\zeta_k$ minimizes F_k, in the Dirichlet class $u_{0,k}-\zeta_k+\mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$. The particular choice $\zeta_k = (u_k)_{\mathrm{B}_{r_0}(x_0)}$ allows for the application of Poincaré's inequality. Using also the triangle inequality, we conclude

$$\sup_{\mathrm{B}_{r_0/2}(x_0)}|u_k-(u_k)_{\mathrm{B}_{r_0/2}(x_0)}| \le Cr_0\bigg[\gamma_k\fint_{\mathrm{B}_{r_0}(x_0)}|\nabla u_k|^2\,\mathrm{d}x + \fint_{\mathrm{B}_{r_0}(x_0)}|\nabla u_k|\,\mathrm{d}x + 1\bigg]\,, \tag{3.17}$$

where C depends only on n, L_f, R, μ, and γ, but not on $k \in \mathbb{N}$.

4 Local Lipschitz regularity for one generalized minimizer

In this section, we prove Theorem 1.2, i.e. we show the existence of one locally Lipschitz continuous generalized minimizer of F in $\mathcal{D}$.

For this purpose, we derive various uniform-in-k estimates for the functions u_k, which were constructed in Section 3 as minimizers of the regularized functionals F_k. First, in Subsection 4.1, we provide certain (weighted) Caccioppoli-type inequalities involving second derivatives. Eventually, in Subsection 4.2, we obtain uniform $\mathrm{W}^{1,p}$ estimates, and finally, in Subsection 4.3, we implement a variant of De Giorgi's level set technique which leads to uniform $\mathrm{W}^{1,\infty}$ estimates. As a straightforward consequence of the $\mathrm{W}^{1,\infty}$ estimates, the limit of $(u_k)_{k\in\mathbb{N}}$ is a locally Lipschitz continuous (generalized) minimizer of F.

4.1 Caccioppoli-type inequalities

Here, we are concerned with the aforementioned Caccioppoli-type inequalities. We will state these inequalities for minimizers of a functional H, defined by

$$H[w] \coloneqq \int_\Omega h(\nabla w)\,\mathrm{d}x \in (-\infty,\infty] \qquad \text{for } w \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)\,, \tag{4.1}$$

where the C^2-integrand $h\colon \mathbb{R}^{Nn} \to \mathbb{R}$ satisfies

$$\lambda|\xi|^2 \le \nabla^2 h(z)(\xi,\xi) \le \Lambda|\xi|^2 \qquad \text{for all } z,\xi \in \mathbb{R}^{Nn} \tag{4.2}$$

with some constants $\Lambda \ge \lambda > 0$. For the moment, it is convenient to work with a general function $h \in \mathrm{C}^2(\mathbb{R}^{Nn})$ satisfying (4.2), but, for later purposes, the only relevant choice is $h \coloneqq f_k$ with f_k from (3.1). We remark, once and for all, that this choice is admissible, as (3.2) implies (4.2) for $h \coloneqq f_k$, with certain k-dependent constants λ and Λ.

We now start by establishing an auxiliary lemma which allows to estimate certain expressions involving second derivatives.

Lemma 4.1. *Suppose that $h\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is a convex C^2 integrand with $h(z) = g(|z|)$ for all $z \in \mathbb{R}^{Nn}$, and consider a function $w \in \mathrm{W}^{2,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $|\nabla w|^2 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$. Then almost everywhere in Ω there holds*

$$\sum_{j=1}^n \nabla^2 h(\nabla w)(\partial_j \nabla w, \partial_j w \otimes \nabla(|\nabla w|^2)) \ge 0\,.$$

Proof. We first observe

$$\nabla^2 h(z) = \frac{g''(|z|)}{|z|^2} z \otimes z + \frac{g'(|z|)}{|z|^3}\big(|z|^2 \mathrm{I}_{Nn} - z \otimes z\big)$$

for $z \in \mathbb{R}^{Nn} \setminus \{0\}$, where g' and g'' are non-negative by the convexity and symmetry of h. Thus (also taking into account that $\nabla(|\nabla w|^2)$ vanishes a.e. on $\{\nabla w = 0\}$), it suffices to verify, for the bilinear forms $\nabla w \otimes \nabla w$ and I_{Nn}, the two inequalities

$$|\nabla w|^2 \sum_{j=1}^n \mathrm{I}_{Nn}(\partial_j \nabla w, \partial_j w \otimes \nabla(|\nabla w|^2)) \ge \sum_{j=1}^n (\nabla w \otimes \nabla w)(\partial_j \nabla w, \partial_j w \otimes \nabla(|\nabla w|^2)) \ge 0\,. \tag{4.3}$$

To this end, we calculate in coordinates

$$\begin{aligned}
2\sum_{j=1}^n (\nabla w \otimes \nabla w)(\partial_j \nabla w, \partial_j w \otimes \nabla(|\nabla w|^2)) &= 2\sum_{i,j,k=1}^n \sum_{\alpha,\beta=1}^N \partial_i w^\beta \partial_k w^\alpha \partial_j \partial_i w^\beta \partial_j w^\alpha \partial_k(|\nabla w|^2) \\
&= \sum_{j,k=1}^n \sum_{\alpha=1}^N \partial_j(|\nabla w|^2) \partial_k w^\alpha \partial_j w^\alpha \partial_k(|\nabla w|^2) \\
&= \sum_{\alpha=1}^N \big[\nabla w^\alpha \cdot \nabla(|\nabla w|^2)\big]^2
\end{aligned}$$

and

$$\begin{aligned}
2\sum_{j=1}^n \mathrm{I}_{Nn}(\partial_j \nabla w, \partial_j w \otimes \nabla(|\nabla w|^2)) &= 2\sum_{i,j=1}^n \sum_{\beta=1}^N \partial_j \partial_i w^\beta \partial_j w^\beta \partial_i(|\nabla w|^2) \\
&= \sum_{i=1}^n \partial_i(|\nabla w|^2) \partial_i(|\nabla w|^2) = |\nabla(|\nabla w|^2)|^2\,.
\end{aligned}$$

In view of these calculations, (4.3) becomes obvious, and the proof of the lemma is complete. □

Next we state the announced Caccioppoli-type inequalities involving weights. These inequalities will be crucial in order to obtain the uniform gradient estimates of the following subsections.

Lemma 4.2 (cf. [7], Lemma 3.2; [4], Lemma 5.1)**.** *Suppose that* $h\colon \mathbb{R}^{Nn} \to \mathbb{R}$ *is a strictly convex* C^2 *function which satisfies* (4.2)*. If* $v \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ *minimizes the functional* H *from* (4.1) *in a Dirichlet class in* $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$*, then we have* $v \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$*, and, for all* $j \in \{1, 2, \ldots, n\}$ *and* $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$*, there holds*

$$\int_\Omega \nabla^2 h(\nabla v)(\partial_j \nabla v, \partial_j \nabla v)\eta^2 \,\mathrm{d}x \leq 4 \int_\Omega \nabla^2 h(\nabla v)(\partial_j v \otimes \nabla\eta, \partial_j v \otimes \nabla\eta) \,\mathrm{d}x. \tag{4.4}$$

If we additionally have $h(z) = g(|z|)$ *for all* $z \in \mathbb{R}^{Nn} \setminus \mathrm{B}_R$*, some function* g*, and a radius* R*, and if* $T \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ *is non-negative and non-decreasing on* $\mathbb{R}$ *and constant on* $[0, R^2]$*, then for every* $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ *there holds*

$$\int_\Omega \sum_{j=1}^n \nabla^2 h(\nabla v)(\partial_j \nabla v, \partial_j \nabla v) T(|\nabla v|^2)\eta^2 \,\mathrm{d}x$$
$$\leq 4 \int_\Omega \sum_{j=1}^n \nabla^2 h(\nabla v)(\partial_j v \otimes \nabla\eta, \partial_j v \otimes \nabla\eta) T(|\nabla v|^2) \,\mathrm{d}x\,. \tag{4.5}$$

Sketch of proof. In principle, the following reasoning is well known, but, for the sake of clarity, we briefly describe the essential arguments. In order to prove the existence of second order derivatives for v, one applies standard difference quotient methods to the the Euler-Lagrange system $\operatorname{div}[\nabla h(\nabla v)] = 0$. In this way, relying on (4.2), one finds $v \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Testing the Euler-Lagrange system with the derivatives $\partial_j \varphi$ of φ and integrating by parts, we obtain

$$\int_\Omega \nabla^2 h(\nabla v)(\partial_j \nabla v, \nabla\varphi) \,\mathrm{d}x = 0\,, \tag{4.6}$$

first for $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$ and then also for $\varphi \in \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$. The choice $\varphi = \eta^2 \partial_j v$ combined with an application of Young's inequality gives (4.4).

