LIMIT OF VISCOUS DYNAMIC PROCESSES IN DELAMINATION AS THE VISCOSITY AND INERTIA VANISH

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Abstract. We introduce a model of dynamic evolution of a delaminated visco-elastic body with viscous adhesive. We prove the existence of solutions of the corresponding system of PDEs and then study the behaviour of such solutions when the data of the problem vary slowly. We prove that a rescaled version of the dynamic evolutions converge to a "local" quasistatic evolution, which is an evolution that satisfies an energy inequality and a semistability condition at all times. In the one-dimensional case we give a more detailed description of the limit evolution and we show that they behaves very similar to the limit of the solutions of the dynamic model in [28], where no viscosity in the adhesive is taken into account.

1. Introduction

Recently the field of contact mechanics is becoming more and more studied, thanks also to the numerous engineering applications and simulations. The problem of delamination is an important part of these modellings. The setting consists of two elastic bodies glued by an adhesive on an interface. External forces and high stresses due to elastic deformations of the bodies may break the macromolecules of the adhesive, weakening its effect. Such process is irreversible, in the sense that the deteriorated adhesive cannot be repristinated. The state of the adhesive is described by the delamination coefficient z, that is a function defined on the interface which takes values in $[0, 1]$ (see Section 2). Untill the glue is effective the movements of the bodies at the interface are costrained. Moreover some constrains at the interface are always considered due to the non-interpenetrability of the two bodies or to the pressure of the system (see Section 2). A rate-independent approach to this problem has been introduced in [16] and carried on by many authors. In the present paper we do not adopt this approach for the delamination process, since we also consider the viscous effects related to the debonding of the adhesive. Such term makes the system no more rate-independent. In the bulk we take into account the viscosity and inertia of the material, neglecting instead the thermal effects. In [25] and [24] it is considered a system where also thermal effects are analysed, while no viscosity of the delamination coefficient is considered. Terms related to friction have been studied in different settings where inertia is neglected (see, e.g., [23], [4]).

In the first part of the paper we prove an existence result for solutions of the PDEs system related to the model described. We also prove the existence of solutions with specific constrains on the strain field at the interface. This is obtained by letting go to infinity the value of a penalization term. Similar existence results for dynamic of delaminated bodies are also provided in several other papers, e.g., [4], [24], [25], where different viscosity terms are studied. We are interested in considering all the dissipative effects due to viscosity in order to develop a vanishing viscosity analysis in the second part of the paper. To be precise, we study the asymptotic behaviour of such solutions when the external forces and boundary data vary in a still slower way. Such analysis coincides to study the asymptotic behaviour of the solutions when the viscosity and inertia vanish in a very specific ratio. If the viscosity goes to zero as ϵ, then the density of the body (inertia) must vanish as ϵ^2. In particular, such analysis is different from the standard vanishing viscosity approach, that is the asymptotic analysis as only the viscosity tends to zero (for similar approach, see, e.g., [12], [8], [31], [21], [15], [17], [18], [20], [22], [28], [29], [27]). We remark that also a vanishing viscosity analysis, keeping fixed the density of the body, is possible in order to obtain a dynamic solution of the problem with neglected viscosity, even if this is not the aim of the present paper. This analysis is done on a similar model in [28]. Analysis like the ours have already been made in different settings (see, e.g., [1], [10]). In many cases the solution corresponding to a dynamic system tends, as ϵ goes to zero, to the quasistatic evolution of the same system. This means that the limiting function is, at every time, a stationary point of the energy functional of the system. In particular, for a quasistatic evolution, a energy balance holds true. We refer to [16] and [30] for the quasistatic model. In [10] dynamic evolutions of a visco-elasto-plastic body are considered and, letting the parameter ϵ vanish, it is proven that such solutions approximate a quasistatic evolution in perfect plasticity. However there are systems whose dynamic solutions do not converge to a quasistatic evolution. The limiting function is proven to satisfy only an energy inequality, showing a lack of energy that disappeared misteriously. The quasistatic limit shows indeed discontinuities in time, where it jumps from a minimum of the energy u^- to another u^+. In some cases it is possible, using an ad-hoc rescaling of the time variable, to prove that at each jump the limiting solution runs istantanously between the initial state u^- and the final state u^+, following the trajectory of a solution of the dynamic problem which has u^- as limit at $-\infty$, and u^+ at $+\infty$. See [1] for a finite dimensional analysis of this fact. In [28] a dynamic model for adhesive contact of visco-elastic body is considered, neglecting viscosity on the interface. When the viscosity tends to zero the solutions of such problem approximate an evolution which satisfies a semistability condition and an energy inequality. An asymptotic analysis for the slow-load limit (the same analysis of ours) is also discussed. In the second part of the present paper we prove, as for the evolutions considered in [28], that the slow-load limit satisfy a semistability condition and an energy inequality. Such conditions turn out to be the same of the corresponding conditions obtained in [28]. However in the presence paper a different argument and proof are needed to prove the semistability condition, due to the presence of the viscosity in the adhesive, that in the passage to the limit yields an additional lost of energy. This result generalizes the analysis in [28] to the case with viscosity. Moreover at least in the one-dimensional case, we give a finer description of the behaviour of the solutions of the limiting system, i.e., of the energy inequality and the semistability condition. We consider the example given in [28, Section 4], where it is proven that under suitable external load, the limit obtained by vanishing viscosity shows a jump where the delamination coefficient switches istantanously from 1 to 0. Theorem 4.14 shows that our limit behaves in the same way, and coincides with the limit in [28] at least in the time interval before the jump occurs.

The final aim of the work is to give the basis for analyzing whether jumps occur for the limiting solution.

