A QUANTITATIVE CHARACTERISATION OF FUNCTIONS WITH LOW AVILES GIGA ENERGY ON CONVEX DOMAINS

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Abstract. Given a connected Lipschitz domain Ω we let $\Lambda(\Omega)$ be the subset of functions in $W^{2,2}(\Omega)$ whose gradient (in the sense of trace) satisfies $\nabla u(x)\cdot\eta_x = 1$ where η_x is the inward pointing unit normal to $\partial\Omega$ at x. The functional $I_\epsilon(u) = \frac{1}{2}\int_\Omega \epsilon^{-1}\left|1-|\nabla u|^2\right|^2 + \epsilon\left|\nabla^2 u\right|^2$ minimised over $\Lambda(\Omega)$ serves as a model in connection with problems in liquid crystals and thin film blisters, it is also the most natural higher order generalisation of the Modica Mortola functional. In [Ja-Ot-Pe 02] Jabin, Otto, Perthame characterised a class of functions which includes all limits of sequences $u_n \in \Lambda(\Omega)$ with $I_{\epsilon_n}(u_n)\to 0$ as $\epsilon_n \to 0$. A corollary to their work is that if there exists such a sequence (u_n) for a bounded domain Ω, then Ω must be a ball and $u := \lim_{n\to\infty} u_n = \mathrm{dist}(\cdot,\partial\Omega)$. We prove a quantitative generalisation of this corollary for the class of bounded convex sets.

There exists positive constant γ_1 such that if Ω is a convex set of diameter 2 and $u \in \Lambda(\Omega)$ with $I_\epsilon(u) = \beta$ then $|B_1(x)\triangle\Omega| \le c\beta^{\gamma_1}$ for some x and

$$\int_\Omega \left|\nabla u(z) - \frac{z-x}{|z-x|}\right|^2 dz \le c\beta^{\gamma_1}.$$

A corollary of this result is that there exists positive constant $\gamma_2 < \gamma_1$ such that if Ω is convex with diameter 2 and C^2 boundary with curvature bounded by $\epsilon^{-\frac{1}{5}}$, then for any minimiser v of I_ϵ over $\Lambda(\Omega)$,

$$\|v-\zeta\|_{W^{2,2}(\Omega)} \le c(\epsilon + \inf_y |\Omega\triangle B_1(y)|)^{\gamma_2}$$

where $\zeta(z) = \mathrm{dist}(z,\partial\Omega)$. Neither of the constants γ_1 or γ_2 are optimal.

1. Introduction

We consider the following functional

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega \epsilon^{-1}\left|1-|\nabla u|^2\right|^2 + \epsilon\left|\nabla^2 u\right|^2 \tag{1}$$

the study of which arises from a number of sources, one of the earliest and most important is the article by Aviles, Giga [Av-Gi 87]. We will refer to the quantity $I_\epsilon(u)$ as the Aviles-Giga energy of functional u. Functional I_ϵ is usually minimised over the space of functions $u \in W^{2,2}(\Omega)$ where $\nabla u(x)\cdot\eta_x = 1$ in $\partial\Omega$ (in the sense of trace) where η_x is the inward pointing unit normal, we will denote this space of functions by $\Lambda(\Omega)$.

Aviles, Giga raised the problem of the study of the limiting behavior of I_ϵ as $\epsilon \to 0$ in connection with the theory of smectic liquid crystals [Av-Gi 87]. In [Gi-Or 97] Gioia, Ortiz studied I_ϵ as a model for thin film blisters. Jin, Kohn [Ji-Ko 00] introduced the by now classic method of estimating the energy by 'divergence of vectorfields'. A related functional arising from micromagnetics was studied by Riviere, Serfaty [Ri-Se 01], in this case the functional acts on vector fields m (in two dimensions) satisfying $|m| = 1$ in Ω and the functional is given by $M_\epsilon(m) = \epsilon\int_\Omega |\nabla m|^2 + \epsilon^{-1}\int_{\mathbb{R}^2}\left|\nabla^{-1}\mathrm{div}\tilde{m}\right|^2$ where $\tilde{m}$ is vectorfield m extended trivially by 0 outside Ω. For the Aviles Giga functional we minimise over curl free vector fields and the

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functional forces the norm of the vector field to be 1 with weighting ϵ^{-1} while constraining an ϵ multiple of the L^2 norm of the gradient, on the other hand the micromagnetics functional is minimised over vectorfields whose norm is taken to be 1 from the outset and the functional forces the vector field to be divergence free with weighting ϵ^{-1} [1] while again constraining an ϵ multiple of the L^2 norm of the gradient. Functional M_ϵ is much more rigid and very much stronger results are known for it than for I_ϵ, see [Al-Ri-Se 02],[Ri-Se 01],[Am-Ki-Ri 02], [Am-Le-Ri 03].

Roughly speaking, the conjecture is that as $\epsilon \to 0$ the energy of minimisers of I_ϵ will converge to a collection of curves on which the gradient of the minimisers make a jump of order $o(1)$ perpendicularly across the curve. This has already been proved for functional M_ϵ [Ri-Se 01]. A way to think about this is the following, given a connected Lipschitz domain Ω let w be the distance from $\partial\Omega$ and let v_ϵ be w convolved by a convolution kernel of diameter ϵ, the regions where $|\nabla v_\epsilon| \not\sim 1$ will be exactly the ϵ neighborhoods of the curves on which ∇w has a jump discontinuity. If Ω is a ball ∇w will have a discontinuity only at one point, in all other cases there will be non trivial curves of singularities and for the specific function v_ϵ, it is exactly in an ϵ neighborhood of these curves that the energy will concentrate. The conjecture is that what we can observe directly for v_ϵ will hold true for the minimisers of I_ϵ

The most natural way to study these questions is within the frame work of Γ convergence. One of the earliest successes of Γ convergence was the characterisation of the Γ limit of the so called Modica Mortola functional $A_\epsilon(w) = \int_\Omega \epsilon |\nabla w|^2 + \epsilon^{-1} \left|1 - |w|^2\right|^2$ which is minimised over scalar functions w satisfying an integral condition of the form $\int_\Omega w = 0$. It was shown by Modica, Mortola [Mo-Mo 00] (confirming a conjecture of DeGiorgi) that the Γ limit of A_ϵ is a constant multiple of the H^{n-1} measure of the jump set J_w minimised over the space of functions $w \in \left\{v \in BV : v \in \{1,-1\}\ a.e. \text{ and } \int v = 0\right\}$. Given the elementary inequality

$$\epsilon |\nabla w|^2 + \epsilon^{-1} \left|1 - |w|^2\right|^2 \geq |\nabla w| \left|1 - |w|^2\right| \tag{2}$$

we have that for any sequence (w_n) of equibounded A_{ϵ_n} energy (for some subsequence $\epsilon_n \to 0$) has a uniform L^1 control of ∇w_n and the measure we obtain as the limit of this L^1 sequence of gradients will naturally be supported on the jump set of the limiting function. In some sense the nature of the Γ limit of A_ϵ could be anticipated from (2).

Functional I_ϵ is the most natural higher order generalisation of A_ϵ, in the case of I_ϵ the conjectured Γ limit is surprising, this is part of the reason that functional I_ϵ has received so much attention. The first works on identifying the Γ limit are by Aviles, Giga [Av-Gi 87] and Jin, Kohn [Ji-Ko 00], later these ideas were developed by Ambrosio, DeLellis, Mantegazza [Am-De-Ma 99], roughly speaking the limiting function space is conjectured to have a structure similar to the space of functions whose gradient is BV and the limiting energy is conjectured to have the form $\int_{J_{\nabla u}} |\nabla u^+ - \nabla u^-|^3 \, dH^1$. Much progress has been made on this conjecture, particularly equi-coercivity of I_ϵ has been shown independently in [Am-De-Ma 99] and in the work of Desimone, Kohn, Muller, Otto [De-Ko-Mu-Ot 00]. A proposed limiting function space $AG(\Omega)$ and limiting functional I as been suggested in [Am-De-Ma 99] and it was shown that all limits of sequences of functions (u_n) with $\sup_n I_{\epsilon_n}(u_n) < \infty$ are such that $u_n \overset{W^{1,3}}{\to} u \in AG(\Omega)$ and $\liminf I_{\epsilon_n}(\nabla u_n) \geq I(u)$. The compactness proofs provided by [De-Ko-Mu-Ot 00] and [Am-De-Ma 99] are different but share some common ideas. The proof by [De-Ko-Mu-Ot 00] identifies the set of functions Φ

$$\int |\nabla \Phi(\nabla u)| \leq c \int \left|\nabla^2 u\right| \left|1 - |\nabla u|^2\right| \text{ for any } C^2 \text{ function } u, \tag{3}$$

[1] the term $\int_{R^2} \left|\nabla^{-1} \text{div} m\right|^2$ is the L^2 norm of the Hodge projection onto curl free vector fields

influenced by ideas of Tartar and Murat on compensated compactness [Ta 79] [Mu 78] the authors are able to prove that this set of Φ is sufficiently rich so as to force ∇u_n to converge strongly. In [Av-Gi 87] the authors (building on work of Jin Kohn [Ji-Ko 00]) found two third order polynomials Σ_1 and Σ_2 such that

$$\int |\mathrm{div}\Sigma_i(\nabla u)| \leq c \int \left|\nabla^2 u\right| \left|1-|\nabla u|^2\right| \text{ for any } C^2 \text{ function } u, \text{ for } i=1,2. \tag{4}$$

Using some elementary and surprising identities satisfied by $\Sigma_1(\nabla u), \Sigma_2(\nabla u)$ a different approach to compactness was found. Rather naturally considering (4), the function space $AG(\Omega)$ proposed by [Am-De-Ma 99] is given by the set of functions v for which $\mathrm{div}(\Sigma_i(\nabla v))$ forms a Radon measure for $i=1,2$ and the limiting energy functional $I(v)$ is given by the total absolute value of this measure on Ω.

Given vector field w let $\chi(\xi,w):=1\!\!1_{\{\xi\cdot w>0\}}$, Jabin, Perthame [Ja-Pe 97] showed that gradients of sequences of bounded Aviles-Giga energy (in fact their method extends to more general functionals) are compact and the limit ∇u satisfies a kinetic equation of the form $\xi\cdot\nabla_x\chi(\xi,R(\nabla u))=m$ where m is a measure on $\mathbb{R}^2_\xi\times\mathbb{R}^2_x$ and R is the rotation given by $R(x,y)=(-y,x)$. By application of kinetic averaging lemmas [Di-Li-Me 91] this leads to some regularity; $\nabla u\in W^{s,q}$ for all $0\leq s<\frac{1}{5}$, $q<\frac{5}{3}$ and using the kinetic equation a different proof of compactness was found. The kinetic equation deduced by [Ja-Pe 97] was motivated by the characterisation of the set of Φ satisfying (3) given in [De-Ko-Mu-Ot 00], indeed defining $\tilde{\Phi}(z)=|z|^2 e$ for $z\cdot e>0$ and 0 otherwise, in [De-Ko-Mu-Ot 00] it was shown that a sequence Φ_n satisfying (2) could be found that approximates $\tilde{\Phi}$ uniformly. Using the kinetic equation deduced in [Ja-Pe 97], Jabin, Otto, Perthame [Ja-Ot-Pe 02] were able to characterise zero energy limits (and the domains that allow them) for I_ϵ, in fact their result is stronger, they showed that if a divergence free vector field m satisfies the kinetic equation $\xi\cdot\nabla\chi(m,\xi)=0$, $|m(x)|=1$ a.e. in Ω and $m(x)\cdot\eta_x=0$ on $\partial\Omega$ then either Ω is a strip and m is a constant or $\Omega=B_r(x)$ for some $r>0$, $x\in\mathbb{R}^2$ and $m(z)=\left(\frac{z-x}{|z-x|}\right)^\perp$. An analogous result for zero energy limits of M_ϵ is stated in [Le-Ri 02] and is a consequence of the main theorem of [Am-Le-Ri 03].

As a corollary, given a sequence $u_n\in\Lambda(\Omega)$ and $\epsilon_n\to 0$ such that $I_{\epsilon_n}(u_n)\to 0$ as $n\to\infty$, letting u be the limit of this sequence, the vector field $R(\nabla u)$ satisfies the hypothesis stated and hence we have a complete description of ∇u.

The main theorem of this paper is a quantitative generalisation of the corollary to Jabin, Otto, Perthame theorem over the class of bounded convex sets.

Theorem 1. *Let $\epsilon>0$ and Ω be a convex domain with diameter 2. Let $u\in W^{2,2}(\Omega)$ with $\nabla u(x)\cdot\eta_x=1$ of $\partial\Omega$ (in the sense of trace) where η_x is the inward pointing unit normal. There exists positive constants $\mathcal{C}>1$ and $\gamma<1$ such that for some $x\in\Omega$,*

$$|\Omega\triangle B_1(x)|\leq\mathcal{C}\left(I_\epsilon(u)\right)^\gamma$$

and

$$\int_\Omega\left|\nabla u(z)-\frac{z-x}{|z-x|}\right|^2 dz\leq\mathcal{C}\left(I_\epsilon(u)\right)^\gamma.$$

Corollary 1. *Let Ω be a convex set with diameter 2, C^2 boundary and curvature bounded above by $\epsilon^{-\frac{1}{5}}$. Let $\Lambda(\Omega):=\left\{u\in W^{2,2}(\Omega):\nabla u(z)\cdot\eta_z=1 \text{ for } z\in\partial\Omega\right\}$. There exists positive constants $\mathcal{C}>1$ and $\lambda<1$ such that if u is a minimiser of I_ϵ over $\Lambda(\Omega)$, then*

$$\|u-\zeta\|_{W^{2,2}(\Omega)}\leq\mathcal{C}\left(\epsilon+\inf_y|\Omega\triangle B_1(y)|\right)^\lambda \tag{5}$$

where $\zeta(z)=\mathrm{dist}(z,\partial\Omega)$.

In Theorem 1 we take $\gamma = 512^{-1}$ and in Corollary 1, $\lambda = 3278^{-1}$. Neither constant is optimal.

1.1. **Background.** Given a sequence $\epsilon_n \to 0$ and $u_n \in \Lambda(\Omega)$ with $\limsup I_{\epsilon_n}(u_n) < \infty$, let u be the limit of u_n, the vector valued measure given by $\nu_u := (\mathrm{div}\Sigma_1(\nabla u), \mathrm{div}\Sigma_2(\nabla u))$ (where Σ_1, Σ_2 are the third order polynomials that satisfy (4)) gives us the expression of the limiting energy, i.e. $I(u) = \|\nu_u\|(\Omega)$. If we consider the 1-dimensional part of the measure

$$\Gamma := \left\{ x : \limsup_{r\to 0} \frac{\|\nu_u(B_r(x))\|}{r} > 0 \right\}$$

it has been shown that Γ is 1-rectifiable [De-Ot 03] (see also [De-Ot-We 03]) and an analogous result has been shown for M_ϵ [Am-Ki-Ri 02]. It was also shown ∇u has jump discontinuous across the rectifiable set Γ exactly as would be the case if ∇u was BV and its jump set was given by Γ. However it is not known (even if u_n are the minimisers of I_{ϵ_n}) if measure $\|\nu_u\|$ is even singular with respect to Lebesgue measure. Note that for the function M_ϵ the minimiser of the limiting energy is known to be rectifiable [Am-Le-Ri 03], for a sequence with only equibounded energy the measure is not known to be singular.

The original motivation for Theorem 1 was to prove a version of it for $\Omega = B_1(0)$ without boundary conditions, under the hypotheses $\int_{B_1} \left|1 - |\nabla u|^2\right| \left|\nabla^2 u\right| = \beta$, $\int_{B_1} \left|1 - |\nabla u|^2\right| \leq \epsilon$ and $\sup \left\{ \|u - A\|_{L^\infty(B_1(0))} : A \text{ is affine with } |\nabla A| = 1 \right\} \leq 1000^{-1}$, the conclusion in this case would be that there exists a smooth function ψ with $|\nabla \psi| = 1$ everywhere such that $\|\nabla u - \nabla \psi\|_{L^2\left(B_{2^{-1}}(0)\right)} \leq c\beta^\gamma$ for some $\gamma > 0$. This is a kind of quantitative version of the main proposition required to prove compactness in [Am-De-Ma 99], (see Proposition 4.6). The hope is to use such a quantitative result to show $\|\nu_u\|$ is singular, or at least that ∇u is continuous at H^1 a.e. point outside Γ, we will address these issues in a forthcoming paper [Lo pr].

The many strong results about measure $\|\nu_u\|$ (and the measure that gives the limiting functional for the micromagnetics function) have been achieved by characterising various kinds of *blow up* of the measure and understanding well the absolute (i.e. non quantitative) situation in the limit [Am-Ki-Ri 02], [De-Ot 03], [De-Ot-We 03], [Ja-Ot-Pe 02], [Am-Le-Ri 03]. In some sense there are only two possibilities, to take a limit and have an absolute situation and to understand the measure from this, or to stop before the limit and have a non-absolute situation and try and understand something about it with a quantitative theorem. Our primary motivation in proving a quantitative version of Jabin-Otto-Perthame Theorem was so as to obtain a result that could be used for the latter approach.

By Poincare's inequality it is easy to see $\inf_{\Lambda(\Omega)} I_\epsilon \geq c\epsilon$ and so Theorem 1 follows from the following slightly more general result.

Theorem 2. *Let Ω be a convex body centered on 0 with $\mathrm{diam}(\Omega) = 2$. Let $\beta > 0$, suppose $u : W^{2,2}(\Omega) \to \mathbb{R}$ is a function satisfying*

$$\int_\Omega \left|1 - |\nabla u|^2\right| \left|\nabla^2 u\right| dz = \beta \tag{6}$$

and

$$\int_\Omega \left|1 - |\nabla u|^2\right|^2 dz \leq \beta^2 \tag{7}$$

and in addition u satisfies $\nabla u(z) \cdot \eta_z = 1$ on $\partial\Omega$ in the sense of trace where η_z is the inward pointing unit normal to $\partial\Omega$ at z.

There exists positive constant $\mathcal{C}_1 > 0$ such that $|B_1(0) \triangle \Omega| < \mathcal{C}_1 \beta^{\frac{1}{256}}$

$$\int_\Omega \left| \nabla u(z) - \frac{z}{|z|} \right|^2 dz \leq \mathcal{C}_1 \beta^{\frac{1}{256}}. \tag{8}$$

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2. Sketch of the proof

2.1. **Sketch of the proof of Theorem 1.** While the proof for convex domains is slightly involved, there are only a couple of ideas that are really central. We will sketch the proof for the case $\Omega = B_1(0)$, ignoring (without comment) many technicalities in order to give an impression of the basic skeleton.

The real engine of the proof is the characterisation in [De-Ko-Mu-Ot 00] of the set of Φ such that (3) is satisfied. As mentioned in the introduction, as consequence of the characterisation it was shown there exists a sequence of Φ_n satisfying (3) that converge uniformly to the function $\tilde{\Phi}(z) = |z|^2 e$ for $z \cdot e > 0$ and 0 otherwise. Following closely the proof of this it is possible to extract the existence of functions Φ_θ and Ψ_θ with $\|\nabla \Phi_\theta\| \leq c\beta^{-\frac{1}{4}}$, $\|\Psi_\theta\| \leq c\beta^{-\frac{1}{4}}$, $\|\nabla \Psi_\theta\| \leq c\beta^{-\frac{1}{2}}$ such that the following two inequalities hold.

Let $\Lambda_\theta(z) := \theta$ for $z \cdot \theta > 0$ and 0 otherwise,

$$|\Phi_\theta(z) - \Lambda_\theta(z)| \leq c\beta^{\frac{1}{4}} \text{ for } z \in N_{\sqrt{\beta}}(S^1) \backslash B_{2\beta^{\frac{1}{4}}}(\theta) \tag{9}$$

and (letting $R(z_1, z_2) = (z_2, -z_1)$)

$$\operatorname{div}\left[\Phi_\theta(R(\nabla w)) - \Psi_\theta(R(\nabla w))\left(1 - |R(\nabla w)|^2\right)\right] \leq c\beta^{-\frac{1}{2}} \left|1 - |\nabla w|^2\right| \left|\nabla^2 w\right| \text{ for any } w \in W^{2,1}. \tag{10}$$

Recall, for simplicity we have taken $\Omega = B_1(0)$, as $\nabla u(z) = -\frac{z}{|z|}$ on $\partial B_1(0)$ then we can extend u to a function $\tilde{u} : B_{11/10}(0) \to \mathbb{R}$ such that

$$\int_{B_{11/10}(0)} \left|1 - |\nabla \tilde{u}|^2\right| \left|\nabla^2 \tilde{u}\right| \leq c\beta, \int_{B_{11/10}(0)} \left|1 - |\nabla \tilde{u}|^2\right|^2 \leq c\beta^2$$

and

$$\nabla \tilde{u}(z) = -\frac{z}{|z|} \text{ for any } z \in B_{11/10}(0). \tag{11}$$

It is more convenient to work with vectorfields that are *almost* curl free instead of *almost* divergence free. So notice that (9) can be rewritten as

$$|R(\Phi_\theta(z)) - R(\Lambda_\theta(z))| \leq c\beta^{\frac{1}{4}} \text{ for } z \in N_{\sqrt{\beta}}(S^1) \backslash B_{2\beta^{\frac{1}{4}}}(\theta) \tag{12}$$

and we have $\int_{B_{11/10}(0)} \left|\operatorname{curl}\left[R(\Phi_\theta(R(\nabla \tilde{u}))) - R(\Psi_\theta(R(\nabla \tilde{u})))\left(1 - |\nabla \tilde{u}|^2\right)\right]\right| \leq c\sqrt{\beta}$. By the quantitative Hodge decomposition type theorem from [Am-De-Ma 99] (Theorem 4.3) we can find a scalar valued function w_θ such that

$$\int_{B_{11/10}(0)} \left|\nabla w_\theta - \left(R(\Phi_\theta(R(\nabla \tilde{u}))) - R(\Psi_\theta(\nabla R(\nabla \tilde{u})))\left(1 - |\nabla \tilde{u}|^2\right)\right)\right| \leq c\sqrt{\beta}. \tag{13}$$

The real power of (13) is that on the annulus $\mathcal{A} := B_{11/10}(0) \backslash B_1(0)$ we know that $\nabla \tilde{u}(z) = -\frac{z}{|z|}$ and hence given inequality (12) (and the fact that $|\nabla \tilde{u}| = 1$ on $\mathcal{A}$) we have a that $\Phi_\theta(R(\nabla \tilde{u}(z))) \in N_{\beta^{\frac{1}{4}}}(\theta)$ for any $z \in \mathcal{A} \cap H(R\theta, 0)$, see figure 1.

In much the same way in the ball $B_1(0)$, by inequalities (12), (13) and $\int_{B_1(0)} \left|1 - |\nabla \tilde{u}|^2\right|^2 \leq \beta^2$ we have that there exists a large set $\mathcal{G} \subset B_1(0) \cap H(0, R\theta)$, with $|B_1(0) \backslash \mathcal{G}| \leq \sqrt{\beta}$ such that if $z \in \mathcal{G}$ then $\nabla w_\theta(z) \in B_{\beta^{\frac{1}{4}}}(R\theta)$ or $\nabla w_\theta(z) \in B_{\beta^{\frac{1}{4}}}(0)$ depending on whether $R(\nabla u(z)) \cdot \theta > 0$ or $R(\nabla u(z)) \cdot \theta \leq 0$.

