ANALYTICAL VALIDATION OF A CONTINUUM MODEL FOR EPITAXIAL GROWTH WITH ELASTICITY ON VICINAL SURFACES

G. DAL MASO, I. FONSECA, AND G. LEONI

Abstract. Within the context of heteroepitaxial growth of a film onto a substrate, terraces and steps self-organize according to misfit elasticity forces. Discrete models of this behavior were developed by Duport, Politi, and Villain [3] and Tersoff, Phang, Zhang, and Lagally [6]. A continuum limit of these was in turn derived by Xiang [7] (see also the work of Xiang and E [8] and Xu and Xiang [9]). In this paper we formulate a notion of weak solution to Xiang's continuum model in terms of a variational inequality that is satisfied by strong solutions. Then we prove the existence of a weak solution.

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Contents

1. Introduction

Within the context of heteroepitaxial growth of a film onto a substrate, terraces and steps self-organize according to misfit elasticity forces. Discrete models of this behavior were developed by Duport, Politi, and Villain [3] and Tersoff, Phang, Zhang, and Lagally [6]. A continuum limit of these was in turn derived by Xiang [7] (see also the work of Xiang and E [8] and Xu and Xiang [9]). In this paper we formulate a notion of weak solution to Xiang's continuum model in terms of a variational inequality that is satisfied by strong solutions. Then we prove the existence of a weak solution.

The evolution equation derived by Xiang in [7, formula (3.62)] is

$$\dot{h} = \left[-H(h_x) - \left(\frac{1}{h_x} + h_x \right) h_{xx} \right]_{xx} , \tag{1.1}$$

where $\dot{h}$ denotes the derivative of h with respect to t. Here, the function h describes the height of the surface of the film, and it is assumed to be monotone. Without loss of generality, we take h to be increasing. Note that in [7] and [8] h is taken to be decreasing with respect to x, therefore in those papers $h(t,x)$ corresponds to our $h(t,-x)$. Moreover, H denotes the periodic Hilbert transform (see (2.25) below).

To exploit the variational structure of equation (1.1), we consider the function $\Phi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ introduced in [7, formula (3.64)] and defined by

$$\Phi(\xi) := \begin{cases} \xi \log \xi + \frac{1}{6}\xi^3 & \text{if } \xi > 0\,, \\ 0 & \text{if } \xi = 0\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that Φ is convex, and that (1.1) can be written as

$$\dot{h} = [-H(h_x) - (\Phi'(h_x))_x]_{xx}\,, \tag{1.3}$$

where the derivative Φ' is given by

$$\Phi'(\xi) = \log \xi + 1 + \tfrac{1}{2}\xi^2 \quad \text{if } \xi > 0\,. \tag{1.4}$$

In Theorem 3.1 we show that the existence of solutions of (1.3) with h_x bounded away from zero is equivalent to the existence of solutions of the parabolic evolution equation

$$\dot{u} = -[H(u_{xx})]_x - [\Phi_a'(u_{xx})]_{xx}\,, \tag{1.5}$$

where u is an appropriate anti-derivative of h, a is a positive constant, and

$$\Phi_a(\xi) := \Phi(\xi + a)\,. \tag{1.6}$$

We study equation (1.5) on a time interval $[0,T]$, for some $T > 0$, and on the space interval $I := (-\pi, \pi)$, with inital boundary condition at $t = 0$ and periodic boundary conditions on ∂I. In this work we use the spaces $L^2_{\mathrm{per}_0}(I)$ and $W^{2,3}_{\mathrm{per}_0}(I)$ of 2π-periodic functions of $L^2_{\mathrm{loc}}(\mathbb{R})$ and $W^{2,3}_{\mathrm{loc}}(\mathbb{R})$, respectively, with average 0 on I. The main difficulty in the analysis of (1.5) is due to the singularity of $\log \xi$ in (1.4) at the origin. To circumvent this problem, we will consider a family of approximating problems (see (5.1) below), and we will prove that their solutions converge to a solution of the variational inequality (1.7). The central result of his paper is the following theorem.

Theorem 1.1. *Let $a > 0$ and let $u^0 \in L^2_{\mathrm{per}_0}(I)$. Then there exists $u \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$ such that*

$$\int_0^T \Big(\langle \dot{w}(t), w(t) - u(t) \rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)} + \mathcal{F}_a(w(t)) \Big) dt \\ \geq \int_0^T \Big(\mathcal{F}_a(u(t)) + \langle \mathcal{H}(u\,(t)), w(t) - u(t) \rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)} \Big) dt \tag{1.7}$$

for every $w \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$, with $\dot{w} \in L^{3/2}([0,T]; (W^{2,3}_{\mathrm{per}_0}(I))')$ and $w(0) = u^0$. Moreover, $\log(u_{xx} + a) \in L^1([0,T]; L^1(I))$.

Here, the functional $\mathcal{F}_a\colon W^{2,3}_{\mathrm{per}_0}(I) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{F}_a(u) := \int_I \Phi_a(u_{xx})\, dx\,, \tag{1.8}$$

and $\mathcal{H}\colon W^{2,3}_{\mathrm{per}_0}(I) \to (W^{2,3}_{\mathrm{per}_0}(I))'$ is the operator given by

$$\langle \mathcal{H}(u), v \rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)} := \int_I H(u_{xx})\, v_x\, dx\,. \tag{1.9}$$

Note that in Proposition 3.4 we prove that strong solutions of (1.5) with $u_{xx} + a$ bounded away from zero satisfy (1.7), so that this variational inequality can be considered as a weak formulation of (1.5).

Let $\mathcal{A}_a\colon D_a \to (W^{2,3}_{\mathrm{per}_0}(I))'$ be the operator defined by

$$\langle \mathcal{A}_a(u), v \rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)} := \int_I \Phi_a'(u_{xx})\, v_{xx}\, dx\,, \tag{1.10}$$

where

$$D_a := \{u \in W^{2,3}_{\mathrm{per}_0}(I) : \log(u_{xx} + a) \in L^{3/2}(I)\} .$$

We prove also the following result, where the variational inequality is more similar to equation (1.5).

Theorem 1.2. *Under the assumptions of Theorem 1.1, the solution u of problem (1.7) satisfies*

$$\int_0^T \Big(\langle \dot w(t), w(t) - u(t)\rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)} + \langle \mathcal{A}_a(w(t)), w(t) - u(t)\rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)} \Big) dt$$

$$\leq \int_0^T \langle \mathcal{H}(u(t)), w(t) - u(t)\rangle_{(W^{2,3}_{\mathrm{per}_0}(I))', W^{2,3}_{\mathrm{per}_0}(I)}\, dt \qquad (1.11)$$

for every $w \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$, with $\dot w \in L^{3/2}([0,T]; (W^{2,3}_{\mathrm{per}_0}(I))')$, $w(0) = u^0$, and $\log(w_{xx} + a) \in L^{3/2}([0,T]; L^{3/2}(I))$.

2. Preliminaries

We begin with a compactness result on Banach spaces, which extends [5, Chapter 1,Theorem 5.1] to the case of $L^1([0,T];B_1)$, without assuming that B_1 is reflexive.

Theorem 2.1. *Let B_0, B, and B_1 be Banach spaces, and let $1 < p < +\infty$. Assume that*

$$B_0 \hookrightarrow B \hookrightarrow B_1 \text{ with continuous embeddings,} \qquad (2.1)$$

$$B_0 \text{ is reflexive,} \qquad (2.2)$$

$$\text{the embedding } B_0 \hookrightarrow B \text{ is compact.} \qquad (2.3)$$

Let $\mathcal{V}$ be the Banach space of all functions $v \in L^p([0,T];B_0)$ whose distributional derivative $\dot v$ belongs to $L^1([0,T];B_1)$ endowed with the norm

$$\|v\|_{\mathcal{V}} := \|v\|_{L^p([0,T];B_0)} + \|\dot v\|_{L^1([0,T];B_1)} . \qquad (2.4)$$

Then the embedding $\mathcal{V} \hookrightarrow L^p([0,T];B)$ is compact.

Proof. Let $\{v_n\}$ be a bounded sequence in $\mathcal{V}$. Using (2.4) we obtain that

$$\{v_n\} \text{ is bounded in } L^p([0,T];B_0) , \qquad (2.5)$$

$$\{\dot v_n\} \text{ is bounded in } L^1([0,T];B_1) . \qquad (2.6)$$

Since $1 < p < +\infty$, by (2.2) the space $L^p([0,T];B_0)$ is reflexive (see, e.g., [4, Theorem 2.112]). By (2.5), extracting a subsequence (not relabeled), we have that

$$v_n \rightharpoonup v \text{ weakly in } L^p([0,T];B_0) \qquad (2.7)$$

for some $v \in L^p([0,T];B_0)$.

By [5, Chapter 1, Lemma 5.1], for every $\eta > 0$ there exists $c_\eta > 0$ such that

$$\|u\|_B \leq \eta\, \|u\|_{B_0} + c_\eta \|u\|_{B_1}$$

for every $u \in B_0$. It follows that

$$\|v_n - v\|_{L^p([0,T];B)} \leq \eta\, \|v_n - v\|_{L^p([0,T];B_0)} + c_\eta \|v_n - v\|_{L^p([0,T];B_1)}$$

for every n. By (2.5), for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\|v_n - v\|_{L^p([0,T];B)} \leq \varepsilon + c_\eta \|v_n - v\|_{L^p([0,T];B_1)}$$

for every n. Therefore, to prove that $v_n \to v$ strongly in $L^p([0,T];B)$ it is enough to show that

$$v_n \to v \text{ strongly in } L^p([0,T];B_1) . \qquad (2.8)$$

Since $\mathcal{V} \hookrightarrow W^{1,1}((0,T);B_1) \hookrightarrow C^0([0,T];B_1)$ with continuous embedding, the sequence $\{v_n\}$ is bounded in $C^0([0,T];B_1)$. By the Dominated Convergence Theorem, to obtain (2.8) it suffices to prove that

$$v_n(t) \to v(t) \text{ strongly in } B_1 \text{ for a.e. } t \in [0,T]\,. \tag{2.9}$$

For $t \in [0,T]$ and $n \in \mathbb{N}$ define

$$V_n(t) := \int_0^t \|\dot{v}_n(s)\|_{B_1}\, ds\,.$$

By (2.6), $\{V_n\}$ is a bounded sequence of monotone functions. By the Helly Theorem there exists a subsequence (not relabeled) that converges pointwise to a monotone function $V\colon [0,T] \to [0,+\infty)$.

