ON THE LOWER SEMICONTINUITY OF QUASICONVEX INTEGRALS

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Abstract. In this paper, dedicated to Neil Trudinger in the occasion of his 70^{th} birthday, we propose a relatively elementary proof of the *weak lower semicontinuity* in $W^{1,p}$ of a general integral of the Calculus of Variations of the type (1.1) below with a *quasiconvex* density function satisfying p-growth conditions. Several comments and references on the related literature, and a subsection devoted to some properties of the *maximal function operator*, are also included.

1. Introduction

A large number of mathematicians working in partial differential equations have been influenced by the researches in the field of mathematical analysis carried out by Neil Trudinger. Sobolev functions are now treated as simply as smooth functions and operations on them, such as − for instance − truncation and composition, are now considered elementary. This is due in large part to the popularity of the book [39], which Neil Trudinger wrote joint with David Gilbarg, and which has been a reference to many of us. With great pleasure this paper is dedicated to Neil Trudinger in the occasion of his 70^{th} birthday.

The subject that we consider here is the *quasiconvexity condition* by Morrey [53] and its connections with *lower semicontinuity*. It is well known that some properties which hold for solutions to partial differential equations, for instance of elliptic type, cannot be extended to systems. Often the lack of the *maximum principle* is one of the main obstructions. If a *variational formulation* exists, from the point of view of an energy functional, a solution $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of an elliptic partial differential equation ($N = 1$) or system ($N > 1$) can be obtained as a minimizer of an integral functional of the type

$$F(u) := \int_\Omega f\big(x, u(x), \nabla u(x)\big)\, dx\,. \tag{1.1}$$

Here a strong distinction occurs depending if $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ is a scalar function (i.e., $N = 1$) or it is a general vector-valued map with values in $\mathbb{R}^N$ with $N > 1$ (for the sake of simplicity we assume $n \geq 2$ throughout this introduction). In the first case, the convexity of the integrand f in (1.1) with respect to the gradient variable ∇u plays a central role in the existence and regularity of minimizers, while convexity is sufficient though not necessary in the vector-valued setting $N > 1$, for which the *first variation* is a system of partial differential equations.

If $N > 1$ the assumption which characterizes the *lower semicontinuity* of the integral F (under some growth conditions and specific topologies, which we will make precise below) is the *quasiconvexity condition* stated at the beginning of the next section (see (2.1)). We emphasize that quasiconvexity reduces to convexity if $N = 1$, while it is really a different − and more general − condition if $N > 1$. For instance, when $n = N > 1$, important examples of quasiconvex, but non convex, functions are given by

$$f(\nabla u) = \det \nabla u\,, \qquad \text{or} \qquad f(\nabla u) = |\det \nabla u|\,,$$

2000 *Mathematics Subject Classification.* 49J45; 49K20.

Key words and phrases. Quasiconvex integrals, lower semicontinuity.

where $\det \nabla u$ represents the *determinant* of the $n \times n$ matrix of the gradient ∇u (*Jacobian determinant*) of a map $u : \mathbb{R}^n \to \mathbb{R}^n$.

As it is well known, the lower semicontinuity of the integral F in (1.1) is one of the main ingredients for the existence of minimizers. The first lower semicontinuity result for quasiconvex integrals has been given by Morrey [53] and then by Meyers [51]. A semicontinuity theorem with some "natural" growth assumptions on the integrand f has been proved by Acerbi and Fusco [3], who used fine properties of the *maximal function* (see also Subsection 3.3 of this paper), and by Marcellini [48], who used some delicate *higher summability properties* of minimizing sequences.

Since then, several generalizations of the result mentioned above have been obtained, we recall some of them in what follows. Weak lower semicontinuity for quasiconvex integrands satisfying p-q growth conditions has been the object of many efforts due to its relevance in modelling elastic cavitation and in connection with the theory of relaxation in Calculus of Variations. The first contributions in this respect are contained in [49], then generalized in the papers [47], [42], [26], [27], [11], [5], [52]. Similar issues are particularly interesting when restricting to the class of *polyconvex* integrands, according to the terminology introduced by Ball in [9], that arise in some problems of continuum mechanics and that, in some instances, do not satisfy natural growth conditions from above. The field is still very active nowadays, the researches by Dacorogna and Marcellini [15] have been pushed forward and developed in several papers (cp. with [46], [2], [12], [38], [17], [37], [24], [25], [31], [32], [33], [6], [23]). References to such subjects are also the books by Dacorogna [14] and by Giusti [40], and the survey paper by Leoni [45].

Even more, densities controlled in terms of suitable convex functions and the weak lower semicontinuity of the associated energies defined on Orlicz-Sobolev spaces are the subjects of investigation in the papers [19], [20], [10], [54], [57]. Similarly, the framework of functionals defined on Sobolev spaces with variable exponent has been studied in [55] and [58].

Extensions of the characterization of weak lower semicontinuity for functionals defined on maps belonging to the kernel of a constant-rank first order linear partial differential operator $\mathscr{A}$ lead to the notion of $\mathscr{A}$-quasiconvexity. In particular, the standard case of the curl operator is included. This topic has been dealt with in the papers [30] and [34], developing the ideas introduced by Dacorogna in [13] in connection with compensated compactness issues and problems in continuum mechanics.