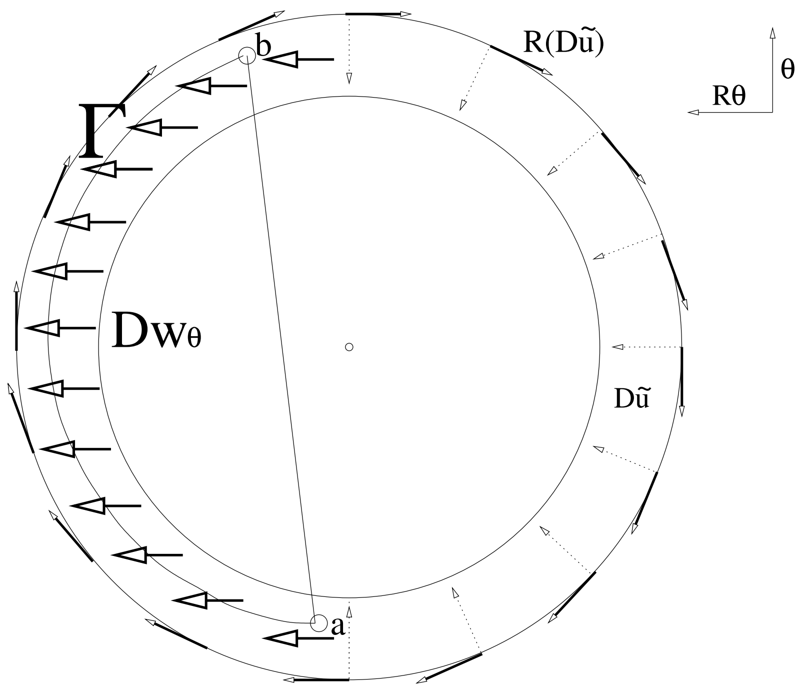

FIGURE 1

It is not hard to see we can find points $a, b \in N_{\beta^{\frac{1}{8}}}(\langle\theta\rangle \cap \partial B_1(0))$ with $|a-b| \sim 2$, $\theta \cdot \frac{b-a}{|b-a|} > 0$, the angle between $\frac{b-a}{|b-a|}$ and θ is at least $\beta^{\frac{1}{8}}$ and $H^1([a,b]\backslash\mathcal{G}) \leq \beta^{\frac{1}{4}}$. Let $\mathcal{G}_1 = \left\{x \in \mathcal{G} : \nabla u(z) \cdot R^{-1}(\theta) > 0\right\}$ and $\mathcal{G}_2 = \mathcal{G}\backslash\mathcal{G}_1$. As can be seen from figure 1 we can connect a to b with a path $\Gamma \subset \mathcal{A}$ so

$$\begin{aligned} |w_\theta(b) - w_\theta(a)| &= \left|\int_\Gamma \nabla w_\theta(z) t_z dH^1 z\right| \geq \left|R\theta \cdot \left(\int_\Gamma t_z dH^1 z\right)\right| - c\beta^{\frac{1}{4}} \\ &= \left|R\theta \cdot \frac{b-a}{|b-a|}\right| |b-a| - c\beta^{\frac{1}{4}}. \end{aligned} \tag{14}$$

On the other hand

$$\begin{aligned} |w_\theta(b) - w_\theta(a)| &= \left|\int_{[a,b]} \nabla w_\theta(z) \frac{b-a}{|b-a|} dH^1 z\right| \leq \left|\int_{[a,b]\cap\mathcal{G}_1} \nabla w_\theta(z) \frac{b-a}{|b-a|} dH^1 z\right| + c\beta^{\frac{1}{4}} \\ &\leq \left|\int_{[a,b]\cap\mathcal{G}_1} R\theta \cdot \frac{b-a}{|b-a|} dH^1 z\right| + c\beta^{\frac{1}{4}} \\ &= \left|R\theta \cdot \frac{b-a}{|b-a|}\right| H^1([a,b]\cap\mathcal{G}_1) + c\beta^{\frac{1}{4}} \end{aligned} \tag{15}$$

and since $\left|R\theta \cdot \frac{b-a}{|b-a|}\right| \geq \beta^{\frac{1}{8}}$ so putting (14) and (15) together

$$|a-b| \leq H^1([a,b]\cap\mathcal{G}_1) + \frac{c\beta^{\frac{1}{4}}}{\left|R\theta \cdot \frac{b-a}{|b-a|}\right|} \leq H^1([a,b]\cap\mathcal{G}_1) + c\beta^{\frac{1}{8}}.$$

So by arguing in the same way for lines parallel to $[a,b]$ by Fubini's theorem we can show $\left|H\left(\frac{a+b}{2}, R\left(\frac{b-a}{|b-a|}\right)\right)\backslash\mathcal{G}_1\right| \leq c\beta^{\frac{1}{8}}$. Thus all but $\beta^{\frac{1}{8}}$ points $z \in B_1(0) \cap H(0, R(\theta))$ are such that $\nabla u(z)\cdot R^{-1}(\theta) > 0$. As θ is arbitrary we can rephrase this the following way. Given $\phi \in S^1$ for all but $\beta^{\frac{1}{8}}$ points $z \in B_1(0) \cap H(0,\phi)$ are such that $\nabla u(z)\cdot(-\phi) > 0$.

Now take $\psi = \begin{pmatrix} \cos\beta^{\frac{1}{16}} \\ \sin\beta^{\frac{1}{16}} \end{pmatrix}$. For all but $\beta^{\frac{1}{8}}$ points in $H(0,e_1)\cap H(0,-\psi)\cap H(0,-e_2)$ we have that $\nabla u(z)\cdot(-e_1) > 0$ and $\nabla u(z)\cdot\psi > 0$, it is not hard to show this implies $|\nabla u(z)\cdot e_1| \leq c\beta^{\frac{1}{16}}$ and since $\nabla u(z)\cdot e_2 > 0$ and $|\nabla u(z)| \sim 1$ we have $\nabla u(z) \in B_{c\beta^{\frac{1}{16}}}(e_2)$ with an exceptional set of measure less than $c\beta^{\frac{1}{8}}$. So integrating a carefully chosen line inside $H(0,e_1)\cap H(0,-\psi)\cap H(0,-e_2)$ and using the fact that $u=0$ on $\partial B_1(0)$ we can show $|u(0)-1| \leq c\beta^{\frac{1}{16}}$.

Now since $|\nabla u|$ is mostly very close to 1 and we have zero boundary condition, so avoiding technicalities assuming the coarea formula we have $\int_{\theta\in S^1}\int_{\mathbb{R}_+\theta\cap B_1(0)}\left||\nabla u(z)|^2-1\right| dH^1 z dH^1\theta \leq c\sqrt{\beta}$ we have

$$\begin{aligned}\int_{\theta\in S^1}\int_{\mathbb{R}_+\theta\cap B_1(0)} & |\nabla u(z)-\theta|^2\, dH^1 z\theta \\ &= \int_{\theta\in S^1}\int_{\mathbb{R}_+\theta\cap B_1(0)} |\nabla u(z)|^2 - 2\nabla u(z)\cdot\theta + |\theta|^2\, dH^1 z dH^1\theta \\ &\leq c\beta^{\frac{1}{16}}. \end{aligned} \tag{16}$$

This concludes the sketch of the proof of Theorem 1.

2.2. **Sketch of the proof of Corollary 1.** In order to deduce Corollary 1 we need to apply Theorem 1 to the minimizer of I_ϵ over $\Lambda(\Omega)$. We can only do this if the minimiser has small energy (and from Theorem 1 we know it can only have small energy if Ω is close to a ball). For this reason it is necessary to construct a function in $\Lambda(\Omega)$ with this property. It turns out this is a surprisingly delicate task, it is achieved in Section 4 of the paper.

The obvious way to attempt the construction is to make some adaption of the function $\zeta(z) = \operatorname{dist}(z,\partial\Omega)$, this function clearly satisfies the correct boundary condition. The first problem is that $\nabla\zeta$ will have its gradient in BV and it is easy to construct examples of convex domains that are close to balls for which the singular part of $\nabla\zeta$ is widely spread over the domain. So it is necessary to convolve ζ, however convolution will destroy the boundary condition. To circumvent this obstacle, in a neighborhood of the boundary we convolve the ζ with a convolution kernel who support decreases in proportionally to the distance to the boundary. We make the assumption that $\partial\Omega$ is C^2 with curvature bounded above by $\epsilon^{-\frac{1}{5}}$ and this allows us estimate the various error terms involved in differentiating a function the convolved with a kernel of varying support.

3. Proof of Theorem

Lemma 1. *Let Ω be a convex body centered on 0. Suppose $u : W^{2,1}(\Omega)\to\mathbb{R}$ satisfies (6) and (7). For each $\theta\in S^1$ define $\Lambda_\theta : \mathbb{R}^2 \to S^1$ be defined by*

$$\Lambda_\theta(z) = \begin{cases} \theta & \text{if } z\cdot\theta > 0, \\ 0 & \text{if } z\cdot\theta \leq 0. \end{cases} \tag{17}$$

Let $R\in SO(2)$ be the rotation defined by $R(z_1,z_2) = (z_2,-z_1)$ and let $m = R(\nabla u)$, we will show there exists a set $\Gamma\subset S^1$ with $H^1(S^1\backslash\Gamma) \leq c\beta^{\frac{1}{8}}$ such that for any $\theta\in\Gamma$ we can find *function* $w_\theta : \Omega \to \mathbb{R}$ *with the property*

$$\int_\Omega |\nabla w_\theta - R(\Lambda_\theta(m))| \leq c\beta^{\frac{1}{8}}. \tag{18}$$

Proof of Lemma 1. Let $M = \left[\frac{\beta^{-\frac{1}{8}}}{4}\right]$, we divide S^1 into M disjoint connected subsets of length $\frac{2\pi}{M}$, denote them $A_1, A_2, \dots A_M$. We assume they have been ordered sequentially, i.e. $\overline{A_i} \cap \overline{A_{i+1}} \neq \emptyset$ for $i = 1, 2, \dots M-1$. Let $\mathcal{B} = \left\{k \in \{1, 2, \dots M\} : \left|\left\{x \in \Omega : \frac{\nabla u(x)}{|\nabla u(x)|} \in \overline{A_k}\right\}\right| \geq \beta^{\frac{1}{8}}\right\}$. Since $\mathrm{Card}(\mathcal{B})\,\beta^{\frac{1}{8}} \leq |\Omega| \leq 2\pi$ we have that $\mathrm{Card}(\mathcal{B}) \leq 2\pi\beta^{-\frac{1}{8}}$.

Let $\mathcal{D} := \{k \in \{2, 3, \dots M-1\} : \{k-1, k, k+1\} \cap \mathcal{B} \neq \emptyset\}$. A simple covering argument shows that $\mathrm{Card}(\mathcal{D}) \leq c\beta^{-\frac{1}{8}}$. Let $\Gamma = \bigcup_{k \in \{2,3,\dots M-1\} \setminus D} \overline{A_k}$. Note that for any $\theta \in \Gamma$ we have

$$\left|\left\{x \in \Omega : \frac{\nabla u(x)}{|\nabla u(x)|} \in B_{2\beta^{\frac{1}{8}}}(\theta)\right\}\right| \leq 3\beta^{\frac{1}{8}}. \tag{19}$$

So pick $\theta \in \Gamma$ without loss of generality we can assume $\theta = e_1$. Let $s : \mathbb{R} \to \mathbb{R}_+$ be a smooth monotone function where $s(x) = 0$ if $x \leq 0$ and $s(x) = x$ if $x > \beta^{\frac{1}{4}}$ and $\|\nabla^2 s\|_{L^\infty} \leq \beta^{-\frac{1}{4}}$ and $\|\nabla^3 s\|_{L^\infty} \leq \beta^{-\frac{1}{2}}$, it is clear such a function exists.

Let $\varphi(z) = s(z \cdot e_1) = s(z_1)$. Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned}\Phi(z) &:= \varphi(z)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \left(\nabla\varphi(z) \cdot \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}\right)\begin{pmatrix} -z_2 \\ z_1 \end{pmatrix} \\ &= \begin{pmatrix} \varphi(z)\,z_1 + z_2^2\varphi_{,1}(z) \\ \varphi(z)\,z_2 - z_2 z_1 \varphi_{,1}(z) \end{pmatrix}. \end{aligned} \tag{20}$$

Define

$$\Psi(z) = \begin{pmatrix} \Psi_1(z) \\ \Psi_2(z) \end{pmatrix} := \begin{pmatrix} -\varphi_{,1}(z) \\ \frac{z_2}{2}\varphi_{,11}(z) \end{pmatrix}.$$

Recall $m(z) := R(\nabla u(z))$ so m is divergence free. Note (using the fact $\varphi_{,2} \equiv 0$ and $\varphi_{,12} \equiv 0$ and $\mathrm{div} m \equiv 0$ for the third inequality, and using $\mathrm{div} m = 0$ for the last inequality)

$$\begin{aligned}\mathrm{div}\,[\Phi(m)] &= \mathrm{div}\begin{pmatrix} \varphi(m)\,m_1 + m_2^2\varphi_{,1}(m) \\ \varphi(m)\,m_2 - m_2 m_1 \varphi_{,1}(m) \end{pmatrix} \\ &= (\varphi_{,1}(m)m_{1,1} + \varphi_{,2}(m)m_{2,1})m_1 + \varphi(m)m_{1,1} + 2m_2 m_{2,1}\varphi_{,1}(m) \\ &\quad + m_2^2(\varphi_{,11}(m)m_{1,1} + \varphi_{,12}(m)m_{2,1}) + (\varphi_{,1}(m)m_{1,2} + \varphi_{,2}(m)m_{2,2})m_2 \\ &\quad + \varphi(m)m_{2,2} - ((m_{1,2}m_2 + m_1 m_{2,2})\varphi_{,1}(m) \\ &\quad + m_1 m_2(\varphi_{,11}(m)m_{1,2} + \varphi_{,12}(m)m_{2,2})) \\ &= m_1\varphi_{,1}(m)m_{1,1} + 2m_2 m_{2,1}\varphi_{,1}(m) + m_2^2 m_{1,1}\varphi_{,11}(m) + m_2 m_{1,2}\varphi_{,1}(m) \\ &\quad - ((m_{1,2}m_2 + m_1 m_{2,2})\varphi_{,1}(m) + m_1 m_2 m_{1,2}\varphi_{,11}(m)) \\ &= 2\varphi_{,1}(m)(m_1 m_{1,1} + m_2 m_{2,1}) - \varphi_{,11}(m)m_2(m_1 m_{1,2} + m_2 m_{2,2}). \end{aligned} \tag{21}$$

Note also that

$$\begin{aligned}\Psi(m) \cdot \nabla(1 - |m|^2) &= -\Psi(m) \cdot \begin{pmatrix} 2(m_1 m_{1,1} + m_2 m_{2,1}) \\ 2(m_1 m_{1,2} + m_2 m_{2,2}) \end{pmatrix} \\ &= 2\varphi_{,1}(m)(m_1 m_{1,1} + m_2 m_{2,1}) - m_2\varphi_{,11}(m)(m_1 m_{1,2} + m_2 m_{2,2}) \end{aligned}$$

so by (21) we have

$$\mathrm{div}\,[\Phi(m)] = \Psi(m) \cdot \nabla(1 - |m|^2). \tag{22}$$

Let $\tilde{\Phi} := R(\Phi)$ and $\tilde{\Psi} := R(\Psi)$ note $\operatorname{curl}\left[\tilde{\Phi}(m)\right] \overset{(22)}{=} \operatorname{div}[\Phi(m)] = \Psi(m)\cdot\nabla(1-|m|^2)$. So

$$\begin{aligned}\operatorname{curl}\left[\tilde{\Psi}(m)(1-|m|^2)\right] &= \operatorname{div}[\Psi(m)](1-|m|^2) - \Psi(m)\cdot\nabla(1-|m|^2)\\ &= \operatorname{div}[\Psi(m)](1-|m|^2) - \operatorname{curl}\left[\tilde{\Phi}(m)\right]. \end{aligned} \tag{23}$$

Thus

$$\begin{aligned}&\operatorname{curl}\left[\tilde{\Phi}(m)+\tilde{\Psi}(m)(1-|m|^2)\right]\\ &\quad\overset{(23)}{=} \operatorname{div}[\Psi(m)](1-|m|^2)\\ &\quad= (\Psi_{1,1}(m)m_{1,1}+\Psi_{1,2}(m)m_{2,1}+\Psi_{2,1}(m)m_{1,2}+\Psi_{2,2}(m)m_{2,2})(1-|m|^2)\\ &\quad\le c\|\nabla\Psi\|_{L^\infty(\Omega)}\left|1-|m|^2\right||\nabla m|. \end{aligned}\tag{24}$$

Hence as $\|\nabla\Psi\|_{L^\infty(\Omega)} \le c\|\nabla^3\varphi\|_{L^\infty(\Omega)} \le c\|\nabla^3 s\|_{L^\infty(\Omega)} \le c\beta^{-\frac{1}{2}}$

$$\begin{aligned}\int_\Omega \left|\operatorname{curl}\left[\tilde{\Phi}(m)+\tilde{\Psi}(m)(1-|m|^2)\right]\right| &\overset{(24)}{\le} c\|\nabla\Psi\|_{L^\infty(\Omega)}\int_\Omega\left|1-|m|^2\right||\nabla m|\\ &\overset{(6)}{\le} c\sqrt{\beta}. \end{aligned}\tag{25}$$

Note that if $z\in N_{\sqrt{\beta}}(S^1)\cap\{z_1>0\}\setminus B_{2\beta^{\frac14}}(e_2)$ then $\varphi(z)=z_1$, $\varphi_{,1}(z)=1$ and so $\Phi(z)\overset{(20)}{=}\begin{pmatrix} z_1^2+z_2^2\\ 0\end{pmatrix}$ on the other hand if $z\in N_{\sqrt{\beta}}(S^1)\cap\{z_1\le 0\}\setminus B_{2\beta^{\frac14}}(e_2)$ then $\varphi(z)=\varphi_{,1}(z)=0$ and so $\Phi(z)=\begin{pmatrix}0\\0\end{pmatrix}$. Now if $z\in N_{\sqrt{\beta}}(S^1)\cap\{z_1>0\}\setminus B_{2\beta^{\frac14}}(e_2)$ we have

$$\begin{aligned}\left|(\tilde{\Phi}(z)+\tilde{\Psi}(z)(1-|z|^2)) - R(\Lambda_{e_1}(z))\right| &\le \left|\tilde{\Phi}(z)-R(\Lambda_{e_1}(z))\right| + c\sqrt{\beta}\sup_{z\in N_{\sqrt{\beta}}(S^1)}\left|\tilde{\Psi}(z)\right|\\ &= \left|R\begin{pmatrix}z_1^2+z_2^2\\0\end{pmatrix} - R\begin{pmatrix}1\\0\end{pmatrix}\right| + c\beta^{\frac14}\\ &\le c\beta^{\frac14}. \end{aligned}\tag{26}$$

And if we have $z\in N_{\sqrt{\beta}}(S^1)\cap\{z_1\le 0\}\setminus B_{2\beta^{\frac14}}(e_2)$ arguing in the same way we can conclude

$$\left|(\tilde{\Phi}(z)+\tilde{\Psi}(z)(1-|z|^2)) - R(\Lambda_{e_1}(z))\right| \le c\beta^{\frac14}. \tag{27}$$

Let $\Pi := \left\{z\in\Omega : |m(z)|\in(1-\sqrt{\beta},1+\sqrt{\beta})\right\}$ and let

$$\mathcal{E} := \left\{x\in\Omega : \frac{\nabla u(x)}{|\nabla u(x)|}\in B_{2\beta^{\frac18}}(e_2)\right\}, \tag{28}$$

note from (19) we know $|\mathcal{E}|\le 3\beta^{\frac18}$. From (26) and (27)

$$\left|\int_{\Pi\setminus\mathcal{E}} (\tilde{\Phi}(m)-\tilde{\Psi}(m)(1-|m|^2)) - R(\Lambda_{e_1}(m))\right| \le c\beta^{\frac14} \tag{29}$$

on the other hand

$$\begin{aligned}&\left|\int_{\Omega\setminus\Pi}\Big((\tilde{\Phi}(m)+\tilde{\Psi}(m)(1-|m|^2)) - R(\Lambda_{e_1}(m))\Big)\right|\\ &\quad\le \|\mathbb{1}_{\Omega\setminus\Pi}((\tilde{\Phi}(m)+\tilde{\Psi}(m)(1-|m|^2)) - R(\Lambda_{e_1}(m)))\|_{L^1(\Omega)}\\ &\quad\le \|\mathbb{1}_{\Omega\setminus\Pi}\|_{L^2(\Omega)}(\|\tilde{\Phi}(m)\|_{L^2(\Omega)} + \|\tilde{\Psi}(m)(1-|m|^2)\|_{L^2(\Omega)} + c). \end{aligned}\tag{30}$$

Note

$$\begin{aligned}\|\tilde{\Phi}(m)\|_{L^2(\Omega)} &= \|\varphi(m)m_1 + m_2^2\varphi_{,1}(m)\|_{L^2(\Omega)} + \|\varphi(m)m_2 - m_2m_1\varphi_{,1}(m)\|_{L^2(\Omega)} \\ &\leq c\| \left|\nabla u\right|^2 \|_{L^2(\Omega)} \leq c + c\|1 - \left|\nabla u\right|^2 \|_{L^2(\Omega)} \leq c. \end{aligned} \tag{31}$$

Similarly

$$\begin{aligned}\|\tilde{\Psi}(m)(1-|m|^2)\|_{L^2(\Omega)} &\leq \left(\int_\Omega (\tilde{\Psi}(m))^2\right)^{\frac{1}{2}} \left(\int_\Omega \left|1-|m|^2\right|^2\right)^{\frac{1}{2}} \\ &\leq c\beta^{-\frac{1}{4}}\beta \leq c\beta^{\frac{3}{4}}. \end{aligned} \tag{32}$$

Thus applying (32), (31) to (30) and using the fact that (7) implies $L^2(\Omega\backslash\Pi) \leq c\sqrt{\beta}$ gives

$$\left|\int_{\Omega\backslash\Pi} \left((\tilde{\Phi}(m) - \tilde{\Psi}(m)(1-|m|^2)) - R\left(\Lambda_{e_1}(m)\right)\right)\right| \leq c\sqrt{\beta}. \tag{33}$$

Together with (29)

$$\left|\int_{\Omega\backslash\mathcal{E}} \left((\tilde{\Phi}(m) - \tilde{\Psi}(m)(1-|m|^2)) - R\left(\Lambda_{e_1}(m)\right)\right)\right| \leq c\beta^{\frac{1}{4}}. \tag{34}$$

Now recall $\|\tilde{\Psi}\|_{L^\infty(\Omega)} \leq c\beta^{-\frac{1}{4}}$, $\left|\tilde{\Phi}(z)\right| \leq c\,|z|$ and note by (19) (recall definition (28)) we have $|\mathcal{E}| \leq 3\beta^{\frac{1}{8}}$

$$\begin{aligned}\left|\int_{\mathcal{E}} \left((\tilde{\Phi}(m) - \tilde{\Psi}(m)(1-|m|^2)) - R\left(\Lambda_{e_1}(m)\right)\right)\right| &\leq c\beta^{-\frac{1}{4}}\beta + |\mathcal{E}| + \left|\int_\Omega \tilde{\Phi}(m)1\!\!1_{\mathcal{E}}\right| \\ &\leq c\beta^{\frac{3}{4}} + c\,|\mathcal{E}| + \int_{\mathcal{E}} |1 - |\nabla u|| \\ &\leq c\beta^{\frac{1}{8}}. \end{aligned} \tag{35}$$

Putting inequality (35) together with (34) gives

$$\left|\int_{\Omega} \left((\tilde{\Phi}(m) - \tilde{\Psi}(m)(1-|m|^2)) - R\left(\Lambda_{e_1}(m)\right)\right)\right| \leq c\beta^{\frac{1}{8}}.$$

Now using (25) and by applying Theorem 4.3 from ([Am-De-Ma 99]) there exists $w_{e_1} \in W^{1,1}(\Omega)$ such that

$$\int_\Omega \left|\nabla w_{e_1} - (\tilde{\Phi}(m) - \tilde{\Psi}(m)(1-|m|^2))\right| \leq c\beta^{\frac{1}{8}} \tag{36}$$

thus putting this together with (34) and gives (18). □

Lemma 2. *Let Ω be a convex body centered on 0 and let $u : W^{2,2}(\Omega) \to \mathbb{R}$ be a function satisfying (6) and (7) and $\nabla u(z)\cdot\eta_z = 1$ is the sense of trace, where η_z is the inward pointing unit normal to $\partial\Omega$ at z.*

For any $r > 0$ define $\Omega_r := N_r(\Omega)$, we will show we can construct a function $\tilde{u} : W^{2,1}(\Omega_r) \to \mathbb{R}$ satisfying

$$\int_{\Omega_r} \left|1 - |\nabla\tilde{u}|^2\right| \left|\nabla^2\tilde{u}\right|^2 \leq \beta^2,\ \int_{\Omega_r} \left|1 - |\nabla\tilde{u}|^2\right| \leq \beta, \tag{37}$$

and

$$\tilde{u}\,(z) = \begin{cases} u(z) + r & \text{for } z \in \overline{\Omega} \\ r - d(z,\Omega) & \text{if } z \in \Omega_r\backslash\Omega \end{cases} \tag{38}$$

Proof of Lemma 2.

Step 1. We will show $\nabla u(x)=\eta_x$ for H^1 a.e. $x\in\partial\Omega$

Proof of Step 1. Recall $\nabla u\in W^{1,1}(\Omega)$ and ∇u is defined on $\partial\Omega$ in the sense of trace, as the trace operator is bounded we know $\int_{\partial\Omega}|\nabla u|\,dH^1<\infty$.

We define

$$v\,(z)=\begin{cases}u(z) & \text{for } z\in\overline{\Omega}\\ 0 & \text{if } z\in\Omega_r\backslash\Omega\end{cases}\tag{39}$$

By Theorem 3.8 [Am-Fu-Pa 00] $\nabla v\in BV(\Omega_r)$ and hence by Theorem 3.76 [Am-Fu-Pa 00] and Theorem 2, Section 5.3 [Ev-Ga 92] for H^1 a.e. $x\in\partial\Omega$ the following limits exist

$$\lim_{\rho\to 0}\fint_{B^+(x,\eta_x)}|\nabla u(z)-\nabla u(x)|\,dz=0\tag{40}$$

and

$$\lim_{\rho\to 0}\fint_{B^-(x,\eta_x)}|\nabla u(z)|\,dz=0.\tag{41}$$

Let $w_x^{\rho}(z)=\frac{u(\rho(z-x))}{\rho}$, by (40) and (41) for any sequence $\rho_n\to 0$ we have $w_x^{\rho_n}(z)\overset{L^1}{\to}w_x$ as $n\to\infty$ where

$$w_x\,(z)=\begin{cases}\nabla u(x)\cdot z & \text{for } z\in H(0,\eta_x)\\ 0 & \text{for } z\in H(0,-\eta_x)\end{cases}\tag{42}$$

however w_x would not be curl free unless $\nabla u(x)=\lambda\eta_x$ for some $\lambda\in\mathbb{R}$. As we know $\nabla u(x)\cdot\eta_x=1$ this implies $\nabla u(x)=\eta_x$ for H^1 a.e. $x\in\partial\Omega$. This completes the proof of Step 1.