Let t_0 be a continuity point of V and a Lebesgue point of v, considered as an integrable function with values in B_1, i.e.,

$$\lim_{s\to 0} \frac{1}{s}\int_{t_0}^{t_0+s} \|v(t) - v(t_0)\|_{B_1}\, dt = 0\,.$$

We want to prove that

$$v_n(t_0) \to v(t_0) \text{ strongly in } B_1\,. \tag{2.10}$$

Fix $\varepsilon > 0$ and $s > 0$ such that

$$V(t_0+s) - V(t_0) < \varepsilon \quad \text{and} \quad \frac{1}{s}\int_{t_0}^{t_0+s} \|v(t) - v(t_0)\|_{B_1}\, dt < \varepsilon\,. \tag{2.11}$$

Using an argument due to R. Temam (see [5, Chapter 1,Theorem 5.1]), we write

$$v_n(t_0) = \frac{1}{s}\int_{t_0}^{t_0+s} v_n(t)\, dt - \frac{1}{s}\int_{t_0}^{t_0+s} (t_0+s-t)\, \dot{v}_n(t)\, dt =: a_n - b_n\,, \tag{2.12}$$

and we define

$$a := \frac{1}{s}\int_{t_0}^{t_0+s} v(t)\, dt\,.$$

By (2.7), the sequence $\{a_n\}$ converges to a weakly in B_0. By (2.1), (2.2), and (2.3), it converges strongly in B_1. Since $\|a - v(t_0)\|_{B_1} < \varepsilon$ by (2.11), we obtain

$$\lim_{n\to\infty} \|a_n - v(t_0)\|_{B_1} < \varepsilon \quad \text{for } n \text{ large enough.} \tag{2.13}$$

On the other hand,

$$\|b_n\|_{B_1} \le \int_{t_0}^{t_0+s} \|\dot{v}_n(t)\|_{B_1}\, dt = V_n(t_0+s) - V_n(t_0)\,,$$

so that, by (2.11),

$$\limsup_{n\to\infty} \|b_n\|_{B_1} \le V(t_0+s) - V(t_0) < \varepsilon\,. \tag{2.14}$$

In view of the arbitrariness of $\varepsilon > 0$, (2.10) follows from (2.12), (2.13), and (2.14). Since the continuity points for V that are Lebesgue points for v form a set of full measure, we have proved (2.9), which concludes the proof of the theorem. □

In what follows, given a Banach space B, we denote by $\langle\cdot,\cdot\rangle_{B',B}$ the duality between B and its dual B'.

Let $I := (-\pi,\pi)$. To introduce the functional setting for the study of equation (1.3), for $k \in \mathbb{Z}$ and $1 \le p < +\infty$, consider the space

$$W^{k,p}_{\mathrm{per}}(I) := \{u \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}) : u \text{ is } 2\pi\text{-periodic}\}\,, \tag{2.15}$$

where the periodicity of u when $k < 0$ is to be understood in the sense of distributions. If $k \ge 0$ then the space $W^{k,p}_{\mathrm{per}}(I)$ is endowed with the norm induced by $W^{k,p}(I)$. If $k < 0$ and

$(1/p)+(1/q)=1$, then $W^{k,q}_{\mathrm{per}}(I)$ is the dual of $W^{-k,p}_{\mathrm{per}}(I)$, and is endowed with the dual norm. For simplicity, we use the notation

$$\langle\cdot,\cdot\rangle_{k,p} := \langle\cdot,\cdot\rangle_{(W^{k,p}_{\mathrm{per}}(I))',W^{k,p}_{\mathrm{per}}(I)}\,. \tag{2.16}$$

Moreover, we also define the space

$$W^{k,p}_{\mathrm{per}_0}(I) := \{u\in W^{k,p}_{\mathrm{per}}(I) : u_I = 0\}\,, \tag{2.17}$$

where u_I is the average of u in one period if $k\geq 0$, and $u_I := \langle u,1\rangle_{k,p}$ when $k<0$. If $k\geq 0$ then the space $W^{k,p}_{\mathrm{per}_0}(I)$ is endowed with the norm induced by $W^{k,p}(I)$. If $k<0$ and $(1/p)+(1/q)=1$, then it can be shown that $W^{k,q}_{\mathrm{per}_0}(I)$ is the dual of $W^{-k,p}_{\mathrm{per}_0}(I)$, and it will be endowed with the dual norm. With an abuse of notation, we continue to use the symbol $\langle\cdot,\cdot\rangle_{k,p}$ to denote the corresponding duality. Finally, set

$$\begin{aligned} W^{1,3}_{\mathrm{per}^*}(I) &:= \{u\in W^{1,3}_{\mathrm{loc}}(\mathbb{R}) : u_x \text{ is } 2\pi\text{-periodic}\}\\ &= \{u\in W^{1,3}_{\mathrm{loc}}(\mathbb{R}) : u(x+2\pi)-u(x) \text{ is constant}\}\,, \end{aligned} \tag{2.18}$$

endowed with the norm induced by $W^{1,3}(I)$.

Let

$$V := W^{2,3}_{\mathrm{per}_0}(I) \quad\text{and}\quad Y := L^2_{\mathrm{per}_0}(I)\,, \tag{2.19}$$

where

$$L^2_{\mathrm{per}_0}(I) := \{u\in L^2_{\mathrm{loc}}(\mathbb{R}) : u \text{ is } 2\pi\text{-periodic and } u_I = 0\}\,.$$

Since every $u\in V$ has mean value zero and is periodic, it follows in particular that u_x has also mean value 0, and so using Poincaré-Wirtinger inequality twice, there exists a constant $\alpha>0$ such that

$$\|u\|_{W^{2,3}(I)} \leq \alpha\|u_{xx}\|_{L^3(I)} \tag{2.20}$$

for every $u\in V$. This allows us to endow V with the norm

$$\|u\|_V := \|u_{xx}\|_{L^3(I)}\,, \tag{2.21}$$

which is equivalent to the norm induced by $W^{2,3}(I)$. The dual space V' coincides with $W^{-2,3/2}_{\mathrm{per}_0}(I)$ and will be endowed with the dual norm to (2.21).

In order to study the evolution equation, we introduce the Banach space

$$\mathcal{W} := \{u : u\in L^3([0,T];V),\ \dot u\in L^{3/2}([0,T];V')\}\,, \tag{2.22}$$

endowed with the norm

$$\|u\|_{\mathcal{W}} := \|u\|_{L^3([0,T];V)} + \|\dot u\|_{L^{3/2}([0,T];V')}\,. \tag{2.23}$$

It is well-known that $\mathcal{W}\subset C^0([0,T];Y)$, with continuous embedding (see, e.g., [10, Proposition 23.23]).

Lemma 2.2. *The embedding $i\colon \mathcal{W}\hookrightarrow L^3([0,T];W^{1,3}_{\mathrm{per}_0}(I))$ is compact.*

Proof. It is enough to apply [5, Chapter 1, Theorem 5.1] with $p_0:=3$, $p_1:=3/2$, $B_0:=V$, $B:=W^{1,3}_{\mathrm{per}_0}(I)$, and $B_1:=V'$. □

Consider the adjoint embedding $i^*\colon L^{3/2}([0,T];W^{-1,3/2}_{\mathrm{per}_0}(I))\to\mathcal{W}'$. By its definition, for every $f\in L^{3/2}([0,T];W^{-1,3/2}_{\mathrm{per}_0}(I))$ and every $v\in\mathcal{W}$ we have

$$\langle i^*(f),v\rangle_{\mathcal{W}',\mathcal{W}} = \int_0^T \langle f(t),i(v)(t)\rangle_{1,3}\,dt = \int_0^T \langle f(t),v(t)\rangle_{V',V}\,dt\,. \tag{2.24}$$

In view of Lemma 2.2 we have the following result.

Lemma 2.3. *The embedding $i^*\colon L^{3/2}([0,T];W^{-1,3/2}_{\mathrm{per}_0}(I))\to\mathcal{W}'$ is compact.*

The periodic Hilbert transform $H(u)$ of a function $u \in L^p_{\rm per}(I)$ is the 2π-periodic function defined by

$$H(u)(x) := \frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} u(x-y)\cot\Big(\frac{y}{2}\Big)\,dy\,, \tag{2.25}$$

where PV denotes the Cauchy principal value.

For a proof of the following proposition, we refer to [2, Proposition 9.1.3].

Proposition 2.4. *Let $1<p<+\infty$ and let $u \in L^p_{\rm per}(I)$. Then $H(u) \in L^p_{\rm per}(I)$ and*

$$\|H(u)\|_{L^p(I)} \le C_p \|u\|_{L^p(I)}$$

for some constant $C_p>0$.

Using the argument in the proof of [2, Proposition 8.3.7], where Proposition 9.3.1 is used in place of Proposition 8.3.1, we obtain the following result.

Proposition 2.5. *Let $1<p<+\infty$ and let $u \in W^{1,p}_{\rm per}(I)$. Then $H(u) \in W^{1,p}_{\rm per}(I)$ and $H(u_x) = (H(u))_x$ a.e. on I.*

Remark 2.6. In view of the latter proposition, it is possible to extend H as a linear bounded operator from $W^{-1,p'}_{\rm per}(I)$ into itself.

3. A Notion of Weak Solution

In this section we derive a notion of weak solution for equation (1.3). In order to transform (1.3) into a parabolic equation, we will derive an evolution equation for an appropriate anti-derivative of h. In what follows we use the notation (2.16).

Theorem 3.1. *The following conditions are equivalent:*

(i) *There exists $h \in L^3([0,T]; W^{1,3}_{\rm per^*}(I))$ with $\dot h \in L^{3/2}([0,T]; W^{-3,3/2}_{\rm per}(I))$ a solution of equation (1.3) satisfying*

$$h_x(t,x) \ge \delta \text{ for a.e. } x\in\mathbb{R} \text{ and } t \in [0,T], \tag{3.1}$$

for some constant $\delta>0$.

(ii) *There exist $a>0$ and $u \in L^3([0,T]; W^{2,3}_{\rm per_0}(I))$ with $\dot u \in L^{3/2}([0,T]; W^{-2,3/2}_{\rm per_0}(I))$ a solution of*

$$\dot u = -[H(u_{xx})]_x - [\Phi'_a(u_{xx})]_{xx} \tag{3.2}$$

satisfying

$$u_{xx}(t,x) \ge -a+\delta \text{ for a.e. } x\in\mathbb{R} \text{ and } t\in[0,T]. \tag{3.3}$$

Observe that, in view of (1.4), the expression $[-H(h_x) - \Phi'(h_x)_x]_{xx}$ belongs to the space $L^{3/2}([0,T]; W^{-3,3/2}_{\rm per}(I))$, so that the equality in (1.3) is well defined, i.e., for a.e. $t\in[0,T]$ we have

$$\langle \dot h, \varphi\rangle_{3,3} = \langle -H(h_x) - (\Phi'(h_x))_x, \varphi_{xx}\rangle_{1,3} \tag{3.4}$$

for every $\varphi \in W^{3,3}_{\rm per}(I)$. Similarly, by (3.3) equation (3.2) is well-defined.

The proof of Theorem 3.1 will use the two lemmas below.