Finally, we mention some recent contributions dealing with energies defined on Sobolev maps taking values into non-flat spaces (see [18] for Almgren's Q-valued maps, [21] for manifold valued maps, and [22] for Q-valued maps modelled upon manifolds). The weak lower semicontinuity of energy functionals is characterized by means of an intrinsic notion of quasiconvexity; the analysis is carried out within the theory of metric space-valued Sobolev maps avoiding any embedding of the target space into Euclidean ones.

None of the proofs in the quoted papers and books can be considered simple, apart from that one valid for the special authonomous case

$$\int_\Omega f\left(\nabla u(x)\right) dx \tag{1.2}$$

given in Section 2 of [48], which is self contained and till now can be considered the most elementary proof for lower semicontinuity of the integral in (1.2).

In this paper we propose a new proof of the lower semicontinuity for general integrals as in (1.1) by combining arguments that are already known in literature. In our opinion the outcome is a more elementary proof than those ones supplied so far. We use some ideas introduced by Acerbi and Fusco [3] concerning truncation techniques via the maximal function operator to gain equi-integrability of sequences of Sobolev maps. We also employ a freezing technique due to Marcellini and Sbordone [50] to reduce the problem to autonomous integrands (see Section 4 for more details). For what the first issue is concerned, we exploit a sharp version of the biting lemma proved by Fonseca, Müller and Pedregal [35]; we include in Subsection 3.2 the elementary proof provided by De Lellis, Focardi and Spadaro [18].

To conclude this introduction we resume briefly the contents of the paper. In the next section we give the main definitions and results, in particular Theorem 2.2, which is proved in Section 4. For the sake of completeness, in Subsection 3.3 we recall some well-known properties of the *maximal function operator* that are used in this paper.

2. Setting of the problem and main result

Let us first recall Morrey's celebrated notion of quasiconvexity.

Definition 2.1 (Quasiconvexity)**.** Let $g : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a locally bounded integrand. We say that g is *quasiconvex* if the following inequality holds for every affine function $u(x) = a + \mathbb{A}\cdot x$, where $a \in \mathbb{R}^N$ and $\mathbb{A} \in \mathbb{M}^{N\times n}$, and any map $w \in u + W^{1,\infty}_0(Q_1,\mathbb{R}^N)$, Q_1 the open unitary cube in $\mathbb{R}^n$,

$$g(\mathbb{A}) \leq \int_{Q_1} g\big(\nabla w(x)\big)dx. \tag{2.1}$$

Let $p \in (1,\infty)$ and $f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to \mathbb{R}$ be a Charathéodory integrand satisfying

$$0 \leq f(x,u,\mathbb{A}) \leq C(1 + |u|^q + |\mathbb{A}|^p) \qquad \text{for all } (u,\mathbb{A}), \tag{2.2}$$

for some constant C, where $q \leq p^* := \frac{np}{n-p}$ if $p < n$ and $q \geq 1$ is any exponent if $p \geq n$. Under the previous assumptions on f it is well-defined the functional $F : W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty)$ given by

$$F(u) := \int_\Omega f\big(x, u(x), \nabla u(x)\big)dx. \tag{2.3}$$

We shall provide below an elementary proof of the ensuing celebrated result due to Morrey [53], Meyers [51], Acerbi and Fusco [3] and Marcellini [48].

Theorem 2.2. *Let f be a Charathéodory integrand satisfying* (2.2)*. The functional F in* (2.3) *is weakly lower semicontinuous in $W^{1,p}(\Omega,\mathbb{R}^N)$ if and only if $f(x,u,\cdot)$ is quasiconvex for $\mathcal{L}^n$ a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^N$.*

In this paper we shall only deal with the sufficiency of quasiconvexity for weak lower semicontinuity, necessity being a less difficult task (see for instance the books [14] and [40]). Our proof mixes several tools developed by different authors in the course of studying the previous result and related problems. We shall quote each contribution trying to be as exhaustive as possible.

Remark 2.3. We can actually prove the same result for integrands satisfying the milder growth condition

$$-C(1 + |u|^r + |\mathbb{A}|^s) \leq f(x,u,\mathbb{A}) \leq C(1 + |u|^q + |\mathbb{A}|^p) \qquad \text{for all } (u,\mathbb{A}),$$

for some positive constant C, where $s \in (1,p)$ and $r \in [1,p^*)$ if $p < n$ and $r \geq 1$ otherwise.

Indeed, one can reduce to the setting of Theorem 2.2 thanks to the equi-integrability of the sequence $(1 + |u_k|^r + |\nabla u_k|^s)$, if (u_k) is bounded in $W^{1,p}(\Omega,\mathbb{R}^N)$ (see for instance [48], [42]).

Counterexamples are well-known in literature if $r = p^*$, and if $s = p$ (see [48, Section 6]).

Remark 2.4. Lower semicontinuity in the case $p = 1$ can be proved along sequences in $W^{1,1}$ that are equi-integrable, as shown first by Fusco [36]. Otherwise, a relaxation phenomenon in the space BV of functions with bounded variation occurs (see for instance [7], [28], [29], [43], [44] and the book [8] for more detailed results and an exhaustive list of references).

3. Preliminary results

The aim of this section is to introduce some notations and to recall some basic definition and results which will be used in the sequel.