Step 2. For any $z\in\Omega_r\backslash\Omega$, $\tilde{u}(z)=d(z,\partial\Omega_r)$.

Proof of Step 2. Note that $\|\nabla\tilde{u}\|_{L^\infty(\Omega_r\backslash\Omega)}\le 1$. Let $x\in\partial\Omega_r$, let $q(x)$ be the metric projection onto a convex set Ω, i.e. the unique point for which $|x-q(x)|=d(x,\Omega)$. Since $x\in\partial\Omega_r=\partial(N_r(\Omega))=\{x\in\Omega^c: d(x,\Omega)=r\}$ so $|x-q(x)|\ge r$, on the otherhand we also know $d(x,\Omega)=r$ so there must exist $y\in\Omega$ such that $|x-y|<r+\delta$ for every $\delta>0$, this implies $|x-q(x)|<r+\delta$ for every $\delta>0$. Thus $|x-q(x)|=r$.

Since $\tilde{u}(x)=0$ and $\tilde{u}(q(x))=r$ and as $\tilde{u}$ is 1-Lipschitz on $\Omega_r\backslash\Omega$ this implies $\tilde{u}((1-\alpha)x+\alpha q(x))=\alpha r$ for any $\alpha\in[0,1]$.

Now let $Q(z):=d(z,\partial\Omega_r)$. For every $x\in\partial\Omega_r$, $Q(q(x))\le|q(x)-x|=r$. As $\partial\Omega_r=\partial(N_r(\Omega))$ so we know $Q(q(x))\ge r$ and thus have $Q(q(x))=r$. We also know Q is 1-Lipschitz and $Q(x)=0$, thus in the same way as before $Q((1-\alpha)x+\alpha q(x))=\alpha r$ for any $\alpha\in[0,1]$. Therefor $Q(z)=\tilde{u}(z)$ for any $z\in[x,q(x)]$, $x\in\partial\Omega_r$ and this completes the proof of Step 1.

Step 3. We will show that $\tilde{u}\in W^{2,1}(\Omega_r)$ and that $\tilde{u}$ satisfies (37).

Proof of Step 3. First we claim that

$$\int_{\Omega_r\backslash\Omega}\left|\nabla^2\tilde{u}\right|dz\le c \text{ and hence } \nabla\tilde{u}\in W^{1,1}(\Omega_r\backslash\Omega).\tag{43}$$

Note $\tilde{u}_{\lfloor\Omega_r\backslash\Omega}$ is a 1-Lipschitz function with $\tilde{u}=0$ on $\partial\Omega_r$ and $\tilde{u}=r$ on $\partial\Omega$. So by the Co-area formula we have $\int_{\Omega_r\backslash\Omega}\left|\nabla^2\tilde{u}\right|dx=\int_0^r\int_{\tilde{u}^{-1}(h)}\left|\nabla^2\tilde{u}\right|dH^1dh$. Recall $\nabla\tilde{u}$ exists everywhere in $\Omega_r\backslash\overline{\Omega}$ and $|\nabla\tilde{u}|=1$ on $\Omega_r\backslash\overline{\Omega}$ and as $\tilde{u}^{-1}(h)$ is connected and the boundary of a smooth convex set and for each $z\in\tilde{u}^{-1}(h)$, $\nabla\tilde{u}(z)$ is normal to the tangent of $\tilde{u}^{-1}(h)$ at z, so $\int_{\tilde{u}^{-1}(h)}\left|\nabla^2\tilde{u}\right|dH^1\le c$ for all $h\in[0,r]$ and hence (43) is shown.

Since Ω is an extension domain by Theorem 1, Section 4.4 [Ev-Ga 92] there exists a function $p: W^{1,2}(\mathbb{R}^2)\to\mathbb{R}^2$ such that $p(z)=\nabla\tilde{u}(z)$ on Ω and Spt$p$ is compact. Similary as $\Omega_r\backslash\Omega$ is an extension domain there exists a function $q: W^{1,1}(\mathbb{R}^2)\to\mathbb{R}^2$ such that $q(z)=\nabla\tilde{u}(z)$ on

$\Omega_r \backslash \Omega$ and Spt$q$ is compact. We define $w : \Omega_r \to \mathrm{I\!R}^2$ by $w := p\mathbb{1}_{\Omega} + q\mathbb{1}_{\Omega_r\backslash\Omega}$, by Theorem 3.83 [Am-Fu-Pa 00] $w \in BV(\Omega_r : \mathrm{I\!R}^2)$ and since p and q agree on $\partial\Omega$ we have that ∇w as a measure is absolutely continuous with respect to Lebesgue measure (and hence $w \in W^{1,1}(\Omega_r : \mathrm{I\!R}^2)$) and $\nabla w = \nabla p \mathbb{1}_{\Omega} + \nabla q \mathbb{1}_{\Omega_r\backslash\Omega}$. Now as $w = \nabla\tilde{u}$ a.e. in Ω_r we have that $\nabla\tilde{u} \in W^{1,1}(\Omega_r)$.

Since $\nabla^2\tilde{u} \in L^1$ we know

$$\begin{aligned}\int_{\Omega_r} \left|1 - |\nabla\tilde{u}|^2\right| \left|\nabla^2\tilde{u}\right| dz &= \int_{\Omega} \left|1 - |\nabla\tilde{u}|^2\right| \left|\nabla^2\tilde{u}\right| dz + \int_{\Omega_r\backslash\Omega} \left|1 - |\nabla\tilde{u}|^2\right| \left|\nabla^2\tilde{u}\right| dz \\ &= \int_{\Omega} \left|1 - |\nabla\tilde{u}|^2\right| \left|\nabla^2\tilde{u}\right| dz \\ &\leq \beta.\end{aligned}$$

Similarly $\int_{\Omega_r} \left|1 - |\nabla\tilde{u}|^2\right| dz = \int_{\Omega} \left|1 - |\nabla\tilde{u}|^2\right| dz \leq \beta$. □

Lemma 3. *Let Ω be a convex body with* $\mathrm{diam}(\Omega) = 2$. *Let $u : W^{2,2}(\Omega) \to \mathrm{I\!R}$ be a function satisfying (6) and (7) and satisfying $\nabla u(z) \cdot \eta_z = 1$ on $\partial\Omega$ in the sense of trace where η_z is the inward pointing unit normal to $\partial\Omega$ at z.*

Let $\Gamma \subset S^1$ be the set constructed in Lemma 1. Let $\mathcal{U} := \Omega_{1/10}$ be the convex body and $\tilde{u} : W^{2,1}(\mathcal{U}) \to \mathrm{I\!R}$ be the function constructed in Lemma 2. Let $R_0 \in \left\{R^{-1}, R\right\}$.

For any $\theta \in \Gamma \cap (-\Gamma)$ there exists unique points $a_\theta, b_\theta \in \partial\mathcal{U}$ with $\eta_{a_\theta} = \theta$ and $\eta_{b_\theta} = -\theta$ the property that if we define $\mathcal{G}_\theta^{R_0} := \left\{z \in \mathcal{U} : \nabla\tilde{u}(z) \cdot R_0^{-1}\theta > 0\right\}$ we have

$$\left|\mathcal{U} \cap H\left(\frac{a_\theta + b_\theta}{2}, R_0\left(\frac{b_\theta - a_\theta}{|b_\theta - a_\theta|}\right)\right) \backslash \mathcal{G}_\theta^{R_0}\right| \leq c\beta^{\frac{1}{24}}. \tag{44}$$

Proof of Lemma 3. Without loss of generality assume Ω is centered on 0, i.e. $\int_\Omega z = 0$. Let $\varphi := RR_0^{-1}\theta$ so note that $\varphi = \theta$ or $\varphi = -\theta$ depending on whether $R_0 = R$ or $R_0 = R^{-1}$.

Since $\partial\mathcal{U}$ is smooth and $\mathcal{U}$ is convex that exists a unique point $a_\varphi \in \partial\mathcal{U}$ with $\eta_{a_\varphi} = \varphi$ and a unique point $b_\varphi \in \partial\mathcal{U}$ with $\eta_{b_\varphi} = -\varphi$. Let $\tilde{m} = R(\nabla\tilde{u})$, it is easy to see that

$$\Pi_\varphi := \{z \in \mathcal{U}\backslash\Omega : \tilde{m}(z) \cdot \varphi > 0\} = \left\{z \in \mathcal{U}\backslash\Omega : \nabla u\,(z) \cdot R^{-1}\varphi > 0\right\} \tag{45}$$

forms a connected set whose boundary is contained in $\partial\mathcal{U}$ and $\partial\Omega$ and in two lines parrallel to φ, see figure 2, also note the endpoints of $\partial\mathcal{U} \cap \overline{\Pi_\varphi}$ are given by a_φ and b_φ.

Now by Lemma 2, (37) function $\tilde{u}$ satisfies (6) and (7). Since either $\varphi = \theta \in \Gamma$ or $\varphi = -\theta$ and so $\theta \in -\Gamma$, thus $\varphi = -\theta \in \Gamma$, thus we can apply Lemma 1, to $\tilde{m}$ and so there exists function $w_\varphi : \mathcal{U} \to \mathrm{I\!R}$ such that

$$\int_{\mathcal{U}} |\nabla w_\varphi - R\left(\Lambda_\varphi\left(\tilde{m}\right)\right)| \leq c\beta^{\frac{1}{8}}. \tag{46}$$

By the Co-area formula and Chebyshev's inequality there exists a set $H \subset [0, 1/10]$ such that $H^1([0,1/10]\backslash H) \leq c\beta^{\frac{1}{24}}$ where

$$\int_{w^{-1}(t)} |\nabla w_\varphi - R\left(\Lambda_\varphi\left(\tilde{m}\right)\right)| \, dH^1 \leq c\beta^{\frac{1}{12}} \text{ for all } r \in H. \tag{47}$$

Pick $s_0 \in \left[1/10 - c\beta^{\frac{1}{24}}, 1/10\right] \cap H$. Define $\tau_\varphi := \frac{b_\varphi - a_\varphi}{|b_\varphi - a_\varphi|}$ and

$$\mathcal{W}_\varphi := \overline{\mathcal{U}} \cap H\left(\frac{a_\varphi + b_\varphi}{2}, R\tau_\varphi\right). \tag{48}$$

We claim that

$$\partial\mathcal{U} \cap \overline{\Pi_\varphi} = \partial\mathcal{U} \cap \overline{\mathcal{W}_\varphi} \tag{49}$$

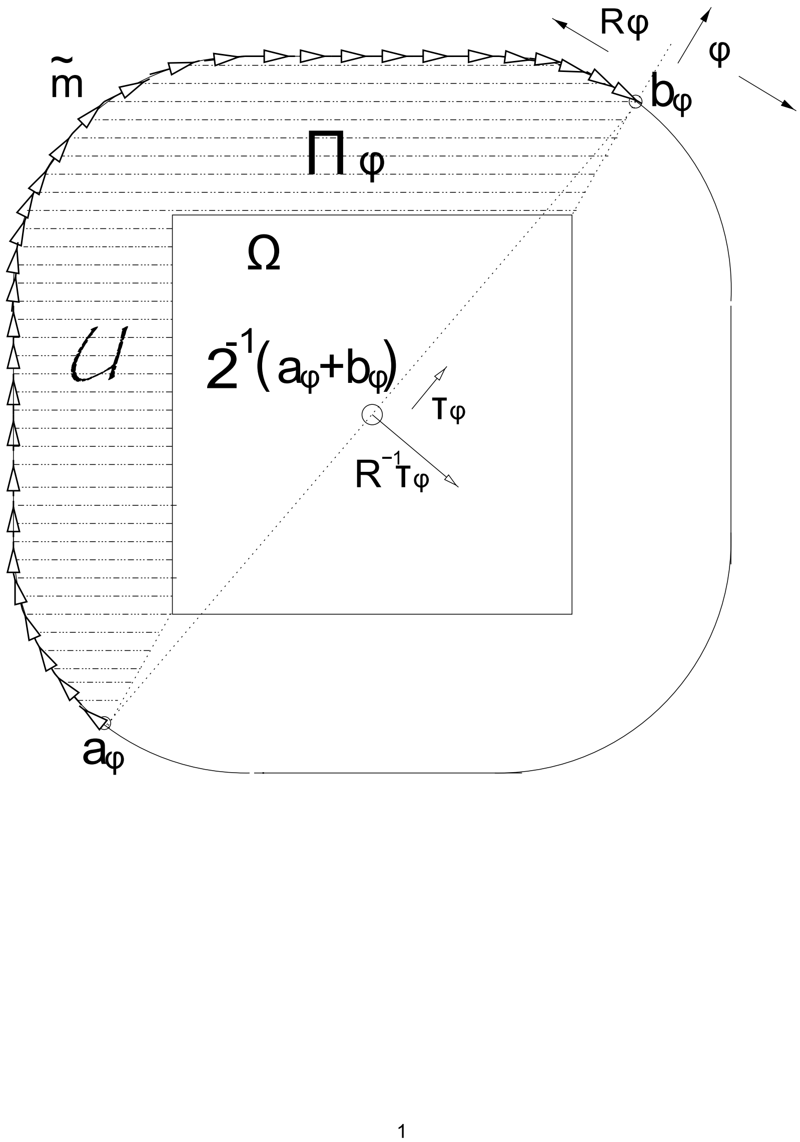

FIGURE 2

Since the endpoints of $\partial\mathcal{U}\cap\overline{\Pi_\varphi}$ are the same as the endpoints of $\partial\mathcal{U}\cap\overline{\mathcal{W}_\varphi}$ it is sufficient to show $H^1\left(\partial\mathcal{U}\cap\overline{\Pi_\varphi}\cap\overline{\mathcal{W}_\varphi}\right)>0$. Let

$$\Lambda=\sup\left\{\lambda>0:\left(\frac{a_\varphi+b_\varphi}{2}+\lambda R\tau_\varphi+\langle\tau_\varphi\rangle\right)\cap\partial\mathcal{U}\neq\emptyset\right\}$$

then let c_φ be the point given by $\left(\frac{a_\varphi+b_\varphi}{2}+\Lambda R\tau_\varphi+\langle\tau_\varphi\rangle\right)\cap\partial\mathcal{U}$, since $\partial\mathcal{U}$ is smooth $-\eta_{c_\varphi}=R^{-1}\tau_\varphi$, so $\nabla u(c_\varphi)=R^{-1}\tau_\varphi$ and thus $\nabla u\left(c_\varphi\right)\cdot R^{-1}\varphi=R^{-1}\tau_\varphi\cdot R^{-1}\varphi>0$ since $|\varphi-\tau_\varphi|<\sqrt{2}/10$. As this inequality is strict, in a neighborhood of c_φ the same inequality will be satisfied. Thus we have $H^1\left(\partial\mathcal{U}\cap\overline{\Pi_\varphi}\cap\overline{\mathcal{W}_\varphi}\right)>0$ and so we have established (49).

By the construction of Π_φ and $\mathcal{W}_\varphi$ by (49) we have

$$H^1\left(\partial\Omega_{s_0}\cap\overline{\Pi_\varphi}\triangle\overline{\mathcal{W}_\varphi}\right)\leq c\beta^{\frac{1}{24}}. \tag{50}$$

There must exist $\psi\in(0,2\beta^{\frac{1}{24}})$ such that defining $Q:=\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}$ we have

$$|R\varphi\cdot Q\tau_\varphi|>\beta^{\frac{1}{24}}. \tag{51}$$

Let $\zeta_\varphi:=\frac{a_\varphi+b_\varphi}{2}+\mathcal{C}_2\beta^{\frac{1}{24}}R\tau_\varphi$. From the construction it is clear that we can chose constant $\mathcal{C}_2$ large enough so that

$$\mathrm{Card}\left(\partial\Omega_{s_0}\cap H\left(\frac{a_\varphi+b_\varphi}{2},R\tau_\varphi\right)\cap\{\zeta_\varphi+\langle Q\tau_\varphi\rangle\}\right)=2.$$

Now for every $t>0$ let ϱ_t^1,ϱ_t^2 be the points defined by $\left\{\varrho_t^1,\varrho_t^2\right\}=\partial\Omega_{s_0}\cap\{\zeta_\varphi+tR\tau_\varphi+\langle Q\tau_\varphi\rangle\}$ and $\varrho_t^2\cdot\varphi\geq\varrho_t^1\cdot\varphi$. By (50) we can assume constant $\mathcal{C}_2$ was chosen large enough so that $\varrho_t^1,\varrho_t^2\in\Pi_\varphi$. Let Σ_t be the connected component of $\partial\Omega_{s_0}\setminus\left\{\varrho_t^1,\varrho_t^2\right\}$ that lies inside Π_φ. Thus

$$\begin{aligned}\left|(w_\varphi(\varrho_t^2)-w_\varphi(\varrho_t^1))-(\varrho_t^2-\varrho_t^1)\cdot R\varphi\right| &= \left|\int_{\Sigma_t}\nabla w_\varphi(z)\cdot t_z dH^1 z-\int_{\Sigma_t}R\varphi\cdot t_z dH^1 z\right|\\ &= \left|\int_{\Sigma_t}(\nabla w_\varphi(z)-R\varphi)\cdot t_z dH^1 z\right|\\ &\overset{(47)}{\leq} c\beta^{\frac{1}{12}}.\end{aligned} \tag{52}$$

Let

$$e_t=\int_{[\varrho_t^1,\varrho_t^2]}\left|\nabla w_\varphi-R\left(\Lambda_\varphi(\tilde{m})\right)\right|, \tag{53}$$

so by the fundamental theorem of Calculus $\left|\left(w_\varphi(\varrho_t^2)-w_\varphi(\varrho_t^1)\right)-\int_{[\varrho_t^1,\varrho_t^2]}R\left(\Lambda_\varphi(\tilde{m})\right)\cdot Q\tau_\varphi\right|\leq e_t$. Thus in combination with (52) we have

$$\left|\left(\varrho_t^2-\varrho_t^1\right)\cdot R\varphi-\int_{[\varrho_t^1,\varrho_t^2]}R\left(\Lambda_\varphi(\tilde{m})\right)\cdot Q\tau_\varphi\right|\leq e_t+c\beta^{\frac{1}{12}}. \tag{54}$$

Given the definition of Λ_φ (see (17)) and of $\mathcal{G}_\theta^{R_0}$ (see the statement of Lemma 3) so

$$R(\Lambda_\varphi(\tilde{m}(x)))=R\varphi\Leftrightarrow\tilde{m}(x)\cdot\varphi>0\Leftrightarrow\nabla u(x)\cdot R^{-1}\varphi>0\Leftrightarrow\nabla u(x)\cdot R_0^{-1}\theta>0\Leftrightarrow x\in\mathcal{G}_\theta^{R_0}.$$

In exactly the same way $\Lambda_\varphi(\tilde{m}(x))=0\Leftrightarrow x\notin\mathcal{G}_\theta^{R_0}$. Hence

$$\int_{[\varrho_t^1,\varrho_t^2]}\Lambda_\varphi(\tilde{m}(x))dH^1x=\varphi H^1\left(\left[\varrho_t^1,\varrho_t^2\right]\cap\mathcal{G}_\theta^{R_0}\right)$$

which from (54)

$$\left|\left(\varrho_t^2-\varrho_t^1\right)\cdot R\varphi-Q\tau_\varphi\cdot R\varphi H^1\left(\left[\varrho_t^1,\varrho_t^2\right]\cap\mathcal{G}_\theta^{R_0}\right)\right|\leq e_t+c\beta^{\frac{1}{12}}$$

since (recall (51)) we chose Q so that $|R\varphi\cdot Q\tau_\varphi|>\beta^{\frac{1}{24}}$ and since $\frac{\varrho_t^2-\varrho_t^1}{|\varrho_t^2-\varrho_t^1|}=Q\tau_\varphi$ so

$$\left|\left|\varrho_t^2-\varrho_t^1\right|-H^1\left(\left[\varrho_t^1,\varrho_t^2\right]\cap\mathcal{G}_\theta^{R_0}\right)\right|\leq c\beta^{-\frac{1}{24}}e_t+c\beta^{\frac{1}{24}}.$$

To simplify notation let $\vartheta=H^1\left(P_{\langle R(Q\tau_\varphi)\rangle}\left(\Omega_{s_0}\cap H(\zeta_\varphi,R(Q\tau_\varphi))\right)\right)$

$$H^1\left(\left[\varrho_t^1,\varrho_t^2\right]\cap\mathcal{G}_\theta^{R_0}\right)\geq\left|\varrho_t^1-\varrho_t^2\right|-c\beta^{-\frac{1}{24}}e_t-c\beta^{\frac{1}{24}}\text{ for any }t\in[0,\vartheta]. \tag{55}$$

So

$$\begin{aligned}\left|\Omega_{s_0}\cap H\left(\zeta_\varphi, R\left(Q\tau_\varphi\right)\right)\cap \mathcal{G}_\theta^{R_0}\right| &= \int_{[0,\vartheta]} H^1\left(\left[\varrho_t^1,\varrho_t^2\right]\cap \mathcal{G}_\varphi^{R_0}\right)dt\\ &\overset{(55)}{\geq} \int_{[0,\vartheta]}\left|\varrho_t^1-\varrho_t^2\right| - c\beta^{-\frac{1}{24}}e_t - c\beta^{\frac{1}{24}}\\ &\overset{(53)}{\geq} \left|\Omega_{s_0}\cap H\left(\zeta_\varphi, R\left(Q\tau_\varphi\right)\right)\right| - c\beta^{\frac{1}{24}}\\ &\quad -c\beta^{-\frac{1}{24}}\int_{\mathcal{W}_\varphi}\left|\nabla w_\varphi - R\left(\Lambda_\varphi\left(\tilde{m}\right)\right)\right|\\ &\overset{(46)}{\geq} \left|\Omega_{s_0}\cap H\left(\zeta_\varphi, R\left(Q\tau_\varphi\right)\right)\right| - c\beta^{\frac{1}{24}}. \end{aligned} \tag{56}$$

Note $|\mathcal{U}\backslash\Omega_{s_0}|\leq c\beta^{\frac{1}{24}}$ and by definition of $\mathcal{W}_\varphi$ (see (48)) $|\mathcal{W}_\varphi\backslash H(\zeta_\varphi, R(Q\tau_\varphi))|\leq c\beta^{\frac{1}{24}}$ this gives $\left|\mathcal{W}_\varphi\backslash G_\theta^{R_0}\right|\leq c\beta^{\frac{1}{24}}$. Now if $R_0=R$ and so $\varphi=\theta$, it is imediate that $\tau_\varphi=\frac{b_\theta-a_\theta}{|b_\theta-a_\theta|}$ and so (again recalling definition (48)) (44) follows. On the other hand if $R_0=R^{-1}$ then $\varphi=-\theta$ and so $a_\varphi=b_\theta$, $b_\varphi=a_\theta$, which implies $\tau_\varphi=-\frac{b_\theta-a_\theta}{|b_\theta-a_\theta|}$ so $R\tau_\varphi=R\left(-\frac{b_\theta-a_\theta}{|b_\theta-a_\theta|}\right)=R^{-1}\left(\frac{b_\theta-a_\theta}{|b_\theta-a_\theta|}\right)=R_0\left(\frac{b_\theta-a_\theta}{|b_\theta-a_\theta|}\right)$ hence (again recalling definition (48)),(44) also follows in this case. □

Lemma 4. *Let Ω be a convex body centered on 0 with $\mathrm{diam}(\Omega)=2$. Let $u: W^{2,2}(\Omega)\to\mathbb{R}$ be a function satisfying (6) and (7) and in addition u satisfies $\nabla u(z)\cdot\eta_z=1$ on $\partial\Omega$ in the sense of trace where η_z is the inward pointing unit normal to $\partial\Omega$ at z. Let $a,b\in\Omega$ be such that $\mathrm{diam}(\Omega)=|a-b|$ and $w=\frac{a+b}{2}$, we will show there exists constant $\mathcal{C}_3>1$ and $r_0\in(\mathcal{C}_3^{-1}\beta^{\frac{1}{256}},\mathcal{C}_3\beta^{\frac{1}{256}})$ such that*

$$|u(x)|\geq 1-\mathcal{C}_3\beta^{\frac{1}{256}} \text{ for any } x\in\partial B_{r_0}(w). \tag{57}$$

Proof of Lemma 4. Let $\mathcal{U}$ be the convex set and $\tilde{u}$ be the function constructed in Lemma 2. It is easy to se we can chose $\tilde{a},\tilde{b}\in\mathcal{U}$ such that $\frac{\tilde{a}-\tilde{b}}{|\tilde{a}-\tilde{b}|}=\frac{a-b}{|a-b|}$ and $\left|\tilde{a}-\tilde{b}\right|=\mathrm{diam}(\mathcal{U})$.