2. Rreliminaries

Reference configuration and notation. We consider an hyperelastic body that occupies a bounded open domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary. We suppose that

$$\Omega = \Omega_1 \cup \Gamma \cup \Omega_2,$$

where Γ is a Lipschitz surface which is the common boundary of the two disjoint connected and open sets Ω_1 and Ω_2. The body is perfectly elastic on $\Omega_1 \cup \Omega_2$ while the surface Γ represents the interface where Ω_1 and Ω_2 are glued and where delamination may occur. We denote by ν the normal to Γ that points from Ω_1 into Ω_2. We also suppose that the boundary $\partial\Omega$ writes as the union

$$\partial\Omega := \partial_D\Omega \cup \partial_N\Omega,$$

where $\partial_D\Omega$ and $\partial_N\Omega$ are the closure in $\partial\Omega$ of two disjoint open sets with common boundary. We assume that $\partial_D\Omega$ has positive $(d-1)$-Hausdorff measure and that it has nonnegligible intersection with both $\partial\Omega_1$ and $\partial\Omega_2$.

In the sequel the symbol $\mathcal{M}_b(A,\mathbb{R}^k)$ denotes the space of Radon measures on the open set A with values in $\mathbb{R}^k$. The symbol $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the space of symmetric $d\times d$ real matrices.

Stress and Strain. The class of admissible displacements of the delamination problem is the space $H^1(\Omega_1\cup\Omega_2,\mathbb{R}^d)$. It is convenient to define

$$H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)) := \{u \in H^1(\Omega_1\cup\Omega_2,\mathbb{R}^d)) : u = 0 \text{ on } \partial_D\Omega\}. \tag{2.1}$$

The jump on Γ of a displacement u is denoted by $[u] = u_2 - u_1$ where u_1 and u_2 are, respectively, the trace on Γ of $u \in H^1(\Omega_1,\mathbb{R}^d)$ and $u \in H^1(\Omega_2,\mathbb{R}^d)$. The continuity of the trace operator from $H^1(\Omega_i,\mathbb{R}^d)$ into $H^{\frac12}(\Gamma,\mathbb{R}^d)$ reads

$$\|u\|_{H^{1/2}(\Gamma)} \le \frac{\gamma}{2}\|u\|_{H^1(\Omega_i)}, \tag{2.2}$$

for a positive constant γ, and then we have

$$\|[u]\|_{H^{1/2}} \le \gamma\|u\|_{H^1_D}. \tag{2.3}$$

The symmetric gradient $e(u)$ of $u\in H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^d))$ is defined as

$$e(u) := \frac12(\nabla u + (\nabla u)^T).$$

In $H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$ the following Korn inequality holds

$$\|u\|_{H^1} \le \beta\|e(u)\|_{L^2} \qquad \text{for every } u\in H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^d)), \tag{2.4}$$

for a positive constant β.

The two elasticity tensors $\mathbb{C}^0$ and $\mathbb{C}^1$ are symmetric and positive definite, there exist positive constants α_i and β_i such that

$$\alpha_0|\eta|^2 \le \langle\mathbb{C}^0\eta,\eta\rangle \le \beta_0|\eta|^2, \tag{2.5a}$$

$$\alpha_1|\eta|^2 \le \langle\mathbb{C}^1\eta,\eta\rangle \le \beta_1|\eta|^2, \tag{2.5b}$$

for all $\eta\in\mathbb{R}^{d\times d}$. It is convenient to introduce the following notations

$$\mathcal{Q}_0(e) = \frac12\langle\mathbb{C}^0 e, e\rangle, \tag{2.6}$$

$$\mathcal{Q}_1(e) = \langle\mathbb{C}^1 e, e\rangle, \tag{2.7}$$

for all $e\in L^2(\Omega_1\cup\Omega_2,\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

The stress σ satisfies the costitutive relation

$$\sigma = \mathbb{C}^0 e(u) + \mu\mathbb{C}^1 e(\dot u), \tag{2.8}$$

where $\mu > 0$ is the viscosity parameter in the bulk. Then the second principle of dynamics reads

$$\rho\ddot u(t) - \mathrm{div}\sigma(t) = f(t) \qquad \text{in } \Omega, \tag{2.9}$$

where we assume that the mass density of the elastic body is the constant $\rho > 0$. Together with (2.9) we require that the following boundary conditions are satisfied

$$u(t) = w(t) \quad \text{on } \partial_D\Omega, \tag{2.10a}$$

$$\sigma(t)\nu = g(t) \quad \text{on } \partial_N\Omega, \tag{2.10b}$$

$$\sigma(t)\nu = -\nabla V([u(t)])z(t) \quad \text{on } \Gamma, \tag{2.10c}$$

where V and z are the potential and the delamination coefficient introduced in the next Section. We can define the total external load of the system $\mathcal{L}(t) \in H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ by

$$\langle \mathcal{L}(t), \varphi\rangle := \langle f(t), \varphi\rangle + \langle g(t), \varphi\rangle_{\partial_N\Omega}, \tag{2.11}$$

for all $\varphi \in H_D^1(\Omega_1\cup\Omega_2, \mathbb{R}^d)$. To deal with (2.9) and (2.10), we define the continuous linear operator $\mathrm{div}_D : L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}) \to H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ by

$$\langle \mathrm{div}_D\sigma, \varphi\rangle := \langle \sigma, e(\varphi)\rangle, \tag{2.12}$$

for every $\sigma \in L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and every $\varphi \in H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$. Hence, if $f(t)$, $g(t)$, $\sigma(t)$, $u(t)$, $\partial_D\Omega$, and $\partial_N\Omega$ are sufficiently regular and $\mathcal{L}(t)$ is the total external load defined by (2.11), then (2.9), (2.10b), and (2.10c) are equivalent to

$$\rho\ddot{u}(t) - \mathrm{div}_D\sigma(t) = \mathcal{L}(t) + T(u, z), \tag{2.13}$$

where equality holds in $H_D^{-1}(\Omega_1\cup\Omega_2, \mathbb{R}^d)$, and where $T(u, z)$ is the linear operator on $H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ defined by (2.17) below. In weak form (2.13) reads as

$$\langle \rho\ddot{u}(t), \varphi\rangle + \langle \sigma(t), e(\varphi)\rangle = \langle \mathcal{L}(t), \varphi\rangle - \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma, \tag{2.14}$$

for every $\varphi \in H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$.