We begin with some algebraic notation.

Let $n, N \geq 2$ and $\mathbb{M}^{N\times n}$ be the linear space of all $N \times n$ real matrices. For $\mathbb{A} \in \mathbb{M}^{N\times n}$, we denote $\mathbb{A} = (\mathbb{A}^i_j)$, $1 \leq i \leq N$, $1 \leq j \leq n$, where upper and lower indices correspond to rows and columns respectively. The euclidean norm of $\mathbb{A}$ will be denoted by $|\mathbb{A}|$.

As usual, $Q_r(x)$, $B_r(x)$ denote the open euclidean cube, ball in $\mathbb{R}^n$, $n \geq 2$, with side r, radius r and center the point x, respectively. The center shall not be indicated explicitly if it coincides with the origin.

3.1. **Equi-integrability.** Let us first recall some definitions and introduce some notation. As usual, in the following $\Omega \subset \mathbb{R}^n$ denotes a Lipschitz set with finite measure.

Definition 3.1. A sequence (g_k) in $L^1(\Omega)$ is *equi-integrable* if one of the following equivalent conditions holds:

(a) for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ with $\mathcal{L}^n(E) \leq \delta$, we have $\sup_k \int_E |g_k| \leq \varepsilon$;

(b) the distribution functions $\varphi_k(t) := \int_{\{|g_k|\geq t\}} |g_k|$ satisfy $\lim_{t\uparrow\infty} \sup_k \varphi_k(t) = 0$;

(c) (De la Vallée Poissin's criterion) there exists a Borel function $\psi : [0,\infty) \to [0,\infty]$ such that

$$\lim_{t\uparrow\infty} \frac{\psi(t)}{t} = \infty \quad \text{and} \quad \sup_k \int_\Omega \psi(|g_k|)dx < \infty.$$

Note that, since Ω has finite measure, an equi-integrable sequence is also equi-bounded.

Our first aim is to prove Chacon's biting lemma, in doing that we shall follow the approach by Fonseca, Müller and Pedregal [35], though the proof presented here is revisited as done by De Lellis, Focardi and Spadaro [18].

Lemma 3.2. *Let (g_k) be a bounded sequence in $L^1(\Omega)$. Then, given any increasing sequence $(t_j) \subset (0,\infty)$ with $t_j \uparrow \infty$, there exists a subsequence (k_j) such that $(g_{k_j} \vee (-t_j) \wedge t_j)$ is equi-integrable.*

Proof. Without loss of generality, assume $g_k \geq 0$ and consider for every $j \in \mathbb{N}$ the functions $h^j_k := g_k \wedge t_j$. Since, for every j, $(h^j_k)_k$ is equi-bounded in $L^\infty(\Omega)$, up to passing to a subsequence (not relabeled), there exists the $L^\infty(\Omega)$ weak* limit f_j of h^j_k for every j. Clearly the limits f_j have the following properties:

(a) $f_j \leq f_{j+1}$ for every j (since $h^j_k \leq h^{j+1}_k$ for every k);

(b) $\|f_j\|_{L^1(\Omega)} = \lim_k \left\|h^j_k\right\|_{L^1(\Omega)}$ (since (h^j_k) is positive and converges to f_j weak* $L^\infty(\Omega)$);

(c) $\sup_j \|f_j\|_{L^1(\Omega)} = \sup_j \lim_k \left\|h^j_k\right\|_{L^1(\Omega)} \leq \sup_k \|g_k\|_{L^1(\Omega)} < \infty$.

By the Lebesgue monotone convergence theorem, (a) and (c), it follows that (f_j) converges in $L^1(\Omega)$ to a function f. Moreover, from (b), for every j we can find a k_j such that

$$\left|\int_\Omega h^j_{k_j}\,dx - \int_\Omega f_j\,dx\right| \leq t_j^{-1}. \tag{3.1}$$

We claim that $h^j_{k_j} = g_{k_j} \wedge t_j$ fulfills the conclusion of the lemma. To see this, it is enough to show that $(h^j_{k_j})$ weakly converges to f in $L^1(\Omega)$, from which the equi-integrability follows. Let

$a \in L^\infty(\Omega)$ be a test function. Since $h^l_{k_j} \le h^j_{k_j}$ for $l \le j$, we have that

$$\int_\Omega \big(\|a\|_{L^\infty} - a\big) h^l_{k_j}\, dx \le \int_\Omega \big(\|a\|_{L^\infty} - a\big) h^j_{k_j}\, dx. \tag{3.2}$$

Taking the limit as j goes to infinity in (3.2), we obtain by the convergence of $(h^l_{k_j})$ to f_l weak* $L^\infty(\Omega)$ and (3.1)

$$\int_\Omega \big(\|a\|_{L^\infty} - a\big) f_l\, dx \le \|a\|_{L^\infty} \int_\Omega f\, dx - \limsup_j \int_\Omega a\, h^j_{k_j}\, dx.$$

From which, passing to the limit in l, we conclude since (f_l) converges to f in $L^1(\Omega)$

$$\limsup_j \int_\Omega a\, h^j_{k_j}\, dx \le \int_\Omega a f\, dx. \tag{3.3}$$

Using $-a$ in place of a, one obtains as well the inequality

$$\int_\Omega a f\, dx \le \liminf_j \int_\Omega a\, h^j_{k_j}\, dx. \tag{3.4}$$

Inqualities (3.3) and (3.4) together conclude the proof of the weak convergence of $h^j_{k_j}$ to f in $L^1(\Omega)$. □

Next we show that concentration effects for critical Sobolev embedding do not show up if equi-integrability of both functions and gradients is assumed. Here we follow De Lellis, Focardi and Spadaro [18].

