Currents with coefficients in groups, applications and other problems in Geometric Measure Theory

Annalisa Massaccesi

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To Euro Sampaolesi,
who made me fall
in love with Math

Contents

Introduction

This thesis collects some of the results I obtained during three years of PhD studies at the Scuola Normale Superiore in Pisa. As the vagueness of the title may lead to suspect, the subjects of this thesis are far from being homogeneous, indeed, the core of this thesis belongs to Geometric Measure Theory, as the number of pages devoted to currents shows. Part I contains classical results about currents and some original results related to problems with a classical flavour, too. Part II deals with currents with coefficients in a group and their applications: they still belong to Geometric Measure Theory, indeed, but are less known and less studied – most of the results presented in this part are new.

Adopting a chronological/heuristic point of view, I should admit that my starting point was the idea of a variational approach to the Steiner tree problem of Section 3.2. From this idea, one has to step back to the theory of currents with coefficients in a group and, at some point, one has to confront with the classical theory of currents. Thus Chapter 1 is motivated by this stratified structure of the thesis and almost every result there is well-known and can be found in the standard literature for Geometric Measure Theory, mainly [24], [43], [36] and [51]. The exceptions are contained in Section 1.3: we are not able to cite any reference for some facts, that belong to folklore anyway.

Hence, Chapter 1 is essentially a long review starting with Measure Theory (Section 1.1) and differential forms (Subsection 1.2.1). Despite the efforts at making this first chapter a systematic and consistent review of the classical theory of currents, one can detect a certain insistence in some notation or some concept: indeed, Chapter 1 serves as a toolbox for the next chapters. For instance, the formalism of differential forms in Subsection 1.2.1 is needed not only for the abstract definition of currents but also in Chapter 2, where we will borrow some notation from Differential Geometry in order to deal with Frobenius Theorem.

Though pretty standard, Sections 1.2, 1.3 and 1.4 have another reason to be in this thesis: they are a sample of the classical theory and they will be a model for the theory of currents with coefficients in a group that we will develop in Section 3.1.

We introduce currents in Subsection 1.2.2, as functionals on the space of compactly supported smooth differential k-forms in $\mathbb{R}^d$. This definition by duality enlightens the analogy with distributions but it is abstract, it can be made milder by the introduction of proper subsets of currents and by approximation theorems, such as the theorems in Subsection 1.2.3. The most important subset of currents is the class of integral currents, introduced in [26] by Federer and Fleming: the integral representation $\int_\Sigma \langle \omega; \tau \rangle \theta \, d\mathscr{H}^k$ (with θ a $\mathbb{Z}$-valued measurable function) allows to think of integral currents as a class of generalized surfaces with good compactness properties, with all the implications concerning the treatment of the Plateau problem in this wider context. The second class of currents we have to keep in mind is the linear subspace of normal currents, that is, currents with an integral representation $\int_{\mathbb{R}^d} \langle \omega; \tau \rangle \, d\mu$ (with a suitable measure μ and real multiplicity) for them and for their boundaries. Integral currents are a subset of normal currents, the latter playing a crucial role whenever a real vector space structure is needed.

Subsections 1.2.3 and 1.2.4 contain the fundamental theorems on (integral) currents: the Deformation Theorem 1.2.49 and the Closure Theorem 1.2.59, respectively. As I previously prompted, integral currents were introduced as a generalization of surfaces for the solution of the Plateau problem, with the mass replacing the area. Since the mass is lower semicontinuous, the Closure Theorem 1.2.59 for integral currents is the (nontrivial) part allowing for the application of the direct method of Calculus of Variations.

The room given to results concerning 1-currents only, in Section 1.3, marks the fact that we are focusing on what will be useful later, namely in Sections 3.2 and 3.3, where the main applications of the theory of currents with coefficients in a group are displayed, both of them needing 1-currents only. Not surprisingly, 1-currents are simpler than k-currents with $k \geq 2$, so, first of all, we have a structure theorem in Subsection 1.3.1: integral 1-currents are union of countably-many loops and finitely-many open curves, with integer multiplicities. Moreover, in Subsection 1.3.2 it appears for the first time a serious issue regarding the commensurability between mass-minima among different subsets of currents. We expose the problem here: as in many variational problems, in the mass-minimization problem one can choose the class of currents among which one looks for the minimizer. Since integral currents are normal currents, in general one might wonder if the minimum of the mass among normal currents is strictly less than the minimum among integral currents (that is, if some sort of Lavrentiev phenomenon occurs). This does not happen for 1-currents, for the two minima coincide. If $k \geq 2$, on the contrary, there are counterexamples to the coincidence of minima, but other problems (e.g., commensurability of minima) are still open.

Finally, Section 1.4 introduces calibrations, a powerful tool for the study of mass-minimizing currents. In this introduction, let me just say that a calibration is a closed differential form (with certain properties) associated with a submanifold or a normal current, guaranteeing, by its mere existence, that this submanifold, or current, is a mass-minimizer among cobordant competitors. Actually, finding a calibration for a candidate minimizer means having solved the mass-minimization problem! As we will see later, the calibration technique plays a prominent role in Part II.

With Chapter 2 we really start with some original results. The starting point was a problem of decomposability of normal currents proposed by F. Morgan in [1]: roughly speaking, we would like to write a normal current T as the integral $T = \int_L R_\lambda \, d\lambda$, on a suitable measure space L, of a family $(R_\lambda)_{\lambda \in L}$ of integer rectifiable currents. In doing this, we do not want to "waste" mass and we ask that $\mathbb{M}(T) = \int_L \mathbb{M}(R_\lambda) \, d\lambda$ (and, possibly, that $\mathbb{M}(\partial T) = \int_L \mathbb{M}(\partial R_\lambda) \, d\lambda$). The problem received some partial answers, which are recalled in Section 2.3, but the turning point is a counterexample to the existence of such a decomposition proposed by M. Zworski in [57]. However the proof proposed by Zworski is not correct, as pointed out by G. Alberti (see Section 4.5 of [43]). In Chapter 2 I give a correct proof of this statement.

Zworski's idea is to prove that there is no decomposition for the normal current $\xi \mathscr{L}^d$, when ξ is a non-integrable vector field. Here, non-integrability is a synonym of non-involutive vector field: in Section 2.1 we recall the Frobenius Theorem (which gives a necessary and sufficient condition on the vector field ξ in order to get a foliation by means of submanifolds having ξ as tangent space) and we explore some other useful ways of writing the Frobenius's involutivity condition on the vectorfield.

In Section 2.2 we show that an integral current behaves like a submanifold with respect to the integrability problem. In fact, if the vectorfield ξ does not fulfill the involutivity condition of Frobenius Theorem, then there is no integral current T to which ξ is tangent (almost everywhere). From this result, we get the expected conclusion in Section 2.3: since the integral currents decomposing the normal current $\xi\mathscr{L}^d$ have ξ as a tangent vectorfield, then ξ has to satisfy the condition of Frobenius Theorem, otherwise this decomposition does not exist.

As I said above, the motivations of Part II and the presentation of the results do not follow the same order, but just here I will privilege the former. The starting point is the idea of giving a variational setting and a notion of calibration to the Steiner tree problem, that is, the problem of finding the shortest connected set containing n given points in $\mathbb{R}^d$. A variational approach is advantageous whenever we are able to find a calibration for the candidate minimizers. Now the Steiner tree problem has become an application of the theory of currents with coefficients in a group of Section 3.1, it is explored in Section 3.2 and some examples of calibrations are given in Subsection 3.2.2. Both the theory of currents with coefficients in a group and the new approach to the Steiner tree problem are contained in a paper in collaboration with Andrea Marchese, see [41].

This project of replacing the Steiner tree problem with a mass-minimization problem for currents (in order to establish a notion of calibration) presents two intertwined issues: first of all, we have to provide a variational setting where the mass-minimization problem is actually equivalent to the Steiner tree problem (and prove this equivalence), secondly we must give a suitable notion of calibration on this setting and use it.

As for the first issue, from some basic examples recalled in Section 3.2 it became clear that we need integral currents with coefficients in a suitable group, different from $\mathbb{Z}$. In Section 3.2 we prove that, for every number of points n of the Steiner tree problem in $\mathbb{R}^d$, there exists a normed group G, depending only on n and generated by $g_1, \dots, g_{n-1}$, such that the support of every mass-minimizing G-current with a suitable boundary[1] is a solution of the classical Steiner problem. Vice versa, every Steiner solutions can be endowed with a structure of 1-current with coefficients in G and this current is a mass-minimizer.

It was the idea of using calibrations that forced us to develop a parallel theory of currents with coefficients in a group, in Section 3.1. In fact, the theory of flat G-chains is well established (see [27] and the subsequent

[1]By "suitable boundary" I mean something very reasonable, that is, the 0-current concentrated on the points of the Steiner tree problem with coefficients $g_1, \dots, g_{n-1}, g_n$, where $g_n = -(g_1 + \dots + g_{n-1})$.

papers [54, 55, 19, 4]) but it lacks of an integral representation device for G-currents. The integral representation provided in Section 3.1 allows us to make computations and give some examples, in Subsection 3.2.2. The notion of calibration and the question of its existence are expanded in Subsections 3.1.1 and 3.1.2. We also devote Subsection 3.2.3 and 3.2.4 to comparisons with (analogues of) the calibration in other contexts. Subsection 3.2.4 has actually some connections with Section 3.3, too.

Once the theory of currents with coefficients in a group with an integral representation was developed, we noticed that these currents are an efficient mathematical description of dislocations, as it is observed in Section 3.3 and in the paper [13] in collaboration with Sergio Conti and Adriana Garroni, which is a sort of theoretical support to [15]. Dislocations are defects occuring in a crystal under the effect of an elasto-plastic stress and currents with coefficients in $\mathbb{Z}^3$ are a good model for them.

After some technical results in Subsection 3.3.1, which are a completion of the earlier Section 1.3, we focus on the problem of minimizing the energy functional of a 1-dimensional dislocation, which is something of the form $\int_\gamma \psi(\theta, \tau)\, d\mathscr{H}^1$ for an integral $\mathbb{Z}^3$-current $[\![\gamma, \tau, \theta]\!]$. Having in mind the direct method in Calculus of Variations, we immediately notice that this functional is not lower semicontinuous, thus we have to characterize its relaxation and indeed Subsection 3.3.2 is devoted to prove that the integral of the so-called $\mathscr{H}^1$-elliptic envelope $\overline{\psi}$ is actually the lower semicontinuous envelope of the energy functional. This result is achieved by the study of a cell problem. Finally, in Subsection 3.3.3 we carry out explicit computations for the $\mathscr{H}^1$-elliptic envelope of a function ψ mentioned in [14].

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Part I

Currents

Chapter 1

A review of the theory of currents

In this first chapter we review some facts about the theory of currents. Clearly this presentation is not supposed to be exhaustive, we refer to [24] for a complete treatise (or [36] and [43] for something more user-friendly).

Our aim here is to fix the notation and recall some important background results, mainly in Section 1.1 and Section 1.2, recall the main theorems about currents (again in Section 1.2) and discuss some additional problems, as in Section 1.3.

1.1 Useful facts from Measure Theory

1.1.1 Measures and representation theorems

For the general Measure Theory see [21],[36] (chapters from 1 to 5) or [49] (chapters from 1 to 8).

Here the letter μ will always denote a positive Borel measure on $\mathbb{R}^d$. If no measure is mentioned in expressions like "almost everywhere", "negligible", "null set" and so on, we are assuming that the measure involved is the Lebesgue measure $\mathscr{L}^d$.

Concerning the notation, if f is a μ-integrable function, then we denote by $f\mu$ the Borel measure defined by

$$f\mu(A) = \int_A f(x)\,\mathrm{d}\mu(x)\,,$$

for every Borel set A. In the special case of $f = \mathbb{1}_S$, where S is a Borel set and $\mathbb{1}_S$ is its characteristic function, we will also denote the restriction of the measure μ to S by $\mu \llcorner S$, i.e.

$$\mu \llcorner S(A) := \mathbb{1}_S\mu(A) = \mu(A \cap S)\,,$$

for every Borel set A.

Definition 1.1.1. A Borel measure μ is called *Borel regular* if for every μ-measurable set A there exists a Borel set $B \supset A$ such that $\mu(B \smallsetminus A) = 0$. Moreover, we say that μ is *locally finite* if every point has a neighborhood of finite measure. A locally finite, Borel measure is called a *Radon measure*.

We endow the space $\mathscr{C}^0_c(\mathbb{R}^d)$ of continuous compactly supported functions on $\mathbb{R}^d$ with the usual topology of uniform convergence on compact sets. A functional F on $\mathscr{C}^0_c(\mathbb{R}^d)$ is called *positive* if $F(\varphi) \geq 0$ for every $\varphi \geq 0$. Thus, if μ is a locally finite positive measure on $\mathbb{R}^d$, then the map

$$\varphi \mapsto \int \varphi(x)\,\mathrm{d}\mu(x) \tag{1.1.1}$$

is a continuous, positive, linear functional on $\mathscr{C}^0_c(\mathbb{R}^d)$. Actually, every continuous, positive, linear functional on $\mathscr{C}^0_c(\mathbb{R}^d)$ admits, for some positive measure μ, a representation like in (1.1.1), as we will see in Theorem 1.1.3. Therefore it is natural to endow the space $\mathscr{M}^+(\mathbb{R}^d)$ of locally finite, positive Borel measures with the weak* topology. In particular, we say that a sequence of locally finite positive measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ converges weakly* to μ, and we write $\mu_n \overset{*}{\rightharpoonup} \mu$, if

$$\lim_n \int \varphi\, d\mu_n = \int \varphi\,\mathrm{d}\mu,$$

for every $\varphi \in \mathscr{C}^0_c(\mathbb{R}^d)$. As usual on a dual space of a separable space, the weak* topology enjoys a sequential compactness property (Banach-Alaoglu Theorem). We say that a family $\{\mu_j\}_{j\in J}$ of measures is *uniformly locally bounded* if for every compact set K there exists a constant C_K such that $\mu_j(K) \leq C_K$ for every j.

Theorem 1.1.2 (Compactness for measures)**.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of uniformly locally bounded positive measures on $\mathbb{R}^d$. Then there exists a subsequence converging to a locally finite measure μ.*

As we mentioned above, the space of locally finite positive Borel measures $\mathscr{M}^+(\mathbb{R}^d)$ coincides with the subspace of positive functionals in the dual space $(\mathscr{C}^0_c(\mathbb{R}^d))^*$, thanks to Riesz Theorem.

Theorem 1.1.3 (Riesz Theorem)**.** *Let F be a continuous positive linear functional on $\mathscr{C}^0_c(\mathbb{R}^d)$. Then there exists a locally finite, positive Borel measure μ on $\mathbb{R}^d$ such that*

$$F(\varphi) = \int \varphi\,\mathrm{d}\mu\,.$$

Since we will need it in Section 1.4, we state also a more general representation theorem. Let us consider a lattice L of functions on a metric space X, that is a real vector space containing constants and closed under infimum.

Theorem 1.1.4 (Representation Theorem). *Let (Λ_*, ν_*) be a separable normed space; we will denote by (Λ^*, ν^*) its dual space, where*

$$\nu^*(\omega) = \sup \{\langle \omega; \lambda \rangle \, : \, \lambda \in \Lambda_*, \nu_*(\lambda) \leq 1\} \ .$$

Let L be a lattice of functions on X containing a countable subset L' such that

$$\sum_{f \in L'} f(x) \geq 1 \quad \forall \, x \in X \ .$$

Finally C_* is a vector space of maps $X \to \Lambda_*$ with the following properties:

$$f \in L, \lambda \in \Lambda_* \Longrightarrow f\lambda \in C_* \tag{1.1.2}$$

$$\tau \in C_*, \alpha \in \Lambda^* \Longrightarrow \alpha \circ \tau \in L, \nu \circ \tau \in L \tag{1.1.3}$$

$$\tau \in C_*, \nu \circ \tau \geq f \in L^+ \Longrightarrow \exists \, t \in C_* \text{ s.t. } \nu \circ t = f, (\nu \circ \tau)\, t = f\tau. \tag{1.1.4}$$

Given a linear map $F : C_* \to \mathbb{R}$ with

$$D(f) := \sup \{F(\tau) \, : \, \nu \circ \tau \leq f\} < \infty$$

and such that

$$\nu \circ t_n \downarrow 0 \implies F(t_n) \to 0$$

for every sequence $(t_n)_{n \geq 0}$ in C_*, we have that D is a monotone Daniell integral and there exists a μ-measurable[1] $\hat{\omega} : X \to \Lambda^*$ such that

$$\nu^*(\hat{\omega}(x)) = 1 \quad \mu - a.e. \ x \in X$$

and

$$F(\tau) = \int_X \langle \hat{\omega}(x); \tau(x) \rangle \, d\mu(x) \quad \forall \, \tau \in C_* \ .$$

Moreover $\hat{\omega}$ is a.e. unique.

For the proof of this theorem, see the second chapter of [24], for instance.

1.1.2 Rectifiable sets

Let k be an integer with $1 \leq k \leq d$. With the symbol $\mathscr{H}^k$ we denote the k-dimensional Hausdorff measure on $\mathbb{R}^d$, i.e.

$$\mathscr{H}^k(A) := \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} \omega_k \left(\frac{\mathrm{diam} B_j}{2} \right)^k \, : \, A \subset \bigcup_{j=1}^{\infty} B_j, \, \mathrm{diam} B_j \leq \delta \right\} ,$$

with ω_k being the volume of the unit k-dimensional ball.

[1] It means that $\nu^* \circ \hat{\omega}$ is μ-measurable.

Definition 1.1.5. A set $S \subset \mathbb{R}^d$ is called *$\mathscr{H}^k$-countably k-rectifiable* (or simply *k-rectifiable*) if $S \subset \bigcup_{j=0}^{\infty} S_j$, where

(i) $\mathscr{H}^k(S_0) = 0$;

(ii) $S_j = F_j(\mathbb{R}^k)$, for $j \geq 1$, where $F_j : \mathbb{R}^k \to \mathbb{R}^d$ is a Lipschitz function.

In the following, we will always assume the rectifiable set S to be $\mathscr{H}^k$-measurable and to have $\mathscr{H}^k$-finite intersection with compact sets.

Definition 1.1.6. A set $U \subset \mathbb{R}^d$ is called *k-purely unrectifiable* if

$$\mathscr{H}^k(U \cap E) = 0\,,$$

for every k-rectifiable set E.

Thanks to the following proposition, rectifiable sets get an handy geometric structure.

Proposition 1.1.7. *An $\mathscr{H}^k$-countably rectifiable set S can be written as*

$$S = \bigcup_{j=1}^{\infty} S_j$$

where

(i) $\mathscr{H}^k(S_0) = 0$*;*

(ii) $S_i \cap S_j = \varnothing$ *if* $i \neq j$*;*

(iii) *for every $j \geq 0$, $S_j \subset \tilde{S}_j$ and $\tilde{S}_j$ is a k-dimensional submanifold of class $\mathscr{C}^1$ in $\mathbb{R}^d$.*

Definition 1.1.8. The map $\beta_{x,r} : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$\beta_{x,r}(y) := \frac{y - x}{r}\,,$$

for every $x \in \mathbb{R}^d$ and $r > 0$. This map will be useful in definitions involving the *blow up technique*, jointly with the dilation map

$$\delta_r(y) := ry\,.$$

We will stick to this notation as far as possible.

Definition 1.1.9. Consider a k-rectifiable set $S \subset \mathbb{R}^d$. An *approximate tangent space* to S at some point $x \in S$ is a k-dimensional linear subspace $W \subset \mathbb{R}^d$ such that, for every $f \in \mathscr{C}_c^0$,

$$\lim_{r \downarrow 0} \int_{\beta_{x,r}(S)} f(y)\, d\mathscr{H}^k(y) = \int_W f(y)\, d\mathscr{H}^k(y)\,.$$

When the approximate tangent space exists at some point $x \in S$, it is unique and we will denote it by $T_x S$.

There are two important facts to remark about the definition of the approximate tangent space:

- If the set S is a submanifold of class $\mathscr{C}^1$, then the tangent space and the approximate tangent space coincide at every point. Thus there is no ambiguity between the classical definition of $T_x S$ in Differential Geometry and Definition 1.1.9, respectively.

- As a matter of fact, the approximate tangent space $T_x S$ exists for $\mathscr{H}^k$-almost every point x of a k-rectifiable set S. See [36] (section 5.4) or [24] (section 3.2) for the proof.

1.1.3 Functions of bounded variation

Definition 1.1.10. Let U be an open subset of $\mathbb{R}^d$.

- A function $f \in L^1(U)$ has *bounded variation* in U if

$$\sup\left\{\int_U f \operatorname{div}\varphi \,\mathrm{d}x : \varphi \in \mathscr{C}_c^1(U;\mathbb{R}^d) \text{ with } |\varphi| \le 1\right\} < \infty\,. \tag{1.1.5}$$

  The space of functions of bounded variation is denoted by $BV(U)$.

- A $\mathscr{L}^d$-measurable set $S \subset \mathbb{R}^d$ has *finite perimeter* in U if $\mathbb{1}_S \in BV(U)$.

- A function $f \in L^1_{\text{loc}}(U)$ has *locally bounded variation* in U if (1.1.5) holds for any $V \subset\subset U$. The space of functions of locally bounded variation is denoted by $BV_{\text{loc}}(U)$.

- A $\mathscr{L}^d$-measurable set $S \subset \mathbb{R}^d$ has *locally finite perimeter* in U if $\mathbb{1}_S \in BV_{\text{loc}}(U)$.

The following theorem states that a BV-function is essentially a function whose weak first partial derivatives are Radon measures. The proof is based on Riesz Representation Theorem 1.1.4 (see, for instance, [21]).

Theorem 1.1.11. *If $f \in BV_{\mathrm{loc}}(U)$, then there exist a Radon measure μ on U and a μ-measurable function $g : U \to \mathbb{R}^d$ such that $|g(x)| = 1$ for μ-a.e. x and*

$$\int_U f \,\mathrm{div}\varphi \,\mathrm{d}x = -\int_U \varphi \cdot g \, d\mu \quad \forall\, \varphi \in \mathscr{C}^1_c(U; \mathbb{R}^d).$$

We will call the measure μ above *total variation* of f (denoted by $\|Df\|$) and we will often denote by Df the measure $g\|Df\|$. Similarly, if $f = \mathbb{1}_S$ for some locally finite perimeter set S, we will denote by $\|\partial S\|$ (*perimeter measure*) its total variation and by $\nu_S := -g$ the outward unit normal[2]. The space $BV(U)$ is naturally endowed with the norm

$$\|f\|_{BV(U)} := \|f\|_{L^1(U)} + \|Df\|(U)\,.$$

Among the properties enjoyed by BV-functions, we recall the lower semicontinuity of the total variation and the sequential compactness of any bounded subset of the space.

Theorem 1.1.12 (Lower semicontinuity of the total variation)**.** *If $f_j \in BV(U)$ and $f_j \to f$ in $L^1_{\mathrm{loc}}(U)$, then*

$$\|Df\|(U) \le \liminf_{j\to\infty} \|Df_j\|(U).$$

Theorem 1.1.13 (Compactness for BV-functions)**.** *Let U be an open, bounded subset of $\mathbb{R}^d$. If $(f_j)_{j\ge1}$ is a bounded sequence in $BV(U)$, i.e.*

$$\sup_{j\ge1} \|f_j\|_{BV(U)} < \infty$$

then there exist a subsequence $(f_{j_h})_{h\ge1}$ and a function $f \in BV(U)$ such that

$$f_{j_h} \overset{h\to\infty}{\longrightarrow} f \text{ in } L^1(U)\,.$$

Since a bounded variation function is an L^1-function, it is not immediate to define its restriction to a lower dimensional set. The following theorem extend the usual notion of trace to BV functions.

[2] ν_S is called outward unit normal because, of course, in the smooth cases, ν_S coincides with the unit normal of S, with the outward orientation.

Theorem 1.1.14. *Let U be an open, bounded subset of $\mathbb{R}^d$ with Lipschitz boundary ∂U and outer unit normal*[3] *ν. There exists a bounded linear mapping*

$$T : BV(U) \to L^1(\partial U)$$

such that

$$\int_U f \operatorname{div}\varphi \, dx = -\int_U \varphi \cdot d[Df] + \int_{\partial U} \varphi \cdot \nu T f \, d\mathscr{H}^{d-1}$$

for every $\varphi \in \mathscr{C}^1(\mathbb{R}^d; \mathbb{R}^d)$.

Definition 1.1.15. The function Tf of Theorem 1.1.14 is called the *trace* of f on ∂U and we will denote it simply by $\operatorname{trace}(f)$.

[3]The outer unit normal of ∂U exists at $\mathscr{H}^{d-1}$-a.e. point thanks to Rademacher's Theorem.

1.2 An overview on currents

1.2.1 Differential forms and Stokes's Theorem

Consider a real vector space V with finite dimension d and its dual space V^*, whose elements we will call *covectors*. The pairing of a covector $w \in V^*$ and a vector $v \in V$ will be denoted by either $\langle w; v\rangle$ or $w \cdot v$, when it is clear we are dealing with the standard Euclidean product in $\mathbb{R}^d$.

Definition 1.2.1. A *k-covector* on V is a multilinear map

$$F : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}\,,$$

the space of k-covectors is denoted by $T^k(V)$. An *l-vector* on V^* is a multilinear map

$$G : \underbrace{V^* \times \ldots \times V^*}_{l \text{ times}} \to \mathbb{R}\,,$$

the space of l-vectors is denoted by $T_l(V)$. It is also possible to consider mixed vectors in the space $T_l^k(V)$, in this case

$$H : \underbrace{V \times \ldots \times V}_{k \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{l \text{ times}} \to \mathbb{R}\,.$$

Remark 1.2.2. Trivially $T^0(V) = \mathbb{R}$, moreover $T^1(V) \cong V^*$ and $T_1(V) \cong V^{**} \cong V$. Since usually we will be considering $\mathbb{R}^d$ as our ambient space, we will often drop the distinction between the original vector space, its dual space and the space of 1-covectors on $\mathbb{R}^d$.

Definition 1.2.3. If $F \in T_l^k(V)$ and $G \in T_q^p(V)$, then $F \otimes G \in T_{l+q}^{k+p}(V)$, evaluated in $(v_1, \ldots, v_{k+p}, w_1, \ldots, w_{l+q})$, is defined as

$$F(v_1, \ldots, v_k, w_1, \ldots, w_l) G(v_{k+1}, \ldots, v_{k+p}, w_{l+1}, \ldots, w_{l+q})\,.$$

If $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ is a basis for V and $(\phi_1, \ldots, \phi_d)$ is the corresponding basis for V^* (that is, $\langle \phi_i; \mathbf{e}_j\rangle = \delta_{ij}$), then a basis for $T_l^k(V)$ is given by

$$\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_k} \otimes \phi_{j_1} \otimes \ldots \otimes \phi_{j_l}$$

with each index varying in $\{1, \ldots, d\}$.

Definition 1.2.4. A k-covector $F \in T^k(V)$ is *alternating* if, for every choice of $v_1, \dots, v_k \in V$ and for every permutation σ of the set of indices $\{1, \dots, k\}$,

$$F(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) F(v_1, \dots, v_k)\,.$$

The vector space of alternating k-covectors is denoted by $\Lambda^k(V)$. The analogous definition applies to l-vectors; the space of alternating l-vectors is denoted by $\Lambda_l(V)$.

When there is no room for misunderstatements, we will omit the word "alternating", calling k-covectors and l-vectors alternating k-covectors and alternating l-vectors, respectively.

Definition 1.2.5. For alternating covectors (and vectors) we define the *exterior product* $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ as the projection of the tensor product $\otimes$ to the space of alternating covectors (and vectors).

The exterior product is a bilinear, associative and alternating map . If $w_1, \dots, w_k \in \Lambda^1(V)$, then

$$w_1 \wedge \dots \wedge w_k(v_1, \dots, v_k) = \det(\langle w_i; v_j \rangle)\,.$$

Definition 1.2.6. Given a k-covector $v \in \Lambda^k(V)$ and an l-vector $w \in \Lambda_l(V)$, with $k \geq l$, the *interior product* is defined as

$$\langle w \lrcorner v; \hat{w} \rangle := \langle v; \hat{w} \wedge w \rangle \quad \forall\, \hat{w} \in \Lambda_{k-l}(V)\,.$$

Viceversa, if $k \leq l$, we can define

$$\langle \hat{v}; w \llcorner v \rangle := \langle v \wedge \hat{v}; w \rangle \quad \forall\, \hat{v} \in \Lambda^{l-k}(V)\,.$$

Remark 1.2.7. On a d-dimensional manifold M, the same linear algebraic definitions are possible: simply $T^k_l M := \coprod_{p \in M} T^k_l(T_p M)$ and $\Lambda^k(M) := \coprod_{p \in M} \Lambda^k(T_p M)$. Possibly, we will change the notation for the standard basis, that is

$$T_p M = \operatorname{span}(\partial_1, \dots, \partial_d),\ T_p M^* = \operatorname{span}(\mathrm{d}x_1, \dots, \mathrm{d}x_d)\,.$$

In the case $M = \mathbb{R}^d$, we keep the basis $(\mathrm{d}x_1, \dots, \mathrm{d}x_d)$ for the dual space.

Proposition 1.2.8. *A basis for $\Lambda_l(\mathbb{R}^d)$ is given by*

$$\{\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_l} : 1 \leq i_1 < \dots < i_l \leq d\}\,.$$

Analogously, a basis for the vector space $\Lambda^k(M)$ is given by

$$\{\mathrm{d}x_{i_1} \wedge \dots \mathrm{d}x_{i_k} : 1 \leq i_1 < \dots < i_k \leq d\}\,.$$

For every positive integer $k \leq d$, we will denote by $\mathcal{I}(d,k)$ the set of multi-indices $\mathbf{i} = (i_1, \ldots, i_k)$, with $1 \leq i_1 < \ldots < i_k \leq d$. Associate with every index $\mathbf{i} \in \mathcal{I}(d,k)$ we will often use the formal expression

$$\mathrm{d}x_{\mathbf{i}} = \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} .$$

Once we fixed a basis, there is canonical way to identify $\Lambda_l \cong \Lambda^{d-l}$ and $\Lambda^k \cong \Lambda_{d-k}$.

Definition 1.2.9. The *Hodge star operator* $\mathbb{D}_l : \Lambda_l(\mathbb{R}^d) \to \Lambda^{d-l}(\mathbb{R}^d)$ is the isomorphism associating

$$\xi \;\mapsto\; \xi \,\lrcorner\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d .$$

Similarly, the canonical isomorphism $\mathbb{D}^k : \Lambda^k(\mathbb{R}^d) \to \Lambda_{d-k}(\mathbb{R}^d)$ associates

$$\omega \;\mapsto\; \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d \,\llcorner\, \omega .$$

Remark 1.2.10. Let us notice that $\mathbb{D}_l$ is the unique linear map such that

$$\mathbb{D}_l \mathbf{e}_{\mathbf{i}} := (-1)^{\sigma_{\mathbf{i}}} \bigwedge_{j \notin \mathbf{i}} \mathrm{d}x_j , \tag{1.2.1}$$

for every $\mathbf{i} \in \mathcal{I}(d,l)$, where the exponent $\sigma_{\mathbf{i}}$ is defined as

$$\sigma_{\mathbf{i}} = \sum_{h=1}^{l} \left((d-l) - (i_h - h)\right) .$$

Therefore

$$(\mathbb{D}_l \mathbf{e}_{\mathbf{i}}) \wedge \mathrm{d}x_{\mathbf{i}} = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d \quad \forall\, \mathbf{i} \in \mathcal{I}(d,l) .$$

Dually

$$\mathbb{D}^k \mathrm{d}x_{\mathbf{i}} := (-1)^{\sigma'_{\mathbf{i}}} \bigwedge_{j \notin \mathbf{i}} \mathbf{e}_j ,$$

for every $\mathbf{i} \in \mathcal{I}(d,k)$, with

$$\sigma'_{\mathbf{i}} := \sum_{h=1}^{k} (i_h - h) .$$

Moreover $\mathbb{D}_l$ and $\mathbb{D}^{d-l}$ are inverse to each other.

Definition 1.2.11. An l-vector v is called *simple* if there exist $v_1, \ldots, v_l \in V$ such that

$$v = v_1 \wedge \ldots \wedge v_l \,.$$

Analogously, a k-covector w is called *simple* if there exist $w_1, \ldots, w_k \in V^*$ such that

$$w = w_1 \wedge \ldots \wedge w_k \,.$$

Remark 1.2.12. In general, k-covectors (and l-vectors, as well) are not simple: for example, the 2-vector

$$v := \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4 \in \Lambda_2(\mathbb{R}^4)$$

is not. If it were simple, then there should be $v_1, v_2 \in \mathbb{R}^4$ such that $v = v_1 \wedge v_2$. But then $v \wedge v = (v_1 \wedge v_2) \wedge (v_1 \wedge v_2) = 0$, on the contrary, an easy computation shows that $v \wedge v = 2(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4) \neq 0$.

Remark 1.2.13. Simple unit vectors are the correct formalism to represent homogeneous k-dimensional oriented planes[4]. In fact, it turns out that the simple vector $v = v_1 \wedge \ldots \wedge v_k$ is null if and only if the v_i are linearly dependent. Moreover if $\mathrm{Span}\{v'_1, \ldots, v'_k\} = \mathrm{Span}\{v_1, \ldots, v_k\}$, then $v'_1 \wedge \ldots \wedge v'_k = \lambda(v_1 \wedge \ldots \wedge v_k)$ for some $\lambda \in \mathbb{R}$.

Definition 1.2.14. In addition to the Euclidean norm $|\cdot|$, on $\Lambda_k(\mathbb{R}^d)$ and $\Lambda^k(\mathbb{R}^d)$ we are going to consider the *mass norm* $\|\cdot\|$ on k-vectors and the *comass norm* $\|\cdot\|^*$ on k-covectors defined as follows:

$$\begin{aligned} \|w\|^* &:= \sup\{|\langle w; v\rangle| : v \text{ is a simple } k-\text{vector with } |v| = 1\}\,, \\ \|v\| &:= \sup\{|\langle w; v\rangle| : \|w\|^* = 1\}\,. \end{aligned}$$

Remark 1.2.15. From the definition above, it is clear that, if a k-vector v is simple, then the Euclidean norm $|v|$ and the mass norm $\|v\|$ coincide.

Definition 1.2.16. A *differential k-form* ω on $\mathbb{R}^d$ is a k-covectorfield, that is a map

$$\omega : \mathbb{R}^d \to \Lambda^k(\mathbb{R}^d)\,.$$

[4]We will call homogeneous or linear k-plane a k-dimensional linear subspace of $\mathbb{R}^d$ and, when V is a linear k-plane and $x \in \mathbb{R}^d$, the set $x + V$ will be called an affine k-plane. We will often use simply the word "k-plane", when there is no ambiguity.

Using the standard basis of $\Lambda^k(\mathbb{R}^d)$, we can write ω as

$$\omega(x) = \sum_{\mathbf{i}\in\mathcal{I}(d,k)} \omega_{\mathbf{i}}(x)\,\mathrm{d}x_{\mathbf{i}}\,,$$

where the coordinates $\omega_{\mathbf{i}}$ are real valued functions on $\mathbb{R}^d$. We will say that a differential k-form has a certain regularity, when the coordinate functions have that regularity. As usual, the *support* of a differential k-form ω is defined as the closure of the set $\{x \in \mathbb{R}^d : \omega(x) \neq 0\}$ and we will denote it as $\mathrm{supp}(\omega)$.

Definition 1.2.17. The *exterior derivative* of a differential k-form ω of class $\mathscr{C}^1$ is the differential $(k+1)$-form:

$$\mathrm{d}\omega(x) = \sum_{\mathbf{i}\in\mathcal{I}(d,k)} \mathrm{d}\omega_{\mathbf{i}} \wedge \mathrm{d}x_{\mathbf{i}},$$

where

$$\mathrm{d}\omega_{\mathbf{i}}(x) = \sum_{j=1}^{d} \frac{\partial \omega_{\mathbf{i}}}{\partial x_j}(x)\,\mathrm{d}x_j\,.$$

Definition 1.2.18. A k-form ω is said to be *closed* if $\mathrm{d}\omega = 0$. Moreover, if there exists a $(k-1)$-form ψ such that $\mathrm{d}\psi = \omega$, then ω is an *exact* k-form.

