Calculus of Variations

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1 History

The calculus of variations is a branch of mathematical analysis that studies extrema and critical points of functionals (or energies). Here, by *functional* we mean a mapping from a function space to the real numbers.

One of the first questions that may be framed within this theory is *Dido's isoperimetric problem* (see Subsection 2.3): to find the shape of a curve of prescribed perimeter that maximizes the area enclosed. Dido was a Phoenician princess who emigrated to North Africa and upon arrival obtained from the native chief as much territory as she could enclose with an ox hide. She cut the hide into a long strip, and used it to delineate the territory later known as Carthage, bounded by a straight coastal line and a semi-circle.

It is commonly accepted that the systematic development of the theory of the calculus of variations began with the *brachistochrone curve problem* proposed by Johann Bernoulli in 1696: consider two points A and B on the same vertical plane but on different vertical lines. Assume that A is higher than B, and that a particle M is moving from A to B along a curve and under the action of gravity. The curve that minimizes the time travelled by M is called the *brachistochrone*. The solution to this problem required the use of infinitesimal calculus and was later found by Jacob Bernoulli, Newton, Leibniz and de l'Hôpital. The arguments thus developed led to the development of the foundations of the calculus of variations by Euler. Important contributions to the subject are attributed to Dirichlet, Hilbert, Lebesgue, Riemann, Tonelli, Weierstrass, among many others.

The common feature underlying Dido's and the brachistochrone problems is that one seeks to maximize or minimize a functional over a class of competitors satisfying given constraints. In both cases the functional is given by an integral of a density depending on an underlying field and some of its derivatives, and this will be the prototype we will adopt in what follows. Precisely, we consider a functional

$$u \in X \mapsto F(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx, \quad (1)$$

where X is a function space (usually a L^p space or a Sobolev-type space), $u : \Omega \to \mathbb{R}^d$, with $\Omega \subset \mathbb{R}^N$ an open set, N and d are positive integers, and the density is a function $f(x, u, \xi)$, with $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$. Here, and in what follows, ∇u stands for the $d \times N$ matrix-valued distributional derivative of u.

The calculus of variations is a vast theory and here we chose to highlight some contemporary aspects of the field, and we conclude this article by mentioning a few forefront areas of application that are driving current research.

2 Extrema

In this section we address fundamental minimization problems and relevant techniques in the calculus of variations. In geometry, the simplest example is the problem of finding the curve of shortest length connecting two points, a *geodesic*. A (continuous) curve joining two points $A, B \in \mathbb{R}^d$ is represented by a (continuous) function $\gamma : [0,1] \to \mathbb{R}^d$ such that $\gamma(0) = A$, $\gamma(1) = B$, and its length is given by

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \right\},$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = 1$, $n \in \mathbb{N}$, of the interval $[0,1]$. If γ is smooth, then $L(\gamma) = \int_0^1 |\gamma'(t)|\, dt$. In the absence of constraints, the geodesic is the straight segment with endpoints A and B, and so $L(\gamma) = |A - B|$. Often in applications the curves are restricted to lie on a given manifold, e.g., a sphere (in this case, the geodesic is the shortest great circle joining A and B).

2.1 Minimal Surfaces

A *minimal surface* is a surface of least area among all those bounded by a given closed curve. The problem of finding minimal surfaces, called the *Plateau problem*, was first solved in three dimensions in the 1930's by Douglas and by Rado, and in the 1960's several authors, including Almgren, De Giorgi, Fleming and Federer, addressed it using geometric measure theoretical tools. This approach gives existence of solutions in a “weak sense”, and their regularity is significantly more involved. De Giorgi proved that minimal surfaces are analytic except on a singular set of dimension at most $N-1$. Later, Federer, based on earlier results by Almgren and Simons, improved the dimension of the singular set to $N-8$. The sharpness of this estimate was confirmed with an example by Bombieri, De Giorgi and Giusti.

Important minimal surfaces are the so-called *non-parametric* minimal surfaces, which are given as graphs of real-valued functions. Precisely, given an open set $\Omega \subset \mathbb{R}^N$ and a smooth function $u : \Omega \to \mathbb{R}$, then the area of the graph of u, $\{(x, u(x)) : x \in \Omega\}$, is given by

$$F(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx. \tag{2}$$

It can be shown that u minimizes the area of its graph subject to prescribed values on the boundary of Ω if

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \Omega.$$

2.2 Willmore Functional

Recently many smooth surfaces, including tori, have been obtained as minima or critical points of certain geometrical functionals in the calculus of variations. An important example is the Willmore (or bending) energy of a compact surface S embedded in $\mathbb{R}^3$, namely the surface integral $\mathcal{W}(S) := \int_S H^2 d\sigma$, where $H := \frac{k_1+k_2}{2}$ and k_1 and k_2 are the principal curvatures of S. This energy has a wide scope of applications, ranging from materials science (e.g., elastic shells, bending energy), to mathematical biology (e.g., cell membranes) to image segmentation in computer vision (e.g., staircasing).