Lemma 3.3. *Let $p \in [1, n)$ and $(g_k) \subset W^{1,p}(\Omega)$ be such that $(|g_k|^p)$ and $(|\nabla g_k|^p)$ are both equi-integrable, then $(|g_k|^{p^*})$ is equi-integrable as well.*

Proof. Since (g_k) is bounded in $W^{1,p}(\Omega)$, Chebychev's inequality implies

$$\sup_j j^p \mathcal{L}^n(\{|g_k| > j\}) \le C < \infty. \tag{3.5}$$

For every fixed $j \in \mathbb{N}$, consider the sequence $g^j_k := g_k - (g_k \vee (-j) \wedge j)$. Then, $(g^j_k) \subset W^{1,p}(\Omega)$ and $\nabla g^j_k = \nabla g_k$ in $\{|g_k| > j\}$ and $\nabla g^j_k = 0$ otherwise.

The Sobolev embedding yields

$$\|g^j_k\|^p_{L^{p^*}(\Omega)} \le c \|g^j_k\|^p_{W^{1,p}(\Omega)} \le c \int_{\{|g_k|>j\}} \big(|g_k|^p + |\nabla g_k|^p\big) dx. \tag{3.6}$$

Therefore, the equi-integrability assumptions and (3.5) imply that for every $\varepsilon > 0$ there exists $j_\varepsilon \in \mathbb{N}$ such that for every $j \ge j_\varepsilon$

$$\sup_k \|g^j_k\|_{L^{p^*}(\Omega)} \le \varepsilon/2. \tag{3.7}$$

Let $\delta > 0$ and consider a generic $\mathcal{L}^n$-measurable sets $E \subseteq \Omega$ with $\mathcal{L}^n(E) \le \delta$. Then, since we have

$$\|g_k\|_{L^{p^*}(E)} \le \|g_k - g^{j_\varepsilon}_k\|_{L^{p^*}(E)} + \|g^{j_\varepsilon}_k\|_{L^{p^*}(E)} \le j_\varepsilon \left(\mathcal{L}^m(E)\right)^{1/p^*} + \|g^{j_\varepsilon}_k\|_{L^{p^*}(\Omega)},$$

by (3.7), to conclude it suffices to choose δ such that $j_\varepsilon \delta^{1/p^*} \le \varepsilon/2$. □

3.2. **A truncation lemma.** In the next lemma we show how a weakly convergent sequence can be truncated in order to obtain an equi-integrable sequence still weakly converging to the same limit. This result is the analog of the decomposition lemma proved by Fonseca, Müller and Pedregal (see [35, Lemma 2.3]) and constitutes the main point in the proof of the sufficiency of quasiconvexity for the lower semicontinuity property. Similar results are well-known in literature and have been obtained by several authors, see for instance the contributions by Acerbi and Fusco [4] and Kristensen [44]. The statement below is in the form presented in [18].

Lemma 3.4. *Let $(v_k) \subset W^{1,p}(\Omega, \mathbb{R}^N)$ be weakly converging to u. Then, there exists a subsequence (v_{k_j}) and a sequence $(u_j) \subset W^{1,\infty}(\Omega, \mathbb{R}^N)$ such that*

(i) *$\mathcal{L}^n(\{v_{k_j} \neq u_j\}) = o(1)$ and $u_j \to u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^N)$;*
(ii) *$(|\nabla u_j|^p)$ is equi-integrable;*
(iii) *if $p \in [1, n)$, $(|u_j|^{p^*})$ is equi-integrable and, if $p \geq n$, $(|u_j|^q)$ is equi-integrable for any $q \geq 1$.*

Proof. Let g_k be the function corresponding to v_k provided by Lemma 3.8, and note that $(|g_k|^p) \subset L^1(\Omega)$ is a bounded sequence thanks to estimate (3.12) and the boundedness of (v_k) in $W^{1,p}(\Omega)$. Applying Chacon's biting lemma (see Lemma 3.2) to $(|g_k|^p)$ with $t_j = j$, we get a subsequence (k_j) and a sequence such that $(|g_{k_j}|^p \wedge j)$ is equi-integrable.

Let $\Omega_j := \{x \in \Omega : |g_{k_j}(x)|^p \leq j\}$ and note that by (3.11), $v_{k_j}|_{\Omega_j}$ is Lipschitz continuous on Ω_j with constant $c\, j^{1/p}$. Let u_j be the Lipschitz extension of $v_{k_j}|_{\Omega_j}$ with Lipschitz constant $c\, j^{1/p}$. Then, it is easy to verify that $\mathcal{L}^n(\Omega \setminus \Omega_j) = O(j^{-1})$, and that

$$|\nabla u_j|^p = |\nabla v_{k_j}|^p \leq |g_{k_j}|^p \wedge j \text{ on } \Omega_j \quad \text{and} \quad |\nabla u_j|^p \leq c\, j = c\,(|g_{k_j}|^p \wedge j) \text{ on } \Omega \setminus \Omega_j.$$

Thus, (ii) follows immediately from these properties and (i) by taking into account the equi-integrability of $(|g_{k_j}|^p \wedge j)$ and the Poincaré type inequality

$$\|v\|_{L^p(\Omega)} \leq c\|\nabla v\|_{L^p(\Omega, \mathbb{R}^{N\times n})}$$

for some constant $c = c(n, p, \Omega)$, for all $v \in W^{1,p}(\Omega)$ such that $\mathcal{L}^n(\{v = 0\}) \geq \sigma$, with $\sigma > 0$ fixed.