Remark 1.2.19. One can check, by means of a simple computation, that $\mathrm{d}^2 \equiv 0$. Since $\mathrm{d}^2 \equiv 0$, an exact form is always closed.

Definition 1.2.20. Given a k-form $\omega \in \Lambda^k(\mathbb{R}^d)$, one can always define the *differential* $D\omega \in \Lambda^1(\mathbb{R}^d) \otimes \Lambda^k(\mathbb{R}^d)$ as

$$D\omega := \sum_{j=1}^{d} \sum_{\mathbf{i}\in\mathcal{I}(d,k)} \frac{\partial \omega_{\mathbf{i}}}{\partial x_j}(x)\,\mathrm{d}x_j \otimes \mathrm{d}x_{\mathbf{i}}\,.$$

Analogously, given an l-vectorfield $\xi \in \Lambda_l(\mathbb{R}^d)$, we define $D\xi \in \Lambda_l(\mathbb{R}^d)\otimes\Lambda^1(\mathbb{R}^d)$ as

$$D\xi := \sum_{j=1}^{d} \sum_{\mathbf{i}\in\mathcal{I}(d,l)} \frac{\partial \xi_{\mathbf{i}}}{\partial x_j}(x)\,\mathbf{e}_{\mathbf{i}} \otimes \mathrm{d}x_j\,.$$

Remark 1.2.21. Given a k-form ω, its exterior derivative $\mathrm{d}\omega$ is the image of $D\omega$ under the linear map induced by the exterior multiplication

$$\Lambda^1(\mathbb{R}^d) \otimes \Lambda^k(\mathbb{R}^d) \xrightarrow{\wedge} \Lambda^{k+1}(\mathbb{R}^d)\,.$$

This is precisely the meaning of Definition 1.2.17.

Definition 1.2.22. Given an l-vectorfield $\xi : \mathbb{R}^d \to \Lambda_l(\mathbb{R}^d)$, we define $\mathrm{div}\xi$ as the image of $D\xi(x)$ under the linear map induced by interior multiplication

$$\Lambda_l(\mathbb{R}^d) \otimes \Lambda^1(\mathbb{R}^d) \xrightarrow{\llcorner} \Lambda_{l-1}(\mathbb{R}^d)\,.$$

This means

$$\mathrm{div}\xi = \sum_{j=1}^{d} \frac{\partial \xi}{\partial x_j} \llcorner \mathrm{d}x_j\,.$$

Remark 1.2.23. We suggest Section 4.1.6 of [24] for a more detailed description of Definition 1.2.22. Nevertheless, it will be useful for coming computations to see how Definition 1.2.22 works in coordinates. Assume the l-vectorfield $\xi \in \Lambda_l(\mathbb{R}^d)$ is written as

$$\xi(x) = \sum_{\mathbf{i} \in \mathcal{I}(d,l)} \xi_{\mathbf{i}}(x) \mathbf{e}_{\mathbf{i}}\,,$$

thus

$$\mathrm{div}\xi(x) = \sum_{\mathbf{i} \in \mathcal{I}(d,l)} \sum_{j=1}^{d} \frac{\partial \xi_{\mathbf{i}}}{\partial x_j}(x)\, \mathbf{e}_{\mathbf{i}} \llcorner \mathrm{d}x_j = \sum_{\mathbf{i} \in \mathcal{I}(d,l)} \sum_{h=1}^{l} (-1)^{h-1} \frac{\partial \xi_{\mathbf{i}}}{\partial x_{i_h}}(x)\, \mathbf{e}_{\hat{\mathbf{i}}_h}\,, \qquad (1.2.2)$$

where $\mathbf{e}_{\hat{\mathbf{i}}_h}$ is a contraction of the $(l-1)$-vector

$$\mathbf{e}_{\hat{\mathbf{i}}_h} := \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{h-1}} \wedge \mathbf{e}_{i_{h+1}} \wedge \ldots \wedge \mathbf{e}_{i_l}\,.$$

Notice that, if $l = 1$, then (1.2.2) coincides with the usual definition of the divergence.

Proposition 1.2.24. *Consider a smooth l-vectorfield $\xi : \mathbb{R}^d \to \Lambda_l(\mathbb{R}^d)$, then*

$$\mathrm{d}\left(\mathbb{D}_l \xi\right) = (-1)^{d-l} \mathbb{D}_{l-1}(\mathrm{div}\xi)\,. \qquad (1.2.3)$$

Proof. Thanks to (1.2.2), the right-hand side of (1.2.3) becomes

$$(-1)^{d-l} \mathbb{D}_{l-1}\left(\mathrm{div}\xi\right) = \sum_{\mathbf{i} \in \mathcal{I}(d,l)} \sum_{h=1}^{l} (-1)^{d-l+h-1} \frac{\partial \xi_{\mathbf{i}}}{\partial x_{i_h}}\, \mathbb{D}_{l-1}(\mathbf{e}_{\hat{\mathbf{i}}_h})\,;$$

on the opposite side we can compute

$$\mathrm{d}\left(\mathbb{D}_l \xi\right) = \sum_{\mathbf{i} \in \mathcal{I}(d,l)} \sum_{h=1}^{l} \frac{\partial \xi_{\mathbf{i}}}{\partial x_{i_h}}\, \mathrm{d}x_{i_h} \wedge \mathbb{D}_l(\mathbf{e}_{\mathbf{i}})$$

and the thesis is equivalent to prove that

$$\mathrm{d}x_{i_h} \wedge \mathbb{D}_l(\mathbf{e_i}) = (-1)^{d-l+h-1}\mathbb{D}_{l-1}(\mathbf{e}_{\hat{\mathbf{i}}_h}), \tag{1.2.4}$$

for every $\mathbf{i} \in \mathcal{I}(d,l)$ and for every $h = 1, \ldots, l$. Clearly, it is just a matter of sign, indeed

$$\begin{aligned} \mathrm{d}x_{i_h} \wedge \mathbb{D}_l(\mathbf{e_i}) &= (-1)^{\alpha_{\mathbf{i}}} \bigwedge_{j \notin (\mathbf{i} \setminus h)} \mathrm{d}x_j \\ \mathbb{D}_{l-1}(\mathbf{e}_{\hat{\mathbf{i}}_h}) &= (-1)^{\beta_{\mathbf{i}}} \bigwedge_{j \notin (\mathbf{i} \setminus h)} \mathrm{d}x_j \,. \end{aligned}$$

Thanks to (1.2.1), we get

$$\alpha_{\mathbf{i}} = \sigma_{\mathbf{i}} + (i_h - h)\,,$$

while

$$\begin{aligned} \beta_{\mathbf{i}} &= \sigma_{\hat{\mathbf{i}}_h} = \sum_{s=1}^{h-1}((d-l+1)-(i_s-s)) + \sum_{s=h+1}^{l}((d-l)-(i_s-s)) \\ &= \sigma_{\mathbf{i}} + (i_h - h) - (d-l) + (h-1)\,. \end{aligned}$$

This proves the claim in (1.2.4). □

Remark 1.2.25. As it happens for usual differentiation, there are useful formulas for the exterior derivative and for the divergence of a product. The most popular one is the following

$$\mathrm{d}\,(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2\,, \tag{1.2.5}$$

for every pair of forms $\omega_1 \in \Lambda^{k_1}(\mathbb{R}^d)$ and $\omega_2 \in \Lambda^{k_2}(\mathbb{R}^d)$. Given a vectorfield $\xi \in \Lambda_l(\mathbb{R}^d)$ and a form $\omega \in \Lambda^k(\mathbb{R}^d)$, with $k > l$, we have that

$$\mathrm{d}(\xi \lrcorner \omega) = \xi \lrcorner \mathrm{d}\omega + (-1)^{k-l}\mathrm{div}\xi \lrcorner \omega\,. \tag{1.2.6}$$

Moreover, consider a pair of vectorfield $\xi_1 \in \Lambda_{l_1}(\mathbb{R}^d)$ and $\xi_2 \in \Lambda_{l_2}(\mathbb{R}^d)$, thus

$$\mathrm{div}(\xi_1 \wedge \xi_2) = \mathrm{div}\xi_1 \wedge \xi_2 + (-1)^{l_1}\xi_1 \wedge \mathrm{div}\xi_2\,.$$

This formula can be easily deduced from the analogous property of the exterior derivative stated in (1.2.5), through (1.2.3): indeed

$$\mathbb{D}_{l_1+l_2-1}(\mathrm{div}(\xi_1 \wedge \xi_2)) = (-1)^{d-(l_1+l_2)}\mathrm{d}\,(\mathbb{D}_{l_1+l_2}(\xi_1 \wedge \xi_2))\,,$$

but, by definition,

$$\mathbb{D}_{l_1+l_2}(\xi_1 \wedge \xi_2) = \xi_1 \lrcorner \mathbb{D}_{l_2}(\xi_2) = -\xi_2 \lrcorner \mathbb{D}_{l_1}(\xi_1)\,,$$

so

$$\begin{aligned} \mathbb{D}_{l_1+l_2-1}(\mathrm{div}(\xi_1 \wedge \xi_2)) &= \mathrm{div}\xi_1 \lrcorner \mathbb{D}_{l_2}\xi_2 + (-1)^{l_1}\xi_1 \lrcorner \mathbb{D}_{l_2-1}(\mathrm{div}\xi_2) \qquad (1.2.7)\\ &= \mathbb{D}_{l_1+l_2-1}(\mathrm{div}\xi_1 \wedge \xi_2) + (-1)^{l_1}\mathbb{D}_{l_1+l_2-1}(\xi_1 \wedge \mathrm{div}\xi_2)\,, \end{aligned}$$

where (1.2.7) is motivated by (1.2.6).

Remark 1.2.13 establishes a one-to-one correspondence between simple k-vectors with unit Euclidean norm and oriented k-dimensional vector subspaces of $\mathbb{R}^d$. This fact motivates the following definition.

Definition 1.2.26. An *orientation* of a k-dimensional surface S of class $\mathscr{C}^1$ is a continuous map $\tau_S : S \to \Lambda_k(\mathbb{R}^d)$ such that $\tau_S(x)$ is a simple unit k-vector spanning T_xS for every $x \in S$. If there exists an orientation of S, then there is a canonical orientation for the boundary of S, namely the one satisfying

$$\tau_S(x) = \nu(x) \wedge \tau_{\partial S}(x) \text{ for every } x \in \partial S\,, \qquad (1.2.8)$$

where ν is the outer normal to ∂S.

Definition 1.2.27. We define the integral of a differential k-form ω on an oriented k-surface S as follows:

$$\int_S \omega = \int_S \langle \omega(x); \tau_S(x)\rangle \,\mathrm{d}\mathscr{H}^k(x)\,.$$

Theorem 1.2.28 (Stokes's Theorem)**.** *For every oriented surface S of dimension k and for every $(k-1)$-form of class at least $\mathscr{C}^1$, the following relation holds:*

$$\int_{\partial S} \omega = \int_S \mathrm{d}\omega\,, \qquad (1.2.9)$$

where the orientation of ∂S has been clarified in (1.2.8).

See [36] or [38] or any other textbook about the integration on manifolds for the proof.

Remark 1.2.29. Stokes's Theorem 1.2.28 is a generalization of the Fundamental Theorem of Calculus and it summarizes, with the notation of differential forms, the Gauss-Green Theorem (for $\omega \in \Lambda^1(\mathbb{R}^2)$), the Curl Theorem (for $\omega \in \Lambda_1(\mathbb{R}^3)$) and the Divergence Theorem, in fact, if $\xi \in \Lambda_1(\mathbb{R}^d)$, then

$$\int_S \langle \nu; \xi \rangle = \int_S \mathbb{D}_1 \xi = \int_{\partial S} \mathrm{d}(\mathbb{D}_1 \xi) = (-1)^{d-1} \int_{\partial S} \mathrm{div} \xi$$

thanks to Proposition 1.2.3.

Finally we define the pull-back of a differential k-form on $\mathbb{R}^{d'}$ under a smooth map $f : \mathbb{R}^d \to \mathbb{R}^{d'}$.

Definition 1.2.30. For any simple k-vector $v = v_1 \wedge \ldots \wedge v_k \in \Lambda_k(\mathbb{R}^d)$ and a point $x \in \mathbb{R}^d$, define the push-forward of v as the simple k-vector

$$\mathrm{d}f_\sharp(v) = Df(x)v_1 \wedge \ldots \wedge Df(x)v_k \, .$$

This map is extended to all k-vectors by linearity. Then, for any differential k-form ω on $\mathbb{R}^{d'}$, define its *pull-back* $f^\sharp \omega$ on $\mathbb{R}^d$ by

$$\langle f^\sharp \omega(x); v \rangle = \langle \omega(f(x)); \mathrm{d}f_\sharp(v) \rangle \quad \forall\, x \in \mathbb{R}^d \ \forall\, v \in \Lambda_k(\mathbb{R}^d) \, . \tag{1.2.10}$$

We conclude this subsection with a clarification on the topology of the *space of smooth compactly supported differential k-forms on U*, denoted by $\mathscr{D}^k(U)$, where $U \subset \mathbb{R}^d$ is an open subset.

Definition 1.2.31. Consider a sequence $(\omega^{(n)})_{n \geq 1}$ of smooth compactly supported k-forms on U. In coordinates we will write

$$\omega^{(n)}(x) = \sum_{\mathbf{i} \in \mathcal{I}(d,k)} \omega^{(n)}_{\mathbf{i}}(x) \, \mathrm{d}x_{\mathbf{i}}$$

for every $n \geq 1$. We have a limit

$$\omega^{(n)} \longrightarrow \omega \quad \text{as } n \to +\infty$$

if the following conditions are satisfied for a fixed compact subset $K \subset U$:

(i) $\mathrm{supp}\, \omega^{(n)}_{\mathbf{i}} \subset K$ for any $\mathbf{i} \in \mathcal{I}(d,k)$ and for any $n \geq 1$;

(ii) $D^{\mathbf{j}} \omega^{(n)}_{\mathbf{i}} \longrightarrow D^{\mathbf{j}} \omega_{\mathbf{i}}$ uniformly in K for every $\mathbf{i} \in \mathcal{I}(d,k)$ and for every choice of the multi-index $\mathbf{j}$.

The topology induced on $\mathscr{D}^k(U)$ by the convergence in Definition 1.2.31 is locally convex and follows the definition of the topology on the space $\mathscr{D}(U)$ of smooth compactly supported functions.

1.2.2 Currents

Currents are the right way to generalize submanifolds, providing compactness properties for some bounded subsets of currents and a "good behavior" in limit processes, namely lower semicountinuity of the mass, which is the concept that generalizes the area of a submanifold. At first, they appeared in [50] and [20], but the theory was fully developed in the sixties with the works of H. Federer and W. Fleming (see [26],[24]).

Definition 1.2.32. Let $U \subset \mathbb{R}^d$ be an open subset, the dual of the space of k-forms $\mathscr{D}^k(U)$, endowed with the topology in Definition 1.2.31, is denoted by $\mathscr{D}_k(U)$ and it is called the space of *k-dimensional currents* (or simply k-currents). As usual $\mathscr{D}_k(U)$ is endowed with its weak* topology. In particular we will say that a sequence of k-currents $(T_n)_{n\in\mathbb{N}}$ converges to a k-current T, and we write $T_n \overset{*}{\rightharpoonup} T$, if they converge in the weak* topology, that is:

$$\langle T_n; \omega\rangle \to \langle T; \omega\rangle$$

for every $\omega \in \mathscr{D}^k(U)$.

Notice that we will use both notations $T(\omega)$ and $\langle T; \omega\rangle$ for the action of $T \in \mathscr{D}_k(U)$ on a test form $\omega \in \mathscr{D}^k(U)$.

Remark 1.2.33. Since a smooth, compactly supported differential 0-form is a function in the class $\mathscr{C}_c^\infty(U)$, the space $\mathscr{D}_0(U)$ of 0-currents coincides with the space of distributions[5] and the topology is the same, as well. Roughly speaking, a k-current is a distribution carrying a higher dimensional geometrical structure.

Example 1.2.34. The simplest example of a k-current on $\mathbb{R}^d$ is the one defined *by integration* of a $\mathscr{L}^d$-measurable map $\xi : U \to \Lambda_k(\mathbb{R}^d)$ with $|\xi| \in L^1(U)$. In fact, we can set

$$\forall\, \omega \in \mathscr{D}^k(U), \quad T_\xi(\omega) := \int_U \langle \omega(x); \xi(x)\rangle\, d\mathscr{L}^d(x)\,.$$

Obviously the Lebesgue measure $\mathscr{L}^d$ can be replaced by a measure μ. Thus, it is also possible to define a *k-current associated with an oriented k-dimensional surface* S of class $\mathscr{C}^1$: we will denote such a current by $[S]$. If τ is the tangent k-vector carrying the orientation of S, then

$$\forall\, \omega \in \mathscr{D}^k(U), \quad [S](\omega) := \int_S \langle \omega(x); \tau(x)\rangle\, d\mathscr{H}^k(x)\,.$$

[5] See [9] for an o verview on distributions.

This motivates some authors to use the terminology "generalized surfaces" when they introduce currents.

By duality, the operations for differential forms in definitions 1.2.5 and 1.2.6 have their counterpart for currents. If $\xi \in \mathscr{C}^\infty(U;\Lambda_h(\mathbb{R}^d))$, then

$$\forall\, \Phi \in \mathscr{D}^{k+h}(U), \quad \langle T \wedge \xi; \Phi\rangle := \langle T; \xi \lrcorner \Phi\rangle\,.$$

Analogously, if $T \in \mathscr{D}_k(U)$ and $\psi \in \mathscr{C}^\infty(\mathbb{R}^d, \Lambda^h(\mathbb{R}^d))$, with $h \le k$, then

$$\forall\, \varphi \in \mathscr{D}^{k-h}(U), \quad \langle T \llcorner \psi; \varphi\rangle := \langle T; \psi \wedge \varphi\rangle\,.$$

As for measures, if $A \subset \mathbb{R}^d$ and $T \in \mathscr{D}_k(U)$, by $T \llcorner A$ we mean $T \llcorner \mathbb{1}_A \in \mathscr{D}_k(U)$.

Moreover, if $f : U \subset \mathbb{R}^d \to U' \subset \mathbb{R}^{d'}$ is a proper smooth map, then it is possible to define the *push-forward* of a k-current T on $U \subset \mathbb{R}^d$ as the k-current $f_\sharp T$ on $\mathbb{R}^{d'}$ defined by

$$\langle f_\sharp T; \omega\rangle = \langle T; f^\sharp \omega\rangle\,,$$

for any $\omega \in \mathscr{D}^k(U')$.

Definition 1.2.35. The *support* of a k-current T in $\mathscr{D}_k(U)$, with $U \subset \mathbb{R}^d$, is the set

$$\mathrm{supp}(T) := \mathbb{R}^d \smallsetminus \bigcup \{W \subset U,\, W \text{ open} : \omega \in \mathscr{D}^k(U),\, \mathrm{supp}(\omega) \subset W \Rightarrow T(\omega) = 0\}.$$

Definition 1.2.36. The *boundary* ∂T of a k-current T is the $(k-1)$-current defined by

$$\langle \partial T; \varphi\rangle = \langle T; \mathrm{d}\varphi\rangle\,,$$

for every $\varphi \in \mathscr{D}^{k-1}(U)$.

Remark 1.2.37. The definition of boundary given in 1.2.36 is a natural one, as the following facts suggest.

- It is immediate to see that $\partial^2 T = 0$, because $\mathrm{d}^2\varphi = 0$ for every $\varphi \in \mathscr{D}^{k-2}(U)$.
- By Stokes's Theorem (see Theorem 1.2.28), Definition 1.2.36 agrees with the usual definition of boundary if S is an oriented surface of class $\mathscr{C}^1$ and $T = [S]$ (meaning that $\partial[S] = [\partial S]$).

- The boundary operator commutes with push-forward, in fact $\partial(f_\sharp T) = f_\sharp(\partial T)$.

Definition 1.2.38. Given an open set $V \subset U$, the *mass* of a current T in V is the quantity

$$\mathbb{M}_V(T) = \sup\{\langle T;\omega\rangle : \omega \in \mathscr{D}^k(V), \|\omega(x)\|^* \le 1 \text{ for every } x\}\,.$$

We recall that, by $\|\cdot\|^*$, we mean the comass norm of Definition 1.2.14. When $U = V$, we just write $\mathbb{M}(T)$.

Remark 1.2.39. If S is an oriented k-dimensional surface of class $\mathscr{C}^1$ and $[S]$ is the associated current, we have $\mathbb{M}([S]) = \mathscr{H}^k(S)$. Therefore the mass can be considered a natural extension to k-currents of the notion of k-volume.

Remark 1.2.40. It is easy to show that the mass is lower semicontinuous with respect to the weak* topology, i.e.

$$T_n \overset{*}{\rightharpoonup} T \quad \Longrightarrow \quad \mathbb{M}_V(T) \le \liminf_{n\to\infty} \mathbb{M}_V(T_n)\,,$$

for any sequence $(T_n)_{n\ge1}$ and for any open subset $V \subset U$.

Remark 1.2.41. In Definition 1.2.32 we endowed $\mathscr{D}^k(U)$ with a topology which is stronger than the one induced by the comass norm

$$\sup_{x\in U} \|\omega(x)\|^*\,.$$

Therefore a current may have (even locally) infinite mass. As an example, consider the 0-current T on $\mathbb{R}$ such that

$$T(\varphi) = \varphi'(x_0) \quad \forall\, \varphi \in \mathscr{D}(U)\,,$$

with $x_0 \in U$. Concerning $\mathscr{D}'(U)$, it turns out that a distribution has finite mass if and only if it has order 0 (in the example above, the distribution has order 1).

If a k-current T has locally finite mass, then, by means of Riesz Theorem 1.1.4, it can be represented by integration, as in Example 1.2.34. So there exist a positive finite measure μ_T on $\mathbb{R}^d$ and a Borel measurable map $\tau : \mathbb{R}^d \to \Lambda_k(\mathbb{R}^d)$ with $\|\tau\| = 1$ μ_T-a.e., such that

$$\langle T;\omega\rangle = \int_{\mathbb{R}^d} \langle \omega(x);\tau(x)\rangle \,\mathrm{d}\mu_T(x)\,,$$

for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$. The mass of T equals the mass[6] of the measure μ_T, which we will often call *total variation measure* associated with T. Sometimes μ_T is also denoted by $\|T\|$. Moreover the orientation τ is the Radon-Nikodym derivative $dT/d\|T\|$ (sometimes τ is also denoted by $\overrightarrow{T}$).

Definition 1.2.42. A k-current $T \in \mathscr{D}_k(U)$ is called *normal* if both T and ∂T have finite mass, i.e. $\mathbb{M}(T) < \infty$ and $\mathbb{M}(\partial T) < \infty$. The space of normal k-currents is denoted by $\mathbb{N}_k(U)$.

Remark 1.2.43. As we observed above, thanks to Riesz Theorem 1.1.4, a normal current admits an integral representation for both the current itself and its boundary.

Now we focus on rectifiable and integral currents, which are the kinds of currents we are really interested in, we postpone to Subsection 1.2.3 the missing definitions about special sets of currents.

Definition 1.2.44. Given a k-rectifiable set Σ, an orientation[7] τ of the set Σ and a real-valued function θ such that $\int_\Sigma \theta(x)\, d\mathscr{H}^k(x) < \infty$, we define $T = [\![\Sigma, \tau_\Sigma, \theta]\!]$ as

$$\langle T;\omega\rangle = \int_\Sigma \langle \omega(x); \tau_\Sigma(x)\rangle\, \theta(x)\, d\mathscr{H}^k(x)\,. \tag{1.2.11}$$

A k-current T is called *rectifiable* if T admits a representation as in (1.2.11). The function θ is often called *multiplicity* of the current $[\![\Sigma, \tau, \theta]\!]$.

A rectifiable current whose multiplicity takes only integral values is called an *integer multiplicity rectifiable current*. The abelian group[8] of integer multiplicity rectifiable k-currents is denoted by $\mathscr{R}_k(U)$.

If both T and ∂T are integer multiplicity rectifiable currents, then T is called an *integral current* and the corresponding space is denoted by $\mathscr{I}_k(U)$.

---

[6] Let us recall that we are considering $\Lambda_k(\mathbb{R}^d)$ endowed with the mass norm of Definition 1.2.14.

[7] An orientation τ_Σ of Σ is a Borel function such that, for every $x \in \Sigma$, $\tau(x)$ is a simple unit k-vector spanning the approximate tangent space $T_x\Sigma$. We recall that every k-rectifiable set Σ admits a weak tangent field.

[8] The set of integer multiplicity rectifiable currents is not a real vector space. We will denote by italic capital letters abelian groups and by blackboard bold (double struck) letters real vector spaces, sticking to the notation in [24].

Remark 1.2.45. If T is a rectifiable k-current with multiplicity $\theta \in L^1(U;\mathbb{R})$, then we can compute its mass through the equality

$$\mathbb{M}(T) = \int_\Sigma |\theta(x)|\, d\mathscr{H}^k(x)\,. \tag{1.2.12}$$

Remark 1.2.46. An integer multiplicity rectifiable 0-current T in $\mathbb{R}^d$ admits the following representation:

$$T = \sum_{i=1}^{k} m_i \delta_{x_i}\,,$$

where x_i are points in $\mathbb{R}^d$, $m_i \in \mathbb{Z}$ and δ_{x_i} is the Dirac mass at x_i. This means that the action of T on a smooth compactly supported function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is

$$\langle T;\varphi\rangle = \sum_{i=1}^{k} m_i \varphi(x_i)\,.$$

1.2.3 Polyhedral chains and approximation theorems

In the previous subsection, we introduced currents in a very abstract way, as the elements of a dual space with a slightly complicated topology. After that, we defined smaller subgroups of $\mathscr{D}_k(U)$ with better representation properties. It is also possible to go through the theory of currents in the other way, starting from the simplest objects (polyhedral chains of Definition 1.2.47) and getting back the to whole abstract space by completion with respect to a suitable norm (the flat norm in Definition 1.2.53). Following this outline, we will complete the picture of the relevant subgroups of $\mathscr{D}_k(\mathbb{R}^d)$ and the ways they relate to each other, exploiting the Deformation Theorem 1.2.49 and Approximation Theorems 1.2.51 and 1.2.57.

Definition 1.2.47. We define $\mathscr{P}_k(\mathbb{R}^d) \subset \mathscr{D}_k(\mathbb{R}^d)$ as the additive subgroup generated by all k-dimensional oriented simplexes in $\mathbb{R}^d$. The currents in $\mathscr{P}_k(\mathbb{R}^d)$ are called *integral polyhedral chains*.

The vector space generated by $\mathscr{P}_k(\mathbb{R}^d)$ is the space of *polyhedral chains* and it is denoted by $\mathbb{P}_k(\mathbb{R}^d)$.

Remark 1.2.48. Connecting Definition 1.2.47 and Definition 1.2.44, a polyhedral current $T \in \mathbb{P}_k(\mathbb{R}^d)$ is nothing but a rectifiable current of the form

$$T = \sum_{i=1}^{n} [\![S_i, \tau_i, \theta_i]\!]\,,$$

where S_i is a k-dimensional simplex in $\mathbb{R}^d$, τ_i is a constant orientation of S_i and θ_i is a constant multiplicity.

Theorem 1.2.49 (Deformation Theorem)**.** *Fix $\varepsilon > 0$ and consider the lattice $\mathcal{L}^d_{k,\varepsilon}$ with step ε and dimension $k \le d$. Consider a normal current $T \in \mathbb{N}_k(\mathbb{R}^d)$. Then we can write*

$$T = P + R + \partial S\,, \tag{1.2.13}$$

where

- *$P \in \mathbb{P}_k(\mathbb{R}^d)$ is not only a polyhedral k-chain with support in $\mathcal{L}^d_{k,\varepsilon}$, but it is written as*
  $$P = \sum_{F \in \mathcal{L}^d_{k,\varepsilon}} [\![F, \tau_F, \theta_F]\!]$$
  with constant coefficients $\theta_F \in \mathbb{R}$. Moreover
  $$\mathbb{M}(P) \le C\mathbb{M}(T) \text{ and } \mathbb{M}(\partial P) \le C\mathbb{M}(\partial T)\,;$$
- *$R \in \mathscr{D}_k(\mathbb{R}^d)$ has $\mathbb{M}(R) \le \varepsilon C\mathbb{M}(\partial T)$;*
- *$S \in \mathbb{N}_{k+1}(\mathbb{R}^d)$ has $\mathbb{M}(S) \le \varepsilon C\mathbb{M}(T)$.*

The constant C depends only on k and d. Moreover

$$\begin{aligned} \operatorname{supp}\partial P \cup \operatorname{supp} R &\subset [\operatorname{supp}\partial T]_{2\varepsilon\sqrt{d}} \\ \operatorname{supp} P \cup \operatorname{supp} S &\subset [\operatorname{supp} T]_{2\varepsilon\sqrt{d}}\,. \end{aligned}$$

Finally, if $T \in \mathscr{R}_k(\mathbb{R}^d)$ is an integer multiplicity rectifiable current, then so are P and S. If $\partial T \in \mathscr{R}_{k-1}(\mathbb{R}^d)$, then $R \in \mathscr{R}_k(\mathbb{R}^d)$ and, if $\partial T \in \mathscr{P}_{k-1}(\mathbb{R}^d)$, then $R \in \mathscr{P}_k(\mathbb{R}^d)$.

See Chapter 7 of [36] for the proof of the Deformation Theorem 1.2.49 and for the proof of the following isoperimetric inequality, as well.

Theorem 1.2.50 (Isoperimetric Inequality)**.** *Consider a compactly supported boundary $\Gamma \in \mathscr{R}_{k-1}(\mathbb{R}^d)$, with $k \ge 2$. Then there exists $T \in \mathscr{I}_k(\mathbb{R}^d)$ such that $\partial T = \Gamma$ and*

$$\mathbb{M}(T)^{k-1} \le C\mathbb{M}(\Gamma)^k\,.$$

The constant C depends only on k, d.

Among the consequences of the Deformation Theorem 1.2.49, we will especially need some polyhedral approximation results. We list them here, starting from the simplest one.

Theorem 1.2.51 (Weak Polyhedral Approximation Theorem). *Consider a normal current $T \in \mathbb{N}_k(\mathbb{R}^d)$. Then there exist a sequence $(\varepsilon_n)_{n\geq 1}$ of real numbers with $\varepsilon_n \to 0$ and a sequence $(P_n)_{n\geq 1}$ of polyhedral chains of the form*

$$P_n = \sum_{F_n \in \mathcal{L}^d_{k,\varepsilon_n}} [\![F_n, \tau_{F_n}, \theta_{F_n}]\!]$$

with constant coefficients $\theta_{F_n} \in \mathbb{R}$, such that

$$P_n \overset{*}{\rightharpoonup} T \text{ in } U$$

and $\partial P_n \overset{*}{\rightharpoonup} \partial T$, too. Moreover, if $T \in \mathscr{R}_k(\mathbb{R}^d)$, then we can choose every polyhedral current P_n with integer multiplicities $\theta_{F_n} \in \mathbb{Z}$.

The proof of this theorem can be found in [36] (Theorem 7.9.2).

Theorem 1.2.52 (Strong Polyhedral Approximation Theorem). *Consider an integral current $T \in \mathscr{I}_k(U)$, with $U \subset \mathbb{R}^d$, and fix $\varepsilon > 0$. Then there exist a bi-Lipschitz map $f \in \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ and an integral polyhedral chain P such that*

$$\mathbb{M}(f_\sharp T - P) + \mathbb{M}(f_\sharp(\partial T) - \partial P) \leq \varepsilon \tag{1.2.14}$$

and

$$|\nabla f(x) - \mathrm{Id}| + |f(x) - x| \leq \varepsilon \quad \forall\, x \in \mathbb{R}^d\ . \tag{1.2.15}$$

Moreover, $f(x) = x$ whenever $\mathrm{dist}(x, \mathrm{supp}T) \geq \varepsilon$.

This theorem can be found in 4.2.20 of [24], but, since we will use it only in the 1-dimensional case, we will prove it in 1.3.1 in the simpler case of 1-dimensional currents.

We interrupt the list of approximation theorems in order to introduce another useful norm on the space of currents, the *flat norm*, whose essence has been anticipated by the decomposition (1.2.13) in Deformation Theorem 1.2.49.

Definition 1.2.53. If T is a k-current, then its *flat norm* is

$$\|T\|_\flat := \inf\{\mathbb{M}(R) + \mathbb{M}(S) : T = R + \partial S, \text{ with } R \in \mathscr{D}_k(\mathbb{R}^d) \text{ and } S \in \mathscr{D}_{k+1}(\mathbb{R}^d)\}\,.$$

Remark 1.2.54. Obviously $\|T\|_\flat \le \mathbb{M}(T)$, so the flat norm is weaker than the mass. In a certain sense, the flat norm gives a more geometric notion of distance between surfaces then the mass norm. For example consider the 1-current $T = [I_1] - [I_2]$ in $\mathbb{R}^2$, where I_1 and I_2 are two parallel segments with same orientation, same length l and ε is the (Hausdorff) distance between them. Then the flat norm of T does not exceed $l\varepsilon$, confirming the intuition that the two segments are close together, while $\mathbb{M}(T) = 2l$, independently from the Hausdorff distance ε.

The importance of the flat norm is due the fact that (at least in the space of integral currents $\mathscr{I}_k(\mathbb{R}^d)$) it metrizes the weak* topology in the ball $\{\mathbb{M}(T) \le 1\}$. See Theorem 3.1.2 of [51] or Section 8.2 of [36] for the proof.

Theorem 1.2.55. *Let $(T_n)_{n\ge1}$ be a sequence of integer multiplicity rectifiable currents with*

$$\sup_n \left(\mathbb{M}(T_n) + \mathbb{M}(\partial T_n)\right) < \infty\,.$$

Then

$$T_n \overset{*}{\rightharpoonup} T \quad \Longleftrightarrow \quad \|T - T_n\|_\flat \to 0\,.$$

Definition 1.2.56. Finally, we define the subgroup of *flat chains* as

$$\mathscr{F}_k(\mathbb{R}^d) := \{T = R + \partial S : R \in \mathscr{R}_k(\mathbb{R}^d), S \in \mathscr{R}_{k+1}(\mathbb{R}^d)\}\,.$$

Moreover

$$\mathbb{F}_k(\mathbb{R}^d) := \overline{\mathbb{P}_k(\mathbb{R}^d)}^\flat\,. \tag{1.2.16}$$

Copying the scheme from [24] 4.1.24, we resume the spaces we introduced with

$$\boxed{\begin{array}{ccccccccc}
\mathscr{P}_k(\mathbb{R}^d) & \subset & \mathscr{I}_k(\mathbb{R}^d) & \subset & & \mathscr{R}_k(\mathbb{R}^d) & & \subset & \mathscr{F}_k(\mathbb{R}^d) \\
\cap & & \cap & & & \cap & & & \cap \\
\mathbb{P}_k(\mathbb{R}^d) & \subset & \mathbb{N}_k(\mathbb{R}^d) & \subset & \mathbb{F}_k(\mathbb{R}^d) \cap \{\mathbb{M}(T) < \infty\} & \subset & \mathbb{F}_k(\mathbb{R}^d) & = & \overline{\mathbb{P}_k(\mathbb{R}^d)}^\flat\,.
\end{array}}$$

We conclude this subsection with an approximation theorem which summarizes the role played by polyhedral chains at the base of the theory of currents, thanks to the flat norm.

Theorem 1.2.57 (Polyhedral Approximation Theorem)**.** *Let T be a normal k-current in $\mathbb{R}^d$ and $\varepsilon > 0$. Then there exists a polyhedral k-current P such* *that* $\|T - P\|_\flat \le \varepsilon$ *and* $\mathbb{M}(P) + \mathbb{M}(\partial P) \le \mathbb{M}(T) + \mathbb{M}(\partial T) + \varepsilon$. *Moreover, if* ∂T *is polyhedral it is possible to take* $\partial P = \partial T$ *and if* T *is integral it is possible to take* P *integral such that*

$$\inf\left\{\mathbb{M}(R) + \mathbb{M}(S) : T - P = R + \partial S,\ R \in \mathscr{R}_k(\mathbb{R}^d) \text{ and } S \in \mathscr{R}_{k+1}(\mathbb{R}^d)\right\} \le \varepsilon\,.$$

The first half of the theorem is stated and proved in [24], 4.2.24: it is a subtle consequence of Deformation Theorem 1.2.49. The second half of the theorem, concerning the approximation of integral currents, proceeds from Strong Polyhedral Approximation Theorem 1.2.52 (see [24], 4.2.21).