Critical points of $\mathcal{W}$ are called *Willmore surfaces*, and satisfy the Euler-Lagrange equation

$$\Delta_S H + 2H(H^2 - K) = 0,$$

where $K := k_1 k_2$ is the Gaussian curvature and Δ_S is the Laplace-Beltrami operator.

In the 1920’s it was shown by Blaschke and by Thomsen that the Willmore energy is invariant under conformal transformations of $\mathbb{R}^3$. Also, the Willmore energy is minimized by spheres, with resulting energy value 4π. Therefore, $\mathcal{W}(S) - 4\pi$ describes how much S differs from a sphere in terms of its bending. The problem of minimizing the Willmore energy among the class of embedded tori T was proposed by Willmore, who conjectured in 1965 that $\mathcal{W}(T) \geq 2\pi^2$. This conjecture has been proved by Marques and Neves in 2012.

2.3 Isoperimetric Problems; the Wulff set

The understanding of the surface structure of crystals plays a central role in many fields of physics, chemistry and materials science. If the dimension of the crystals is sufficiently small, then the leading morphological mechanism is driven by the minimization of surface energy. Since the work of Herring in the 1950’s, a classical question in this field is to determine the crystalline shape that has smallest surface energy for a given volume. Precisely, we seek to minimize the surface integral

$$\int_{\partial E} \psi(\nu(x))\, d\sigma \tag{3}$$

over all smooth sets $E \subset \mathbb{R}^N$ with prescribed volume, and where $\nu(x)$ is the outward unit normal to ∂E at x. The right variational framework for this problem is within the class of sets of finite perimeter. The solution, which exists and is unique up to translations, is called the Wulff shape. A key ingredient in the proof is the Brunn-Minkowski inequality

$$(\mathcal{L}^N(A))^{1/N} + (\mathcal{L}^N(B))^{1/N} \leq (\mathcal{L}^N(A+B))^{1/N} \tag{4}$$

which holds for all Lebesgue measurable sets $A, B \subset \mathbb{R}^N$ such that $A+B$ is also Lebesgue measurable. Here $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure.

3 The Euler Lagrange Equation

Consider the functional (1), in the scalar case $d = 1$, and where f of class C^1 and X is the

Sobolev space $X = W^{1,p}(\Omega)$, $1 \le p \le +\infty$, of all functions $u \in L^p(\Omega)$ whose distributional gradient ∇u belongs to $L^p(\Omega; \mathbb{R}^N)$. Let $u \in X$ be a *local minimizer* of the functional F, that is,

$$\int_U f(x, u(x), \nabla u(x))\, dx \le \int_U f(x, v(x), \nabla v(x))\, dx$$

for every open subset Ucompactly contained in Ω, and all v such that $u - v \in W_0^{1,p}(U)$, where $W_0^{1,p}(U)$ is the space of all functions in $W^{1,p}(U)$ "vanishing" on the boundary of ∂U. Note that v will then coincide with u outside the set U. If $\varphi \in C_c^1(\Omega)$ then $u + t\varphi$, $t \in \mathbb{R}$, are admissible, and thus

$$t \in \mathbb{R} \mapsto g(t) := F(u + t\varphi)$$

has a minimum at $t = 0$. Therefore, under appropriate growth conditions on f, we have that $g'(0) = 0$, i.e.,

$$\int_\Omega \left(\sum_{i=1}^N \frac{\partial f}{\partial \xi_i}(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + \frac{\partial f}{\partial u}(x, u, \nabla u)\varphi \right) dx = 0. \quad (5)$$

A function $u \in X$ satisfying (5) is said to be a *weak solution of the Euler Lagrange equation* associated to (1).

Under suitable regularity conditions on f and u, (5) can be written in the strong form

$$\mathrm{div}(\nabla_\xi f(x, u, \nabla u)) = \frac{\partial f}{\partial u}(x, u, \nabla u), \quad (6)$$

where $\nabla_\xi f(x, u, \xi)$ is the gradient of the function $f(x, u, \cdot)$.

In the vectorial case $d > 1$ the same argument leads to a system of partial differential equations (PDEs) in place of (5).