Delamination parameter and energy stored by the adhesive. As in the modelling approach by M. Frémond (see [13], [14]), at a fixed time the state of the glue on the interface Γ is described by the variable $z : \Gamma \to [0, 1]$. The state $z(x) = 1$ means that the adhesive is completely undestroyed, while $z(x) = 0$ means that the molecular links are all broken and the interface is totally debonded at $x \in \Gamma$. The deterioration of the glue is considered irreversible, that is the variable z is a nonincreasing function of the time. This turns into the condition

$$\dot{z} \leq 0.$$

The class of admissible delamination parameters is denoted by

$$\mathcal{Z} := \{z \in L^2(\Gamma) : 0 \leq z \leq 1\}.$$

During the evolution of the system the energy needed to delaminate is denoted by $\alpha \in L^\infty(\Gamma)$, and such energy is dissipated in two ways, by heat production, whose cost we denote by $a_1 = a_1(x) > 0$, $x \in \Gamma$, and by creation of new delaminated surfaces, whose cost we denote by $a - a_1 := a_0 = a_0(x) > 0$, $x \in \Gamma$. Hence the dissipation due to these effects in the time interval $[t_1, t_2]$ reads

$$\mathcal{D}_a(t_1, t_2) := -\int_{t_1}^{t_2} \langle a_0 + a_1, \dot{z}(s)\rangle_\Gamma ds. \tag{2.15}$$

When evolution is quite fast we also consider the dissipation due to the viscosity of the glue. We consider a parameter $\mu = \mu(x) > 0$, $x \in \Gamma$, for which the energy dissipated by viscosity effects during the delamination process in the interval $[t_1, t_2]$ reads

$$\mathcal{D}_\tau(t_1, t_2) := \int_{t_1}^{t_2} \langle \mu\dot{z}(s), \dot{z}(s)\rangle_\Gamma ds. \tag{2.16}$$

In the sequel we will adopt the simplier (but not retrictive) hypothesis that μ is constant on Γ and coincides with the friction μ introduced in (2.8).

The energy stored in Γ by the adhesive is modelled as follows: let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonnegative and covex map such that

(i) $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. In particular $x = 0$ is the only local minimum of V.
(ii) $\nabla V : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with constant $L > 0$.
(iii) There exists $1 \leq \delta \leq \delta^*$ and $C > 0$ such that $|V(x)| \leq C(|x| + 1)^\delta$ for all $x \in \mathbb{R}^d$.

Here $\delta^* = +\infty$ for $d \leq 2$ and $\delta^* = \frac{d-1}{d-2}$ for $d > 2$. Since from (i) ∇V must vanish at the origin, property (ii) has the following consequence

(iv) For all $x \in \mathbb{R}^n$ it holds $|\nabla V(x)| \leq L|x|$.

The energy stored on Γ at a fixed time then reads:

$$\mathcal{E}_\Gamma(u, z) := \langle V([u]), z\rangle_\Gamma.$$

We remark that in dimension $d \leq 3$ we can take $V([u]) := \frac{1}{2}\mathbb{K}[u] \cdot [u]$ where $\mathbb{K}$ is called elastic coefficient of the adhesive. Such matrix is supposed positive definite and symmetric. With this choice we see that the growth of V in (iii) above is $\delta = 2$. In higher dimension such a choice cannot be done for compactness reasons that will be clear in the proof of Theorem 3.1.

For all $u \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ and $z \in L^\infty(\Gamma)$ we define $T(u, z) \in H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ as

$$\langle T(u, z), \varphi\rangle := \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma, \tag{2.17}$$

for every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$, so that, from (2.3), one has

$$|\langle T(u, z), \varphi\rangle| \leq \|\nabla V([u])\|_{L^2}\|[\varphi]\|_{L^2}\|z\|_{L^\infty} \leq 2L\gamma\|u\|_{H^1_D}\|\varphi\|_{H^1_D}\|z\|_{L^\infty},$$

which implies that there exists a positive constant C such that

$$\|T(u, z)\|_{H^{-1}_D} \leq C\|u\|_{H^1_D}\|z\|_{L^\infty}. \tag{2.18}$$

Mechanical constraints and delamination process. When delamination occurs on the interface Γ it may happen that the two parts Ω_1 and Ω_2 of the body separates. In particular cavitation phenomena or shear movements may occur. Such phenomenon arises by the appearance of a non-zero jump of the displacement on Γ. Since interpenetration of Ω_1 and Ω_2 must be avoid, classically such jump is constrained to have a nonnegative normal component. Such condition is known in literature as Signorini contact condition. A generalization of the Signorini condition is usually considered, in the following way. Let $D(x) \subset \mathbb{R}^d$ be a convex and closed cone, possibly depending on $x \in \Gamma$. This induces an ordering relation on the set of functions $v : \Gamma \to \mathbb{R}^d$, as follows,

$$v_1 \preceq v_2 \text{ if and only if } v_2(x) - v_1(x) \in D(x) \text{ for a.e. } x \in \Gamma.$$

The dual ordering $\preceq^*$ induced by the negative polar cone to D is given by

$$\zeta \preceq^* 0 \text{ if and only if } \zeta(x) \leq 0 \text{ for all } w \in D(x), \text{ for a.e. } x \in \Gamma.$$

Possible choices for the cone $D(x)$ are the following,

$$D(x) = \{v \in \mathbb{R}^d : v \cdot \nu(x) \geq 0\}, \tag{2.19a}$$

$$D(x) = \{v \in \mathbb{R}^d : v \cdot \nu(x) = 0\}, \tag{2.19b}$$

the first case being the classical unilateral Signorini contact condition, the latter being considered when cavitation cannot occur, for instance in systems under high pressure. The delamination mode (2.19a) and (2.19b) are usually refered to as

Mode I and *Mode II* respectively. The constraint on the jump $[u]$ and the normal stress $t(\sigma) := \sigma\nu$ on Γ than reads

$$[u] \succeq 0, \tag{2.20a}$$
$$t(\sigma) + T(u,z) \succeq^* 0, \tag{2.20b}$$
$$(t(\sigma) + T(u,z)) \cdot [u] = 0. \tag{2.20c}$$