1.2.4 Compactness results for currents

We conclude this section of recap on currents with some essential results of compactness.

The compactness is an immediate consequence of the compactness theorem for Radon measures together with the fact that, given a sequence of normal currents $(T_n)_{n\geq 1}$ in $\mathbb{N}_k(U)$, if $T_n \overset{*}{\rightharpoonup} T$, then $\partial T_n \overset{*}{\rightharpoonup} \partial T$.

Theorem 1.2.58 (Compactness theorem for normal currents)**.** *Let $(T_n)_{n\geq 1}$ be a sequence of normal k-currents on $U \subset \mathbb{R}^d$ such that $\mathbb{M}_V(T_n) + \mathbb{M}_V(\partial T_n)$ is uniformly bounded for every open subset $V \subset\subset U$. Then there exists a subsequence $(T_{n_j})_{j\geq 1}$ weakly converging to a normal k-current.*

The main theorem for integral currents is the Closure Theorem. Actually the theorem is stated and exploited as a compactness result for $\mathscr{I}_k(U)$, but let us point out that, given a sequence of integral currents $(T_n)_{n\geq 1}$, the existence of a converging subsequence $\left(T_{n_j}\right)_{j\geq 1}$ and a limit current $T \in \mathbb{F}_k(U)$ is a consequence of Theorem 1.2.58. Thus the nontrivial point is that the limit T is an integral current and this is why the following result is known with the name of Closure Theorem.

Theorem 1.2.59 (Closure Theorem)**.** *Let $(T_n)_{n\geq 1}$ be a sequence of integral k-currents on $U \subset \mathbb{R}^d$ such that $\mathbb{M}_V(T_n) + \mathbb{M}_V(\partial T_n)$ is uniformly bounded for any open subset $V \subset\subset U$. Then there exist an integral k-current $T \in \mathscr{I}_k(U)$ and a subsequence $(T_{n_j})_{j\geq 1}$ such that*

$$T_{n_j} \overset{*}{\rightharpoonup} T \quad \textit{as } j \to \infty\,.$$

This theorem was firstly proved in [26] by means of the structure theory for sets of finite Hausdorff measure. Among the alternative proofs which came after, let us mention [53] and [5], the latter introducing the slicing technique. The same proof by slicing can be found in [36], Section 8.1.

As a consequence of the Closure Theorem 1.2.59, plus the Weak Polyhedral Approximation Theorem 1.2.51, we are able to prove that $\mathscr{R}_k(U) \cap \mathbb{N}_k(U) = \mathscr{I}_k(U)$, that is, an integer multiplicity rectifiable current turns out to be an integral current, when its boundary has finite mass. See also Theorem 7.9.3 in [36].

Theorem 1.2.60 (Boundary rectifiability Theorem)**.** *Let T be an integer multiplicity rectifiable current with $\mathbb{M}(\partial T) < \infty$. Then ∂T is an integer multiplicity rectifiable current.*

Historically, integral currents were introduced by Federer and Fleming as a framework for the Plateau's problem, that is, find the k-dimensional area-minimizing surface spanning a prescribed boundary. Thus currents are meant as a generalization of surfaces and the Closure Theorem above guarantees the existence of a mass-minimizing solution.

Theorem 1.2.61. *Let $\Gamma \subset U$ be the boundary of an integral k-current in $U \subset \mathbb{R}^d$, with $1 \leq k \leq d$. Then there exists a current minimizing the mass among all integral currents $T \in \mathscr{I}_k(U)$ satisfying $\partial T = \Gamma$.*

Proof. We apply the direct method of Calculus of Variations to integral k-currents. Indeed, let m be the infimum of $\mathbb{M}(T)$ among integral k-currents with $\partial T = \Gamma$ and let $(T_n)_{n\geq 1}$ be a minimizing sequence, that is $\lim_{n\to\infty}(T_n) = m$. Since $\mathbb{M}(T_n)$ is bounded and $\mathbb{M}(\partial T_n)$ is constant, we can apply Theorem 1.2.59 to the sequence[9] $(T_n)_{n\geq 1}$ and find a subsequence converging to an integral current T. By the continuity of the boundary operator we still have $\partial T = \Gamma$ and by lower semicontinuity of the mass (see Remark 1.2.40) we have $\mathbb{M}(T) \leq m$. □

It is worth mentioning that, in [30], the Plateau's problem for hypersurfaces is solved without the Closure Theorem above, using a result of decomposition of normal currents, instead. We will come back to the subject of decomposition of normal currents in Chapter 2.

[9] Actually, we could state this theorem with Γ being the boundary of an integer multiplicity rectifiable k-current S. Indeed, we could apply Theorem 1.2.59 to the sequence $T_n - S$, which has no boundary.

1.3 Some results for 1-currents

We have no choice but to notice that 1-dimensional currents are nicer than k-dimensional currents, with $k \geq 2$. In some cases it is just a matter of simplicity: as we underline in Theorem 1.3.2, 1-dimensional rectifiable currents have a handy structure, and some statements are easier to prove. But there are also some facts, holding for 1-currents, which are no longer valid in higher dimension, as we will see in Subsection 1.3.2. The aim of this Section is to establish these special results for 1-currents only. Their usefulness will be clear in Chapter 3.

1.3.1 The structure of 1-currents

We start with the proof of the Strong Polyhedral Approximation Theorem 1.2.52: having at least the 1-dimensional case proof is important for the possible extension of the result to 1-dimensional currents with coefficients in a group. We formulate this density result on $\mathbb{R}^d$, the local version can be easily deduced using an extension lemma. It will be important that a current T without boundary can be approximated by polygonal currents without boundary, so we insert the possibility of the boundary being preserved in the statement, with (1.3.2). This possibility holds true only in dimension 1, because of the peculiar structure of rectifiable 0-currents (see Example 1.2.46).

Theorem 1.3.1 (1d Strong Polyhedral Approximation Theorem)**.** *Consider a* 1*-current* $T \in \mathscr{I}_1(\mathbb{R}^d)$ *and fix* $\varepsilon > 0$*. Then there exist a bi-Lipschitz map* $f \in \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ *and an integral polyhedral* 1*-current* $P \in \mathscr{P}_1(\mathbb{R}^d)$ *such that*

$$\mathbb{M}(f_\sharp T - P) \leq \varepsilon \tag{1.3.1}$$

and

$$\partial P = \partial T\,. \tag{1.3.2}$$

Moreover

$$|\nabla f(x) - \mathrm{Id}| + |f(x) - x| \leq \varepsilon \quad \forall\, x \in \mathbb{R}^d\,, \tag{1.3.3}$$

and $f(x) = x$ *whenever* $\mathrm{dist}(x, \mathrm{supp}\, T) \geq \varepsilon$*.*

The proof follows closely the one in [24] 4.2.19.

Proof. By standard arguments on rectifiable sets, we are able to cover $\mathbb{R}^d$ by a countable family $\mathcal{F}$ of $\mathscr{C}^1$-curves, up to a $\|T\|$-null set. We denote by λ a real parameter in the interval $(0,1)$, which will be chosen at the end of the proof.

Step 1: We fix a point $x_0 \in \gamma \in \mathcal{F}$ such that $x_0 \notin \operatorname{supp} \partial T$ and that, for some $\theta_0 \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{r\to 0} \frac{\|T - S\|(Q_r^\tau(x_0))}{r} = 0\,, \tag{1.3.4}$$

where S is the current defined by $\langle S, \varphi\rangle = \int_\gamma \theta_0 \langle \varphi(x), \tau(x)\rangle d\mathscr{H}^1(x)$, and $Q_r^\tau(x_0)$ is the cube of side $2r$, center in x_0 and one side parallel to the vector τ, which is the tangent to γ in x_0.

Without loss of generality we can assume $x_0 = 0$ and $\mathrm{Tan}_0\gamma = \mathbb{R}\mathbf{e}_1$, where $\mathbf{e}_1$ is the first unit vector of the canonical basis of $\mathbb{R}^d$. We denote by Q_r the cube of center 0, side $2r$ and sides parallel to the coordinate directions. Let $\varepsilon' > 0$ be a small parameter chosen later. For r sufficiently small, we have that $\overline{Q_r} \cap \operatorname{supp} \partial T = \varnothing$ and that the set $\gamma \cap Q_r$ is the graph of a $\mathscr{C}^1$ function $g : (-r, r) \to \mathbb{R}^{d-1}$ with $g(0) = 0$ and $\|g\|_{\mathscr{C}^1} < \varepsilon'$. The function $\tilde g : (-r, r) \to \mathbb{R}^d$ defined as $\tilde g(x_1) = (0, g(x_1))$ obeys

$$\|D\tilde g\|_{L^\infty((-r,r))} < \varepsilon' \quad \text{and} \quad \|\tilde g\|_{L^\infty((-r,r))} < \varepsilon' r\ .$$

We define the function $f \in \mathscr{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ as

$$f(x) = x - \psi(x)\tilde g(x_1)\,,$$

where $\psi \in \mathscr{C}_c^\infty(Q_r; [0,1])$ obeys $\psi \equiv 1$ on $Q_{\lambda r}$ and

$$\|\nabla\psi\|_{L^\infty} \le \frac{2}{(1-\lambda)r}\ .$$

For $2\varepsilon' < 1 - \lambda$ the function f is bi-Lipschitz and maps $\gamma \cap Q_{\lambda r}$ into the segment $(\mathbb{R}e_1) \cap Q_{\lambda r}$. Moreover for sufficiently small ε' (on a scale set by λ and ε) one has

$$\begin{aligned} |f(x) - x| + |\nabla f(x) - \mathrm{Id}| &\le |\psi(x)\tilde g(x_1)| + |\psi(x)\nabla\tilde g(x_1) \otimes \mathbf{e}_1| \\ &+ |\tilde g(x_1) \otimes \nabla\psi(x)| \\ &< \varepsilon'\left(r + 1 + \frac{2}{(1-\lambda)}\right) < \varepsilon \end{aligned} \tag{1.3.5}$$

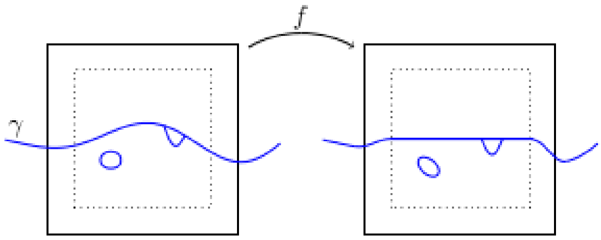

Figure 1.1: The action of f on T in the proof of Theorem 1.3.1. The inner cube is $Q_{\lambda r}$, the outer one Q_r.

and

$$\|f^{-1}\|_{\mathscr{C}^1} \le 1+\varepsilon \ . \tag{1.3.6}$$

Step 2: We let P be the polyhedral current defined by

$$\langle P, \varphi\rangle = \theta_0 \int_{(-\lambda r, \lambda r)\mathbf{e}_1} \langle \varphi, \mathbf{e}_1\rangle \, d\mathscr{H}^1 \, .$$

With S as in (1.3.4), by definition of P and f we have

$$\mathbb{M}(S \llcorner Q_r - f_\sharp^{-1} P) = |\theta_0| \, \mathscr{H}^1 \left(\gamma \cap (Q_r \setminus Q_{\lambda r})\right) \, .$$

Since γ is a $\mathscr{C}^1$ curve,

$$\lim_{r\to 0} \frac{\mathscr{H}^1(\gamma \cap (Q_r \setminus Q_{\lambda r}))}{2r} = (1-\lambda) \, .$$

Using a triangle inequality and (1.3.5) we obtain

$$\begin{aligned} \mathbb{M}\left(f_\sharp (T \llcorner Q_r) - P\right) &\le \mathbb{M}\left(f_\sharp((T-S)\llcorner Q_r)\right) + \mathbb{M}\left(f_\sharp(S \llcorner Q_r - f_\sharp^{-1}P)\right) \\ &\le (1+\varepsilon)\mathbb{M}\left((T-S)\llcorner Q_r\right) + (1+\varepsilon)\mathbb{M}\left(S\llcorner Q_r - f_\sharp^{-1}P\right) \end{aligned}$$

and, recalling (1.3.4),

$$\limsup_{r\to 0} \frac{\mathbb{M}\left(f_\sharp (T\llcorner Q_r) - P\right)}{2r} \le (1+\varepsilon)(1-\lambda)|\theta_0| \, .$$

Since, again by (1.3.4), $\|T\|(Q_r)/(2r) \to |\theta_0|$, for r sufficiently small

$$\mathbb{M}\left(f_\sharp (T\llcorner Q_r) - P\right) < 2(1-\lambda)\|T\|(Q_r) \ . \tag{1.3.7}$$

Step 3: By [24, Th. 4.3.17] for $\mathscr{H}^1$-almost every point in $\operatorname{supp} T \setminus \operatorname{supp} \partial T$ there is a θ_0 with the property (1.3.4), and therefore an $r_x \in (0, \varepsilon/\sqrt{d})$ satisfying both

$$\overline{Q_{r_x}^{\tau(x)}(x)} \cap \operatorname{supp} \partial T = \varnothing$$

and the property (1.3.7) with Q_r replaced by $Q_{r_x}^{\tau(x)}(x)$. Using Morse's covering Theorem, we cover $\|T\|$-almost all the set $\mathbb{R}^d$ with a countable family of disjoint cubes $Q_{r_n}^{\tau_n}(x_n)$ with $\tau_n = \tau(x_n)$ and sides $2r_n$, with $r_n < r_{x_n}$; each of these cubes has positive distance from $\operatorname{supp} \partial T$. Then we have a polyhedral 1-current P_n with support in $Q_{r_n}^{\tau_n}(x_n)$ and a bi-Lipschitz map $f_n : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (1.3.5), (1.3.6) and (1.3.7).

We choose a finite subfamily such that

$$\sum_{n=1}^{N(\lambda)} \|T\|(Q_{r_n}^{\tau_n}(x_n)) \geq \lambda \mathbb{M}(T). \tag{1.3.8}$$

and define

$$f = f_1 \circ \ldots \circ f_{N(\lambda)} \ .$$

Since $f_n(x) = x$ outside $Q_{r_n}^{\tau_n}(x_n)$ for all n and the cubes are disjoint the condition (1.3.5) still holds and $f(x) = x$ outside an ε-neighbourhood of $\operatorname{supp} T$. Moreover, we have

$$\partial f_\sharp T = f_\sharp \partial T = \partial T \,,$$

because f is the identity map in a neighbourhood of ∂T.

We define the polyhedral current

$$\overline{P} = \sum_{n=1}^{N(\lambda)} P_n \,,$$

write

$$f_\sharp T - \overline{P} = \sum_{n=1}^{N(\lambda)} \left(f_\sharp \left(T \llcorner Q_{r_n}^{\tau_n}(x_n) \right) - P_n \right) + f_\sharp \Big(T \llcorner \bigcup_{n > N(\lambda)}^{\infty} Q_{r_n}^{\tau_n}(x_n) \Big)$$

and, recalling (1.3.7) and (1.3.8), conclude that

$$\mathbb{M}(f_\sharp T - \overline{P}) < 2(1-\lambda)\mathbb{M}(T) + (1-\lambda)\mathbb{M}(T) = 3(1-\lambda)\mathbb{M}(T) \ . \tag{1.3.9}$$

Step 4: We did not obtain the thesis yet: indeed $\overline{P}$ is polyhedral, but in general $\partial\overline{P}$ does not coincide with ∂T. Notice that the current $f_\sharp T - \overline{P}$ has multiplicity in $\mathbb{Z}$ and hence it is an integral 1-current. We can apply the Deformation Theorem in [24, Th. 4.2.9] to $f_\sharp T - \overline{P}$ in order to represent it as

$$f_\sharp T - \overline{P} = \hat{P} + R + \partial S \ .$$

Here $\hat{P}, R$ are polyhedral 1-currents satisfying

$$\mathbb{M}(\hat{P}) \leq \hat{c}(\mathbb{M}(f_\sharp T - \overline{P}) + \tilde{\varepsilon}\mathbb{M}(\partial(f_\sharp T - \overline{P})))$$

and

$$\mathbb{M}(R) \leq \tilde{\varepsilon} c_R \mathbb{M}\left(\partial(f_\sharp T - \overline{P})\right),$$

for some $\tilde{\varepsilon}$ arbitrarily small, where $\hat{c}, c_R > 0$ are geometric constants.

Then $P = \overline{P} + \hat{P} + R$ is a polyhedral 1-current with $\partial P = \partial f_\sharp T = \partial T$ and

$$\begin{aligned}
\mathbb{M}(f_\sharp T - P) &\leq \mathbb{M}(f_\sharp T - \overline{P}) + \mathbb{M}(P - \overline{P}) \\
&\leq 3(1-\lambda)\mathbb{M}(T) + \mathbb{M}(\hat{P} + R) \\
&\leq 3(1+\hat{c})(1-\lambda)\mathbb{M}(T) + \tilde{\varepsilon}(\hat{c} + c_R)\mathbb{M}\left(\partial(f_\sharp T - \overline{P})\right).
\end{aligned}$$

We first choose a $\lambda \in (0,1)$ such that the first term is less than $\frac{1}{2}\varepsilon$, then $\tilde{\varepsilon}$ such that the second term is also less than $\frac{1}{2}\varepsilon$, and conclude. □

As a consequence of the Theorem 1.3.1 we easily recover Theorem 1.2.57: any current $T \in \mathscr{I}_1(\mathbb{R}^d)$ can be approximated by sequences of polyhedral currents[10] $(P_n)_{n\geq 1}$ in the weak* topology for currents. Moreover $\mathbb{M}(P_n) \to \mathbb{M}(T)$ and, if T is closed, then we can choose a sequence with $\partial P_n = 0$ for every $n \geq 1$.

The approximation results proved above allow us to characterize the support of 1-currents without boundary as the countable union of loops. This characterization can be found in [24], subsection 4.2.25.

Theorem 1.3.2 (Structure of closed integral 1-currents)**.** *Let $T \in \mathscr{I}_1(\mathbb{R}^d)$ with $\partial T = 0$. Then there are countably many oriented Lipschitz closed curves γ_i with tangent vector fields $\tau_i : \gamma_i \to \mathbb{S}^{d-1}$ and multiplicities $\theta_i \in \mathbb{Z}$ such that $T = [\![\gamma_i, \tau_i, \theta_i]\!]$, that is*

$$\langle T; \omega \rangle = \sum_{i=1}^{\infty} \theta_i \int_{\gamma_i} \langle \omega; \tau_i \rangle \, d\mathscr{H}^1 \,, \tag{1.3.10}$$

[10] We recall that the currents P_n have support on a finite number of segments.

for every $\omega \in \mathscr{D}^1(\mathbb{R}^d)$. *Further,*

$$\sum_{i=1}^{\infty} |\theta_i| \mathscr{H}^1(\gamma_i) = \mathbb{M}(T)\,. \tag{1.3.11}$$

Proof. From the density of polyhedral currents there is a sequence of closed integral polyhedral currents (with a finite number of segments) $P_n \in \mathscr{P}_1(\mathbb{R}^d)$ such that

$$P_n \overset{*}{\rightharpoonup} T \text{ and } \mathbb{M}(P_n) \to \mathbb{M}(T)\,. \tag{1.3.12}$$

Each P_n can be decomposed into the sum of finitely many polyhedral loops,

$$P_n = \sum_{j=1}^{J_n} C_{j,n}\,, \tag{1.3.13}$$

such that

$$\sum_{j=1}^{J_n} \mathbb{M}(C_{j,n}) = \mathbb{M}(P_n) \leq M\,, \tag{1.3.14}$$

for some $M > 0$.

We can assume these loops $C_{j,n}$ to be ordered by mass, starting with the biggest one. Moreover we can assume (up to extracting a subsequence) that the currents $C_{j,n}$ have multiplicity 1 and that for every j they weakly converge to some closed 1-current C_j. Parametrizing each polygonal curve by arc length, and eventually passing to a further subsequence, we see that each converges to a closed Lipschitz curve. Let us denote by $\tilde{T}$ the current

$$\tilde{T} = \sum_{j=1}^{\infty} C_j$$

We need to show that $\tilde{T} = T$. If $\mathbb{M}(T) = 0$ there is nothing to prove. Otherwise we fix $\delta > 0$ and observe that by (1.3.14) we have $\mathbb{M}(C_{i,n}) < \delta$ for all $i > M/\delta$. We write

$$\langle P_n; \omega\rangle = \sum_{i \leq \frac{M}{\delta}} \langle C_{i,n}; \omega\rangle + \sum_{i > \frac{M}{\delta}} \langle C_{i,n}; \omega\rangle\,. \tag{1.3.15}$$

In the first sum of the right hand side we can take the limit as $n \to \infty$ and get $\sum_{i \leq \frac{M}{\delta}} \langle C_i; \omega\rangle$.

The second sum in (1.3.15) can be estimated as follows. For every $i > M/\delta$ and for every n we fix a point $x_i^n \in \operatorname{supp} C_{i,n} = \gamma_{i,n}$ and using the fact that $\gamma_{i,n}$ is a closed curve we have

$$\begin{aligned} \Big| \sum_{i>\frac{M}{\delta}} \langle C_{i,n}; \omega \rangle \Big| &= \Big| \sum_{i>\frac{M}{\delta}} \int_{\gamma_{i,n}} \langle \omega - \omega(x_i^n); \tau_i^n \rangle \, d\mathscr{H}^1 \Big| \\ &\le \sum_{i>\frac{M}{\delta}} \sup_{\gamma_{i,n}} |\omega - \omega(x_i^n)| \mathbb{M}(C_{i,n}) \\ &\le \delta \|\omega\|_{\text{Lip}} \sum_{i>\frac{M}{\delta}} \mathbb{M}(C_{i,n}) \le \delta M \|\omega\|_{\text{Lip}} \,. \end{aligned} \tag{1.3.16}$$

Then we get

$$\Big| \langle T - \sum_{i \le \frac{M}{\delta}} \langle C_i; \omega \rangle \Big| \le o(1) + \Big| \langle P_n - \sum_{i \le \frac{M}{\delta}} \langle C_{i,n}; \omega \rangle \Big| \le o(1) + \delta M \|\omega\|_{\text{Lip}}$$

which implies $T = \tilde{T}$ and hence

$$T = \sum_{j=1}^{\infty} \tau_j \mathscr{H}^1 \llcorner \gamma_j \,,$$

with $\gamma_j = \operatorname{supp} L_j$ and τ_j the corresponding tangent vector. □

This theorem is saying that we are allowed to consider an integral 1-current as the formal sum of a finite number of open Lipschitz curves and a countable number of Lipschitz loops, with multiplicities in $\mathbb{Z}$. So we conclude the subsection with its useful corollary.

Corollary 1.3.3. *Let T be an integral 1-current in $\mathbb{R}^d$, then*

$$T = \sum_{k=1}^{K} T_k + \sum_{\ell \ge 1} C_\ell \,, \tag{1.3.17}$$

with

(i) T_k and C_ℓ are integral 1-currents associated to oriented simple Lipschitz curves with finite length, for $k = 1, \dots, K$ and $\ell \ge 1$;

(ii) $\partial C_\ell = 0$ for every $\ell \ge 1$.

Moreover

$$\mathbb{M}(T) = \sum_{k=1}^{K} \mathbb{M}(T_k) + \sum_{\ell \geq 1} \mathbb{M}(C_\ell) \tag{1.3.18}$$

and

$$\mathbb{M}(\partial T) = \sum_{k=1}^{K} \mathbb{M}(\partial T_k) \ . \tag{1.3.19}$$

1.3.2 Relating normal and rectifiable 1-currents

There is a series of open and very interesting problems concerning currents: in the 1-dimensional case we have more advanced answers, thanks to the simpler structure of 1-currents. Among them, we find the problem of the decomposition of a normal current in a sum of integral currents which preserves the mass and, related to the existence of a decomposition, the problem of commensurability among mass-minimizing currents belonging to different subgroups of $\mathscr{D}_k(U)$. Here we do not insist on the k-dimensional formulation of these problems, because we will examine them afterwards, in Chapter 2 and in Chapter 3, respectively. We limit ourselves to a self-contained exposition of the 1-dimensional results.

Given a compact measure space (L, λ) and a family of 1-currents $\{T_x\}_{x \in L}$ in $\mathbb{R}^d$, such that

$$\int_L \mathbb{M}(T_x)\, d\lambda(x) < +\infty\,,$$

we denote by

$$T := \int_L T_x\, d\lambda(x)$$

the 1-current T satisfying

$$\langle T; \omega \rangle = \int_L \langle T_x, \omega \rangle\, d\lambda(x)\,,$$

for every smooth compactly supported 1-form ω.

Proposition 1.3.4. *Every normal* 1*-current* T *in* $\mathbb{R}^d$ *can be written as*

$$T = \int_0^M T_x\, dx\,,$$

where T_x *is an integral current with* $\mathbb{M}(T_x) \leq 2$ *and* $\mathbb{M}(\partial T_x) \leq 2$ *for every* x*, and* M *is a positive number depending only on* $\mathbb{M}(T)$ *and* $\mathbb{M}(\partial T)$*. Moreover*

$$\mathbb{M}(T) = \int_0^M \mathbb{M}(T_x)\, dt\,.$$

The first result of this kind has been stated and proved in [52]: the so-called solenoidal 1-currents[11] can be decomposed in elementary solenoids. See [45] for the generalization of Proposition 1.3.4 to metric spaces.

The following fact is probably in the folklore, unfortunately we were not able to find any literature on it. In the proof, it will be made clear where a decomposition result for normal currents is needed.

Theorem 1.3.5. *Consider the boundary of an integral* 1*-current in* $\mathbb{R}^d$*, represented as*

$$B_0 = -\sum_{i=1}^{N_-} a_i\delta_{x_i} + \sum_{j=1}^{N_+} b_j\delta_{y_j}, \quad a_i, b_j \in \mathbb{N} \ . \tag{1.3.20}$$

If we denote

$$\mathscr{M}_N(B_0) := \min\{\mathbb{M}(T) : T \text{ is a normal current } , \partial T = B_0\}$$

and

$$\mathscr{M}_I(B_0) := \min\{\mathbb{M}(T) : T \text{ is an integral current } \partial T = B_0\} \geq \mathscr{M}_N(B_0) \ ,$$

then the minima of the mass of 1*-currents with boundary* B_0 *among normal 1-currents and among integral 1-currents coincide, that is*

$$\mathscr{M}_N(B_0) = \mathscr{M}_I(B_0) \ .$$

Proof. Let us assume that the minimum among normal currents is attained at some current T_0, that is

$$\mathbb{M}(T_0) = \mathscr{M}_N(B_0) \, .$$

Let $\{T_h\}_{h\in\mathbb{N}}$ be an approximation of T_0 made by polyhedral 1-currents, such that

- $\mathbb{M}(T_h) \to \mathbb{M}(T_0)$ as $h \to \infty$,
- $\partial T_h = B_0$ for all $h \in \mathbb{N}$,
- the multiplicities allowed in T_h are only integer multiples of $\frac{1}{h}$.

[11] Roughly speaking, a solenoidal 1-current is a normal current which can be represented by a divergence-free Borel vectorfield.

The existence of such a sequence is a consequence of the Polyhedral Approximation Theorem 1.2.57. Thanks to Corollary 1.3.3, it is possible to decompose such a T_h as a sum of two addenda:

$$T_h = P_h + C_h \,, \tag{1.3.21}$$

so that

$$\mathbb{M}(T_h) = \mathbb{M}(P_h) + \mathbb{M}(C_h) \quad \forall\, h \geq 1$$

and

- $\partial C_h = 0$, so C_h collects the cyclical part of T_h;
- P_h does not admit any decomposition $P_h = A + B$ satisfying $\partial A = 0$ and $\mathbb{M}(P_h) = \mathbb{M}(A) + \mathbb{M}(B)$.

It is clear that P_h is the sum of a certain number of polyhedral currents $P_h^{i,j}$ each one having boundary a non-negative multiple of $-\frac{1}{h}\delta_{x_i} + \frac{1}{h}\delta_{y_j}$ and satisfying

$$\mathbb{M}(P_h) = \sum_{i,j} \mathbb{M}(P_h^{i,j}) \,.$$

We replace each $P_h^{i,j}$ with the oriented segment $Q_h^{i,j}$, from x_i to y_j having the same boundary as $P_h^{i,j}$ (therefore having multiplicity a non-negative multiple of $\frac{1}{h}$). This replacement is represented in Figure 1.2.

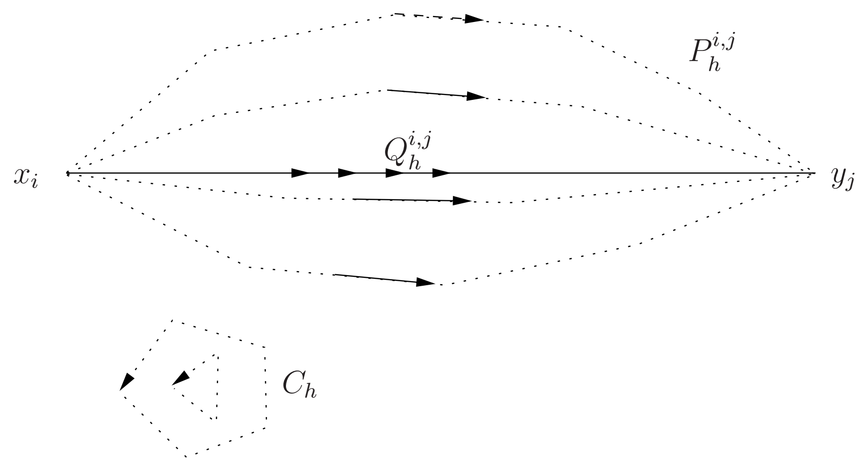

Figure 1.2: Replacement with a segment

Since this replacement obviously does not increase the mass, there holds $\mathbb{M}(P_h) \geq \mathbb{M}(Q_h)$, where $Q_h = \sum_{i,j} Q_h^{i,j}$. In other words we can write $Q_h = \int_L Q_h^\lambda \, d\lambda_h$, as an integral of currents, with respect to a discrete measure λ_h supported on the finite set L of unit multiplicity oriented segments with the first extreme among the points $x_1, \dots, x_{N_-}$ and second extreme among the points $y_1, \dots, y_{N_+}$. It is also easy to see that the total variation of λ_h has eventually the following bound from above

$$\|\lambda_h\| \leq \frac{\mathbb{M}(T_h)}{\min_{i \neq j} d(x_i, y_j)} \leq \frac{\mathbb{M}(T_0) + 1}{\min_{i \neq j} d(x_i, y_j)}\,.$$

Hence, up to subsequences, λ_h converges to some positive measure λ on L and so the normal 1-current

$$Q = \int_L Q^\lambda \, d\lambda$$

satisfies

$$\partial Q = B_0 \tag{1.3.22}$$

and

$$\mathbb{M}(Q) \leq \mathbb{M}(T_0) = \mathscr{M}_N(B_0)\,.$$

In order to conclude the proof of the theorem, we need to show that Q can be replaced by an integral current R with same boundary and mass $\mathbb{M}(R) = \mathbb{M}(Q) \leq \mathscr{M}_N(B_0)$. Since L is the set of unit multiplicity oriented segments Σ^{ij} from x_i to y_j, we can obviously represent

$$Q = \sum_{i,j} k^{ij} \Sigma^{ij} \quad \text{with } k^{ij} \in \mathbb{R}\,,$$

and, again, thanks to (1.3.22),

$$\sum_{i=1}^{N_-} k^{ij} = b_j \quad \text{and} \quad \sum_{j=1}^{N_+} k^{ij} = a_i\,.$$

If $k^{ij} \in \mathbb{Z}$ for any i, j, then Q itself is integral and then we are done; if not, let us consider the finite set of non-integer multiplicities

$$K_{\mathbb{R}\setminus\mathbb{Z}} := \left\{k^{ij} : i = 1, \dots, N_-,\ j = 1, \dots, N_+\right\} \setminus \mathbb{Z} \neq \varnothing\,.$$

We fix $k \in K_{\mathbb{R}\setminus\mathbb{Z}}$ and we choose an index (i_0, j_0), such that k is the multiplicity of the oriented segment $\Sigma^{i_0 j_0}$ in Q. It is possible to track down a non-trivial cycle $\overline{Q}$ in Q with the following algorithm: after $\Sigma^{i_0 j_0}$, choose a segment from $x_{i_1} \neq x_{i_0}$ to y_{j_0} with non-integer multiplicity, it must exist because $B_0 = \partial Q$ is integral. Then choose a segment from x_{i_1} to $y_{j_1} \neq y_{j_0}$ with non-integer multiplicity and so on. Since $K_{\mathbb{R}\setminus\mathbb{Z}}$ is finite, at some moment we will get a cycle. Up to reordering the indices i and j we can write

$$\overline{Q} = \sum_{l=1}^{n} (\Sigma^{i_l j_l} - \Sigma^{i_{l+1} j_l}) \,.$$

We will denote by

$$\begin{aligned} \alpha &:= \min_l (k^{i_l j_l} - \lfloor k^{i_l j_l} \rfloor) > 0 \\ \beta &:= \min_l (k^{i_{l+1} j_l} - \lfloor k^{i_{l+1} j_l} \rfloor) > 0 \,. \end{aligned}$$

Finally notice that both $Q - \alpha\overline{Q}$ and $Q + \beta\overline{Q}$ have lost at least one non-integer coefficient; in addition, we claim that either

$$\mathbb{M}(Q - \alpha\overline{Q}) \leq \mathbb{M}(Q) \quad \text{or} \quad \mathbb{M}(Q + \beta\overline{Q}) \leq \mathbb{M}(Q) \,. \tag{1.3.23}$$

In fact we can define the linear auxiliary function

$$F(t) := \mathbb{M}(Q) - \mathbb{M}(Q - t\overline{Q}) = \sum_l (k^{i_l j_l} - t) d(x_{i_l}, y_{j_l}) + (k^{i_{l+1} j_l} + t) d(x_{i_{l+1}}, y_{j_l})$$

for which $F(0) = 0$, so either

$$F(\alpha) \geq 0 \quad \text{or} \quad F(-\beta) \geq 0 \,.$$

Since $K_{\mathbb{R}\setminus\mathbb{Z}}$ is a finite set, iterating the procedure we illustrated for the removal of $\overline{Q}$ finitely many times we obtain an integral current without increasing the mass. □

1.4 Calibrations

We introduce here a very important tool in the study of area-minimizing currents. Classically a calibration ω associated with a given oriented k-submanifold $S \subset \mathbb{R}^d$ is a unit closed k-form taking value 1 on the tangent space of S. As it will be clear from Theorem 1.4.3, the very aim of the definition of a calibration is to provide a sufficient condition of minimality.

1.4.1 Definition and usefulness of calibrations

We give the definitions in a very general setting, our purpose is to give some interesting examples not only in $\mathbb{R}^d$, but also in the complex space $\mathbb{C}^d$.

Definition 1.4.1. Given a d-dimensional Riemannian manifold X, we say that a k-dimensional smooth differential form ω is a *calibration* associated with a k-dimensional integral current $S \in \mathscr{I}_k(X)$ if the following properties hold:

(i) the form ω restricted to the (a.e. defined) tangent plane of S coincides with its volume form, that is[12] $\langle \omega; \tau_S \rangle = \|\tau_S\|$;

(ii) the form ω is closed, that is $\mathrm{d}\omega = 0$;

(iii) for every other k-section $\tau \in TX$, the form ω does not exceed the volume form, that is $\langle \omega; \tau \rangle \leq \|\tau_S\|$.

Definition 1.4.2. Given a d-dimensional Riemannian manifold X and a pair of k-dimensional integral currents S and T, we say that S is *cobordant* to T if their sum is the boundary of a $k+1$-dimensional integral current $R \in \mathscr{I}_{k+1}(X)$, that is $\partial R = S - T$.

Theorem 1.4.3. *Consider a d-dimensional Riemannian manifold X and a k-dimensional integral current $S \in \mathscr{I}_k(X)$, if there exists a calibration ω associated with S, then S is a mass-minimizing current among integral currents in its cobordism class.*

---

[12]We stick to the notation introduced in Section 1.2, so $\|\cdot\|$ is the mass norm of Definition 1.2.14.