The estimate (4.5) is derived similarly. Therefore, we only comment on the additional arguments which are needed to allow the quantity $T(|\nabla v|^2)$ in the test function, and moreover, we explain how the structure condition on h enters. The differentiated Euler-Lagrange system (4.6) is now tested with $\varphi = \eta^2 \partial_j v T_K(|\nabla v|^2)$, where $T_K(\tau) \coloneqq \min\{T(\tau), K\}$ denotes the truncation of T at level $K > 0$ (by the chain rule for Sobolev functions, φ is in $\mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$ and hence admissible in (4.6)). In order to verify that the term arising from the differentiation of $T_K(|\nabla v|^2)$ is non-negative, we observe that, by the monotonicity assumption on T and Lemma 4.1, we have almost everywhere

$$2T_K'(|\nabla v|^2) \sum_{j=1}^n \nabla^2 h(\nabla v)(\partial_j \nabla v, \partial_j v \otimes \nabla(|\nabla v|^2))\eta^2 \geq 0\,.$$

Arguing with a Young-type inequality as in the derivation of (4.4), we arrive at the estimate (4.5) with T replaced by the truncated version T_K. The final form of (4.5) is then obtained by the passage to the limit $K \to \infty$, via the monotone convergence theorem. □

4.2 $\mathbf{W^{1,p}}$ estimates

We now return to the study of the minimization problem for the functional F in the Dirichlet class $\mathcal{D} = u_0 + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N)$ with a given function $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. For the remainder of this section, we will permanently assume that the integrand f is convex and satisfies (H1) and (H2), and we fix the regularized functionals F_k with integrands $f_k(z) \coloneqq \chi_k * f(z) + \gamma_k|z|^2/2$ defined in (3.1), the approximating Dirichlet classes $\mathcal{D}_k$, the sequence $(\gamma_k)_{k\in\mathbb{N}}$ according to Lemma 3.3, and the minimizers u_k of F_k in $\mathcal{D}_k$.

The Caccioppoli-type inequality (4.5) can be employed similarly as in [7, Theorem 5.1] to obtain the following uniform L^p estimates for the sequence $(\nabla u_k)_{k\in\mathbb{N}}$. These estimates depend crucially on our assumption that μ is strictly less than 3.

Lemma 4.3. *For every $p \in [1,\infty)$ the sequence $(\nabla u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^{Nn})$. Moreover, for every ball $\mathrm{B}_{4r_0}(x_0) \Subset \Omega$, we have*

$$\fint_{\mathrm{B}_{r_0}(x_0)} |\nabla u_k|^p \,\mathrm{d}x \le C\left[\gamma_k \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k|^2 \,\mathrm{d}x + \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k| \,\mathrm{d}x + 1\right]^{\frac{2}{3-\mu}(p-1)+1}, \tag{4.7}$$

with a constant C, which depends only on n, L_f, R, μ, γ, Γ, and p, but not on $k \in \mathbb{N}$.

Proof. Since the claim is obvious in the case $p = 1$ (compare Lemma 3.3), in what follows we assume $p > 1$. To justify the subsequent computations, we first record that we have $u_k \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ for every $k \in \mathbb{N}$. This can be inferred, for instance, from [12, Theorem 1.1] and Lemma 4.2 (applied with $h = f_k$). Now we set $\zeta_k := (u_k)_{\mathrm{B}_{2r_0}(x_0)}$ and introduce the quantity

$$M_k := 1 + r_0^{-1}\|u_k - \zeta_k\|_{\mathrm{L}^\infty(\mathrm{B}_{2r_0}(x_0),\mathbb{R}^N)}\,,$$

for which (3.17) yields the estimate

$$M_k \le C\left[\gamma_k \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k|^2 \,\mathrm{d}x + \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k| \,\mathrm{d}x + 1\right]. \tag{4.8}$$

We will employ the test function

$$\varphi := (|\nabla u_k| - \ell)_+^p (u_k - \zeta_k)\eta^s\,,$$

where we have fixed the level $\ell := R{+}2$, a cut-off function $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\mathbb{1}_{\mathrm{B}_{r_0}(x_0)} \le \eta \le \mathbb{1}_{\mathrm{B}_{2r_0}(x_0)}$ and $|\nabla\eta| \le 2/r_0$ on Ω, and $s := 2p/(3-\mu) > 1$ (the last choice is made for later convenience). In order to test the Euler-Lagrange equation (3.15) with φ, we compute

$$\begin{aligned}\nabla\varphi = s(|\nabla u_k| - \ell)_+^p (u_k - \zeta_k) \otimes \eta^{s-1}\nabla\eta + (|\nabla u_k| - \ell)_+^p \nabla u_k \eta^s \\ + p(|\nabla u_k| - \ell)_+^{p-1}(u_k - \zeta_k) \otimes \nabla|\nabla u_k|\eta^s =: \Phi_1 + \Phi_2 + \Phi_3\,.\end{aligned}$$

Then, via (3.5), (3.15), and (3.3), we obtain

$$\begin{aligned}
&\int_\Omega \big(\gamma_k|\nabla u_k| + \nu\big)|\nabla u_k|(|\nabla u_k| - \ell)_+^p \eta^s \,\mathrm{d}x \\
&\quad\le \int_\Omega \nabla f_k(\nabla u_k)\cdot\Phi_2 \,\mathrm{d}x + C\int_\Omega (|\nabla u_k| - \ell)_+^p \eta^s \,\mathrm{d}x \\
&\quad= -\int_\Omega \nabla f_k(\nabla u_k)\cdot(\Phi_1 + \Phi_3) \,\mathrm{d}x + C\int_\Omega (|\nabla u_k| - \ell)_+^p \eta^s \,\mathrm{d}x \\
&\quad\le s\int_\Omega (\gamma_k|\nabla u_k| + L_f)(|\nabla u_k| - \ell)_+^p |u_k - \zeta_k|\eta^{s-1}|\nabla\eta| \,\mathrm{d}x \\
&\qquad + p\int_\Omega (\gamma_k|\nabla u_k| + L_f)(|\nabla u_k| - \ell)_+^{p-1}|u_k - \zeta_k||\nabla^2 u_k|\eta^s \,\mathrm{d}x + C\int_\Omega (|\nabla u_k| - \ell)_+^p \eta^s \,\mathrm{d}x \\
&\quad=: sI_1 + pI_2 + CI_3
\end{aligned}$$

with the obvious labeling. Next we exploit that we have $1 + |u_k - \zeta_k||\nabla\eta| \le 2M_k$ and $\eta \le 1$ and that $|\nabla u_k| \ge \ell \ge 1$ holds wherever $(|\nabla u_k| - \ell)_+\eta^s$ does not vanish. In this way, we can estimate

$$I_1 + I_3 \le 2(L_f + 1)M_k\left[\int_\Omega \big(\gamma_k|\nabla u_k| + 1\big)|\nabla u_k|^{p+\frac{\mu-1}{2}}\eta^{s-1} \,\mathrm{d}x\right].$$