Step 1. Let $P:\left[0,H^1(\partial\mathcal{U})\right)\to\partial\mathcal{U}$ be a 'clockwise' parameterisation of $\mathcal{U}$ by arclength with $P(0)=\tilde{a}$. Let $\sigma_1=P(H^1(\partial\mathcal{U})-\beta^{\frac{1}{256}})$ and $\sigma_2=P(\beta^{\frac{1}{256}})$, see figure 3. The points σ_1,σ_2 satisfy the following properties, firstly

$$\eta_{\sigma_i}\cdot e_2\geq 1-c\beta^{\frac{1}{128}} \text{ for } i=1,2. \tag{58}$$

Secondly

$$|\sigma_1-\sigma_2|\leq 2\beta^{\frac{1}{256}}. \tag{59}$$

Thirdly

$$\sigma_1\cdot(-e_1)\geq\frac{\beta^{\frac{1}{256}}}{2} \text{ and } \sigma_2\cdot e_1\geq\frac{\beta^{\frac{1}{256}}}{2}. \tag{60}$$

Proof of Step 1. Firstly note that for any $x\in\partial\mathcal{U}$ we can inscribe a ball $B_{\frac{1}{10}}(z_x)\subset\mathcal{U}$ with $\partial B_{\frac{1}{10}}(z_x)\cap\partial\mathcal{U}=\{x\}$ so the curvature of $\partial\mathcal{U}$ is bounded above by 10 which is equivalent to the bound $\|\ddot{P}\|_{L^\infty(\partial\mathcal{U})}\leq 10$. Since $\dot{P}(\tilde{a})=e_2$, $\left|\dot{P}(\tilde{a})-\dot{P}(\beta^{\frac{1}{256}})\right|\leq 10\beta^{\frac{1}{256}}$ and as $\eta_{\omega_2}=R(\dot{P}(\beta^{\frac{1}{256}}))$ this proves (58) for $i=2$. The proof for $i=1$ follows the same way.

Inequality (59) follows instantly since σ_1 and σ_2 is connected by a path of length less than $2\beta^{\frac{1}{256}}$. Now from (58) for any $t\in[0,\beta^{\frac{1}{256}}]$ we have

$$\left|e_1-\dot{P}(t)\right|=\left|\dot{P}(0)-\dot{P}(t)\right|\leq\int_0^t\left|\ddot{P}(s)\right|ds\leq 10\beta^{\frac{1}{256}}.$$

Thus $\sigma_2\cdot e_1=(\sigma_2-\tilde{a})\cdot e_1=\int_0^{\beta^{\frac{1}{256}}}\dot{P}(s)\cdot e_1 ds\geq(1-10\beta^{\frac{1}{256}})\beta^{\frac{1}{256}}$. Arguing in exactly the same way $(-e_1)\cdot\sigma_1\geq(1-10\beta^{\frac{1}{256}})\beta^{\frac{1}{256}}$ which establishes (60).

Step 2. We will show there exists positive constant $\mathcal{C}_4$ and $x_0\in N_{\mathcal{C}_4\beta^{\frac{1}{256}}}\left(\left[\tilde{a},\tilde{b}\right]\right)\cap\mathcal{U}$ such that for some $\psi_0\in B_{\mathcal{C}_4\beta^{\frac{1}{256}}}(e_2)$ the following inequality holds

$$\left|X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\setminus\left\{x:\left|\nabla u\left(x\right)\cdot e_1\right|<\mathcal{C}_4\beta^{\frac{1}{256}}\right\}\right|\leq\mathcal{C}_4\beta^{\frac{1}{24}}. \tag{61}$$

Proof of Step 2. We know $\eta_{\tilde{a}}=-e_2$ and $\eta_{\tilde{b}}=e_2$. Let $\omega_1\in\partial\mathcal{U}$ be the unique point for which $-\eta_{\omega_1}=\eta_{\sigma_1}$ and let $\omega_2\in\partial\mathcal{U}$ be the unique point for which $-\eta_{\omega_2}=\eta_{\sigma_2}$, see figure 3.

Define

$$\Pi_2:=H\left(\frac{\sigma_2+\omega_2}{2},R\left(\frac{\omega_2-\sigma_2}{\left|\omega_2-\sigma_2\right|}\right)\right)\cap H\left(\frac{\sigma_1+\omega_1}{2},R^{-1}\left(\frac{\omega_1-\sigma_1}{\left|\omega_1-\sigma_1\right|}\right)\right) \tag{62}$$

and

$$\Pi_1:=H\left(\frac{\sigma_2+\omega_2}{2},R^{-1}\left(\frac{\omega_2-\sigma_2}{\left|\omega_2-\sigma_2\right|}\right)\right)\cap H\left(\frac{\sigma_1+\omega_1}{2},R\left(\frac{\omega_1-\sigma_1}{\left|\omega_1-\sigma_1\right|}\right)\right) \tag{63}$$

and let $\Pi=\Pi_1\cup\Pi_2$ and let $x_0:=\overline{\Pi_1}\cap\overline{\Pi_2}$, see figure 3.

First we will show $(x_0+\mathbb{R}e_2)\subset\Pi$ however this inclusion is relatively easy to see because firstly

$$e_2\cdot R\left(\frac{\omega_1-\sigma_1}{\left|\omega_1-\sigma_1\right|}\right)=e_1\cdot\left(\frac{\omega_1-\sigma_1}{\left|\omega_1-\sigma_1\right|}\right)\geq\frac{\beta^{\frac{1}{256}}}{4}$$

thus $l_0^{e_2}\subset H\left(0,R\left(\frac{\omega_1-\sigma_1}{|\omega_1-\sigma_1|}\right)\right)$. And secondly as $x_0\in\partial H\left(\frac{\sigma_1+\omega_1}{2},R\left(\frac{\omega_1-\sigma_1}{|\omega_1-\sigma_1|}\right)\right)$

$$l_{x_0}^{e_2}\subset H\left(x_0,R\left(\frac{\omega_1-\sigma_1}{\left|\omega_1-\sigma_1\right|}\right)\right)=H\left(\frac{\sigma_1+\omega_1}{2},R\left(\frac{\omega_1-\sigma_1}{\left|\omega_1-\sigma_1\right|}\right)\right).$$

In exactly the same way $l_{x_0}^{e_2}\subset H\left(\frac{\sigma_2+\omega_2}{2},R^{-1}\left(\frac{\omega_2-\sigma_2}{|\omega_2-\sigma_2|}\right)\right)$. Hence $l_{x_0}^{e_2}\subset\Pi_1$. Arguing in the same manner we have $l_{x_0}^{-e_2}\subset\Pi_2$ and thus we have established the claim.

Let $\gamma=l_{x_0}^{e_2}\cap\partial\mathcal{U}$, by construction we have that γ lies in the component of $\partial\mathcal{U}$ between σ_1 and σ_2 and hence we know $d\left(\gamma,l_{x_0}^{e_2}\right)\leq c\beta^{\frac{1}{256}}$ and so it follows $x_0\in N_{c\beta^{\frac{1}{256}}}\left(\left[\tilde{a},\tilde{b}\right]\right)\cap\mathcal{U}$

Since $\eta_{\tilde{a}}=-e_2$, $\eta_{\tilde{b}}=e_2$ and $\mathcal{U}$ is convex we know $\omega_2\in H\left(0,-e_1\right)$ and for the same reasons $\omega_1\in H\left(0,e_1\right)$ see figure 3. So $(\sigma_2-\omega_2)\cdot e_1\geq\sigma_2\cdot e_1\geq c\beta^{\frac{1}{256}}$ and for exactly the same reason $(\sigma_1-\omega_1)\cdot(-e_1)\geq\sigma_2\cdot(-e_1)\geq c\beta^{\frac{1}{256}}$. Thus as $|\sigma_1-\omega_1|\leq 2\mathrm{diam}\left(\mathcal{U}\right)$ and $|\sigma_2-\omega_2|\leq 2\mathrm{diam}\left(\mathcal{U}\right)$ we have $\frac{\sigma_2-\omega_2}{|\sigma_2-\omega_2|}\cdot e_1\geq c\beta^{\frac{1}{256}}$ and $\frac{\sigma_1-\omega_1}{|\sigma_1-\omega_1|}\cdot(-e_1)\geq c\beta^{\frac{1}{256}}$. Hence

$$\begin{aligned}\left(\frac{\sigma_1-\omega_1}{\left|\sigma_1-\omega_1\right|}\right)\cdot\left(\frac{\sigma_2-\omega_2}{\left|\sigma_2-\omega_2\right|}\right) &= \left(\frac{\sigma_1-\omega_1}{\left|\sigma_1-\omega_1\right|}\cdot e_1\right)\left(\frac{\sigma_2-\omega_2}{\left|\sigma_2-\omega_2\right|}\cdot e_1\right)\\ &\quad+\left(\frac{\sigma_1-\omega_1}{\left|\sigma_1-\omega_1\right|}\cdot e_2\right)\left(\frac{\sigma_2-\omega_2}{\left|\sigma_2-\omega_2\right|}\cdot e_2\right)\\ &\leq -c\beta^{\frac{1}{128}}+1.\end{aligned}$$

In other words the angle between $\frac{\sigma_1-\omega_1}{|\sigma_1-\omega_1|}$ and $\frac{\sigma_2-\omega_2}{|\sigma_2-\omega_2|}$ is greater than $\mathcal{C}_4\beta^{\frac{1}{256}}$ for some positive constant $\mathcal{C}_4$. Thus there exists $\psi_0\in B_{c\beta^{\frac{1}{256}}}(e_2)$ such that $X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\subset\Pi$. By Lemma 3 we know that

$$\left|\mathcal{U}\cap H\left(\frac{\sigma_2+\omega_2}{2},R^{-1}\left(\frac{\omega_2-\sigma_2}{\left|\omega_2-\sigma_2\right|}\right)\right)\setminus\mathcal{G}_{\eta_{\sigma_2}}^{R^{-1}}\right|\leq c\beta^{\frac{1}{24}}$$

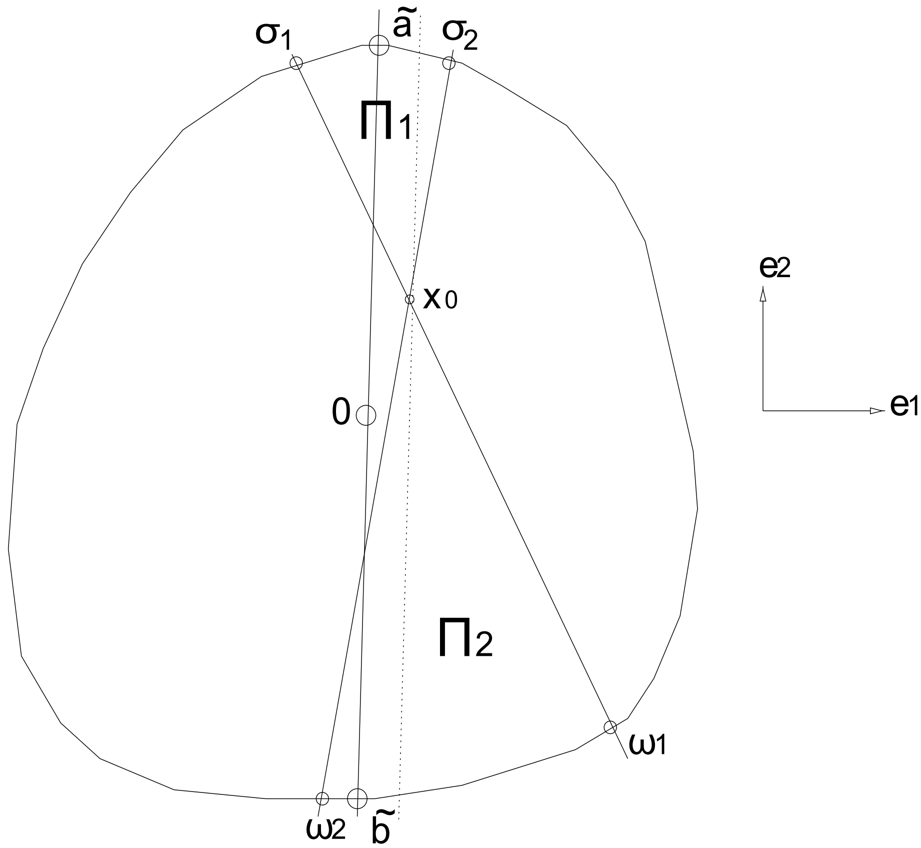

FIGURE 3

and

$$\left|\mathcal{U}\cap H\left(\frac{\sigma_1+\omega_1}{2},R\left(\frac{\omega_1-\sigma_1}{|\omega_1-\sigma_1|}\right)\right)\backslash\mathcal{G}^R_{\eta_{\sigma_1}}\right|\leq c\beta^{\frac{1}{24}}.$$

Thus (recalling the definition of Π_1, (63))

$$\left|\Pi_1\cap\mathcal{U}\backslash\mathcal{G}^{R^{-1}}_{\eta_{\sigma_2}}\cap\mathcal{G}^R_{\eta_{\sigma_1}}\right|\leq c\beta^{\frac{1}{24}}. \tag{64}$$

In exactly the same way we have (recall (62))

$$\left|\Pi_2\cap\mathcal{U}\backslash\mathcal{G}^{R^{-1}}_{\eta_{\sigma_1}}\cap\mathcal{G}^R_{\eta_{\sigma_2}}\right|\leq c\beta^{\frac{1}{24}}. \tag{65}$$

Now for any $x\in\mathcal{G}^{R^{-1}}_{\eta_{\sigma_2}}\cap\mathcal{G}^R_{\eta_{\sigma_1}}$ we have $\nabla u(x)\cdot R\eta_{\sigma_2}\geq 0$ and $\nabla u(x)\cdot R^{-1}\eta_{\sigma_1}\geq 0$. Since from (58) $\eta_{\sigma_i}\in X^+\left(0,e_2,c\beta^{\frac{1}{256}}\right)$ for $i=1,2$ we know $R\eta_{\sigma_2}\in X^+\left(0,-e_1,c\beta^{\frac{1}{256}}\right)$ and $R^{-1}\eta_{\sigma_1}\in X^+\left(0,e_1,\beta^{\frac{1}{256}}\right)$, from this it is easy to see (assuming we chose $\mathcal{C}_4$ large enough)$|\nabla u(x)\cdot e_1|\leq$

$\mathcal{C}_4\beta^{\frac{1}{256}}$. And in the same way for any $x \in \mathcal{G}^{R^{-1}}_{\eta_{\sigma_1}} \cap \mathcal{G}^{R}_{\eta_{\sigma_2}}$ we also have $|\nabla u(x) \cdot e_1| \leq \mathcal{C}_1\beta^{\frac{1}{256}}$.

$$\begin{aligned}&\left|X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\setminus\left\{x:|\nabla u(x)\cdot e_1|<\mathcal{C}_4\beta^{\frac{1}{256}}\right\}\right|\\&\quad\leq c\left|\Pi_1\cap\mathcal{U}\backslash\mathcal{G}^{R}_{\eta_{\sigma_1}}\cap\mathcal{G}^{R^{-1}}_{\eta_{\sigma_2}}\right|+c\left|\Pi_2\cap\mathcal{U}\backslash\mathcal{G}^{R}_{\eta_{\sigma_2}}\cap\mathcal{G}^{R^{-1}}_{\eta_{\sigma_1}}\right|\\&\quad\leq\mathcal{C}_4\beta^{\frac{1}{24}}\end{aligned}$$

which establishes (61).

Step 3. There exists constant $\mathcal{C}_5$ such that for any $w \in \mathbb{R}^2$ define

$$\mathbb{V}_w := \left\{x \in \mathcal{U} : \nabla u(x) \in N_{\mathcal{C}_5\beta^{\frac{1}{256}}}(w)\right\},$$

we will show there exists $v_1 \in \{e_2, -e_2\}$ such that

$$\left|X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\cap H\left(\mathcal{C}_5\beta^{\frac{1}{256}}v_1,v_1\right)\cap\mathcal{U}\backslash\mathbb{V}_{-v_1}\right|\leq\mathcal{C}_5\beta^{\frac{1}{24}}. \tag{66}$$

Proof of Step 3. Let $\varpi_0 = l_0^{-e_1} \cap \partial\mathcal{U}$. Note since $\mathcal{U}$ is convex $\eta_{\varpi_0} \cdot e_1 > 0$. We claim

$$\eta_{\varpi_0}\cdot e_1>\frac{1}{10}. \tag{67}$$

Suppose this were not the case, then $\eta_{\varpi_0} \cdot e_1 \leq \frac{1}{10}$. Since $\mathcal{U}$ is convex and $\text{diam}(\mathcal{U}) < 2$ we know $\mathcal{U} \subset \overline{H(\varpi_0, \eta_{\varpi_0})} \subset H(-2e_1, \eta_{\varpi_0})$ which implies $(\tilde{b} + 2e_1) \cdot \eta_{\varpi_0} > 0$ and thus

$$\tilde{b}\cdot e_2\sqrt{\frac{99}{100}}\geq\left(\left(\tilde{b}+2e_1\right)\cdot e_2\right)(\eta_{\varpi_0}\cdot e_2)>-\left(\left(\tilde{b}+2e_1\right)\cdot e_1\right)(\eta_{\varpi_0}\cdot e_1)=-2\eta_{\varpi_0}\cdot e_1\geq-\frac{1}{5}$$

however as $\left|\tilde{a}-\tilde{b}\right| = \text{diam}(\mathcal{U}) = \frac{22}{10}$, $\frac{\tilde{a}+\tilde{b}}{2} = 0$ and $\frac{\tilde{a}-\tilde{b}}{|\tilde{a}-\tilde{b}|} = e_2$ this is a contradiction.

Let $\varpi_1 \in \partial\mathcal{U}$ be the unique point for which $\eta_{\varpi_1} = -\eta_{\varpi_0}$. Since $\eta_{\varpi_1} \cdot (-e_1) \geq \frac{1}{10}$ we must have that $\varpi_1 \in H(0, e_1) \cap \partial\mathcal{U}$. Now let $l \in \left(\frac{\varpi_1-\varpi_0}{|\varpi_1-\varpi_0|}\right)^{\perp} \cap S^1$ be such that

$$H^1\left([a,b]\cap H\left(\frac{\varpi_1+\varpi_0}{2},l\right)\right)\geq\frac{|a-b|}{2}. \tag{68}$$

Chose $S \in \left\{R^{-1}, R\right\}$ so that $S\left(\frac{\varpi_1-\varpi_0}{|\varpi_1-\varpi_0|}\right) = l$, by Lemma 3 we have

$$\left|\mathcal{U}\cap H\left(\frac{\varpi_1+\varpi_0}{2},l\right)\backslash\mathcal{G}^{S}_{\eta_{\varpi_0}}\right|\leq c\beta^{\frac{1}{24}}. \tag{69}$$

From (7) and (61) we know

$$\left|X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\setminus\{x:\nabla u(x)\in N_{100^{-1}}(\{e_2,-e_2\})\}\right|\leq c\beta^{\frac{1}{24}}. \tag{70}$$

Since so $\left|S^{-1}\eta_{\varpi_0} \cdot e_2\right| \overset{(67)}{>} 10^{-1}$ if $x \in \mathcal{G}^{S}_{\eta_{\varpi_0}} \cap \{x : \nabla u(x) \in N_{100^{-1}}(\{e_2, -e_2\})\}$ then $\nabla u(x) \in B_{100^{-1}}(v_0)$ for some $v_0 \in \{e_2, -e_2\}$ and so using (69) and (70)

$$\left|\mathcal{U}\cap X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\cap H\left(\frac{\varpi_1+\varpi_0}{2},l\right)\setminus\{x:\nabla u(x)\in B_{100^{-1}}(v_0)\}\right|\leq c\beta^{\frac{1}{24}}. \tag{71}$$

Now for any $w \in H(0, v_0)$ we have the elementary inequality $|w - v_0| \leq 4d(w, S^1) + |w \cdot e_1|$, so using (7), (61) and (71) we have

$$\left|\mathcal{U}\cap X\left(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}}\right)\cap H\left(\frac{\varpi_1+\varpi_0}{2},l\right)\backslash\mathbb{V}_{v_0}\right|\leq c\beta^{\frac{1}{24}}. \tag{72}$$

Recall $\varpi_0 = l_0^{-e_1} \cap \partial\mathcal{U}$ and $\varpi_1 \in H(0, e_1) \cap \partial\mathcal{U}$, so $\left|\frac{\varpi_0-\varpi_1}{|\varpi_0-\varpi_1|} \cdot e_1\right| \geq \frac{1}{10}$ and thus $|l \cdot e_2| \geq \frac{1}{10}$ so there for by the fact that $\psi_0 \in B_{\mathcal{C}_4\beta^{\frac{1}{256}}}(e_2)$ and that inequality (68) implies $0 \in \overline{H(\frac{\varpi_1+\varpi_0}{2}, l)}$ there exists $v_1 \in \{e_2, -e_2\}$ such that for some constant $\mathcal{C}_5$ we have

$$X\left(x_0, \psi_0, \mathcal{C}_4\beta^{\frac{1}{256}}\right) \cap H\left(\mathcal{C}_5\beta^{\frac{1}{256}}v_1, v_1\right) \subset H\left(\frac{\varpi_1 + \varpi_0}{2}, l\right) \tag{73}$$

putting this together with (72) gives

$$\left|\mathcal{U} \cap X\left(x_0, \psi_0, \mathcal{C}_4\beta^{\frac{1}{256}}\right) \cap H\left(\mathcal{C}_5\beta^{\frac{1}{256}}v_1, v_1\right) \backslash \mathbb{V}_{v_0}\right| \leq c\beta^{\frac{1}{24}}.$$

Let $x \in \mathcal{U}\backslash\overline{\Omega} \cap X\left(x_0, \psi_0, \mathcal{C}_4\beta^{\frac{1}{256}}\right) \cap H(\mathcal{C}_5\beta^{\frac{1}{256}}v_1, v_1)$ so as $\tilde{u}(x) = d(x, \partial\mathcal{U})$ and since $\psi_0 \in B_{c\beta^{\frac{1}{256}}}(e_2)$ so $\nabla\tilde{u}(x) \in N_{\mathcal{C}_5\beta^{\frac{1}{256}}}(-v_1)$ thus we must have $v_0 = -v_1$, this gives (66).

Step 4. We will show there exists a positive constant $\mathcal{C}_6$ such that

$$l_x^\theta \backslash B_{\mathcal{C}_6\beta^{\frac{1}{128}}}(x) \subset X(x_0, \psi_0, \mathcal{C}_4\beta^{\frac{1}{256}}) \text{ for all } x \in B_{\frac{1}{128}}(x_0), \theta \in S^1 \cap B_{\beta^{\frac{1}{128}}}(\psi_0) \tag{74}$$

Proof of Step 4. Without loss of generality we assume $x_0 = 0$, $\psi_0 = e_2$ and $\mathcal{C}_5 = 1$. To begin with to take point $x = \beta^{\frac{1}{128}}e_1$, we will show later the general case follows from this. See figure 4.

Let $\theta = \begin{pmatrix} \sin\beta^{\frac{1}{128}} \\ \cos\beta^{\frac{1}{128}} \end{pmatrix}$ and let $y = \partial X(0, e_2, \beta^{\frac{1}{256}}) \cap l_x^\theta$. We will get an upper bound on $|y|$. Let $z = y \cdot e_1 e_1$. We have two triangles to calculate with, triangle T_1 with corners on $0, x, y$ which is a subset of triangle T_2 with corners on $0, z, y$. Note that by applying the law of sins we have $|y|^{-1}\sin(\frac{\pi}{2}+\beta^{\frac{1}{128}}) = |x-y|^{-1}\sin(\frac{\pi}{2}-\beta^{\frac{1}{256}})$. Note that $T_3 = T_2\backslash T_1$ is also a right angle triangle and since $|z| = \beta^{\frac{1}{128}} + |x-z|$ we have $|y|\cos(\frac{\pi}{2}-\beta^{\frac{1}{256}}) = \beta^{\frac{1}{128}} + |y-x|\cos(\frac{\pi}{2}-\beta^{\frac{1}{128}})$. Putting this together with the previous equation we have $|y|\sin\beta^{\frac{1}{256}} = \beta^{\frac{1}{128}} + |y|\frac{\cos\beta^{\frac{1}{256}}}{\cos\beta^{\frac{1}{128}}}\sin\beta^{\frac{1}{128}}$ which gives $|y|\left(\sin\beta^{\frac{1}{256}} - \frac{\cos\beta^{\frac{1}{256}}}{\cos\beta^{\frac{1}{128}}}\sin\beta^{\frac{1}{128}}\right) = \beta^{\frac{1}{128}}$. Now by taking the Taylor series approximating sin and cos we have $|y|\left(\beta^{\frac{1}{256}} + O\left(\beta^{\frac{1}{128}}\right)\right) = \beta^{\frac{1}{128}}$. Thus $|y| \sim \beta^{\frac{1}{256}}$ and thus the existence of constant $\mathcal{C}_6$ such that (74) holds follows instantly for the case $x = \beta^{\frac{1}{128}}e_1$.

In the general case where $x \neq \beta^{\frac{1}{128}}e_1$ define $x_0 = (x + \langle\theta\rangle) \cap \langle e_1\rangle$, since the angle between θ and e_1 is with $\beta^{\frac{1}{256}}$ of $\frac{\pi}{2}$ it is easy to see $x_0 \in B_{2\beta^{\frac{1}{128}}}(0)$ and of course $l_{x_0}^\theta \cap \partial X(0, e_2, \beta^{\frac{1}{256}}) = l_x^\theta \cap \partial X(0, e_2, \beta^{\frac{1}{256}})$ so the argument for the special case $x = \beta^{\frac{1}{128}}e_1$ can be applied to show the existence of constant $\mathcal{C}_6$ satisfying (74).