Lemma 3.2. *The following conditions are equivalent:*

(a) *$h \in L^3([0,T]; W^{1,3}_{\rm per^*}(I))$ and $\dot h \in L^{3/2}([0,T]; W^{-3,3/2}_{\rm per}(I))$;*

(b) *there exist $a\in\mathbb{R}$ and $h^a \in L^3([0,T]; W^{1,3}_{\rm per}(I))$, with $\dot h^a \in L^{3/2}([0,T]; W^{-3,3/2}_{\rm per}(I))$, such that $h(t,x) = h^a(t,x) + ax$ for a.e. $t\in[0,T]$ and every $x\in\mathbb{R}$.*

Proof. The implication (b) $\Rightarrow$ (a) is straightforward, so we only prove that (a) implies (b). Assume that h satisfies (a) and, for a.e. $t \in [0,T]$, let

$$a(t) := \frac{1}{2\pi} \int_I h_x(t,x)\,dx\,.$$

If $h \in C_c^\infty((0,T); C^\infty_{\mathrm{per}^*}(I))$ with $\dot{h} \in C_c^\infty((0,T); C^\infty_{\mathrm{per}}(I))$, then

$$\dot{a}(t) = \frac{1}{2\pi} \int_I (\dot{h})_x(t,x)\,dx = 0$$

by the periodicity of $\dot{h}\,(t,\cdot)$ for every t. It now follows by density that a is constant. Hence, $h^a(t,x) := h(t,x) - ax$ satisfies (b). □

Recall that the function Φ_a is defined in (1.6).

Lemma 3.3. *Let $a > 0$ and let $h^a \in L^3([0,T]; W^{1,3}_{\mathrm{per}}(I))$, with $\dot{h}^a \in L^{3/2}([0,T]; W^{-3,3/2}_{\mathrm{per}}(I))$, be a solution of*

$$\dot{h}^a = [-H(h^a_x) - (\Phi'_a(h^a_x))_x]_{xx} \tag{3.5}$$

satisfying

$$h^a_x(t,x) \geq -a + \delta \text{ for a.e. } x \in \mathbb{R} \text{ and } t \in [0,T] \tag{3.6}$$

for some $\delta > 0$. Then there exists a constant b, depending on the solution h^a, such that

$$\frac{1}{2\pi} \int_I h^a(t,x)\,dx = b \quad \text{for a.e. } t \in [0,T]\,. \tag{3.7}$$

Proof. Let

$$b(t) := \frac{1}{2\pi} \int_I h^a(t,x)\,dx\,, \tag{3.8}$$

for a.e. $t \in [0,T]$. We want to show that b is constant. By (3.8), we have $b(t) = \langle h^a(t), 1/(2\pi)\rangle_{1,3}$, so that its distributional derivative satisfies $\dot{b}(t) = \langle \dot{h}^a(t), 1/(2\pi)\rangle_{3,3}$. Therefore (3.5) yields

$$\begin{aligned}\dot{b}(t) &= \langle [-H(h^a_x) - (\Phi'_a(h^a_x))_x]_{xx}, 1/(2\pi)\rangle_{3,3} \\ &= \langle [-H(h^a_x) - (\Phi'_a(h^a_x))_x]_x, 0\rangle_{2,3} = 0\,.\end{aligned}$$

□

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. Assume (i). Let h^a be the function given by Lemma 3.2. Since $H(a) = 0$ it follows that h^a is a solution of (3.5). Moreover, using the fact that $h^a(t,\cdot)$ is 2π-periodic, we have that the mean value of $h^a_x(t,\cdot)$ on a period is 0, so that (3.6) implies $a > 0$. Define

$$u(t,x) := \int_{-\pi}^{x} (h^a(t,y) - b)\,dy\,, \tag{3.9}$$

where b is the number given in Lemma 3.3. Then $x \mapsto u(t,x)$ is 2π-periodic and has mean value zero for a.e. $t \in [0,T]$. Moreover, $u_x = h^a - b$, $u_{xx} = h^a_x$, and $\dot{u}_x = \dot{h}^a$. It follows that $u \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$ and $\dot{u} \in L^{3/2}([0,T]; W^{-2,3/2}_{\mathrm{per}_0}(I))$. In turn, equation (3.5) can be written as

$$\dot{u}_x = [-H(u_{xx}) - (\Phi'_a(u_{xx}))_x]_{xx}\,.$$

Hence, for a.e. $t \in [0,T]$ there exists a constant $c(t) \in \mathbb{R}$ such that

$$\dot{u} = -[H(u_{xx})]_x - [\Phi'_a(u_{xx})]_{xx} + c(t)\,,$$

that is, for a.e. $t \in [0,T]$ we have

$$\begin{aligned}\tfrac{d}{dt}\langle u(t), \varphi\rangle_{0,3} &= \langle \dot{u}(t), \varphi\rangle_{2,3} \\ &= \langle H(u_{xx}), \varphi_x\rangle_{0,3} - \langle \Phi'_a(u_{xx}), \varphi_{xx}\rangle_{0,3} + \langle c(t), \varphi\rangle_{0,3}\end{aligned}$$

for every test function $\varphi \in W^{2,3}_{\mathrm{per}}(I)$. Taking $\varphi = 1$ we get $c(t) = 0$ for a.e. $t \in [0,T]$.

Therefore u satisfies the equation (3.2) in the sense that, for a.e. $t \in [0,T]$, we have

$$\langle \dot u(t), \varphi\rangle_{2,3} = \langle H(u_{xx}), \varphi_x\rangle_{0,3} - \langle \Phi'_a(u_{xx}), \varphi_{xx}\rangle_{0,3}$$

for every test function $\varphi \in W^{2,3}_{\mathrm{per}}(I)$.

Conversely, assume (ii). Then the function $h\,(t,x) := u_x(t,x) + ax$ satisfies (i). □

In this paper we establish the existence and uniqueness of a solution of a variational inequality satisfied by all solutions considered in Theorem 3.1(ii). More precisely, we have the following result.

Proposition 3.4. *Let $u^0 \in L^2_{\mathrm{per}_0}(I)$ and let u satisfy (ii) in Theorem 3.1 with $u(0) = u^0$. Then*

$$\log(u_{xx} + a) \in L^{3/2}([0,T]; L^{3/2}(I)) \tag{3.10}$$

and

$$\int_0^T \Big(\langle \dot w(t), w(t) - u(t)\rangle_{V',V} + \mathcal{F}_a(w(t))\Big)dt \\ \geq \int_0^T \Big(\mathcal{F}_a(u(t)) + \langle \mathcal{H}(u\,(t)), w(t) - u(t)\rangle_{V',V}\Big)dt \tag{3.11}$$

for every $w \in \mathcal{W}$ with $w(0) = u^0$, where $\mathcal{F}_a$ is defined in (1.8).

Proof. Since $u_{xx} \in L^3([0,T]; L^3(I))$ and (3.3) holds, we obtain (3.10). By (1.4), this implies that $\Phi'_a(u_{xx}) \in L^{3/2}([0,T]; L^{3/2}(I))$. Let w be as in the statement. For a.e. $t \in [0,T]$ we multiply (3.2) by $w(t) - u(t)$ and add $\langle \dot w(t), w(t) - u(t)\rangle_{V',V}$ to both sides to obtain

$$\langle \dot w(t), w(t) - u(t)\rangle_{V',V} + \langle \Phi'_a(u_{xx}(t)), w_{xx}(t) - u_{xx}(t)\rangle_{0,3} \\ = \langle \dot w(t) - \dot u(t), w(t) - u(t)\rangle_{V',V} + \langle \mathcal{H}(u\,(t)), w(t) - u(t)\rangle_{V',V}\,.$$

Since Φ_a is convex, we have $\mathcal{F}_a(w(t)) - \mathcal{F}_a(u(t)) \geq \langle \Phi'_a(u_{xx}(t)), w_{xx}(t) - u_{xx}(t)\rangle_{0,3}$. Therefore

$$\int_0^T \Big(\langle \dot w(t), w(t) - u(t)\rangle_{V',V} + \mathcal{F}_a(w(t)) - \mathcal{F}_a(u(t))\Big)dt \\ \geq \int_0^T \tfrac{1}{2}\tfrac{d}{dt}\|w(t) - u(t)\|^2_{L^2(I)}\,dt + \int_0^T \langle \mathcal{H}(u\,(t)), w(t) - u(t)\rangle_{V',V}\,dt \\ = \tfrac{1}{2}\|w(T) - u(T)\|^2_{L^2(I)} + \int_0^T \langle \mathcal{H}(u\,(t)), w(t) - u(t)\rangle_{V',V}\,dt \\ \geq \int_0^T \langle \mathcal{H}(u\,(t)), w(t) - u(t)\rangle_{V',V}\,dt\,,$$

where we used [10, Proposition 23.23] and the fact that $w(0) = u(0) = u^0$. □

4. Bounded Monotone Problems

In order to overcome the difficulties due to the fact that Φ_a takes infinite values, we consider a suitable finite valued approximation, denoted by $\Phi_{a,\delta}$. Let

$$\Psi(\xi) := \begin{cases} \xi \log \xi & \text{if } \xi > 0\,, \\ 0 & \text{if } \xi = 0\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

For $a \geq 0$ and $0 < \delta < 1/e$ we define

$$\Psi_a(\xi) := \Psi(\xi + a)\,, \quad \Psi_{a,\delta}(\xi) := \begin{cases} (\xi + a)\log(\xi + a) & \text{if } \xi \geq -a + \delta\,, \\ (\xi + a)\log\delta + \xi + a - \delta & \text{if } \xi \leq -a + \delta\,, \end{cases} \tag{4.2}$$

and

$$\Phi_{a,\delta}(\xi) := \Psi_{a,\delta}(\xi) + \tfrac{1}{6}|\xi + a|^3 . \tag{4.3}$$

Note that $\Phi_{a,\delta} \colon \mathbb{R} \to \mathbb{R}$ is a convex function. In the following lemma we give some estimates on $\Phi'_{a,\delta}$, which is given by

$$\Phi'_{a,\delta}(\xi) = \begin{cases} \log(\xi + a) + 1 + \frac{1}{2}|\xi + a|(\xi + a) & \text{if } \xi \geq -a + \delta , \\ \log \delta + 1 + \frac{1}{2}|\xi + a|(\xi + a) & \text{if } \xi \leq -a + \delta . \end{cases} \tag{4.4}$$

Lemma 4.1. *Assume $a > 0$ and $0 < \delta < 1/e$. Then*

$$|\Phi'_{a,\delta}(\xi)| \leq |\log \delta| + 2a^2 + 2|\xi|^2 , \tag{4.5}$$

$$(\Phi'_{a,\delta}(\xi_2) - \Phi'_{a,\delta}(\xi_1))(\xi_2 - \xi_1) \geq \tfrac{1}{4}|\xi_2 - \xi_1|^3 , \tag{4.6}$$

$$\Phi'_{a,\delta}(\xi)\xi \geq \tfrac{1}{4}|\xi|^3 - 2 - 4a^3 , \tag{4.7}$$

for every ξ, ξ_1, $\xi_2 \in \mathbb{R}$. Moreover

$$\Phi'_{a,\delta}(\xi)\xi \geq a|\log(\xi + a)| + \tfrac{1}{2}|\xi|^3 - 2 - 2a^3 \quad \textit{if } \delta \leq \xi + a \leq 1 , \tag{4.8}$$

$$\Phi'_{a,\delta}(\xi)\xi \geq a|\log \delta| + \tfrac{1}{4}|\xi|^3 - 2 - 4a^3 \quad \textit{if } \ \xi + a \leq \delta , \tag{4.9}$$

$$\Phi'_{a,\delta}(\xi)\xi \geq c_1|\Phi'_{a,\delta}(\xi)| - c_2 \quad \textit{for every } \ \xi \in \mathbb{R} , \tag{4.10}$$

where the constants $c_1 > 0$ and $c_2 > 0$ depend on a, but not on δ.