Similarly, using also $|u_k - \zeta_k| \le r_0 M_k$ on $\operatorname{spt}\eta$, we find

$$I_2 \le (L_f+1)M_k \int_\Omega \Big[r_0\gamma_k |\nabla u_k|^{1+\frac{\mu-1}{2}} |\nabla^2 u_k|\eta^s + r_0|\nabla^2 u_k|\eta^s \Big] (|\nabla u_k|-\ell)_+^{p-1}\,\mathrm{d}x\,.$$

The terms on the right-hand side of the last formula can further be controlled via the Young inequalities

$$r_0\gamma_k|\nabla u_k|^{1+\frac{\mu-1}{2}}|\nabla^2 u_k|\eta^s \le r_0^2\gamma_k|\nabla^2 u_k|^2|\nabla u_k|^{\frac{\mu-1}{2}}\eta^{s+1} + \gamma_k|\nabla u_k|^{2+\frac{\mu-1}{2}}\eta^{s-1}\,,$$
$$r_0|\nabla^2 u_k|\eta^s \le r_0^2\nu|\nabla^2 u_k|^2|\nabla u_k|^{-\mu+\frac{\mu-1}{2}}\eta^{s+1} + \nu^{-1}|\nabla u_k|^{1+\frac{\mu-1}{2}}\eta^{s-1}\,.$$

Collecting the estimates for I_1, I_2, I_3, we arrive at

$$\begin{aligned}
&\int_\Omega \big(\gamma_k|\nabla u_k|+1\big)|\nabla u_k|(|\nabla u_k|-\ell)_+^p\eta^s\,\mathrm{d}x \\
&\quad\le CM_k\bigg[r_0^2\int_\Omega \big(\gamma_k+\nu|\nabla u_k|^{-\mu}\big)|\nabla^2 u_k|^2(|\nabla u_k|-\ell)_+^{p-1}|\nabla u_k|^{\frac{\mu-1}{2}}\eta^{s+1}\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(\gamma_k|\nabla u_k|+1\big)|\nabla u_k|^{p+\frac{\mu-1}{2}}\eta^{s-1}\,\mathrm{d}x\bigg]\,, \qquad (4.9)
\end{aligned}$$

where C depends only on L_f, R, μ, γ, and p. Next we deal with the crucial term in (4.9), namely the one involving $\nabla^2 u_k$. We emphasize that, thanks to the specific choice or our test function φ, this term contains the factor $(|\nabla u_k|-\ell)_+^{p-1}$ and vanishes on $\{|\nabla u_k|\le\ell\} = \big\{\nabla u_k \in \overline{\mathrm{B}_{R+2}}\big\}$. Therefore, the uncontrolled behavior of f_k on $\overline{\mathrm{B}_{R+2/k}}$ does not interfere with the subsequent estimation of $\nabla^2 u_k$, which is based on the Caccioppoli-type inequality (4.5) from Lemma 4.2. This inequality is now employed, with $h \coloneqq f_k$ and the non-decreasing function $T\colon [0,\infty)\to[0,\infty)$ which is defined by

$$T(\tau) \coloneqq \big(\sqrt{\tau}-\ell\big)_+^{p-1}\tau^{\frac{\mu-1}{4}}$$

and which vanishes on $[0,\ell^2]$. Using also (3.2), we obtain

$$\begin{aligned}
&\int_\Omega \big(\gamma_k+\nu|\nabla u_k|^{-\mu}\big)|\nabla^2 u_k|^2(|\nabla u_k|-\ell)_+^{p-1}|\nabla u_k|^{\frac{\mu-1}{2}}\eta^{s+1}\,\mathrm{d}x \\
&\quad\le \int_\Omega \sum_{j=1}^n \nabla^2 f_k(\nabla u_k)(\partial_j\nabla u_k,\partial_j\nabla u_k)T(|\nabla u_k|^2)\eta^{s+1}\,\mathrm{d}x \\
&\quad\le (s+1)^2\int_\Omega \sum_{j=1}^n \nabla^2 f_k(\nabla u_k)(\partial_j u_k\otimes\nabla\eta,\partial_j u_k\otimes\nabla\eta)T(|\nabla u_k|^2)\eta^{s-1}\,\mathrm{d}x \\
&\quad\le (s+1)^2\int_\Omega \big(\gamma_k|\nabla u_k|+2\Gamma\big)|\nabla u_k|T(|\nabla u_k|^2)\eta^{s-1}|\nabla\eta|^2\,\mathrm{d}x \\
&\quad\le Cr_0^{-2}\int_\Omega \big(\gamma_k|\nabla u_k|+1\big)|\nabla u_k|(|\nabla u_k|-\ell)_+^{p-1}|\nabla u_k|^{\frac{\mu-1}{2}}\eta^{s-1}\,\mathrm{d}x\,.
\end{aligned}$$

Now we plug this estimate into (4.9) and recall $\operatorname{spt}\eta\subset\overline{\mathrm{B}_{2r_0}(x_0)}$. This yields

$$\int_{\mathrm{B}_{2r_0}(x_0)} \big(\gamma_k|\nabla u_k|+1\big)|\nabla u_k|(|\nabla u_k|-\ell)_+^p\eta^s\,\mathrm{d}x \le CM_k\int_{\mathrm{B}_{2r_0}(x_0)} \big(\gamma_k|\nabla u_k|+1\big)|\nabla u_k|^{p+\frac{\mu-1}{2}}\eta^{s-1}\,\mathrm{d}x\,.$$

Then, via the inequality $|\nabla u_k|^p \le 2^p[(|\nabla u_k|-\ell)_+^p+\ell^p]$ and the choice $\ell = R{+}2$, we arrive at

$$\begin{aligned}
&\int_{\mathrm{B}_{2r_0}(x_0)} (\gamma_k|\nabla u_k|+1)|\nabla u_k|^{p+1}\eta^s\,\mathrm{d}x \\
&\quad\le C\bigg[M_k\int_{\mathrm{B}_{2r_0}(x_0)} (\gamma_k|\nabla u_k|+1)|\nabla u_k|^{p+\frac{\mu-1}{2}}\eta^{s-1}\,\mathrm{d}x + \int_{\mathrm{B}_{2r_0}(x_0)} (\gamma_k|\nabla u_k|+1)|\nabla u_k|\eta^s\,\mathrm{d}x\bigg]\,,
\end{aligned}$$

with C depending only on L_f, R, μ, γ, Γ, and p. We next use Young's inequality, with exponents $2p/(2p{+}\mu{-}3)$, $2p/(3{-}\mu)$ and an arbitrary $\varepsilon > 0$, to find

$$M_k|\nabla u_k|^{p+\frac{\mu-1}{2}}\eta^{s-1} \le \varepsilon|\nabla u_k|^{p+1}\eta^{(s-1)\frac{2p}{2p+\mu-3}} + C(\varepsilon,\mu,p)M_k^{\frac{2p}{3-\mu}}|\nabla u_k|\,.$$

As we have taken $s = 2p/(3{-}\mu)$, we have the equality $(s{-}1)2p/(2p{+}\mu{-}3) = s$. Therefore, choosing ε suitably small, employing an absorption argument, and recalling $\eta \le 1 \le M_k$, we deduce

$$\int_{\mathrm{B}_{2r_0}(x_0)}(\gamma_k|\nabla u_k|+1)|\nabla u_k|^{p+1}\eta^s\,\mathrm{d}x \le CM_k^{\frac{2p}{3-\mu}}\int_{\mathrm{B}_{2r_0}(x_0)}(\gamma_k|\nabla u_k|+1)|\nabla u_k|\,\mathrm{d}x\,.$$