Proof. Consider a competitor $T \in \mathscr{I}_k(X)$ with $\partial R = S - T$ for some $R \in \mathscr{I}_{k+1}(X)$. Therefore[13]

$$\mathbb{M}(S) \overset{\text{(i)}}{=} S(\omega) = R(\mathrm{d}\omega) + T(\omega) \overset{\text{(ii)}}{=} T(\omega) \overset{\text{(iii)}}{\leq} \mathbb{M}(T)\,. \tag{1.4.1}$$

□

Remark 1.4.4. We have a simpler case of the theorem above when S is a genuine submanifold of X and we look for area-minimizing submanifolds in the cobordism class of S. Thus (1.4.1) becomes

$$\mathrm{vol}(S) \overset{\text{(i)}}{=} \int_S \omega \overset{\text{(ii)}}{=} \int_T \omega \overset{\text{(iii)}}{\leq} \int_T \mathrm{d}\,\mathrm{vol}_T = \mathrm{vol}(T)\,,$$

for every submanifold $T \subset X$ with $\partial R = S \sqcup T$ for some compact submanifold $R \subset X$ one dimension higher. Here the equality $\int_S \omega = \int_T \omega$ is no longer a consequence of the definition of boundary of a current, but truly a consequence of the Stokes's Theorem.

Remark 1.4.5. From line (1.4.1), it is clear that $\mathbb{M}(S) = \mathbb{M}(T)$ if and only if $T(\omega) = \mathbb{M}(T)$. Thus, a calibration associated with an integral current S "calibrates" simultaneously all the mass-minimizers for the given cobordism class of S.

Remark 1.4.6. From the proof of Theorem 1.4.3 it is clear that a calibrated integral current S is a mass-minimizer not only among integral currents, but also among normal currents in its cobordism class. Indeed, the inequality in (1.4.1) still holds if the competitor $T \in \mathbb{N}_k(X)$ has an integral representation.

The remark above is not trivial, since mass-minima among normal currents and among integral currents do not coincide, in general, as [] and [] show. See also Subsection 3.1.2 for a detailed discussion of the matter in the context of currents with coefficients in a group.

An outstanding example of use of the calibration technique is the following observation, due to H. Federer (see [24], 5.4.19).

Theorem 1.4.7. *In $\mathbb{C}^d$ a complex submanifold S is always area-minimizing.*

[13]We think that line (1.4.1) is more incisive with a direct reference to the reason of each equality or inequality: Roman numerals above each equality or inequality correspond to those in Definition 1.4.1.

Proof. As a real orthonormal basis for the tangent bundle of $\mathbb{C}^d \cong \mathbb{R}^d + i\mathbb{R}^d$ we choose $\{\mathrm{d}x_1, \mathrm{d}y_1, \dots, \mathrm{d}x_d, \mathrm{d}y_d\}$. We define the Kähler form

$$\omega := \sum_{j=1}^{d} \mathrm{d}x_j \wedge \mathrm{d}y_j \,. \tag{1.4.2}$$

We claim that the form $\omega^k/k!$ is a calibration for every complex submanifold $S \subset \mathbb{C}^d$ of real dimension $2k$ and the equality in (1.4.1) holds if and only if S is complex. Property (ii) is trivial, since ω has constant coefficients. Properties (i) and (iii) - and the characterization of the equality case - are a consequence of the Wirtinger's inequality, which is stated in the following lemma. $\square$

Lemma 1.4.8 (Wirtinger's inequality). *Consider the Kähler form ω defined in* (1.4.2)*, then for every $2k$-dimensional real subspace $V \subset \mathbb{C}^d$ we have that*

$$\frac{\omega^k}{k!}_{|V} \le \mathrm{d}\,\mathrm{vol}_V \,. \tag{1.4.3}$$

Moreover, equality in (1.4.3) *holds if and only if V is a complex subspace.*

Proof. If we denote by J the multiplication by $i := \sqrt{-1}$ in $\mathbb{C}^d$, we can write

$$\omega(u,v) = \langle Ju; v\rangle \quad \forall\, u, v \in \mathbb{C}^d \cong \mathbb{R}^d + i\mathbb{R}^d \,.$$

Thanks to Cauchy-Scharz inequality

$$|\omega(u,v)| \le |u||v| \quad \forall\, u, v \in \mathbb{C}^d \cong \mathbb{R}^d + i\mathbb{R}^d \,, \tag{1.4.4}$$

thus we have to characterize the equality case. Fix a $2k$-dimensional real subspace $V \subset \mathbb{C}^d$, the matrix representing $\omega_{|V}$ is skew-symmetric, so $\omega_{|V}$ has a canonical form: we can represent it with a $2k \times 2k$ block diagonal real matrix, with k blocks of the form

$$\begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix} .$$

We will call $\{\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{e_k}, \mathbf{f}_k\}$ the basis of V for such a representation of $\omega_{|V}$, and $\{\mathrm{d}\xi_1, \mathrm{d}\eta_1 \dots, \mathrm{d}\xi_k, \mathrm{d}\eta_k\}$ will denote the associated dual basis. With this notation

$$\frac{\omega^k}{k!}_{|V} = \frac{1}{k!}\left(\lambda_1 \mathrm{d}\xi_1 \wedge \mathrm{d}\eta_1 + \dots + \lambda_k \mathrm{d}\xi_k \wedge \mathrm{d}\eta_k\right)^k = \lambda_1 \dots \lambda_k \,\mathrm{d}\,\mathrm{vol}_V \,.$$

Notice that (1.4.4) implies $|\lambda_j| \le 1$ for every $j = 1, \dots, k$, thus

$$\frac{\omega^k}{k!} = \pm \mathrm{d}\,\mathrm{vol}_V \iff |\lambda_j| = 1 \ \forall\, j = 1, \dots, k\,.$$

The latter condition holds if and only if $J(V) = V$, meaning that V is a complex subspace. □

The proof above is a reinterpretation of [24], 1.8.2. See also [23].

Remark 1.4.9. Theorem 1.4.7 can be generalized to every Kähler manifold X. Indeed, by definition, a complex manifold X is a Kähler manifold if the Kähler form $\omega := \sum_{j=1}^{d} \mathrm{d}x_j \wedge \mathrm{d}y_j$ (in the coordinates of each chart) is closed. Moreover, Wirtinger's inequality (1.4.3) remains true in a general Kähler manifold.

Remark 1.4.10. Actually, in the case of the Kähler structure for complex manifolds, it is possible to make a link between complex submanifolds and currents calibrated by the Kähler form: structure theorems by King ([35]) and by Harvey and Shiffman (see [32]) state that a $2k$-dimensional current T with a.e. complex tangent planes can we written as

$$T = \sum_j [\![V_j, \tau_j, n_j]\!]\,,$$

where $n_j \in \mathbb{N}$ and V_j are complex submanifolds.

As we just saw, calibrations are a powerful tool for the theory of mass-minimizing currents (and area-minimizing submanifolds, respectively). Nonetheless, it can be very hard – or even impossible! – to find a calibration for a given current, whom we suspect to be a minimizer. The approach suggested by Harvey and Lawson in [31] is reversed: it is not difficult to find a closed k-form ω with comass norm less or equal than 1 (choose a form with constant coefficients and possibly rescale it, for instance!), then the problem becomes to find a submanifold, or a current, for which ω is a calibration. The following definition can help to figure out the new problem: $\mathscr{G}$ is the set of candidate calibrations, provided we endow it with some properties.

Definition 1.4.11. Consider a d-dimensional Riemannian manifold X and the Grassmannian of k-dimensional tangent planes $\mathrm{Gr}(k, TX)$. Fix a subset $\mathscr{G} \subset \mathrm{Gr}(k, TX)$. We say that a k-dimensional submanifold $S \subset X$ is a $\mathscr{G}$-submanifold if, for every $x \in S$, $T_x S \in \mathscr{G}_x$, where $\mathscr{G}_x$ is the intersection of $\mathscr{G}$ with the fiber on x if $\mathrm{Gr}(k, TX)$. Analogous definitions may be given for integer rectifiable currents, integral currents, chains, ecc.

Remark 1.4.12. With the formalism of the definition above, we can see complex submanifolds of a Kähler manifold as $\mathscr{G}$-submanifolds, where

$$\mathscr{G} = \bigcup_{x\in X} \mathscr{G}_x = \bigcup_{x\in X} \{\tau_x \in \mathrm{Gr}(2k, TX) : \langle\omega(x);\tau(x)\rangle = 1\}$$

and ω is the Kähler form.

1.4.2 Existence of calibrations

The existence of a calibration is a sufficient condition for a manifold to be a minimizer; one could wonder whether this condition is necessary as well. In general, a smooth (or piecewise smooth) calibration might not exist; nevertheless, one can still search for some weak calibration, for instance a differential form with bounded measurable coefficients.

A duality argument due to H. Federer ensures that a weak calibration exists for mass-minimizing normal currents. Therefore an equivalence principle between minima among normal and integral 1-currents is sufficient to conclude that a calibration exists. This equivalence principle has been stated in Theorem 1.3.5 for 1-currents only. As we already noticed, for k-currents with $k \geq 2$ this equivalence principle does not hold in general.

The remainder of this section is devoted to Federer's argument on the existence of a generalized calibration for normal currents, in the sense of Theorem 1.4.16. See [25] for the original and detailed proof.

Definition 1.4.13. Fix an arbitrary subset $\Gamma \subset \mathbb{N}_k(\mathbb{R}^d)$, for any $T \in \mathbb{N}_k(\mathbb{R}^d)$ we define

$$\mathscr{M}_\Gamma(T) := \inf\left\{\mathbb{M}(S) : S \in \mathbb{N}_k(\mathbb{R}^d) \text{ and } S - T \in \Gamma\right\} .$$

Heuristically, we fixed a family Γ of admissible differences – for our purposes Γ will be the family of k-boundaries – and we look for the "smallest" mass $\mathbb{M}(S)$, where S is close – in the Γ sense – to the current $T \in \mathbb{N}_k(\mathbb{R}^d)$.

Remark 1.4.14. If Γ is a convex cone, then $\mathscr{M}_\Gamma : \mathbb{N}_k(\mathbb{R}^d) \to [0,\infty]$ is convex and positively 1-homogeneous.

Definition 1.4.15. Given a subset $\Gamma \subset \mathbb{N}_k(\mathbb{R}^d)$, as in Definition 1.4.13, we consider the dual space $\mathrm{Hom}(\mathbb{N}_k(\mathbb{R}^d);\mathbb{R})$ with the following norm

$$\mathscr{M}_\Gamma(\alpha) := \inf\{r : \alpha(T) \leq r\mathscr{M}_\Gamma(T) \text{ with } T \in \mathbb{N}_k(\mathbb{R}^d)\} .$$

From now on we will set $\Gamma = \mathbb{B}_k(\mathbb{R}^d)$, the set of boundaries in $\mathbb{N}_k(\mathbb{R}^d)$, so that the infimum of the mass is calculated among cobordant currents. Thus we get

$$\mathscr{M}_\Gamma(T) = \mathbb{M}(T_0)$$

where T_0 is a mass-minimizing normal current, cobordant to T.

Theorem 1.4.16. *If $T \in \mathbb{N}_k(U)$ is a mass-minimizer in its cobordism class, then there exists a functional $\phi \in \mathrm{Hom}(\mathbb{N}_k(\mathbb{R}^d), \mathbb{R})$ such that*

(i) $\phi(T) = \mathbb{M}(T)$*;*

(ii) ϕ *is "closed", that is*

$$\phi(\partial R) = 0 \quad \forall\, R \in \mathbb{N}_{k+1}(U)\ ;$$

(iii) ϕ *has unit norm with respect to Definition 1.4.15.*

Proof. Notice that $\mathscr{M}_\Gamma$ is convex and positively 1-homogeneous as we stated in Remark 1.4.14, so, by Hahn-Banach Theorem, there exists

$$\phi \in \mathrm{Hom}(\mathbb{N}_k(\mathbb{R}^d), \mathbb{R})$$

such that

(i) $\phi(T) = \mathbb{M}(T)$;

(ii) $\phi \le \mathscr{M}_\Gamma$.

Therefore

$$\forall\, R \in \mathbb{N}_{k+1}(\mathbb{R}^d) \quad \phi(\partial R) \le \mathscr{M}_\Gamma(\partial R) = \inf\left\{\mathbb{M}(S) : S \in \mathbb{Z}_k(\mathbb{R}^d)\right\} = \mathbb{M}(0) = 0\,.$$

Finally, property (iii) comes from the bound $\phi \le \mathscr{M}_\Gamma$ together with the definition of $\mathscr{M}_\Gamma(\phi)$. □

Remark 1.4.17. We stated Theorem 1.4.16 in an abstract way, proving the existence of a so-called *flat cochain* (see [24], 4.1.19, [25] and [56]) playing the role of a generalized calibration. This was done in order to preserve the analogy with Definition 3.1.15 in Subsection 3.1.1. Nonetheless it is possible to improve the result of Theorem 1.4.16 by means of the Representation Theorem 1.1.4: there exist bounded Lebesgue measurable differential forms $\omega_\phi \in \Lambda^k(\mathbb{R}^d)$ and $\hat{\omega}_\phi \in \Lambda^{k-1}(\mathbb{R}^d)$ such that

1. $\phi(\xi\mathscr{L}^d) = \int_{\mathbb{R}^d}\langle\omega_\phi(x);\xi(x)\rangle\,dx$ for every $\xi \in \mathscr{C}_c(\mathbb{R}^d;\Lambda_k(\mathbb{R}^d))$;

2. $\phi\,(\partial(\zeta\mathscr{L}^d)) = \int_{\mathbb{R}^d}\langle\hat{\omega}_\phi(x);\zeta(x)\rangle\,dx$ for every $\zeta \in \mathscr{C}_c(\mathbb{R}^d;\Lambda_{k+1}(\mathbb{R}^d))$;

3. $\int_{\mathbb{R}^d}\langle\omega_\phi;\mathrm{div}\zeta\rangle + \langle\hat{\omega}_\phi;\zeta\rangle = 0$ for every $\zeta \in \mathscr{C}_c^1(\mathbb{R}^d;\Lambda_{k+1}(\mathbb{R}^d))$.

This cannot be considered as an actual representation for the generalized calibration ϕ, since it works only for very special currents, but it is the best we can expect.

Chapter 2

Decomposition of currents

2.1 Frobenius Theorem on the integrability of vector fields

One of the most important theorems in Differential Geometry fully answers to this question: under which conditions a given k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ represents the tangent vector field of a smooth manifold M?

Definition 2.1.1. A k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, is *integrable* if, for every point $x_0 \in \mathbb{R}^d$, there exist an open neighborhood $U \ni x_0$ and a k-dimensional submanifold $M \ni x_0$ such that

$$T_x M = \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\}$$

for every $x \in U \cap M$. We will say that ξ is *completely integrable* if, for every $x_0 \in \mathbb{R}^d$, there exist an open neighborhood $U \ni x_0$ and a $\mathscr{C}^2$-function[1] $F : U \to \mathbb{R}^{d-k}$ such that its level sets $\{F = p\}$ are k-submanifolds with $\operatorname{span}\{\tau_1, \ldots, \tau_k\}$ as tangent space.

Roughly speaking, a completely integrable vector field is a tangent field for a local foliation of $\mathbb{R}^d$ in $\mathscr{C}^2$-submanifolds. Obviously, a completely integrable vector field is integrable.

We recall the definition of the *Lie bracket* in (2.1.1) in a convenient coordinate version.

Definition 2.1.2. Given a vector field $X : U \subset \mathbb{R}^d \to \mathbb{R}^d$ of class $\mathscr{C}^1$, we represent its action on a smooth function $f : U \to \mathbb{R}$ as $X(f) = \nabla_X f := \langle \nabla f; X \rangle = \sum_{i=1}^d \frac{\partial f}{\partial x_i} X_i$. If $X \equiv \mathbf{e}_i$ for some $i \in \{1, \ldots, d\}$, then we also write $\nabla_i f = \frac{\partial f}{\partial x_i}$.

Given two $\mathscr{C}^1$-vector fields $X, Y : U \to \mathbb{R}^d$ their Lie bracket is $[X, Y] := XY - YX$ and $[X, Y](f) = \nabla_X \nabla_Y f - \nabla_Y \nabla_X f$. In coordinates

$$[X, Y]_h = \langle \nabla Y_h; X \rangle - \langle \nabla X_h; Y \rangle = \sum_{i=1}^d \left(\frac{\partial Y_h}{\partial x_i} X_i - \frac{\partial X_h}{\partial x_i} Y_i \right), \quad \text{with } h = 1, \ldots, d.$$

[1]From this perspective, it is clear that this problem and the problem of the existence of a potential $F : U \subset \mathbb{R}^d \to \mathbb{R}$ for a given map $f : U \to \mathbb{R}^d$, with $\nabla F = f$, are related and we refer to them as integrability problems.

The proof of the following theorem characterizing integrable vector fields can be found in [38] or in any other book about the basics of smooth manifolds.

Theorem 2.1.3 (Frobenius Theorem)**.** *A k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, is completely integrable if and only if*

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\} \tag{2.1.1}$$

for every $x \in \mathbb{R}^d$ and for every $m, n = 1, \ldots, k$.

Let us remark that condition (2.1.1) does not depend on the choice of $\tau_1, \ldots, \tau_k$, but only on their product $\xi = \tau_1 \wedge \ldots \wedge \tau_k$. This fact will be a clear consequence of Corollary 2.1.5. We will say that a vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$ is *involutive* if (2.1.1) is satisfied and we will spend the remainder of this section characterizing the involutivity condition in convenient ways.

Lemma 2.1.4. *Consider a k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, then*[2]

$$\operatorname{div}\xi \wedge \tau_m \wedge \tau_n = -\xi \wedge [\tau_m, \tau_n], \tag{2.1.2}$$

for every $m, n = 1, \ldots, k$.

Proof. We need to find a connection between the Lie bracket $[\tau_m, \tau_n]$ and $\operatorname{div}\xi \wedge \tau_m \wedge \tau_n$. Using the isomorphism $\mathbb{D}$ of Definition 1.2.9 and Proposition 1.2.24, we can write

$$\begin{aligned}
\mathbb{D}_{k+1}(\operatorname{div}\xi \wedge \tau_m \wedge \tau_n) &= (\operatorname{div}\xi \wedge \tau_m \wedge \tau_n) \lrcorner \, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d \\
&= (\tau_m \wedge \tau_n) \lrcorner \, \mathbb{D}_{k-1}(\operatorname{div}\xi) \\
&= (-1)^{d-k} (\tau_m \wedge \tau_n) \lrcorner \, \mathrm{d}(\mathbb{D}_k \xi).
\end{aligned}$$

Moreover, for any multi-index[3] $\mathbf{i} \in \mathcal{I}(d, d-k-1)$, we can write

$$\begin{aligned}
&\langle \tau_m \wedge \tau_n \lrcorner \, \mathrm{d}(\mathbb{D}_k \xi); \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{d-k-1}} \rangle \\
= \;& \langle \mathrm{d}(\mathbb{D}_k \xi), \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{d-k-1}} \wedge \tau_m \wedge \tau_n \rangle \\
= \;& \sum_{h=1}^{d-k-1} (-1)^{h-1} \langle \nabla_{i_h}(\mathbb{D}_k \xi); \mathbf{e}_{\hat{\mathbf{i}}_h} \wedge \tau_m \wedge \tau_n \rangle && (2.1.3) \\
+ \;& (-1)^{d-k-1} \langle \nabla_{\tau_m}(\mathbb{D}_k \xi); \mathbf{e}_{\mathbf{i}} \wedge \tau_n \rangle && (2.1.4) \\
+ \;& (-1)^{d-k} \langle \nabla_{\tau_n}(\mathbb{D}_k \xi); \mathbf{e}_{\mathbf{i}} \wedge \tau_m \rangle. && (2.1.5)
\end{aligned}$$

[2] The divergence $\operatorname{div}\xi$ has been defined in Definition 1.2.22.

[3] Remember that by $\mathbf{e}_{\hat{\mathbf{i}}_h}$ we mean the $(k-1)$-vector $\mathbf{e}_{\hat{\mathbf{i}}_h} = \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{h-1}} \wedge \mathbf{e}_{i_{h+1}} \wedge \ldots \wedge \mathbf{e}_{i_k}$, with $\mathbf{i} = \{i_1, \ldots, i_k\}$.

Since $\langle \mathbb{D}_k\xi; \mathbf{e}_{\hat{\mathbf{i}}_h} \wedge \tau_m \wedge \tau_n \rangle \equiv 0$ for any $h = 1, \ldots, d-k-1$, then (2.1.3) becomes

$$\sum_{h=1}^{d-k-1} (-1)^{h-1} \Big\langle \mathbb{D}_k\xi; \mathbf{e}_{\hat{\mathbf{i}}_h} \wedge \left(\frac{\partial \tau_m}{\partial x_{i_h}} \wedge \tau_n - \tau_m \wedge \frac{\partial \tau_n}{\partial x_{i_h}} \right) \Big\rangle$$
$$= \sum_{h=1}^{d-k-1} (-1)^{h-1} \langle \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d; \mathbf{e}_{\hat{\mathbf{i}}_h} \wedge \left(\frac{\partial \tau_m}{\partial x_{i_h}} \wedge \tau_n - \tau_m \wedge \frac{\partial \tau_n}{\partial x_{i_h}} \right) \wedge \xi \rangle = 0\,.$$

Also $\langle \mathbb{D}_k\xi; \mathbf{e}_{\mathbf{i}} \wedge \tau_m \rangle \equiv 0 \equiv \langle \mathbb{D}_k\xi; \mathbf{e}_{\mathbf{i}} \wedge \tau_n \rangle$, then we conclude through (2.1.4) and (2.1.5) that

$$\begin{aligned} -\langle \mathbb{D}_{k+1}(\mathrm{div}\xi \wedge \tau_m \wedge \tau_n); \mathbf{e}_{\mathbf{i}} \rangle &= \langle \mathbb{D}_k\xi; \mathbf{e}_{\mathbf{i}} \wedge (\nabla_{\tau_m}\tau_n - \nabla_{\tau_n}\tau_m) \rangle \\ &= \langle \mathbb{D}_k\xi; \mathbf{e}_{\mathbf{i}} \wedge [\tau_m, \tau_n] \rangle \\ &= \langle \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d; \mathbf{e}_{\mathbf{i}} \wedge [\tau_m, \tau_n] \wedge \xi \rangle \\ &= \langle \mathbb{D}_{k+1} \left([\tau_m, \tau_n] \wedge \xi \right); \mathbf{e}_{\mathbf{i}} \rangle\,, \end{aligned}$$

for every multi-index $\mathbf{i} \in \mathcal{I}(d, d-k-1)$. The thesis follows, since $\mathbb{D}_{k+1}$ is an isomorphism. □

Corollary 2.1.5. *Consider a k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$. Involutivity holds if and only if*

$$\mathrm{div}\xi \wedge \tau_m \wedge \tau_n = 0 \quad \textit{for every } m, n = 1, \ldots, d\,. \tag{2.1.6}$$

Lemma 2.1.6. *Given a non-involutive simple k-vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, there exist an open subset $U \subset \mathbb{R}^d$ and a $(k-1)$-form α such that, in U, we have*

$$\langle \mathrm{d}\alpha; \xi \rangle \neq 0 \tag{2.1.7}$$

and

$$\langle \alpha; \eta \rangle = 0 \tag{2.1.8}$$

whenever η is a simple $(k-1)$-vectorfield representing at each point a linear subspace of ξ.

Proof. Since ξ is non-involutive, Lemma 2.1.4 provides us a pair of indices (m, n) and an open subset $U \subset \mathbb{R}^d$ where

$$-\mathrm{div}\xi \wedge \tau_m \wedge \tau_n = \xi \wedge [\tau_m, \tau_n] \neq 0\,.$$

For every point $x \in U$, we get a basis for $\mathbb{R}^d$ by completing the set of linearly independent vectors $\{\tau_1, \dots, \tau_k, [\tau_m, \tau_n]\}$ with unit orthonormal vectors $\nu_1, \dots, \nu_{d-(k+1)}$. For the sake of brevity, we denote by N the $(d-k-1)$-vectorfield $\nu_1 \wedge \dots \wedge \nu_{d-(k+1)}$. We claim that

$$\alpha := \mathbb{D}_{d-k+1}(\tau_m \wedge \tau_n \wedge \nu_1 \wedge \dots \wedge \nu_{d-(k+1)})$$

satisfies (2.1.7) and (2.1.8) at the same time and we devote the remainder of this proof to prove this claim.

Concerning (2.1.7), we can compute

$$\langle \mathrm{d}\alpha; \xi \rangle = (-1)^{k-1} \langle \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_d; \xi \wedge \operatorname{div}(\tau_m \wedge \tau_n \wedge N) \rangle \tag{2.1.9}$$

$$= \langle \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_d; \operatorname{div}\xi \wedge \tau_m \wedge \tau_n \wedge N \rangle \neq 0\,; \tag{2.1.10}$$

where (2.1.9) is a consequence of (1.2.3) and the equality in (2.1.10) is allowed by Proposition 1.2.25, because $\xi \wedge \tau_m = \xi \wedge \tau_n \equiv 0$.

Finally, if η is a $(k-1)$-vectorfield with $\operatorname{span}\eta \subset \operatorname{span}\xi$, then

$$\langle \alpha; \eta \rangle = \langle \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_d; \eta \wedge \tau_m \wedge \tau_n \wedge N \rangle = 0$$

because $\eta \wedge \tau_m \wedge \tau_n \equiv 0$. □

2.2 Non-integrability for integral currents

Consider a non-involutive k-vectorfield ξ in $\mathbb{R}^d$, as above. We may wonder if the non-involutivity property is strong enough to prevent not only the existence of a surface with tangent field ξ, but also the existence of an integral current with such a tangent field. The answer is affirmative, as we state in Theorem 2.2.6. We begin with the key theorem.

Theorem 2.2.1. *Let ξ be a continuous k-dimensional vector field on $\mathbb{R}^d$ and let $R \in \mathscr{I}_k(\mathbb{R}^d)$ be a k-dimensional integral current with $R = [\![\Sigma, \xi, \theta]\!]$. If $\partial R = [\![\Sigma', \eta, \theta']\!]$, then*

$$\operatorname{span}\eta(x) \subset \operatorname{span}\xi(x) \quad \mathscr{H}^{k-1}\text{-a.e. } x \in \Sigma'\,. \tag{2.2.1}$$

The proof of Theorem 2.2.1 is based on the blow up technique and can be essentially split in two lemmata: Lemma 2.2.4 deals with the possibility of doing the blow up at $\mathscr{H}^{k-1}$-a.e. point of Σ' (remember that $\partial R = [\![\Sigma', \eta, \theta']\!]$), while Lemma 2.2.5 draws some conclusions about the structure of the blow up current. Lemma 2.2.4 is proved by means of a family of "projections" on a k-dimensional space and Lemma 2.2.3 deals with the blow up of these projected currents. We recall that we denote

$$\beta_{x,r}(y) := r^{-1}(y-x)\,.$$

Moreover, we will repeatedly need Theorem 4.3.17 of [24]. We briefly recall it here, adapting the notation to ours.

Theorem 2.2.2. *Let $S = [\![\Gamma, \gamma, g]\!]$ be a h-dimensional rectifiable current, then $\mathscr{H}^h$-a.e. $y \in \Gamma$ we have that*

$$\lim_{r\to 0} \beta_{y,r\,\sharp} S = \overline{\gamma}\,\mathscr{H}^h \llcorner T_y\Gamma\,,$$

where

$$\overline{\gamma} \equiv g\gamma$$

is the orientation of S at y, multiplied by the multiplicity g of the current S at y.

Lemma 2.2.3. *Let P be a normal k-current in $\mathbb{R}^k$, then, for $\mathscr{H}^{k-1}$-almost every $y_0 \in \mathbb{R}^k$ we have that*

$$\mathbb{M}(P \llcorner B_r(y_0)) = O(r^k)\,. \tag{2.2.2}$$

We will denote by G the set of points where (2.2.2) holds, i.e.,

$$G := \left\{ y_0 : \limsup_{r\to 0} \mathbb{M}\left(\beta_{y_0,r\,\sharp} P\right) < +\infty \right\}\,.$$

Proof. First of all, let us remark that Theorem 2.2.2 cannot be applied straight to P, because we need a result for $\mathscr{H}^{k-1}$-a.e. y, $\mathscr{H}^k$-almost everywhere is not enough!

Since P is a k-dimensional normal current in $\mathbb{R}^k$, then its multiplicity p is a BV-function and we have that[4]

$$⨍_{B_r(y_0)} p(y)\,dy \xrightarrow{r\to 0} \tilde{p}(y_0) \quad \mathscr{H}^{k-1} - \text{a.e. } y_0 \in \mathbb{R}^k\,, \tag{2.2.3}$$

[4]See Section 5.3 in [21] for the results concerning approximated limits and traces of BV-functions.

where $\tilde{p}$ is the *precise representative* of p. Moreover, since $|\mathrm{d}p|$ is a finite measure, we have that[5]

$$|\mathrm{d}p|\,(B_r(y_0)) = O(r^h) \tag{2.2.4}$$

for $\mathscr{H}^h$-a.e. y_0.

Thus, let us fix a point y_0 such that both (2.2.3) and (2.2.4) (with $h = k-1$) hold. We claim that y_0 is a point in G. In fact, thanks to the Poincaré inequality, we get

$$\begin{aligned}
\mathbb{M}(P \llcorner B_r(y_0)) &\leq \int_{B_r(y_0)} |p(y)|\,dy \\
&\leq \int_{B_r(y_0)} \left| p(y) - \left(\fint_{B_r(y_0)} p \right) \right| dy + \omega_k \,|\tilde{p}(y_0)|\, r^k + o(r^k) \\
&\leq cr \int_{B_r(y_0)} |\mathrm{d}p| + O(r^k) = O(r^k)
\end{aligned}$$

Finally notice that, by construction, $\mathscr{H}^{k-1}(\mathbb{R}^k \setminus G) = 0$. $\square$

Lemma 2.2.4. *Let ξ be a continuous k-dimensional vector field on $\mathbb{R}^d$ and let $R \in \mathscr{I}_k(\mathbb{R}^d)$ be a k-dimensional integral current with $R = [\![\Sigma, \xi, \theta]\!]$ and $\partial R = [\![\Sigma', \eta, \theta']\!]$. Then for $\mathscr{H}^{k-1}$-a.e. $x \in \Sigma'$ we have that*

$$\limsup_{r \to 0} \mathbb{M}(\beta_{x,r_\sharp} R) < +\infty\,. \tag{2.2.5}$$

Proof. We cover Σ' with a family of balls $B_{\rho_i}(x_i) = V_i$ such that there exists an orientation preserving linear map whose restriction $\pi_i : V_i \to \mathbb{R}^k$ has the following properties:

(i) π_i has maximal rank k on $\operatorname{span}\xi(x)$ for every $x \in V_i$;

(ii) π_i has rank $k-1$ on $\operatorname{span}\eta(x)$ for every $x \in \Sigma' \cap V_i$;

(iii) the pushforward $\pi_{i\sharp}(R \llcorner B_{\rho_i}(x_i))$ is an integral current.

The first property is easy to fulfill, since ξ is continuous. Property (ii) can be obtained by a Lusin-type argument on η, possibly getting the thesis (2.2.5) $\mathscr{H}^{k-1}$-a.e. $x \in \Sigma' \cap K_\varepsilon$, where the compact set K_ε is arbitrarily large in Σ'

[5] In some sense, this is equivalent to apply Theorem 2.2.2 to ∂P, getting $\mathbb{M}(\partial P \llcorner B_r(y_0)) = O(r^{k-1})$ for $\mathscr{H}^{k-1}$-a.e. y_0.

(that is $\mathscr{H}^{k-1}(\Sigma' \smallsetminus K_\varepsilon) < \varepsilon$ for any $\varepsilon > 0$), and then passing to the limit. The existence of a linear map with maximal rank on both $\xi(x_i)$ and $\eta(x_i)$ is a trivial fact in Linear Algebra. The third property holds, provided we chose the right ρ_i: indeed, the slicing process $(\pi_i)_\sharp (R \llcorner B_\rho(x_i))$ gives an integral current for almost every ρ (see Chapter 7.6 for this kind of results).

We will denote by P_i the integral current $(\pi_i)_\sharp (R \llcorner V_i) = [\![\pi_i(\Sigma \cap V_i), \pi_i, \theta_i^\sharp]\!]$, with boundary $\partial P_i = [\![\Gamma_i, \gamma_i, p_i']\!]$. Let us remark that, since π_i is not changing the orientation, then

$$\theta_i^\sharp(y) = \sum_{x \in \pi_i^{-1}(y)} \theta(x) > 0 \,. \tag{2.2.6}$$

Thanks to Lemma 2.2.3, we have that

$$\mathbb{M}(P_i \llcorner B_r(y_0)) = O(r^k)$$

for every $y_0 \in G_i$, with $\mathscr{H}^{k-1}(\Gamma_i \smallsetminus G_i) = 0$.

Now we have to step back from $\mathbb{R}^k$ to the original space $\mathbb{R}^d$ with the original current $R \llcorner V_i$ to prove that $\mathbb{M}(R \llcorner B_r(x)) = O(r^k)$ for $\mathscr{H}^{k-1}$-a.e. $x \in \Sigma'$. Firstly, let us notice that

$$\mathscr{H}^{k-1}\left(\pi_i^{-1}(\Gamma_i \smallsetminus G_i) \cap \Sigma'\right) = 0 \,.$$

Assume by contradiction that $\mathscr{H}^{k-1}\left(\pi_i^{-1}(\Gamma_i \smallsetminus G_i) \cap \Sigma'\right) > 0$. Since by construction π_i has rank $k-1$ on the approximate tangent space to Σ' in $\mathscr{H}^{k-1}$-a.e. point of $\Sigma' \cap V_i$, then we can apply the Area Formula (see Section 3.3 in [21]) to π_i, and we get that

$$0 < \mathscr{H}^{k-1}\left(\pi_i\left(\pi_i^{-1}(\Gamma_i \smallsetminus G_i) \cap \Sigma'\right)\right) \leq \mathscr{H}^{k-1}(\Gamma_i \smallsetminus G_i) \,,$$

contradicting the fact that $\mathscr{H}^{k-1}(\Gamma_i \smallsetminus G_i) = 0$.

Moreover, thanks to (2.2.6), there exists a number $\delta_i > 0$ such that for every $x_0 \in \pi_i^{-1}(y_0) \cap \Sigma'$ and for every r with $\delta_i r < \rho_i$, we have that

$$\mathbb{M}(R \llcorner B_{\delta_i r}(x_0)) \leq \mathbb{M}(P_i \llcorner B_r(y_0)) = O(r^k) \,. \qquad \square$$

Lemma 2.2.5. *Consider a normal k-current in $\mathbb{R}^d$, written as $R = \xi\mu$, where ξ is an orientation and μ is a finite measure, and assume that there exist a point $x_0 \in \Sigma$ and a sequence $(r_n)_{n\geq 1}$, with $r_n \to 0$ as $n \to \infty$, such that there exists the limit*

$$R_0 := \lim_{n\to\infty} \beta_{x_0, r_n \sharp} R \,.$$

Then R_0 has constant orientation $\xi(x_0)$.

Proof. We simply notice that, for every k-dimensional covector ν orthogonal to $\xi(x_0)$ (that is, $\langle \nu, \xi(x_0)\rangle = 0$) and for every $\varphi \in \mathscr{D}$, we get

$$R_0 \llcorner \nu(\varphi) = R_0(\varphi \wedge \nu) = \lim_{n\to\infty} \beta_{x_0,r_n\sharp} R(\varphi \wedge \nu) = 0\,.$$

□

We are now ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Fix a point $x_0 \in \Sigma'$ satisfying (2.2.5) in Lemma 2.2.4. Thus there exists a sequence $(r_n)_{n\geq 1}$, with $r_n \to 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \beta_{x_0,r_n\sharp} R = R_0 \in \mathscr{I}_k(\mathbb{R}^d)\,.$$

Recalling Lemma 2.2.5, we get a constant orientation $\xi(x_0)$ for R_0. We claim that $\partial R_0 \llcorner \nu = 0$ for every $\nu \in \Lambda^1(\mathbb{R}^d)$ such that $\xi(x_0) \llcorner \nu = 0$, indeed

$$\partial R_0 \llcorner \nu(\varphi) = \partial R_0(\varphi \wedge \nu) = -R_0(\nu \wedge \mathrm{d}\varphi) = -R_0 \llcorner \nu(\mathrm{d}\varphi) = 0\,. \tag{2.2.7}$$

On the other hand the orientation of ∂R_0 is $\eta(x_0)$ by means of Theorem 2.2.2, thus (2.2.7) means that $\operatorname{span}\eta(x_0) \subset \operatorname{span}\xi(x_0)$. □

Theorem 2.2.6. *Let $\xi = \tau_1 \wedge \ldots \wedge \tau_k$ be a k-dimensional simple vector field on $\mathbb{R}^d$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d)$, and let $T \in \mathscr{I}_k(\mathbb{R}^d)$ be a k-dimensional integral current with $R = [\![\Sigma, \xi, \theta]\!]$, then*

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\}$$

for every pair $m, n = 1, \ldots, k$ and for every x in the closure of the set of points of positive density of Σ.