Step 5. We will establish (57).
Proof of Step 5. Let

$$h(z) := \mathbb{1}_{X\left(x_0, \psi_0, \mathcal{C}_4\beta^{\frac{1}{256}}\right) \cap H\left(\mathcal{C}_5\beta^{\frac{1}{256}}v_1, v_1\right) \cap \mathcal{U}\backslash\mathbb{V}_{-v_1}} \tag{75}$$

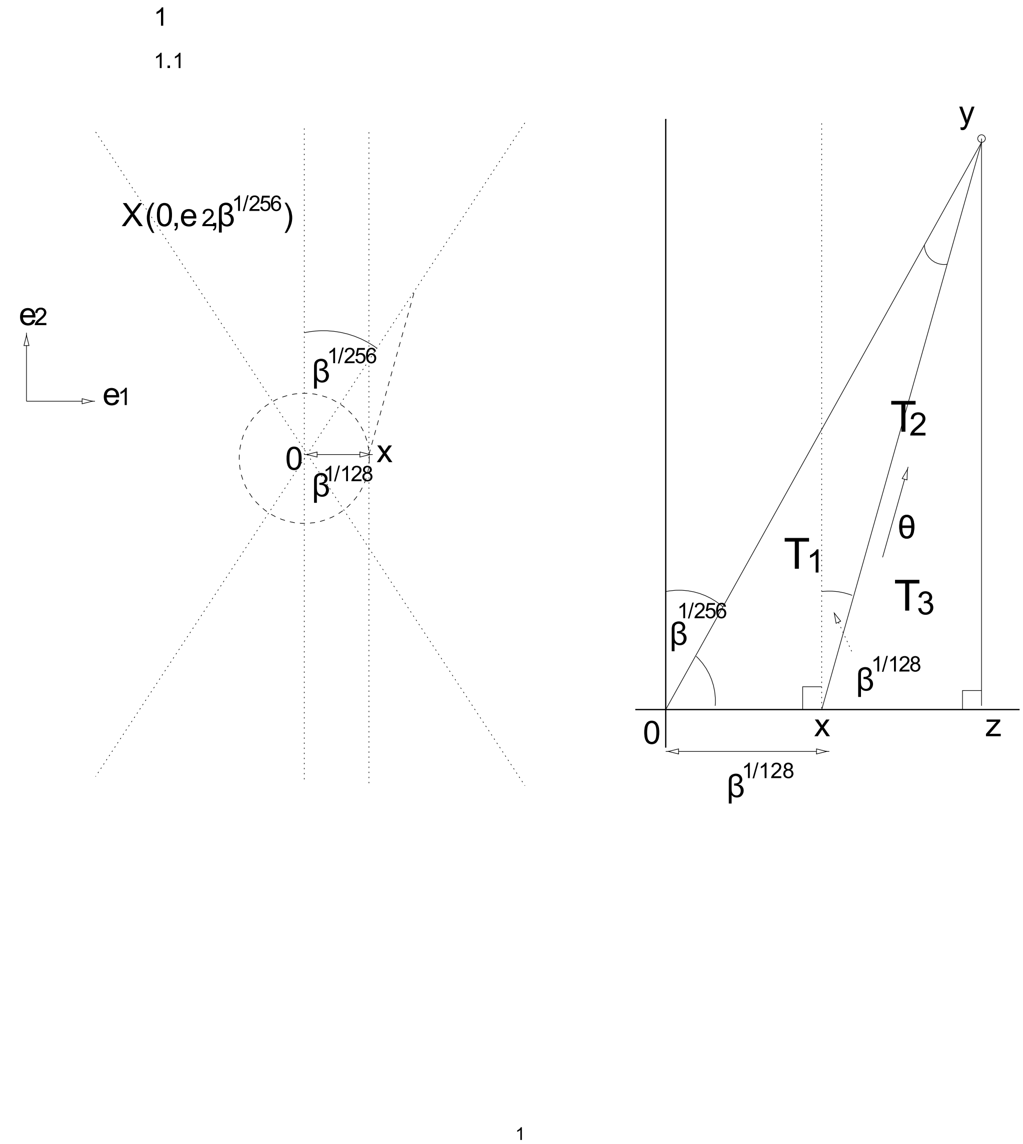

FIGURE 4

so we know $\int h \overset{(66)}{\leq} c\beta^{\frac{1}{24}}$. So by the Fubini's Theorem

$$
\begin{aligned}
&\int_{\mathcal{U}}\int_{\mathcal{U}} \left(h(z)+\beta^{-1}\left|1-|\nabla\tilde{u}(z)|^2\right|\right)|z-x|^{-1}\,dzdx \\
&\qquad\leq \int_{\mathcal{U}} \left(h(z)+\beta^{-1}\left|1-|\nabla\tilde{u}(z)|^2\right|\right)\left(\int |z-x|^{-1}\,dx\right)dz \\
&\qquad\leq c\int_{\mathcal{U}} \left(h(z)+\beta^{-1}\left|1-|\nabla\tilde{u}(z)|^2\right|\right)dz \\
&\qquad\overset{(7)}{\leq} c\beta^{\frac{1}{24}}. \qquad (76)
\end{aligned}
$$

Let

$$G:=\left\{x\in B_{\beta^{\frac{1}{128}}}(x_0):\int_{\mathcal{U}}\left(h(z)+\beta^{-1}\left|1-|\nabla\tilde{u}(z)|^2\right|\right)|z-x|^{-1}\,dz\leq c\beta^{\frac{1}{256}}\right\}$$

so we know $\beta^{\frac{1}{256}}\left|B_{\beta^{\frac{1}{128}}}(x_0)\backslash G\right|\leq c\beta^{\frac{1}{24}}$, thus $\left|B_{\beta^{\frac{1}{128}}}(x_0)\backslash G\right|\leq c\beta^{\frac{7}{202}}$, since $\beta^{\frac{7}{207}}<\beta^{\frac{1}{64}}$ assuming β is small enough $|G|\geq 2^{-1}\beta^{\frac{1}{64}}$. By Step 4, (74) for any $x\in B_{\beta^{\frac{1}{128}}}(x_0)$, $\theta\in B_{\beta^{\frac{1}{128}}}(\psi_0)\cap S^1$ we have $l_x^{\theta}\backslash B_{\mathcal{C}_6\beta^{\frac{1}{256}}}(x)\subset X(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}})$.

Since $X(0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}})=X(0,-\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}})$ we can assume without loss generality that $\psi_0\cdot v_1>0$. Pick $x\in G$, by the Co-area formula we must be able to find $\theta_1\in B_{\beta^{\frac{1}{128}}}(\psi_0)\cap S^1$ such that

$$\int_{l_x^{\theta_1}\cap\mathcal{U}}h(z)+\beta^{-1}\left|1-|\nabla\tilde{u}(z)|^2\right|dH^1z\leq c\beta^{\frac{1}{256}} \tag{77}$$

Recall inequality (73), we will assume $\mathcal{C}_6$ is large enough so that

$$l_x^{\theta_1}\backslash B_{\mathcal{C}_6\beta^{\frac{1}{256}}}(x)\subset H(\mathcal{C}_5\beta^{\frac{1}{256}}v_1,v_1).$$

So let d,e be the endpoints of the segments $l_x^{\theta_1}\cap\mathcal{U}\backslash B_{\mathcal{C}_6\beta^{\frac{1}{256}}}(x)$ where we chose $d\in\partial B_{\mathcal{C}_6\beta^{\frac{1}{256}}}(x)$ and $e\in\partial\mathcal{U}$, note $[d,e]\subset X(x_0,\psi_0,\mathcal{C}_4\beta^{\frac{1}{256}})\cap H(\mathcal{C}_5\beta^{\frac{1}{256}}v_1,v_1)$. Now since $\theta_1\in B_{\beta^{\frac{1}{128}}}(\psi_0)\subset B_{\mathcal{C}_4\beta^{\frac{1}{256}}}(v_1)$ and for any $z\in[d,e]$ with $h(z)=0$ by (75) we have $z\in\mathbb{V}_{-v_1}$ and so $\nabla\tilde{u}(z)\in B_{\mathcal{C}_5\beta^{\frac{1}{256}}}(-v_1)$ then $\nabla\tilde{u}(z)\cdot(-\theta_1)\geq 1-c\beta^{\frac{1}{128}}$. Thus by the fundamental theorem of Calculus

$$\begin{aligned}
|\tilde{u}(d)-\tilde{u}(e)| &= \left|\int_{[d,e]}\nabla u(z)\cdot\theta_1 dH^1z\right| \\
&\geq \left|\int_{\mathbb{V}_{-v_1}\cap[d,e]}\nabla\tilde{u}(z)\cdot\theta_1 dH^1z\right|-\int_{[d,e]\backslash\mathbb{V}_{-v_1}}|\nabla\tilde{u}(z)|\,dH^1z \\
&\geq \left(1-c\beta^{\frac{1}{128}}\right)H^1\left(\mathbb{V}_{-v_1}\cap[d,e]\right)-H^1\left([d,e]\backslash\mathbb{V}_{-v_1}\right) \\
&\quad -c\int_{[d,e]}\left|1-|\nabla\tilde{u}|^2\right|dH^1 \\
&\overset{(77)}{\geq} |d-e|\,(1-c\beta^{\frac{1}{256}}).
\end{aligned} \tag{78}$$

Since the curvature of $\partial\mathcal{U}$ is bounded above by 10 it is easy to see that

$$|e-\tilde{a}|\leq c\beta^{\frac{1}{256}}, \tag{79}$$

it is also easy to see $[e,\tilde{a}]\subset\mathcal{U}\backslash\Omega$ and $\tilde{u}$ is 1-Lipschitz on $\mathcal{U}\backslash\Omega$ so

$$|\tilde{u}(e)-\tilde{u}(\tilde{a})|\leq c\beta^{\frac{1}{256}}. \tag{80}$$

Thus we have

$$\begin{aligned}
|\tilde{u}(d)| &= |\tilde{u}(d)-\tilde{u}(\tilde{a})| \\
&\overset{(78),(79),(80)}{\geq} |d-\tilde{a}|-c\beta^{\frac{1}{256}} \\
&\geq |\tilde{a}|-c\beta^{\frac{1}{256}}=2^{-1}\mathrm{diam}(\mathcal{U})-c\beta^{\frac{1}{256}}.
\end{aligned} \tag{81}$$

Pick $r_0\in[|d|,2|d|]$ such that $\int_{\partial B_{r_0}(0)}\left|1-|\nabla\tilde{u}(z)|^2\right|dH^1z\leq c\beta^{-\frac{1}{256}}\beta$. Now fix $y\in\partial B_{r_0}(0)$, let $s=[d,e]\cap\partial B_{r_0}(0)$ and Γ_1 denote a connected component of $\partial B_{r_0}(0)\backslash\{s,y\}$. So we know $\int_{\Gamma_1\cup[d,s]}|\nabla\tilde{u}(z)|\,dH^1z\leq cH^1(\Gamma_1\cup[d,s])\leq c\beta^{\frac{1}{256}}$ so we can apply the fundamental theorem of

Calculus we have that $|u(y)-u(d)|\le c\beta^{\frac{1}{256}}$ and since y is an arbitrary fixed point in $\partial B_{r_0}(0)$, using (81) this gives

$$\inf\{|\tilde{u}(z)|: z\in\partial B_{r_0}(0)\}\ge 2^{-1}\mathrm{diam}(\mathcal{U})-c\beta^{\frac{1}{256}}. \tag{82}$$

By definition (see (39)) $\tilde{u}(z)=u(z)+10^{-1}$ for any $z\in\partial B_{r_0}(0)$. Since $\mathrm{diam}(\mathcal{U})=\frac{22}{10}$ putting this with (82) we have (57). □

Proof of Theorem 2. Let $r_0\in(\mathcal{C}_3^{-1}\beta^{\frac{1}{256}},\mathcal{C}_3\beta^{\frac{1}{256}})$ be a number we obtain from Lemma 4 that satisfies (57). By Fubini's Theorem we know $\int_\Omega\int_\Omega\left|1-|\nabla u(z)|^2\right||z-y|^{-1}\,dzdy\le\mathcal{C}_7\beta^2$ for some constant $\mathcal{C}_7>0$. Let

$$G_0:=\left\{y\in\Omega:\int_\Omega\left|1-|\nabla u(z)|^2\right|^2|z-y|^{-1}\,dz\le\beta\right\}. \tag{83}$$

Note that $|\Omega\backslash G_0|\le\mathcal{C}_7\beta$.

Since $r_0>\mathcal{C}_3^{-1}\beta^{\frac{1}{256}}$ we can pick $x_0\in B_{\sqrt{\beta}}(0)\cap G_0\subset B_{r_0}(0)$. So by the Co-area formula there exists $\Psi\subset S^1$ such that $H^1(S^1\backslash\Psi)\le\sqrt{\beta}$ and

$$\int_{l^\theta_{x_0}}\left|1-|\nabla u|^2\right|dH^1z\le c\sqrt{\beta}\text{ for each }\theta\in\Psi. \tag{84}$$

For any $\theta\in S^1$ define $P(\theta):=l^\theta_{x_0}\cap\partial\Omega$, we will show

$$|P(\theta)-x_0|\ge 1-c\beta^{\frac{1}{256}}\text{ for any }\theta\in\Psi. \tag{85}$$

To see this we argue as follows, for each $z\in[x,P(\theta)]$ let $\theta_z\in S^1$ be such that $|\nabla u(z)-\theta_z|=d(\nabla u(z),S^1)$. Note $d(\nabla u(z),S^1)\le c\,||\nabla u(z)|-1|$

$$\begin{aligned}\int_{[x_0,P(\theta)]}|\nabla u(z)-\theta_z|\,dH^1z &\le c\left(\int_{[x_0,P(\theta)]}\left||\nabla u|^2-1\right|^2dH^1z\right)^{\frac{1}{2}}\\ &\overset{(84)}{\le} c\beta^{\frac{1}{2}}.\end{aligned} \tag{86}$$

So

$$\begin{aligned}|u(x_0)| &= |u(x_0)-u(P(\theta))|\\ &= \left|\int_{[x_0,P(\theta)]}\nabla u(z)\cdot\theta dH^1z\right|\\ &\overset{(86)}{\le}\left|\int_{[x_0,P(\theta)]}\theta_z\cdot\theta dH^1z\right|+c\beta^{\frac{1}{2}}\\ &\le |x_0-P(\theta)|+c\beta^{\frac{1}{2}}.\end{aligned} \tag{87}$$

Let $y_\theta:=[x_0,P(\theta)]\cap\partial B_{r_0}(0)$. In exactly the same way we have

$$|u(y_\theta)-u(x_0)|\le c\beta^{\frac{1}{256}}. \tag{88}$$

So

$$|u(x_0)|\ge|u(y_\theta)|-|u(y_\theta)-u(x_0)|\overset{(88)}{\ge}|u(y_\theta)|-c\beta^{\frac{1}{256}}\overset{(57)}{\ge}1-c\beta^{\frac{1}{256}} \tag{89}$$

this together with (87) establishes (85).

Let $N=\left[2^{-1}\beta^{-\frac{1}{2}}\right]$, we can divide S^1 into N disjoint pieces of equal length, denote them $I_1,I_2,\dots I_N$. Formally; $\bigcup_{k=1}^N I_k=S^1$ and $H^1(I_k)=\frac{2\pi}{N}$ for each $k=1,2,\dots N$. We can pick $\theta_k\in I_k\cap\Psi$ for each $k=1,2,\dots N$.

Let

$$h=\min\{|P(\theta_k)-x_0|:k\in\{1,2,\dots N\}\}. \tag{90}$$

We define Π to be the convex hull of the points $x_0+h\theta_1,x_0+h\theta_2,\dots x_0+h\theta_N$. Now by the construction of Π, for any $y\in\partial\Pi$ we can find $k\in\{1,2,\dots N\}$ such that $|y-(x_0+h\theta_k)|\leq c\sqrt{\beta}$ and thus $|y-x_0|\geq h-c\sqrt{\beta}$ and so

$$B_{h-c\sqrt{\beta}}(x_0)\subset\Pi. \tag{91}$$

Note that by using (85) we know $h>1-c\beta^{\frac{1}{256}}$ and since $|x_0|\leq\sqrt{\beta}$ (recalling also that Ω is convex and so $\Pi\subset\Omega$) there exists positive constant $\mathcal{C}_8$ such that

$$B_{1-\mathcal{C}_8\beta^{\frac{1}{256}}}(0)\subset\Omega. \tag{92}$$

We claim

$$\Omega\subset B_{1+2\mathcal{C}_8\beta^{\frac{1}{256}}}(0). \tag{93}$$

Suppose not, so there exists $y\in\partial\Omega$ such that $|y|\geq 1+2\mathcal{C}_8\beta^{\frac{1}{256}}$. By inequality (92) we know $-\frac{y}{|y|}\left(1-\mathcal{C}_8\beta^{\frac{1}{256}}\right)\subset\Omega$ and as by convexity of Ω, $\left[y,-\frac{y}{|y|}\left(1-\mathcal{C}_8\beta^{\frac{1}{256}}\right)\right]\subset\Omega$ thus

$$H^1\left(\left[y,-\frac{y}{|y|}\left(1-\mathcal{C}_8\beta^{\frac{1}{256}}\right)\right]\right)=2+\mathcal{C}_8\beta^{\frac{1}{256}}$$

which contradicts the fact $\mathrm{diam}(\Omega)=2$ hence (93) is established. Now $|x_0-P(\theta)|\leq|P(\theta)|+|x_0|\overset{(93)}{\leq}1+c\beta^{\frac{1}{256}}$ so putting this together with (89) we have

$$|u(x_0)-u(P(\theta))|=|u(x_0)|\geq|x_0-P(\theta)|-c\beta^{\frac{1}{256}}. \tag{94}$$

Thus

$$\begin{aligned}\int_{[x_0,P(\theta)]}|\nabla u(z)-\theta|^2\,dH^1z &= \int_{[x_0,P(\theta)]}\left(|\nabla u(z)|^2-2\nabla u(z)\cdot\theta+1\right)dH^1z\\ &\overset{(84)}{\leq} 2(1+c\beta^{\frac{1}{4}})\,|x_0-P(\theta)|-2\,|u(P(\theta))-u(x_0)|\\ &\overset{(94)}{\leq} c\beta^{\frac{1}{256}}\text{ for any }\theta\in\Psi.\end{aligned} \tag{95}$$

Now using the elementary fact that $\left|\nabla u(z)-\frac{z-x_i}{|z-x_i|}\right|^2\leq\left||\nabla u(z)|^2-1\right|^2+4$ since $x_0\in G_0$ we have

$$\begin{aligned}&\int_{\theta\in S^1\backslash\Psi}\int_{l^\theta_{x_0}}\left|\nabla u(z)-\frac{z-x_0}{|z-x_0|}\right|^2dH^1zdH^1\theta\\ &\qquad\leq 4H^1(S^1\backslash\Psi)+\int_{\theta\in S^1}\int_{l^\theta_{x_0}}\left||\nabla u(z)|^2-1\right|^2dH^1zdH^1\theta\\ &\qquad\overset{(83)}{\leq}5\sqrt{\beta}.\end{aligned} \tag{96}$$

And thus

$$\begin{aligned}\int_\Omega\left|\nabla u(z)-\frac{z-x_0}{|z-x_0|}\right|^2dz &\leq \int_\Omega\left|\nabla u(z)-\frac{z-x_0}{|z-x_0|}\right|^2|z-x_0|^{-1}\,dz\\ &\leq \int_{\theta\in S^1}\int_{l^\theta_{x_0}}\left|\nabla u(z)-\frac{z-x_0}{|z-x_0|}\right|^2dH^1zdH^1\theta\\ &\overset{(96),(95)}{\leq} c\beta^{\frac{1}{256}}.\end{aligned}$$

By Holder's inequality this gives

$$\left(\int_\Omega \left|\nabla u(z)-\frac{z-x_0}{|z-x_0|}\right|^2\right)^{\frac{1}{2}}\leq c\beta^{\frac{1}{512}}. \tag{97}$$

Now for any $z\in\Omega\backslash B_{\beta^{\frac{1}{256}}}(0)$, since $|x_0|\leq c\sqrt{\beta}$

$$\begin{aligned}\left|\frac{z}{|z|}-\frac{z-x_0}{|z-x_0|}\right| &= \left|\frac{z\,|z-x_0|-(z-x_0)\,|z|}{|z|\,|z-x_0|}\right|\\ &= \left|\frac{z(|z-x_0|-|z|)+x_0\,|z|}{|z|\,|z-x_0|}\right|\\ &\leq \left|\frac{|z-x_0|-|z|}{|z-x_0|}\right|+\frac{|x_0|}{|z-x_0|}\\ &\leq c\beta^{\frac{1}{4}}.\end{aligned}$$

So

$$\left(\int_\Omega \left|\frac{z}{|z|}-\frac{z-x_0}{|z-x_0|}\right|^2\right)^{\frac{1}{2}}\leq c\beta^{\frac{1}{128}}+\left(\int_{\Omega\backslash B_{\beta^{\frac{1}{256}}}}\left|\frac{z}{|z|}-\frac{z-x_0}{|z-x_0|}\right|^2\right)^{\frac{1}{2}}\overset{(97)}{\leq} c\beta^{\frac{1}{512}}.$$

Putting this together with (97) we have (8). □

4. Construction of upper bound for nearly circular domains

In this section we will show that given a convex domain Ω with C^2 boundary with curvature bounded above by $\epsilon^{-\frac{1}{5}}$ and that satisfies $|B_1(0)\triangle\Omega|\leq\beta$ we will construct a function u with $I_\epsilon(u)\leq\beta^{\frac{3}{16}}$, this is the contents of Proposition 1 below. The proof of Corollary 1 will follows easily from this.

Proposition 1. *Let Ω be a convex body with C^2 boundary and with curvature bounded above by $\epsilon^{-\frac{1}{5}}$ and $|\Omega\triangle B_1(0)|\leq\beta$. Let $\epsilon\in(0,\beta^{\frac{1}{8}}]$, there exists a function C^∞ function $\xi:\Omega\to\mathbb{R}$ which satisfies $\nabla u(z)\cdot\eta_z=1$ (where η_z is the inward pointing unit normal to $\partial\Omega$ at z) and for which*

$$\int_\Omega \epsilon^{-1}\left|1-|\nabla\xi|^2\right|^2+\epsilon\left|\nabla^2\xi\right|^2dz\leq c\beta^{\frac{3}{32}}. \tag{98}$$

4.1. **Proof of Proposition 1.** We begin with a preliminary lemma

Lemma 5. *Let $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function. Let ρ denote the standard convolution kernel, i.e. $\int\rho=1$ and $\mathrm{Spt}\rho\subset B_{\frac{3}{2}}(0)$ and define $\rho_h(z):=h^{-2}\rho(h^{-1}z)$.*

Suppose $f:\mathbb{R}^n\to\mathbb{R}$ be an affine function with $\eta=\nabla f$, let $g(x)=f*\rho_{\phi(x)}(x)$ then

$$g(x)=f(x)\ \text{for all}\ x\in\mathbb{R}^n. \tag{99}$$

Proof. So

$$
\begin{aligned}
g(x) &= \int f(x-y)(\phi(x))^{-2}\rho(\phi(x)^{-1}y)dy \\
&= \int (f(x)+\eta\cdot(x-y))(\phi(x))^{-2}\rho(\phi(x)^{-1}y)dy \\
&= f(x)+\int \eta\cdot(x-y)(\phi(x))^{-2}\rho(\phi(x)^{-1}y)dy \\
&= f(x)+\int_{[-\phi(x)+x\cdot\eta,\phi(x)+x\cdot\eta]}\int_{\lambda\eta+\eta^{\perp}}(x\cdot\eta-\lambda)(\phi(x))^{-2}\rho(\phi(x)^{-1}z)dH^{n-2}zd\lambda \\
&= f(x)+\int_{[-\phi(x),0]}\int_{(x\cdot\eta+\lambda)\eta+\eta^{\perp}}-\lambda(\phi(x))^{-2}\rho(\phi(x)^{-1}z)dH^{n-2}zd\lambda \\
&\quad +\int_{[0,\phi(x)]}\int_{(x\cdot\eta+\lambda)\eta+\eta^{\perp}}-\lambda(\phi(x))^{-2}\rho(\phi(x)^{-1}z)dH^{n-2}zd\lambda
\end{aligned} \tag{100}
$$

Since for any $\lambda\in[0,\phi(x)]$ we have

$$
\int_{(x\cdot\eta-\lambda)\eta+\eta^{\perp}}\lambda(\phi(x))^{-2}\rho(\phi(x)^{-1}z)dH^{n-2}z=\int_{(x\cdot\eta+\lambda)\eta+\eta^{\perp}}\lambda(\phi(x))^{-2}\rho(\phi(x)^{-1}z)dH^{n-2}z
$$

thus the last two term of (100) cancel and so $g(x)=f(x)$. This completes the proof of the lemma. □

Lemma 6. *Suppose Ω is a convex and $|\Omega\triangle B_1|=\beta$. Let $a_\theta=\partial\Omega\cap l_0^\theta$ we have*

$$
||a_\theta|-1|\le c\sqrt{\beta}\ \text{and so}\ \partial\Omega\subset N_{c\sqrt{\beta}}(\partial\Omega). \tag{101}
$$

In addition for any θ for which a unique normal exist at a_θ, then

$$
|\eta_{a_\theta}-\theta|\le\beta^{\frac{1}{4}} \tag{102}
$$

Proof of Lemma.

Step 1. We will show $B_{\frac{1}{2}}(0)\subset\Omega$.

Proof of Step 1. Suppose not, so we can pick $x\in\partial\Omega\cap B_{\frac{1}{2}}(0)$. Let η_x be an inward pointing unit normal to $\partial\Omega$ at x, by convexity of Ω we have $\Omega\subset\overline{H(x,\eta_x)}$ and so $B_1(0)\cap H(x,-\eta_x)\cap\Omega=\emptyset$ which implies $|B_1(0)\backslash\Omega|\ge|B_1(0)\cap H(x,-\eta_x)|>\frac{1}{8}$ which contradicts that $|\Omega\triangle B_1|\le\beta$.