The behaviour of the variable z is strictly connected to the evolution of $[u]$. Whenever $[u]$ varies this has the effect of destroying molecular links on Γ, that turns into a decrease of the corresponding glue state z. When the glue is completely erased, that is $z = 0$, any change of $[u]$ will not require energetic cost due to delamination. This is expressed by the costitutive equations

$$\dot z \leq 0, \tag{2.21a}$$
$$d \leq -\mu \dot z, \tag{2.21b}$$
$$\dot z(d + \mu \dot z) = 0, \tag{2.21c}$$
$$d \in \partial I_{[0,1]} + V([u]) - \alpha, \tag{2.21d}$$

where $\partial I_{[0,1]}$ is the subdifferential of the function $I_{[0,1]}$, that is the function with equals 0 on $[0,1]$ and $+\infty$ on $\mathbb{R} \setminus [0,1]$. The paramenter $\mu > 0$ is the viscosity of the adhesive. Let us remark that as soon as $z = 0$ equations (2.21b)-(2.21d) lose their significance and system (2.21) reduces to $z \equiv 0$, and no restriction to the evolution of $[u]$ is prescribed. At the same time, when $z > 0$ system (2.21) reads

$$\dot z \leq 0, \tag{2.22a}$$
$$\dot z(V([u]) + \mu \dot z - \alpha) = 0. \tag{2.22b}$$

Since z is a function defined on the interface Γ, equations (2.21) and (2.22) must be intended to hold everywhere on Γ.

3. Existence of unconstrained dynamic solutions

Theorem 3.1. *Let $\mathcal{L} \in L^2([0,T], H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$, $u_0, v_0 \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$, and $z_0 \in \mathcal{Z}$. Then there exists a triple (u, σ, z) with*

$$u \in L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.1a}$$
$$\dot u \in L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)) \cap L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.1b}$$
$$\ddot u \in L^2([0,T], H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.1c}$$
$$\sigma \in L^2([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}_{sym}), \tag{3.1d}$$
$$z \in H^1([0,T], L^2(\Gamma)) \cap L^\infty([0,T], L^\infty(\Gamma)), \tag{3.1e}$$

satisfying, for a.e. $t \in [0,T]$,

$$\rho \ddot u(t) - \operatorname{div} \sigma(t) = \mathcal{L}(t) + T(u,z), \tag{3.2a}$$
$$\sigma(t) = \mathbb{C}^0 E u(t) + \mu \mathbb{C}^1 E \dot u(t), \tag{3.2b}$$

on Ω,

$$\dot z(t) \leq 0, \tag{3.3a}$$
$$\dot z(t)(kV([u(t)]) + \mu \dot z(t) - \alpha) = 0, \tag{3.3b}$$

on Γ,

$$V([u(t)]) + \mu \dot z(t) - \alpha \leq 0, \tag{3.4}$$

on $\Gamma \cap \{z(t) > 0\}$*, and with initial data*

$$u(0) = u_0, \qquad \dot u(0) = v_0, \qquad z(0) = z_0. \tag{3.5}$$

Remark 3.2. Let us remark that, when $\mathcal{L}$ takes the form (2.11), in the regular case, (3.2a) means that

$$\sigma(t)\nu = -\nabla V([u(t)])z(t), \tag{3.6}$$

on Γ, and

$$\rho\ddot u(t) - \mathrm{div}\sigma(t) = f(t) \quad \text{ in } \Omega, \tag{3.7a}$$

$$\sigma(t)\nu = g(t) \quad \text{on } \partial_N\Omega. \tag{3.7b}$$

This is proved as follows. Integrating by parts (2.14) we get

$$\begin{aligned}&\langle\rho\ddot u,\varphi\rangle - \langle\mathrm{div}\sigma,\varphi\rangle - \langle\mathcal{L},\varphi\rangle = \\ &-\langle\nabla V([u])\cdot[\varphi],z\rangle_\Gamma - \langle\sigma\nu,[\varphi]\rangle_\Gamma - \langle\sigma\nu,\varphi\rangle_{\partial_N\Omega},\end{aligned} \tag{3.8}$$

where ν represents both the normal versor to Γ pointing from Ω_1 into Ω_2 and the outer normal to $\partial_N\Omega$. If we set $[\varphi] = 0$ we obtain the strong form (3.7), which together with (3.8) implies

$$\langle\nabla V([u])\cdot[\varphi],z\rangle_\Gamma = -\langle\sigma\nu,[\varphi]\rangle_\Gamma,$$

that is (3.6).

To prove Theorem 3.1 we procede by time discretization, and solve a minimal problem at every dicrete time. For all integer $n > 0$ we divide the interval $[0,T]$ in n equal subintervals of length $\tau := T/n$. We set $t_i^n := i\tau$,

$$u_0^n = u_0, \qquad u_{-1}^n := u_0 - \tau v_0, \qquad z_0^n := z_0,$$

and define $\mathcal{L}_i^n := \frac{1}{\tau}\int_{t_i^n}^{t_{i+1}^n}\mathcal{L}(s)ds$ for all $n > 0$. Then for $1 \le i \le n$ we recursively define $u_i^n \in H_D^1(\Omega_1\cup\Omega_2;\mathbb{R}^d))$ as a minimizer of

$$\begin{aligned}U_i^n(u) :=&\frac{\rho}{2}\|\frac{u-u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2 + \mathcal{Q}_0(e(u)) + \langle V([u]),z_{i-1}^n\rangle_\Gamma \\ &+\frac{\mu}{2}\langle\mathbb{C}^1e(u-u_{i-1}^n),e(u-u_{i-1}^n)\rangle - \langle\mathcal{L}_i^n,u\rangle,\end{aligned} \tag{3.9}$$

and $z_i^n \in \mathcal{Z}$ as the minimizer of

$$W_i^n(z) := \frac{\mu}{2\tau}\|z - z_{i-1}^n\|_{L^2}^2 + \langle V([u_i^n]),z\rangle_\Gamma - \langle\alpha,z\rangle_\Gamma. \tag{3.10}$$

Computing variations in the variable u at the minimum u_i^n of (3.9) we get

$$\begin{aligned}&\frac{\rho}{\tau}\langle\frac{u-u_{i-1}^n}{\tau} - \frac{u_{i-1}^n-u_{i-2}^n}{\tau},\varphi\rangle + \langle\mathbb{C}^0e(u_i^n),e(\varphi)\rangle \\ &+\frac{\mu}{\tau}\langle\mathbb{C}^1(e(u_i^n)-e(u_{i-1}^n)),e(\varphi)\rangle = \langle\mathcal{L}_i^n,\varphi\rangle - \langle\nabla V([u_i^n])\cdot[\varphi],z_{i-1}^n\rangle_\Gamma,\end{aligned} \tag{3.11}$$

for every $\varphi \in H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d)$. Instead taking variations η of the minimum z_i^n of (3.10), and taking into account that z_i must be non-negative, we get

$$\langle V([u_i^n]),\eta\rangle_{\Gamma\cap\{z_i>0\}} + \frac{\mu}{\tau}\langle z_i^n - z_{i-1}^n,\eta\rangle_\Gamma - \langle\alpha,\eta\rangle_\Gamma \ge 0, \tag{3.12}$$

for every $\eta \le 0$.