Proof. **Step 1:** We first prove (4.5). Using the inequality $0 \leq \log s \leq s^2/2$ for $s \geq 1$, from (4.4) we obtain

$$|\Phi'_{a,\delta}(\xi)| \leq 1 + |\xi + a|^2 \quad \text{for } \xi + a \geq 1 . \tag{4.11}$$

Since $|\log \delta| \geq 1$, this implies

$$|\Phi'_{a,\delta}(\xi)| \leq |\log \delta| + |\xi + a|^2 . \tag{4.12}$$

On the other hand, if $\delta \leq \xi + a \leq 1$ we have $-|\log \delta| \leq \log(\xi + a) + 1 \leq 1 \leq |\log \delta|$, so that (4.12) holds also in this case. Finally, (4.12) follows immediately from (4.4) if $\xi + a \leq \delta$. Hence, (4.5) is a consequence of (4.12) and Cauchy's Inequality.
Step 2: To show (4.6), note that, since $\Phi_{a,\delta}(\xi) - \frac{1}{6}|\xi + a|^3$ is convex, we have

$$[\Phi'_{a,\delta}(\xi_2) - \Phi'_{a,\delta}(\xi_1)](\xi_2 - \xi_1) \geq \tfrac{1}{2}[|\xi_2 + a|(\xi_2 + a) - |\xi_1 + a|(\xi_1 + a)](\xi_2 - \xi_1) \geq \tfrac{1}{4}|\xi_2 - \xi_1|^3$$

for every ξ_1, $\xi_2 \in \mathbb{R}$, where the last inequality follows from a straightforward calculation.
Step 3: To prove (4.7) we consider several cases.
Case 1: Assume first that $\xi + a \leq 0$. Then $\xi(\log \delta + 1) \geq -a(\log \delta + 1) = a|\log \delta| - a$ and $|\xi + a|(\xi + a)\xi = (\xi + a)^2|\xi| \geq |\xi|^3 - 2a|\xi|^2 \geq \frac{1}{2}|\xi|^3 - 5a^3$, so that (4.4) implies

$$\Phi'_{a,\delta}(\xi)\xi \geq a|\log \delta| - a + \tfrac{1}{4}|\xi|^3 - 3a^3 \geq a|\log \delta| + \tfrac{1}{4}|\xi|^3 - 1 - 4a^3 . \tag{4.13}$$

Case 2: Consider next the case $\xi + a > 0$.

If $-a < \xi \leq 0$, then $(\xi + a)^2\xi \geq -a^3 \geq |\xi|^3 - 2a^3$, while if $\xi > 0$, then $(\xi + a)^2\xi \geq |\xi|^3$. Therefore

$$\xi + \tfrac{1}{2}(\xi + a)^2\xi \geq -a + \tfrac{1}{2}(\xi + a)^2\xi \geq -a + \tfrac{1}{2}|\xi|^3 - a^3 \geq \tfrac{1}{2}|\xi|^3 - 1 - 2a^3 . \tag{4.14}$$

To estimate the logarithmic terms we consider three ranges of $\xi + a$. If $\xi + a \leq \delta$, then by (4.4) and (4.14),

$$\begin{aligned} \Phi'_{a,\delta}(\xi)\xi &= \xi \log \delta + \xi + \tfrac{1}{2}(\xi + a)^2\xi \\ &\geq (\delta - a)\log \delta + \tfrac{1}{2}|\xi|^3 - 1 - 2a^3 \geq a|\log \delta| - \frac{1}{e} + \tfrac{1}{2}|\xi|^3 - 1 - 2a^3 \\ &\geq a|\log \delta| + \tfrac{1}{2}|\xi|^3 - 2 - 2a^3 . \end{aligned} \tag{4.15}$$

In the case $\delta < \xi + a \leq 1$ we have $(\xi + a)\log(\xi + a) \geq -1/e$, hence $\xi \log(\xi + a) \geq a|\log(\xi + a)| - 1$. It follows from (4.4) and (4.14) that

$$\Phi'_{a,\delta}(\xi)\xi \geq a|\log(\xi + a)| - 1 + \xi + \tfrac{1}{2}(\xi + a)^2\xi \geq a|\log(\xi + a)| + \tfrac{1}{2}|\xi|^3 - 2 - 2a^3 . \quad (4.16)$$

Finally, if $1 < \xi + a$ and $\xi \leq 0$, then $0 < \log(\xi + a) \leq \xi + a$, hence $\xi \log(\xi + a) \geq \xi(\xi + a) \geq -a^2$, while if $1 < \xi + a$ and $0 < \xi$, then $\xi \log(\xi + a) > 0$. In both cases, (4.4) and (4.14) give

$$\Phi'_{a,\delta}(\xi)\xi \geq -a^2 + \xi + \tfrac{1}{2}(\xi + a)^2\xi \geq \tfrac{1}{2}|\xi|^3 - 2 - 3a^3. \quad (4.17)$$

Step 4: Note that (4.8) is exactly (4.16). Inequality (4.9) follows from (4.13) and (4.15). To prove (4.10), again we consider three ranges of $\xi + a$. In the case $\xi + a \leq \delta$ inequality (4.10) follows from (4.5) and (4.9). When $\delta < \xi + a \leq 1$ the same inequality is a consequence of (4.4) and (4.8). Finally, for $1 < \xi + a$ inequality (4.10) can be obtained from (4.7) and (4.11). □

We recall that $V := W^{2,3}_{\mathrm{per}_0}(I)$, $Y := L^2_{\mathrm{per}_0}(I)$, and $\|u\|_V := \|u_{xx}\|_{L^3(I)}$. We introduce the operator $\mathcal{A}_{a,\delta} \colon V \to V'$ defined by

$$\langle \mathcal{A}_{a,\delta}(u), v\rangle_{V',V} := \int_I \Phi'_{a,\delta}(u_{xx})\, v_{xx}\, dx \quad (4.18)$$

for every u, $v \in V$. Note that by (4.5) and Hölder's and Minkowski's Inequalities, the operator $\mathcal{A}_{a,\delta}$ is well-defined and

$$\|\mathcal{A}_{a,\delta}(u)\|_{V'} \leq (2\pi)^{2/3}(|\log\delta| + 2a^2) + 2\,\|u\|_V^2 \quad (4.19)$$

for every $u \in V$. Moreover, by (4.7) we have

$$\langle \mathcal{A}_{a,\delta}(u), u\rangle_{V',V} \geq \tfrac{1}{4}\|u\|_V^3 - (4 + 8a^3)\pi . \quad (4.20)$$

Finally, (4.6) gives

$$\langle \mathcal{A}_{a,\delta}(u^2) - \mathcal{A}_{a,\delta}(u^1), u^2 - u^1\rangle_{V',V} \geq \tfrac{1}{4}\|u^2 - u^1\|_V^3 \quad (4.21)$$

for every u^1, $u^2 \in V$.

Therefore $\mathcal{A}_{a,\delta}$ is a bounded monotone operator. Moreover, by the continuity properties of Nemitski operators we deduce from (4.5) that $\mathcal{A}_{a,\delta}$ is continuous from V to V'.

Next we state the main theorem of this section. We recall that the operator $\mathcal{H}$ is defined in (1.9).

Theorem 4.2. *Let $a > 0$ and let $u^0 \in L^2_{\mathrm{per}_0}(I)$. For every $0 < \delta < 1/e$ there exists a solution u^δ of the problem*

$$\begin{cases} \dot u^\delta(t) + \mathcal{A}_{a,\delta}(u^\delta(t)) = \mathcal{H}(u^\delta(t)) \quad \text{for a.e. } t \in [0,T]\,, \\ u^\delta \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))\,,\ \dot u^\delta \in L^{3/2}([0,T]; W^{-2,3/2}_{\mathrm{per}_0}(I))\,, \\ u^\delta(0) = u^0\,. \end{cases} \quad (4.22)$$

Moreover, there exists a constant $c = c\left(a, T, \|u^0\|_Y\right) > 0$ such that for every $0 < \delta < 1/e$,

$$\|u^\delta\|_{L^3([0,T];V)} \leq c\,, \qquad \|(H(u^\delta_{xx}))_x\|_{L^{3/2}([0,T];V')} \leq c\,. \quad (4.23)$$

Proof. **Step 1:** We first obtain apriori estimates for the solution of the auxiliary problem

$$\begin{cases} \dot u(t) + \mathcal{A}_{a,\delta}(u(t)) = f(t) \quad \text{for a.e. } t \in [0,T], \\ u \in L^3([0,T]; V)\,,\ \dot u \in L^{3/2}([0,T]; V')\,, \\ u(0) = u^0\,, \end{cases} \quad (4.24)$$

where $f \in L^{3/2}([0,T]; V')$ and $u^0 \in Y$. In order to guarantee the existence and uniqueness of the solution of problem (4.24), we use a slight extension of [5, Chapter 2, Theorem 1.2], where the estimates in (1.34) and (1.36) in this reference are replaced by (4.19) and (4.20), respectively. The proof is essentially the same.

Substep 1 a: Let $f^1,\ f^2 \in L^{3/2}([0,T];V')$, and let $u^1,\ u^2$ be the corresponding solutions of (4.24) with the same initial value u^0. We claim that

$$\|u^2 - u^1\|^2_{L^3([0,T];V)} \le 4\,\|f^2 - f^1\|_{L^{3/2}([0,T];V')}\,. \tag{4.25}$$

We multiply both equations by $u^2(t) - u^1(t)$ and subtract the first from the second to get

$$\begin{aligned}&\langle \dot u^2(t) - \dot u^1(t), u^2(t) - u^1(t)\rangle_{V',V} + \langle \mathcal{A}_{a,\delta}(u^2(t)) - \mathcal{A}_{a,\delta}(u^1(t)), u^2(t) - u^1(t)\rangle_{V',V}\\ &= \langle f^2(t) - f^1(t), u^2(t) - u^1(t)\rangle_{V',V}\end{aligned}$$

for a.e. $t \in [0,T]$. Since $t \mapsto \frac{1}{2}\|u^2(t) - u^1(t)\|^2_Y$ is absolutely continuous and its derivative equals $\langle \dot u^2(t) - \dot u^1(t), u^2(t) - u^1(t)\rangle_{V',V}$ for a.e. $t \in [0,T]$ (see, e.g., [10, Proposition 23.23]), integrating the previous equality from 0 to T we deduce that

$$\begin{aligned}&\frac{1}{2}\|u^2(T) - u^1(T)\|^2_Y + \int_0^T \langle \mathcal{A}_{a,\delta}(u^2(t)) - \mathcal{A}_{a,\delta}(u^1(t)), u^2(t) - u^1(t)\rangle_{V',V}\,dt\\ &= \int_0^T \langle f^2(t) - f^1(t), u^2(t) - u^1(t)\rangle_{V',V}\,dt\,.\end{aligned} \tag{4.26}$$

By (4.21) we obtain

$$\tfrac{1}{4}\|u^2 - u^1\|^3_{L^3([0,T];V)} \le \|f^2 - f^1\|_{L^{3/2}([0,T];V')}\,\|u^2 - u^1\|_{L^3([0,T];V)}\,,$$

and so (4.25) holds.