4 Variational Inequalities, Free Boundary and Free Discontinuity Problems

We now add a constraint to the minimization problem considered in the previous section. Precisely, let $d = 1$ and let ϕ be a function in Ω. If u is a local minimizer of (1) among all functions $v \in W^{1,p}(\Omega)$ subject to the constraint $v \ge \phi$ in Ω, then the variation $u + t\varphi$ is admissible if $\varphi \ge 0$ and $t \ge 0$. Therefore, the function g satisfies $g'(0) \ge 0$, and the Euler-Lagrange equation (5) becomes the variational inequality

$$\int_\Omega \left(\sum_{i=1}^N \frac{\partial f}{\partial \xi_i}(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + \frac{\partial f}{\partial u}(x, u, \nabla u)\varphi \right) dx \ge 0$$

for all nonnegative $\varphi \in C_c^1(\Omega)$. This is called the *obstacle problem*, and the coincidence set $\{u = \phi\}$ is not known a priori and is called the *free boundary*. This is an example of a broad class of variational inequalities and free boundary problems that have applications in a variety of contexts, including the modeling of the melting of ice (the *Stefan problem*), lubrication, and the filtration of a liquid through a porous medium.

A related class of minimization problems in which the unknowns are both an underlying field u and a subset E of Ω, is the class of *free discontinuity problems* that are characterized by the competition between a volume energy of the type (1) and a surface energy, e.g., as in (3). Important examples are in the study of liquid crystals, optimal design of composite materials in continuum mechanics (see Subsection 13.3), and image segmentation in computer vision (see Subsection 13.4).

5 Lagrange Multipliers

The method of *Lagrange multipliers in Banach spaces* is used to find extrema of functionals $G : X \to \mathbb{R}$ subject to a constraint

$$\{x \in X : \Psi(x) = 0\}, \quad (7)$$

where $\Psi : X \to Y$ is another functional and X and Y are Banach spaces. It can be shown that if G and Ψ are of class C^1 and $u \in X$ is an extremum of G subject to (7), and if the derivative $D\Psi(u) : X \to Y$ is surjective, then there exists a continuous, linear functional $\lambda : Y \to \mathbb{R}$ such that

$$DG(u) + \lambda \circ D\Psi(u) = 0, \quad (8)$$

where $\circ$ stands for the composition operator between functions. The functional λ is called a *Lagrange multiplier*.

In the special case in which $Y = \mathbb{R}$, λ may be identified with a scalar, still denoted by λ, and (8) takes the familiar form

$$DG(u) + \lambda D\Psi(u) = 0.$$

Therefore, candidates for extrema may be found among all critical points of the family of functionals $G + \lambda\Psi$, $\lambda \in \mathbb{R}$.

If G has the form (1) and $X = W^{1,p}(\Omega;\mathbb{R}^d)$, $1 \le p \le +\infty$, then typical examples of Ψ are

$$\Psi(u) := \int_\Omega |u|^s \, dx - c_1 \quad \text{or} \quad \Psi(u) := \int_\Omega u \, dx - c_2$$

for some constants $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}^d$, and $1 \le s < +\infty$.

6 Minimax Methods

Minimax methods are used to establish the existence of *saddle points* of the functional (1), i.e., critical points that are not extrema. More generally, for C^1 functionals $G : X \to \mathbb{R}$ where X is an infinite dimensional Banach space, as introduced in Section 5, the *Palais-Smale compactness condition* (P.-S.) plays the role of compactness in the finite-dimensional case. Precisely, G satisfies the (P.-S.) condition if whenever $\{u_n\} \subset X$ is such that $\{G(u_n)\}$ is a bounded sequence in $\mathbb{R}$ and $DG(u_n) \to 0$ in the dual of X, X', then $\{u_n\}$ admits a convergent subsequence.

An important result for the existence of saddle points that uses the (P.-S.) condition is the *Mountain Pass Lemma* of Ambrosetti and Rabinowitz, which states that if G satisfies the (P.-S.) condition, if $G(0) = 0$ and there are $r > 0$ and $u_0 \in X \setminus \overline{B(0,r)}$ such that

$$\inf_{\partial B(0,r)} G > 0 \quad \text{and} \quad G(u_0) \le 0,$$

then

$$\inf_{\gamma \in \mathcal{C}} \sup_{u \in \gamma} G(u)$$

is a critical value, where $\mathcal{C}$ is the set of all continuous curves from $[0,1]$ into X joining 0 to u_0.

In addition, minimax methods can be used to prove the existence of multiple critical points of functionals G that satisfy certain symmetry properties, for example, the generalization of the result by Ljusternik and Schnirelmann for symmetric functions to the infinite dimensional case.