Now we can drop the γ_k-term on the left-hand side. When we divide by r_0^n, exploit $\eta \ge \mathbb{1}_{\mathrm{B}_{r_0}(x_0)}$, and employ (4.8), we thus end up with

$$\begin{aligned}\fint_{\mathrm{B}_{r_0}(x_0)}|\nabla u_k|^{p+1}\,\mathrm{d}x &\le CM_k^{\frac{2p}{3-\mu}}\left[\gamma_k\fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|^2\,\mathrm{d}x + \fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|\,\mathrm{d}x\right]\\ &\le C\left[\gamma_k\fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|^2\,\mathrm{d}x + \fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|\,\mathrm{d}x + 1\right]^{\frac{2}{3-\mu}p+1},\end{aligned} \tag{4.10}$$

where C depends only on n, L_f, R, μ, γ, Γ, and p. A final application of Hölder's inequality yields the claimed estimate

$$\begin{aligned}\fint_{\mathrm{B}_{r_0}(x_0)}|\nabla u_k|^p\,\mathrm{d}x &\le C\left(\fint_{\mathrm{B}_{r_0}(x_0)}|\nabla u_k|^{p+1}\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\fint_{\mathrm{B}_{r_0}(x_0)}|\nabla u_k|\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le C\left[\gamma_k\fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|^2\,\mathrm{d}x + \fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|\,\mathrm{d}x + 1\right]^{\frac{2}{3-\mu}(p-1)+1}\end{aligned}$$

with the asserted dependencies of the constant. In particular, taking into account (3.14), we infer that the sequence $(\nabla u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^{Nn})$. □

4.3 A De Giorgi type lemma and the proof of Theorem 1.2

Roughly speaking, De Giorgi's technique allows to derive L^∞ estimates from Caccioppoli inequalities on superlevel sets; compare [22, Chapter 7]. The following lemma — which resembles some arguments in [8, Section 3.3.3] — yields the same conclusion in the presence of certain additional weight functions.

Lemma 4.4. *Consider non-negative exponents θ, σ with $\theta{+}\sigma \ge 4$, a constant $C_{\mathrm{Cacc}} \ge 1$, and some function $w \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $p > (\theta{+}\sigma)n/2$. If w satisfies, for all $\ell \ge \ell_0 \ge 1$ and every $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, the Caccioppoli-type inequality with weights*

$$\int_\Omega|\nabla w|^2|w|^{-\theta}(|w|-\ell)_+^2\eta^2\,\mathrm{d}x \le C_{\mathrm{Cacc}}\int_\Omega|w|^{\sigma-2}(|w|-\ell)_+^2|\nabla\eta|^2\,\mathrm{d}x\,, \tag{4.11}$$

then we have $w \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and for all balls $\mathrm{B}_{r_0}(x_0) \Subset \Omega$ there holds

$$\sup_{\mathrm{B}_{r_0/2}(x_0)}|w| \le \ell_0 + C\left(\fint_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\right)^{\frac{\theta+\sigma}{4p}}, \tag{4.12}$$

where C depends only on n, θ, σ, p, and C_{Cacc}.

Remark 4.5. *In the case $\theta{+}\sigma = 4$, the estimates* (4.11) *and* (4.12) *are non-degenerate in the sense that the leading terms on both sides exhibit the same homogeneity in w. However, under our assumptions we can verify the inequality* (4.11) *only for $\theta = \mu > 1$ and $\sigma = 3$, so that we have $\theta{+}\sigma > 4$. This reflects, to some extent, the fact that we deal with non-uniformly elliptic situations.*

Proof of Lemma 4.4. We fix a ball $\mathrm{B}_{r_0}(x_0) \Subset \Omega$, we set

$$A(\ell,r) \coloneqq \big\{x \in \mathrm{B}_r(x_0)\colon |w(x)| > \ell\big\}$$

for $\ell \geq \ell_0$ and $0 < r \leq r_0$, and we define a function φ on $[\ell_0,\infty) \times (0,r_0]$ by

$$\varphi(\ell,r) \coloneqq \int_{A(\ell,r)} |w|^{\sigma-2}(|w|-\ell)^2\,\mathrm{d}x\,.$$

We notice that φ is finite-valued, since we have assumed $w \in \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $p > (\theta{+}\sigma)n/2 \geq \sigma$. Moreover, φ is non-increasing in ℓ for fixed r, and it is non-decreasing in r for fixed ℓ. Next, in order to apply Lemma 2.5, we verify that φ fulfills (2.2). For this purpose, we consider arbitrary levels $m > \ell \geq \ell_0$ and radii $0 < \rho < r \leq r_0$, and we choose a cut-off function $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ which satisfies $\mathbb{1}_{\mathrm{B}_\rho(x_0)} \leq \eta \leq \mathbb{1}_{\mathrm{B}_r(x_0)}$ and $|\nabla\eta| \leq 2/(r{-}\rho)$. Then we infer from Sobolev's inequality

$$\begin{aligned}
\varphi(m,\rho) &\leq \int_{\mathrm{B}_r(x_0)} |w|^{\sigma-2}(|w|-m)_+^2\eta^2\,\mathrm{d}x\\
&\leq C\bigg[\bigg(\int_{A(m,r)} |w|^{\sigma\frac{n}{n+2}}|\nabla\eta|^{\frac{2n}{n+2}}\,\mathrm{d}x\bigg)^{\frac{n+2}{n}} + \bigg(\int_{A(m,r)} |\nabla w|^{\frac{2n}{n+2}}|w|^{(\sigma-2)\frac{n}{n+2}}\eta^{\frac{2n}{n+2}}\,\mathrm{d}x\bigg)^{\frac{n+2}{n}}\bigg]\\
&\eqqcolon C\big[I + II\big]
\end{aligned}$$

with the obvious labeling and with a constant C which depends only on n and σ. Hölder's inequality (applied with exponents $(n{+}2)/2$, $(n{+}2)/n$) yields for the first term

$$\begin{aligned}
I &\leq \bigg(\int_{A(m,r)} |w|^n(|w|-\ell)^n\,\mathrm{d}x\bigg)^{\frac{2}{n}}\int_{A(m,r)} |w|^{\sigma-2}(|w|-\ell)^{-2}|\nabla\eta|^2\,\mathrm{d}x\\
&\leq (m-\ell)^{-4}\bigg(\int_{A(\ell,r)} |w|^n(|w|-\ell)^n\,\mathrm{d}x\bigg)^{\frac{2}{n}}\int_{A(\ell,r)} |w|^{\sigma-2}(|w|-\ell)^2|\nabla\eta|^2\,\mathrm{d}x\,.
\end{aligned}$$

For the second term we obtain, via a similar application of Hölder's inequality and the Caccioppoli-type inequality (4.11),

$$\begin{aligned}
II &\leq \bigg(\int_{A(m,r)} |w|^{(\theta+\sigma-2)\frac{n}{2}}(|w|-\ell)^n\,\mathrm{d}x\bigg)^{\frac{2}{n}}\int_{A(m,r)} |\nabla w|^2|w|^{-\theta}(|w|-\ell)^{-2}|\eta|^2\,\mathrm{d}x\\
&\leq (m-\ell)^{-4}\bigg(\int_{A(\ell,r)} |w|^{(\theta+\sigma-2)\frac{n}{2}}(|w|-\ell)^n\,\mathrm{d}x\bigg)^{\frac{2}{n}}\int_{A(m,r)} |\nabla w|^2|w|^{-\theta}(|w|-\ell)^2|\eta|^2\,\mathrm{d}x\\
&\leq C_{\mathrm{Cacc}}(m-\ell)^{-4}\bigg(\int_{A(\ell,r)} |w|^{(\theta+\sigma-2)\frac{n}{2}}(|w|-\ell)^n\,\mathrm{d}x\bigg)^{\frac{2}{n}}\int_{A(\ell,r)} |w|^{\sigma-2}(|w|-\ell)^2|\nabla\eta|^2\,\mathrm{d}x\,.
\end{aligned}$$