Moreover, if the variation $\eta \le 0$ is such that $z_i \pm \epsilon\eta \in [0, z_{i-1}]$ for some $\epsilon > 0$, then we will have equality. Denoting by $\mathcal{V}(z_i)$ the set of such variations, we have

$$\langle V([u_i^n]),\eta)\rangle_\Gamma + \frac{\mu}{\tau}\langle z_i^n - z_{i-1}^n,\eta\rangle_\Gamma - \langle\alpha,\eta\rangle_\Gamma = 0, \tag{3.13}$$

for all $\eta \in \mathcal{V}(z_i)$. Now we set $v_i^n := \frac{u_i^n - u_{i-1}^n}{\tau}$ and define the following piecewise linear (or constant) functions

$$
\begin{aligned}
&u_\tau(t) := u_i^n + (t - t_i^n)\frac{u_{i+1}^n - u_i^n}{\tau} && \text{for } t \in [t_i^n, t_{i+1}^n),\\
&z_\tau(t) := z_i^n + (t - t_i^n)\frac{z_{i+1}^n - z_i^n}{\tau} && \text{for } t \in [t_i^n, t_{i+1}^n),\\
&v_\tau(t) := v_i^n + (t - t_i^n)\frac{v_{i+1}^n - v_i^n}{\tau} && \text{for } t \in [t_i^n, t_{i+1}^n),\\
&\mathcal{L}_\tau(t) := \mathcal{L}_i^n && \text{for } t \in [t_i^n, t_{i+1}^n),
\end{aligned}
\tag{3.14}
$$

for $i = 0, \dots, n-1$. The fact that

$$
\mathcal{L}_\tau \to \mathcal{L} \quad \text{strongly in } L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.15}
$$

is standard and will often be tacitely used in the sequel. The following statement holds.

Proposition 3.3. *There are a function $u \in H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ and a function $z \in L^\infty([0,T], L^2(\Gamma)) \cap \mathcal{Z}$ such that*

$$
u_\tau \rightharpoonup u \quad \textit{weakly in } H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.16a}
$$

$$
u_\tau(t) \rightharpoonup u(t) \quad \textit{weakly in } H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d), \textit{ for every } t \in [0,T], \tag{3.16b}
$$

$$
z_\tau \rightharpoonup z \quad \textit{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{3.16c}
$$

$$
z_\tau(t) \rightharpoonup z(t) \quad \textit{weakly* in } L^\infty(\Gamma) \textit{ for every } t \in [0,T],, \tag{3.16d}
$$

as $\tau \to 0$. Moreover $\dot{u} \in H^1([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$, $z \in H^1([0,T], L^2(\Gamma))$, and

$$
v_\tau \rightharpoonup \dot{u} \quad \textit{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.16e}
$$

$$
\dot{v}_\tau \rightharpoonup \ddot{u} \quad \textit{weakly in } L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.16f}
$$

$$
\dot{z}_\tau \rightharpoonup \dot{z} \quad \textit{weakly in } L^2([0,T], L^2(\Gamma)). \tag{3.16g}
$$

Proof. We choose $\varphi = u_i^n - u_{i-1}^n$ and $\eta = z_i^n - z_{i-1}^n$ in (3.11) and sum it with (3.13), we get

$$
\begin{aligned}
&\frac{\rho}{2}\|\frac{u_i^n - u_{i-1}^n}{\tau}\|_{L^2}^2 - \frac{\rho}{2}\|\frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2 + \frac{\rho}{2}\|\frac{u_i^n - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2\\
&+ \mathcal{Q}_0(e(u_i^n)) - \mathcal{Q}_0(e(u_{i-1}^n)) + \frac{1}{2}\langle \mathbb{C}^0(e(u_i^n) - e(u_{i-1}^n)), e(u_i^n) - e(u_{i-1}^n)\rangle\\
&+ \frac{\mu}{\tau}\langle \mathbb{C}^1(e(u_i^n) - e(u_{i-1}^n)), e(u_i^n) - e(u_{i-1}^n)\rangle - \langle \mathcal{L}_i^n, u_i^n - u_{i-1}^n\rangle\\
&- \langle \alpha, (z_i^n - z_{i-1}^n)\rangle_\Gamma + \langle \nabla V([u_i^n]) \cdot [u_i^n - u_{i-1}^n], z_{i-1}^n\rangle_\Gamma\\
&+ \langle V([u_i^n]), (z_i^n - z_{i-1}^n)\rangle_\Gamma + \frac{\mu}{\tau}\|z_i^n - z_{i-1}^n\|_{L^2}^2 \leq 0.
\end{aligned}
\tag{3.17}
$$