Substep 1 b: Let $f \in L^{3/2}([0,T];V')$, let $u^0 \in Y$, and let u be the corresponding solution of (4.24). We will prove that

$$\begin{aligned}\|u\|_{\mathcal{W}} \le{}& (2\pi T)^{2/3}(|\log\delta| + 2a^2) + 25\|f\|_{L^{3/2}([0,T];V')} + 2\,\|f\|^{1/2}_{L^{3/2}([0,T];V')}\\ &+ 24\,\|u^0\|^{4/3}_Y + 2\,\|u^0\|^{2/3}_Y + 8(1+a)T^{1/3} + 384(1+a)^2T^{2/3}\,,\end{aligned} \tag{4.27}$$

where $\mathcal{W}$ is defined in (2.22).

We multiply the equation by $u(t)$ and argue as in the previous substep. Using (4.20) we obtain

$$\tfrac{1}{4}\|u\|^3_{L^3([0,T];V)} \le \|f\|_{L^{3/2}([0,T];V')}\,\|u\|_{L^3([0,T];V)} + (4 + 8a^3)\pi T + \tfrac{1}{2}\|u^0\|^2_Y\,,$$

which, together with Young's Inequality, gives

$$\|u\|_{L^3([0,T];V)} \le 2\,\|f\|^{1/2}_{L^{3/2}([0,T];V')} + 8(1+a)T^{1/3} + 2\,\|u^0\|^{2/3}_Y\,. \tag{4.28}$$

Since $\dot u(t) = f(t) - \mathcal{A}_{a,\delta}(u(t))$ for a.e. $t \in [0,T]$, from (4.19) we get

$$\|\dot u\|_{L^{3/2}([0,T];V')} \le \|f\|_{L^{3/2}([0,T];V')} + (2\pi T)^{2/3}(|\log\delta| + 2a^2) + 2\,\|u\|^2_{L^3([0,T];V)}\,,$$

which, together with (4.28), yields

$$\begin{aligned}\|\dot u\|_{L^{3/2}([0,T],V')} \le{}& (2\pi T)^{2/3}(|\log\delta| + 2a^2) + 25\|f\|_{L^{3/2}([0,T];V')}\\ &+ 384(1+a)^2T^{2/3} + 24\,\|u^0\|^{4/3}_Y\,.\end{aligned} \tag{4.29}$$

Inequality (4.27) follows from (2.23), (4.28), and (4.29).

Step 2: Fix $u^0 \in Y$. We will prove the existence of a solution of (4.22) using a fixed point argument. We begin by observing that given $u \in L^3([0,T];V)$, the function $t \mapsto (H(u(t)_{xx}))_x$ belongs to $L^{3/2}([0,T];V')$. Therefore, in view of Step 1 there exists a unique solution v of the problem

$$\begin{cases}\dot v(t) + \mathcal{A}_{a,\delta}(v(t)) = \mathcal{H}(u(t)) & \text{for a.e. } t \in [0,T]\,,\\ v \in \mathcal{W}, \quad v(0) = u^0\,.\end{cases} \tag{4.30}$$

Let $\mathcal{T}_\delta\colon L^3([0,T];V)\to L^3([0,T];V)$ be the operator defined by $\mathcal{T}_\delta(u):=v$. In order to apply Schauder's Fixed Point Theorem we need to establish the following properties:

$$\mathcal{T}_\delta \text{ is continuous;} \tag{4.31}$$

$$\mathcal{T}_\delta \text{ is compact;} \tag{4.32}$$

$$\mathcal{T}_\delta \text{ has an invariant ball.} \tag{4.33}$$

To prove (4.31) we show that $\mathcal{T}$ is Hölder's continuous. Indeed, let u^1, $u^2\in L^3([0,T];V)$. Then by Hölder's Inequality,

$$\|u^2_{xx}-u^1_{xx}\|_{L^2([0,T];L^2(I))}\le(2\pi T)^{1/6}\|u^2-u^1\|_{L^3([0,T];V)}\,,$$

and so, by Proposition 2.4, we have that

$$\|H(u^2_{xx})-H(u^1_{xx})\|_{L^2([0,T];L^2(I))}\le C_2(2\pi T)^{1/6}\|u^2-u^1\|_{L^3([0,T];V)}\,.$$

Therefore, there exists a constant $c=c(T)>0$ such that

$$\|(H(u^2_{xx}))_x-(H(u^1_{xx}))_x\|_{L^{3/2}([0,T];W^{-1,3/2}_{\mathrm{per}_0}(I))}\le c\,\|u^2-u^1\|_{L^3([0,T];V)}\,, \tag{4.34}$$

hence

$$\|(H(u^2_{xx}))_x-(H(u^1_{xx}))_x\|_{L^{3/2}([0,T];V')}\le c\,\|u^2-u^1\|_{L^3([0,T];V)}\,, \tag{4.35}$$

so that (4.25) yields

$$\|\mathcal{T}_\delta(u^2)-\mathcal{T}_\delta(u^1)\|^2_{L^3([0,T];V)}\le 4c\,\|u^2-u^1\|_{L^3([0,T];V)}\,,$$

which establishes (4.31).

Let us prove (4.32). Let $\{u^n\}$ be a bounded sequence in $L^3([0,T];V)$ and, for every n, let

$$v^n:=\mathcal{T}_\delta(u^n)\quad\text{and}\quad g^n:=(H(u^n_{xx}))_x\,.$$

By (4.34) the sequence $\{g^n\}$ is bounded in $L^{3/2}([0,T];W^{-1,3/2}_{\mathrm{per}_0}(I))$, so, in particular, it is also bounded in $L^{3/2}([0,T];V')$. Hence, by (4.27) the sequence $\{v^n\}$ is bounded in $\mathcal{W}$, and so, passing to a subsequence, we may assume that $\{v^n\}$ converges weakly in $\mathcal{W}$. On the other hand, by Lemma 2.3, passing to a further subsequence, we may assume that $\{i^*(g^n)\}$ converges strongly in $\mathcal{W}'$, where $i^*\colon L^{3/2}([0,T];W^{-1,3/2}_{\mathrm{per}_0}(I))\to\mathcal{W}'$ is the embedding defined by (2.24). Arguing as in (4.26), we obtain

$$\int_0^T\langle\mathcal{A}_{a,\delta}(v^n(t))-\mathcal{A}_{a,\delta}(v^m(t)),v^n(t)-v^m(t)\rangle_{V',V}\,dt\le\int_0^T\langle g^n(t)-g^m(t),v^n(t)-v^m(t)\rangle_{V',V}\,dt$$

for every n and m, and by (4.21) and (2.24) we have

$$\|v^n-v^m\|^3_{L^3([0,T];V)}\le 4\,\langle i^*(g^n)-i^*(g^n),v^n-v^m\rangle_{\mathcal{W}',\mathcal{W}}\,.$$

Since the right-hand side of the previous inequality converges to zero as n, $m\to\infty$, it follows that $\{v^n\}$ is a Cauchy sequence in $L^3([0,T];V)$. This concludes the proof of (4.32).

To prove (4.33), we fix $R>0$. By (4.35), for every $u\in L^3([0,T];V)$ with $\|u\|_{L^3([0,T];V)}\le R$ we have

$$\|(H(u_{xx}))_x\|_{L^{3/2}([0,T];V')}\le c\,\|u\|_{L^3([0,T];V)}\le c\,R\,, \tag{4.36}$$

Therefore, using (4.28) we get

$$\|\mathcal{T}_\delta(u)\|_{L^3([0,T];V)}\le 2(c\,R)^{1/2}+8(1+a)T^{1/3}+2\,\|u^0\|^{2/3}_Y\,. \tag{4.37}$$

By taking $R=R\big(a,T,\|u^0\|_Y\big)$ sufficiently large we obtain that the right-hand side of the previous inequality is less than R. This concludes the proof of (4.33).

In view of (4.31), (4.32), and (4.33), by Schauder's Fixed Point Theorem the operator $\mathcal{T}_\delta$ has a fixed point, which is a solution of problem (4.22). Moreover, (4.23) follows from (4.36) and (4.37). $\square$

5. An Auxiliary Problem

Let $a>0$ and let $u^0\in L^2_{\mathrm{per}_0}(I)$. For every $0<\delta<1/e$ let $f^\delta\in L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))$ and let u^δ be the solution of the problem

$$\begin{cases}\dot u^\delta(t)+\mathcal{A}_{a,\delta}(u^\delta(t))=f^\delta(t) \quad \text{for a.e. } t\in[0,T]\,,\\ u^\delta\in L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))\,,\ \dot u^\delta\in L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))\,,\\ u^\delta(0)=u^0\,,\end{cases} \tag{5.1}$$

Lemma 5.1. *Let $a>0$ and let $u^0\in L^2_{\mathrm{per}_0}(I)$. Assume that $\{f^\delta\}_{0<\delta<1/e}$ is bounded in $L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))$, and for every $0<\delta<1/e$ let u^δ be the solution of problem* (5.1). *Then there exists a constant M such that*

$$\|u^\delta\|_{L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I)))}\le M\,, \tag{5.2}$$

$$\int_0^T\langle\mathcal{A}_{a,\delta}(u^\delta(t)),u^\delta(t)\rangle_{V',V}\,dt\le M \tag{5.3}$$

for every $0<\delta<1/e$.