As for (iii), note that the functions $w_j = |u_j|$ are in $W^{1,p}(\Omega)$, with $|\nabla w_j| \leq |\nabla u_j|$. Moreover, (w_j) converge weakly to $|u|$ by (i). Hence, $(|w_j|^p)$ and $(|\nabla w_j|^p)$ are equi-integrable. In case $p \in [1, n)$, this implies (see Lemma 3.3) the equi-integrability of $(|u_j|^{p^*})$. In case $p \geq n$, the property follows from Hölder inequality and Sobolev embedding. □

3.3. **The maximal function operator.** For the sake of completeness we recall some results well-known in literature that are employed to prove the truncation Lemma 3.4. We refer to the classical book by Stein [56] for further references and results, and to the expository paper by Aalto and Kinnunen [1] for a more up-to-date state of the art on the subject.

We start off introducing the maximal function operator.

Definition 3.5. Let $v \in L^1_{loc}(\mathbb{R}^n)$, the *Hardy-Littlewood maximal function* $\mathscr{M}v : \mathbb{R}^n \to [0, \infty]$ of v is defined as

$$\mathscr{M}v(x) := \sup_{r>0} \fint_{B_r(x)} |v(y)|\, dy \qquad \text{for all } x \in \mathbb{R}^n.$$

It is elementary to check that $\mathscr{M}v$ is a positive lower semicontinuous function being the supremum of positive continuous functions.

To be self-contained we present the proof of the well-known Hardy-Littlewood's maximal theorem about the continuity of the maximal function operator. We establish first the so called weak type estimate.

Theorem 3.6. *If $v \in L^1(\mathbb{R}^n)$, then for all $\lambda > 0$*

$$\mathcal{L}^n(\{\mathscr{M}v > \lambda\}) \leq \frac{5^n}{\lambda}\|v\|_{L^1(\mathbb{R}^n)}.$$

Proof. Given $\lambda > 0$ consider the superlevel set $E_\lambda := \{\mathscr{M}v > \lambda\}$, that turns out to be open thanks to the quoted lower semicontinuity of $\mathscr{M}v$. Then, for all $x \in E_\lambda$ we can find a radius $r_x > 0$ such that

$$⨍_{B_{r_x}(x)} |v|\, dy > \lambda.$$

Then $r_x < (\lambda^{-1}\omega_n^{-1}\|v\|_{L^1(\mathbb{R}^n)})^{1/n}$, and by Vitali's $5r$-covering theorem (see for instance [8] and [16]) we can find disjoint balls $B_{r_i}(x_i)$, $i \in \mathbb{N}$, such that

$$E_\lambda \subseteq \cup_{i\in\mathbb{N}} B_{5r_i}(x_i), \quad \text{and} \quad ⨍_{B_{r_i}(x_i)} |v|\, dy > \lambda.$$

Therefore, we conclude that

$$\mathcal{L}^n(E_\lambda) \leq \sum_{i\geq 0} \mathcal{L}^n(B_{5r_i}(x_i)) \leq \frac{5^n}{\lambda}\sum_{i\geq 0}\int_{B_{r_i}(x_i)} |v|\, dy \leq \frac{5^n}{\lambda}\|v\|_{L^1(\mathbb{R}^n)}.$$ □

Theorem 3.7. *Let $p \in (1, \infty]$, there exists a positive constant $c = c(n, p)$ such that for all $v \in L^p(\mathbb{R}^n)$*

$$\|\mathscr{M}v\|_{L^p(\mathbb{R}^n)} \leq c\|v\|_{L^p(\mathbb{R}^n)}.$$

Proof. In case $p = \infty$, estimate

$$\|\mathscr{M}v\|_{L^\infty(\mathbb{R}^n)} \leq \|v\|_{L^\infty(\mathbb{R}^n)} \tag{3.8}$$

easily follows from the very definition of the maximal operator.