Using the notations introduced in (3.14) and keeping into account that

$$
\begin{aligned}
&\langle \nabla V([u_i^n]) \cdot [u_i^n - u_{i-1}^n], z_{i-1}^n\rangle_\Gamma + \langle V([u_i^n]), (z_i^n - z_{i-1}^n)\rangle_\Gamma =\\
&= \langle V([u_i^n]), z_i^n\rangle_\Gamma - \langle V([u_{i-1}^n]), z_{i-1}^n\rangle_\Gamma\\
&- \langle \int_{t_{i-1}}^{t_i} \nabla V([u_\tau]) \cdot [\dot{u}_\tau] - \nabla V([u_i^n]) \cdot [\dot{u}_\tau] dt, z_{i-1}^n\rangle_\Gamma,
\end{aligned}
\tag{3.18}
$$

we can rewrite (3.17) as follows

$$\frac{\rho}{2}\|v_\tau(t_i^n)\|_{L^2}^2 - \frac{\rho}{2}\|v_\tau(t_{i-1}^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_{t_{i-1}^n}^{t_i^n} \|\dot v_\tau\|_{L^2}^2 dt + \mathcal{Q}_0(e(u_\tau(t_i^n)))$$
$$- \mathcal{Q}_0(e(u_\tau(t_{i-1}^n))) + \tau\int_{t_{i-1}^n}^{t_i^n} \mathcal{Q}_0(e(\dot u_\tau))dt + \mu\int_{t_{i-1}^n}^{t_i^n} \mathcal{Q}_1(e(\dot u_\tau))dt + \mu\int_{t_{i-1}^n}^{t_i^n} \|\dot z_\tau\|_{L^2}^2 dt$$
$$- \int_{t_{i-1}^n}^{t_i^n} \langle \alpha, \dot z_\tau\rangle dt + \langle V([u_\tau(t_i^n)]), z_\tau(t_i^n)\rangle - \langle V([u_\tau(t_{i-1}^n)]), z_\tau(t_{i-1}^n)\rangle$$
$$\le \int_{t_{i-1}^n}^{t_i^n} \langle \mathcal{L}_\tau, \dot u_\tau\rangle dt + \int_{t_{i-1}^n}^{t_i^n} \langle \nabla V([u_\tau]) \cdot [\dot u_\tau] - \nabla V([u_i^n]) \cdot [\dot u_\tau], z_{i-1}^n\rangle_\Gamma dt. \tag{3.19}$$

Using the Lipschitzianity of ∇V, the continuity of the trace operator (2.3), and the Korn inequality (2.4) we write

$$|\int_{t_{i-1}^n}^{t_i^n} \langle \nabla V([u_\tau]) \cdot [\dot u_\tau] - \nabla V([u_i^n]) \cdot [\dot u_\tau], z_{i-1}^n\rangle_\Gamma dt| \le \tau k L\int_{t_{i-1}^n}^{t_i^n} \|[\dot u_\tau]\|_2^2 dt$$
$$\le \tau L\gamma^2\int_{t_{i-1}^n}^{t_i^n} \|\dot u_\tau\|_{H^1}^2 dt \le \tau L\gamma^2\beta^2\int_{t_{i-1}^n}^{t_i^n} \|e(\dot u_\tau)\|_{L^2}^2 dt. \tag{3.20}$$

Summing over $i = 1, \ldots, j$ expression (3.19) and then using (2.5), one gets

$$\frac{\rho}{2}\|v_\tau(t_j^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_0^{t_j^n} \|\dot v_\tau\|_{L^2}^2 dt + \frac{\alpha_0}{2}\|e(u_\tau(t_j^n))\|_{L^2}^2$$
$$+ \frac{\alpha_0\tau}{2}\int_0^{t_j^n} \|e(\dot u_\tau)\|_{L^2}^2 dt + \alpha_1\mu\int_0^{t_j^n} \|e(\dot u_\tau)\|_{L^2}^2 dt + \mu\int_0^{t_j^n} \|\dot z_\tau\|_{L^2}^2 dt$$
$$- \int_0^{t_j^n} \langle \alpha, \dot z_\tau\rangle dt + \langle V([u_\tau(t_j^n)]), z_\tau(t_j^n)\rangle$$
$$\le \int_0^{t_j^n} \langle \mathcal{L}_\tau, \dot u_\tau\rangle dt + \tau L\gamma^2\beta^2\int_0^{t_j^n} \|e(\dot u_\tau)\|_{L^2}^2 dt + C, \tag{3.21}$$

for a constant $C > 0$ depending on u_0, v_0, z_0, μ, ρ, but independent of τ. Now we write

$$\int_0^{t_j^n} \langle \mathcal{L}_\tau, \dot u_\tau\rangle dt \le \frac{\lambda^{-1}}{2}\int_0^{t_j^n} \|\mathcal{L}_\tau\|_{H_D^{-1}}^2 dt + \frac{\lambda}{2}\int_0^{t_j^n} \|\dot u_\tau\|_{H^1}^2 dt$$
$$\le \frac{\lambda\beta^2}{2}\int_0^{t_j^n} \|e(\dot u_\tau)\|_{L^2}^2 dt + C, \tag{3.22}$$

where we have used the Korn inequality (2.4), $C > 0$ is a constant depending on the squared norm of $\mathcal{L} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ and on a fixed arbitrary positive number λ, but independent of τ. Then (3.21) implies

$$\frac{\rho}{2}\|v_\tau(t_j^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_0^{t_j^n} \|\dot v_\tau\|_{L^2}^2 dt + \frac{\alpha_0}{2}\|e(u_\tau(t_j^n))\|_{L^2}^2 + \mu\int_0^{t_j^n} \|\dot z_\tau\|_{L^2}^2 dt$$
$$+ (\alpha_1\mu - \delta)\int_0^{t_j^n} \|e(\dot u_\tau)\|_{L^2}^2 dt - \int_0^{t_j^n} \langle \alpha, \dot z_\tau\rangle dt + \langle V([u_\tau(t_j^n)]), z_\tau(t_j^n)\rangle \le C, \tag{3.23}$$

where $\delta := \frac{\lambda\beta^2}{4} + \tau L\gamma^2\beta^2$ and C is a positive. Since for λ sufficiently small and τ small enough all the terms in the left hand side are positive, we entail that all such terms are bounded. In particular there is a constant $L > 0$ such that

$$\|e_\tau(t)\|_{L^2}^2 \le L, \tag{3.24}$$

for all $t \in [0,T]$, n and $\tau = \tau(n)$. So that there are an increasing sequence n_k and a function $e \in L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ such that

$$e(u_{\tau(n_k)}) \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{3.25a}$$

as $k \to \infty$. We will write $\tau \to 0$ for $k \to \infty$. Using the Korn inequality, from (3.24) we get for all $t \in [0,T]$

