A Remark on the Compactness for the Cahn-Hilliard Functional

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November 2, 2013

Abstract

In this note we prove compactness for the Cahn-Hilliard functional without assuming coercivity of the multi-well potential.

1 Introduction

The purpose of this note is to prove compactness for the Cahn–Hilliard functional (see [5], [8], [9]) without assuming coercivity of the multi-well potential W. Precisely, for $\varepsilon > 0$ consider the functional

$$F_\varepsilon : W^{1,2}\left(\Omega; \mathbb{R}^d\right) \to [0, \infty]$$

defined by

$$F_\varepsilon\left(u\right) := \int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx,$$

where $d \geq 1$ and the potential W satisfies the following hypotheses:

(H_1) $W : \mathbb{R}^d \to [0, \infty)$ is continuous, $W\left(z\right) = 0$ if and only if $z \in \{\alpha_1, \ldots, \alpha_\ell\}$ for some $\alpha_i \in \mathbb{R}^d$, $i = 1, \ldots, \ell$, with $\alpha_i \neq \alpha_j$ for $i \neq j$.

(H_2) There exists $L > 0$ such that

$$\inf_{|z| \geq L} W\left(z\right) > 0.$$

Then the following result holds.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded connected set with Lipschitz boundary. Assume that the multi-well potential W satisfies conditions (H_1) and (H_2). Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}\left(\Omega; \mathbb{R}^d\right)$ be such that*

$$M := \sup_n F_{\varepsilon_n}\left(u_n\right) < \infty \tag{1.1}$$

and

$$\frac{1}{|\Omega|}\int_{\Omega} u_n(x)\,dx = m \quad \textit{for all } n \in \mathbb{N} \tag{1.2}$$

and for some $m \in \mathbb{R}^d$*. Then there exist* $u \in BV(\Omega; \{\alpha_1, \ldots, \alpha_\ell\})$ *and a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that*

$$u_{n_k} \to u \textit{ in } L^1\left(\Omega; \mathbb{R}^d\right).$$

For a two-well potential ($\ell = 2$), Theorem 1.1 has been proved in the scalar case $d = 1$ by Modica [8] under the assumption

$$\frac{1}{C}|z|^p \leq W(z) \leq C|z|^p$$

for all $|z|$ large and for some $p > 2$, and by Sternberg [9] for $p \geq 2$; while in the vectorial case $d \geq 2$, it has been proved by Fonseca and Tartar [4] under the assumption

$$\frac{1}{C}|z| \leq W(z)$$

for all $|z|$ large. The case of a multi-well potential $\ell \geq 3$ has been studied by Baldo (see Propositions 4.1 and 4.2 in [2])), who proved compactness of a sequence of minimizers bounded in $L^\infty(\Omega)$.

An example of a double-well potential satisfying (H_1) and (H_2) but not coercive is

$$W(z) = \arctan\left[(z-\alpha)^2(z-\beta)^2\right],$$

while an example of a potential satisfying (H_1) but not (H_2) is

$$W(z) = (z-\alpha)^2(z-\beta)^2 e^{-|z|^2}.$$

In the one dimensional case $N = 1$, the hypothesis (1.2) is not needed. Indeed, we have the following elementary result.

Theorem 1.2 *Assume that the multi-well potential* W *satisfies conditions* (H_1) *and* (H_2)*. Let* $\varepsilon_n \to 0^+$ *and let* $\{u_n\} \subset W^{1,2}\left((a,b);\mathbb{R}^d\right)$ *be such that* (1.1) *holds. Then there exist* $u \in BV((a,b);\{\alpha_1,\ldots,\alpha_\ell\})$ *and a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that*

$$u_{n_k} \to u \textit{ in } L^1\left((a,b);\mathbb{R}^d\right).$$

On the other hand, when (1.2) holds, then condition (H_2) can be weakened to:

(H_3) $\int_0^\infty \sqrt{V(s)}\,ds = \infty$, where for every $s \geq 0$,

$$V(s) := \min_{|z|=s} W(z). \tag{1.3}$$

Note that (H_2) implies that $\sqrt{V(s)} \geq \inf_{|z|\geq L} \sqrt{W(z)} > 0$ for all $s \geq L$, and so (H_3) is satisfied. On the other hand, if

$$W(z) \sim \frac{c}{|z|^q}$$

as $|z| \to \infty$ for some $c > 0$ and $0 < q \leq 2$, then (H_3) holds but not (H_1).