Proof. Choose a $(k-1)$-form α satisfying (2.1.7) and (2.1.8) of Lemma 2.1.6 for some open set $U \subset \mathbb{R}^d$. Therefore,

$$0 \neq \langle T \llcorner U; \mathrm{d}\alpha\rangle = \langle \partial T \llcorner U; \alpha\rangle\,,$$

because of 2.1.7. But then Theorem 2.2.1 and condition (2.1.8) imply

$$\langle \partial T \llcorner U; \alpha\rangle = 0$$

and this is a contradiction. □

Remark 2.2.7. The spirit of the question arised in this section and answered by Theorem 2.2.6 is more or less this: consider a non-involutive k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, for which weak notion of k-dimensional surface (weaker than a submanifold, of course) the conclusions of the Frobenius Theorem hold? We provided the answer for integral k-currents. When the vector field in analysis is the horizontal distribution of the Heisenberg group $\mathbb{H}^d$, i.e.

$$\tau_i = \partial_{x_i} - x_{d+i}\partial_{x_{2d+1}} \quad \tau_{d+i} = \partial_{x_{d+i}} + x_i\partial_{x_{2d+1}} \quad i = 1, \ldots, d$$

in $\mathbb{R}^{2d+1}$, there is a rich literature on the subject starting from [7]. Being an important example of space endowed with a non-involutive vector field, we try to sketch the state of the art in the case of the first Heisenberg group $\mathbb{H}^1$. In [7] it is proved that, if Σ is a 2-dimensional Lipschitz graph in $\mathbb{H}^1$, then Σ satisfies a suitable non-horizontality[6] condition which brings us back to Frobenius Theorem. Moreover, if Σ is the image of a function in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$ (with $\Omega \subset \mathbb{R}^2$) with maximal rank Jacobian matrix, then we have non-horizontality, too, and this is proved (in the more general case of k-dimensional horizontal distributions in $\mathbb{H}^d$ with $d < k \leq 2d$) in [40]. The picture is completed by a different result of horizontality in [7]: it is proved that there exists a BV-function defined in the square $(0,1)^2 \subset \mathbb{R}^2$ whose graph is contained in the so-called Heisenberg square Q_H, which is an horizontal fractal introduced by Balogh and Tyson in [8]. Moreover, this graph is 2-dimensional in a suitable sense.

2.3 Decomposition of normal currents

As we anticipated in Subsection 1.3.2, an interesting problem in the theory of currents concerns the *decomposition of a normal current* by means of a family of integral currents. This problem firstly appeared in [1], formulated by F. Morgan. More precisely, given a normal current $T \in \mathbb{N}_k(\mathbb{R}^d)$, we ask whether there exists a family of integral currents $(R_\lambda)_{\lambda \in L}$, where L is a suitable measure space, such that

[6] By "non-horizontality" we mean a suitable property, too technical for the scope of this remark, catching the fact that Σ cannot have the horizontal distribution as a tangent distribution.

(i) $T = \int_L R_\lambda \, d\lambda$, i.e., for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$, we can write

$$T(\omega) = \int_L R_\lambda(\omega)\, d\lambda\,;$$

(ii) $\mathbb{M}(T) = \int_L \mathbb{M}(R_\lambda)\, d\lambda$;

(iii) $\mathbb{M}(\partial T) = \int_L \mathbb{M}(\partial R_\lambda)\, d\lambda$.

Condition (ii) and (iii) recall (1.3.18) and (1.3.19), respectively, from Corollary 1.3.3: they express the requirement of a decomposition where no mass is wasted someway. In the analysis below, we will discuss also weaker versions of the problem: we can drop condition (iii), and we can also change the type of "decomposing" currents, saying we are satisfied with a family of rectifiable currents, instead of the family of integral ones. We agree that, when we do not specify the type of currents to which the decomposing family belongs, we will always be looking for a decomposition into integral currents; otherwise, we will always specify the type of the decomposing currents.

When the dimension of the normal current is 1, it is known that there exists a decomposition satisfying (i) and (ii): see Proposition 1.3.4 and [52] or [45] for the proof (while a decomposition satisfying (i), (ii) and (iii) may not always exist). When the codimension of the normal current is 1, there exists a decomposition in rectifiable currents satisfying (i) and (ii) (as pointed out by G. Alberti), too. In the special case of codimension 1 with an integer rectifiable boundary, there exists a decomposition satisfying (i), (ii) and (iii), thanks to an observation by M. Zworski in [57], and we will see the core of this argument at the end of this section.

The existence of a decomposition for normal currents with rectifiable boundary in codimension 1 is an isolated result. Indeed, in general the search for a decomposition by means of integral currents is actually too strong: we claim that any normal current of the form $\xi\mathscr{L}^d$ cannot be decomposed into integral currents satisfying (i) and (ii), provided ξ is non-involutive. This claim follows from Lemma 2.3.1 and Theorem 2.2.6.

In [57], M. Zworski exhibited this very same counterexample, claiming that, in general, a normal current has no decomposition satisfying (i) and (ii), even if we allow the decomposing currents to be rectifiable only. However the proof in [57] does not work, as pointed out by Alberti (see Section 4.5 of [43]). There is a gap in the argument, possibly due to a misunderstanding when referring to the Federer Flatness Theorem 4.1.15 in [24]. We propose the same counterexample for the problem of decomposing a normal current with a family of integral currents satisfying (i) and (ii) (at page 66) and the results we got in Section 2.2 fill the aforementioned gap. Lemma 2.3.2 is a piece of Zworski's argument, but it can also be read and proved as a classical result on the decomposition of vector-valued measures, as we do in Lemma 2.3.1.

Lemma 2.3.1. *Consider a vector-valued measure $\mu = \xi|\mu|$, where*

$$\xi \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$$

is a smooth k-vector field and assume that

(i) $\mu = \int_L \mu_\lambda \, d\lambda$, *where (L, λ) is a measure space;*

(ii) $\|\mu\| = \int_L \|\mu_\lambda\| \, d\lambda$.

Then, for λ-a.e. μ_λ, we have that

$$\mu_\lambda = \xi|\mu_\lambda| \, .$$

Proof. For every $\lambda \in L$ we can write

$$\mu_\lambda = (f_\lambda \xi + \nu_\lambda)\,|\mu_\lambda| \, ,$$

where f_λ is a real-valued function and ν_λ is an orthogonal (with respect to ξ) k-vector field. Consider the family of measures

$$\hat{\mu}_\lambda := f_\lambda \xi |\mu_\lambda| \quad \text{for every } \lambda \in L \, ,$$

we claim that

$$\mu = \int_L \hat{\mu}_\lambda \, d\lambda \, .$$

Indeed, given any smooth and compactly supported test covector field $w \in \mathscr{C}_c^\infty(\mathbb{R}^d; T^k(\mathbb{R}^d))$, we can split it as $w = \hat{w}\xi^* + w_\nu$, where $\xi^* \in T_k(\mathbb{R}^d)$ is the dual element of ξ in the biorthogonal system of $T_k(\mathbb{R}^d)$ and $T^k(\mathbb{R}^d)$, that is, $\langle \xi^*, \xi \rangle = \|\xi\|^2$, $\langle \xi^*, \nu \rangle = 0$ whenever $\nu \in T_k(\mathbb{R}^d)$ is orthogonal to ξ and $\langle w_\nu; \xi \rangle = 0$, then we can compute

$$\begin{aligned}
\langle \mu, w \rangle &= \int \langle \hat{w}\xi^* + w_\nu; \xi \rangle \, d|\mu| = \langle \mu; \hat{w}\xi^* \rangle \\
&\overset{\text{(i)}}{=} \int_L \langle \mu_\lambda; \hat{w}\xi^* \rangle \, d\lambda = \int_L \int \langle \hat{w}\xi^*; f_\lambda \xi + \nu_\lambda \rangle \, d|\mu_\lambda| \, d\lambda \\
&= \int_L \int \langle \hat{w}\xi^* + w_\nu; f_\lambda \xi \rangle \, d|\mu_\lambda| \, d\lambda = \int_L \langle \hat{\mu}_\lambda; w \rangle \, d\lambda \, .
\end{aligned}$$

Therefore, we can estimate

$$\|\mu\| \leq \int_L \|\hat{\mu}_\lambda\| \, d\lambda \leq \int_L \|\mu_\lambda\| \, d\lambda \overset{\text{(ii)}}{=} \|\mu\| \,,$$

meaning that $\|\hat{\mu}_\lambda\| = \|\mu_\lambda\|$ for almost every λ. We can conclude that $\nu_\lambda(x) = 0$ for μ_λ-a.e. x and a.e. λ. □

For the sake of completeness, we also state the analog of Lemma 2.3.1 in the version given in [57].

Lemma 2.3.2. *Consider the normal current $T := \xi \mathscr{L}^d \in \mathbb{N}_k(\mathbb{R}^d)$ given by the smooth vector field $\xi \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ and assume that*

(i) *$T = \int_L R_\lambda \, d\lambda$, where (L, λ) is a measure space and $\{R_\lambda\}_{\lambda \in L} \subset \mathscr{R}_k(\mathbb{R}^d)$;*

(ii) *$\mathbb{M}(T) = \int_L \mathbb{M}(R_\lambda) \, d\lambda$.*

Then, for λ-a.e. R_λ, we have that

$$R_\lambda = [\![\Sigma_\lambda, \xi, \theta_\lambda]\!] \,,$$

for some rectifiable set $\Sigma_\lambda \subset \mathbb{R}^d$ and some $\theta_\lambda \in L^1(\mathbb{R}^d)$.

Theorem 2.3.3. *Consider the normal current $T := \xi \mathscr{L}^d \in \mathbb{N}_k(\mathbb{R}^d)$ given by the smooth non-involutive vector field $\xi \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$. Then there exist no measure space L and no family of integral currents $(R_\lambda)_{\lambda \in L}$ such that* (i) *and* (ii) *at page 66 hold.*

The proof of this theorem is a consequence of Lemma 2.3.1 and Theorem 2.2.6.

As we said before, we conclude this section with an explanation of the existence of the decomposition for a $(d-1)$-dimensional normal current with integer rectifiable boundary (see also [57]). Indeed, if N is a closed normal $(d-1)$-current, then N is the boundary of a normal d-current associated with a suitable function[7] $f \in BV(\mathbb{R}^d)$, that is $N = \partial\left(f(x)\, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d\right)$. Applying the Coarea Formula[8], we get

$$N = \int \partial\left(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d \llcorner \{f \geq s\}\right) \, ds \,,$$

[7] This result is related to the comparison between currents and distributions in Remark 1.2.33: the function f has bounded variation because N has finite mass, implying that $\mathrm{div} f$ is a distribution of order 0.

[8] For the Coarea Formula see [21], or [28] for the original paper.

with $L = L^1(\mathbb{R})$. Moreover

$$\mathbb{M}(N) = \int \mathbb{M}\left(\partial\left(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d \llcorner \{f \geq s\}\right)\right) ds \,.$$

More generally, if $\partial N \in \mathscr{R}_{d-2}(\mathbb{R}^d)$, then we can reduce to the previous case thanks to this result due to R. Hardt and J. Pitts (see [30]).

Theorem 2.3.4 (Hardt-Pitts decomposition)**.** *Consider a compactly supported normal current $N \in \mathbb{N}_{d-1}(\mathbb{R}^d)$ with rectifiable boundary $\partial N \in \mathscr{R}_{d-2}(\mathbb{R}^d)$. Then there exists an integral current $R \in \mathbf{I}_{d-1}(\mathbb{R}^d)$ with the same boundary $\partial R = \partial N$ and*

$$\mathbb{M}(N - R) = \mathbb{M}(N) + \mathbb{M}(R) \,.$$

Moreover, for $\|R\|$-a.e. $x \in \mathbb{R}^d$, we have that the orientation of $R - N$ at x coincides with the orientation of R at x. Finally, if $N - R = \partial\left(f(x)\ \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d\right)$, then

$$D^{d-1}\|N - R\|(x) = D^{d-1}\|R\|(x) = \overline{f}(x) - \underline{f}(x) \in \mathbb{Z}$$

for $\|R\|$-a.e. $x \in \mathbb{R}^d$, where $\underline{f}$ and $\overline{f}$ are the approximate lower and upper limits of f, respectively.

As we announced at the end of Subsection 1.2.4, Theorem 2.3.4 provides an alternative solution to the Plateau's problem for integral currents in codimension 1, without invoking the Closure Theorem 1.2.59.

Part II

Currents with coefficients in a group

Chapter 3

Currents with coefficients in a group

Currents with coefficients in a group were introduced by W. Fleming. There is a vast literature on the subject: let us mention only the seminal paper [27], the work of B. White [54, 55], and the more recent papers by T. De Pauw and R. Hardt [19] and by L. Ambrosio and M. G. Katz [4].

In the aforementioned works, the set of *flat G-chains* is obtained by completion: they define *polyhedral G-chains* as the additive group generated by oriented simplexes with multiplicities in G (compare with Definition 1.2.47). The group G is required to be a normed abelian group such that its norm makes it a complete metric space. The group of flat G-chains is the smallest group containing the polyhedral G-chains and closed under the flat norm (see Definition 1.2.53). The most important theorems about currents still hold in the context of G-chains: in [54] we can find the analog of Deformation Theorem 1.2.49. In [55] a closure theorem for flat G-chains is proved, provided G satisfies the following property: any non-constant continuous path in G has infinite length. Let us remark that discrete groups fulfill this property.

We will need a different approach to G-chains, or currents with coefficients in G. Since we are interested in the calibration technique, we need an integral representation for currents with coefficients in a group. In Section 3.1 we provide definitions for currents over a coefficient group, with some basic examples. We also restate the main theorems in this new framework and we conclude the section by investigating the existence of calibrations for mass-minimizing currents with coefficients in G.

In Section 3.2 we analyze the Steiner tree problem by means of currents with coefficients in a suitably chosen group G. Section 3.1 and Section 3.2 are both drawn from [41].

In Section 3.3, following [13], we discuss an application of currents with group coefficients to models of crystals dislocations.

3.1 Definitions and main theoretical results

This first part of the section may seem a pretty tedious repetition of Section 1.2, but we actually need it in order to fix the notation and clarify some choices.

Fix an open set $U \subset \mathbb{R}^d$ and a normed vector space $(E, \|\cdot\|_E)$ with finite dimension $m \geq 1$. We will denote by $(E^*, \|\cdot\|_{E^*})$ its dual space endowed with the dual norm

$$\|f\|_{E^*} := \sup_{\|v\|_E \le 1} \langle f; v\rangle \ .$$

Definition 3.1.1. We say that a map

$$\omega : \Lambda_k(\mathbb{R}^d) \times E \to \mathbb{R}$$

is an E^*-valued k-covector in $\mathbb{R}^d$ if

(i) $\forall\, \tau \in \Lambda_k(\mathbb{R}^d), \quad \omega(\tau, \cdot) \in E^*$, that is $\omega(\tau, \cdot) : E \to \mathbb{R}$ is a linear function.

(ii) $\forall\, v \in E, \quad \omega(\cdot, v) : \Lambda_k(\mathbb{R}^d) \to \mathbb{R}$ is a (classical) k-covector.

Sometimes we will use $\langle \omega; \tau, v\rangle$ instead of $\omega(\tau, v)$, in order to simplify the notation. The space of E^*-valued k-covectors in $\mathbb{R}^d$ is denoted by $\Lambda^k_E(\mathbb{R}^d)$ and it is endowed with the comass norm

$$\|\omega\| := \sup\left\{\|\omega(\tau, \cdot)\|_{E^*} : |\tau| \le 1, \tau \text{ simple}\right\} \ . \tag{3.1.1}$$

Remark 3.1.2. Fix an orthonormal system of coordinates in $\mathbb{R}^d$, $(\mathbf{e}_1, \dots, \mathbf{e}_d)$, as usual, with its corresponding dual base in $(\mathbb{R}^d)^*$ given by $(\mathrm{d}x_1, \dots, \mathrm{d}x_d)$. Consider a complete biorthonormal system for E, i.e., a pair

$$(v_1, \dots, v_m) \in E^m; \ (w_1, \dots, w_m) \subset (E^*)^m$$

such that $\|v_i\|_E = 1$, $\|w_i\|_{E^*} = 1$ and $\langle w_i; v_j\rangle = \delta_{ij}$. Given an E^*-valued k-covector ω, we denote

$$\omega^j := \omega(\cdot, v_j).$$

For each $j \in \{1, \dots, m\}$, ω^j is a k-covector in the usual sense. Hence the biorthonormal system $(v_1, \dots, v_m)$, $(w_1, \dots, w_m)$ allows to write ω in "components"

$$\omega = (\omega^1, \dots, \omega^m) \,,$$

in fact we have

$$\omega(\tau, v) = \sum_{j=1}^m \langle \omega^j; \tau\rangle \langle w_j; v\rangle \ .$$

In particular ω^j admits the usual representation

$$\omega^j = \sum_{1 \le i_1 < \ldots < i_k \le d} a^j_{i_1 \ldots i_k} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \,, \qquad j = 1, \dots, m.$$

Definition 3.1.3. An E^*-valued differential k-form in $U \subset \mathbb{R}^d$, or just a k-form when it is clear which group we are referring to, is a map

$$\omega : U \to \Lambda^k_E(\mathbb{R}^d);$$

we say that ω is $\mathscr{C}^\infty$-regular if every component ω^j is so (see Remark 3.1.2). We denote by $\mathscr{C}^\infty_c(U, \Lambda^k_E(\mathbb{R}^d))$ the vector space of $\mathscr{C}^\infty$-regular E^*-valued k-forms with compact support in U.

We are mainly interested in E^*-valued 1-forms, nevertheless we analyze k-forms in wider generality, in order to ease other definitions, such as the differential of an E^*-valued form and the boundary of an E-current.

Definition 3.1.4. We define the differential $\mathrm{d}\omega$ of a $\mathscr{C}^\infty$-regular E^*-valued k-form ω by components:

$$\mathrm{d}\omega^j = \mathrm{d}(\omega^j) : U \to \Lambda^{k+1}(\mathbb{R}^d)\,, \qquad j = 1, \dots, m\ ,$$

Moreover, $\mathscr{C}^\infty_c(U, \Lambda^1_E(\mathbb{R}^d))$ has a norm, denoted by $\| \cdot \|$, given by the supremum of the comass norm of the form defined in (3.1.1). Hence we mean

$$\|\omega\| := \sup_{x \in U} \|\omega(x)\|\ . \tag{3.1.2}$$

Definition 3.1.5. A k-dimensional current T in $U \subset \mathbb{R}^d$, with coefficients in E, or just an E-current when there is no doubt on the dimension, is a linear and continuous function

$$T : \mathscr{C}^\infty_c(U, \Lambda^k_E(\mathbb{R}^d)) \longrightarrow \mathbb{R}\ ,$$

where the continuity is meant with respect to the locally convex topology on the space $\mathscr{C}^\infty_c(U, \Lambda^k_E(\mathbb{R}^d))$, built in analogy with the topology on $\mathscr{C}^\infty_c(\mathbb{R}^n)$, with respect to which distributions are dual. This defines the weak* topology on the space of k-dimensional E-currents. Convergence in this topology is equivalent to the convergence of all the "components" in the space of classical[1] k-currents, by which we mean the following. We define for every k-dimensional E-current T its components T^j, for $j = 1, \dots m$, and we write

$$T = (T^1, \dots, T^m),$$

[1]In the sequel we will use "classical" to refer to the usual currents, with coefficients in $\mathbb{R}$ or possibly in $\mathbb{Z}$.

denoting

$$\langle T^j; \varphi \rangle := \langle T; \widetilde{\varphi}_j \rangle \ ,$$

for every (classical) compactly supported differential k-form φ on $\mathbb{R}^d$. Here $\widetilde{\varphi}_j$ denotes the E^*-valued differential k-form on $\mathbb{R}^d$ such that

$$\widetilde{\varphi}_j(\cdot, v_j) = \varphi, \tag{3.1.3}$$

$$\widetilde{\varphi}_j(\cdot, v_i) = 0 \quad \text{for } i \neq j \ . \tag{3.1.4}$$

It turns out that a sequence of k-dimensional E-currents T_h weakly* converges to an E-current T (in this case we write $T_h \overset{*}{\rightharpoonup} T$) if and only if the sequence of the components T_h^j converge to T^j in the space of classical k-currents, for $j = 1, \dots, m$.

Definition 3.1.6. For a k-current T over E we define the boundary operator

$$\langle \partial T; \varphi \rangle := \langle T; \mathrm{d}\varphi \rangle \quad \forall\, \varphi = (\varphi^1, \dots, \varphi^m) \in \mathscr{C}_c^\infty(U, \Lambda_E^{k-1}(\mathbb{R}^d))$$

and the mass

$$\mathbb{M}(T) := \sup_{\|\omega\| \le 1} \langle T; \omega \rangle.$$

As one can expect, the boundary $\partial(T^j)$ of every component T^j is the relative component $(\partial T)^j$of the boundary ∂T.

Definition 3.1.7. A k-dimensional normal E-current in $U \subset \mathbb{R}^d$ is an E-current T with $\mathbb{M}(T) < +\infty$ and $\mathbb{M}(\partial T) < +\infty$. Thanks to the Riesz Theorem, T admits the following representation:

$$\langle T; \omega \rangle = \int_U \langle \omega(x); \tau(x), v(x) \rangle \, \mathrm{d}\mu_T(x) \ , \quad \forall\, \omega \in \mathscr{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d)) \ .$$

where μ_T is a Radon measure on U and $v : U \to E$ is summable with respect to μ_T and $|\tau| = 1$, μ_T-a.e. A similar representation holds for the boundary ∂T.

Definition 3.1.8. A rectifiable k-current T in $U \subset \mathbb{R}^d$, over E, or a rectifiable E-current is an E-current admitting the following representation:

$$\langle T; \omega \rangle := \int_\Sigma \langle \omega(x); \tau(x), \theta(x) \rangle \, \mathrm{d}\mathscr{H}^k(x), \quad \forall\, \omega \in \mathscr{C}_c^\infty(\mathbb{R}^d, \Lambda_E^k(U))$$

where Σ is an $\mathscr{H}^k$-rectifiable set contained in U, $\tau(x) \in T_x\Sigma$ with $|\tau(x)| = 1$ for $\mathscr{H}^k$-a.e. $x \in \Sigma$ and $\theta \in L^1(U; E)$. We will refer to such a current as $T = [\![\Sigma, \tau, \theta]\!]$. If B is a Borel set and $[\![\Sigma, \tau, \theta]\!]$ is a rectifiable E-current, we denote by $T \llcorner B$ the current $[\![\Sigma \cap B, \tau, \theta]\!]$.

Consider now a discrete subgroup $G < E$, endowed with the restriction of the norm $\|\cdot\|_E$. If the multiplicity θ takes only values in G, and if the same representation holds for ∂T, we call T a rectifiable G-current. Pay attention to the fact that, in the framework of currents over the coefficient group E, rectifiable E-currents play the role of (classical) rectifiable current, while rectifiable G-currents correspond to (classical) integral currents. Actually this correspondence is an equality, when E is the group $\mathbb{R}$ (with the euclidean norm) and G is $\mathbb{Z}$.

Example 3.1.9. Let $E = \mathbb{R}^d$ and let G be the additive subgroup generated by m elements $g_1, \dots, g_m$. Given $m+1$ points $p_1, \dots, p_m, p_{m+1} \in \mathbb{R}^2$, consider the cone C over $(p_1, \dots, p_m)$ with respect to p_{m+1}: if Σ_r is the oriented segment from p_r to p_{m+1}, $r = 1, \dots, m$, then

$$C = \bigcup_{r=1}^{m} \Sigma_r \ .$$

We can define a rectifiable G-current supported on C as

$$\langle T; \omega \rangle := -\sum_{r=1}^{m} \int_{\Sigma_r} \langle \omega(x); \tau_r(x), g_r \rangle \, \mathrm{d}\mathscr{H}^1(x),$$

where τ_r is the unit tangent vector to Σ_r, pointing towards p_{m+1}. It is easy to see that, denoting $g_{m+1} = -(g_1 + \ldots + g_m)$ we can represent the 0-dimensional rectifiable G-current ∂T with the points $p_1, \dots, p_{m+1}$ with multiplicities $g_1, \dots, g_{m+1}$, respectively. From now on we will denote such a current as $g_1\delta_{p_1} + \ldots + g_{m+1}\delta_{p_{m+1}}$.

Proposition 3.1.10. *Let $T = [\![\Sigma, \tau, \theta]\!]$ be a rectifiable E-current, then*

$$\mathbb{M}(T) = \int_{\Sigma} \|\theta(x)\|_G \, \mathrm{d}\mathscr{H}^1(x) \ .$$

Since the mass is lower semicontinuous, we can apply the direct method of the Calculus of Variations for the existence of minimizers with given boundary, once we provide the following compactness result. Here we assume for simplicity that G is the subgroup of E generated by $v_1, \dots, v_m$. A similar argument works for every discrete subgroup G.

Theorem 3.1.11. *Let $(T_h)_{h \geq 1}$ be a sequence of rectifiable G-currents such that there exists a positive finite constant C satisfying*

$$\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leq C \quad \text{for every } h \geq 1 \ .$$

Then there exists a subsequence $(T_{h_i})_{i\geq 1}$ *and a rectifiable* G*-current* T *such that*

$$T_{h_i} \overset{*}{\rightharpoonup} T.$$

Proof. The statement of the theorem can be proved component by component. In fact, let $T_h^1, \dots, T_h^m$ be the components of T_h. Since $(v_1, \dots, v_m)$ and $(w_1, \dots, w_m)$ is a biorthonormal system, we have

$$\mathbb{M}(T_h^j) + \mathbb{M}(\partial T_h^j) \leq \mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leq C \ ,$$

hence, after a diagonal procedure, we can find a subsequence $(T_{h_i})_{i\geq 1}$ such that $\left(T_{h_i}^j\right)_{i\geq 1}$ weakly* converges to some integral current T^j, for every $j = 1, \dots, m$. Denoting by T the rectifiable G-current, whose components are $T^1, \dots, T^m$, we have

$$T_{h_i} \overset{*}{\rightharpoonup} T.$$

□

3.1.1 Generalized calibrations

As we recalled in the introduction to this chapter, our interest in calibrations is the reason why we have chosen to provide an integral representation for E-currents, in fact the existence of a calibration guarantees the minimality of the associated current, as we will see in Proposition 3.1.13. As it happened for the definitions on currents, the model for this theory is the well-known standard case ($E = \mathbb{R}$ and $G = \mathbb{Z}$), which can be found in Section 1.4.

Definition 3.1.12. A smooth calibration associated with a k-dimensional rectifiable G-current $[\![\Sigma, \tau, \theta]\!]$ is a smooth compactly supported E^*-valued differential k-form ω, with the following properties:

(i) $\langle \omega(x); \tau(x), \theta(x)\rangle = \|\theta(x)\|_G$ for $\mathscr{H}^k$-a.e. $x \in \Sigma$;

(ii) $\mathrm{d}\omega = 0$;

(iii) $\|\omega\| \le 1$, i.e., $\|\langle \omega; \tau\rangle\|_{E^*} \le 1$, for every simple k-vector τ with $|\tau| = 1$.

Proposition 3.1.13. *A rectifiable G-current T which admits a smooth calibration ω is a minimizer for the mass among the normal E-currents with boundary ∂T.*

Proof. Fix a competitor T' which is a normal E-current associated with the vectorfield τ', the multiplicity θ' and the measure $\mu_{T'}$, with $\partial T' = \partial T$. Since $\partial(T - T') = 0$, then $T - T'$ is a boundary of some current S in $\mathbb{R}^d$, and then

$$
\begin{aligned}
\mathbb{M}(T) &= \int_\Sigma \|\theta\|_G \, \mathrm{d}\mathscr{H}^k && (3.1.5)\\
&\overset{\text{(i)}}{=} \int_\Sigma \langle \omega(x); \tau(x), \theta(x)\rangle \, \mathrm{d}\mathscr{H}^k = \langle T; \omega\rangle && (3.1.6)\\
&\overset{\text{(ii)}}{=} \langle T'; \omega\rangle = \int_{\mathbb{R}^d} \langle \omega(x); \tau'(x), \theta'(x)\rangle \, \mathrm{d}\mu_{T'} && (3.1.7)\\
&\overset{\text{(iii)}}{\le} \int_{\mathbb{R}^d} \|\theta'\|_G \, \mathrm{d}\mu T' = \mathbb{M}(T')\ , && (3.1.8)
\end{aligned}
$$

where each equality (respectively inequality) holds because of the corresponding property of ω, as established in Definition 3.1.12. In particular, equality in (ii) follows from

$$\langle T - T'; \omega\rangle = \langle \partial S; \omega\rangle = \langle S; \mathrm{d}\omega\rangle = 0.$$

□

Remark 3.1.14. If T is a rectifiable G-current calibrated by ω, then every mass minimizer with boundary ∂T is calibrated by the same form ω. In fact, choose a mass minimizer $T' = [\![\Sigma', \tau', \theta']\!]$ with boundary $\partial T' = \partial T$: obviously we have $\mathbb{M}(T) = \mathbb{M}(T')$, then equality holds in (3.1.8), which means

$$\langle \omega(x); \tau'(x), \theta'(x) \rangle = \|\theta'(x)\|_G \quad \text{for } \mathscr{H}^k - \text{a.e. } x \in \Sigma' \ .$$

In Definition 3.1.12 we intentionally kept vague the regularity of the form ω. Indeed ω has to be a compactly supported[2] smooth form, a priori, in order to fit Definition 3.1.5. Nevertheless, in some situations it will be useful to consider calibrations with lower regularity, for instance piecewise constant forms. As long as (3.1.6)-(3.1.8) remain valid, it is meaningful to do so; for this reason we introduce the following very general definition.

Definition 3.1.15. A *generalized calibration* associated with a k-dimensional normal E-current T is a linear and bounded functional ϕ on the space of normal E-currents satisfying the following conditions:

(i) $\phi(T) = \mathbb{M}(T)$;

(ii) $\phi(\partial R) = 0$ for any $(k+1)$-dimensional normal E-current R;

(iii) $\|\phi\| \le 1$.

Remark 3.1.16. The thesis in Proposition 3.1.13 is still true, since for every competitor T' with $\partial T = \partial T'$, there holds

$$\mathbb{M}(T) = \phi(T) = \phi(T') + \phi(\partial R) \le \mathbb{M}(T') \ ,$$

where R is chosen such that $T - T' = \partial R$. Such R exists because T and T' are in the same homology class.

As we will show by some examples (see 3.2.6 and 3.2.7), an easy way to "build" a calibration is to search it among piecewise constant forms. So we have to establish a compatibility condition which brings piecewise constant forms back to Definition 3.1.15. Since this definition has a practical aim, we allow ourselves to treat the case of 1-currents in $\mathbb{R}^2$ only.

[2]Since we will always deal with currents that are compactly supported, we can easily drop the assumption that ω has compact support.

Definition 3.1.17. Fix a 1-dimensional rectifiable G-current T in $\mathbb{R}^2$, $T = [\![\Sigma, \tau, \theta]\!]$. Assume we have a collection $\{C_r\}_{r\geq 1}$ which is a locally finite, Lipschitz partition of $\mathbb{R}^2$, i.e., $\bigcup_{r\geq 1} C_r = \mathbb{R}^2$, the boundary of every set C_r is a Lipschitz curve and $C_r \cap C_s = \varnothing$ whenever $r \neq s$. Assume moreover that ∂C_r is a connected set for every r and that C_r contains the connected non-empty interior of its closure. Let us consider a compactly supported piecewise constant E^*-valued 1-form ω with

$$\omega \equiv \omega_r \quad \text{on } C_r$$

where $\omega_r \in \Lambda^1_E(\mathbb{R}^2)$ for every r. In particular $\omega \neq 0$ only on finitely many elements of the partition. Then we say that ω represents a *compatible calibration* for T if the following conditions hold:

(i) for almost every $x \in \Sigma$, $\langle \omega(x); \tau(x), \theta(x) \rangle = \|\theta(x)\|_G$;

(ii) for $\mathscr{H}^1$-almost every point $x \in \partial C_r \cap \partial C_s$ we have

$$\langle \omega_r - \omega_s; \tau(x), \cdot \rangle = 0,$$

where τ is tangent to ∂C_r;

(iii) $\|\omega_r\| \leq 1$ for every r.

We will refer to condition (ii) with the expression of *compatibility condition* for a piecewise constant form.

Proposition 3.1.18. *Let ω be a compatible calibration for the rectifiable G-current T. Then T minimizes the mass among the normal E-currents with boundary ∂T.*

Proof. Firstly we see that a suitable counterpart of Stokes's Theorem 1.2.28 holds. Namely, given a component ω^j of ω and a classical integral 1-current $T = [\![\Sigma, \tau, 1]\!]$ in $\mathbb{R}^2$, without boundary, then the quantity

$$\langle \omega^j; T \rangle := \int_\Sigma \langle \omega^j(x); \tau(x) \rangle \mathrm{d}\mathscr{H}^1(x)$$

is well defined, and we claim that it is equal to zero. The fact that it is well defined is a direct consequence of the compatibility condition (ii) in Definition 3.1.17. To prove that it is equal to zero, note that it is possible to find at most countably many unit multiplicity integral 1-currents $T_i = [\![\Sigma_i, \tau_i, 1]\!]$ in

$\mathbb{R}^2$, without boundary, each one supported in a single set C_r, such that $\sum_i T_i = T$. Since

$$\int_{\Sigma_i} \langle \omega^j(x); \tau_i(x) \rangle \mathrm{d}\mathscr{H}^1(x) = 0$$

for every i, then the claim follows from (ii). As a consequence we have that there exists a family of Lipschitz functions $\phi_j : \mathbb{R}^2 \to \mathbb{R}$ such that for every (classical) integral 1-current S with $\mathbb{M}(\partial S) \leq 2$ (in particular $\partial S = \delta_{x_S} - \delta_{y_S}$, and $x_S = y_S$ if and only if $\partial S = 0$) there holds:

$$\langle \omega^j; S \rangle = \phi_j(x_S) - \phi_j(y_S), \quad \text{for every } j.$$

In fact it is sufficient to choose $\phi_j(0) = 0$ and

$$\phi_j(x) = |x| \int_0^1 \langle \omega^j(tx); \frac{x}{|x|} \rangle \, \mathrm{d}t.$$

Moreover it is easy to see that every ϕ_j is constant outside of the support of ω^j, so we can assume, possibly subtracting a constant, that ϕ_j is compactly supported.

Now, take a 2-dimensional normal E-current T. Let $\{T^j\}_j$ be the components of T. For every j, use the decomposition given by Proposition 1.3.4, in order to write $S^j := \partial T^j = \int_0^{M_j} S_t^j \, \mathrm{d}t$. Then we have

$$\langle \omega; \partial T \rangle = \sum_j \int_0^{M_j} \langle \omega^j; S_t^j \rangle \, \mathrm{d}t = \sum_j \int_0^{M_j} \phi_j(x_{S_t^j}) - \phi_j(y_{S_t^j}) \, \mathrm{d}t.$$

Since for every j we have

$$0 = \partial(\partial T^j) = \int_0^{M_j} \delta_{x_{S_t^j}} - \delta_{y_{S_t^j}} \, \mathrm{d}t,$$

then, for every j, we must have

$$\int_0^{M_j} g(x_{S_t^j}) - g(y_{S_t^j}) \, \mathrm{d}t = 0,$$

for every compactly supported Lipschitz function g, in particular for every ϕ_j. Hence we have $\langle \omega; \partial T \rangle = 0$. $\square$

We postpone the examples of calibrations to Section 3.2.

3.1.2 The equivalence between minima and the existence of a calibration

Once we established that the existence of a calibration is a sufficient condition for a rectifiable G-current to be a mass-minimizer, we may wonder if the converse is also true: does a calibration (of some sort) exist for every mass-minimizing rectifiable G-current?

Let us step backward: does it occur for classical integral currents? The answer is quite articulate, but we can briefly summarize the state of the art we will rely upon. See also [12] for an overview of this problem of duality between minima and calibrations.