7 Lower Semicontinuity

7.1 The Direct Method

The *direct method in the calculus of variations* provides conditions on the function space X and on a functional G, as introduced in Section 5, that guarantee the existence of minimizers of G. The method consists of the following steps:

Step 1. Consider a *minimizing sequence* $\{u_n\} \subset X$, i.e., $\lim_{n\to\infty} G(u_n) = \inf_{u\in X} G(u)$.

Step 2. Prove that $\{u_n\}$ admits a subsequence $\{u_{n_k}\}$ converging to some $u_0 \in X$ with respect to some (weak) topology τ in X. When G has an integral representation of the form (1), this is usually a consequence of a priori coercivity conditions on the integrand f.

Step 3. Establish the *sequential lower semicontinuity of G with respect to τ*, i.e., $\liminf_{n\to\infty} G(v_n) \ge G(v)$ whenever the sequence $\{v_n\} \subset X$ converges weakly to $v \in X$ with respect to τ.

Step 4. Conclude that u_0 minimizes G. Indeed,

$$\begin{aligned}\inf_{u\in X} G(u) &= \lim_{n\to\infty} G(u_n) = \lim_{k\to\infty} G(u_{n_k}) \\ &\ge G(u_0) \ge \inf_{u\in X} G(u).\end{aligned}$$

7.2 Integrands: convex, polyconvex, quasiconvex, rank-one convex

In view of Step 3 above, it is important to characterize the class of integrands f in (1) for which the corresponding functional F is sequentially lower semicontinuous with respect to τ. In the case in which X is the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^d)$, $1 \le p \le +\infty$, and τ is the weak topology (weak-$\star$ if $p = +\infty$), this is related to convexity-type properties of $f(x,u,\cdot)$. If $\min\{d,N\} = 1$ then under appropriate growth and regularity conditions, it can be shown that convexity of $f(x,u,\cdot)$ is necessary and sufficient. More generally, if $\min\{d,N\} > 1$ then the corresponding condition is called *quasiconvexity*; precisely, $f(x,u,\cdot)$ is said to be quasiconvex if

$$f(x,u,\xi) \le \int_{(0,1)^N} f\big(x,u,\xi + \nabla\varphi(y)\big)\, dy$$

for all $\xi \in \mathbb{R}^{d\times N}$ and all $\varphi \in W_0^{1,\infty}((0,1)^N;\mathbb{R}^d)$, whenever the right-hand side in this inequality is well defined. Since this condition is nonlocal, in applications in mechanics often one studies related classes of integrands, such as polyconvex and rank-one convex functions, for which there are algebraic criteria.

8 Relaxation

In most applications Step 3 in Subsection 7.1 fails, and this leads to an important topic at the core of the calculus of variations, namely the introduction of a *relaxed*, or *effective*, energy $\mathcal{G}$ that is related to G, as introduced in Section 5, as follows:

(a) $\mathcal{G}$ is sequentially lower semicontinuous with respect to τ;

(b) $\mathcal{G} \leq G$ and $\mathcal{G}$ inherits coercivity properties from G;

(c) $\min\limits_{u \in X} \mathcal{G} = \inf\limits_{u \in X} G$.

When G is of the type (1), a central problem is to understand if $\mathcal{G}$ has an integral form of the type (1) for some new integrand h, and if so then what is the relation between h and the original integrand f.

If $X = W^{1,p}(\Omega)$, $p \geq 1$, and τ is the weak topology, then under appropriate growth and regularity conditions it can be shown that $h(x, u, \cdot)$ is the *convex envelope* of $f(x, u, \cdot)$, i.e., the greatest convex function less than $f(x, u, \cdot)$. In the vectorial case $d > 1$ the convex envelope is replaced by a similar notion of quasiconvex envelope (see Subsection 7.2).

9 Γ-Convergence

Often in physical problems the behavior of a system is described in terms of a sequence $\{G_n\}$, $n \in \mathbb{N}$, of energy functionals $G_n : X \to [-\infty, +\infty]$ where X is a metric space with a metric d. Is it possible to identify a limiting energy G_∞ that captures qualitative properties of this family, and such that minimizers of G_n converge to minimizers of G_∞?