Theorem 1.3 *Assume that the multi-well potential W satisfies conditions (H_1) and (H_3). Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}\left((a,b);\mathbb{R}^d\right)$ be such that (1.1) and (1.2) hold. Then there exist $u \in BV((a,b);\{\alpha_1,\ldots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and such that*

$$u_{n_k} \to u \text{ in } L^1\left((a,b);\mathbb{R}^d\right).$$

The next simple example shows that compactness fails without (1.2) or (H_2).

Example 1.4 *If condition (H_2) does not hold, then there exists $\{z_n\} \subset \mathbb{R}^d$ such that $|z_n| \to \infty$ and*

$$\lim_{n\to\infty} W(z_n) = 0.$$

Find a sequence $\varepsilon_n \to 0$ such that

$$\frac{1}{\varepsilon_n} W(z_n) \to 0,$$

(e.g. $\varepsilon_n := \sqrt{W(z_n)}$) and consider the sequence of functions $u_n(x) :\equiv z_n$. Then

$$F_{\varepsilon_n}(u_n) = \frac{1}{\varepsilon_n} W(z_n)(b-a) \to 0$$

but no subsequence of $\{u_n\}$ converge in $L^1((a,b))$.

Remark 1.5 *I have not been able to determine if Theorems 1.2 and 1.3 hold in dimension $N \geq 2$ or if (H_3) is needed in Theorem 1.3.*

2 Proof of Theorems 1.1 and 1.2

The proof of Theorem 1.1 will make use of the following auxiliary results. For a proof of the following theorem see, e.g., Proposition 16.21 in [6].

Theorem 2.1 *Let $u \in W^{1,1}\left(\mathbb{R}^N\right)$, $N \geq 2$. Then*

$$\sup_{s>0} s\left[\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : |u(x)| \geq s\right\}\right)\right]^{\frac{N-1}{N}} \leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla u(x)|\, dx.$$

For a proof of the next theorem, see Lemma 2.6 in [1].

Theorem 2.2 *Let $A, \Omega \subset \mathbb{R}^N$ be open sets and let $1 \leq p < \infty$. Assume that A is bounded and that Ω is connected and has Lipschitz boundary at each point of $\partial\Omega \cap \overline{A}$. Then there exists a linear and continuous operator $T : W^{1,p}(\Omega) \to W^{1,p}(A)$ such that, for every $u \in W^{1,p}(\Omega)$,*

$$T(u)(x) = u(x) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \cap A,$$
$$\int_A |T(u)(x)|^p \, dx \leq C \int_\Omega |u(x)|^p \, dx,$$
$$\int_A |\nabla T(u)(x)|^p \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx,$$

where $C > 0$ depends only on N, p, A, and Ω.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of (1.1) and (H_2) for every $n \in \mathbb{N}$, we have

$$\begin{aligned} M &\geq \frac{1}{2}\int_\Omega \sqrt{W(u_n(x))}|\nabla u_n(x)|\,dx \\ &\geq c\int_{\{|u_n|\geq L\}} |\nabla u_n(x)|\,dx, \end{aligned} \tag{2.1}$$

where $c := \frac{1}{2}\inf_{|z|\geq L}\sqrt{W(z)} > 0$. Construct a C^1 function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(z) = z$ if $|z| \geq 2L$ and $f(z) = 0$ if $|z| < L$. By the chain rule, for every $n \in \mathbb{N}$ the function $v_n := f \circ u_n$ belongs to $W^{1,2}(\Omega; \mathbb{R}^d)$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $x \in \Omega$,

$$\frac{\partial v_n}{\partial x_i}(x) = \sum_{j=1}^{d} \frac{\partial f}{\partial z^{(j)}}(u_n(x))\,\frac{\partial (u_n)^{(j)}}{\partial x_i}(x),$$

where we write $z = \left(z^{(1)}, \ldots, z^{(d)}\right)$. Since $\frac{\partial f}{\partial z^{(j)}}(z) = 0$ if $|z| < L$, it follows that

$$\begin{aligned} \int_\Omega |\nabla v_n(x)|\,dx &= \int_{\{|u_n|\geq L\}} |\nabla v_n(x)|\,dx \\ &\leq \operatorname{Lip} f \int_{\{|u_n|\geq L\}} |\nabla u_n(x)|\,dx \leq c^{-1} M \operatorname{Lip} f. \end{aligned} \tag{2.2}$$