In view of $\theta{+}\sigma \geq 4$ and $C_{\mathrm{Cacc}} \geq 1$, the last bound for II is also a bound for I. To estimate further, we use in the first step the inequalities $|\nabla\eta| \leq 2/(r{-}\rho)$ and

$$(|w|-\ell)^n \leq |w|^{(\sigma-2)\frac{2p-n(\theta+\sigma)}{2(p-\sigma)}+p\frac{n(\theta+\sigma)-2\sigma}{2(p-\sigma)}-(\theta+\sigma-2)\frac{n}{2}}(|w|-\ell)^{\frac{2p-n(\theta+\sigma)}{p-\sigma}} \qquad \text{on } A(\ell,r)$$

(the two exponents on the right-hand side are non-negative and sum up to n), and in the second step we apply Hölder's inequality (with exponents $2(p{-}\sigma)/[2p{-}n(\theta{+}\sigma)]$ and $2(p{-}\sigma)/[n(\theta{+}\sigma){-}2\sigma]$). In this way we infer

$$\begin{aligned}\varphi(m,\rho) &\le C(m-\ell)^{-4}(r-\rho)^{-2}\biggl(\int_{A(\ell,r)}\bigl[|w|^{\sigma-2}(|w|-\ell)^2\bigr]^{\frac{2p-n(\theta+\sigma)}{2(p-\sigma)}}|w|^{p\frac{n(\theta+\sigma)-2\sigma}{2(p-\sigma)}}\,\mathrm{d}x\biggr)^{\frac{2}{n}}\\
&\qquad\times\int_{A(\ell,r)}|w|^{\sigma-2}(|w|-\ell)^2\,\mathrm{d}x\\
&\le C(m-\ell)^{-4}(r-\rho)^{-2}\biggl(\int_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{n(\theta+\sigma)-2\sigma}{n(p-\sigma)}}\biggl(\int_{A(\ell,r)}|w|^{\sigma-2}(|w|-\ell)^2\,\mathrm{d}x\biggr)^{1+\frac{2p-n(\theta+\sigma)}{n(p-\sigma)}}\\
&= C\biggl(\int_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{n(\theta+\sigma)-2\sigma}{n(p-\sigma)}}(m-\ell)^{-4}(r-\rho)^{-2}[\varphi(\ell,r)]^{1+\frac{2p-n(\theta+\sigma)}{n(p-\sigma)}}\,,\end{aligned}$$

where C still depends only on n, σ, and C_{Cacc}. We have thus obtained an estimate of the type (2.2), with

$$\alpha_1=4\,,\ \alpha_2=2\,,\ \delta=\frac{2p-n(\theta+\sigma)}{n(p-\sigma)}\,,\ \text{and } K=C\biggl(\int_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{n(\theta+\sigma)-2\sigma}{n(p-\sigma)}}\,.$$

The application of Lemma 2.5 yields

$$\varphi(\ell_0+d,r_0/2)=0\,, \tag{4.13}$$

where d is controlled by

$$\begin{aligned}d^4 &= C\biggl(\int_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{n(\theta+\sigma)-2\sigma}{n(p-\sigma)}}r_0^{-2}\bigl[\varphi(\ell_0,r_0)\bigr]^{\frac{2p-n(\theta+\sigma)}{n(p-\sigma)}}\\
&\le C\biggl(\fint_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{n(\theta+\sigma)-2\sigma}{n(p-\sigma)}}\biggl(\fint_{\mathrm{B}_{r_0}(x_0)}|w|^\sigma\,\mathrm{d}x\biggr)^{\frac{2p-n(\theta+\sigma)}{n(p-\sigma)}}\\
&\le C\biggl(\fint_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{\theta+\sigma}{p}}\end{aligned} \tag{4.14}$$

for a constant C depending only on n, θ, σ, p, and C_{Cacc}. In other words, (4.13) and (4.14) show $w\in\mathrm{L}^\infty(\mathrm{B}_{r_0/2}(x_0),\mathbb{R}^N)$ with

$$\sup_{\mathrm{B}_{r_0/2}(x_0)}|w|\le\ell_0+C\biggl(\fint_{\mathrm{B}_{r_0}(x_0)}|w|^p\,\mathrm{d}x\biggr)^{\frac{\theta+\sigma}{4p}}\,,$$

and the proof of the lemma is complete. □

Relying on both Lemma 4.2 and Lemma 4.4, we now improve the uniform L^p estimates of Lemma 4.3 to uniform L^∞ estimates for the sequence $(\nabla u_k)_{k\in\mathbb{N}}$.

Lemma 4.6. *The sequence $(\nabla u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^{Nn})$, and, for every ball $\mathrm{B}_{4r_0}(x_0)\Subset\Omega$, we have*

$$\limsup_{k\to\infty}\sup_{\mathrm{B}_{r_0/2}(x_0)}|\nabla u_k|\le C\biggl(1+\limsup_{k\to\infty}\fint_{\mathrm{B}_{4r_0}(x_0)}|\nabla u_k|\,\mathrm{d}x\biggr)^{1+\frac{3(\mu-1)}{2(3-\mu)}}\,,$$

where C depends only on n, L_f, R, μ, γ, and Γ.

Proof. We first combine the Caccioppoli-type inequality (4.5) (applied with $h := f_k$, $v := u_k$, and $T(\tau) := (\sqrt{\tau} - \ell)_+^2$ for $\ell \geq \ell_0 := R+2$) and the growth condition (3.2) for $\nabla^2 f_k$. In this way we infer

$$\int_\Omega |\nabla^2 u_k|^2 |\nabla u_k|^{-\mu} (|\nabla u_k| - \ell)_+^2 \eta^2 \,\mathrm{d}x \leq C \int_\Omega \big[\gamma_k |\nabla u_k|^2 + |\nabla u_k|\big] (|\nabla u_k| - \ell)_+^2 |\nabla \eta|^2 \,\mathrm{d}x \tag{4.15}$$

for all $\ell \geq \ell_0$ and every $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, where C depends only on R, μ, γ, and Γ. In particular, this means (in view of $\gamma_k \leq 1 \leq \ell$) that the weighted Caccioppoli inequality (4.11) is satisfied for the choices $w = \nabla u_k$, $\theta = \mu$, $\sigma = 4$, and with Nn in place of N. Thus, for every $k \in \mathbb{N}$, Lemma 4.4 gives

$$\sup_{\mathrm{B}_{r_0/2}(x_0)} |\nabla u_k| \leq R + 2 + C \bigg(\fint_{\mathrm{B}_{r_0}(x_0)} |\nabla u_k|^p \,\mathrm{d}x \bigg)^{\frac{\mu+4}{4p}} \tag{4.16}$$