Remark 3.1.19. An actual calibration cannot exist for every minimizer. In fact there are currents which minimize the mass among integral currents with a fixed boundary, but not among normal currents (in some cases the two problems have different minima). This means that such integral currents cannot be calibrated, in fact the existence of a calibration proves the minimality among normal currents.

Remark 3.1.20. For every mass-minimizing classical normal k-current T, there exists a generalized calibration ϕ in the sense of Definition 3.1.15. Moreover, by means of the Riesz Representation Theorem, ϕ can be represented by a measurable map from U to $\Lambda^k(\mathbb{R}^d)$. This result is contained in [25].

In particular, Remark 3.1.20 provides a positive answer to the question of the existence of a generalized calibration for mass-minimizing integral currents of dimension $k = 1$, because minima among both normal and integral currents coincide, as we proved in Theorem 1.3.5. It is possible to apply the same technique in the class of normal E-currents, therefore we have the following proposition.

Proposition 3.1.21. *For every mass minimizing normal E-current T, there exists a generalized calibration.*

In order to guarantee the existence of a generalized calibration also for 1-dimensional mass-minimizing rectifiable G-currents, we need an analog of Proposition 1.3.5 in the framework of G-currents. Namely, we need to prove that the minimum of the mass among 1-dimensional normal E currents with the same boundary coincides with the minimum calculated among rectifiable

G-currents. Here the boundary is of course a 0-dimensional rectifiable G-current. This is a well-known issue for classical k-dimensional currents: for $k \geq 2$ it is not even known whether the two minima are commensurable, i.e., whether or not there exists a constant C such that, for every fixed $(k-1)$-dimensional integral boundary B, the minimum of the mass among integral k-currents with boundary B is less then C times the minimum among normal k-currents with the same boundary. From the argument used in the proof of Proposition 1.3.5 we realize that the equality of the two minima in the framework of 1-dimensional E-currents is equivalent to the homogeneity property in Remark 3.1.22. This property, which is trivially verified for classical integral currents, seems to be an interesting issue in the class of rectifiable G-currents. In Example 3.1.23 we exhibit a subset $M \subset \mathbb{R}^2$ such that, if our currents are forced to be supported on M, then the homogeneity property does not hold. In other words, we can say that equality of the two minima does not hold in the framework of 1-dimensional E-currents on the metric space M. We can see the same phenomenon if we substitute the metric space M with the metric space $\mathbb{R}^2$ endowed with a density, which is unitary on the points of M and very high outside.

Thus, we want to know whether the analog of Theorem 1.3.5 holds also in the framework of 1-dimensional E-currents. Fix a 0-dimensional rectifiable G-current R in $U \subset \mathbb{R}^d$. Do the minima for the mass among 1-dimensional normal E-currents and rectifiable G-currents with boundary R coincide?

Remark 3.1.22. The answer to the previous question is positive if and only if the following is true: given $R = \sum_{i=1}^n g_i \delta_{x_i}$ with $\|g_i\|_G = 1$ and T a rectifiable G-current which is mass-minimizer with $\partial T = R$, then for every $k \in \mathbb{N}$ we have that

$$\min \{\mathbb{M}(S) : S \text{ rectifiable } G-\text{current}, \partial S = kR\} = k\mathbb{M}(T) \ . \tag{3.1.9}$$

Notice that, using the notation introduced in Theorem 1.3.5, (3.1.9) can be meaningfully written as

$$\mathscr{M}_I(kR) = k\mathscr{M}_I(R) \ . \tag{3.1.10}$$

The condition 3.1.10 is clearly necessary to have the equality of the two minima. It is also sufficient, in fact one can approximate a normal E-current with polyhedral currents with coefficients in $\mathbb{Q}G$.

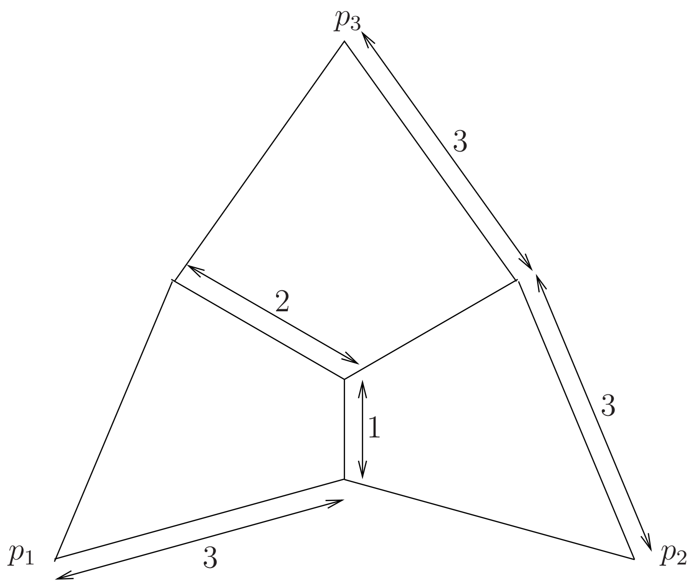

Figure 3.1: Metric space in the Example 3.1.23

Example 3.1.23. Consider the metric space[3] $M \subset \mathbb{R}^2$ given[4] in Figure 3.1. Consider a group $G \subset \mathbb{R}^2$ which has three elements $g_1, g_2, g_3 \in G$ such that $\|g_1\|_E = \|g_2\|_E = \|g_3\|_E = 1$ and $g_1 + g_2 + g_3 = 0$. Such a group exists and we will analyze with all the details in Section 3.2. Let $R := g_1\delta_{p_1} + g_2\delta_{p_2} + g_3\delta_{p_3}$. We will show that (3.1.10) does not hold even when $k = 2$. In fact it is trivial to prove that

$$\mathscr{M}_I(R) = 12 \ .$$

Nevertheless, concerning $\mathscr{M}_I(2R)$, it is shown in Figure 3.2 that

$$\mathscr{M}_I(2R) \leq 23 < 24 = 2\mathscr{M}_I(R) \ .$$

Remark 3.1.24. One can expect a behavior like that in Example 3.1.23 in the metric space $\mathbb{R}^2$ endowed with a density which is very high outside of the subset $M \subset \mathbb{R}^2$. To be precise, let us consider a bounded continuous function

[3] For currents in metric spaces, see [5].

[4] The length of each segment is explicitly declared in Figure 3.1, note that the set is symmetric with respect to the vertical axis.

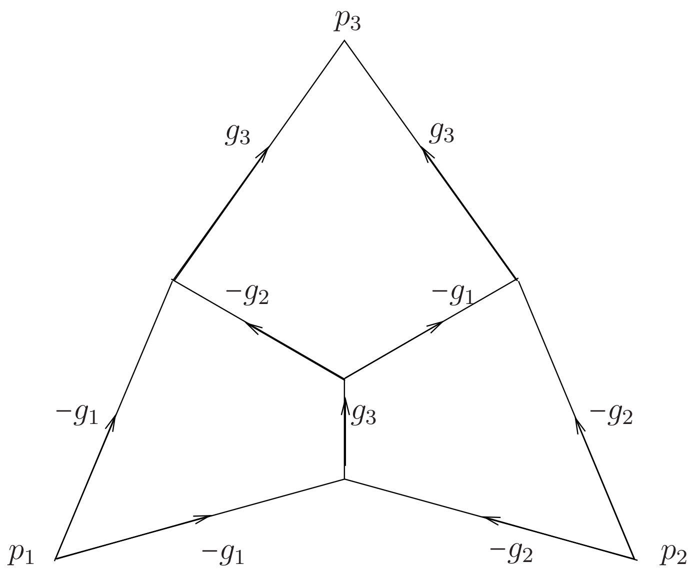

Figure 3.2: Counterexample to (3.1.10)

$W : \mathbb{R}^2 \to \mathbb{R}$, with $W \equiv 1$ on M and $W >> 1$ out of a small neighborhood of M. For any couple $(x_0, x_1) \in \mathbb{R}^2$, the distance on $(\mathbb{R}^2, W)$ is given by

$$d(x_0, x_1) = \inf \left\{ \int_0^1 |\gamma'(t)| W(\gamma(t))\, \mathrm{d}t : \gamma(0) = x_0 \text{ and } \gamma(1) = x_1 \right\}.$$

3.2 The Steiner tree problem revisited

The classical *Steiner tree problem* consists in finding the shortest connected set containing n given distinct points $p_1, \dots, p_n$ in $\mathbb{R}^d$. Some very well-known examples are shown in Figure 3.3.

The problem is completely solved in $\mathbb{R}^2$ and there exists a wide literature on the subject, mainly devoted to improving the efficiency of algorythms for the construction of solutions: see, for instance, [29] and [16] for a survey of the problem. The recent papers [46] and [47] witness the current studies on the problem and its generalizations.

Our aim is to rephrase the Steiner tree problem with an equivalent mass minimization problem by replacing connected sets with 1-currents with coef-

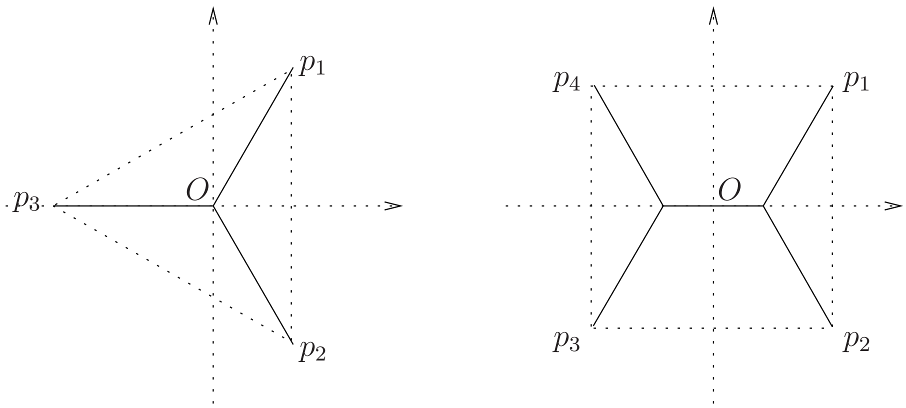

Figure 3.3: Solutions for the vertices of an equilateral triangle and a square.

ficients in a more suitable group than $\mathbb{Z}$, in such a way that solutions of one problem correspond to solutions of the other, and vice-versa. The use of currents allows to exploit techniques and tools from the Calculus of Variations and the Geometric Measure Theory, such as calibrations.

Let us briefly point out a few facts suggesting that classical integral polyhedral chains might not be the correct environment for our problem. First, one should make the given points $p_1, \dots, p_n$ in the Steiner problem correspond to some integral polyhedral 0-chain supported on $p_1, \dots, p_n$, with suitable multiplicities $m_1, \dots, m_n$. One has to impose that $m_1 + \dots + m_n = 0$ in order for this 0-chain to be the boundary of a compactly supported 1-chain. In the example of the equilateral triangle, see Figure 3.3, the condition $m_3 = -(m_1 + m_2)$ forces to break symmetry, leading to the minimizer in Figure 3.4. The desired solution is instead depicted in Figure 3.3. In the second example from Figure 3.3, we get the "wrong" non-connected minimizer even though all boundary multiplicities have modulus 1; see Figure 3.4.

These examples show that $\mathbb{Z}$ is not the right group of coefficients. In this section we recast Steiner problem in terms of a mass minimization problem over currents with coefficients in a discrete group G, chosen only on the basis of the number of boundary points. As we already said, this construction provides a way to pass from a mass minimizer to a Steiner solution and viceversa.

This new formulation will permit to initiate a study of calibrations as a

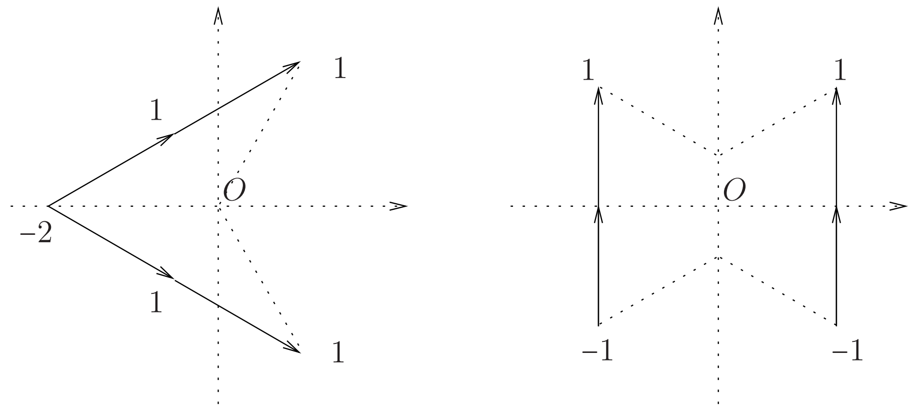

Figure 3.4: Solutions for the mass minimization problems among polyhedral chains with integer coefficients

sufficient condition for minimality; this is the subject of Subsection 3.2.2. In examples 3.2.6 and 3.2.7, we will exhibit calibrations for the problem on the right of Figure 3.3 and for the Steiner tree problem on the vertices of a regular hexagon plus the center. It is worthwhile to note that our theory works for the Steiner tree problem in $\mathbb{R}^d$ and for currents supported in $\mathbb{R}^d$; we made explicit computations only on 2-dimensional configurations for simplicity reasons.

3.2.1 The equivalence between the Steiner problem and a mass-minimization problem

We now establish the equivalence between the Steiner tree problem and a mass minimization problem in a family of G-currents. We first need to choose the right group of coefficients G. Once we fix the n points in the Steiner problem, we look for a subgroup $(G, \|\cdot\|_G)$, of a normed vector space $(E, \|\cdot\|_E)$, (where $\|\cdot\|_G$ is the restriction to G of the norm $\|\cdot\|_E$) satisfying the following properties:

(P1) there exist $g_1, \dots, g_{n-1} \in G$ and $h_1, \dots, h_{n-1} \in E^*$ such that $(g_1, \dots, g_{n-1})$ with $(h_1, \dots, h_{n-1})$ is a complete biorthonormal system for E and G is additively generated by $g_1, \dots, g_{n-1}$;

(P2) $\|g_{i_1} + \ldots + g_{i_k}\|_G = 1$ whenever $1 \le i_1 < \ldots < i_k \le n-1$ and $k \le n-1$;

(P3) $\|g\|_G \geq 1$ for every $g \in G \setminus \{0\}$.

For the moment we will assume the existence of G and E. The proof of their existence and an explicit representation, useful for the computations, will be given later in this section.

The next lemma has a fundamental role: through it, we can give a nice structure of 1-dimensional rectifiable G-current to every suitable competitor for the Steiner tree problem. From now on we will denote $g_n := -(g_1 + \ldots + g_{n-1})$.

Lemma 3.2.1. *Let B be a connected 1-rectifiable set with finite length in $\mathbb{R}^d$, containing $p_1, \ldots, p_n$. Then there exists a connected set $B' \subset B$ containing $p_1, \ldots, p_n$ and a 1-dimensional rectifiable G-current $T_{B'} = [\![B', \tau, \theta]\!]$, such that*

(i) $\|\theta(x)\|_E = 1$ *for a.e.* $x \in B'$,

(ii) $\partial T_{B'}$ *is the* 0*-dimensional G-current* $g_1\delta_{p_1} + \ldots + g_n\delta_{p_n}$

Proof. Since B is a connected set of finite length, then B is connected by paths of finite length (see Lemma 3.12 of [22]). Consider a simple path B_1 contained in B going from p_1 to p_n. In analogy with Example 3.1.9, associate it with a current T_1 with constant multiplicity $-g_1$ and orientation going from p_1 to p_n. Repeat this procedure keeping the ending point p_n and replacing at each step p_1 with $p_2, \ldots, p_{n-1}$. To be precise, in this procedure, as soon as a new path B_i intersects an other path B_j $(i > j)$, then the remaining part of B_j must coincide with the remaining part of B_i. The set $B' = B_1 \cup \ldots \cup B_{n-1} \subset B$ is a connected set containing $p_1, \ldots, p_n$ and the 1-dimensional rectifiable G-current $T = T_1 + \ldots + T_{n-1}$ satisfies the requirements of the lemma, in particular condition (i) comes from (P2). □

Via the next lemma, we can say that mass minimizers for our problem have connected supports. In its proof we will need Corollary 1.3.3 on the structure of integral 1-currents. As we said in Section 1.3, it allows us to consider an integral 1-current as a countable sum of oriented simple Lipschitz curves with integer multiplicities.

Lemma 3.2.2. *Let T be a* 1*-dimensional rectifiable G-current, such that ∂T is the* 0*-current* $g_1\delta_{p_1} + \ldots + g_n\delta_{p_n}$. *Then there exists a rectifiable G-current* $\widetilde{T} = [\![\widetilde{\Sigma}, \widetilde{\tau}, \widetilde{\theta}]\!]$ *such that*

(i) $\partial\widetilde{T} = \partial T = g_1\delta_{p_1} + \ldots + g_n\delta_{p_n}$;

(ii) $\mathbb{M}(\widetilde{T}) \leq \mathbb{M}(T)$ *and the equality holds only if* $\widetilde{T} = T$*;*

(iii) the support of $\widetilde{T}$ *is a connected* 1*-rectifiable set containing* $\{p_1, \ldots, p_n\}$ *and it is contained in the support of* T*;*

(iv) $\mathscr{H}^1(\mathrm{supp}(\widetilde{T}) \setminus \widetilde{\Sigma}) = 0$.

Proof. Let $T^j = [\![\Sigma^j, \tau^j, \theta^j]\!]$ be the components of T, for $j = 1, \ldots, n-1$ (with respect to the biorthonormal system $(g_1, \ldots, g_{n-1}), (h_1, \ldots, h_{n-1})$).

For every j, we can use Corollary 1.3.3 and write

$$T^j = \sum_{k=1}^{K_j} T^j_k + \sum_{\ell \geq 1} C^j_\ell \ .$$

Notice that, for every $j = 1, \ldots, n-1$, if θ^j_k denotes the multiplicity of T^j_k, then, by (1.3.18), we have

$$\sum_{k=1}^{K_j} \left|\theta^j_k\right| \leq |\theta^j| \quad \mathscr{H}^1\text{–a.e. on } \mathrm{supp}(T^j). \tag{3.2.1}$$

We choose $\widetilde{T}$ the rectifiable G-current whose components are

$$\widetilde{T}^j := \sum_{k=1}^{K_j} T^j_k \ .$$

Again, because of (1.3.18), we have $\mathrm{supp}(\widetilde{T}) \subset \mathrm{supp}(T)$ (the cyclic part of T^j never cancels the acyclic one).

Property (i) is easy to check. Property (ii) is a consequence of (3.2.1) and of the following property of the norm $\|\cdot\|_G$: if $\theta = \sum_{j=1}^{n-1} \theta^j g_j$ and $\widetilde{\theta} = \sum_{j=1}^{n-1} \widetilde{\theta}^j g_j$ (with $0 \leq \widetilde{\theta}^j \leq \theta^j$ when $\theta^j \geq 0$ and $0 \geq \widetilde{\theta}^j \geq \theta^j$ otherwise), then $\|\widetilde{\theta}\|_G \leq \|\theta\|_G$ (this property follows from the fact that $(g_1, \ldots, g_{n-1}), (h_1, \ldots, h_{n-1})$ is a complete biorthonormal system for E).

Property (iv) is also easy to check, because the corresponding property holds for every T^j_k and therefore for every component $\widetilde{T}^j$.

It remains to prove property (iii). By construction $\widetilde{T}$ is a finite sum of oriented curves with multiplicities; since we are considering curves with ending points (closed sets), $\mathrm{supp}(\widetilde{T})$ has a finite number of (closed) connected components far apart: consider S a connected component of $\mathrm{supp}(\widetilde{T})$ and the related restriction $\widetilde{T} \llcorner S$. Notice that S has positive distance from any other connected component of $\operatorname{supp}(\widetilde{T})$. We want to prove that either S contains all the p_i's or none of them. Assume by contradiction that S contains a proper subset of $\{p_1, \ldots, p_n\}$, let us relabel the points such that $S \supset \{p_1, \ldots, p_{\widetilde{n}}\}$, with $1 \le \widetilde{n} < n$, and $p_j \notin S$ if $j > \widetilde{n}$. Thus $\partial(\widetilde{T} \llcorner S)$ is the 0-current associated with $p_1, \ldots, p_{\widetilde{n}}$ with multiplicities $g_1, \ldots, g_{\widetilde{n}}$. Then we can choose an element $w \in E^*$ such that $w(g_j) = 1$ for $j = 1, \ldots, \widetilde{n}$ and take $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d, \Lambda_E^1(\mathbb{R}^d))$ a smooth E^*-valued 1-form such that

$$\begin{aligned} \varphi &\equiv w && \text{on } S \\ \varphi &\equiv 0 && \text{on } \operatorname{supp}(\widetilde{T}) \setminus S \,. \end{aligned}$$

Then $0 = \widetilde{T} \llcorner S(d\varphi) = \partial(\widetilde{T} \llcorner S)(\varphi) = \widetilde{n}$, which is clearly a contradiction. Therefore there is no boundary for the restriction of $\widetilde{T}$ to every connected component of its support, but one. Possibly replacing $\widetilde{T}$ by its restriction to this non-trivial connected component, we get the thesis. □

Theorem 3.2.3. *Assume that $T_0 = [\![\Sigma_0, \tau_0, \theta_0]\!]$ is a mass-minimizer among rectifiable 1-dimensional G-currents with boundary*

$$B = g_1\delta_{p_1} + \ldots + g_n\delta_{p_n} \,.$$

Then $S_0 := \operatorname{supp}(T_0)$ is a solution of the Steiner tree problem. Conversely, given a set C which is a solution of the Steiner problem for the points $p_1, \ldots, p_n$, there exists a canonical 1-dimensional G-current, supported on C, minimizing the mass among the currents with boundary B.

Proof. The existence of T_0 is a direct consequence of Theorem 3.1.11. Moreover, since T_0 is a mass minimizer, then it must coincide with the current $\widetilde{T_0}$ given by Lemma 3.2.2. In particular, Lemma 3.2.2 guarantees that S_0 is a connected set.

Let S be a competitor for the Steiner tree problem and let S' and $T_{S'}$ be the connected set and the rectifiable 1-current given by Lemma 3.2.1, respectively. Hence we have

$$\mathscr{H}^1(S) \ge \mathscr{H}^1(S') \overset{(i)}{=} \mathbb{M}(T_{S'}) \overset{(ii)}{\ge} \mathbb{M}(T_0) \overset{(iii)}{\ge} \mathscr{H}^1(\Sigma_0) \overset{(iv)}{=} \mathscr{H}^1(S_0) \,,$$

in fact

(i) thanks to the second property of Lemma 3.2.1 and Proposition 3.1.10, we obtain

$$\mathbb{M}(T_{S'}) = \int_{S'} \|\theta_{S'}(x)\|_G \, \mathrm{d}\mathscr{H}^1(x) = \mathscr{H}^1(S') \,;$$

(ii) we assumed that T_0 is a mass-minimizer;

(iii) from property (P3), we get

$$\mathbb{M}(T_0) = \int_{\Sigma_0} \|\theta_0(x)\|_G \, \mathrm{d}\mathscr{H}^1(x) \geq \int_{\Sigma_0} 1 \, \mathrm{d}\mathscr{H}^1(x) = \mathscr{H}^1(\Sigma_0) \ ;$$

(iv) is property (iv) in Lemma 3.2.2.

To prove the second part of the Theorem, apply Lemma 3.2.1 to the set C. Notice that with the procedure described in the lemma, the rectifiable G-current $T_{C'}$ is uniquely determined, because for every point p_i, C contains exactly one path from p_i to p_n, in fact it is well-known that solutions of the Steiner tree problem cannot contain cycles. By Lemma 3.2.2 $T_{C'}$ is a solution of the mass minimization problem. □

Eventually, we give an explicit representation for G and E. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$; we consider on $\mathbb{R}^n$ the seminorm

$$\|u\|_\star := \max_{i=1,\ldots,n} u \cdot \mathbf{e}_i - \min_{i=1,\ldots,n} u \cdot \mathbf{e}_i \ .$$

We now take the quotient

$$E := \frac{\mathbb{R}^n}{\mathrm{Span}\{\mathbf{e}_1 + \ldots + \mathbf{e}_n\}}$$

and denote by π the projection of $\mathbb{R}^n$ onto E. According to the relation in the quotient, we get $[(u_1, \ldots, u_n)] = [(u_1 + c, \ldots, u_n + c)]$, for every $c \in \mathbb{R}$ and for every $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ (here $[u]$ denotes the element of the quotient associated with the vector $u \in \mathbb{R}^n$). Since $\|u\|_\star = \|u + v\|_\star$ for every $u \in \mathbb{R}^n, v \in \mathrm{Span}\{\mathbf{e}_1 + \ldots + \mathbf{e}_n\}$, then it is well defined the corresponding seminorm $\|\cdot\|_E$ induced on E and it is actually a norm. Moreover $\|\cdot\|_\star$ is constant on every fibre. For the sake of completeness, we remark that, with this notation, the dual space E^* can be represented as $E^* = \{(z_1, \ldots, z_n) \in \mathbb{R}^n : \sum_{i=1}^n z_i = 0\}$ and its dual norm $\|\cdot\|_{E^*}$ coincides with $\frac{1}{2}\|\cdot\|_1$. In fact, for every $[u] \in E$ with $\|[u]\|_E = 1$ we can choose a representative u, such that $|u_i| \leq \frac{1}{2}$, $i = 1, \ldots, n$ and then

$$\|z\|_{E^*} = \sup_{\|[u]\|_E = 1} \sum_{i=1}^n z_i u_i = \frac{1}{2} \sum_{i=1}^n |z_i| \ .$$

The choice of E as a quotient is motivated by the idea that the sum of the coefficients $\mathbf{e}_i$ must be zero, for boundary reasons. Anyway, we find that a slightly different representation of E, would ease computations later and we would rather introduce G with this new representation. Consider

$$F := \{v \in \mathbb{R}^n \,:\, v \cdot \mathbf{e}_n = 0\} \subset \mathbb{R}^n$$

and the homomorphism $\phi : \mathbb{R}^n \to F$ such that

$$\phi(u_1, \ldots, u_n) := (u_1 - u_n, \ldots, u_{n-1} - u_n, 0)\ ; \tag{3.2.2}$$

the seminorm $\|\cdot\|_\star$ is a norm on F.

The homomorphism ϕ in (3.2.2) induces an isometrical isomorphism $\widetilde{\phi} : E \to F$ defined by the relation $\widetilde{\phi} \circ \pi = \phi$: in fact, if $v \in E$ and $u \in \pi^{-1}(v)$, then $\|v\|_E = \|u\|_\star = \|\phi(u)\|_\star = \|\widetilde{\phi}(v)\|_\star$. For every $i = 1, \ldots, n-1$, define $g_i = \widetilde{\phi}^{-1}(\mathbf{e}_i)$ and define $g_n = -(g_1 + \ldots + g_{n-1})$. Let G be the subgroup of E generated by $g_1, \ldots, g_{n-1}$. For every $i = 1, \ldots, n-1$ denote by h_i the element of E^* satisfying $h_i(g_j) = \delta_{ij}$. The pair $(g_1, \ldots, g_{n-1})$, $(h_1, \ldots, h_{n-1})$ is a biorthonormal system. With these coordinates, an element $v \in E$ has unit norm $\|v\|_E = 1$ if and only if

$$\|v\|_E = \|\widetilde{\phi}(v)\|_\star = \max_{i=1,\ldots,n-1}(v_i \vee 0) - \min_{i=1,\ldots,n-1}(v_i \wedge 0) = 1\ . \tag{3.2.3}$$

The norm $\|\cdot\|_{E^*}$ of an element $w = w_1 h_1 + \ldots w_{n-1} h_{n-1} \in E^*$ can be characterized in the following way: let us abbreviate $w^P := \sum_{i=1}^{n-1}(w_i \vee 0)$ and $w^N := -\sum_{i=1}^{n-1}(w_i \wedge 0)$ and $\lambda(v) = \max_{i=1,\ldots,n-1}(v_i \vee 0) \in [0,1]$, then

$$\begin{aligned}\|w\|_{E^*} = \sup_{\|v\|_E=1} \sum_{i=1}^{n-1} w_i v_i &= \sup_{\|v\|_E=1} \left[\lambda(v) w^P + (1-\lambda(v)) w^N\right] \\ &= \sup_{\lambda\in[0,1]} \left[(\lambda w^P + (1-\lambda) w^N\right] = w^P \vee w^N\ . \end{aligned} \tag{3.2.4}$$

Notice that, recalling the notation of Section 3.1, $m = n-1$. Properties (P1), (P2) and (P3) are easy to check.

In the sequel, we will fix both the normed space E and the group G, where n is the number of points in the corresponding Steiner tree problem that we want to solve.

Remark 3.2.4. We already know that the elements $g_1, \ldots, g_n$ are the multiplicities of the n points in the boundary, for the Steiner tree problem. The definition we just gave does not seem to be "symmetric", in fact g_n has, in a certain sense, a privileged role, while the n points in the Steiner tree problem have of course all the same importance. To restore this lost symmetry, one may note that the group E is represented in $\mathbb{R}^n$ as the hyperplane $P := \{x_1 + \ldots + x_n = 0\}$ with a norm which is a multiple of the norm induced on P by the seminorm $\|\cdot\|_\star$ on $\mathbb{R}^n$. Here $g_1, \ldots, g_n$ are the orthogonal projections on P of $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ and $-(\mathbf{e}_1 + \ldots + \mathbf{e}_{n-1})$ respectively. It is easy to see that these points of π are the vertices of an $(n-1)$-dimensional regular tetrahedron. In particular the unit elements of G are the vertices of a convex $(n-1)$-dimensional polyhedron which is symmetric with respect to the origin. The vertices of the polyhedron are all the points of the form $g_{i_1} + \ldots + g_{i_k}$ with $1 \le i_1 < \ldots < i_k \le n-1$ and their inverses. It is clear that in this representation the role of the p_i's is perfectly symmetric.

3.2.2 Examples

After the theoretical preliminaries, we can give some examples: we solve some Steiner tree problems converting them in mass-minimization problems and then building suitable calibrations.

We need a short digression on the representation of a E^*-valued 1-form ω; we will consider $d = 2$, all our examples being for the Steiner tree problem in $\mathbb{R}^2$. Remember that in Section 3.2 we fixed a basis $(h_1, \ldots, h_{n-1})$ for E^*, dual to the basis $(g_1, \ldots, g_{n-1})$ for E. We will represent

$$\omega = \begin{pmatrix} \omega_{1,1}\,\mathrm{d}x_1 + \omega_{1,2}\,\mathrm{d}x_2 \\ \vdots \\ \omega_{n-1,1}\,\mathrm{d}x_1 + \omega_{n-1,2}\,\mathrm{d}x_2 \end{pmatrix},$$

so that, if $\tau = \tau_1\mathbf{e}_1 + \tau_2\mathbf{e}_2 \in \Lambda_1(\mathbb{R}^2)$ and $v = v_1 g_1 + \ldots + v_{n-1}g_{n-1} \in E$, then

$$\langle \omega; \tau, v\rangle = \sum_{i=1}^{n-1} v_i(\omega_{i,1}\tau_1 + \omega_{i,2}\tau_2)\ .$$

Example 3.2.5. Consider the vector space E and the group G defined in Section 3.2 with $n = 3$; let

$$p_0 = (0,0), p_1 = (1/2, \sqrt{3}/2), p_2 = (1/2, -\sqrt{3}/2), p_3 = (-1,0)$$

(see Figure 3.3). Consider the rectifiable G-current T supported in the cone over (p_1, p_2, p_3), with respect to p_0, with piecewise constant weights $g_1, g_2, g_3 = -(g_1 + g_2)$ on $\Sigma_1, \Sigma_2, \Sigma_3$ respectively (recall Example 3.1.9 for notation and orientation). This current T is a minimizer for the mass. In fact, a constant G-calibration ω associated with T can be represented as

$$\omega := \begin{pmatrix} \frac{1}{2}\,\mathrm{d}x_1 + \frac{\sqrt{3}}{2}\,\mathrm{d}x_2 \\ \frac{1}{2}\,\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\,\mathrm{d}x_2 \end{pmatrix} .$$

Condition (i) is easy to check and condition (ii) is trivially verified because ω is constant. To check condition (iii) we note that, for the general vector $\tau = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2$, we have

$$\langle\omega;\tau,\cdot\rangle = \begin{pmatrix} \frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha \\ \frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha \end{pmatrix} .$$

In order to calculate the comass norm of ω, we could stick to the method explained in Section 3.2, but for $n = 3$ computations are simpler. Since the unit ball of E is convex, and its extreme points are the unit points of G, then it is sufficient to evaluate $\langle\omega;\tau,\cdot\rangle$ on $\pm g_1, \pm g_2, \pm(g_1+g_2)$(remember that $\|g_1 - g_2\|_E = 2$). We have

$$\begin{aligned}
|\langle\omega;\tau,g_1\rangle| = |\langle\omega;\tau,-g_1\rangle| &= \left|\sin\left(\alpha + \frac{\pi}{6}\right)\right| \le 1 \ , \\
|\langle\omega;\tau,g_2\rangle| = |\langle\omega;\tau,-g_2\rangle| &= \left|\sin\left(\alpha + \frac{5}{6}\pi\right)\right| \le 1 \ , \\
|\langle\omega;\tau,g_1+g_2\rangle| = |\langle\omega;\tau,-(g_1+g_2)\rangle| &= |\cos\alpha| \le 1 \ .
\end{aligned}$$

One may notice that, for every $x \in \mathbb{R}^2$, the E^*-covector $\omega(x)$ can be represented as a map from $\mathbb{R}^2$ to itself, since E^* and $\mathbb{R}^2$ coincide as vector spaces. Moreover, with a suitable choice of a basis for E^* this map is the identity. It turns out that the form ω has unit norm because the Euclidean unit ball is contained in the unit ball of E^*. One may be led to believe that for the same reason the cone on the vertices of a regular tetrahedron centered at the baricenter is the Steiner minimizer for the 4 vertices. This is not the case, and the analog of the form ω is not a calibration in this case, because its norm is bigger than one. In fact the Euclidean unit ball is not contained in the unit ball of E^* in dimension larger than 2.

An interesting way to generalize this result will be recalled in Remark 3.2.10.

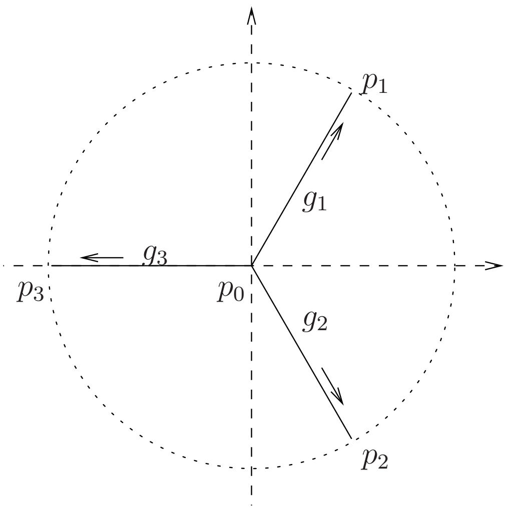

Figure 3.5: Solution for the problem with boundary on the vertex of an equilateral triangle

Both calibrations in Example 3.2.6 and in Example 3.2.7 are piecewise constant 1-forms (with values in normed vector spaces of dimension 3 and 6, respectively), that is why we established a compatibility condition in Definition 3.1.17.

Example 3.2.6. Consider the points

$$p_1 = (1,1), p_2 = (1,-1), p_3 = (-1,-1), p_4 = (-1,1) \in \mathbb{R}^2.$$

The length-minimizer graphs for the classical Steiner tree problem[5] are those represented in Figure 3.3. We associate with each point p_j with $j = 1,\dots,4$ the coefficients $g_j \in G$, where G has "dimension" $m = 3$: let us call

$$B := g_1\delta_{p_1} + g_2\delta_{p_2} + g_3\delta_{p_3} + g_4\delta_{p_4}\ .$$

This 0-dimensional current is our boundary. Intuitively our mass-minimizing candidates among 1-dimensional rectifiable G-currents are those represented

[5]In dimension $d > 2$, an interesting question related to this problem is the following: is the cone over the $(d-2)$-skeleton of the hypercube in $\mathbb{R}^d$ area minimizing, among hypersurfaces separating the faces? The question has a positive answer if and only if $d \geq 4$ (see [11] for the proof).

in Figure 3.6: these currents $T_{\mathrm{hor}}, T_{\mathrm{ver}}$ are supported in the sets drawn, respectively, with continuous and dashed lines in Figure 3.6 and have piecewise constant coefficients intended to satisfy the boundary condition $\partial T_{\mathrm{hor}} = B = \partial T_{\mathrm{ver}}$.