Let $r > 0$ be so large that $\overline{\Omega} \subset B(0, r)$ and set $A := B(0, 2r)$. By Theorem 2.2 we may extend each function v_n to a function in $W^{1,1}(A; \mathbb{R}^d)$, still denoted v_n, in such a way that

$$\int_A |v_n(x)|\,dx \leq C \int_\Omega |v_n(x)|\,dx, \tag{2.3}$$

$$\int_A |\nabla v_n(x)|\,dx \leq C \int_\Omega |\nabla v_n(x)|\,dx \leq C c^{-1} M \operatorname{Lip} f, \tag{2.4}$$

where C depends only on r, N, and Ω. By the Poincaré inequality,

$$\int_A |v_n(x) - c_n| \, dx \le C \int_A |\nabla v_n(x)| \, dx, \tag{2.5}$$

where $c_n := \frac{1}{|\Omega|} \int_\Omega v_n(x) \, dx$ and again C depends only on r, N, and Ω. Note that, since $f(z) = z$ if $|z| \ge 2L$,

$$\begin{aligned} |c_n| &= \frac{1}{|\Omega|} \left| \int_\Omega f \circ u_n \, dx \right| = \frac{1}{|\Omega|} \left| \int_{\{|u_n| > 2L\}} u_n \, dx + \int_{\{|u_n| \le 2L\}} f \circ u_n \, dx \right| \\ &= \left| m + \frac{1}{|\Omega|} \int_{\{|u_n| \le 2L\}} (f \circ u_n - u_n) \, dx \right| \le |m| + 4L. \end{aligned}$$

Consider a cut-off function $\varphi \in C_c^\infty(A; [0,1])$ such that $\varphi = 1$ in $B(0, r)$ and define

$$w_n := \varphi (v_n - c_n).$$

Then $w_n \in W^{1,1}(\mathbb{R}^N)$ and by (2.5),

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_n(x)| \, dx &\le \operatorname{Lip} \varphi \int_A |v_n - c_n| \, dx + \int_A |\nabla v_n(x)| \, dx \\ &\le (C \operatorname{Lip} \varphi + 1) \int_A |\nabla v_n(x)| \, dx. \end{aligned} \tag{2.6}$$

Applying Theorem 2.1 to $|w_n|$, we obtain

$$\begin{aligned} \sup_{s>0} s \left[\mathcal{L}^N \left(\left\{ x \in \mathbb{R}^N : |w_n|(x) \ge s \right\} \right) \right]^{\frac{N-1}{N}} &\le \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla |w_n|(x)| \, dx \\ &\le C_1 \int_{\{|u_n| \ge L\}} |\nabla u_n(x)| \, dx \le C_2, \end{aligned}$$

where we have used (2.2), (2.4), and (2.6).

Fix $s_1 > 2(|m| + 4L) + 1$. Using the facts that $\varphi = 1$ in $B(0, r)$, that $f(z) = z$ if $|z| \ge 2L$, and that $|c_n| \le |m| + 4L$, for $s \ge s_1$ we have

$$\begin{aligned} \{x \in \Omega : |u_n(x)| \ge s\} = \{x \in \Omega : |v_n(x)| \ge s\} &\subset \left\{ x \in \Omega : |v_n(x) - c_n| \ge \frac{s}{2} \right\} \\ &\subset \left\{ x \in \mathbb{R}^N : |w_n|(x) \ge \frac{s}{2} \right\}, \end{aligned}$$

and so

$$\mathcal{L}^N(\{x \in \Omega : |u_n(x)| \ge s\}) \le \frac{C}{s^{\frac{N}{N-1}}}$$

for all $s \ge s_1$. Hence,

$$\begin{aligned} \int_{\{|u_n| > s_1\}} |u_n(x)| \, dx &= \int_{s_1}^\infty \mathcal{L}^N(\{x \in \Omega : |u_n(x)| \ge s\}) \, ds \\ &\le C \int_{s_1}^\infty \frac{1}{s^{\frac{N}{N-1}}} \, d\tau = \frac{N-1}{s_1^{\frac{1}{N-1}}}, \end{aligned}$$

which shows that $\{u_n\}$ is bounded in $L^1\left(\Omega;\mathbb{R}^d\right)$ and equi-integrable.