$$\|u_\tau(t)\|_{H^1} \leq C, \tag{3.25b}$$

for some constant $C > 0$. This implies that, up to a subsequence, there is a function $u \in L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ such that

$$u_\tau \rightharpoonup u \quad \text{weakly* in } L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.25c}$$

as $\tau \to 0$. (3.25c) also implies that $e(u(t)) = e(t)$ for a.e. $t \in [0,T]$. Moreover (3.23) gives, up to passing to another subsequece,

$$e(\dot u_\tau) \rightharpoonup l \quad \text{weakly in } L^2([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{3.25d}$$

$$v_\tau \rightharpoonup v \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.25e}$$

$$z_\tau \rightharpoonup \hat z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{3.25f}$$

$$\dot z_\tau \rightharpoonup h \quad \text{weakly in } L^2([0,T], L^2(\Gamma)), \tag{3.25g}$$

as $\tau \to 0$, for functions $l \in L^2([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^{d\times d}_{\mathrm{sym}}))$, $v \in L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$, $\hat z \in L^\infty([0,T], \mathcal{Z})$, and $h \in L^2([0,T], L^2(\Gamma))$. Moreover z_τ are all functions with bounded variation on $[0,T]$, and their variations are bounded by the same constant. A generalization of Helly Theorem (see Lemma 7.2 of [7]) then implies that

$$z_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma), \tag{3.25h}$$

for all $t \in [0,T]$ as $\tau \to 0$, for a function $z \in L^2([0,T], \mathcal{Z})$.

Writing $z_\tau(t) = z_0 + \int_0^t \dot z_\tau(s)ds$ and multiplying by a test function in $L^2(\Gamma)$ we see that $h(t) = \dot z(t)$. Multiplying z_τ by a test function in $L^1([0,T], L^2(\Gamma))$ it is easily seen that it must be $\hat z = z$. A similar argument shows that $l(t) = e(\dot u(t))$ for a.e. $t \in [0,T]$. The Korn inequality and (3.25d) implies that there is a function $\hat u \in L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ such that

$$\dot u_\tau \rightharpoonup \hat u \quad \text{weakly in } L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.25i}$$

and writing $u_\tau(t) = u_0 + \int_0^t \dot u_\tau(s)ds$, arguing as before, we entail that u in (3.25c) belongs to $L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$, that $\hat u = \dot u$, and also that

$$u_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d), \tag{3.26}$$

for all $t \in [0,T]$.

From (3.11) it follows

$$\rho \dot v_\tau(t) = -\mathrm{div}_D(\mathbb{C}^0 e_\tau(t_i^n) + \mu \mathbb{C}^1 \dot e_\tau(t)) + \mathcal{L}_i^n - T(u_\tau(t_i^n), z_\tau(t_i^n)), \tag{3.27}$$

for all $t \in [t_i^n, t_{i+1}^n]$ and all i. From the continuity of the operators div_D and T, and from the convergences (3.25) we see that the right-hand side of the last expression is uniformly bounded in $L^2([0,T], H^{-1}_D(\Omega; \mathbb{R}^d))$, so that the same is true for $\dot v_\tau$ and, up to subsequences, there exists $\hat v \in L^2([0,T], H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ such that

$$\dot v_\tau \rightharpoonup \hat v \quad \text{weakly in } L^2([0,T], H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)). \tag{3.28}$$

Now, $v_\tau(t) - \dot u_\tau(t) = (\tau - (t - t_i^n))\dot v_\tau(t)$ when $t \in [t_i^n, t_{i+1}^n]$, for all i, so that $\int_0^T \|v_\tau - \dot u_\tau\|^2_{H^{-1}_D} ds = \frac{\tau^2}{3} \int_0^T \|\dot v_\tau\|^2_{H^{-1}_D} ds$, which, for the boundedness of $\dot v_\tau$, tends to zero. In particular, by (3.25i), since $\hat{u} = \dot{u}$, we find out that $\dot{u}(t) = v(t)$ for a.e. $t \in [0,T]$ and

$$v_\tau, \dot{u}_\tau \rightharpoonup \dot{u} \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^d)). \tag{3.29}$$

Finally we write $v_\tau(t) = v_0 + \int_0^t \dot{v}_\tau(s)ds$ and multiplying by a test function in $L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ we get $\dot{u}(t) = v_0 + \int_0^t \hat{v}(s)ds$, and then we get that $\dot{u}$ is differentiable in time and $\ddot{u} = \hat{v} \in L^2([0,T], H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$. This concludes the proof. □

Corollary 3.4. *For the same subsequence of Theorem 3.1, it holds*

$$[u_\tau] \rightharpoonup [u] \quad \text{weakly* in } L^\infty([0,T], H^{\frac{1}{2}}(\Gamma)), \tag{3.30a}$$

$$[u_\tau(t)] \rightharpoonup [u(t)] \quad \text{weakly in } H^{\frac{1}{2}}(\Gamma), \text{ for every } t \in [0,T], \tag{3.30b}$$

$$[u_\tau(t)] \rightharpoonup [u(t)] \quad \text{strongly in } L^q(\Gamma), \text{ for every } t \in [0,T], \tag{3.30c}$$

for every $1 \le q < q^*$ *with* $\frac{1}{q^*} = \frac{d-2}{2(d-1)}$ *if* $d > 2$, $q^* = +\infty$ *otherwise.*

Proof. (3.30a) and (3.30b) are straightforward consequence of (3.16a), (3.16b), and continuity of the trace operator. (3.30c) follows instead from (3.16b) and the fact that the embedding $H^{\frac{1}{2}} \hookrightarrow L^q$ is compact for all $q < q^*$. □

Let us introduce the piecewise constant functions

$$\begin{aligned} \tilde{u}_\tau &= u_\tau(t_i^n) && \text{for } t \in [t_i^n, t_{i+1}^n), \\ \tilde{z}_\tau &= z_\tau(t_i^n) && \text{for } t \in [t_i^n, t_{i+1}^n), \end{aligned} \tag{3.31}$$

for all $i \le (n-1)$. It is easy to show that convergences (3.16a), (3.16b), and (3.16d) holds true also for $\tilde{u}_\tau$ and $\tilde{z}_\tau$ in place of u_τ and z_τ. Now we are ready to prove the first Euler condition.