The notion of Γ-*convergence*, introduced by De Giorgi, provides a tool for answering these questions. To motivate this concept with an example, consider a fluid confined into a container $\Omega \subset \mathbb{R}^N$. Assume that the total mass of the fluid is m, so that admissible density distributions $u : \Omega \to \mathbb{R}$ satisfy the constraint $\int_\Omega u(x)\, dx = m$. The total energy is given by the functional $u \mapsto \int_\Omega W(u(x))\, dx$, where $W : \mathbb{R} \to [0, \infty)$ is the energy per unit volume. Assume that W supports two phases $a < b$, that is, W is a *double-well potential*, with $\{u \in \mathbb{R} : W(u) = 0\} = \{a, b\}$. Then any density distribution u that renders the body stable in the sense of Gibbs is a minimizer of the following problem $(\mathcal{P}_0)$

$$\min \left\{ \int_\Omega W(u(x))\, dx : \int_\Omega u(x)\, dx = m \right\}. \quad (9)$$

If $\mathcal{L}^N(\Omega) = 1$ and $a < m < b$, then given any measurable set $E \subset \Omega$ with $\mathcal{L}^N(E) = \frac{b-m}{b-a}$, the function $u = a\chi_E + b\chi_{\Omega \setminus E}$ is a solution of problem (9). This lack of uniqueness is due the fact that interfaces between the two phases a and b are not penalized by the total energy. The physically preferred solutions should be the ones that arise as limiting cases of a theory that penalizes interfacial energy, so it is expected that these solutions should minimize the surface area of $\partial E \cap \Omega$.

In the van der Walls–Cahn–Hilliard theory of phase transitions, the energy depends not only on the density u but also on its gradient, precisely,

$$\int_\Omega W(u(x))\, dx + \varepsilon^2 \int_\Omega |\nabla u(x)|^2\, dx.$$

Note that the gradient term penalizes rapid changes of the density u, and thus it plays the role of an interfacial energy. Stable density distributions u are now solutions of the minimization problem $(\mathcal{P}_\varepsilon)$

$$\min \left\{ \int_\Omega W(u(x))\, dx + \varepsilon^2 \int_\Omega |\nabla u(x)|^2\, dx \right\},$$

where the minimum is taken over all smooth functions u satisfying $\int_\Omega u(x)\, dx = m$. In 1983 Gurtin conjectured that the limits, as $\varepsilon \to 0$, of solutions of $(\mathcal{P}_\varepsilon)$ are solutions of $(\mathcal{P}_0)$ with minimal surface area. Using results of Modica and Mortola, this conjecture was proved independently by Modica and by Sternberg in the setting of Γ-convergence.

The Γ-*limit* $G_\infty : X \to [-\infty, +\infty]$ of $\{G_n\}$ with respect to a metric d, when it exists, is defined uniquely by the following properties:

(i) (liminf inequality) for every sequence $\{u_n\} \subset X$ converging to $u \in X$ with respect to d

$$G_\infty(u) \leq \liminf_{n\to\infty} G_n(u_n);$$

(ii) (limsup inequality) for every $u \in X$ there exists a sequence $\{u_n\} \subset X$ converging to $u \in X$ with respect to d such that

$$G_\infty(u) \geq \limsup_{n\to\infty} G_n(u_n).$$

This notion may be extended to the case in which the convergence of the sequences is taken with respect to some weak topology rather than the topology induced by the metric d. In this context, we remark that when the sequence $\{G_n\}$ reduces to a single energy functional $\{G\}$, under appropriate growth and coercivity assumptions, G_∞ coincides with the relaxed energy $\mathcal{G}$, as discussed in Section 8.

Other important applications of Γ-convergence include the Ginzburg-Landau theory for superconductivity (see Subsection 13.5), homogenization of variational problems (see Subsection 13.3), dimension reduction problems in elasticity (see Subsection 13.2), free-discontinuity problems in image segmentation in computer vision (see Subsection 13.4) and in fracture mechanics.

10 Regularity

Optimal regularity of minimizers and local minimizers of the energy (1) in the vectorial case $d \geq 2$, and when $X = W^{1,p}(\Omega;\mathbb{R}^d)$, $1 \leq p \leq +\infty$, is mostly open. In the scalar case $d = 1$ there is an extensive body of literature on the regularity of weak solutions of the Euler Lagrange equation (5), stemming from a fundamental result of De Giorgi, independently obtained by Nash, in the late 1950's. For $d \geq 2$, in general (local) minimizers of (1) are not everywhere smooth. On the other hand, and under suitable hypotheses on the integrand f, it can be shown that *partial regularity* holds, i.e., if u is a local minimizer then there exists an open subset of Ω, Ω_0, of full measure such that $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^d)$, for some $\alpha \in (0,1)$. Sharp estimates on α and on the Hausdorff dimension of the *singular set* $\Sigma_u := \Omega \setminus \Omega_0$ are still unknown.

11 Symmetrization

Rearrangements of sets preserve their measure while modifying their geometry to achieve specific symmetries. In turn, rearrangements of a function u yield new functions with desired symmetry properties, and which are obtained via suitable rearrangements of the *t-superlevel sets* of u, $\Omega_t := \{x \in \Omega : u(x) > t\}$. These tools are used in a variety of contexts, from harmonic analysis and PDEs to spectral theory of differential operators. In the calculus of variations, they may be often found in the study of extrema of functionals of type (1). Among the most common rearrangements, we mention the directional monotone decreasing rearrangement, the star-shaped rearrangement, the directional Steiner symmetrization, the Schwarz symmetrization, the circular and spherical symmetrization, and the radial symmetrization.