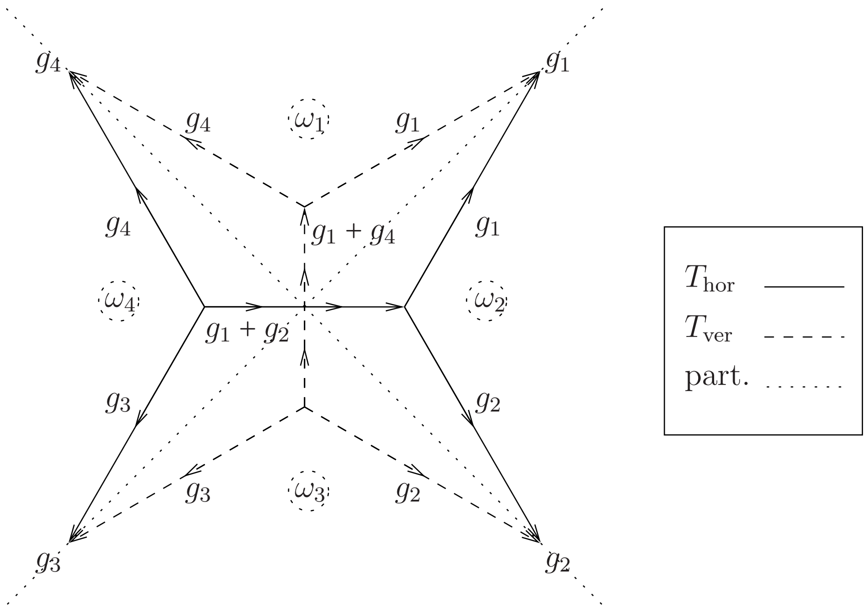

Figure 3.6: Solution for the mass minimization problem

In this case, a compatible calibration for both T_{hor} and T_{ver} is defined piecewise as follows (the notation is the same as in Example 3.2.5 and the partition is delimited by the dotted lines):

$$\omega_1 \equiv \begin{pmatrix} \frac{\sqrt{3}}{2}\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ \left(-1 + \frac{\sqrt{3}}{2}\right)\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \qquad \omega_2 \equiv \begin{pmatrix} \frac{1}{2}\mathrm{d}x_1 + \frac{\sqrt{3}}{2}\mathrm{d}x_2 \\ \frac{1}{2}\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\mathrm{d}x_2 \\ -\frac{1}{2}\mathrm{d}x_1 - \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{d}x_2 \end{pmatrix}$$

$$\omega_3 \equiv \begin{pmatrix} \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \qquad \omega_4 \equiv \begin{pmatrix} \frac{1}{2}\mathrm{d}x_1 + \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{d}x_2 \\ \frac{1}{2}\mathrm{d}x_1 - \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{d}x_2 \\ -\frac{1}{2}\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\mathrm{d}x_2 \end{pmatrix}$$

It is easy to check that ω satisfies both condition (i) and the compatibility condition of Definition 3.1.17. To check that condition (iii) is satisfied, we can use formula (3.2.4).

Example 3.2.7. Consider the vertices of a regular hexagon plus the center, namely

$$p_1 = (1/2, \sqrt{3}/2), \quad p_2 = (1,0), \quad p_3 = (1/2, -\sqrt{3}/2),$$
$$p_4 = (-1/2, -\sqrt{3}/2), \quad p_5 = (-1,0), \quad p_6 = (-1/2, \sqrt{3}/2), \quad p_7 = (0,0)$$

and associate with each point p_j the corresponding multiplicity $g_j \in G$, where G is the group with dimension $m = 6$. A mass-minimizer for the problem with boundary

$$B = \sum_{j=1}^{7} g_j \delta_{p_j}$$

is illustrated in Figure 3.7, the other one can be obtained with a $\pi/3$-rotation of the picture.

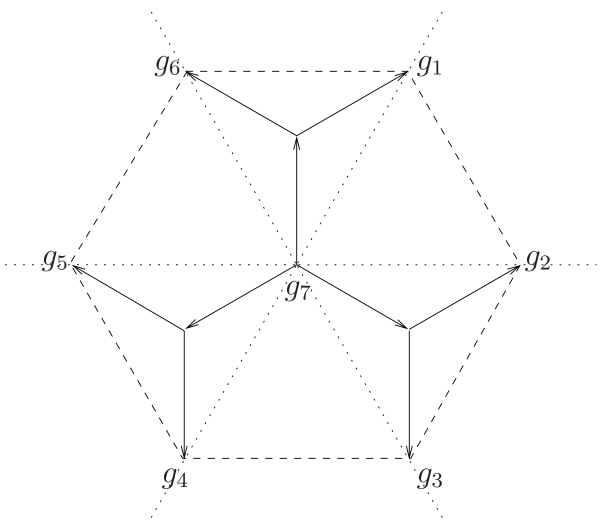

Figure 3.7: Solution for the mass minimization problem

Let us divide $\mathbb{R}^2$ in 6 cones of angle $\pi/3$, as in Figure 3.7; we will label each cone with a number from 1 to 6, starting from that containing $(0,1)$

and moving clockwise. A compatible calibration for the two minimizers is the following

$$\omega_1 = \begin{pmatrix} -\frac{\sqrt{3}}{2}\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ \mathrm{d}x_2 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \omega_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ -\mathrm{d}x_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\omega_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ 0 \end{pmatrix} \quad \omega_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mathrm{d}x_2 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \quad \omega_6 = \begin{pmatrix} \mathrm{d}x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \tag{3.2.5}$$

Again, it is not difficult to check that ω satisfies both condition (i) and the compatibility condition of Definition 3.1.17. To check that condition (iii) is satisfied, we use formula (3.2.4).

Remark 3.2.8. We may wonder whether or not the calibration given in Example 3.2.7 can be adjusted so to work for the set of the vertices of the hexagon (without the seventh point in the center): the answer is negative, in fact the support of the current in Figure 3.7 is not a solution for the Steiner tree problem on the six points, the perimeter of the hexagon minus one side being the shortest graph, as proved in [34].

Remark 3.2.9. In both Examples 3.2.6 and 3.2.7, once we fixed the group G and we decided to look for a piecewise constant calibration for our candidates, the construction of ω was forced by both conditions (i) of Definition 3.1.12 and the compatibility condition of Definition 3.1.17. Notice that the calibration for the Example 3.2.7 has evident analogies with the one exhibited in the Example 3.2.5. Actually we obtained the first one simply pasting suitably "rotated" copies of the second one.

3.2.3 Comparisons with calibrations in other settings

With the following remarks we intend to underline the analogies and the connections with calibrations in similar contexts. See Chapter 6 of [43] for an overview on the subject of calibrations.

Remark 3.2.10 (Clusters with multiplicities)**.** In [42], F. Morgan applies flat chains with coefficients in a group G to soap bubble clusters and immiscible fluids, following the idea of B. White in [53]. The model (in $\mathbb{R}^d$ for m immiscible fluids) associates to each fluid a coefficient $f_i \in G$, where $G \cong \mathbb{Z}^m \subset \mathbb{R} \otimes G \cong \mathbb{R}^m$ throughout the paper. Naturally, we are looking for least-energy interfaces, that is a minimizing $(d-1)$-dimensional flat chain with coefficient in G. The mass norm is induced by the largest norm in $\mathbb{R}\otimes G$ such that

$$\|f_i\|_G = a_i \quad \forall\, i \in \{1, \dots, m\}$$

and

$$\|f_i - f_j\| = a_{ij} \quad \forall\, i, j\ .$$

Concerning soap bubble clusters, we choose $a_i = a_{ij} = 1$; hence, if $m = 2$, the unit ball is pictured in Figure 3.8.

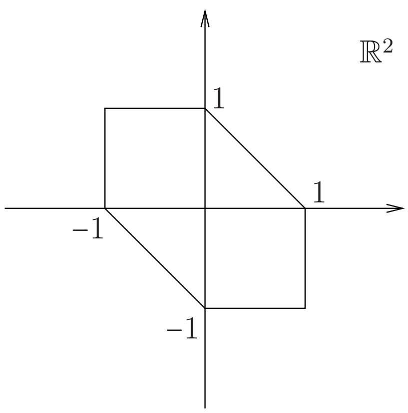

Figure 3.8: Unit ball in F. Morgan's model for soap bubble clusters

Following the idea in [42], a calibration for a rectifiable m-chain T in $\mathbb{R}^d$ is a homomorphism

$$\omega : G \to \mathscr{C}^\infty(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d))$$

with the following properties:

(i) $\langle \vec{T}(x); \omega(g)(x) \rangle = \|g\|_G$ for a.e. $x \in \mathrm{supp}(T)$;

(ii) $\omega(g)$ is a closed differential m-form for every $g \in G$;

(iii) $\|\omega(g)\| \leq \|g\|_G$ for every $g \in G$.

These properties guarantee that T is a mass-minimizer among flat chains with the same boundary; the proof is by all means analogous to the one given in Proposition 3.1.13. Notice that this definition for the calibration works truly well in the case of a free abelian group $\mathbb{Z}^m$, because we are considering homomorphisms with values in a vector space and every finite order subgroup is trivialized by such a homomorphism. As F. Morgan shows in Proposition 4.5 of [42], in this framework it is easy to prove a generalization of Example 3.2.5: consider a cone $C = \sum_{i=1}^n g_i v_i$ in $\mathbb{R}^d$ of unit vectors v_i with coefficients in $G = \mathrm{span}\{g_i\}$ and assume that

$$\left| \sum_{i=1}^n \lambda_i \|g_i\|_G v_i \right| \leq \left\| \sum_{i=1}^n \lambda_i g_i \right\|_G \qquad \forall\, \lambda_i \geq 0 \ ,$$

then C is a minimizer because it admits a calibration with constant coefficients.

Remark 3.2.11 (Paired calibrations for the tetrahedron)**.** It is worth mentioning another analogy between the technique of calibrations (for currents with coefficients in a group) illustrated in this paper and the technique of paired calibrations in [37]. We confine our attention on a specific example: consider the 1-skeleton of the tetrahedron in $\mathbb{R}^3$, centered in the origin, then the truncated cone over the skeleton is the surface with least area among those separating the faces of the tetrahedron. In [37] this is obtained through paired calibrations.

We sketch here a way to get this result through currents with coefficients in a group[6]. Put

$$\begin{array}{lll} g_1 := p_2 - p_1 & g_2 := p_3 - p_2 & g_3 := p_4 - p_3 \\ g_4 := p_4 - p_2 & g_5 := p_4 - p_1 & g_6 := p_3 - p_1 \ , \end{array}$$

[6] Notice that the theory of currents with coefficients in a group has been stated for every dimension k. Of course, the equivalence with the Steiner tree problem of Section 3.2 has no meaning in dimension $k \geq 2$.

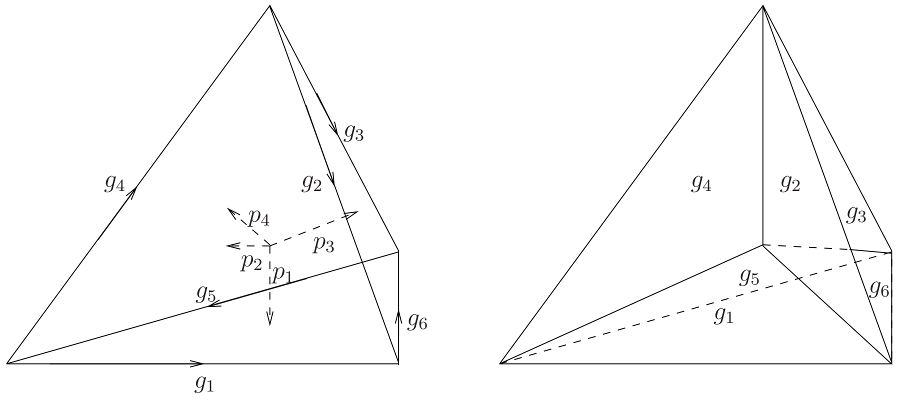

Figure 3.9: 1-skeleton of the tetrahedron and mass-minimizing current

where $p_i \in \mathbb{R}^3$ are the vectors directed from the baricenter of the tetrahedron to the centers of the faces, with labels as in Figure 3.9. The points p_i are the vertices of the dual tetrahedron with unit edges. Notice that the Euclidean norm of $p_j - p_i$ is 1 for any $i \neq j$. This choice of g_i will be made clear in few lines, but let us remark that the coefficients g_i coincide with those of the paired calibration in [37]. If $G \subset \mathbb{R}^3$ is the additive group generated by $\{p_1, p_2, p_3\}$ and it is endowed with the Euclidean norm in $\mathbb{R}^3$, let us assign a (constant) coefficient g_i to each segment of the skeleton, as illustrated in Figure 3.9. Thus the identity is a calibration for the 2-current in the right side of Figure 3.9, that is the truncated cone on the 1-skeleton of the tetrahedron with coefficients $g_1, \dots, g_6$ on each piece of plane[7].

Following an idea of Federer (see [25]), in [42] and [37] (and in [11] and [12], as well) one can observe the exploitment of the duality between minimal surfaces and maximal flows through the same boundary. We examined this very same duality in Subsection 3.1.2, but we conclude with a remark closely related to this idea.

Remark 3.2.12 (Covering spaces and calibrations for soap films)**.** In [12] Brakke develops new tools in Geometric Measure Theory for the analysis of

[7] The orientation in each flat piece of the cone is determined by the orientation of the corresponding segment in the 1-skeleton of the tetrahedron, as shown in Figure 3.9. Let us finally point out that the boundary of the truncated cone with coefficients $g_1, \dots, g_6$ has no support out of the 1-skeleton of the tetrahedron because the sum of the coefficients on the edges of the cone gives 0.

soap films: as the underlying physical problem suggests, one can represent a soap film as the superposition of two oppositely oriented currents. In order to avoid cancellations of multiplicities, the currents are defined in a covering space and, as stated in [12], the calibration technique holds valid.

Let us remark that cancellations between multiplicities were a significant obstacle for the Steiner tree problem, too. The representation of currents in a covering space goes in the same direction of currents with coefficients in a group, though, as in Remark 3.2.11, a sort of Poincaré duality occurs in the formulation of the Steiner tree problem (1-dimensional currents in $\mathbb{R}^d$) with respect to the soap film problem (currents of codimension 1 in $\mathbb{R}^d$).

3.2.4 Comparison with the BV theory

This subsection has to be considered as a completion of Subsection 3.2.3. With the same intent of comparing the existing literature and our way to treat the Steiner tree problem, this subsection is devoted to the theory of partitions of an open set Ω in a finite number of sets of finite perimeter. This theory was developed by Ambrosio and Braides in the papers [2, 3], which we refer to for a complete exposition.

The interest in minimizing interface energies depending on the partition of a domain Ω arises from Physics. Consider, as examples, crystals or immiscible fluids, where the energy may depend, respectively, on the orientation of the interface between fluids and their densities, or some other constant of proportionality varying for each pair of fluids.

Our motivation for this parenthesis is twofold: on the one hand we would like to stress the analogy between minimal partitions and the Steiner problem discussed above; on the other hand, for section 3.3, it will be useful to bear in mind some results about relaxation of integral functionals depending on the partition of Ω.

If $H = \{\eta_1, \dots, \eta_n\}$ is a finite subset of $\mathbb{R}^m$, we will consider BV functions $u : \Omega \to H$, that is, u is a function separating Ω in a finite number of finite perimeter sets, namely $\{u = \eta_i\}$ with $i = 1, \dots, n$.

We define the jump set of a function $u \in BV(\Omega, H)$ as the union of essential boundaries[8] of level sets, that is

$$J(u) := \bigcup_{i=1}^{n} \partial^* (\{u = \eta_i\}) \,.$$

Since for each point x in $\Omega \setminus J(u)$ there exists a unique value η_i for which x has density 1 in $\{u = \eta_i\}$, we will refer to this essential value η_i of u in x with the same notation $u(x)$. Moreover, by standard results on sets of finite perimeter, for $\mathscr{H}^{d-1}$-almost every point x in the jump set $J(u)$ there exist $u^+, u^- \in H$ and a normal direction $\nu \in \mathbb{S}^{d-1}$ such that

$$\lim_{r \downarrow 0} r^{-d} \mathscr{H}^d \left(\{ y \in B_r(x) \,|\, \langle y - x, \nu \rangle > 0,\ u^+ \neq u(y) \} \right) = 0$$

and analogously

$$\lim_{r \downarrow 0} r^{-d} \mathscr{H}^d \left(\{ y \in B_r(x) \,|\, \langle y - x, \nu \rangle < 0,\ u^- \neq u(y) \} \right) = 0 \,.$$

[8]By definition, the essential boundary $\partial^\star A$ of a Borel set $A \subset \Omega$ is the set of points of Ω satisfying $\limsup_{r\to 0} r^{-d}\mathscr{H}^d(A \cap B_r(x)) > 0$ and $\limsup_{r\to 0} r^{-d}\mathscr{H}^d((\Omega \setminus A) \cap B_r(x)) > 0$ at the same time.

We are interested in the study of variational problems defined by a functional depending on an integral on the interfaces of a partition of the domain $\Omega \subset \mathbb{R}^d$: since the (finite perimeter) sets of the partition are the levels of a BV function $u : \Omega \to H$, we can represent such a functional as

$$\mathcal{F}(u) = \int_{J(u)} f(x, u^+, u^-, \nu) \, d\mathscr{H}^{d-1}(x) \,. \tag{3.2.6}$$

Naturally, we want to minimize the functional (3.2.6) among BV functions with values in H, thus, in sight of the direct method of Calculus of Variations, a first important result in [2] is a closure result under Γ-convergence[9] - under a reasonable equicontinuity condition on the sequence of functions (integrands).

Paper [3] is devoted to characterize the lower semicontinuity of the functional (3.2.6). Let us open a small parenthesis to enlighten the efforts to characterize the lower semicontinuity in other classical contexts: for variational functionals of the type $\mathcal{L}(u) = \int_\Omega L(x, u(x), Du(x)) \, dx$ it is well known that sequential lower semicontinuity (in the weak* $W^{1,\infty}$-topology) is equivalent to quasiconvexity of the Lagrangian L, that is

$$L(A) \leq \fint_\Omega L(A + D\varphi) \, dx \qquad \forall \, \varphi \in C_c^\infty(\Omega; \mathbb{R}^m) \,.$$

Concerning functionals defined on partitions of Ω, Ambrosio and Braides introduced the BV-ellipticity, as it is recalled in the following definition.

Definition 3.2.13. *Fix a point $\overline{x} \in \Omega$, $i \neq j \in \{1, \dots, n\}$ and $\overline{\nu} \in \mathbb{S}^{d-1}$, we define*

$$\overline{u}(x) := \begin{cases} \eta_i & \text{if} \langle x - \overline{x}, \overline{\nu} \rangle > 0 \\ \eta_j & \text{if} \langle x - \overline{x}, \overline{\nu} \rangle \leq 0 \,. \end{cases}$$

A function $f : H \times H \times \mathbb{S}^{d-1} \to [0, +\infty)$ is BV-elliptic if, for every $i \neq j$ and every $\nu \in \mathbb{S}^{d-1}$, we have

$$\int_{J(\overline{u}) \cap \Omega} f(i, j, \nu) \leq \int_{J(u) \cap \Omega} f(u^+, u^-, \nu_u)$$

for every $u \in BV(\Omega; H)$ with the same trace of $\overline{u}$ on $\partial\Omega$. A function f : $\Omega \times H \times H \times \mathbb{S}^{d-1} \to [0, +\infty)$ is BV-elliptic if $f(x, \cdot, \cdot, \cdot)$ is so for every $x \in \Omega$.

[9] For an overview on Γ-convergence, see [10].

As we suggested above, BV-ellipticity holds if and only if the functional is lower semicontinuous with respect to the convergence in measure (see [3], Thm. 2.1, for further details).

If the integrand f is not BV-elliptic, one can naturally define the BV-elliptic envelope $E(f)$ as the greatest BV-elliptic function less or equal than f. It is possible to see that not only $E(f)$ coincides with

$$\inf\left\{\int_{J(u)} f(u^+,u^-,\nu_u)\,d\mathscr{H}^{d-1}\,|\,u\in BV(\Omega,H),\text{ same trace of }\overline{u}\text{ on }\partial\Omega\right\},$$

but also $E(f)$ provides a representation of the integrand for the relaxed functional

$$\inf\left\{\liminf_{h\to+\infty}\int_{\Omega\cap J(u_h)} f(u_h^+,u_h^-,\nu_{u_h})\,d\mathscr{H}^{d-1}(x)\,|\,u_h\to u\text{ in measure}\right\},$$

that is, the greatest lower semicontinuous functional less than $\mathcal{F}(u)$.

We conclude this short excursus on functionals defined on partitions with the changes for the Dirichlet problem: in fact, assume $g:\partial\Omega\to H$ to be a Borel function and consider

$$\tilde{\mathcal{F}}(u):=\begin{cases}\mathcal{F}(u) & \text{if trace}(u)=g\\ +\infty & \text{otherwise}.\end{cases}$$

This case does not distance itself from the original one: in [3] it is proved that, under reasonable equicontinuity assumptions on the integrands f_h, if $\mathcal{F}_h$ do Γ-converge to $\int f(x,u^+,u^-,\nu_u)\,d\mathscr{H}^{d-1}$, then the functionals $\tilde{\mathcal{F}}_h$ do Γ-converge to

$$\int_{J(u)} f(x,u^+,u^-,\nu_u)\,d\mathscr{H}^{d-1}(x)+\int_{\partial\Omega} f(x,\text{trace}(u),g,\nu_\Omega)\,d\mathscr{H}^{d-1}(x),$$

for every $u\in BV(\Omega,H)$.

Once we treated the Dirichlet problem it is easy to link the theory of minimization of functionals defined on partitions with the Steiner problem, albeit only in dimension $d=2$ (later we will see why).

Consider a Steiner problem on points $x_1,\dots,x_n\in\partial\Omega$, where Ω is a convex open subset of $\mathbb{R}^2$. We can choose $H=\{\eta_1,\dots,\eta_n\}$, with a distance[10] d in H

[10] We will often assume that $H\subset\mathbb{R}^m$, with $\text{dist}(\eta_i,\eta_j)=|\eta_i-\eta_j|$. Clearly (3.2.7) holds if $\{\eta_1,\dots,\eta_n\}$ are the vertices of a regular m-dimensional tetrahedron in $\mathbb{R}^m$. See Remark 3.2.4 to deepen the analogy with our group G in Section 3.2.

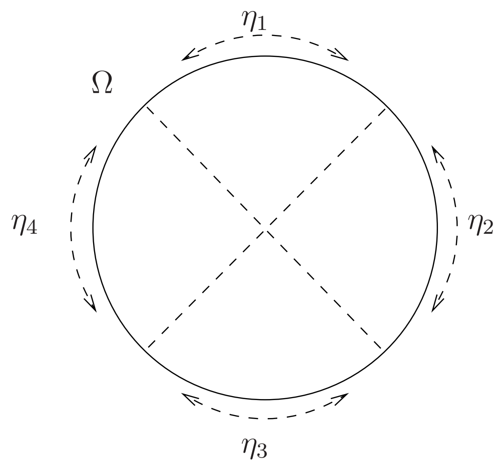

Figure 3.10: Boundary data

and the property

$$\mathrm{d}(\eta_i,\eta_j) = 1 \quad \forall\, i \neq j,\, i,j \in \{1,\dots,n\}\,. \tag{3.2.7}$$

The boundary datum $g : \partial\Omega \to H$ is defined as follows: the set $\partial\Omega \smallsetminus \{x_1,\dots,x_n\}$ consists of n connected components, namely $C_1,\dots,C_n$, we set $g \equiv \eta_i$ for each C_i.

If the integrand $f = f(x,\eta_i,\eta_j,\nu)$ is nothing but $1 = |\eta_i - \eta_j|$, then the problem of minimization of the functional $\mathcal{F}$ in (3.2.6) is equivalent to the Steiner problem, whose solution coincides with the jump set $J(u)$ of a minimizer u; in fact it is immediate to notice that $\mathscr{H}^1(J(u)) = \mathcal{F}(u)$. Therefore we find again the existence of the minimizer for the Steiner problem via Ambrosio-Braides result in [2].

Let us point out analogies and differences between the two methods:

- condition (3.2.7) is a transcription of condition (P2) with $k = 1$;
- the other conditions on the group coefficients are useless here, connection being "forced" by the boundary data.

Actually the relationship between the Steiner problem and the theory of functionals on partitions is more complex in dimension $d > 2$, where a sort of Poincaré duality occurs. The latter is a problem in codimension 1 and its actual counterpart in minimal surfaces is the problem illustrated in [37]: given a polyhedron $\Omega \subset \mathbb{R}^d$ (the convex set of the partition problem, roughly speaking) what about the surface with boundary in the $(d-2)$-skeleton of Ω (the edges) minimizing the area and separating the faces at the same time? Let us notice that separation of faces is a way to select the homology class to work in. For this kind of "dual" problem (with respect to ours) the existence of a minimizer is guaranteed by the Ambrosio-Braides partition theory. Moreover, in [37], calibration-like method is proposed and applied in some interesting examples.

The equivalent of a calibration for functionals on partitions is given by a special class of Null Lagrangians, that is, a functional depending only on the boundary behavior of a BV function (see [17] for further explanations).

Proposition 3.2.14. *Consider a set $H = \{\eta_1, \dots, \eta_n\} \subset \mathbb{R}^m$ satisfying condition* (3.2.7)*, then consider a functional $\mathcal{F}$ defined on partitions of Ω, $\mathcal{F}(u) = \int_{J(u)} |u^+ - u^-| \, d\mathcal{H}^1$, and a BV-function $u : \Omega \to H$. Assume that $V : \Omega \times H \to \mathbb{R}^d$ is a vector field with the following properties:*

(i) for every $x \in J[u]$,

$$[V(x, u^+) - V(x, u^-)] \cdot \nu(x) = 1 \ ;$$

(ii) marking $v_i(x) := V(x, \eta_i)$, $i = 1, \dots, n$,

$$\mathrm{div}_x V(x, \eta_i) = \mathrm{div} v_i(x) = 0 \ ;$$

(iii) for every $i, j = 1, \dots, n$,

$$|v_i(x) - v_j(x)| \leq 1 \ .$$

Then V defines a Null Lagrangian and, as it happens in Proposition 3.1.13, u is a minimizer for $\mathcal{F}$ among BV functions with values in H and the same trace of u on $\partial\Omega$.

Proof. Thanks to properties of V listed above

$$\begin{aligned}
\mathcal{F}(u) &\overset{(i)}{=} \int_\Omega \mathrm{div}(V(x, u(x))) \, dx = \int_\Omega \mathrm{div}(V(x, u'(x))) \, dx \\
&\overset{(ii)}{=} \int_\Omega V_u(x, u'(x)) \cdot \nabla u'(x) \, dx \\
&= \int_{J[u']} (V(x, (u')^+(x)) - V(x, (u')^-(x))) \cdot \nu(x) \, d\mathcal{H}^{d-1}(x) \\
&\overset{(iii)}{\leq} \int_{J[u']} |(u')^+ - (u')^-| \, d\mathcal{H}^{d-1}(x) = \mathcal{F}(u') \, .
\end{aligned}$$

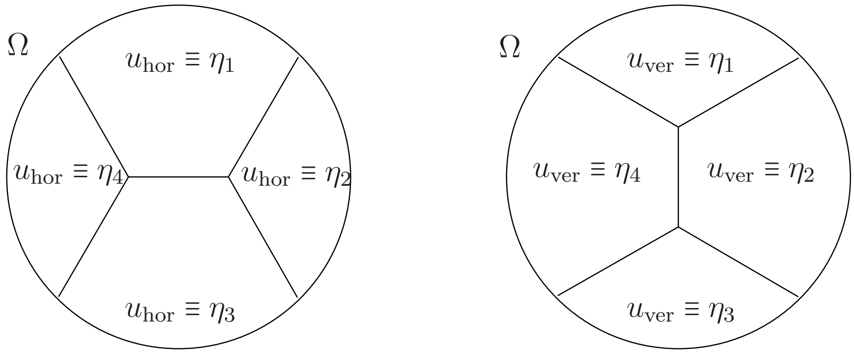

Figure 3.11: Minimizers

where u' is a competitor in $BV(\Omega; H\})$ with the same trace as u on $\partial\Omega$. $\square$

In order to seal the similarity of the Null Lagrangian problem with the calibration for the Steiner tree problem, consider the trace u_0 in Figure 3.10. The minimizers of the functional $\tilde{\mathcal{F}}$ are showed in Figure 3.11. As a matter of fact, the minimizers $u_{\text{or}}, u_{\text{ver}}$ admit a Null Lagrangian vector field, satisfying a compatibility condition and clearly related to the calibration ω defined above.

3.3 Dislocations of crystals

Currents with coefficients in a group turned out to be a useful tool in the study of some models for dislocations in crystals. The purpose of the following lines is only to sketch a rough picture of the context, the subject being really wide. For an introduction to crystal dislocations from both physical and mathematical point of view, see [44, 33, 48].

A crystal is a solid (a metal, for instance, but there are many other classes of nonmetallic solids) whose atoms are arranged in a 3d-periodic pattern. It is a quite complex material, due to the structures at many different scales. As we will see throughout this section, dealing with crystals needs methods built around the explicit consideration of multiple scales simultaneously: the microscopic scale (atoms), the mesoscopic scale and the macroscopic scale.

Under the effect of a stress, a crystal undergoes a deformation, which can be either elastic or elastoplastic. In the latter, the (plastic) main mechanism competing with the elastic deformation is the slip on the so-called slip planes. The slip not being uniform causes topological defects in the crystal lattice, called dislocations. By definition, a dislocation is a line on the slip plane separating regions undergoing different slips. The attention is often restricted to a single slip plane.

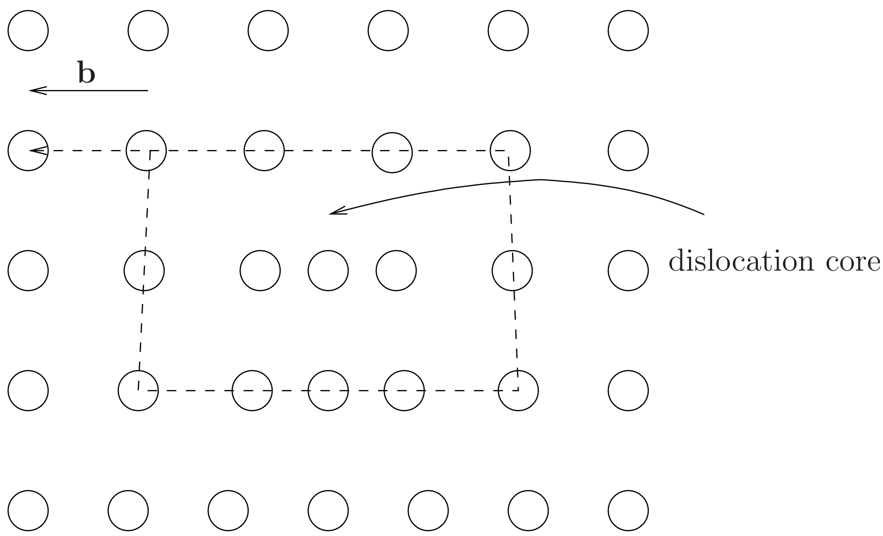

Figure 3.12: Burgers circuit and dislocation core in section.

For each dislocation we can identify a dislocation core and a Burgers vector, often denoted by **b**, representing the magnitude and direction of the lattice distortion (see Figure 3.12). Roughly speaking, the Burgers vector is a label, recording the topological information about the distortion of the crystal lattice. Since the deformation is represented by means of a lattice, then $\mathbf{b} \in \mathbb{Z}^3$. In formulas, if $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a deformation including the dislocation phenomenon, we can study the Lebesgue decomposition of the deformation gradient

$$Du = \nabla u \, \mathscr{L}^3 + [u] \otimes \mathbf{n} \, \mathrm{d}\mathscr{H}^2 \llcorner \Sigma = \beta^{\mathrm{elast}} + \beta^{\mathrm{plast}} \,, \tag{3.3.1}$$

where the absolutely continuous part ∇u is the diffuse elastic distortion and the singular part $[u] \otimes \mathbf{n}$ is the slip along slip planes. Decomposition (3.3.1) means that, from a mathematical point of view, dislocations may be understood as singularities of the dislocation density $\mu = \mathrm{curl} \nabla u$.

A fundamental goal in the study of crystal dislocations is to get a continuum model taking into account the underlying discrete or hybrid semidiscrete structure (possible models in [6, 39]): choosing a variational approach , this may be faced with the Γ-convergence[11] of the energy functional F_ε (ε is microscale) to some functional F, preserving the same relevant features of F_ε.

For better clarity let us mention that the plastic slip for a straight dislocation, along the direction $t \in \mathbb{S}^2$ and with Burgers vector **b**, is

$$\beta_0 = \frac{1}{r} \Gamma_0(\theta) \,,$$

with $\mathrm{curl}\beta_0 = \mathbf{b} \otimes t \, \mathscr{H}^1 \llcorner \gamma$. We omit, in these circumstances, the linear elasticity term of the energy.

In general, an admissible plastic slip β has the form

$$\beta = \mathbf{b} \otimes \nu \, \mathscr{H}^2 \llcorner \Sigma \,,$$

where Σ is a 2-dimensional rectifiable subset of Ω, ν is (almost everywhere) its normal and **b** is Borel measurable. The distribution of dislocations is described by

$$\mu = s \otimes t \, \mathrm{d}\mathscr{H}^1 \llcorner \gamma \,,$$

[11] See [10] and [18] for an introduction to Γ-convergence.

where γ is a 1-dimensional rectifiable subset of Ω, $t : \gamma \to \mathbb{S}^2$ is (almost everywhere) its tangent vector and s belongs to the set of Burgers vectors.

Such plastic slip may be interpreted as an $\mathcal{L}$-valued 1-currents, where $\mathcal{L}$ is a lattice in $\mathbb{R}^3$, and the variational problem has the form

$$E_0(\mu) = \begin{cases} \int_\gamma \psi(\mathbf{b} \otimes t)\, \mathrm{d}\mathscr{H}^1 & \mu \in \mathcal{M}(\mathbb{R}^3) \\ +\infty & \text{otherwise} \end{cases} \tag{3.3.2}$$

The approach via 1-currents needs some generalization of well-known facts about classical currents. This generalizations and adjustments are treated in [13] and are fully exploited in [15].

3.3.1 Some technical results on $\mathbb{Z}^m$-valued 1-currents

In the case of crystal dislocations, we will consider currents with coefficients in $\mathbb{Z}^m \subset \mathbb{R}^m$, which is the group where the Burgers vector takes values in. Actually, a significant part of the theory of $\mathbb{Z}^m$-valued currents can be done componentwise, reducing to the classical theory, nevertheless we already established some useful theorems in Section 3.1.

For the sake of brevity, we will denote by $\mathscr{R}_1(\mathbb{R}^d, \mathbb{Z}^m)$ the set of rectifiable 1-currents. Consistently with Section 3.1, we say that a rectifiable 1-current is polyhedral if its support γ is the union of finitely many segments and θ is constant on each of them. We denote by $\mathscr{P}_1(\mathbb{R}^d; \mathbb{Z}^m)$ the set of polyhedral 1-currents. Alternatively, one can interpret rectifiable 1-currents as measures: we denote by $\mathcal{M}_{\mathrm{df}}(\Omega)$ the set of all divergence-free measures $\mu \in \mathscr{M}(\Omega; \mathbb{R}^{n\times n})$ of the form

$$\mu = \theta \otimes \tau d\mathscr{H}^1 \llcorner \gamma$$

where γ is a 1-rectifiable set contained in Ω, $\tau : \gamma \to \mathbb{S}^{d-1}$ its tangent vector, and $\theta : \gamma \to \mathbb{Z}^m$. With this identification the total variation of μ coincides with the mass of T, $\mathbb{M}(T) = \|\mu\|(\Omega)$.

We start with an extension lemma, that can be found in various forms in the literature. We sketch here the argument for the case of interest, in which the closedness is preserved.