Proof. Inequality (5.2) follows from (4.28). To prove the other inequality, for a.e. $t\in[0,T]$ we multiply the equation in (5.1) by $u_\delta(t)$, obtaining

$$\langle\dot u^\delta(t),u^\delta(t)\rangle_{V',V}+\langle\mathcal{A}_{a,\delta}(u^\delta(t)),u^\delta(t)\rangle_{V',V}=\langle f^\delta(t),u^\delta(t)\rangle_{V',V}\,.$$

Since $t\mapsto\frac12\|u(t)\|^2_{L^2(I)}$ is absolutely continuous and its derivative equals $\langle\dot u(t),u(t)\rangle_{V',V}$ for a.e. $t\in[0,T]$ (see, e.g., [10, Proposition 23.23]), we may integrate the previous equality from 0 to T to deduce that

$$\begin{aligned}&\tfrac12\|u^\delta(T)\|^2_{L^2(I)}+\int_0^T\langle\mathcal{A}_{a,\delta}(u^\delta(t)),u^\delta(t)\rangle_{V',V}\,dt\\ &\qquad\le\|f^\delta\|_{L^{3/2}([0,T];V')}\|u^\delta\|_{L^3([0,T];V)}+\tfrac12\|u^0\|^2_{L^2(I)}\,.\end{aligned} \tag{5.4}$$

By (5.2), this implies (5.3). □

By (5.2) there exist a subsequence (not relabeled) and a function $u\in L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I)))$, such that

$$u^\delta\rightharpoonup u\quad\text{weakly in } L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I)))\,. \tag{5.5}$$

In the sequel (see Lemma 5.4) we will prove that, if $\{f^\delta\}_{0<\delta<1/e}$ converges strongly to some function f in $L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))$, then u is a *weak solution* of the limit problem, in the sense that

$$\int_0^T\Big(\langle\dot v(t),v(t)-u(t)\rangle_{V',V}+\mathcal{F}_a(v(t))\Big)dt\ge\int_0^T\Big(\mathcal{F}_a(u(t))+\langle f(t),v(t)-u(t)\rangle_{V',V}\Big)dt \tag{5.6}$$

for every $v\in\mathcal{W}$ with $v(0)=u^0$, where $\mathcal{F}_a$ is the functional defined in (1.8).

Lemma 5.2. *Under the assumptions of Lemma 5.1, for every $0<\delta<1/e$ and a.e. $t\in[0,T]$ let*

$$E^\delta(t):=\{x\in I: u^\delta_{xx}(t,x)+a\le\delta\}\,, \tag{5.7}$$

$$F^\delta(t):=\{x\in I:\delta\le u^\delta_{xx}(t,x)+a\le 1\}\,. \tag{5.8}$$

Then there exists a constant M such that

$$\int_0^T\mathcal{L}^1(E_\delta(t))\,dt\le M|\log\delta|^{-1}\,, \tag{5.9}$$

$$\int_0^T\Big(\int_{F^\delta(t)}\big|\log(u^\delta(t)_{xx}+a)\big|\,dx\Big)dt\le M \tag{5.10}$$

for every $0 < \delta < 1/e$.

Proof. By (4.7) and (4.9), for a.e. $t \in [0,T]$ we have

$$\langle \mathcal{A}_{a,\delta}(u^\delta(t)), u^\delta(t) \rangle_{V',V} \geq a |\log \delta| \, \mathcal{L}^1(E^\delta(t)) - 4\pi - 8\pi a^3 .$$

Integrating in time and using (5.3) we get (5.9).

On the other hand, by (4.7) and (4.8) for a.e. $t \in [0,T]$ we have

$$\langle \mathcal{A}_{a,\delta}(u^\delta(t)), u^\delta(t) \rangle_{V',V} \geq a \int_{F_\delta(t)} \big| \log(u^\delta(t)_{xx} + a) \big| \, dx - 4\pi - 8\pi a^3 .$$

Integrating in time and using (5.3) we get (5.10). □

For $\delta > 0$ consider the functionals $\mathcal{F}_a$, $\mathcal{F}_{a,\delta} \colon W^{2,3}_{\mathrm{per}_0}(I) \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{G}_a$, $\mathcal{G}_{a,\delta} \colon W^{2,3}_{\mathrm{per}_0}(I) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{F}_a(u) := \int_I \Phi_a(u_{xx}) \, dx , \qquad \mathcal{F}_{a,\delta}(u) := \int_I \Phi_{a,\delta}(u_{xx}) \, dx , \tag{5.11}$$

$$\mathcal{G}_a(u) := \int_I \Psi_a(u_{xx}) \, dx , \qquad \mathcal{G}_{a,\delta}(u) := \int_I \Psi_{a,\delta}(u_{xx}) \, dx , \tag{5.12}$$

where Φ_a, $\Phi_{a,\delta}$, and Ψ_a, $\Psi_{a,\delta}$ are given in (1.6), (4.3), (1.6), and (4.2), respectively.

Lemma 5.3. *Under the assumptions of Lemma 5.1, for every* $0 < \delta < 1/e$ *we have*

$$\int_0^T \Big(\langle \dot v(t), v(t) - u^\delta(t) \rangle_{V',V} + \mathcal{F}_{a,\delta}(v(t)) - \mathcal{F}_{a,\delta}(u^\delta(t)) \Big) dt \geq \int_0^T \langle f^\delta(t), v(t) - u^\delta(t) \rangle_{V',V} \, dt \tag{5.13}$$

for every $v \in \mathcal{W}$ *with* $v(0) = u^0$.

Proof. Fix $0 < \delta < 1/e$ and $v \in \mathcal{W}$ with $v(0) = u^0$. For a.e. $t \in [0,T]$ we multiply equation (5.1) by $v(t) - u^\delta(t)$. Adding $\langle \dot v(t), v(t) - u^\delta(t) \rangle_{V',V}$ to both sides we get

$$\begin{aligned} &\langle \dot v(t), v(t) - u^\delta(t) \rangle_{V',V} + \langle \mathcal{A}_{a,\delta}(u^\delta(t)), v(t) - u^\delta(t) \rangle_{V',V} \\ &\quad = \langle \dot v(t) - \dot u^\delta(t), v(t) - u^\delta(t) \rangle_{V',V} + \langle f^\delta(t), v(t) - u^\delta(t) \rangle_{V',V} . \end{aligned}$$

Since $\mathcal{A}_{a,\delta} = \partial \mathcal{F}_{a,\delta}$ and $\mathcal{F}_{a,\delta}$ is convex, we obtain

$$\begin{aligned} &\langle \dot v(t), v(t) - u^\delta(t) \rangle_{V',V} + \mathcal{F}_{a,\delta}(v(t)) - \mathcal{F}_{a,\delta}(u^\delta(t)) \\ &\quad \geq \tfrac{1}{2} \tfrac{d}{dt} \| v(t) - u^\delta(t) \|^2_{L^2(I)} + \langle f^\delta(t), v(t) - u^\delta(t) \rangle_{V',V} , \end{aligned}$$

where we have also used [10, Proposition 23.23]. Integrating with respect to t we obtain (5.13). □

Lemma 5.4. *Under the hypotheses of Lemma 5.1, let* u *be the function defined in* (5.5). *Assume that* $\{f^\delta\}_{0<\delta<1/e}$ *converges strongly to some function* f *in* $L^{3/2}([0,T]; W^{-2,3/2}_{\mathrm{per}_0}(I))$. *Then* u *satisfies* (5.6).

Proof. By (5.5) we have

$$\int_0^T \langle \dot v(t), v(t) - u^\delta(t) \rangle_{V',V} \, dt \to \int_0^T \langle \dot v(t), v(t) - u(t) \rangle_{V',V} \, dt , \tag{5.14}$$

$$\int_0^T \langle f^\delta(t), v(t) - u^\delta(t) \rangle_{V',V} \, dt \to \int_0^T \langle f(t), v(t) - u(t) \rangle_{V',V} \, dt . \tag{5.15}$$

On the other hand,

$$\int_0^T \mathcal{F}_a(v(t)) \, dt \geq \int_0^T \mathcal{F}_{a,\delta}(v(t)) \, dt \tag{5.16}$$

because $\mathcal{F}_a \geq \mathcal{F}_{a,\delta}$.

Finally, in order to study the term $\mathcal{F}_{a,\delta}(u^\delta(t))$ we fix $0 < \eta < 1/e$ and we use the inequality

$$\mathcal{G}_{a,\delta}(u^\delta(t)) \geq \mathcal{G}_{a,\eta}(u^\delta(t)) \tag{5.17}$$

for every $0 < \delta < \eta$. Since the functional

$$w \mapsto \int_0^T \mathcal{G}_{a,\delta}(w)\, dt$$

is convex and continuous for the strong topology of $L^3([0,T];V)$, it is also lower semicontinuous for the weak topology of $L^3([0,T];V)$. By (5.5) and (5.17) this implies

$$\liminf_{\delta\to 0} \int_0^T \mathcal{G}_{a,\delta}(u^\delta(t))\, dt \geq \liminf_{\delta\to 0} \int_0^T \mathcal{G}_{a,\eta}(u^\delta(t))\, dt \geq \int_0^T \mathcal{G}_{a,\eta}(u(t))\, dt\,.$$

Taking the limit as $\eta \to 0$ and using the Monotone Convergence Theorem, we get

$$\liminf_{\delta\to 0} \int_0^T \mathcal{G}_{a,\delta}(u^\delta(t))\, dt \geq \int_0^T \mathcal{G}_{a}(u(t))\, dt\,. \tag{5.18}$$

Adding the cube of the norm in $L^3([0,T]);V)$ we obtain

$$\liminf_{\delta\to 0} \int_0^T \mathcal{F}_{a,\delta}(u^\delta(t))\, dt \geq \int_0^T \mathcal{F}_{a}(u(t))\, dt\,. \tag{5.19}$$

Inequality (5.6) follows now from (5.13), (5.14), (5.15), (5.16), and (5.19). $\square$

From now on we assume that there exists $w^0 \in \mathcal{W}$ such that $w^0(0) = u^0$ and

$$\int_0^T \mathcal{F}_a(w^0(t))\, dt < +\infty\,. \tag{5.20}$$

Then (5.6) implies that

$$\int_0^T \mathcal{F}_a(u(t))\, dt < +\infty\,. \tag{5.21}$$

Lemma 5.5. *Let $u^0 \in L^2_{\mathrm{per}_0}(I)$ and let $w \in L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))$ be such that for a.e. $t \in [0,T]$ the function $w_{xx}(t) + a$ is nonnegative. Then there exists a sequence $\{w^n\}$ in $\mathcal{W}$ such that*

$$w^n(0) = u^0 \quad \textit{for every } n\,, \tag{5.22}$$

$$\textit{for every } t \in [0,T] \textit{ and every } n \in \mathbb{N} \textit{ the function } w^n_{xx}(t) + a \textit{ is nonnegative}\,, \tag{5.23}$$

$$\limsup_{n\to\infty} \int_0^T \langle \dot w^n(t), w^n(t) - w(t)\rangle_{V',V}\, dt \leq 0\,, \tag{5.24}$$

$$w^n \to w \quad \textit{strongly in } L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))\,, \tag{5.25}$$

$$\lim_{n\to\infty} \int_0^T \mathcal{F}_a(w^n(t))\, dt = \int_0^T \mathcal{F}_a(w(t))\, dt\,. \tag{5.26}$$