for every $p > (\mu{+}4)n/2$, where C depends only on n, R, μ, γ, Γ, and p. By Lemma 4.3, the right-hand side of the last inequality is k-uniformly bounded, and thus we have shown that the sequence $(\nabla u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$. In particular, we have $\gamma_k |\nabla u_k| \leq 1$ on $\mathrm{B}_{4r_0}(x_0) \Subset \Omega$ for $k \geq k_0 = k_0(\mathrm{dist}(\mathrm{B}_{4r_0}(x_0), \partial\Omega))$, and when we exploit this insight on the right-hand side of (4.15), we can slightly improve on the estimate (4.16). Indeed, we can reapply Lemma 4.4 on $\mathrm{B}_{4r_0}(x_0)$, with $\sigma = 3$ instead of $\sigma = 4$; then we get (4.16) with the refined exponent $(\mu{+}3)/(4p)$ instead of $(\mu{+}4)/(4p)$ and for every $p > (\mu{+}3)n/2$ — but only for $k \geq k_0$. When we combine this partial refinement of (4.16) with (4.7), we arrive at

$$\sup_{\mathrm{B}_{r_0/2}(x_0)} |\nabla u_k| \leq C \bigg(1 + \gamma_k \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k|^2 \,\mathrm{d}x + \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k| \,\mathrm{d}x \bigg)^{\left(\frac{2}{3-\mu}(p-1)+1\right)\frac{\mu+3}{4p}}$$

for $k \geq k_0$. In view of $\gamma_k |\nabla u_k| \leq 1$ on $\mathrm{B}_{4r_0}(x_0)$ for $k \geq k_0$, we can omit the γ_k-term on the right-hand side. Therefore, passing to the limit, fixing an arbitrary admissible p, and observing

$$\Big(\frac{2}{3-\mu}(p-1) + 1 \Big) \frac{\mu+3}{4p} < \frac{2}{3-\mu} p \cdot \frac{\mu+3}{4p} = 1 + \frac{3(\mu-1)}{2(3-\mu)}\,,$$

we arrive at the claimed estimate. □

At this stage, we can finally prove the existence of a locally Lipschitz continuous generalized minimizer:

Proof of Theorem 1.2. As a consequence of Lemma 3.3, a subsequence of $(u_k)_{k\in\mathbb{N}}$ converges in $\mathrm{L}^1(\Omega, \mathbb{R}^N)$ to a generalized minimizer u of F in the Dirichlet class $\mathcal{D}$. Moreover, for every $x_0 \in \Omega$, setting $r_0 := \mathrm{dist}(x_0, \partial\Omega)/5$, we have $\mathrm{B}_{4r_0}(x_0) \Subset \Omega$, and Lemma 4.6 gives

$$|\nabla u(x_0)| \leq \limsup_{k\to\infty} \sup_{\mathrm{B}_{r_0/2}(x_0)} |\nabla u_k| \leq C \bigg(1 + \limsup_{k\to\infty} \fint_{\mathrm{B}_{4r_0}(x_0)} |\nabla u_k| \,\mathrm{d}x \bigg)^{1+\frac{3(\mu-1)}{2(3-\mu)}}, \tag{4.17}$$

where C depends only on n, L_f, R, μ, γ, and Γ. In order to estimate the right-hand side of (4.17), we observe on the one hand that the minimality of u_k, combined with (3.13), the convergence $\lim_{k\to\infty} f_k(0) = f(0)$, and Lemma 2.4 (i), yields

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega \big[f_k(\nabla u_k) - f_k(0) \big] \mathrm{d}x &= \lim_{k\to\infty} \Big[\inf_{\mathcal{D}_k} F_k - (\chi_k * f)(0) \mathcal{L}^n(\Omega) \Big] = \inf_{\mathcal{D}} F - f(0)\mathcal{L}^n(\Omega) \\ &\leq \int_\Omega \big[f(\nabla u_0) - f(0) \big] \mathrm{d}x \leq L_f \int_\Omega |\nabla u_0| \,\mathrm{d}x\,. \end{aligned} \tag{4.18}$$

On the other hand, we have

$$\int_\Omega \nabla f_k(0) \cdot \nabla u_k \,\mathrm{d}x = \int_\Omega \nabla f_k(0) \cdot \nabla u_{0,k} \,\mathrm{d}x\,,$$

and in view of (3.11) and (3.3) this gives

$$\limsup_{k\to\infty}\int_\Omega \big[-\nabla f_k(0)\cdot\nabla u_k\big]\,\mathrm{d}x \le L_f\int_\Omega |\nabla u_0|\,\mathrm{d}x\,. \tag{4.19}$$

Now we take advantage of (3.7), and the estimates (4.18), (4.19). We infer

$$\begin{aligned}\limsup_{k\to\infty} \fint_{\mathrm{B}_{4r_0}(x_0)} \nu|\nabla u_k|\,\mathrm{d}x &\le \limsup_{k\to\infty}\fint_{\mathrm{B}_{4r_0}(x_0)}\big[C+f_k(\nabla u_k)-f_k(0)-\nabla f_k(0)\cdot\nabla u_k\big]\,\mathrm{d}x\\ &\le C+Cr_0^{-n}\limsup_{k\to\infty}\int_\Omega\big[f_k(\nabla u_k)-f_k(0)-\nabla f_k(0)\cdot\nabla u_k\big]\,\mathrm{d}x\\ &\le C+Cr_0^{-n}\int_\Omega|\nabla u_0|\,\mathrm{d}x\,,\end{aligned}$$

where C depends only on n and L_f. Plugging the last estimate into the right-hand side of (4.17), we conclude

$$|\nabla u(x_0)| \le C\bigg[1+r_0^{-n}\int_\Omega|\nabla u_0|\,\mathrm{d}x\bigg]^{1+\frac{3(\mu-1)}{2(3-\mu)}}\,.$$

Recalling $r_0=\operatorname{dist}(x_0,\partial\Omega)/5$, we arrive at the claim (1.3). Since $x_0\in\Omega$ is arbitrary, this implies in particular $u\in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$, and the proof of the theorem is complete. □

5 Local $\mathrm{C}^{1,\alpha}$ regularity

In this section we establish the regularity and uniqueness results of Theorem 1.1 and Theorem 1.3. Indeed, we will show that Theorem 1.2 makes it possible to deduce these assertions from standard regularity results for integrands with q-growth and the uniqueness statement in [5, Corollary 2.5]. The details of this reasoning are implemented below.

Proof of Theorem 1.3. In the following we consider the case $q\ne 2$, while the case $q=2$ will only be addressed briefly at the end of the proof.

We first observe that assumption (H3) implies (H2) with γ replaced by $\gamma/2$ and with $R=1$. Hence, by Theorem 1.2 there exists a generalized minimizer u of F in $\mathcal{D}$ which satisfies $u\in\mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. For the following, we fix a subdomain $\Omega'\Subset\Omega$ and a number $M\ge 1+\|\nabla u\|_{\mathrm{L}^\infty(\Omega',\mathbb{R}^{Nn})}$ which will be specified later.