Step 2. $a_\theta\in B_{1+c\sqrt{\beta}}(0)$.

Proof of Step 2. Suppose not. Since Ω is convex we have conv $\left(\{a_\theta\}\cup B_{\frac{1}{2}}(0)\right)\subset\Omega$ and

$$
\left|\text{conv}\left(\{a_\theta\}\cup B_{\frac{1}{2}}(0)\right)\backslash B_1(0)\right|>c\beta,
$$

thus we have $|\Omega\backslash B_1(0)|>c\beta$ which contradicts the fact that $|\Omega\triangle B_1(0)|=\beta$.

Step 3. We will show $a_\theta\in B_{1-c\sqrt{\beta}}(0)$.

Proof of Step 3. Note $|B_1(0)\backslash H(a_\theta,\eta_{a_\theta})|\ge c\beta$ and $\Omega\subset H(a_\theta,\eta_{a_\theta})$ so $|B_1(0)\backslash\Omega|\ge c\beta$ which gives a contradiction.

Proof of Lemma completed. Suppose (102) is false, since $|a_\theta-\theta|\le\sqrt{\beta}$ we have

$$
|B_1(0)\backslash H(a_\theta,\eta_{a_\theta})|\ge c\sqrt{\beta}.
$$

as before this implies $|B_1(0)\backslash\Omega|>c\sqrt{\beta}$ which is a contradiction. □

Lemma 7. *Let Ω be convex and define $u(x):=d(z,\partial\Omega)$ for any $z\in\Omega$ then function u is concave*

Proof of Lemma. Let $a, b \in \Omega$. Since Ω is convex $\operatorname{conv}\left(B_{u(a)}(a) \cup B_{u(b)}(b)\right) \subset \Omega$. Now suppose there exists $\lambda \in (0,1)$ such that

$$u\left(\lambda a+(1-\lambda) b\right)<\lambda u(a)+(1-\lambda) u\left(b\right)$$

then as this implies $B_{u(\lambda a+(1-\lambda)b)}\left(\lambda a+(1-\lambda)\, b\right) \subset \operatorname{int}\left(\operatorname{conv}\left(B_{u(a)}(a) \cup B_{u(b)}(b)\right)\right)$ we must be able to find $x \in \partial\Omega$ with $x \in \partial\Omega \cap \operatorname{conv}\left(B_{u(a)}(a) \cup B_{u(b)}(b)\right)$ which is a contradiction. □

Lemma 8. *Let $\epsilon > 0$, suppose Ω is a convex body with C^2 boundary and with curvature bounded above by $\epsilon^{-\frac{1}{5}}$. We will construct a function $\psi : \Omega \backslash (1-3\sqrt{\epsilon})\Omega \to \mathbb{R}$ with the following properties*

$$\int_{\Omega \backslash \left(1-3\sqrt{\epsilon}\right)\Omega} \left|1-|\nabla \psi|^2\right|^2 \leq c\epsilon^{\frac{11}{10}}, \tag{103}$$

$$\int_{\Omega \backslash \left(1-3\sqrt{\epsilon}\right)\Omega} \left|\nabla^2 \psi\right|^2 \leq c\epsilon^{\frac{1}{10}}, \tag{104}$$

$$\psi(z) = \left[\rho_{2\sqrt{\epsilon}} * u\right](z) \text{ for any } z \in (1-\sqrt{\epsilon})\Omega \backslash (1-3\sqrt{\epsilon})\Omega \tag{105}$$

and

$$\nabla \psi(z) = \eta_z \text{ for each } z \in \partial\Omega. \tag{106}$$

Proof. Let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth monotonic function with the following properties

$$w\left(z\right) = \begin{cases} z & \text{for } z \in [0, \sqrt{\epsilon}) \\ 2\sqrt{\epsilon} & \text{for } z \geq 3\sqrt{\epsilon} \end{cases} \tag{107}$$

and $\sup|\ddot{w}| \leq \epsilon^{-\frac{1}{2}}$.

Let $u(x) = d(x, \partial\Omega)$. For any $x \in \Omega \backslash \Omega_{(1-3\sqrt{\epsilon})}$ define $\phi(x) = w(u(x))$. Let ρ be the standard convolution kernel, i.e. as defined in Lemma 5. We will convolve the function u with convolution kernel $\rho_{\phi(x)}(z) := \rho\left(\frac{z}{\phi(x)}\right) / \left(\phi(x)\right)^2$. Since the convulsion kernel varies with x, when we differentiate $u * \rho_{\phi(x)}$, the derivative will involve a term with the derivative of $\rho_{\phi(x)}$. For this reason we need to calculate various partial derivatives of $\rho_{\phi(x)}$.

For each $x \in \Omega \backslash \Omega_{(1-3\sqrt{\epsilon})}$ let $b_x \in \partial\Omega$ be defined by $|x - b_x| = u(x)$. We define $\varsigma_x = \frac{x-b_x}{|x-b_x|}$, let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and define $\omega_x = R\varsigma_x$.

Note $\varsigma_x = \eta_{b_x}$, the inward pointing unit normal to $\partial\Omega$ at b_x. Note also that for all small enough h, $b_x = b_{x+h\varsigma_x}$ so $u(x + h\varsigma_x) = h + u(x)$. Thus

$$\begin{aligned} \phi_{,\varsigma_x}(x) &= \lim_{h\to 0} \frac{\phi(x+h\varsigma_x)-\phi(x)}{h} \\ &= \lim_{h\to 0} \frac{w(u(x)+h)-w(u(x))}{h} \\ &= \dot{w}(u(x)). \end{aligned}$$

Note also that since $|\nabla u(x)| = 1$ and $u_{,\varsigma_x}(x) = \lim_{h\to 0} \frac{u(x+h\varsigma_x)-u(x)}{h} = 1$ so

$$u_{,\omega_x}(x) = \lim_{h\to 0} \frac{u(x+h\omega_x)-u(x)}{h} = 0.$$

Thus

$$\phi_{,\omega_x}(x) = \dot{w}(u(x)) u_{,\omega_x}(x) = 0. \tag{108}$$

So

$$\begin{aligned} \frac{\partial}{\partial \varsigma_x}\left(\rho_{\phi(x)}(z)\right) &= \frac{\partial}{\partial \varsigma_x}\left(\rho\left(\frac{z}{\phi(x)}\right)\left(\phi(x)\right)^{-2}\right) \\ &= -\nabla\rho\left(\frac{z}{\phi(x)}\right) \cdot z \frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^4} - 2\rho\left(\frac{z}{\phi(x)}\right) \frac{\phi_{,\varsigma_x}(x)}{\left(\phi(x)\right)^3} \end{aligned} \tag{109}$$

and

$$\frac{\partial}{\partial \omega_x}\left(\rho_{\phi(x)}(z)\right) = 0. \tag{110}$$

Define

$$\psi(x) := \int u(x-z)\rho_{\phi(x)}(z)dz = \int u(x-z)\rho\left(\frac{z}{\phi(x)}\right)(\phi(x))^{-2}dz.$$

Now

$$\begin{aligned}
\psi_{,\varsigma_x}(x) &= \int u_{,\varsigma_x}(x-z)\rho\left(\frac{z}{\phi(x)}\right)(\phi(x))^{-2}dz \\
&\quad + \int u(x-z)\partial_{\varsigma_x}\left(\rho\left(\frac{z}{\phi(x)}\right)(\phi(x))^{-2}\right)dz \\
&\overset{(109)}{=} \int u_{,\varsigma_x}(x-z)\rho_{\phi(x)}(z)dz \\
&\quad - \int u(x-z)\left(\nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^4} + 2\rho\left(\frac{z}{\phi(x)}\right)\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^3}\right)
\end{aligned} \tag{111}$$

In the same way it is easy to see $\psi_{,\omega_x}(x) = 0$ and so

$$\psi_{,\varsigma_x\omega_x}(x) = 0. \tag{112}$$

We also know that $u_{,\varsigma_x\varsigma_x}(x) = 0$ so

$$\begin{aligned}
&\psi_{,\varsigma_x\varsigma_x}(x) \\
&= \int u(x-z)\partial_{\varsigma_x}\left(\sum_{k=1}^{2} -\rho_{,k}\left(\frac{z}{\phi(x)}\right)\frac{z_k\phi_{,\varsigma_x}(x)}{(\phi(x))^4} - 2\rho\left(\frac{z}{\phi(x)}\right)\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^3}\right)dz \\
&= \int u(x-z)\left(\sum_{k,l=1}^{2}\rho_{,kl}\left(\frac{z}{\phi(x)}\right)\frac{(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^6}z_k z_l - \sum_{k=1}^{2}\rho_{,k}\left(\frac{z}{\phi(x)}\right)z_k\partial_{\varsigma_x}\left(\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^4}\right)\right. \\
&\qquad \left. + 2\sum_{m=1}^{2}\rho_{,m}\left(\frac{z}{\phi(x)}\right)z_m\frac{(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^4} - 2\rho\left(\frac{z}{\phi(x)}\right)\partial_{\varsigma_x}\left(\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^3}\right)\right)dz
\end{aligned}$$

Note

$$\partial_{\varsigma_x}\left(\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^3}\right) = \frac{-3(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^4} + \frac{\phi_{,\varsigma_x\varsigma_x}(x)}{(\phi(x))^3}$$

and

$$\partial_{\varsigma_x}\left(\frac{\phi_{,\varsigma_x}(x)}{(\phi(x))^4}\right) = \frac{-4(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^5} + \frac{\phi_{,\varsigma_x\varsigma_x}(x)}{(\phi(x))^4} \tag{113}$$

So

$$\begin{aligned}
\psi_{,\varsigma_x\varsigma_x}(x) &= \int u(x-z)\left(\left(\nabla^2\rho\left(\frac{z}{\phi(x)}\right):z\otimes z\right)\frac{(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^6}\right. \\
&\quad + \left(-\frac{\phi_{,\varsigma_x\varsigma_x}(x)}{(\phi(x))^4} + \frac{4(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^5} + \frac{2(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^4}\right)\nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z \\
&\quad \left. + \left(\frac{6(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^4} - \frac{2\phi_{,\varsigma_x\varsigma_x}(x)}{(\phi(x))^3}\right)\rho\left(\frac{z}{\phi(x)}\right)\right)dz.
\end{aligned} \tag{114}$$

Step 1. For any $x \in \Omega\backslash(1-3\sqrt{\epsilon})\Omega$ we have

$$\sup\left\{|\nabla u(z) - \varsigma_x| : z \in B_{4u(x)}(x)\cap\Omega\right\} \leq c\epsilon^{-\frac{1}{5}}u(x). \tag{115}$$

Proof of Step 1. Since $\partial\Omega$ has curvature less than $\epsilon^{-\frac{1}{5}}$ for any $x_1, x_2 \in \partial\Omega$, $\left[x_1, x_1 + \epsilon^{\frac{1}{5}}\eta_{x_1}\right] \cap \left[x_2, x_2 + \epsilon^{\frac{1}{5}}\eta_{x_2}\right] = \emptyset$. So for any $x_1, x_2 \in B_{4u(x)}(x) \cap \partial\Omega$, $|\eta_{x_1} - \eta_{x_2}| \leq \epsilon^{-\frac{1}{5}} H^1(B_{4u(x)}(x) \cap \partial\Omega)$. Note as $\Omega \cap B_{4u(x)}(x)$ is convex and $\partial\Omega \cap B_{4u(x)}(x) \subset \partial(\Omega \cap B_{4u(x)}(x))$ so $H^1(\partial\Omega \cap B_{4u(x)}(x)) \leq cu(x)$. Hence $|\eta_{x_1} - \eta_{x_2}| \leq c\epsilon^{-\frac{1}{5}} u(x) \leq c\epsilon^{\frac{3}{10}}$ so it is clear that

$$B_{4u(x)}(x) \cap \Omega \subset \bigcup_{z \in \partial\Omega \cap B_{4u(x)}(x)} \left[z, z + \epsilon^{\frac{1}{5}}\eta_z\right]. \tag{116}$$

For any $z \in B_{4u(x)}(x) \cap \Omega$ we have $\nabla u(z) = \frac{z - b_z}{|z - b_z|} = \eta_{b_z}$ where b_z is such that $|z - b_z| = d(z, \partial\Omega)$. So for any $z_1, z_2 \in B_{4u(x)}(x) \cap \Omega$ by (116) we have that $b_{z_1}, b_{z_2} \in \partial\Omega \cap B_{4u(x)}(x)$, so $|\nabla u(z_1) - \nabla u(z_2)| = \left|\eta_{b_{z_1}} - \eta_{b_{z_2}}\right| \leq c\epsilon^{-\frac{1}{5}} u(x)$.

Step 2. For any $x \in \Omega \backslash (1 - 3\sqrt{\epsilon})\Omega$ we have

$$||\nabla\psi(x)| - 1| \leq c\epsilon^{\frac{3}{10}}. \tag{117}$$

And

$$\lim_{y \to z} \nabla\psi(y) = \eta_z. \tag{118}$$

Proof of Step 2. From (111) we have

$$\begin{aligned} &|\psi_{,\varsigma_x}(x) - 1| \\ &\leq \overbrace{\left|\int (u_{,\varsigma_x}(x - z) - 1)\rho\left(\frac{z}{\phi(x)}\right)(\phi(x))^{-2}\, dz\right|}^{B} \\ &\quad + \overbrace{\left|\int \frac{-u(x - z)\phi_{,\varsigma_x}(x)}{(\phi(x))^3}\left(\nabla\rho\left(\frac{z}{\phi(x)}\right) \cdot \frac{z}{\phi(x)} + 2\rho\left(\frac{z}{\phi(x}\right)\right) dz\right|}^{C}. \end{aligned} \tag{119}$$

Now from (115), for any $z \in \mathrm{Spt}\rho_{\phi(x)}$ we have that $\nabla u(x - z) = u_{,\varsigma_x}(x - z)\varsigma_x + u_{,\omega_x}(x - z)\omega_x$ now since $\mathrm{Spt}\rho_{\phi(x)} \subset B_{\phi(x)}(0) \subset B_{u(x)}(0)$ so for any $z \in \mathrm{Spt}\rho_{\phi(x)}$ by (115) from Step 1 we have $|\nabla u(x - z) - \varsigma_x| \leq c\epsilon^{-\frac{1}{5}} u(x)$ and thus

$$|u_{,\varsigma_x}(x - z) - 1| \leq c\epsilon^{-\frac{1}{5}} u(x) \tag{120}$$

so (noting $u(x) \leq c\phi(x)$ for any $x \in \Omega \backslash (1 - 3\sqrt{\epsilon})\Omega$)

$$B \leq cu(x)\epsilon^{-\frac{1}{5}} < c\phi(x)\epsilon^{-\frac{1}{5}}. \tag{121}$$

Also defining $w = \fint_{B_{\phi(x)}} \nabla u$

$$|w - \varsigma_x| = \left|\fint_{B_{\phi(x)}} (\nabla u(z) - \varsigma_x)\, dz\right| \overset{(115)}{\leq} c\epsilon^{-\frac{1}{5}}\phi(x). \tag{122}$$

So by Poincare inequality there exists affine function l_w with $\nabla l_w = w$

$$\fint_{B_{\phi(x)}(x)} |u(z) - l_w(z)|\, dz \leq c\phi(x) \fint_{B_{\phi(x)}} |\nabla u(z) - \varsigma_x|\, dz \overset{(115)}{\leq} c\epsilon^{-\frac{1}{5}}(\phi(x))^2. \tag{123}$$

Now from (122), again for the appropriate choice of affine function l_{ς_x} with $\nabla l_{\varsigma_x} = \varsigma_x$ we have

$$\fint_{B_{\phi(x)}(x)} |l_{\varsigma_x}(z) - l_w(z)|\, dz \leq c\phi(x) \fint_{B_{\phi(x)}} |w - \varsigma_x|\, dz \overset{(122)}{\leq} c\epsilon^{-\frac{1}{5}}(\phi(x))^2$$

with (123) gives

$$\fint_{B_{\phi(x)}(x)} |l_{\varsigma_x}(z) - u(z)|\, dz \leq c\epsilon^{-\frac{1}{5}} (\phi(x))^2. \tag{124}$$

Let g be defined by $g(x) = l_{\varsigma_x} * \rho_{\rho(x)}(x)$, note by Lemma 5 we have $\nabla g(z) = \varsigma_x$ for any $z \in \mathbb{R}^2$ and hence $g_{,\varsigma_x}(x) = 1$ and as

$$\begin{aligned} g_{\varsigma_x}(x) &= \int \rho\left(\frac{z}{\phi(x)}\right)(\phi(x))^{-2}\, dz \\ &\quad - \int \frac{l_{\varsigma_x}(x-z)}{(\phi(x))^3} \phi_{,\varsigma_x}(x) \left(\nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z\,(\phi(x))^{-1} + 2\rho\left(\frac{z}{\phi(x)}\right)\right) dz \\ &= 1 - \int \frac{l_{\varsigma_x}(x-z)}{(\phi(x))^3} \phi_{,\varsigma_x}(x) \left(\nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z\,(\phi(x))^{-1} + 2\rho\left(\frac{z}{\phi(x)}\right)\right) dz. \end{aligned}$$

Thus

$$0 = \int \frac{l_{\varsigma_x}(x-z)}{(\phi(x))^3} \phi_{,\varsigma_x}(x) \left(\nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z\,(\phi(x))^{-1} + 2\rho\left(\frac{z}{\phi(x)}\right)\right) dz \tag{125}$$

So

$$\begin{aligned} C &\leq \int \frac{|l_{\varsigma_x}(x-z) - u(x-z)|}{(\phi(x))^3} \phi_{,\varsigma_x}(x) \left(\nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z\,(\phi(x))^{-1} + 2\rho\left(\frac{z}{\phi(x)}\right)\right) dz \\ &\leq c(\phi(x))^{-3} \int_{B_{\phi(x)}(x)} |l_{\varsigma_x}(z) - u(z)|\, dz \\ &\overset{(124)}{\leq} c\epsilon^{-\frac{1}{5}} \phi(x). \end{aligned} \tag{126}$$

Since $x \in \Omega\backslash(1-3\sqrt{\epsilon})\Omega$ we know $\phi(x) \leq c\sqrt{\epsilon}$ applying (126) and (121) to (119) gives

$$|\psi_{,\varsigma_x}(x) - 1| \leq c\epsilon^{-\frac{1}{5}} \phi(x) \leq c\epsilon^{\frac{3}{10}}. \tag{127}$$

As $\psi_{,\omega_x}(x) = 0$, so $|\nabla\psi(x) - \eta_x| \leq c\epsilon^{\frac{3}{10}}$ and (117) follows easily. Also for (127) we know $|\nabla\psi(x) - \eta_x| \leq c\epsilon^{-\frac{1}{5}} u(x)$ and (118) follows. This completes the proof of Step 2.

Step 3. For any $x \in \Omega\backslash(1-3\sqrt{\epsilon})\Omega$ we have $\left|\nabla^2\psi(x)\right| \leq c\epsilon^{-\frac{1}{4}}$.

Proof Step 3. From Step 2 (124) we know the existence of an affine function l_{ς_x} with $\nabla l_{\varsigma_x} = \varsigma_x$ with $\fint_{B_{\phi(x)}(x)} |u - l_{\varsigma_x}|\, dz \leq c\epsilon^{\frac{3}{10}} \phi(x)$.

Let $g(x) := l_{\varsigma_x} * \rho_{\phi(x)}(x)$ so by Lemma 5 we know $g_{,\varsigma_x\varsigma_x}(x) = 0$. By following through the same calculation that gave (114) we have

$$\begin{aligned} 0 = \int l_{\varsigma_x}(x-z) \Bigg(&\left(\nabla^2\rho\left(\frac{z}{\phi(x)}\right) : z\otimes z\right) \frac{(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^6} \\ &+ \left(-\frac{\phi_{,\varsigma_x\varsigma_x}(x)}{(\phi(x))^4} + \frac{4(\phi_{,\varsigma_x}(x))^4}{(\phi(x))^5} + \frac{2(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^4}\right) \nabla\rho\left(\frac{z}{\phi(x)}\right)\cdot z \\ &+ \left(\frac{6(\phi_{,\varsigma_x}(x))^2}{(\phi(x))^4} - \frac{2\phi_{,\varsigma_x\varsigma_x}(x)}{(\phi(x))^3}\right)\rho\left(\frac{z}{\phi(x)}\right)\Bigg) dz. \end{aligned} \tag{128}$$

So applying this to (114)

$$\begin{aligned}
&\left|\psi_{,\varsigma_x\varsigma_x}(x)\right| \\
&\leq \int \left|u\left(x-z\right)-l_{\varsigma_x}\left(x-z\right)\right| \left| \left(\nabla^2\rho\left(\frac{z}{\phi\left(x\right)}\right): z\otimes z\right)\frac{(\phi_{,\varsigma_x}(x))^2}{\left(\phi\left(x\right)\right)^6}\right. \\
&\quad + \left(\frac{-\phi_{,\varsigma_x\varsigma_x}(x)}{\left(\phi\left(x\right)\right)^4}+\frac{4(\phi_{,\varsigma_x}(x))^4}{\left(\phi\left(x\right)\right)^5}+\frac{2(\phi_{,\varsigma_x}(x))^2}{\left(\phi\left(x\right)\right)^4}\right)\nabla\rho\left(\frac{z}{\phi\left(x\right)}\right)\cdot z \\
&\quad \left. + \left(\frac{6(\phi_{,\varsigma_x}(x))^2}{\left(\phi\left(x\right)\right)^4}-\frac{2\phi_{,\varsigma_x\varsigma_x}(x)}{\left(\phi\left(x\right)\right)^3}\right)\rho\left(\frac{z}{\phi\left(x\right)}\right)\right| dz \\
&\leq c\int_{B_{\phi(x)}(0)} \frac{\left|u\left(x-z\right)-l_{\varsigma_x}\left(x-z\right)\right|}{\left(\phi\left(x\right)\right)^4} dz\left(\left\|\nabla^2\rho\right\|_\infty+\left\|\nabla\rho\right\|_\infty+\left\|\rho\right\|_\infty\right) \\
&\leq c\int_{B_{\phi(x)}(x)} \left|u\left(z\right)-l_{\varsigma_x}\left(z\right)\right|\left(\phi\left(x\right)\right)^{-4} dz \\
&\overset{(124)}{\leq} c\epsilon^{-\frac{1}{5}}. \qquad (129)
\end{aligned}$$

Proof of Lemma completed. From Step 2, (117), for any $x\in\Omega\backslash(1-3\sqrt{\epsilon})\Omega$ we have $\left|\left|\nabla\psi(x)\right|^2-1\right|^2\leq c\epsilon^{\frac{6}{10}}$ so

$$\int_{\Omega\backslash(1-3\sqrt{\epsilon})}\left|\left|\nabla\psi\right|^2-1\right|^2 dx\leq c\epsilon^{\frac{11}{10}}.$$

In the same way from Step 3, for any $x\in\Omega\backslash(1-3\sqrt{\epsilon})\Omega$ we have

$$\int_{\Omega\backslash(1-3\sqrt{\epsilon})}\left|\nabla^2\psi\right|^2\leq c\epsilon^{\frac{1}{10}}. \qquad \square$$

Lemma 9. *Let $\beta>0$, suppose Ω is a convex set with $|\Omega\triangle B_1(0)|\leq\beta$. Let $u(z)=d(z,\partial\Omega)$. For any $x\in\Omega\backslash B_{\beta^{\frac{1}{8}}}(0)$ for which the approximate derivative ∇u exists*

$$\left|\nabla u(x)-\frac{x}{|x|}\right|\leq c\beta^{\frac{3}{16}}. \qquad (130)$$

Proof. For any $x\in\Omega\backslash B_{\beta^{\frac{1}{8}}}(0)$ let $b_x\in\partial\Omega$ be such that $|b_x-x|=u\left(x\right)$. Recall $a_{\frac{x}{|x|}}=\partial\Omega\cap l_0^{\frac{x}{|x|}}$, we begin by showing

$$\left|b_x-\frac{x}{|x|}\right|\leq c\beta^{\frac{3}{16}}. \qquad (131)$$

Using (101) from Lemma 6

$$|x-b_x|\leq\left|x-a_{\frac{x}{|x|}}\right|\leq 1-|x|+\sqrt{\beta}. \qquad (132)$$

Hence

$$\left|x-b_x\right|^2=\left|x\right|^2-2x\cdot b_x+\left|b_x\right|^2\overset{(132)}{\leq}1-2\left|x\right|+\left|x\right|^2+c\sqrt{\beta} \qquad (133)$$

Therefor

$$\begin{aligned}
-2x\cdot b_x &\overset{(133)}{\leq} 1-2\left|x\right|+c\sqrt{\beta}-\left|b_x\right|^2 \\
&\overset{(101)}{\leq} -2\left|x\right|+c\sqrt{\beta}.
\end{aligned}$$

Thus $2\left|x\right|\leq 2x\cdot b_x+c\sqrt{\beta}$. Since $|x|>\beta^{\frac{1}{8}}$ we have

$$1\leq\frac{x}{|x|}\cdot b_x+c\frac{\sqrt{\beta}}{|x|}\leq 1+c\beta^{\frac{3}{8}}. \qquad (134)$$

Hence

$$\left|b_x - \frac{x}{|x|}\right|^2 = |b_x|^2 + 1 - 2\frac{x}{|x|} \cdot b_x \overset{(134),(101)}{\leq} c\beta^{\frac{3}{8}}$$

which gives

$$\left|\frac{x}{|x|} - b_x\right| \leq c\beta^{\frac{3}{16}}. \tag{135}$$

Let $\theta_x = \frac{b_x}{|b_x|}$ so using Lemma 6 $\left|\eta_{b_x} - \frac{b_x}{|b_x|}\right| = \left|\eta_{a_{\theta_x}} - \theta_x\right| \overset{(102)}{\leq} \beta^{\frac{1}{4}}$ and by (101) this easily implies

$$|\eta_{b_x} - b_x| \leq c\beta^{\frac{1}{4}}. \tag{136}$$

Now since $\nabla u(x) = \frac{x-b_x}{|x-b_x|} = \eta_{b_x}$ and so

$$\left|\nabla u(x) - \frac{x}{|x|}\right| \leq |\eta_{b_x} - b_x| + \left|b_x - \frac{x}{|x|}\right| \overset{(135),(136)}{\leq} c\beta^{\frac{3}{16}}$$

thus we have established (130). □

Lemma 10. *Let Ω be a convex set and $|\Omega \triangle B_1| \leq \beta$. Define $u(x) = d(x, \partial\Omega)$, note that since u is convex ∇u is BV. Let $V(\nabla u, \cdot)$ denotes the total total variation of the measure ∇u. Firstly we have*

$$V(\nabla u, \Omega \backslash B_{2\beta^{\frac{1}{8}}}(0)) \leq c. \tag{137}$$

Secondly for any $\epsilon \in (0, \beta^{\frac{1}{8}}]$, for any $x \in \Omega \backslash B_{2\beta^{\frac{1}{8}}}(0)$ we have

$$V(\nabla u, B_\epsilon(x)) \leq c\beta^{\frac{3}{16}}\epsilon. \tag{138}$$

Proof. Step 1. Let $\tau \in (0, \epsilon)$ be some small number. For any $x \in \Omega \backslash N_{4\tau}(\partial\Omega) =: \Omega_\tau$. Let $w_\tau(x) = u * \rho_\tau(x)$ and $v^\tau = \frac{\nabla_x w_\tau}{|\nabla_x w_\tau|}$.