In view of Vitali's convergence theorem, it remains to show that a subsequence converges in measure to some function $u \in BV(\Omega;\{\alpha_1,\ldots,\alpha_\ell\})$. This is classical (see e.g. [2] or [4]). ■

Remark 2.3 *Theorem 1.1 continues to hold if in place of* (1.2) *we assume that*

$$u_n = g \quad \text{on } \partial\Omega \tag{2.7}$$

for all $n \in \mathbb{N}$ and for some function $g \in L^1(\partial\Omega;\{\alpha_1,\ldots,\alpha_\ell\})$. In this case, by Gagliardo's trace theorem (see, e.g. Theorem 15.10 in [6]) there exists a function $w \in W^{1,1}\left(\mathbb{R}^N \setminus \Omega;\mathbb{R}^d\right)$ such that $w = g$ on $\partial\Omega$. Extend each u_n to be w outside Ω. We can now apply Theorem 2.1 directly to $f \circ u_n \in W^{1,1}\left(\mathbb{R}^N;\mathbb{R}^d\right)$ without introducing the constants c_n, the function φ, and without using Theorem 2.2.

We now turn to the proof of Theorem 1.2. The argument below is likely well-known. We present it for the convenience of the reader.

Proof of Theorem 1.2. Without loss of generality, we can assume that each function u_n is absolutely continuous.

Since the set $A_n := \{x \in (a,b) : |u_n(x)| > L\}$ is open, we may write it as

$$A_n = \bigcup_k (a_{k,n}, b_{k,n}).$$

Moreover, by (1.1) and (H_2), for every $n \in \mathbb{N}$, we have

$$M\varepsilon_n \geq \int_a^b W(u_n(x))\,dx \geq |A_n| \inf_{|z|\geq L} W(z),$$

and so its complement $(a,b)\setminus A_n$ is nonempty for all n sufficiently large. Fix any such n. If A_n is empty, then $|u_n(x)| \leq L$ for all $x \in (a,b)$. Otherwise, let $x \in (a_{k,n}, b_{k,n})$. Then at least one of the endpoints, say $a_{k,n}$, is not an endpoint of (a,b) and so $|u_n(a_{k,n})| = L$. By the fundamental theorem of calculus,

$$u_n(x) = u_n(a_{k,n}) + \int_{a_{k,n}}^x u'(t)\,dt.$$

Hence,

$$\sup_{x\in(a_{k,n},b_{k,n})} |u_n(x)| \leq L + \int_{\{|u_n|\geq L\}} |u_n'(t)|\,dt \leq L + c^{-1}M,$$

where we have used (2.1). This shows that $\{u_n\}$ is bounded in $L^\infty\left((a,b);\mathbb{R}^d\right)$. We can now continue as in Lemma 6.2 of [3]. ■

Finally, we prove Theorem 1.3.

Proof of Theorem 1.2. Without loss of generality, we can assume that each function u_n is absolutely continuous. In view of (1.1) and (1.3), for every $n \in \mathbb{N}$ we have

$$M \geq \frac{1}{2}\int_a^b \sqrt{W(u_n(x))}|u_n'(x)|\,dx \geq \frac{1}{2}\int_a^b \sqrt{V(|u_n|(x))}\,|u_n|'(x)\,|\,dx.$$

Using the area formula for absolutely continuous functions (see, e.g., Theorem 3.65 in [6]), we obtain

$$\begin{aligned} M &\geq \frac{1}{2}\int_a^b \sqrt{V(|u_n|(x))}\,|\,|u_n|'(x)\,|\,dx = \frac{1}{2}\int_{\mathbb{R}} \sqrt{V(s)}\,\mathrm{card}\,|u_n|^{-1}(\{s\})\,ds \\ &\geq \frac{1}{2}\int_{\min|u_n|}^{\max|u_n|} \sqrt{V(s)}\,ds, \end{aligned}$$

where card is the cardinality and $|u_n|^{-1}(\{s\}) = \{x \in (a, b : |u_n(x)| = s)\}$. By (1.2) and the intermediate value theorem, there exists $x_n \in (a, b)$ such that