Now we *construct a rotationally symmetric comparison integrand with q-growth* which coincides with $f-f(0)$ on B_M. For this purpose, we choose a rotationally symmetric, smooth cut-off function $\eta_M\in\mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^{Nn})$ with $\mathbb{1}_{\mathrm{B}_{2M}}\le\eta_M\le\mathbb{1}_{\mathrm{B}_{4M}}$ and $|\nabla\eta_M|\le M^{-1}$ on $\mathbb{R}^{Nn}$, and with $|\nabla^2\eta_M(z)(\xi,\xi)|\le 2M^{-2}|\xi|^2$ for all $z,\xi\in\mathbb{R}^{Nn}$. Setting[7]

$$h_M(z) := \big(\sqrt{|z|}-\sqrt{M}\big)_+^{2q}\,,$$

we define a $\mathrm{C}^{2,\min\{2q-2,1\}}$ function h_M on $\mathbb{R}^{Nn}$ with

$$\nabla^2 h_M(z) = q\big(\sqrt{|z|}-\sqrt{M}\big)_+^{2q-2}\bigg[(2q-1)\frac{z\otimes z}{2|z|^3}+\big(\sqrt{|z|}-\sqrt{M}\big)_+\Big(\frac{\mathrm{I}_{Nn}}{|z|^{3/2}}-\frac{3z\otimes z}{2|z|^{7/2}}\Big)\bigg]$$

for $z\in\mathbb{R}^{Nn}\setminus\{0\}$. Moreover, we introduce the rotationally symmetric integrand

$$f_M := \eta_M(f-f(0))+h_M \in \mathrm{C}^2(\mathbb{R}^{Nn}\setminus\{0\})\cap\mathrm{C}^1(\mathbb{R}^{Nn})\,,$$

[7] In the case $q>2$ we could alternatively take $h_M(z)=(|z|-M)_+^q$, but for $q<2$ this choice would not be of class C^2.

and we claim that $\nabla^2 f_M$ satisfies the following growth and Hölder continuity conditions

$$\gamma_M|z|^{q-2}|\xi|^2 \le \nabla^2 f_M(z)(\xi,\xi) \le \Gamma_M|z|^{q-2}|\xi|^2 \qquad \text{for } \xi\in\mathbb{R}^{Nn} \text{ and } z\in\mathbb{R}^{Nn}\setminus\{0\}\,, \tag{5.1}$$

$$|\nabla^2 f_M(\widetilde{z})-\nabla^2 f_M(z)| \le L_M\mathrm{S}_{q,\kappa}(|z|,|\widetilde{z}|)|\widetilde{z}-z|^\kappa \qquad \text{for } z,\widetilde{z}\in\mathbb{R}^{Nn}\setminus\{0\} \tag{5.2}$$

with positive M-dependent constants γ_M, Γ_M, L_M, and for every κ such that

$$0<\kappa\le\min\{\beta,2q-2\}\,.$$

In order to check (5.1), we first observe that (H3) gives

$$\gamma(4M)^{-q-2}|z|^{q-2}|\xi|^2 \le \nabla^2 f(z)(\xi,\xi) \le \Gamma|z|^{q-2}|\xi|^2 \qquad \text{for } |z|\le 4M\,,$$

while direct computation shows

$$2\gamma_0|z|^{q-2}|\xi|^2\mathbb{1}_{\mathbb{R}^n\setminus\mathrm{B}_{2M}}(z) \le \nabla^2 h_M(z)(\xi,\xi) \le \Gamma_0|z|^{q-2}|\xi|^2 \qquad \text{for arbitrary } z\in\mathbb{R}^{Nn}$$

with certain positive constants γ_0 and Γ_0, which depend[8] only on q. From these two observations we directly get (5.1) in the cases $|z|\le 2M$ and $|z|\ge 4M$, using that $\nabla^2 f_M(z)=\nabla^2 f(z)+\nabla^2 h_M(z)$ holds in the first case and $\nabla^2 f_M(z)=\nabla^2 h_M(z)$ in the second one. Turning to the remaining case $2M<|z|<4M$, we first estimate the terms

$$X(z,\xi) \coloneqq (\nabla f(z)\cdot\xi)(\nabla\eta_M(z)\cdot\xi)+(f(z)-f(0))\nabla^2\eta_M(z)(\xi,\xi)\,,$$

which contain derivatives of η_M. With the help of Lemma 2.4 (i) we find

$$|X(z,\xi)| \le L_f\big[M^{-1}+2|z|M^{-2}\big]|\xi|^2 \le 9L_fM^{-1}|\xi|^2 \le \frac{1}{4}\gamma_0M^{q-2}|\xi|^2 \le \gamma_0|z|^{q-2}|\xi|^2\,,$$

where we have finally fixed the parameter M sufficiently large that the third inequality in the preceding formula holds true (this is indeed possible, as the relevant constants L_f, γ_0 are M-independent and as $q{-}2>-1$). Altogether we can thus bound

$$\nabla^2 f_M(z)(\xi,\xi)=\eta_M(z)\nabla^2 f(z)(\xi,\xi)+X(z,\xi)+\nabla^2 h_M(z)(\xi,\xi)$$

from below by $\gamma_0|z|^{q-2}|\xi|^2$ and from above by $(\Gamma{+}\gamma_0{+}\Gamma_0)|z|^{q-2}|\xi|^2$, so that we arrive at (5.1) also in the last case.

In order to verify (5.2), we proceed as follows. First of all, if $|\widetilde{z}{-}z|\ge|z|/2$, we get (5.2) directly from the upper bound in (5.1) via the estimate

$$|\nabla^2 f_M(\widetilde{z})-\nabla^2 f_M(z)| \le 2\Gamma_M\max\{|z|^{q-2},|\widetilde{z}|^{q-2}\} \le 10\Gamma_M\mathrm{S}_{q,\kappa}(|z|,|\widetilde{z}|)|\widetilde{z}-z|^\kappa\,.$$

Otherwise, if $|\widetilde{z}{-}z|<|z|/2$, we distinguish three subcases: in the subcase $|z|<M/2$, we have $\nabla^2 f_M(z)=\nabla^2 f(z)$ and $\nabla^2 f_M(\widetilde{z})=\nabla^2 f(\widetilde{z})$, so that the inequality in (5.2) follows from (H3); in the subcase $M/2\le|z|\le 8M$, the scaling factor $\mathrm{S}_{q,\kappa}(|z|,|\widetilde{z}|)$ is bounded (away from 0 and ∞), and it suffices to use the $\mathrm{C}^{2,\kappa}$ regularity of the involved functions; in the last subcase $|z|>8M$, we have $\nabla^2 f_M(z)=\nabla^2 h_M(z)$ and $\nabla^2 f_M(\widetilde{z})=\nabla^2 h_M(\widetilde{z})$, and (5.2) follows once more, as $\nabla^2 h_M$ is of class C^1 outside B_{4M} with derivatives of $(q{-}3)$-growth.

Exploiting the above properties of f_M (which are sometimes referred to as Uhlenbeck structure), we will next *deduce regularity of u from classical results* in the spirit of Uhlenbeck's famous paper [36]. To this end, we recall that u is a generalized minimizer of F, which is $\mathrm{W}^{1,\infty}$ on Ω'. In particular, u then minimizes $w\mapsto\int_{\Omega'}f(\nabla w)\,\mathrm{d}x$ in $u+\mathrm{W}^{1,\infty}_0(\Omega',\mathbb{R}^N)$, and in view of $\|\nabla u\|_{\mathrm{L}^\infty(\Omega',\mathbb{R}^{Nn})}<M$ and $f_M=f-f(0)$ on B_M, the corresponding Euler equation gives

$$\int_{\Omega'}\nabla f_M(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x=\int_{\Omega'}\nabla f(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x=0\,,$$

[8]The M-independence of γ_0 and Γ_0 can also be inferred from the formula $\nabla^2 h_M(z)=M^{q-2}\nabla^2 h_1(z/M)$.

first for all $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega', \mathbb{R}^N)$ and then also for all $\varphi \in \mathrm{W}_0^{1,q}(\Omega', \mathbb{R}^N)$. As f_M is convex, u minimizes $w \mapsto \int_{\Omega'} f_M(\nabla w)\,\mathrm{d}x$ in $u + \mathrm{W}_0^{1,q}(\Omega', \mathbb{R}^N)$, and we can rely on classical regularity results for integrands with q-growth and rotational symmetry. Precisely, we observe that (5.1) implies suitable growth conditions for f_M and ∇f_M, and then we apply[9] [19, Theorem 3.1] in the case $q > 2$ and [1, Theorem 1.1] in the case $q < 2$. From these results we infer the existence of an exponent $\alpha > 0$ (which in general may depend on M and thereby on Ω') such that $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega', \mathbb{R}^N)$ holds. Since $\Omega' \Subset \Omega$ was arbitrary, this proves the claim of Theorem 1.3 regarding $\mathrm{C}^{1,\alpha}$ regularity. In particular, u is of class C^1 on Ω, and therefore the claimed uniqueness statement follows from [5, Corollary 2.5][10].