Of these, we highlight the Schwarz symmetrization, which is the most frequently used in the calculus of variations. If u is a nonnegative measurable function with compact support in $\mathbb{R}^N$, then its *Schwartz symmetric rearrangement* is the (unique) spherically symmetric and decreasing function $u^\star$ such that for all $t > 0$ the t-superlevel sets of u and $u^\star$ have the same measure.

When $\Omega = \mathbb{R}^N$, it can be shown that $u^\star$ preserves the L^p norm of u and the regularity of u up to first order, that is, if u belongs to $W^{1,p}(\mathbb{R}^N)$ then so does $u^\star$, $1 \leq p \leq +\infty$. Moreover, by the Pólya-Szegö inequality, $||\nabla u^\star||_p \leq ||\nabla u||_p$, and we remark that for $p = \infty$ this is obtained using the Brunn-Minkowski inequality discussed in Subsection 2.3.

Another important inequality relating u and $u^\star$ is the Riesz inequality, and the Faber-Krahn inequality compares eigenvalues of the Dirichlet problems in Ω and in $\Omega^\star$. Classical applications of rearrangements include the derivation of the sharp constant in the Sobolev-Gagliardo-Nirenberg inequality in $W^{1,p}(\mathbb{R}^N)$, $1 < p < N$, as well as in the Young inequality and the Hardy-Littlewood-Sobolev inequality.

Finally, we remark that the first and most important application of Steiner symmetrization is the isoperimetric property of balls (see Dido's problem in Section 1).

12 Duality Theory

Duality theory associates to a minimization problem $(\mathcal{P})$ a maximization problem $(\mathcal{P}^*)$, called the *dual problem*, and studies the relation between these two. It has important applications in several disciplines, including economics and mechanics, and different areas of mathematics, such as the calculus of variations, convex analysis, and numerical analysis.

The theory of dual problems is inspired by the notion of duality in convex analysis and by the *Fenchel transform* $f^\star$ of a function $f : \mathbb{R}^N \to [-\infty, +\infty]$, defined as

$$f^\star(\eta) := \sup\{\eta \cdot \xi - f(\xi) : \xi \in \mathbb{R}^N\} \quad \text{for } \eta \in \mathbb{R}^N.$$

As an example, consider the minimization problem

$$(\mathcal{P}) \qquad \inf \left\{ \int_\Omega f(\nabla u)\, dx : u \in W_0^{1,p}(\Omega) \right\},$$

with $f : \mathbb{R}^N \to \mathbb{R}$. If f satisfies appropriate growth and convexity conditions, then the dual problem $(\mathcal{P}^\star)$ is given by

$$\sup \left\{ -\int_\Omega f^\star(v(x))\, dx : v \in L^q(\Omega; \mathbb{R}^N), \right. \\ \left. \operatorname{div} v = 0 \text{ in } \Omega \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The latter problem may be simpler to handle in specific situations, e.g., for nonparametric minimal surfaces and with f given as in (2), where, due to lack of coercivity, $(\mathcal{P})$ may not admit a solution in X.

13 Some Contemporary Applications

There is a plethora of applications of the calculus of variations. Classical ones include Hamiltonians and Lagrangians, the Hamilton-Jacobi equation, conservation laws, Noether's theorem, optimal control. Below we focus on a few contemporary applications that are pushing the frontiers of the theory in novel directions.

13.1 Elasticity

Consider an elastic body that occupies a domain $\Omega \subset \mathbb{R}^3$ in a given reference configuration. The deformations of the body can be described by maps $u : \Omega \to \mathbb{R}^3$. If the body is homogeneous then the total elastic energy corresponding to u is given by the functional

$$F(u) := \int_\Omega f(\nabla u(x))\, dx, \tag{10}$$

where f is the *stored-energy density* of the material. In order to prevent interpenetration of matter, the deformations should be invertible and it should require an infinite amount of energy to violate this property, i.e.,

$$f(\xi) \to +\infty \quad \text{as } \det\xi \to 0^+. \tag{11}$$

Also, f needs to be *frame indifferent*, i.e.,

$$f(R\xi) = f(\xi) \tag{12}$$

for all rotations R and all $\xi \in \mathbb{R}^{3\times3}$.