$$u_n(x_n) = \frac{1}{b-a}\int_a^b u_n(x)\,dx = \frac{m}{b-a}.$$

Hence, $|u_n(x_n)| = \frac{|m|}{b-a}$, which implies that

$$M \geq \frac{1}{2}\int_{\frac{|m|}{b-a}}^{\max|u_n|} \sqrt{V(s)}\,ds.$$

By (H_3) there exists $R > 0$ such that $\int_{\frac{|m|}{b-a}}^{R} \sqrt{V(s)}\,ds > 2M$. In turn, $|u_n(x)| < R$ for all $x \in (a, b)$ and all $n \in \mathbb{N}$. This shows that $\{u_n\}$ is bounded in $L^\infty((a, b); \mathbb{R}^d)$. ■

Remark 2.4 *Observe that in Theorems 1.2 and 1.3 we can replace* (H_1) *with the weaker hypothesis*

(H_4) $W : \mathbb{R}^d \to [0, \infty)$ *is continuous and for every* $r > 0$ *the set*

$$\{z \in B(0, r) : W(z) = 0\}$$

has finitely many elements.

Indeed, if $\{u_n\} \subset W^{1,2}((a, b); \mathbb{R}^d)$ *is such that* (1.1) *holds, then by Theorem 1.2 or 1.3, there exists* $R > 0$ *such that* $|u_n(x)| < R$ *for all* $x \in (a, b)$ *and all* $n \in \mathbb{N}$*. Find* $S \in (R, 2R)$ *such that* $V(S) > 0$*. Note that such* S *exists, since otherwise we would have* $V(s) = 0$ *for all* $s \in (R, 2R)$*, which would imply that* $\{z \in B(0, 2R) : W(z) = 0\}$ *has infinitely many elements and would contradict* (H_4)*. Define*

$$W_1(z) := \begin{cases} W(z) & \text{if } |z| < S, \\ W\left(\frac{z}{|z|}S\right) & \text{if } |z| \geq S. \end{cases}$$

Since $|u_n(x)| < R < S$ *for all* $x \in (a, b)$ *and all* $n \in \mathbb{N}$*, we have that*

$$M \geq F_{\varepsilon_n}(u_n) = \int_a^b \left(\frac{1}{\varepsilon_n}W_1(u_n) + \varepsilon_n|u_n'|^2\right)\,dx.$$

The function W_1 satisfies hypotheses (H_1) and (H_2). Hence, we may now apply Theorem 1.2 to find $u \in BV((a,b);\{\alpha_1,\dots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \to u \text{ in } L^1\left((a,b);\mathbb{R}^d\right).$$

Here $\{\alpha_1,\dots,\alpha_\ell\}$ are the zeros of W in $B(0,s)$.

In view of the previous remark, we can prove a compactness result for $N = 1$ and bounded domains for the functional studied in the classical paper of Modica and Mortola [7].

Corollary 2.5 *Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}\left((a,b);\mathbb{R}^d\right)$ be such that*

$$\int_a^b \left(\frac{1}{\varepsilon_n}\sin^2(\pi u_n) + \varepsilon_n |u_n'(x)|^2\right) dx \le M$$

and (1.2) hold. Then there exist $u \in BV((a,b);\{\alpha_1,\dots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \to u \text{ in } L^1(a,b).$$

Here, $\{\alpha_1,\dots,\alpha_\ell\} \subset \mathbb{Z}$.

Proof. It is enough to observe that the function $W(z) = \sin^2(\pi z)$ satisfies (H_3) and (H_4). ■

Remark 2.6 *I am not aware of any compactness result for $N \ge 2$ for the functional*

$$\int_\Omega \left(\frac{1}{\varepsilon}\sin^2(\pi u) + \varepsilon|\nabla u|^2\right) dx,$$

when (1.2) holds. Note that $W(z) = \sin^2(\pi z)$ satisfies (H_3) and (H_4) but not (H_1) and (H_2).

Acknowledgments

The research of G. Leoni was partially funded by the National Science Foundation under Grant No. DMS-1007989. G. Leoni also acknowledges the Center for Nonlinear Analysis (NSF Grant No. DMS-0635983, PIRE Grant No. OISE-0967140), where part of this research was carried out. The author wishes to thank Massimiliano Morini for useful conversations on the subject of this paper.

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