Lemma 3.3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz open set. For every closed 1-current $T \in \mathscr{R}_1(\Omega; \mathbb{Z}^m)$ with finite mass there is a closed 1-current $ET \in \mathscr{R}_1(\mathbb{R}^d; \mathbb{Z}^m)$ with $ET \llcorner \Omega = T$ and $\mathbb{M}(ET) \le C\mathbb{M}(T)$. The constant depends only on Ω.*

Proof. Step 1. We first extend T to a neighborhood of Ω.

Choose a function $N \in C^1(\partial\Omega; \mathbb{S}^{d-1})$ such that $N(x) \cdot \nu(x) \geq \alpha > 0$ for almost all $x \in \partial\Omega$, where ν is the outer normal. For $\rho > 0$ sufficiently small the function $g : \partial\Omega \times (-\rho, \rho) \to \mathbb{R}^d$, $g(x,t) = x + tn(x)$, is bi-Lipschitz. Let $D_\rho = g(\partial\Omega \times (-\rho,\rho))$ and $f : D_\rho \to D_\rho$ be defined by $f(g(x,t)) = g(x,-t)$. Then f is bi-Lipschitz and coincides with its inverse.

We define $\tilde{T} = T - f_\# T \llcorner (D_\rho \setminus \Omega)$. Let $\varphi \in W_0^{1,\infty}(\Omega \cup D_\rho)$. Then

$$\langle \tilde{T}, D\varphi \rangle = \langle T, D\varphi \rangle - \langle f_\sharp T, D\varphi \rangle = \langle T, D\varphi - D(\varphi \circ f) \rangle = 0 \tag{3.3.3}$$

since $\varphi - \varphi \circ f \in W_0^{1,\infty}(\Omega)$.

Step 2. Let $\tilde{\gamma}$ and $\tilde{\theta}$ be the support and the multiplicity of $\tilde{T}$. Since

$$\mathbb{M}(\tilde{T}) \geq \int_0^\rho \Big(\sum_{x \in \tilde{\gamma} \cap \partial(\Omega_s)} |\tilde{\theta}(x)| \Big) ds \tag{3.3.4}$$

we can choose[12] $s \in (0, \rho)$ such that

$$\sum_{x \in \tilde{\gamma} \cap \partial(\Omega_s)} |\tilde{\theta}(x)| \leq C \mathbb{M}(T)\,, \tag{3.3.5}$$

with a constant depending only on Ω. In particular, the sum runs over finitely many points $x_1, \ldots, x_M$. The points $x_1, \ldots, x_M$, with multiplicity $\tilde{\theta}(x_1), \ldots, \tilde{\theta}(x_M)$ and positive orientation if $\tilde{\gamma}$ exists Ω_s at x_i, is the boundary of $\tilde{T} \llcorner \Omega_s$. For each $i = 2, \ldots, M$, let γ_i be a Lipschitz curve in $\mathbb{R}^d \setminus \Omega$ which joins x_1 with x_i and has length bounded by $C(\Omega)$. Let τ_i be the tangent vector, with the same orientation as $\tilde{\gamma}$ in x_i. We set

$$\langle ET, \varphi \rangle = \langle \tilde{T} \llcorner \Omega_s, \varphi \rangle + \sum_{i=2}^{M} \tilde{\theta}(x_i) \int_{\gamma_i} \langle D\varphi, \tau_i \rangle d\mathscr{H}^1 \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d). \tag{3.3.6}$$

Since T was closed one can see that ET is also closed and the proof is concluded. $\square$

Another result we need to recall is a strong polyhedral approximation result, like Theorem 1.3.1: an integral current T can be approximated through a current $f_\sharp T$, where f is a bi-Lipschitz map and it is close to the identity (see (1.3.3)) and P is a polyhedral current with the same boundary of T. The same density result of Theorem 1.3.1 can be obtained for 1-currents

[12] As usual, $\Omega_s = \bigcup_{x \in \Omega} B_s(x)$.

with vector-valued multiplicity working componentwise. We formulated this density result on $\mathbb{R}^d$, the local version can be easily deduced using the above extension lemma.

Finally, let us remind that we already proved compactness in Theorem 3.1.11. With the same argument of Theorem 1.3.2, we get that a rectifiable 1-current with coefficients in $\mathbb{Z}^m$ can be represented as the sum of finitely many paths and countably many loops with multiplicities in $\mathbb{Z}^m$.

3.3.2 The energy relaxation and the $\mathscr{H}^1$-ellipticity

In this section we consider the *relaxation of functionals* of the form

$$E(\mu) = \begin{cases} \displaystyle\int_\gamma \psi(\theta,\tau)\,d\mathscr{H}^1 & \text{if } \mu = \theta\otimes\tau\mathscr{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathrm{df}}(\mathbb{R}^d,\mathbb{Z}^m) \\ +\infty & \text{otherwise.} \end{cases}$$

We shall show that the relaxation is

$$\bar{E}(\mu) = \begin{cases} \displaystyle\int_\gamma \bar{\psi}(\theta,\tau)\,d\mathscr{H}^1 & \text{if } \mu = \theta\otimes\tau\mathscr{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathrm{df}}(\mathbb{R}^d,\mathbb{Z}^m) \\ +\infty & \text{otherwise.} \end{cases}$$

where, for $b\in\mathbb{Z}^m$, $t\in\mathbb{S}^{d-1}$,

$$\begin{aligned}\bar{\psi}(b,t) = \inf\Big\{ \int_\gamma \psi(\theta,\tau)\,d\mathscr{H}^1 \;:\; &\mu\in\mathcal{M}_{\mathrm{df}}(\mathbb{R}^d)\,, \\ &\mathrm{supp}(\mu - b\otimes t\mathscr{H}^1 \llcorner (\mathbb{R}t)) \subset B_{1/2}(0)\Big\}. \end{aligned} \tag{3.3.7}$$

We call the function $\bar{\psi}$ the $\mathscr{H}^1$*-elliptic envelope* of ψ and say that ψ is $\mathscr{H}^1$-elliptic if $\bar{\psi}=\psi$. We use for the localization the notation

$$E(\mu,\Omega) = \int_{\gamma\cap\Omega} \psi(\theta,\tau)d\mathscr{H}^1 \tag{3.3.8}$$

where $\mu = \theta\otimes\tau\mathscr{H}^1\llcorner\gamma \in \mathcal{M}_{\mathrm{df}}(\mathbb{R}^d)$, and the same for $\bar{E}$.

Lemma 3.3.2 (Truncated energy)**.** *For every $h,l>0$, let us define the truncated energy on the parallelepiped $R_{l,h} = \left(-\frac{l}{2},\frac{l}{2}\right)\times\left(-\frac{h}{2},\frac{h}{2}\right)^{n-1}$ as*

$$\begin{aligned} e(l,h) := \inf\Big\{ \liminf_{k\to\infty} l^{-1}E(\mu_k,R_{l,h}) \,:\, &\mu_k\in\mathcal{M}_{\mathrm{df}}(B_{1/2}), \\ &\mu_k \overset{*}{\rightharpoonup} b\otimes t\mathcal{H}^1\llcorner(\mathbb{R}t\cap R_{l,h})\Big\}, \end{aligned} \tag{3.3.9}$$

with $t=\mathbf{e}_1$, for simplicity. Then e does not depend on l and h.

Proof. The thesis is obtained through the following remarks.

(i) With a rescaling argument we get that

$$e(l,h) = e(\lambda l, \lambda h) \quad \forall\, \lambda \in (0,1)\,. \tag{3.3.10}$$

(ii) It is also immediate to notice that

$$e(l,h) \le e(l,H) \quad \text{whenever } h \le H\,, \tag{3.3.11}$$

by definition.

(iii) Moreover

$$e\left(\frac{l}{p}, h\right) \le e(l,h) \quad \forall\, p \in \mathbb{N} \setminus \{0\} \tag{3.3.12}$$

with a selection argument[13].

Thus our claim is proved, because

$$e\left(\frac{l}{p}, h\right) \overset{(3.3.12)}{\le} e(l,h) \overset{(3.3.10)}{=} e\left(\frac{l}{p}, \frac{h}{p}\right) \overset{(3.3.11)}{\le} e\left(\frac{l}{p}, h\right) \quad \forall\, p \in \mathbb{N} \setminus \{0\}\,.$$

□

In the following we will simply denote by e the truncated energy in (3.3.9).

Proposition 3.3.3 (Cell problem)**.** *Given $\psi : \mathbb{Z}^m \times \mathbb{S}^{d-1} \to [0,\infty)$ Borel measurable with $\psi(b,\tau) \ge c|b|$ and $\psi(0,\cdot) = 0$, the energy density $\bar\psi$ defined by (3.3.7) satisfies:*

(i) For every sequence $\mu_k \in \mathcal{M}_{\mathrm{df}}(B_{1/2})$ with $\mu_k \overset{*}{\rightharpoonup} b \otimes t\mathscr{H}^1 \llcorner (\mathbb{R}t \cap B_{1/2})$ one has

$$\bar\psi(b,t) \le \liminf_{k\to\infty} \bar E(\mu_k, \overline{B}_{1/2})\,. \tag{3.3.13}$$

(ii) The function $\bar\psi$ is subadditive in its first argument, i.e.,

$$\bar\psi(b+b',t) \le \bar\psi(b,t) + \bar\psi(b',t). \tag{3.3.14}$$

[13] Assume $p = 2$, then (3.3.12) is obtained choosing for each k the half part of $R_{l,h}$ with energy less than $\frac{1}{2}E(\mu_k, R_{l,h})$.

(iii) *The function* $\bar{\psi}$ *obeys*

$$\bar{\psi}(b,t) \le C|b| \tag{3.3.15}$$

for all b *and* t.

(iv) *The function* $\bar{\psi}$ *is Lipschitz-continuous in the sense that*

$$|\bar{\psi}(b,t) - \bar{\psi}(b',t')| \le c|b-b'| + c|b|\,|t-t'|\,. \tag{3.3.16}$$

with C *depending only on* ψ;

Proof. 1:

Consider a sequence $\mu_k \overset{*}{\rightharpoonup} \mu = b \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}t \cap B_{1/2})$. Without loss of generality we can assume $t = \mathbf{e}_1$.

We begin modifying the sequence, represented as $\mu_k = \theta_k \otimes \tau_k \mathcal{H}^1 \llcorner \gamma_k$.

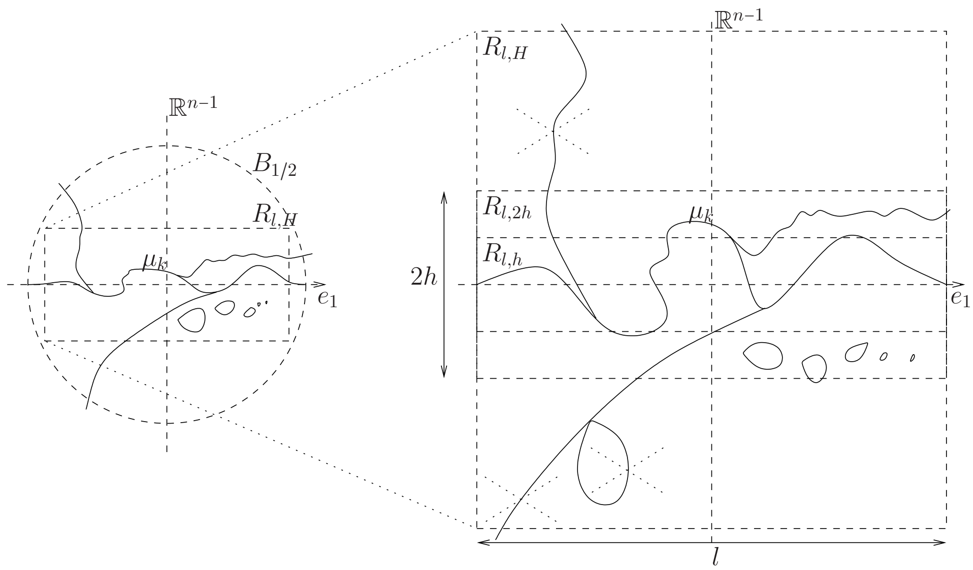

Figure 3.13: The construction for the truncated energy.

Choose[14] some parameters $h << H$ and $l \in (0,1)$. Since $\mu_k \overset{*}{\rightharpoonup} \mu$ then

$$\lim_{k\to\infty} E(\mu_k, R_{l,H} \setminus R_{l,h}) = 0.$$

[14]In order to remain in $B_{1/2}$ we imply $l^2 + (n-1)h^2 \le 1$.

We denote by γ_k° the union of connected components of the rectifiable set γ_k (that is, the support of μ_k) with a non-trivial intersection with $R_{l,h}$.
Since

$$\mathcal{H}^1 \left(\gamma_k \cap R_{l,H} \setminus R_{l,h}\right) \longrightarrow 0 \quad \text{as } k \to +\infty\,,$$

then $\gamma_k^\circ \subset R_{l,2h}$, definitely.
Consider $\mu_k^\circ := \mu_k \llcorner \gamma_k^\circ$: this new sequence of vector-valued measures satisfies

$$\mu_k^\circ \overset{*}{\rightharpoonup} \mu$$

with $\mathrm{supp}\mu_k^\circ \subset R_{l,2h}$ and $\partial\mu_k^\circ = 0$.
As a consequence of the definition of the truncated energy in Lemma 3.3.2 we get

$$\liminf_{k\to\infty} E(\mu_k^\circ, R_{l-2h}) \geq (l-2h)e\,,$$

thus

$$\liminf_{k\to\infty} E(\mu_k^\circ, S_h) \leq 2he\,,$$

with $S_h := R_{l,2h} \setminus R_{l-2h,2h}$.

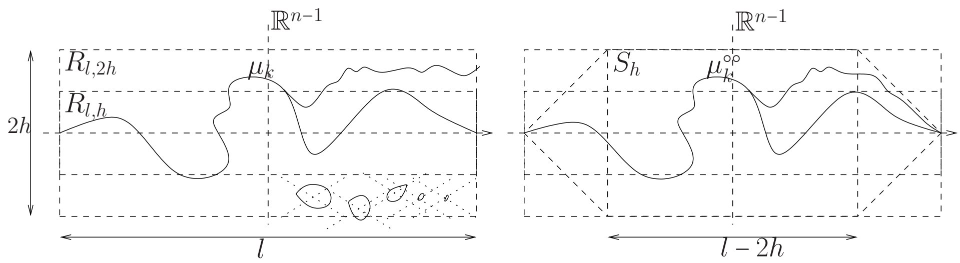

Figure 3.14: Passing from μ_k to $\mu_k^{\circ\circ}$.

As we drew in Figure 3.14, we head to the conclusion squeezing the measure μ_k° out of S_h through the projection $f : R_{l,2h} \to R_{l,2h}$. We mean that $f_{|R_{l-2h,2h}} \equiv \mathrm{id}$, while, writing $R_{l,2h} \ni x = (x_1, x')$,

$$f_{|S_h}(x_1, x') = \left(x_1, \left(\frac{l}{2h} - \frac{1}{h}x_1\right)x'\right)\,.$$

Let us define

$$\mu_k^{\circ\circ} := f_\sharp(\mu_k^\circ)\,.$$

Thus
$$E(\mu_k^{\circ\circ}, S_h) \le CE(\mu_k^{\circ}, S_h)$$
and
$$E(\mu_k^{\circ\circ}, R_{l,h}) \longrightarrow e + O\left(\frac{h}{l}\right). \tag{3.3.17}$$

Finally we deal with the boundary which we possibly changed through the projection: this is the last step, because (3.3.17) is what we needed as $h/l \to 0$. We write
$$\partial\mu_k^{\circ\circ} = \theta'\left(\delta_{l/2\mathbf{e}_1} - \delta_{-l/2\mathbf{e}_1}\right).$$
Indeed, the measure
$$\mu_k^{\circ\circ\circ} := \mu_k^{\circ\circ} + \theta' \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}\mathbf{e}_1 \setminus R_{l,h})$$
satisfies $\partial\mu_k^{\circ\circ\circ} = 0$, but at the same time
$$\mu_k^{\circ\circ\circ} \overset{*}{\rightharpoonup} b \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}\mathbf{e}_1 \cap B_{1/2}) + \theta' \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}\mathbf{e}_1 \setminus R_{l,h}),$$
thus $\theta' = b$.

2: This property follows easily from 1 by approximating the measure $(b+b') \otimes t\mathscr{H}^1 \llcorner (\mathbb{R}t \cap B_{1/2})$ with to measures supported on disjoint segments converging to $\mathbb{R}t \cap B_{1/2}$ and multiplicity b and b' respectively.

3: Set
$$M = \sum_{i=1}^n \sum_{j=1}^n (\psi(e_j, e_i) + \psi(e_j, -e_i) + \psi(-e_j, e_i) + \psi(-e_j, -e_i)). \tag{3.3.18}$$

Let $t \in \mathbb{S}^{d-1}$. For notational simplicity assume $t \cdot e_i \ge 0$, $b \cdot e_j \ge 0$ for all i, j. Choose a piecewise affine curve γ joining $-t/2$ with $t/2$ such that its tangent vector takes values in $\{e_i\}$. From the definition one obtains
$$\bar{\psi}(e_j, t) \le \int_\gamma \psi(e_j, \tau)\, d\mathscr{H}^1 \le \sum_{i=1}^n \psi(e_j, e_i) \tag{3.3.19}$$

From 2 one then obtains
$$\bar{\psi}(b, t) \le \sum_{j=1}^n |b \cdot e_j| \bar{\psi}(e_j, t) \le \sum_{j=1}^n |b \cdot e_j| \sum_{i=1}^n \psi(e_j, e_i) \le n|b|M. \tag{3.3.20}$$

4: From 2 and 3 we obtain
$$\bar{\psi}(b, t) \le \bar{\psi}(b', t) + c|b - b'|. \tag{3.3.21}$$

Construct now γ as the curve that joins $-t'/2$ with $t'/2$, then $t-t'/2$, extended t-periodic. Let γ_j be a copy of γ scaled down by a factor j, τ_j its tangent vector. Then $\mu_j = b \otimes \tau_j \mathscr{H}^1 \llcorner \gamma_j \overset{*}{\rightharpoonup} b \otimes t \mathscr{H}^1 \llcorner (t\mathbb{R})$. By 1 we obtain

$$\bar{\psi}(b,t) \leq \bar{\psi}(b,t') + |t-t'|\bar{\psi}(b, \frac{t-t'}{|t-t'|}) \leq \bar{\psi}(b,t') + c|b|\,|t-t'|\,. \tag{3.3.22}$$

$\square$

Lemma 3.3.4. *Assume that* $\psi : \mathbb{R}^m \times \mathbb{S}^{d-1} \to [0,\infty)$ *obeys*

$$|\psi(b,t) - \psi(b',t')| \leq c|b-b'| + c(1+|b|)\,|t-t'|\,. \tag{3.3.23}$$

Let $\mu = \theta \otimes \tau \mathscr{H}^1 \llcorner \gamma$, $\mu' = \theta' \otimes \tau' \mathscr{H}^1 \llcorner \gamma'$, *with* τ *and* τ' *the tangent vectors and* γ, γ' *one-rectifiable. Then*

$$|E(\mu) - E(\mu')| \leq C|\mu - \mu'|(\mathbb{R}^d)\,, \tag{3.3.24}$$

where $E(\mu) = \int_\gamma \psi(\theta,\tau) d\mathscr{H}^1$. *Further, if* $f : \mathbb{R}^d \to \mathbb{R}^d$ *is bi-Lipschitz then*

$$|E(\mu) - E(f_\# \mu)| \leq C E(\mu) \|Df - \mathrm{Id}\|_{L^\infty} \tag{3.3.25}$$

Proof. The first estimate follows from the properties of ψ, considering separately the sets $\gamma \cap \gamma'$, $\gamma \setminus \gamma'$ and $\gamma' \setminus \gamma$. The second one follows from the change of variables formula. $\square$

Theorem 3.3.5 (Relaxation)**.** *Let* ψ *and* $\bar{\psi}$ *be as in Proposition 3.3.3. Then the energy*

$$\bar{E}_0(\mu) = \begin{cases} \displaystyle\int_\gamma \bar{\psi}(\theta,\tau)\, d\mathscr{H}^1 & \text{if } \mu \in \mathcal{M}_{\mathrm{df}}(\mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

is the lower semicontinuous envelope of

$$E_0(\mu) = \begin{cases} \displaystyle\int_\gamma \psi(\theta,\tau)\, \mathscr{H}^1 & \text{if } \mu \in \mathcal{M}_{\mathrm{df}}(\mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

with respect to the weak convergence in $\mathcal{M}_{\mathrm{df}}(\mathbb{R}^3)$, *i.e.,*

$$\int_\gamma \bar{\psi}(b,t)\, d\mathscr{H}^1 = \inf\left\{ \liminf_{j\to\infty} \int_{\gamma_j} \psi(\theta_j,\tau_j)\, d\mathscr{H}^1 : \mu_j \overset{*}{\rightharpoonup} \mu \right\} \tag{3.3.26}$$

where $\mu = \theta \otimes \tau \mathscr{H}^1 \llcorner \gamma$ *and* $\mu_j = \theta_j \otimes \tau_j \mathscr{H}^1 \llcorner \gamma_j \in \mathcal{M}_{\mathrm{df}}$.

Proof. Upper bound. Let $\mu \in \mathcal{M}_{\mathrm{df}}$. By Theorem 1.3.1 there is a sequence of polygonal measures μ_k and a sequence of bi-Lipschitz maps f_k such that

$$\mu_k \overset{*}{\rightharpoonup} \mu\,, \quad \|Df_k - \mathrm{Id}\|_{L^\infty} \to 0\,, \quad \text{and} \quad \|\mu_k - (f_k)_\# \mu\| \to 0\,. \tag{3.3.27}$$

By Lemma 3.3.4 one easily obtains

$$\lim_{k\to\infty} \bar{E}(\mu_k) = \bar{E}(\mu)\,. \tag{3.3.28}$$

Therefore it suffices to prove the upper bound for polygonal measures.

Let $\mu \in \mathcal{M}_{\mathrm{df}}$, polygonal. Then γ is a finite union of disjoint segments γ_i, $i = 1, \dots, N$. Let b_i, t_i be the corresponding vector multiplicity and orientation. For each i consider a sequence $\mu_i : \mathbb{N} \to \mathcal{M}_{\mathrm{df}}(\mathbb{R}^3)$ such that

$$\lim_{k\to\infty} E(\mu_{i,k}, B_{1/2}(0)) = \bar{\psi}(b_i, t_i)\,, \quad \mathrm{supp}(\mu_{i,k} - b_i \otimes t_i \mathscr{H}^1 \llcorner (\mathbb{R}t_i)) \subset B_{1/2}(0)\,. \tag{3.3.29}$$

The recovery sequence is obtained covering each γ_i with an increasing number of small balls and gluing scaled and translated copies of $\mu_{i,k}$.

Lower bound. Let $\mu \in \mathcal{M}_{\mathrm{df}}(\mathbb{R}^3)$, and $\mu_k \overset{*}{\rightharpoonup} \mu$. We identify μ and μ_k with the corresponding currents $T, T_k \in \mathscr{R}_1(\mathbb{R}^d; \mathbb{Z}^m)$.

We shall approximate the limit T by a polygonal current, and show that there is a small modification of the sequence (T_k) which converges to the polygonal current. On this sequence the lower semicontinuity will follow from (3.3.13).

We fix $\varepsilon > 0$ and apply the density theorem to T, let f and P be the resulting bi-Lipschitz map and polygonal current. We define

$$\tilde{T}_k = (f^{-1})_\# (T_k - Q), \qquad \text{where } Q = T - f_\# P\,. \tag{3.3.30}$$

It is easy to see that $\tilde{T}_k \overset{*}{\rightharpoonup} P$ and that $\partial \tilde{T}_k = 0$. From Lemma 3.3.4 we have

$$\bar{E}(\tilde{T}_k) \le (1 + C\varepsilon)\bar{E}(T_k) + C\varepsilon\,. \tag{3.3.31}$$

We can cover a $(1-\varepsilon)$-fraction of the support of P with finitely many disjoint balls, centered on it. In each ball P is supported on a segment and has constant multiplicity and $\tilde{T}_k \overset{*}{\rightharpoonup} P$. By (3.3.13) the statement follows. □

Remark 3.3.6. Notice that for $d = 2$ this reduces to the concept of BV-ellipticity for vector-valued partition problems in [2],[3].

As in that case, the conditions of Proposition 3.3.3 do not completely characterize the $\mathscr{H}^1$-ellipticity.

3.3.3 An example

We focus here on the 2-dimensional case, with an explicit function ψ borrowed from [14] (see (1.8) for the definition).
Consider multiplicities of the form $\theta = (\theta_1, \theta_2) \in \mathbb{Z}^2$ and a tangent vector field $\tau = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1$, we define

$$\psi^c(\theta,\tau) := \frac{1}{4\pi(1-\nu)}\theta \begin{pmatrix} 2-2\nu\sin^2\alpha & \nu\sin(2\alpha) \\ \nu\sin(2\alpha) & 2-2\nu\cos^2\alpha \end{pmatrix} \theta\,,$$

with a parameter $\nu \in (-1, 1/2)$. First of all, we observe that

$$\begin{aligned} \psi^c(\theta,\tau) &= \frac{1}{4\pi}\theta\left[(2-2\nu)\mathrm{Id} + 2\nu\begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha = \frac{1}{2\pi}\psi(\theta,\tau) \end{pmatrix}\right] \\ &= \frac{1}{4\pi(1-\nu)}\left(2(1-\nu)|\theta|^2 + 2\nu(\theta\cdot\tau)^2\right) = \frac{1}{2\pi}\psi(\theta,\tau)\,, \end{aligned} \tag{3.3.32}$$

where

$$\psi(\theta,\tau) = |\theta|^2 + \eta(\theta\cdot\tau)^2$$

and $\eta = \frac{\nu}{1-\nu} < 1$ (because $\nu < 1/2$).

Without loss of generality we can assume $\eta \in [0,1)$: indeed, if $\nu < 0$, we can rewrite (3.3.32) as

$$\psi^c(\theta,\tau) = \frac{1}{2\pi(1-\nu)}\left(|\theta|^2 - \nu(\theta\cdot\tau^\perp)^2\right) = \frac{1}{2\pi(1-\nu)}\psi'(\theta,\tau)\,,$$

where $\psi'(\theta,\tau) = |\theta|^2 + \eta'(\theta\cdot\tau^\perp)^2$ differs from ψ for the constant $\eta' := -\nu \in [0,1)$ and for the rotation on τ. In what follows, we focus on ψ with $\eta \in [0,1)$ and we neglect the differences between ψ and ψ'.

Remark 3.3.7. If $\theta_1 \geq |\theta_2|$ and $\theta_1 \geq 2$, then

$$\psi(\theta,\tau) \geq \psi(\theta - e_1,\tau) + \psi(e_1,\tau)\,.$$

Proof. A simple computation leads to the estimate

$$\begin{aligned} & \psi(\theta,\tau) - \psi(\theta-e_1,\tau) - \psi(e_1,\tau) \\ =\ & |\theta|^2 - |\theta-e_1|^2 - 1 + \eta\left((\theta\cdot\tau)^2 - (\theta\cdot\tau-\tau_1)^2 - \tau_1^2\right) \\ =\ & 2(\theta_1-1) + 2\eta\left(\theta_1\tau_1^2 + \theta_2\tau_1\tau_2 - \tau_1^2\right) \\ =\ & 2(\theta_1-1)(1+\eta\tau_1^2) + 2\eta\theta_2\tau_1\tau_2 \\ \geq\ & 2(\theta_1-1) - |\theta_2| = 2\theta_1 - |\theta_2| - 2 \geq 0\,, \end{aligned}$$

where the hypothesis on θ_1 have been used in the last line together with $\eta \in [0,1)$. $\square$

If $b \in \mathbb{Z}^2$ and $t \in \mathbb{S}^1$, we want to compute the relaxation[15]

$$\bar{\psi}(b,t) = \inf\left\{ \int_\gamma \psi(\theta,\tau)\, d\mathscr{H}^1 : \operatorname{curl}\left(\theta \otimes \tau \mathscr{H}^1 \llcorner \gamma - b \otimes t \mathscr{H}^1 \llcorner (-1/2, 1/2)t\right) = 0 \right\}$$

The computation of the infimum above can be considerably simplified by the following remarks.

(1) Thanks to the density result in Subsection 3.3.1, we can assume γ to be the union of finitely many segments, with θ constant on each of them.

(2) We can further assume that $\theta_i \in \{-1, 0, 1\}$ $(i = 1, 2)$ almost everywhere. In fact, suppose $\theta_1 \geq 2$ and $\theta_1 \geq |\theta_2|$ on the segment $[s', s''] \times \{0\}$. The measure $\theta \otimes \tau \mathscr{H}^1 \llcorner ([s', s''] \times \{0\})$ has energy $\psi(\theta,\tau)(s'' - s')$. We replace it by the sum

$$(\theta - \mathbf{e}_1) \otimes \tau \mathscr{H}^1 \llcorner ([s', s''] \times \{0\}) + \mathbf{e}_1 \otimes \tau' \mathscr{H}^1 \llcorner \sigma' + \mathbf{e}_1 \otimes \tau'' \mathscr{H}^1 \llcorner \sigma'', \quad (3.3.33)$$

where σ' and σ'' are the segments connecting $(s', 0)$ and $((s' + s'')/2, \epsilon)$ and $((s' + s'')/2, \epsilon)$ and $(s'', 0)$, respectively, and τ', τ'' are their tangent vectors. Thanks to Remark 3.3.7 the energy of the new measure (3.3.33) is not higher than $\psi(\theta,\tau)(s'' - s')$: in fact the new energy is

$$\begin{aligned} & \psi(\theta - \mathbf{e}_1, \tau)(s'' - s') + \psi(e_1, \tau')\mathscr{H}^1(\sigma') + \psi(\mathbf{e}_1, \tau'')\mathscr{H}^1(\sigma'') \\ = \; & \psi(\theta - \mathbf{e}_1, \tau)(s'' - s') + \psi(e_1, \tau)(s'' - s') + O(\epsilon)\,. \end{aligned}$$

(3) Finally we can assume that

$$\begin{aligned} \theta_1\theta_2 \geq 0 & \qquad \text{on segments with} \quad \tau_1\tau_2 < 0 \\ \theta_1\theta_2 \leq 0 & \qquad \text{on segments with} \quad \tau_1\tau_2 > 0\,. \end{aligned}$$

In fact, if $\theta_1\theta_2\tau_1\tau_2 > 0$, then

$$\begin{aligned} \psi(\theta,\tau) &= |\theta|^2 + \eta|\theta \cdot \tau|^2 = \psi((\theta_1, 0), \tau) + \psi((0, \theta_2), \tau) + 2\eta\theta_1\theta_2\tau_1\tau_2 \\ &> |\theta|^2 + \eta|\theta \cdot \tau|^2 = \psi((\theta_1, 0), \tau) + \psi((0, \theta_2), \tau) - 2\eta\theta_1\theta_2\tau_1\tau_2 \\ &= \psi((-\theta_1, \theta_2), \tau)\,. \end{aligned}$$

[15] It is implied that, in the infimum of below, γ is a 1-dimensional rectifiable set in $B_{1/2}$ with tangent τ almost everywhere and the multiplicity θ belongs to $L^1(\gamma, \mathbb{Z}^2)$.

We decompose the support γ of a measure $\theta \otimes \tau \mathscr{H}^1 \llcorner \gamma$ in

$$\begin{aligned}
\gamma_1 &:= \{x \in \gamma : \theta_1 = 0\} \\
\gamma_2 &:= \{x \in \gamma : \theta_2 = 0\} \\
\gamma_+ &:= \{x \in \gamma : \theta_1\theta_2 > 0\} \\
\gamma_- &:= \{x \in \gamma : \theta_1\theta_2 < 0\} .
\end{aligned}$$

Let us notice that the curves above are pairwise disjoint. We define[16]

$$\begin{aligned}
T_1 &:= \int_{\gamma_1} \theta_1 \tau \, d\mathscr{H}^1 \\
T_2 &:= \int_{\gamma_2} \theta_2 \tau \, d\mathscr{H}^1 \\
T_+ &:= \int_{\gamma_+} \theta_1 \tau \, d\mathscr{H}^1 \\
T_- &:= \int_{\gamma_-} \theta_1 \tau \, d\mathscr{H}^1
\end{aligned}$$

and

$$\Theta \otimes T := \int_\gamma \theta \otimes \tau \, d\mathscr{H}^1 = \begin{pmatrix} T_1 + T_+ + T_- \\ T_2 + T_+ - T_- \end{pmatrix} \in \mathbb{R}^{2\times 2} .$$

With this notation

$$\begin{aligned}
E(\theta \otimes \tau \mathscr{H}^1 \llcorner \gamma) &= \int_{\gamma_1} \psi(e_1, \tau) \, d\mathscr{H}^1 + \int_{\gamma_2} \psi(e_2, \tau) \, d\mathscr{H}^1 \\
&+ \int_{\gamma_+} \psi(e_1 + e_2, \tau) \, d\mathscr{H}^1 + \int_{\gamma_-} \psi(e_1 - e_2, \tau) \, d\mathscr{H}^1 \qquad (3.3.34)
\end{aligned}$$

Remark 3.3.8. Consider $b \in \mathbb{R}^2$ and $\hat{\gamma}$ a union of simple curves with tangent vector $\tau(x)$, then

$$\int_{\hat{\gamma}} \psi(b, \tau(x)) \, d\mathscr{H}^1(x) \geq \psi\left(b, \frac{\hat{\tau}}{|\hat{\tau}|}\right) |\hat{\tau}|$$

where $\hat{\tau} := \int_{\hat{\gamma}} \tau(x) \, d\mathscr{H}^1(x)$.

[16]The illusory asymmetry in the definition of N_+, N_- is motivated by the fact that $\theta_1 = \theta_2$ on γ_+ and $\theta_1 = -\theta_2$ on γ_-.

Proof. Since the map $\tau \mapsto \eta(b\cdot\tau)^2$ is convex and $\fint_{\hat\gamma}\tau = \hat\tau\mathscr{H}^1(\hat\gamma)^{-1}$, then, by Jensen inequality,

$$\int_{\hat\gamma}\psi(b,\tau)\,d\mathscr{H}^1 \geq \mathscr{H}^1(\hat\gamma)\left(|b|^2+\eta\left(b\cdot\frac{\hat\tau}{\mathscr{H}^1(\hat\gamma)}\right)^2\right) =: h(L)\,, \tag{3.3.35}$$

with $L = \mathscr{H}^1(\hat\gamma)$ and $h(l) := l|b|^2 + l^{-1}\eta b\cdot\hat\tau$.
This function h is increasing in our interval of interest, since it has a global minimum at $l_0 = \sqrt{\eta}\frac{|b\cdot\hat\tau|}{|b|} \leq |\hat\tau|$ (and it is increasing afterwards) and we have that

$$L = \int_{\hat\gamma}|\tau| \geq \left|\int_{\hat\gamma}\tau\right| = |\hat\tau| \geq l_0\,. \tag{3.3.36}$$

Thus we can conclude that

$$\int_{\hat\gamma}\psi(b,\tau)\,d\mathscr{H}^1 \overset{(3.3.35)}{\geq} h(L) \overset{(3.3.36)}{\geq} h(|\hat\tau|) = |\hat\tau|\psi\left(b,\frac{\hat\tau}{|\hat\tau|}\right).$$

The equality holds if and only if τ is constant along γ. $\square$

Coupling this remark with (3.3.34) we get

$$E(\theta\otimes\tau\mathscr{H}^1 \llcorner \gamma) \geq f(T_1,T_2,T_+,T_-)\,, \tag{3.3.37}$$

where $f:\mathbb{R}^2\times\ldots\times\mathbb{R}^2\to\mathbb{R}$ is the function

$$\begin{aligned} f(z_1,z_2,z_3,z_4) &= |z_1|\psi\left(e_1,\frac{z_1}{|z_1|}\right)+|z_2|\psi\left(e_2,\frac{z_2}{|z_2|}\right)\\ &+ |z_3|\psi\left(e_1+e_2,\frac{z_3}{|z_3|}\right)+|z_4|\psi\left(e_1-e_2,\frac{z_4}{|z_4|}\right). \end{aligned}$$

and equality in (3.3.37) holds if and only if τ is constant in each of the four subsets of γ.

We conclude that

$$\bar\psi(b,t) = \min\{f:\, T_1+T_2+T_- = b_1T,\ T_2+T_+-T_- = b_2T\}\,.$$

This is a 4-dimensional minimization problem with constraint.

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