Under appropriate coercivity and convex-type conditions on f (see Subsection 7.2), and boundary conditions, it can be shown that F admits a global minimizer u_0. However, the regularity of u_0 is still an open problem and so the Euler-Lagrange equation cannot be derived (see Section 3). In addition, the existence of local minimizers remains unsolved.

13.2 Dimension Reduction

An important problem in elasticity is the derivation of models for thin structures, such as membranes, shells, plates, rods, beams, etc., from the three dimensional elasticity theory. The mathematical rigorous analysis was initiated by Acerbi, Buttazzo and Percivale in the 90's for rods, followed by the work of Le Dret and Raoult for membranes. Recent contributions by Friesecke, James and Müller allowed to handle the physical requirements (11) and (12). The main tool underlying these works is Γ-convergence (see Section 9).

To illustrate the deduction in the case of membranes, consider a thin cylindrical elastic body of thickness $2\varepsilon > 0$ occupying the reference configuration $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, with $\omega \subset \mathbb{R}^2$. Using the typical rescaling $(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3) := (x_1, x_2, x_3/\varepsilon)$, the deformations u of Ω_ε now correspond to deformation v of the fixed domain Ω_1, through the formula $v(y_1, y_2, y_3) = u(x_1, x_2, x_3)$. Therefore

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f(\nabla u)\,dx = \int_{\Omega_1} f\left(\frac{\partial v}{\partial y_1}, \frac{\partial v}{\partial y_2}, \frac{1}{\varepsilon}\frac{\partial v}{\partial y_3}\right)\,dy.$$

The right-hand side of the previous equality yields a family of functionals to which the theory of Γ-convergence is applied.

13.3 Homogenization

Homogenization theory is used to describe the macroscopic behavior of heterogeneous composite materials, which are characterized by having two or more finely mixed material components. Composite materials have important technological and industrial applications as their effective properties are often better than the corresponding properties of the individual constituents. The study of these materials falls within the so-called *multiscale problems*, with the two relevant scales here being the *microscopic scale* at the level of the heterogeneities, and the *macroscopic scale* describing the resulting "homogeneous" material. Mathematically, the properties of composite materials can be described in terms of PDEs with fast oscillating coefficients, or energy functionals depending on a small parameter ε. As an example, consider a material matrix A with corresponding stored energy density f_A, with periodically distributed inclusions of another material B with stored energy density f_B, whose periodicity cell has side-length ε. Then the total energy of the composite is given by

$$\int_\Omega \left[(1-\chi(x/\varepsilon))f_A(\nabla u) + \chi(x/\varepsilon)f_B(\nabla u)\right]\,dx,$$

where χ is the characteristic function of the locus of material B contained in the unit cube Q of material A, extended periodically to $\mathbb{R}^3$ with period Q. The goal here its to characterize the "homogenized" energy when $\varepsilon \to 0^+$ using Γ-convergence (see Section 9).

13.4 Computer Vision

Several problems in computer vision can be treated variationally, including image segmentation (e.g., the Mumford-Shah and the Blake-Zisserman models), image morphing, image denoising (e.g., the Perona-Malik scheme and the Rudin-Osher-Fatemi total variation model), inpainting (e.g., recolorization).

The Mumford-Shah model provides a good example of the use of calculus of variations to treat free discontinuity problems. Let Ω be a rectangle in the plane, representing the locus of the image with grey levels given by a function $g : \Omega \to [0,1]$. We want to find an approximation of g that is smooth outside a set K of sharp contours related to the set of discontinuities of g. This leads to the minimization of the functional

$$\int_{\Omega\setminus K} (|\nabla u|^2 + \alpha(u-g)^2)\,dx + \beta\text{length}(K\cap\Omega),$$

over all contour curves K and functions $u \in C^1(\Omega \setminus K)$, and where the first term forces u to not vary much outside K, the second term asks that u stays close to the original grey level g, and the last term ensures that K has length as short as possible. The existence of a minimizing pair (u, K) was established by De Giorgi, Carriero and Leaci, with u in a class of functions larger than $C^1(\Omega \setminus K)$, the so-called functions of special bounded variation. The full regularity of these solutions u and the structure of K remain an open problem.

13.5 Ginzburg-Landau Theory for Superconductivity

In the 1950's Ginzburg and Landau proposed a mathematical theory to study phase transition problems in superconductivity; there are similar formulations to address problems in superfluids, e.g. helium II, and in XY-magnetism. In its simplest form, the Ginzburg-Landau functional reduces to

$$F_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2-1)^2\,dx,$$

where $\Omega \subset \mathbb{R}^2$ is a star-shaped domain, the *condensate wave function* $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ is an order parameter with two degrees of freedom, and the parameter ε is a (small) characteristic length. Given $g : \partial\Omega \to \mathbb{S}^1$, with $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$ centered at the origin, we are interested in characterizing the limits of minimizers u_ε of F_ε subject to the boundary condition $u_\varepsilon = g$ on $\partial\Omega$. Under suitable geometric conditions on g (related to the winding number), Bethuel, Brezis and Hélein have shown that there are no limiting functions u in $C(\overline{\Omega}, \mathbb{S}^1)$ satisfying the boundary condition. Rather, the limiting functions are smooth outside a finite set of singularities, called *vortices*. Γ-convergence techniques may be used to study this family of functionals (see Section 9).