For any $t \in (4\tau, \sup_\Omega v^\tau)$

$$\int_{w_\tau^{-1}(t)} \left|v^\tau_{1,1}(z) + v^\tau_{2,2}(z)\right| dH^1 z \leq c \tag{139}$$

Proof of Step 1. We define the 'angle' function by

$$A(x) := \begin{cases} \cos^{-1}\left(\frac{x_1}{|x|}\right) & \text{for } x_2 \geq 0 \\ 2\pi - \cos^{-1}\left(\frac{x_1}{|x|}\right) & \text{for } x_2 < 0 \end{cases} \tag{140}$$

Note that A is smooth expect at the half line $\{(x_1, x_2) : x_2 = 0, x_1 > 1\}$. For $x \in \Omega_\tau$ we have $|v^\tau(x)|^2 = 1$, so

$$\partial_1(|v^\tau(x)|^2) = v^\tau_1(x)v^\tau_{1,1}(x) + v^\tau_2(x)v^\tau_{2,1}(x) = 0 \tag{141}$$

and

$$\partial_2(|v^\tau(x)|^2) = v^\tau_1(x)v^\tau_{1,2}(x) + v^\tau_2(x)v^\tau_{2,2}(x) = 0. \tag{142}$$

Since u is the 1-Lipschitz, $\|w_\tau - u\|_{L^\infty(\Omega_\tau)} \leq \tau$ and so for any $t > 4\tau$, $w_\tau^{-1}(t) \subset \Omega_\tau$ and hence v^τ is well defined along this level set. We also know that for any $x \in w_\tau^{-1}(t)$ the tangent to curve $w_\tau^{-1}(t)$ is given by $\begin{pmatrix} -v^\tau_2(x) \\ v^\tau_1(x) \end{pmatrix}$.

Now there exists a point $x_0 \in w_\tau^{-1}(t)$ such that $A\begin{pmatrix} -v^\tau_2(x_0) \\ v^\tau_1(x_0) \end{pmatrix} = 0$. Let $\Phi^t : \left[0, H^1(w_\tau^{-1}(t))\right) \to w_\tau^{-1}(t)$ denote that parameterisation of Γ_t by arc-length with $\Phi^t(0) = x_0$, so $\dot{\Phi}^t(s) = \begin{pmatrix} -v^\tau_2(\Phi^t(s)) \\ v^\tau_1(\Phi^t(s)) \end{pmatrix}$.

Define $\Theta_t : [0, H^1(w_\tau^{-1}(t))) \to \mathbb{R}$ by $\Theta_t(s) = A(\dot{\Phi}^t(s))$. Now pick $s \in \left[0, H^1(w_\tau^{-1}(t))\right)$, suppose $v_1^\tau\left(\Phi^t(s)\right) \geq 0$, then

$$\begin{aligned}
\dot{\Theta}_t(s) &= \dot{\cos}^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\frac{\partial}{\partial t}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\\
&= \dot{\cos}^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\left(-v_{2,1}^\tau\left(\Phi^t(s)\right)\dot{\Phi}_1^t(t) - v_{2,2}^\tau\left(\Phi^t(s)\right)\dot{\Phi}_2^t(t)\right)\\
&= \dot{\cos}^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\left(v_{2,1}^\tau\left(\Phi^t(s)\right)v_2^\tau\left(\Phi^t(s)\right) - v_{2,2}^\tau\left(\Phi^t(s)\right)v_1^\tau\left(\Phi^t(s)\right)\right)\\
&\overset{(141)}{=} \dot{\cos}^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\left(-v_{1,1}^\tau\left(\Phi^t(s)\right)v_1^\tau\left(\Phi^t(s)\right) - v_{2,2}^\tau\left(\Phi^t(s)\right)v_1^\tau\left(\Phi^t(s)\right)\right)\\
&= -\dot{\cos}^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)v_1^\tau\left(\Phi^t(s)\right)\left(v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right)\right). \qquad (143)
\end{aligned}$$

Now for any $w \in (-1, 1)$, $\dot{\cos}^{-1}(w) = -(\sin(\cos^{-1}(w)))^{-1}$ so

$$\dot{\Theta}_t(t) = \frac{v_1^\tau\left(\Phi^t(s)\right)}{\sin(\cos^{-1}(-v_2^\tau\left(\Phi^t(s)\right)))}\left(v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right)\right). \qquad (144)$$

Recall $\left|\begin{pmatrix} -v_2^\tau\left(\Phi^t(s)\right) \\ v_1^\tau\left(\Phi^t(s)\right)\end{pmatrix}\right| = 1$ and we supposed $v_1^\tau\left(\Phi^t(s)\right) \geq 0$, so

$$\begin{aligned}
v_1^\tau\left(\Phi^t(s)\right) &= \sqrt{1 - \left(v_2^\tau\left(\Phi^t(s)\right)\right)^2}\\
&= \sqrt{1 - \left(\cos\left(\cos^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\right)\right)^2}\\
&= \sin\left(\cos^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\right). \qquad (145)
\end{aligned}$$

Thus from (144)

$$\dot{\Theta}_t(s) = \left(v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right)\right) \text{ for any } s \in \left[0, H^1(w_\tau^{-1}(t))\right] \text{ with } v_1^\tau\left(\Phi^t(s)\right) \geq 0. \quad (146)$$

Suppose we have $s \in \left[0, H^1(w_\tau^{-1}(t))\right]$ with $v_1^\tau\left(\Phi^t(s)\right) < 0$, then in the same way as (145) we have

$$v_1^\tau\left(\Phi^t(s)\right) = -\sqrt{1 - \left(\cos\left(\cos^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\right)\right)^2} = -\sin\left(\cos^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\right). \qquad (147)$$

And since $v_1^\tau(\Phi^t(s)) < 0$, by definition of A (see (140)) arguing as in (144) we have

$$\begin{aligned}
\dot{\Theta}_t(s) &= \frac{-v_1^\tau\left(\Phi^t(s)\right)}{\sin\left(\cos^{-1}\left(-v_2^\tau\left(\Phi^t(s)\right)\right)\right)}\left(v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right)\right)\\
&\overset{(147)}{=} v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right) \text{ for } s \in \left[0, H^1(w_\tau^{-1}(t))\right) \text{ with } v_1^\tau\left(\Phi^t(s)\right) < 0.
\end{aligned}$$

Thus we have

$$\dot{\Theta}_t(s) = v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right) \text{ for } s \in \left[0, H^1(w_\tau^{-1}(t))\right). \qquad (148)$$

Now since ψ is concave, w_τ is concave and so the set $w_\tau^{-1}([t, \infty))$ is a convex set, hence

$$v_{1,1}^\tau\left(\Phi^t(s)\right) + v_{2,2}^\tau\left(\Phi^t(s)\right) = \dot{\Theta}_t(s) \geq 0 \text{ for any } s \in \left[0, H^1(w_\tau^{-1}(t))\right). \qquad (149)$$

Hence

$$\int_{w_\tau^{-1}(t)} \left|v_{1,1}^\tau(z) + v_{2,2}^\tau(z)\right| dH^1 z = \int_0^{H^1(w_\tau^{-1}(t))} \Theta_t(s) ds \leq c$$

Step 2. Let $x \in \Omega_\tau$ and define

$$t_1 = \inf\left\{s \in \mathbb{R} : w_\tau^{-1}(s) \cap B_\epsilon(x) \neq \emptyset\right\} \text{ and } t_2 = \sup\left\{s \in \mathbb{R} : w_\tau^{-1}(s) \cap B_\epsilon(x) \neq \emptyset\right\}. \quad (150)$$

For any $t \in (t_1, t_2)$

$$\sup\left\{\Theta_t(s_1) - \Theta_t(s_2) : \Phi^t(s_1), \Phi^t(s_2) \in B_\epsilon(x)\right\} \leq c\beta^{\frac{3}{16}}. \qquad (151)$$

Proof of Step 2. Let $s_1, s_2 \in \left[0, H^1(w_\tau^{-1}(t))\right)$ such that $\Phi^t(s_1), \Phi^t(s_2) \in B_\epsilon(x)$, since Φ^t is parameterisation of Γ_t by arclength $\dot{\Phi}^t(s)$ is the unit tangent to $w_\tau^{-1}(t)$ at $\Phi^t(s)$. Thus

$$R\left(\frac{\nabla w_\tau\left(\Phi^t(s_i)\right)}{\left|\nabla w_\tau\left(\Phi^t(s_i)\right)\right|}\right) = \dot{\Phi}^t(s_i) \text{ for } i = 1, 2.$$

However by Lemma 9 (and using the fact that $|\Phi^t(s_1)| > \frac{\beta^{\frac{1}{8}}}{2}$ and $|\Phi^t(s_2)| > \frac{\beta^{\frac{1}{8}}}{2}$ for the last inequality)

$$\begin{aligned}
\left|\nabla w_\tau\left(\Phi^t(s_1)\right) - \nabla w_\tau\left(\Phi^t(s_2)\right)\right| &= \left|\int \left(\nabla u\left(\Phi^t(s_1) - z\right) - \nabla u\left(\Phi^t(s_2) - z\right)\right)\rho_\tau(z)dz\right| \\
&\overset{(130)}{\leq} c\int_{B_\tau(0)} \left|\frac{\Phi^t(s_1) - z}{|\Phi^t(s_1) - z|} - \frac{\Phi^t(s_2) - z}{|\Phi^t(s_2) - z|}\right| \rho_\tau(z)dz + c\beta^{\frac{3}{16}} \\
&\leq \int_{B_\tau(0)} \left|\frac{\Phi^t(s_1)}{|\Phi^t(s_1) - z|} - \frac{\Phi^t(s_2)}{|\Phi^t(s_2) - z|}\right| \rho_\tau(z)dz + c\beta^{\frac{3}{16}} \\
&\leq c\beta^{\frac{3}{16}}. \qquad (152)
\end{aligned}$$

Again from Lemma 9 for any $x \in \Omega \backslash B_{2\beta^{\frac{1}{8}}}(0)$

$$\begin{aligned}
\left|\nabla w_\tau(x) - \frac{x}{|x|}\right| &= \left|\int \left(\nabla u(x - z) - \frac{x}{|x|}\right)\rho_\tau(z)dz\right| \\
&\leq \int \left|\left(\nabla u(x - z) - \frac{x - z}{|x - z|}\right)\rho_\tau(z)\right| dz + \int \left|\frac{x - z}{|x - z|} - \frac{x}{|x|}\right| \rho_\tau(z)dz \\
&\overset{(130)}{\leq} c \sup_{z \in B_\tau(0)} \left|\frac{(x - z)|x| - |x - z|\, x}{|x - z|\,|x|}\right| + c\beta^{\frac{3}{16}} \\
&\leq c\beta^{\frac{3}{16}}. \qquad (153)
\end{aligned}$$

As a consequence we know

$$\left|\left|\nabla w_\tau(x)\right| - 1\right| \leq c\beta^{\frac{3}{16}}, \qquad (154)$$

so

$$\begin{aligned}
\left|\dot{\Phi}(s_1) - \dot{\Phi}(s_2)\right| &\overset{(152)}{\leq} \left|R\left(\frac{\nabla w_\tau\left(\Phi^t(s_1)\right)}{\left|\nabla w_\tau\left(\Phi^t(s_1)\right)\right|}\right) - R\left(\nabla w_\tau\left(\Phi^t(s_1)\right)\right)\right| \\
&\quad + \left|R\left(\frac{\nabla w_\tau\left(\Phi^t(s_2)\right)}{\left|\nabla w_\tau\left(\Phi^t(s_2)\right)\right|}\right) - R\left(\nabla w_\tau\left(\Phi^t(s_2)\right)\right)\right| + c\beta^{\frac{3}{16}} \\
&\overset{(154)}{\leq} c\beta^{\frac{3}{16}}. \qquad (155)
\end{aligned}$$

Thus

$$|\Theta_t(s_1) - \Theta_t(s_2)| = \left|A\left(\dot{\Phi}^t(s_1)\right) - A\left(\dot{\Phi}^t(s_2)\right)\right| \overset{(155)}{\leq} c\beta^{\frac{3}{16}} \qquad (156)$$

and so (151) is established.

Proof of Lemma completed. For any $t \in (t_1, t_2)$. Let $s_1^t = \inf\{s : \Phi^t(s) \in B_\epsilon(x)\}$, $s_2^t = \sup\{s : \Phi^t(s) \in B_\epsilon(x)\}$. So $[s_1^t, s_2^t] = \{s : \Phi^t(s) \in B_\epsilon(x)\}$. Now

$$\begin{aligned}
\int_{[s_1^t, s_2^t]} \left|v_{1,1}^\tau(\Phi^t(s)) + v_{1,1}^\tau(\dot{\Phi}_t(s))\right| ds &\overset{(149)}{=} \int_{[s_1^t, s_2^t]} \dot{\Theta}_t(s)ds \\
&\overset{(156)}{\leq} c\beta^{\frac{3}{16}}. \qquad (157)
\end{aligned}$$

Thus

$$\begin{aligned}\int_{B_\epsilon(x)} \left|v^\tau_{1,1}(z)+v^\tau_{1,1}(z)\right| \left|\nabla w_\tau(z)\right| dz &= \int_{t_1}^{t_2}\int_{w_\tau^{-1}(t)} \left|v^\tau_{1,1}(z)+v^\tau_{1,1}(z)\right| dH^1 z dt\\ &= \int_{t_1}^{t_2}\int_{\left[s_1^t,s_2^t\right]} \left|v^\tau_{1,1}(\Phi^t(s))+v^\tau_{1,1}(\Phi^t(s))\right| ds dt\\ &\overset{(157)}{\leq} c\left|t_1-t_2\right|\beta^{\frac{3}{16}}.\end{aligned}$$

Now recall from (153) we know $|\nabla w_\tau(z)| \geq 1-c\beta^{\frac{3}{16}}$ for any $z\in B_\epsilon(x)$, so recalling the definition (150) of Step 2 we must have $|t_1-t_2|\leq c\epsilon$. Putting these things together we have

$$\int_{B_\epsilon(x)} \left|v^\tau_{1,1}(z)+v^\tau_{1,1}(z)\right| dz \leq c\epsilon\beta^{\frac{3}{16}}. \tag{158}$$

Since $v^\tau_{2,1}=v^\tau_{1,2}$ we have

$$\begin{pmatrix} v^\tau_{1,1} & v^\tau_{2,1}\\ v^\tau_{2,1} & v^\tau_{2,2}\end{pmatrix}\begin{pmatrix} v^\tau_1\\ v^\tau_2\end{pmatrix} \overset{(141),(142)}{=} \begin{pmatrix}0\\0\end{pmatrix} \text{ and } \begin{pmatrix} v^\tau_{1,1} & v^\tau_{2,1}\\ v^\tau_{2,1} & v^\tau_{2,2}\end{pmatrix}\begin{pmatrix} -v^\tau_2\\ v^\tau_1\end{pmatrix} \overset{(141),(142)}{=} (v^\tau_{1,1}+v^\tau_{2,2})\begin{pmatrix} -v^\tau_2\\ v^\tau_1\end{pmatrix}.$$

Letting $\|\cdot\|$ denote the operator norm of a matrix, since $\begin{pmatrix} v^\tau_1 & -v^\tau_2\\ v^\tau_2 & v^\tau_1\end{pmatrix}\in O(2)$ thus

$$\begin{aligned}\left\|\begin{pmatrix} v^\tau_{1,1} & v^\tau_{2,1}\\ v^\tau_{2,1} & v^\tau_{2,2}\end{pmatrix}\right\| &= \left\|\begin{pmatrix} v^\tau_{1,1} & v^\tau_{2,1}\\ v^\tau_{2,1} & v^\tau_{2,2}\end{pmatrix}\begin{pmatrix} v^\tau_1 & -v^\tau_2\\ v^\tau_2 & v^\tau_1\end{pmatrix}\right\|\\ &= \left\|\begin{pmatrix} 0 & -(v^\tau_{1,1}+v^\tau_{2,2})v^\tau_2\\ 0 & (v^\tau_{1,1}+v^\tau_{2,2})v^\tau_1\end{pmatrix}\right\|\\ &\leq c\left|v^\tau_{1,1}+v^\tau_{2,2}\right|\end{aligned}$$

As the operator norm and Euclidean norm are equivalent $|Dv^\tau|\leq c\left|v^\tau_{1,1}+v^\tau_{2,2}\right|$. Hence from (158)

$$\int_{B_\epsilon(x)} |Dv^\tau| \leq c\beta^{\frac{3}{16}}\epsilon. \tag{159}$$

Additionally for any $t\in(4\tau,\sup_\Omega v^\tau)$, we have $\int_{w_\tau^{-1}(t)} |Dv^\tau|\, dH^1 z \overset{(139)}{\leq} c$. Note that $\Omega_{4\tau}\subset w_\tau^{-1}([\tau,\infty))$ by using the Co-area formula

$$\int_{\Omega_{4\tau}} |Dv^\tau|\,|\nabla w^\tau| \leq \int_\tau^{\sup_\Omega v^\tau}\int_{w_\tau^{-1}(s)} |Dv^\tau|\, dH^1 z ds \leq c.$$

As $|\nabla w^\tau(x)| = \left|\int \rho_\tau(x-z)\frac{z}{|z|}dz\right| \overset{(130)}{\geq} \frac{1}{2}$ for any $x\subset \Omega_{4\tau}\backslash B_{\beta^{\frac{1}{8}}}(0)$. Thus

$$\int_{\Omega_\tau\backslash B_{2\beta^{\frac{1}{8}}}(0)} |Dv^\tau| \leq c \tag{160}$$

Let $\tau_n = 2^{-n}\epsilon$. By definition of w_{τ_n} we know $\nabla w_{\tau_n}\overset{L^1}{\to}\nabla u$ and so

$$\begin{aligned}\int_{B_\epsilon(x)} |1-|\nabla w_{\tau_n}|| &= \int_{B_\epsilon(x)} ||\nabla u|-|\nabla w_{\tau_n}||\\ &\leq \int_{B_\epsilon(x)} |\nabla u-\nabla w_{\tau_n}| \to 0 \text{ as } n\to\infty.\end{aligned}$$

So we have

$$\begin{aligned}\|v^{\tau_n}-\nabla u\|_{L^1(B_{2\epsilon}(x))} &\leq \| |\nabla w^{\tau_n}|^{-1}\nabla w_{\tau_n}-\nabla w_{\tau_n}\|_{L^1(B_{2\epsilon}(x))}+\|\nabla w_{\tau_n}-\nabla u\|_{L^1(B_{2\epsilon}(x))}\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Also as by (159) v^{τ_n} is bounded in BV, by Proposition 3.13 we have that

$$v^{\tau_n} \overset{BV(B_{2\epsilon}(x))}{\rightharpoonup} \nabla u.$$

Hence by Remark 3.5 [Am-Fu-Pa 00] total variation is lower semicontinuous with respect to weak convergence we have $V(\nabla u, B_{2\epsilon}(x)) \leq c\beta^{\frac{3}{16}}\epsilon$. By arguing in exactly the same way we see (160) implies $V(\Omega\backslash B_{2\beta^{\frac{1}{8}}}(0)) \leq c$. □

Lemma 11. *Let Ω be a convex domain and $|\Omega\triangle B_1(0)| \leq \beta$. Let $u(x) = d(x, \partial\Omega)$ and for $\epsilon \in (0, \beta^{\frac{1}{8}}]$ define $u_\epsilon := u * \rho_{2\epsilon}$. For any a for which $a \in \Omega\backslash B_{5\beta^{\frac{1}{8}}}(0)$ we have*

$$||\nabla u_\epsilon(x)| - 1| \leq c\min\left\{\epsilon^{-1}V(\nabla u, B_{4\epsilon}(a)), \beta^{\frac{3}{8}}\right\} \text{ for any } x \in B_{2\epsilon}(a). \tag{161}$$

Proof. Firstly recall that since u is concave and hence ∇u is BV, so by Poincare's inequality

$$\left(\int_{B_{4\epsilon}(a)} |\nabla u - w|^2 \, dz\right)^{\frac{1}{2}} \leq cV\left(\nabla u, B_{4\epsilon}(a)\right). \tag{162}$$

Using Holder's inequality we get another form of Poincare's inequality

$$\int_{B_{4\epsilon}(a)} |\nabla u - w| \leq c\|\nabla u - w\|_{L^2(B_{4\epsilon}(a))}\epsilon \leq c\epsilon V\left(\nabla u, B_{4\epsilon}(a)\right). \tag{163}$$

Now by the Co-area formula we can find $h \in (3\epsilon, 4\epsilon)$ such that

$$\int_{\partial B_h(a)} |\nabla u - w| \, dz \leq cV\left(\nabla u, B_{4\epsilon}(a)\right).$$

So there exists affine function l_w with ∇l_w with $\nabla l_w = w$ such that

$$\|u - l_w\|_{L^\infty(\partial B_h(a))} \leq cV\left(\nabla u, B_{4\epsilon}(a)\right) \tag{164}$$

Now

$$\int_{B_h(a)} \nabla u \cdot w dz = \int_{s\in P_{w^\perp}(B_h(a))}\int_{l_s^{\frac{w}{|w|}}\cap B_h(a)} \nabla u \cdot w dH^1 z ds.$$

For each $s \in P_{w^\perp}(B_h(a))$ let z_s, w_s be the endpoints of $l_s^{\frac{w}{|w|}} \cap B_h(a)$ with $(z_s - w_s)\cdot w > 0$, so

$$\begin{aligned}\int_{l_s^{\frac{w}{|w|}}\cap B_h(a)} \nabla u \cdot w dH^1 &= |w|\int_{l_s^{\frac{w}{|w|}}\cap B_h(a)} \nabla u \cdot \frac{w}{|w|} dH^1\\ &= |w|\left(u(z_s) - u(w_s)\right).\end{aligned}$$

Thus putting this together with (164) we have

$$\left|\int_{l_s^{\frac{w}{|w|}}\cap B_h(a)} (\nabla u \cdot w - |w|^2) dH^1 z\right| \leq cV\left(\nabla u, B_{4\epsilon}(a)\right) \text{ for each } s \in P_{w^\perp}(B_h(a)). \tag{165}$$

So integrating in s gives us

$$\left|\int_{B_h(a)} \nabla u \cdot w - |w|^2 \, dx\right| \leq c\epsilon V\left(\nabla u, B_{4\epsilon}(a)\right). \tag{166}$$

Hence

$$\begin{aligned}\left|\int_{B_h(a)}|\nabla u|^2-|w|^2\,dx\right| &\leq \left|\int_{B_h(a)}|\nabla u|^2-2\nabla u\cdot w+|w|^2\,dx\right|\\ &\quad+\left|\int_{B_h(a)}2\nabla u\cdot w-2\,|w|^2\,dx\right|\\ &\overset{(166),(162)}{\leq} c\left(V\left(\nabla u,B_{4\epsilon}(a)\right)\right)^2+c\epsilon V\left(\nabla u,B_{4\epsilon}(a)\right) \end{aligned} \tag{167}$$

Thus

$$\begin{aligned}\left||w|^2-1\right| &= \fint_{B_h(a)}\left||w|^2-1\right|dx\\ &\leq c\epsilon^{-2}\left(\int_{B_h(a)}\left||w|^2-|\nabla u|^2\right|dx\right)\\ &\overset{(167)}{\leq} c\left(\frac{\left(V\left(\nabla u,B_{4\epsilon}(a)\right)\right)^2}{\epsilon^2}+\epsilon^{-1}V\left(\nabla u,B_{4\epsilon}(a)\right)\right)\end{aligned}$$

So there exists a vector $v\in S^1$ such that

$$|w-v|\leq c\left(\frac{\left(V\left(\nabla u,B_{4\epsilon}(a)\right)\right)^2}{\epsilon^2}+\epsilon^{-1}V\left(\nabla u,B_{4\epsilon}(a)\right)\right)$$

Thus from (163)

$$\fint_{B_{4\epsilon}(a)}|\nabla u-v|\,dx\leq c\frac{\left(V\left(\nabla u,B_{4\epsilon}(a)\right)\right)^2}{\epsilon^2}+c\epsilon^{-1}V\left(\nabla u,B_{4\epsilon}(a)\right). \tag{168}$$

Hence for any $w\in B_\epsilon(a)$, using Lemma 10 for the last inequality

$$\begin{aligned}|\nabla u_\epsilon(w)-v| &= \left|\int\left(\nabla u(z)-v\right)\rho_\epsilon(w-z)dz\right|\\ &\leq c\epsilon^{-2}\left|\int\left(\nabla u(z)-v\right)\rho(\epsilon^{-1}(z-w))dz\right|\\ &\leq c\epsilon^{-2}\int_{B_{2\epsilon}(w)}|\nabla u(z)-v|\,dz\\ &\overset{(168)}{\leq} c\frac{\left(V\left(\nabla u,B_{4\epsilon}(a)\right)\right)^2}{\epsilon^2}+c\epsilon^{-1}V\left(\nabla u,B_{4\epsilon}(a)\right)\\ &\overset{(138)}{\leq} c\min\left\{\beta^{\frac{3}{8}},\epsilon^{-1}V\left(\nabla u,B_{4\epsilon}(a)\right)\right\}\end{aligned}$$

This completes the proof of Lemma 11. □

Lemma 12. *Let Ω be a convex domain and $|\Omega\triangle B_1(0)|\leq\beta$. Let $u(x)=d(x,\partial\Omega)$ and define $u_\epsilon:=u*\rho_{2\epsilon}$. Define $\Lambda:=\Omega_{4\epsilon}\backslash B_{5\beta^{\frac{1}{8}}}(0)$, we will show that for any $\epsilon\in(0,\beta^{\frac{1}{8}}]$*

$$\int_\Lambda \epsilon^{-1}\left|1-|\nabla u_\epsilon|^2\right|^2+\epsilon\left|\nabla^2 u_\epsilon\right|^2\leq c\beta^{\frac{3}{16}}. \tag{169}$$

Proof of Lemma. By the 5r Covering Theorem ([Ma 95], Theorem 2.1) them we can find a finite collection of balls $J:=\left\{B_{\frac{2\epsilon}{5}}(x_i):i=1,2,\dots m\right\}$ that are piecewise disjoint and $\Lambda\subset\bigcup_{i=1}^m B_{2\epsilon}(x_i)$.