Next fix $p \in (1, \infty)$. For any $\lambda > 0$ decompose v as the sum of v_λ and w_λ, where

$$v_\lambda(x) := v(x)\chi_{\{|v|>\lambda/2\}}(x).$$

Hence, $v_\lambda \in L^1(\mathbb{R}^n)$ with

$$\|v_\lambda\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |v_\lambda|^{1-p}|v_\lambda|^p\, dx \leq \left(\frac{\lambda}{2}\right)^{1-p}\|v\|^p_{L^p(\mathbb{R}^n)},$$

and $w_\lambda \in L^\infty(\mathbb{R}^n)$ with

$$\|w_\lambda\|_{L^\infty(\mathbb{R}^n)} \leq \frac{\lambda}{2}. \tag{3.9}$$

By the very definition, it turns out that the maximal function defines a positive sublinear operator, so that

$$\mathscr{M}v \leq \mathscr{M}v_\lambda + \mathscr{M}w_\lambda.$$

In particular, by taking into account estimate (3.8) for w_λ and the inequality in (3.9), we can apply Theorem 3.6 to v_λ and deduce that

$$\mathcal{L}^n\left(\{\mathscr{M}v > \lambda\}\right) \leq \mathcal{L}^n\left(\{\mathscr{M}v_\lambda > \lambda/2\}\right) \leq \frac{2}{\lambda}5^n\int_{\{|v|>\lambda/2\}} |v|\, dx. \tag{3.10}$$

Therefore, by using the layer-cake representation formula and Fubini's theorem we find

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathscr{M}v|^p dx &= p\int_0^\infty \lambda^{p-1}\mathcal{L}^n\left(\{\mathscr{M}v > \lambda\}\right) d\lambda\\ &\overset{(3.10)}{\leq} 2\,5^n\,p\int_0^\infty \lambda^{p-2}\int_{\{|v|>\lambda/2\}} |v|\, dx\, d\lambda\end{aligned}$$

$$= 2^p\, 5^n\, p \int_0^{\infty} \lambda^{p-2} \int_{\{|v|>\lambda\}} |v|\, dx\, d\lambda = \frac{2^p\, 5^n\, p}{p-1} \int_{\mathbb{R}^n} |v|^p\, dx. \qquad \square$$

We are now ready to prove the ensuing result, instrumental for the truncation technique used in Lemma 3.4. Actually, the pointwise estimate proved below provides a characterization of Sobolev functions as shown by Hajłasz [41].

Lemma 3.8. *Let $v \in W^{1,p}(\Omega)$, then there exist a function $g \in L^p(\Omega)$ and a positive constant $c = c(n, \Omega)$ such that for $\mathcal{L}^n$ a.e. x, $y \in \Omega$*

$$|v(x) - v(y)| \le c\,(g(x) + g(y))\,|x - y| \tag{3.11}$$

and

$$\|g\|_{L^p(\Omega)} \le c\|v\|_{W^{1,p}(\Omega)}. \tag{3.12}$$

Proof. We first prove the result for functions $v \in W^{1,p}(\mathbb{R}^n)$.

To this aim, let x be a Lebesgue point of v and $r \in (0, \infty)$ be any radius to be chosen suitably along the proof. Set $B_i := B_{r/2^i}(x)$, $i \in \mathbb{N}$, and

$$v_{B_i} := \fint_{B_i} v(y)\, dy.$$

Being x a Lebesgue point of v, from the trivial estimate

$$|v(x) - v_{B_0}| \le |v(x) - v_{B_j}| + \sum_{i=0}^{j-1} |v_{B_{i+1}} - v_{B_i}| \quad \text{for all } j \in \mathbb{N},$$

and Poincaré inequality, we deduce that

$$\begin{aligned} |v(x) - v_{B_0}| &\le \sum_{i\ge 0} |v_{B_{i+1}} - v_{B_i}| = \sum_{i\ge 0} \left| \fint_{B_{i+1}} (v - v_{B_i})\, dy \right| \\ &\overset{B_{i+1}\subset B_i}{\le} 2^n \sum_{i\ge 0} \fint_{B_i} |v - v_{B_i}|\, dy \le c(n) \sum_{i\ge 0} \frac{r}{2^i} \fint_{B_i} |\nabla v|\, dy \le c(n)\, r\, \mathscr{M}(|\nabla v|)(x). \end{aligned} \tag{3.13}$$

Let $z \in B_r(x)$ be another Lebesgue point of v, since by definition of r we have $B_r(x) \subset B_{2r}(z)$, by estimate (3.13) applied in z and Poincaré inequality we get

$$\begin{aligned} |v(z) - v_{B_r(x)}| &\le |v(z) - v_{B_{2r}(z)}| + |v_{B_{2r}(z)} - v_{B_r(x)}| \\ &\le 2c(n) r\, \mathscr{M}(|\nabla v|)(z) + \fint_{B_r(x)} |v - v_{B_{2r}(z)}| dy \\ &\le 2c(n) r\, \mathscr{M}(|\nabla v|)(z) + c(n) \fint_{B_{2r}(z)} |v - v_{B_{2r}(z)}| dy \le 4c(n) r\, \mathscr{M}(|\nabla v|)(z). \end{aligned} \tag{3.14}$$

Finally, we choose $r = 2|x - z|$ so that z is in $B_r(x)$, then from estimates (3.13) and (3.14) we conclude that

$$\begin{aligned} |v(x) - v(z)| &\le |v(x) - v_{B_r(x)}| + |v(z) - v_{B_r(x)}| \\ &\le 4c(n) \Big(\mathscr{M}(|\nabla v|)(x) + \mathscr{M}(|\nabla v|)(z) \Big) |x - z|. \end{aligned} \tag{3.15}$$

The conclusion follows at once with $g := \mathscr{M}(|\nabla v|)$ thanks to the Hardy-Littlewood's maximal theorem (see Theorem 3.7).