Finally, we *comment on the particular case* $q = 2$, in which the basic reasoning is very similar to the one just given. However, in order to quote a suitably formulated result from the literature, it seems advantageous to work with an auxiliary integrand f_M of superquadratic growth. This can, for instance, be accomplished by choosing

$$h_M(z) \coloneqq \big(|z| - M\big)_+^3 \,.$$

Then one obtains for f_M the non-degenerate growth condition

$$\gamma_M(1 + |z|)|\xi|^2 \le \nabla^2 f_M(z)(\xi, \xi) \le \Gamma_M(1 + |z|)|\xi|^2$$

and a correspondingly scaled Hölder condition with exponent β, so that $\mathrm{C}^{1,\alpha}$ regularity of u on Ω' still follows from [19, Theorem 3.1]. □

Remark 5.1. *In non-degenerate situations, one can improve on the regularity result of Theorem 1.3. Indeed, if, under the assumptions of the theorem with* $\beta < 1$, u *is a generalized minimizer of* F, *then* $\Omega^* \coloneqq \{x \in \Omega : \nabla u(x) \neq 0\}$ *is an open set and we have* $u \in \mathrm{C}^{2,\beta}_{\mathrm{loc}}(\Omega^*, \mathbb{R}^N)$. *Furthermore, in the non-degenerate case* $q = 2$ *we even have* $u \in \mathrm{C}^{2,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

The statements of Remark 5.1 are routine consequences of Theorem 1.3. Nevertheless, we comment briefly on the proof.

Sketch of proof for Remark 5.1. Since we already know that u is of class C^1, the set Ω^* of non-degenerate points is open. Now we consider a subdomain $\Omega' \Subset \Omega$ and the corresponding exponent $\alpha \in (0, 1)$ (which exists by Theorem 1.3) such that we have $u \in \mathrm{C}^{1,\alpha}(\Omega', \mathbb{R}^N)$. Then, it is not difficult to show that the partial derivatives $\partial_i u$ with $i \in \{1, 2, \ldots, n\}$ are $\mathrm{W}^{1,2}$ solutions of the differentiated Euler equation

$$\int_{\Omega^* \cap \Omega'} \nabla^2 f(\nabla u)(\nabla \partial_i u, \nabla \varphi)\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega^* \cap \Omega', \mathbb{R}^N)\,,$$

where the coefficients $\nabla^2 f(\nabla u)$ are uniformly elliptic, bounded, and of class $\mathrm{C}^{0,\beta\alpha}$ on compact subsets of $\Omega^* \cap \Omega'$. By means of the Schauder estimates [18, Theorem 5.19] it follows that the $\partial_i u$ are locally $\mathrm{C}^{1,\beta\alpha}$ on $\Omega^* \cap \Omega'$. Thus, the coefficients $\nabla^2 f(\nabla u)$ are even locally $\mathrm{C}^{0,\beta}$ on Ω^*, and then, by a second application of the Schauder estimates, the $\partial_i u$ are $\mathrm{C}^{1,\beta}$ on Ω^*. In the case $q = 2$, the same reasoning works on all of Ω. □

Proof of Theorem 1.1. We argue that the choice $f = \mathrm{m}_p$, with a fixed $1 < p < 2$, does indeed satisfy the assumption (H3) — as already claimed in the introduction. Indeed, the conditions in the first line

[9] Both papers quoted here build on an application of the De Giorgi-Nash-Moser theorem to a certain subsolution $H(\nabla u)$ of an elliptic equation, but it is not accurately justified why $H(\nabla u)$ is of class $\mathrm{W}^{1,2}$ or why it is a subsolution. However, these difficulties disappear in our situation, since we have the extra information $u \in \mathrm{W}^{1,\infty}(\Omega', \mathbb{R}^N)$ at our disposal. With this knowledge, it is straightforward to adapt the arguments of [19, 1] and to obtain the relevant $\mathrm{W}^{1,2}$ regularity of $H(\nabla u)$.

[10] In [5, Definition 3.1], we introduced a notion of generalized minimizers which makes sense even for possibly unbounded and irregular Ω. It follows from [2, Corollary 3.89] that this notion coincides with the one of Definition 2.1 in the case of bounded Lipschitz domains Ω considered here.

of (H3) are obviously valid, and the ones in the second line can be verified — for $q = p$, $\mu = 1{+}p$, and constants γ, Γ, which depend only on p — with the help of the explicit formula

$$\nabla^2 \mathrm{m}_p(z) = (1+|z|^p)^{\frac{1}{p}-1}|z|^{p-2}\mathrm{I}_{Nn} + (1+|z|^p)^{\frac{1}{p}-2}(p-2-|z|^p)|z|^{p-4} z \otimes z\,. \tag{5.3}$$

In order to verify the last requirement in (H3), we read off from (5.3) that $\nabla^2 \mathrm{m}_p = \ell_1 H_1 + \ell_2 H_2$ holds for fixed functions ℓ_1, H_1, ℓ_2, H_2 such that ℓ_1 and ℓ_2 are bounded Lipschitz functions on $\mathbb{R}^{Nn}$, while H_1 and H_2 are homogeneous of degree $p{-}2$ and smooth on $\mathbb{R}^{Nn} \setminus \{0\}$. From the homogeneity of H_1 we infer

$$|H_1(z)| \le C|z|^{p-2}\,, \qquad |H_1(\widetilde{z}) - H_1(z)| \le C\mathrm{S}_{p;1}(|z|,|\widetilde{z}|)|\widetilde{z} - z|$$

for all $z, \widetilde{z} \in \mathbb{R}^{Nn} \setminus \{0\}$, and for the product $\ell_1 H_1$, taking into account that ℓ_1 is bounded and Lipschitz-continuous, it is not difficult to show

$$|\ell_1(\widetilde{z})H_1(\widetilde{z}) - \ell_1(z)H_1(z)| \le C_1(1+|z|+|\widetilde{z}|)\mathrm{S}_{p;1}(|z|,|\widetilde{z}|)|\widetilde{z} - z|\,,$$

still for all $z, \widetilde{z} \in \mathbb{R}^{Nn} \setminus \{0\}$, and with a constant $C_1 \in \mathbb{R}$. Evidently, the same reasoning works for $\ell_2 H_2$, and thus the third line of (H3) is verified for $q = p$, for $\beta = 1$, and — assuming that for $\ell_2 H_2$ we have an analogous estimate with a constant C_2 — for $\Psi(t) = (C_1{+}C_2)(1{+}t)$.

Consequently, Theorem 1.3 is applicable to the model integrals M_p with $1 < p < 2$, and it yields the claimed uniqueness statement and C^1 regularity on Ω for every generalized minimizer u of M_p. Moreover, by Remark 5.1, we even have C^2 regularity of u on Ω^*, and it just remains to reason that u is locally $\mathrm{C}^{1,\alpha}$ on a neighborhood of $\Omega \setminus \Omega^*$ in Ω, with a fixed Hölder exponent $\alpha(n,N,p) > 0$ (which in particular does not depend on the distance to $\partial\Omega$). However, this last requirement is ensured, when we notice that every point of Ω is a Lebesgue point of ∇u (by C^1 regularity) and then apply the local regularity result [28, Theorem 2.5] in every point of $\Omega \setminus \Omega^*$. □

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