Note that for any $i = 1, 2, \dots n$ since they are pairwise disjoint there are at most C_1 balls from J inside $B_{5\epsilon}(x_i)$. Thus $\| \sum_{i=1}^m 1\!\!1_{B_{5\epsilon}(x_i)}\|_{L^\infty(\Omega)} \leq C_1$ and this obviously implies $\| \sum_{i=1}^m 1\!\!1_{B_{2\epsilon}(x_i)}\|_{L^\infty(\Omega)} \leq C_1$.

Now given $a \in \Lambda$ if $x \in B_{2\epsilon}(a)$, let $w = \fint_{B_\epsilon(x)} \nabla u$

$$\begin{aligned}
\left|\nabla^2 u_\epsilon(x)\right| &= \left|\int \nabla u(z) \cdot \nabla \rho_\epsilon(x-z)dz\right| \\
&\leq \left|\int (\nabla u(z) - w) \cdot \nabla \rho\left(\frac{x-z}{\epsilon}\right)\epsilon^{-3}dz\right| \\
&\leq c\epsilon^{-3}\left|\int_{B_\epsilon(x)} (\nabla u - w)dz\right| \\
&\overset{(163)}{\leq} c\epsilon^{-2} V(\nabla u, B_{4\epsilon}(a)). \qquad (170)
\end{aligned}$$

So

$$\begin{aligned}
\int_\Lambda \left|\nabla^2 u_\epsilon\right|^2 dx &\leq \sum_{i=1}^m c\int_{B_{2\epsilon}(x_i)} \left|\nabla^2 u_\epsilon\right|^2 dx \\
&\leq c\sum_{i=1}^m \epsilon^2 \|\nabla^2 u_\epsilon\|^2_{L^\infty(B_{2\epsilon}(x_i))} \\
&\overset{(170)}{\leq} c\epsilon^2\left(\sum_{i=1}^m \epsilon^{-4}\left(V(\nabla u, B_{4\epsilon}(x_i))\right)^2\right) \\
&\overset{(138)}{\leq} c\beta^{\frac{3}{16}}\epsilon^{-1}\left(\sum_{i=1}^m V(\nabla u, B_{4\epsilon}(x_i))\right) \\
&\leq c\beta^{\frac{3}{16}}\epsilon^{-1} V\left(\nabla u, \Omega \backslash B_{2\beta^{\frac{1}{8}}}(0)\right) \\
&\overset{(137)}{\leq} c\epsilon^{-1}\beta^{\frac{3}{16}}. \qquad (171)
\end{aligned}$$

Now

$$\begin{aligned}
\int_\Lambda \left|1 - |\nabla u_\epsilon|^2\right|^2 dz &\leq c\sum_{i=1}^m \int_{B_{2\epsilon}(x_i)} \left|1 - |\nabla u_\epsilon|\right|^2 dz \\
&\overset{(161)}{\leq} \sum_{i=1}^m c\epsilon^2\beta^{\frac{3}{8}} \|\, |1 - |\nabla u_\epsilon||\, \|_{L^\infty(B_{2\epsilon}(x_i))} \\
&\overset{(161)}{\leq} \sum_{i=1}^m c\epsilon\beta^{\frac{3}{8}} V(\nabla u, B_{4\epsilon}(x_i)) \\
&\leq c\epsilon\beta^{\frac{3}{8}} V(\nabla u, \Omega \backslash B_{3\beta^{\frac{1}{8}}}(0)) \\
&\overset{(137)}{\leq} c\beta^{\frac{3}{8}}\epsilon. \qquad (172)
\end{aligned}$$

Putting (172) together with (171) establishes (169). □

Lemma 13. *Let* $\eta(x) = \frac{x}{|x|}$, $\epsilon > 0$ *and define* $\eta_\epsilon(x) := \int \eta(z)\rho_\epsilon(x-z)dz$. *So*

$$\int_{B_1(0)\backslash B_{2\epsilon}(0)} \left|1 - |\nabla \eta_\epsilon|^2\right|^2 dz \leq c\log(\epsilon^{-1})\epsilon^2 \qquad (173)$$

and

$$\int_{B_1(0)\backslash B_{2\epsilon}(0)} \left|\nabla^2\eta_\epsilon\right|^2 dz \leq c\log(\epsilon^{-1}). \tag{174}$$

Proof. Note for $x \notin B_{2\epsilon}(0)$, $z \in B_\epsilon(x)$

$$\begin{aligned}\left|\frac{z}{|z|}-\frac{x}{|x|}\right| &\leq \left|\frac{z\,|x|-x\,|z|}{|z|\,|x|}\right| \\ &\leq \left|\frac{z\,|x|-x\,|x|}{|z|\,|x|}\right|+\left|\frac{x\,|x|-x\,|z|}{|z|\,|x|}\right| \\ &\leq \frac{c\epsilon}{|x|-\epsilon}. \end{aligned} \tag{175}$$

So for $x \notin B_{2\epsilon}(0)$

$$\begin{aligned}\left|\nabla\eta_\epsilon(x)-\frac{x}{|x|}\right| &= \left|\int \rho_\epsilon(x-z)\left(\frac{x}{|x|}-\frac{z}{|z|}\right)dz\right| \\ &\overset{(175)}{\leq} \frac{c\epsilon}{|x|-\epsilon}. \end{aligned} \tag{176}$$

Now for any $x \notin B_{2\epsilon}(0)$, since $\int \frac{x}{|x|}\otimes\nabla\rho_\epsilon(x-z)dz = 0$

$$\begin{aligned}\nabla^2\eta_\epsilon(x) &= \left|\int \nabla\eta_\epsilon(z)\otimes\nabla\rho_\epsilon(x-z)dz\right| \\ &= \left|\int\left(\nabla\eta_\epsilon(z)-\frac{z}{|z|}\right)\otimes\nabla\rho_\epsilon(x-z)dz\right|+\left|\int\left(\frac{x}{|x|}-\frac{z}{|z|}\right)\otimes\nabla\rho_\epsilon(x-z)dz\right| \\ &\overset{(175),(176)}{\leq} \frac{c\epsilon}{|x|-\epsilon}\left|\int\nabla\rho_\epsilon(x-z)dz\right| \\ &\leq \frac{c}{|x|-\epsilon}. \end{aligned} \tag{177}$$

Hence

$$\begin{aligned}\int_{B_1(0)\backslash B_{2\epsilon}(0)}\left|\nabla^2\eta_\epsilon(x)\right|^2 dx &= c\int_{2\epsilon}^1\int_{\partial B_h(0)}\left(\frac{1}{|z|-\epsilon}\right)^2 dH^1 z dr \\ &\leq c\int_\epsilon^1 \frac{1}{r}dr \\ &\leq c\log(\epsilon^{-1})\end{aligned}$$

which establish (174). □

Lemma 14. *Let Ω be a convex domain and $|\Omega\triangle B_1(0)| \leq \beta$.*

Let $u(x) = d(x,\partial\Omega)$ and $\eta(x) = 1-\beta^{\frac{3}{32}}+|x|$. Define $\Gamma := \{x : u(x) = \eta(x)\}$, we will show Γ is the boundary of a convex set with $H^1(\Gamma) \leq c\beta^{\frac{3}{32}}$,

$$\Gamma \subset N_{c\beta^{\frac{3}{16}}}(\partial B_{2^{-1}\beta^{\frac{3}{32}}}(0)) \tag{178}$$

and for any $\epsilon \in (0,\beta^{\frac{1}{8}}]$

$$|N_{2\epsilon}(\Gamma)| \leq c\epsilon\beta^{\frac{3}{32}}. \tag{179}$$

Proof of Lemma.

Step 1. We will show $\Pi := \{x \in \Omega : \eta(x) \leq u(x)\}$ is convex.

Proof of Step 1. Take $a,b \in \Pi$ and pick $\lambda \in [0,1]$. Since u is concave $u(\lambda a+(1-\lambda)b) \geq \lambda u(a)+(1-\lambda)u(b)$ and since η is convex $\eta(\lambda a+(1-\lambda)b) \leq \lambda\eta(a)+(1-\lambda)\eta(b)$. Hence as

$a, b \in \Pi$, $u(\lambda a+(1-\lambda)b) \geq \eta(\lambda a+(1-\lambda)b)$. Thus $[a,b] \subset \Pi$ and thus the set Π is convex.

Step 2. We will establish (178).

Proof of Step 2. Let $x \in \Gamma$ and let $b_x \in \partial\Omega$ be such that $|x-b_x| = u(x)$. So

$$1-\beta^{\frac{3}{32}}+|x| = |b_x - x|. \tag{180}$$

And thus $1-\beta^{\frac{3}{32}}+|x| \geq |b_x|-|x|$, so using (101)

$$2|x| \geq |b_x|-1+\beta^{\frac{3}{32}} \geq \beta^{\frac{3}{32}}-\sqrt{\beta}. \tag{181}$$

Also from (180) we have

$$|x| = |b_x - x| - (1-\beta^{\frac{3}{32}}) \overset{(101)}{\leq} \beta^{\frac{3}{32}}+\sqrt{\beta}. \tag{182}$$

Now using Lemma 9, since $\nabla u(x) = \frac{x-b_x}{|x-b_x|}$ so

$$\begin{aligned}\left|\frac{x}{|x|}-\frac{b_x}{|b_x|}\right| &\leq \left|\frac{b_x-x}{|b_x-x|}-\frac{b_x}{|b_x|}\right|+\left|\frac{x-b_x}{|x-b_x|}-\frac{x}{|x|}\right| \\ &\overset{(182),(130)}{\leq} c\beta^{\frac{3}{32}} \end{aligned} \tag{183}$$

so

$$\left|1-\frac{b_x}{|b_x|}\cdot\frac{x}{|x|}\right| = 2^{-1}\left|\frac{b_x}{|b_x|}-\frac{x}{|x|}\right|^2 \leq c\beta^{\frac{3}{16}}. \tag{184}$$

Again by Lemma 9 we have $\left|\nabla u(x)-\frac{x}{|x|}\right| = \left|\frac{x-b_x}{|x-b_x|}-\frac{x}{|x|}\right| \overset{(130)}{\leq} c\beta^{\frac{3}{16}}$ and thus

$$\begin{aligned}\left|2x\cdot\frac{x}{|x|}-\beta^{\frac{3}{32}}\right| &\overset{(184)}{\leq} \left|-\beta^{\frac{3}{32}}+1-\frac{b_x}{|b_x|}\cdot\frac{x}{|x|}+2x\cdot\frac{x}{|x|}\right|+c\beta^{\frac{3}{16}} \\ &= \left|1-\beta^{\frac{3}{32}}+|x|-\left(\frac{b_x}{|b_x|}-x\right)\cdot\frac{x}{|x|}\right|+c\beta^{\frac{3}{16}} \\ &\overset{(180)}{=} \left||b_x-x|+\left(\frac{b_x}{|b_x|}-x\right)\cdot\frac{x}{|x|}\right|+c\beta^{\frac{3}{16}} \\ &\leq \left||b_x-x|+(b_x-x)\cdot\frac{x}{|x|}\right|+c\beta^{\frac{3}{16}} \\ &\overset{(130)}{\leq} c\beta^{\frac{3}{16}}\end{aligned}$$

hence $\left|2|x|-\beta^{\frac{3}{32}}\right| \leq c\beta^{\frac{3}{16}}$ for any $x \in \Gamma$, so (178) is established.

Since (178) implies the diameter of Π is bounded by $c\beta^{\frac{3}{32}}$ and since Π is a convex set it follows immediately that $H^1(\Pi) \leq c\beta^{\frac{3}{32}}$.

We claim

$$H^1\left(l_0^\theta \cap N_\epsilon(\partial\Pi)\right) \leq c\epsilon \text{ for all } \theta \in S^1. \tag{185}$$

Define $[\![x]\!] := \inf\left\{\lambda : x \in \lambda\widetilde{\Pi}\right\}$. Note that $\widetilde{\Pi}$ is not centrally symmetric and thus does not form a norm, however for any $a, b \in l_0^\theta$ (for some $\theta \in S^1$) we have $[\![a+b]\!] \leq [\![a]\!]+[\![b]\!]$.

Now for any $x \notin (1+\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}$, $y \in (1-\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}$ with $x, y \in l_0^\theta$ we have $[\![x]\!] \geq 1+\beta^{-\frac{3}{32}}\epsilon$, $[\![y]\!] < 1-\beta^{-\frac{3}{32}}\epsilon$ and so $[\![x-y]\!] \geq [\![x]\!]-[\![y]\!] \geq 2\beta^{-\frac{3}{32}}\epsilon$ and this implies $N_{c\epsilon\beta^{-\frac{3}{32}}}(\partial\Omega) \subset (1+\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}\backslash(1-\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}$. Let $a, b \in l_0^\theta \cap (1+\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}\backslash(1-\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}$, with $b\cdot\theta \geq a\cdot\theta$

$$(1+\beta^{-\frac{3}{32}}\epsilon) \geq [\![a]\!] \geq [\![b]\!]-[\![b-a]\!] \geq (1-\beta^{-\frac{3}{32}}\epsilon)-[\![b-a]\!]$$

which gives

$$|b-a| = |b\cdot\theta - a\cdot\theta| \leq 2\,|b\cdot\theta - a\cdot\theta|\,\lceil\!\lceil\theta\rceil\!\rceil = 2\lceil\!\lceil b-a\rceil\!\rceil \leq 2\beta^{-\frac{3}{32}}\epsilon.$$

Hence

$$H^1\left(l_0^\theta \cap N_{c\beta^{-\frac{3}{32}}\epsilon}(\partial\widetilde{\Pi})\right) \leq H^1\left(l_0^\theta \cap (1+\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}\backslash(1-\beta^{-\frac{3}{32}}\epsilon)\widetilde{\Pi}\right) \leq c\beta^{-\frac{3}{32}}\epsilon$$

and as $\beta^{\frac{3}{32}} l_0^\theta \cap N_{c\beta^{-\frac{3}{32}}\epsilon}(\partial\widetilde{\Pi}) = l_0^\theta \cap N_{c\epsilon}(\partial\Pi)$ this implies (185).

Let $\mathcal{A} : \mathbb{R}^2 \to S^1$ be defined by $\mathcal{A}(x) = \frac{x}{|x|}$, note that for some positive constant $\mathcal{C}_7 > 1$

$$\mathcal{C}_7^{-1}\,|x|^{-1} \leq |D\mathcal{A}(x)| \leq \mathcal{C}_7\,|x|^{-1} \text{ for all } x \neq 0 \tag{186}$$

so by the Co-area formula

$$\begin{aligned}\int |D\mathcal{A}(z)|\,\mathbb{1}_{N_\epsilon(\partial\Pi)}(z)dz \quad &\leq \quad \int_{\theta\in S^1}\int_{l_0^\theta} \mathbb{1}_{N_\epsilon(\partial\Pi)}(x)dH^1 x dH^1\theta \\ &\overset{(185)}{\leq} \quad c\epsilon.\end{aligned}$$

However from (186) and the fact that by (181) we know $B_{2^{-1}\beta^{\frac{3}{32}}}(0) \subset \Pi$ we know

$$\inf\{|D\mathcal{A}(z)| : z \in N_\epsilon(\partial\Pi)\} \geq c\beta^{-\frac{3}{32}}$$

so (179) follows. $\Box$.

4.2. **Proof of Proposition 1.** Let $u(x) = d(x, \partial\Omega)$, let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be the smooth monotonic function from the proof of Lemma 8, so w satisfies (107) and $\sup|\ddot{w}| \leq \epsilon^{-\frac{1}{2}}$ as in Lemma 8 for $x \in \Omega\backslash\Omega_{(1-3\sqrt{\epsilon})}$ define $\phi(x) = w(u(x))$. Let

$$w(x) := \min\left\{u(x), 1 - \beta^{\frac{3}{32}} + |x|\right\}. \tag{187}$$

and define

$$\xi(x) = \int w(x-z)\rho_{\phi(x)}(z)dz. \tag{188}$$

Let $\Pi = \left\{x : u(x) > 1 - \beta^{\frac{3}{32}} + |x|\right\}$, and define $\Lambda_0 = \Omega_{4\epsilon}\backslash N_\epsilon(\Pi)$, note that $\xi(x) = u_\epsilon(x)$ for any $x \in \Lambda_0$.

Recall the function ψ defined in Lemma 8, note that $\xi(x) = \psi(x)$ for any $x \in \Omega\backslash(1-3\sqrt{\epsilon})\Omega$ thus

$$\int_{\Omega\backslash(1-3\sqrt{\epsilon})\Omega} \epsilon^{-1}\left|1 - |\nabla\xi|^2\right|^2 + \epsilon\left|\nabla^2\xi\right|^2 \overset{(103),(104)}{\leq} 2\epsilon^{\frac{1}{10}}$$

Since $\psi = u_\epsilon$ in Λ_0 (169) we have $\int_{\Lambda_0} \epsilon^{-1}\left|1 - |\nabla\xi|^2\right|^2 + \epsilon\left|\nabla^2\xi\right|^2 \leq c\beta^{\frac{3}{16}}$ and so putting this two inequalities together we have

$$\int_{\Omega\backslash N_\epsilon(\Pi)} \epsilon^{-1}\left|1 - |\nabla\xi|^2\right|^2 + \epsilon\left|\nabla^2\xi\right|^2 \leq c\beta^{\frac{3}{16}} \tag{189}$$

Now as for any $x \in \Pi\backslash N_\epsilon(\partial\Pi)$, $w(x) = 1 - \beta^{\frac{3}{32}} + |x|$ and so $u_\epsilon(x) = \eta_\epsilon(x) + (1 - \beta^{\frac{3}{32}})$ where $\eta(x) = |x|$ and $\eta_\epsilon = \eta * \rho_\epsilon$. So $\nabla\xi(x) = \nabla\eta_\epsilon(x)$ and $\nabla^2\xi(x) = \nabla^2\eta_\epsilon(x)$ thus applying Lemma 13 we have

$$\int_{\Pi\backslash N_\epsilon(\partial\Pi)} \epsilon^{-1}\left|1 - |\nabla\xi|^2\right|^2 + \epsilon\left|\nabla^2\xi\right|^2 \overset{(173),(174)}{\leq} c\epsilon\log(\epsilon^{-1}). \tag{190}$$

Since w is Lipschitz, so ξ is Lipschitz and we have

$$\int_{N_\epsilon(\partial\Pi)} \epsilon^{-1}\left|1 - |\nabla\xi|^2\right|^2 \leq c\beta^{\frac{3}{32}}. \tag{191}$$

And note for any $x \in \Omega_\epsilon$

$$\left|\nabla^2 \xi(x)\right| = \epsilon^{-3} \left| \int \nabla w(z) \cdot \nabla \rho \left(\frac{x-z}{\epsilon} \right) dz \right| \leq c\epsilon^{-1}$$

so using the fact ξ is Lipschitz

$$\begin{aligned} \int_{N_\epsilon(\partial\Pi)} \epsilon \left|\nabla^2 \xi\right|^2 &\leq c\epsilon^{-1} \left|N_\epsilon(\partial\Pi)\right| \\ &\leq c\beta^{\frac{3}{32}}. \qquad (192) \end{aligned}$$

Putting these inequalities together we have

$$\int_{N_\epsilon(\partial\Pi)} \epsilon^{-1} \left|1 - |\nabla\xi|^2\right|^2 + \epsilon \left|\nabla^2 \xi\right|^2 \leq c\beta^{\frac{3}{32}}. \qquad (193)$$

Now inequalities (189), (190) and (193) give us that ξ satisfies (98). And since $\xi(x) = \psi(x)$ on $\Omega \backslash (1 - 3\sqrt{\epsilon})\Omega$ from (106) satisfies $\nabla \xi(x) \cdot \eta_x = 1$ for any $x \in \partial\Omega$. This completes the proof of Proposition 1. □

4.3. **Proof Corollary 1.** Let $\alpha = \inf_y |\Omega \triangle B_1(y)|$, without loss of generality assume $|\Omega \triangle B_1(0)| = \alpha$. Let $\beta = \alpha + \epsilon$, note that for $\epsilon \leq \beta^{\frac{1}{8}}$ and $|\Omega \triangle B_1(0)| \leq \beta$.

Since $\xi \in \Lambda(\Omega)$ from (98) we have that $\inf_{u \in \Lambda(\Omega)} I_\epsilon(u) \leq c\beta^{\frac{3}{16}}$. Let $v \in \Lambda(\Omega)$ be the minimiser of I_ϵ and let $\tilde{\beta} = \beta^{\frac{3}{16}}$, since v satisfies

$$\int_\Omega \left|1 - |\nabla v|^2\right| \left|\nabla^2 v\right| dz \leq \int_\Omega \epsilon^{-1} \left|1 - |\nabla v|^2\right|^2 + \epsilon \left|\nabla^2 v\right|^2 \leq c\beta^{\frac{3}{16}}$$

and as $\epsilon \in (0, \beta^{\frac{1}{8}})$

$$\int_\Omega \left|1 - |\nabla v|^2\right|^2 dz \leq c\beta^{\frac{5}{16}}.$$

So defining $\tilde{\beta} = \beta^{\frac{5}{32}}$, we have that (6), (7) are satisfied and hence

$$\int_\Omega \left| \nabla v(z) - \frac{z}{|z|} \right|^2 dz \leq c\tilde{\beta}^{\frac{1}{256}} \leq c\beta^{\frac{1}{1639}}.$$

Applying Lemma 9 we have

$$\int_{\Omega \backslash B_{\beta^{\frac{1}{8}}}(0)} |\nabla v - \nabla \zeta|^2 \leq c\beta^{\frac{1}{1639}}. \qquad (194)$$

Now

$$\begin{aligned} \int_{B_{\beta^{\frac{1}{8}}}(0)} |\nabla v - \nabla \zeta|^2 dz &\leq \int_{B_{\beta^{\frac{1}{8}}}(0)} |\nabla v|^2 + |\nabla \zeta| + 1 dz \\ &\leq \left(\left|B_{\beta^{\frac{1}{8}}}(0)\right| \left\|1 - |\nabla v|^2\right\|_{L^2(\Omega)} \right)^{\frac{1}{2}} + c\left|B_{\beta^{\frac{1}{8}}}(0)\right| \\ &\leq c\beta^{\frac{1}{4}} \end{aligned}$$

together with (194) this gives $\|v - \zeta\|_{W^{2,2}(\Omega)} \leq c\beta^{\frac{1}{3278}} \leq c(\epsilon + \alpha)^{\frac{1}{3278}}$. □

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