Proof. Let $w^0 \in \mathcal{W}$ be the function in (5.20). Since $w^0 \in L^3([0,T];V) \cap C^0([0,T];H)$, there exists a sequence $\delta_n \searrow 0$ such that

$$\delta_n \|w^0(\delta_n)\|^3_V \to 0\,. \tag{5.27}$$

Define

$$w^n(t) := \begin{cases} w^0(t) & \text{for } t \in [0,\delta_n]\,, \\ \dfrac{1}{\delta_n} \displaystyle\int_{\delta_n}^t e^{-(t-s)/\delta_n} w(s)\, ds + e^{-(t-\delta_n)/\delta_n} w^0(\delta_n) & \text{for } t \in [\delta_n, T]\,. \end{cases} \tag{5.28}$$

Then $w^n \in \mathcal{W}$ and (5.22) holds. Since $w_{xx}(t) + a \geq 0$ for a.e. $t \in [0,T]$ by hypothesis, and $w^0_{xx}(t) + a \geq 0$ for every $t \in [0,T]$ by (5.20) (recall that w^0 is continuous with values in $L^2_{\mathrm{per}_0}(I)$), we have that (5.23) is satisfied. Moreover, we have

$$\delta_n \dot{w}^n(t) = w(t) - w^n(t) \quad \text{for a.e. } t \in [\delta_n, T] .$$

Hence,

$$\delta_n \langle \dot{w}^n(t), w^n(t) - w(t) \rangle_{V',V} = - \langle w^n(t) - w(t), w^n(t) - w(t) \rangle_{V',V} \leq 0$$

for a.e. $t \in [\delta_n, T]$. Therefore,

$$\int_0^T \langle \dot{w}^n(t), w^n(t) - w(t) \rangle_{V',V} \, dt \leq \int_0^{\delta_n} \langle \dot{w}^0(t), w^0(t) - w(t) \rangle_{V',V} \, dt .$$

Since $w^0 \in \mathcal{W}$ and $w \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$, inequality (5.24) follows.

Let us prove (5.25). We observe that

$$w^n(t) - w(t) = a^n(t) + b^n(t) , \tag{5.29}$$

with

$$a^n(t) := \frac{1}{\delta_n} \int_{\delta_n}^t e^{-(t-s)/\delta_n} (w(s) - w(t)) \, ds ,$$
$$b^n(t) := e^{-(t-\delta_n)/\delta_n} (w^0(\delta_n) - w(t)) ,$$

for every $t \in [\delta_n, T]$. From standard properties of convolutions, we deduce that

$$\int_{\delta_n}^T \|a^n(t)\|^3_V dt \to 0 , \tag{5.30}$$

while

$$\int_{\delta_n}^T \|b^n(t)\|^3_V dt \to 0 \tag{5.31}$$

by (5.27) and by the Dominated Convergence Theorem. Since $w^n(t) - w(t) = w^0(t) - w(t)$ for every $t \in [0, \delta_n]$, we have also

$$\int_0^{\delta_n} \|w^n(t) - w(t)\|^3_V dt \to 0 . \tag{5.32}$$

Property (5.25) follows from (5.29)-(5.32). Equality (5.26) is a consequence of (5.25), in view of (5.23). □

Theorem 5.6. *Let $a > 0$, let $f \in L^{3/2}([0,T]; W^{-2,3/2}_{\mathrm{per}_0}(I))$, and let $u^0 \in L^2_{\mathrm{per}_0}(I)$. Assume that* (5.20) *is satisfied. Then there exists a unique solution of* (5.6) *in $L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$.*

Proof. The existence follows from (5.5) and Lemma 5.4 with $f_\delta := f$ for all $\delta > 0$. To prove uniqueness, let u^1 and $u^2 \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$ be solutions of (5.6), and let $w := \frac{1}{2}(u^1 + u^2)$. By (5.20) we have

$$\int_0^T \mathcal{F}_a(u^1(t)) \, dt < +\infty \quad \text{and} \quad \int_0^T \mathcal{F}_a(u^2(t)) \, dt < +\infty ,$$

so that for a.e. $t \in [0,T]$ the functions $u^1_{xx}(t) + a$ and $u^2_{xx}(t) + a$ are nonnegative. Let $\{w^n\}$ be the sequence given by Lemma 5.5. We write (5.6) for u^1 and u^2 with w^n in place of v. Adding the resulting inequalities, we obtain

$$\int_0^T \Big(\langle \dot{w}^n(t), w^n(t) - w(t) \rangle_{V',V} + \mathcal{F}_a(w^n(t)) \Big) dt$$
$$\geq \int_0^T \Big(\frac{1}{2} \mathcal{F}_a(u^1(t)) + \frac{1}{2} \mathcal{F}_a(u^2(t)) + \langle f(t), w^n(t) - w(t) \rangle_{V',V} \Big) dt .$$

Passing to the limit and using (5.24), (5.25), and (5.26) we obtain

$$\int_0^T \mathcal{F}_a(w(t))\,dt \geq \frac{1}{2}\int_0^T \mathcal{F}_a(u^1(t))\,dt + \frac{1}{2}\int_0^T \mathcal{F}_a(u^2(t))\,dt\,.$$

Since $\mathcal{F}_a$ is strictly convex in $L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))$ (see (1.2) and (2.20)), we conclude that $u^1 = u^2$. □

Remark 5.7. Under the assumptions of this theorem, it follows that if u^δ is the solution of problem (5.1), $0 < \delta < 1/e$, and if $\{f^\delta\}_{0<\delta<1/e}$ converges strongly to some function f in $L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))$, then the entire sequence $\{u^\delta\}_{0<\delta<1/e}$ weakly converges in $L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))$ to the unique solution of (5.6).

Proposition 5.8. *Let $a > 0$ and let $u^0 \in L^2_{\mathrm{per}_0}(I)$. Assume that $\{f^\delta\}_{0<\delta<1/e}$ converges strongly to some function f in $L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))$. For every $0 < \delta < 1/e$ let u^δ be the solution of problem (5.1) and let u be the solution of (5.6). Then*

$$u^\delta \to u \quad \textit{strongly in } L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))\,, \tag{5.33}$$

$$\lim_{\delta\to 0}\int_0^T \mathcal{G}_{a,\delta}(u^\delta(t))\,dt = \int_0^T \mathcal{G}_a(u(t))\,dt\,. \tag{5.34}$$

Proof. We will prove that

$$\lim_{\delta\to 0}\int_0^T \mathcal{F}_{a,\delta}(u^\delta(t))\,dt = \int_0^T \mathcal{F}_a(u(t))\,dt\,. \tag{5.35}$$

In view of (5.19) it is enough to show that

$$\limsup_{\delta\to 0}\int_0^T \mathcal{F}_{a,\delta}(u^\delta(t))\,dt \leq \int_0^T \mathcal{F}_a(u(t))\,dt\,. \tag{5.36}$$

Let $\{w^n\}$ be the sequence given by Lemma 5.5 with $w = u$. By (5.13) we have

$$\int_0^T \Big(\langle \dot w^n(t), w^n(t) - u^\delta(t)\rangle_{V',V} + \mathcal{F}_{a,\delta}(w^n(t))\Big)dt \geq \int_0^T \Big(\mathcal{F}_{a,\delta}(u^\delta(t)) + \langle f^\delta(t), w^n(t) - u^\delta(t)\rangle_{V',V}\Big)dt$$

for every $0 < \delta < 1/e$ and every n. Taking the limit as $\delta \to 0$ and using the fact that $\mathcal{F}_a \geq \mathcal{F}_{a,\delta}$, we obtain

$$\begin{aligned}\int_0^T &\Big(\langle \dot w^n(t), w^n(t) - u(t)\rangle_{V',V} + \mathcal{F}_a(w^n(t))\Big)dt \\ &\geq \limsup_{\delta\to 0}\int_0^T \mathcal{F}_{a,\delta}(u^\delta(t))\,dt + \int_0^T \langle f(t), w^n(t) - u(t)\rangle_{V',V}\,dt\,.\end{aligned}$$

Taking the limit as $n \to \infty$ and using (5.24), (5.23), and (5.25) we obtain (5.36), which gives (5.35). Hence, (5.35) holds, or, equivalently,

$$\begin{aligned}\lim_{\delta\to 0}&\Big(\int_0^T \mathcal{G}_{a,\delta}(u^\delta(t))\,dt + \tfrac{1}{6}\|u^\delta + a\|^3_{L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))}\Big) \\ &= \int_0^T \mathcal{G}_a(u(t))\,dt + \tfrac{1}{6}\|u + a\|^3_{L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))}\,.\end{aligned}$$

In view of (5.18) and of the weak lower semicontinuity of the norm, we obtain (5.34) and the convergence of $\|u^\delta + a\|_{L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))}$ to $\|u + a\|_{L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))}$. By the uniform convexity of the norm of $L^3([0,T];W^{2,3}_{\mathrm{per}_0}(I))$, we deduce (5.33). □

Proposition 5.9. *Under the assumptions of Theorem 5.6, the solution u of problem* (5.6) *satisfies*

$$\log(u_{xx}+a) \in L^1([0,T];L^1(I))\,. \tag{5.37}$$

In particular, $u_{xx}+a>0$ a.e. on $[0,T]\times I$.

Proof. For every $0<\delta<1/e$ we define

$$\tilde{E}^\delta := \{(t,x)\in[0,T]\times I : u^\delta_{xx}(t,x)+a\le\delta\}\,.$$

By Fubini's Theorem and by Lemma 5.2,

$$\int_{[0,T]\times I} 1_{\tilde{E}^\delta}(t,x)\,dxdt = \int_0^T \mathcal{L}^1(E_\delta(t))\,dt \to 0\,,$$

and so there exists a subsequence (not relabeled) such that $1_{\tilde{E}^\delta}\to 0$ pointwise a.e. on $[0,T]\times I$. Since $u^\delta_{xx}\to u_{xx}$ strongly in $L^3([0,T];L^3_{\mathrm{per}_0}(I))$, passing to a further subsequence (still not relabeled) we obtain that $u^\delta_{xx}\to u_{xx}$ pointwise a.e. on $[0,T]\times I$.