Next consider a continuous extension operator $\mathscr{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$. By (3.15) we have for all pairs x and z of Lebesgue points of v in Ω

$$|v(x) - v(z)| \leq 4c(n)\Big(\mathscr{M}(|\nabla(\mathscr{E}v)|)(x) + \mathscr{M}(|\nabla(\mathscr{E}v)|)(z)\Big)|x - z|.$$

Therefore, the conclusion follows by setting $g := \mathscr{M}(|\nabla(\mathscr{E}v)|)$, since by Theorem 3.7 and the continuity of the extension operator $\mathscr{E}$ we have for a positive constant $c = c(n, \Omega)$

$$\|g\|_{L^p(\Omega)} \leq c\,\|\nabla(\mathscr{E}v)\|_{L^p(\mathbb{R}^n)} \leq c\,\|v\|_{W^{1,p}(\Omega)}.$$

□

4. Proof of Theorem 2.2

In this section we prove Theorem 2.2 following a by now well-established strategy. We first reduce ourselves to check the inequality on equi-integrable sequences by taking advantage of Lemma 3.4 and thanks to (2.2). Equi-integrability, in fact, allows us to perform several operations on the given sequence paying only an infinitesimal energetic error. More precisely, by equi-integrability we can freeze the lower order variables and reduce ourselves to autonomous functionals and target affine functions. The last setting is easily dealt with, again thanks to equi-integrability, by changing the boundary data in order to exploit directly the very definition of quasiconvexity to conclude.

4.1. **Sufficiency of quasiconvexity.** We are now ready to show that the direct implication of Theorem 2.2. That is, given a sequence $(v_k) \subset W^{1,p}(\Omega, \mathbb{R}^N)$ weakly converging to $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and f as (2.2), then

$$F(u) \leq \liminf_{k\to\infty} F(v_k). \tag{4.1}$$

Clearly, we may suppose that the right hand term above is finite since otherwise there is nothing to prove. Moreover, up to extracting a subsequence, we may assume that the inferior limit in (4.1) is actually a limit. In what follows, for the sake of convenience, subsequences will never be relabeled.

The proof is divided into several steps.

Step 1. *Reduction to an equi-integrable sequence*

Using Lemma 3.4, up to a subsequence, there exists (u_k) such that (i)-(iii) in Lemma 3.4 hold. Therefore, (4.1) follows provided we prove

$$F(u) \leq \lim_{k\to\infty} F(u_k), \tag{4.2}$$

since, by the positivity of f and the equi-integrability properties (ii) and (iii),

$$\begin{aligned} F(u_k) &= \int_{\{v_k = u_k\}} f(x, v_k, \nabla v_k) + \int_{\{v_k \neq u_k\}} f(x, u_k, \nabla u_k) \\ &\leq F(v_k) + C\int_{\{v_k\neq u_k\}} (1 + |u_k|^q + |\nabla u_k|^p) = F(v_k) + o(1). \end{aligned}$$

Step 2. *Reduction to autonomous integrands.*

Once the equi-integrability of the relevant sequence is guaranteed we can end the proof following some arguments introduced by Marcellini and Sbordone in [50].

We start off fixing $\varepsilon > 0$ and noting that by the equi-integrability of (u_k) and (∇u_k) we can find $\delta = \delta_\varepsilon > 0$ such that for any set measurable E with $\mathcal{L}^n(E) \leq \delta$ it holds

$$\sup_k \int_E (1 + |u|^q + |\nabla u|^p + |u_k|^q + |\nabla u_k|^p)\,dx \leq \varepsilon. \tag{4.3}$$

Then, for all $m \in \mathbb{N}$ consider a partition of Ω into cubes Q^i_m of side m^{-1} and centre x^i_m. Denote by Ω_m the union of all those cubes Q^i_m, $i \in I$, contained in Ω, and set

$$[u]^m := \sum_{i\in I}\left(\fint_{Q^i_m} u\,dx\right)\chi_{Q^i_m}, \text{ and } \quad [\nabla u]^m := \sum_{i\in I}\left(\fint_{Q^i_m} \nabla u\,dx\right)\chi_{Q^i_m}.$$

Note that $[u]^m$ and $[\nabla u]^m$ are constant on each cube Q^i_m and null on $\Omega \setminus \Omega_m$. Furthermore, it is a standard fact that $([u]^m)$ converges strongly to u in $L^p(\Omega)$, and that $([\nabla u]^m)$ converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^{N\times n})$.

Hence, up to the extraction of subsequences as usual not relabeled, we may assume also that (u_k) and $([u]^m)$ both converge to u $\mathcal{L}^n$ a.e. on Ω, and $([\nabla u]^m)$ to ∇u $\mathcal{L}^n$ a.e. on Ω.