13.6 Mass Transport

Mass transportation was introduced by Monge in 1781, studied by the Nobel prize winner Kantorovich in the 1940's and, and revived by Brenier in 1987. Since then, it has surfaced in a variety of areas, from economics to optimization. Given a pile of sand of mass one and a hole of volume one, we want to fill the hole with the sand while minimizing the cost of transportation. This problem is formulated using probability theory as follows. The pile and the hole are represented by probability measures μ and ν, with supports in measurable spaces X and Y, respectively. If $A \subset X$ and $B \subset Y$ are measurable sets, then $\mu(A)$ measures the amount of sand in A, and $\nu(B)$ measures the amount of sand that can fill B. The cost of transportation is modeled by a measurable cost function $c : X \times Y \to \mathbb{R} \cup +\infty$. Kantorovich's optimal transportation problem consists of minimizing

$$\int_{X \times Y} c(x, y)\, d\pi(x, y)$$

over all probability measures on $X \times Y$ such that $\pi(A \times Y) = \mu(A)$ and $\pi(X \times B) = \nu(B)$, for all measurable sets $A \subset X$ and $B \subset Y$. The main problem is to establish existence of minimizers and to obtain their characterization. This depends strongly on the cost function c and on the regularity of the measures μ and ν. There is a multitude of applications of this theory, and here we mention that it can be used to give a simple proof of the Brunn-Minkowski inequality (see (4)).

13.7 Gradient Flows

Given a function $h : \mathbb{R}^m \to \mathbb{R}$ of class C^2, the *gradient flow of* h is the family of maps $S_t : \mathbb{R}^m \to \mathbb{R}^m$, $t \geq 0$, satisfying the following property: for every $w_0 \in \mathbb{R}^m$, $S_0(w_0) := w_0$ and the curve $w_t := S_t(w_0)$, $t > 0$, is the unique C^1 solution of the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} w_t = -\nabla h(w_t) \quad \text{for } t > 0, \qquad \lim_{t \to 0^+} w_t = w_0, \tag{13}$$

if it exists.

If $D^2 h \geq \alpha \mathbb{I}$ for some $\alpha \in \mathbb{R}$, then it can be shown that the gradient flow exists, it is unique, and it satisfies a semigroup property, i.e.,

$$S_{t+s}(w_0) = S_t(S_s(w_0)), \qquad \lim_{t \to 0^+} S_t(w_0) = w_0$$

for every $w_0 \in \mathbb{R}^m$.

A common way used to approximate discretely the solution of (13) is the *implicit Euler scheme*: given a time step $\tau > 0$, consider the partition of $[0, +\infty)$

$$\{0 = t_\tau^0 < t_\tau^1 < \cdots < t_\tau^n < \cdots\},$$

where $t_\tau^n := n\tau$. Define recursively a discrete sequence $\{W_\tau^n\}$ as follows: assuming that W_τ^{n-1} has already been defined, let W_τ^n be the unique minimizer of the function

$$w \mapsto \frac{1}{2\tau} |w - W_\tau^{n-1}|^2 + h(w). \tag{14}$$

Introduce the piecewise linear function $W_\tau : [0, +\infty) \to \mathbb{R}^m$ given by

$$W_\tau(t) = \frac{t - t_\tau^{n-1}}{\tau} W_\tau^{n-1} + \frac{t_\tau^n - t}{\tau} W_\tau^n$$

for $t \in [t_\tau^{n-1}, t_\tau^n]$. If $W_\tau^0 \to w_0$ as $\tau \to 0^+$ then it can be shown that $\{W_\tau\}_{\tau > 0}$ converges to the solution of (13) as $\tau \to 0^+$.

This approximation scheme, here described for the finite dimensional vector space $\mathbb{R}^d$, may be extended to the case in which $\mathbb{R}^d$ is replaced by an infinite dimensional metric space X, the function h is replaced by a functional $G : X \to \mathbb{R}$, and the minimization procedure in (14) is now a variational minimization problem of the type addressed in Section 7, known as *De Giorgi's minimizing movements*. Important applications include the study of a large class of parabolic PDEs.

Further Reading

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