Define

$$G^\delta := \{(t,x)\in[0,T]\times I : u^\delta_{xx}(t,x)+a<1\} \quad\text{and}\quad G := \{(t,x)\in[0,T]\times I : u_{xx}(t,x)+a<1\}\,.$$

Then, for a.e. $(t,x)\in[0,T]\times I$,

$$1_G(t,x) \le \liminf_{\delta\to 0} 1_{G^\delta}(t,x)\,.$$

Since $1_{G^\delta}(t,x)\le 1_{\tilde{E}^\delta}(t,x)+1_{F^\delta(t)}(x)$, we conclude that

$$1_G(t,x) \le \lim_{\delta\to0} 1_{\tilde{E}^\delta}(t,x) + \liminf_{\delta\to0} 1_{F^\delta(t)}(x) = \liminf_{\delta\to0} 1_{F^\delta(t)}(x)\,,$$

which gives

$$1_G(t,x)\,|\log(u_{xx}(t,x)+a)| \le \liminf_{\delta\to0} 1_{F^\delta(t)}(x)\,|\log(u^\delta_{xx}(t,x)+a)|\,.$$

By Fatou Lemma and Fubini's Theorem it follows from (5.10) that

$$\begin{aligned}\int_G |\log(u_{xx}(t,x)+a)|\,dxdt &\le \int_{[0,T]\times I} \liminf_{\delta\to0} 1_{F^\delta(t)}(x)\,|\log(u^\delta_{xx}(t,x)+a)|\,dxdt\\ &\le \liminf_{\delta\to0}\int_{[0,T]\times I} 1_{F^\delta(t)}(x)\,|\log(u^\delta_{xx}(t,x)+a)|\,dxdt \le M\,.\end{aligned}$$

Using the fact that $u_{xx}\in L^3([0,T];L^3_{\mathrm{per}_0}(I))$, we deduce that $|\log(u_{xx}+a)|$ is integrable on the set $\{(t,x)\in[0,T]\times I : u_{xx}(t,x)+a\ge 1\}$, which, together with the previous inequality, gives the result. □

The following proposition shows that the solution of (5.6) is also a solution of a variational inequality involving the operator defined in (1.10).

Proposition 5.10. *Let $a>0$, let $u^0\in L^2_{\mathrm{per}_0}(I)$, let $f\in L^{3/2}([0,T];W^{-2,3/2}_{\mathrm{per}_0}(I))$, and let u be the solution of* (5.6). *Then*

$$\int_0^T\Big(\langle \dot{w}(t), w(t)-u(t)\rangle_{V',V} + \langle \mathcal{A}_a(w(t)), w(t)-u(t)\rangle_{V',V}\Big)dt \le \int_0^T \langle f(t), w(t)-u(t)\rangle_{V',V}\,dt \tag{5.38}$$

for every $w\in\mathcal{W}$, with $w(0)=u^0$, such that

$$\log(w_{xx}+a)\in L^{3/2}([0,T];L^{3/2}(I))\,. \tag{5.39}$$

Proof. Let $w \in \mathcal{W}$ be as in the statement and, for every $0 < \delta < 1/e$, let u^δ be the solution of problem (5.1) with $f^\delta = f$. For a.e. $t \in [0,T]$ we multiply equation (5.1) by $u^\delta(t) - w(t)$. Adding and subtracting $\langle \dot{w}(t), u^\delta(t) - w(t)\rangle_{V',V}$ and $\langle \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}$, we get

$$\begin{aligned}&\langle \dot{u}^\delta(t) - \dot{w}(t), u^\delta(t) - w(t)\rangle_{V',V} + \langle \mathcal{A}_{a,\delta}(u^\delta(t)) - \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}\\ &+ \langle \dot{w}(t), u^\delta(t) - w(t)\rangle_{V',V} + \langle \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}\\ &\quad = \langle f(t), u^\delta(t) - w(t)\rangle_{V',V}\,.\end{aligned}$$

Since $t \mapsto \frac{1}{2}\|u^\delta(t) - w(t)\|^2_{L^2(I)}$ is absolutely continuous and its derivative equals $\langle \dot{u}^\delta(t) - \dot{w}(t), u^\delta(t) - w(t)\rangle_{V',V}$ for a.e. $t \in [0,T]$ (see, e.g., [10, Proposition 23.23]), we may integrate both sides of the previous equality with respect to t and use the fact that $u^\delta(0) = w(0) = u^0$ to obtain

$$\begin{aligned}&\tfrac{1}{2}\|u^\delta(T) - w(T)\|^2_{L^2(I)} + \int_0^T \langle \mathcal{A}_{a,\delta}(u^\delta(t)) - \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}\,dt\\ &+ \int_0^T \langle \dot{w}(t), u^\delta(t) - w(t)\rangle_{V',V}\,dt + \int_0^T \langle \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}\,dt\\ &\quad = \int_0^T \langle f(t), u^\delta(t) - w(t)\rangle_{V',V}\,dt\,.\end{aligned}$$

By (4.6) and (4.18), we deduce

$$\begin{aligned}&\int_0^T \langle \dot{w}(t), u^\delta(t) - w(t)\rangle_{V',V}\,dt + \int_0^T \langle \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}\,dt\\ &\quad \le \int_0^T \langle f(t), u^\delta(t) - w(t)\rangle_{V',V}\,dt\,.\end{aligned} \tag{5.40}$$

Note that if $0 < \xi + a \le \delta < 1$, then $|\log \delta| \le |\log(\xi + a)|$. Hence, by (4.4),

$$|\Phi'_{a,\delta}(\xi)| \le |\log(\xi + a)| + 1 + \tfrac{1}{2}|\xi + a|^2$$

for all $\xi + a > 0$. Moreover, in view of (5.39), $w_{xx} + a > 0$ a.e. on $[0,T] \times I$. It follows that,

$$|\Phi'_{a,\delta}(w_{xx}(t,x))| \le |\log(w_{xx}(t,x) + a)| + 1 + \tfrac{1}{2}|w_{xx}(t,x) + a|^2 \tag{5.41}$$

for all $0 < \delta < 1/e$ and a.e. on $[0,T] \times I$. By (1.4), (1.6), (4.4), and (5.41) we obtain that $\Phi'_{a,\delta}(w_{xx}) \to \Phi'_a(w_{xx})$ strongly in $L^{3/2}([0,T]; L^{3/2}_{\mathrm{per}_0}(I))$. Since $u^\delta_{xx} \to u_{xx}$ strongly in $L^3([0,T]; L^3_{\mathrm{per}_0}(I))$ by (5.33), from (1.10) and (4.18) we obtain

$$\lim_{\delta\to 0} \int_0^T \langle \mathcal{A}_{a,\delta}(w(t)), u^\delta(t) - w(t)\rangle_{V',V}\,dt = \int_0^T \langle \mathcal{A}_a(w(t)), u(t) - w(t)\rangle_{V',V}\,dt\,, \tag{5.42}$$

and we may let $\delta \to 0$ in (5.40) to get (5.38). □

6. Proof of the Main Theorems

In this section we prove Theorems 1.1 and 1.2. Let $a > 0$ and let $u^0 \in L^2_{\mathrm{per}_0}(I)$. For every $0 < \delta < 1/e$ let u^δ be the solution to problem (4.22) (see Theorem 4.2). Define

$$f^\delta(t,x) := -(H(u^\delta_{xx}(t)))_x\,.$$

By (4.23), we may apply Lemma 5.1 to obtain that (5.2) and (5.3) are satisfied. Using (4.10) and (5.3) we deduce that

$$\|\mathcal{A}_{a,\delta}(u^\delta(t))\|_{L^1([0,T];(W^{2,\infty}_{\mathrm{per}_0}(I))')} \le M\,. \tag{6.1}$$

In turn, by (4.22), (5.2), (6.1), and Proposition 2.4 we conclude that

$$\|\dot{u}^\delta\|_{L^1([0,T];(W^{2,\infty}_{\mathrm{per}_0}(I))')} \le M\,. \tag{6.2}$$

By (5.2) there exist a subsequence (not relabeled) and a function $u \in L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I))$, such that

$$u^\delta \rightharpoonup u \quad \text{weakly in } L^3([0,T]; W^{2,3}_{\mathrm{per}_0}(I)) \,. \tag{6.3}$$

In view of (6.2) we can now apply Theorem 2.1 with $B_0 = W^{2,3}_{\mathrm{per}_0}(I)$, $B = W^{1,3}_{\mathrm{per}_0}(I)$, $B_1 = (W^{2,\infty}_{\mathrm{per}_0}(I))'$, and $p = 3$, to obtain that

$$u^\delta \to u \quad \text{strongly in } L^3([0,T]; W^{1,3}_{\mathrm{per}_0}(I)) \,, \tag{6.4}$$

and hence

$$u^\delta_{xx} \to u_{xx} \quad \text{strongly in } L^3([0,T]; W^{-1,3}_{\mathrm{per}_0}(I)) \,. \tag{6.5}$$

Since H is a continuous linear operator from $W^{-1,3}_{\mathrm{per}_0}(I))$ into itself (see Remark 2.6), we deduce that $(H(u^\delta_{xx}))_x \to (H(u_{xx}))_x$ strongly in $L^3([0,T]; W^{-2,3}_{\mathrm{per}_0}(I)))$, and so,

$$(H(u^\delta_{xx}))_x \to (H(u_{xx}))_x \quad \text{strongly in } L^{3/2}([0,T]; W^{-2,3/2}_{\mathrm{per}_0}(I))) \,. \tag{6.6}$$

Hence, we can apply Lemma 5.4 to establish that u satisfies (1.7) for every $w \in \mathcal{W}$ with $w(0) = u^0$. By Proposition 5.9 we have $\log(u_{xx} + a) \in L^1([0,T]; L^1(I))$. This concludes the proof of Theorem 1.1.

Theorem 1.2 follows from Proposition 5.10.

Remark 6.1. As in the proof of Theorem 2.1, it follows from (5.2) and (6.2) that

$$\|u^\delta\|_{W^{1,1}([0,T];(W^{2,\infty}_{\mathrm{per}_0}(I))')} \le M$$

and that

$$u^\delta(t) \to u(t) \quad \text{strongly in } (W^{2,\infty}_{\mathrm{per}_0}(I))' \text{ for a.e. } t \in [0,T] \,. \tag{6.7}$$

Let $0 < t_1 < \cdots < t_n < T$ be a partition of $[0,T]$ with the property that (6.7) holds for every t_i, $i = 1, \dots, n$. Since u^δ is absolutely continuous, it follows that

$$\begin{aligned}\sum_{i=2}^n \|u^\delta(t_i) - u^\delta(t_{i-1})\|_{(W^{2,\infty}_{\mathrm{per}_0}(I))'} &= \sum_{i=2}^n \left\| \int_{t_{i-1}}^{t_i} \dot u^\delta(t)\, dt \right\|_{(W^{2,\infty}_{\mathrm{per}_0}(I))'} \\ &\le \sum_{i=2}^n \int_{t_{i-1}}^{t_i} \left\| \dot u^\delta(t) \right\|_{(W^{2,\infty}_{\mathrm{per}_0}(I))'} dt \le M \,.\end{aligned}$$

Letting $\delta \to 0$ and using (6.7), we obtain

$$\sum_{i=2}^n \|u(t_i) - u(t_{i-1})\|_{(W^{2,\infty}_{\mathrm{per}_0}(I))'} \le M \,.$$

This shows that the essential pointwise variation of $u : [0,T] \to (W^{2,\infty}_{\mathrm{per}_0}(I))'$ is finite.

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E-mail address, Gianni Dal Maso: `dalmaso@sissa.it`
E-mail address, Irene Fonseca: `fonseca@andrew.cmu.edu`
E-mail address, Giovanni Leoni: `giovanni@andrew.cmu.edu`

(G. Dal Maso) SISSA, via Bonomea 265, 34136 Trieste, Italy

(I. Fonseca and G. Leoni) Carnegie Mellon University, Pittsburgh PA 15213-3890, USA