Therefore, by Egoroff-Severini and Scorza-Dragoni theorems we may find a compact subset K in Ω with $\mathcal{L}^n(\Omega\setminus K) \le \delta/2$, δ for which (4.3) holds, such that (u_k) and $([u]^m)$ converge to u in $L^\infty(K)$, $([\nabla u]^m)$ converges to ∇u in $L^\infty(K, \mathbb{R}^{N\times n})$, and f is continuous on $K \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$. In addition, we fix $M = M_\varepsilon > 0$ such that

$$\sup_k \mathcal{L}^n(\{|u| + |\nabla u| + |u_k| + |\nabla u_k| \ge M\}) \le \delta. \tag{4.4}$$

Hence, for every $\eta > 0$, supposing that m and k are sufficiently big so that

$$m^{-1} + \|u_k - u\|_{L^\infty(K)} + \|u - [u]^m\|_{L^\infty(K)} + \|\nabla u - [\nabla u]^m\|_{L^\infty(K)} \le \eta, \tag{4.5}$$

thanks to (4.3)-(4.5), we can finally estimate as follows

$$\begin{aligned}\int_\Omega f(x, u_k,\nabla u_k)\,dx &= \int_{K\cap\Omega_m} f(x, u_k, \nabla u_k)\,dx + O(\varepsilon)\\
&= \sum_{i\in I}\int_{\{K\cap Q^i_m:\,|u_k|+|\nabla u_k|\le M\}} f(x, u_k, \nabla u_k)\,dx + O(\varepsilon)\\
&= \sum_{i\in I}\int_{\{K\cap Q^i_m:\,|u_k|+|\nabla u_k|\le M\}} f\left(x^i_m, [u]^m, [\nabla u]^m + \nabla(u_k - u)\right)dx - \mathcal{L}^n(\Omega)\,\omega_M(\eta) + O(\varepsilon)\\
&= \sum_{i\in I}\int_{Q^i_m} f\left(x^i_m, [u]^m, [\nabla u]^m + \nabla(u_k - u)\right)dx - \mathcal{L}^n(\Omega)\,\omega_M(\eta) + O(\varepsilon),\end{aligned} \tag{4.6}$$

where ω_M is a modulus of continuity of f on the compact subset $K \times \{(u, \mathbb{A}) : |u| + |\mathbb{A}| \le M\}$.

To conclude it suffices to show that for all indices $i \in I$ we have

$$\liminf_k \int_{Q^i_m} f\left(x^i_m, [u]^m, [\nabla u]^m + \nabla(u_k - u)\right)dx \ge \int_{Q^i_m} f\left(x^i_m, [u]^m, [\nabla u]^m\right)dx. \tag{4.7}$$

Indeed, given this for granted, from (4.6) we infer that

$$\begin{aligned}\liminf_k \int_\Omega f(x, u_k, \nabla u_k)\,dx &\\
&\ge \sum_{i\in I}\int_{Q^i_m} f\left(x^i_m, [u]^m, [\nabla u]^m\right)dx - \mathcal{L}^n(\Omega)\,\omega_M(\eta) + O(\varepsilon)\\
&= \int_\Omega f(x, u, \nabla u)\,dx - \mathcal{L}^n(\Omega)\,\omega_M(\eta) + O(\varepsilon),\end{aligned}$$

where in the last equality we have used again (4.3)-(4.5). The lower semicontinuity inequality in (4.2) then follows by letting first $\eta \downarrow 0$ and then $\varepsilon \downarrow 0$.

Step 3. *Conclusion.*

To finish the proof we need to establish inequality (4.7) in the previous step. To this aim we note that for all $i \in I$ the integral functional to be considered is autonomous, i.e. it depends only on the gradient variable being $[u]^m$ constant on each cube Q^i_m. Moreover, by taking into account that $[\nabla u]^m$ satisfies the same property, the inequality has to be checked for affine target functions. Therefore, we can rephrase inequality (4.7) as

$$\int_Q g(\nabla w)dx \leq \liminf_k \int_Q g(\nabla w_k)dx$$

for a quasiconvex function g satisfying

$$0 \leq g(\mathbb{A}) \leq C(1+|\mathbb{A}|^p) \quad \text{for all } \mathbb{A}, \tag{4.8}$$

and an equi-integrable sequence (w_k) weakly converging to an affine function $w(x) := a + \mathbb{A}\cdot x$ on a cube Q.

With fixed $\varepsilon > 0$, by equi-integrability we can find an open subcube $Q' \subset\subset Q$ such that

$$\sup_k \int_{Q\setminus Q'} (1+|\nabla w|^p + |\nabla w_k|^p)\, dx \leq \varepsilon.$$

Let then $\varphi \in C^\infty_c(Q,[0,1])$ be such that $\varphi|_{Q'} = 1$, and define the functions $\varphi_k := (1-\varphi)w + \varphi\, w_k$. Clearly, (φ_k) converges weakly to w in $W^{1,p}$, and being $\varphi_k \in w + W^{1,p}_0(Q,\mathbb{R}^N)$ one can use it to test the quasiconvexity of g at $\mathbb{A}$ and get

$$\mathcal{L}^n(Q)\, g(\mathbb{A}) \leq \int_Q g(\nabla\varphi_k)\, dx \leq \int_{Q'} g(\nabla w_k)\, dx + C\int_{Q\setminus Q'} (1+|\nabla w|^p + |\nabla w_k|^p)\, dx,$$

for some positive constant C depending on $\|\nabla\varphi\|_\infty$. Therefore, the choice of Q' and the growth condition on g in (4.8) give

$$\mathcal{L}^n(Q)\, g(\mathbb{A}) \leq \liminf_k \int_Q g(\nabla w_k)\, dx + O(\varepsilon),$$

and the arbitrariness of $\varepsilon > 0$ provides the conclusion.

Acknowledgments

Authors have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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