Proposition 3.5. *Let u and z be the functions obtained in Proposition 3.3. Then it holds*

$$\langle \rho \ddot{u}, \varphi \rangle + \langle \mathbb{C}^0 e(u) + \mu \mathbb{C}^1 e(\dot{u}), e(\varphi) \rangle - \langle \mathcal{L}, \varphi \rangle + \langle \nabla V([u]) \cdot [\varphi], z \rangle_\Gamma = 0, \tag{3.32}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ *and for a.e.* $t \in [0,T]$.

Proof. We start from (3.11), that with the notation introduced above reads

$$\rho \langle \dot{v}_\tau, \varphi \rangle + \langle \mathbb{C}^0 e(\tilde{u}_\tau) + \mu \mathbb{C}^1 e(\dot{u}_\tau), e(\varphi) \rangle - \langle \mathcal{L}_\tau, \varphi \rangle + \langle \nabla V([\tilde{u}_\tau]) \cdot [\varphi], \tilde{z}_\tau \rangle_\Gamma = 0. \tag{3.33}$$

For $\psi \in C_c^\infty((0,T))$ we write

$$\begin{aligned} &\int_0^T \big(\langle \mathbb{C}^0 e(\tilde{u}_\tau) + \mu \mathbb{C}^1 e(\dot{u}_\tau), e(\varphi) \rangle - \langle \mathcal{L}_\tau, \varphi \rangle + \langle \nabla V([\tilde{u}_\tau]) \cdot [\varphi], \tilde{z}_\tau \rangle_\Gamma \big) \psi dt \\ &= -\int_0^T \rho \langle v_\tau, \varphi \rangle \dot{\psi} dt, \end{aligned} \tag{3.34}$$

and letting $\tau \to 0$, thanks to (3.25) we get

$$\begin{aligned} &\int_0^T \big(\langle \mathbb{C}^0 e(u) + \mu \mathbb{C}^1 e(\dot{u}), e(\varphi) \rangle - \langle \mathcal{L}, \varphi \rangle + \langle \nabla V([u]) \cdot [\varphi], z \rangle_\Gamma \big) \psi dt \\ &= -\int_0^T \rho \langle \dot{u}, \varphi \rangle \dot{\psi} dt. \end{aligned} \tag{3.35}$$

Arbitraryness of ψ then implies (3.32). □

In order to prove the next Lemma we need to recall the Fréchet-Kolmogorov Theorem. For all $h \in \mathbb{R}^d$ we introduce the h-translation in $\mathbb{R}^d$, that is the function $s_h : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ defined by $s_h(f)(x) := f(x+h)$ for all $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d)$. Then the following Theorem holds true.

Theorem 3.6 (Fréchet-Kolmogorov)**.** *Let B be a subset of $L^1(\mathbb{R}^d)$ such that for all $f \in B$ it holds $f = 0$ out of a bounded set $U \subset \mathbb{R}^d$. Then B is a relatively compact set in $L^1(\mathbb{R}^d)$ if and only if there exists a continuous non-negative function $\omega : \mathbb{R}^d \to \mathbb{R}$ such that $\omega(0) = 0$ and $\|f - s_h(f)\|_1 \leq \omega(h)$, for all $f \in B$ and for all $h \in \mathbb{R}^d$.*

See, e.g., [5] for a proof.

Lemma 3.7. *For all $q \geq 1$ and $t \in [0,T]$ we have*

$$z_\tau(t) \to z(t) \quad \textit{strongly in } L^q(\Gamma). \tag{3.36}$$

Proof. Since

$$z_i = \operatorname{argmin}_{\,0 \leq z \leq z_{i-1}} \langle V([u_i]) - \alpha, z\rangle_\Gamma + \frac{\mu}{2\tau}\|z - z_{i-1}\|^2_{L^2(\Gamma)},$$

we see that the value of $z_i(x)$ in $x \in \Gamma$ is exactly the minimizer in $[0, z_{i-1}(x)]$ of

$$\langle V([u_i(x)]) - \alpha(x), z\rangle_\Gamma + \frac{\mu}{2\tau}|z - z_{i-1}(x)|^2, \tag{3.37}$$

so that, denoting $a(x) := V([u_i(x)]) - \alpha(x)$, we can explicitely compute the value of $z_i(x)$. If $\hat{z}(x) := -\frac{\tau}{\mu}a(x) + z_{i-1}(x)$ is the minimizer of (3.37) on $\mathbb{R}$, then we have, omitting the symbol x,

$$\begin{cases} \hat{z} > z_{i-1} & \Leftrightarrow & a < 0 & \Rightarrow & z_i = z_{i-1}, \\ 0 \leq \hat{z} \leq z_{i-1} & \Leftrightarrow & 0 \leq a < \frac{\mu}{\tau} z_{i-1} & \Rightarrow & z_i = -\frac{\tau}{\mu}a + z_{i-1}, \\ \hat{z} < 0 & \Leftrightarrow & a > \frac{\mu}{\tau}z_{i-1} & \Rightarrow & z_i = 0, \end{cases} \tag{3.38}$$

from which it follows

$$\mu \dot{z}_\tau = -(a \wedge \frac{\mu}{\tau} z_{i-1})^+, \quad \text{and } z_i = z_{i-1} - (\frac{\tau}{\mu}a \wedge z_{i-1})^+. \tag{3.39}$$

From (3.30c) and the definition of V we see that $V([u_\tau])(t)$ is a converging sequence in $L^1(\Gamma, \mathbb{R}^d)$. So that from Theorem 3.6 we get a function $\omega : \Gamma \cong \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\omega(0) = 0$ and

$$\|[u_\tau]^2(t) - s_h([u_\tau]^2(t))\|_1 \leq \omega(h), \tag{3.40}$$

for all $h \in \mathbb{R}^{d-1}$ and for all τ and $t \in [0,T]$. Without lose of generality we can also suppose that $\|a - s_h(a)\|_1 \leq \omega(h)$, since $\alpha \in L^\infty(\Gamma)$.

For fixed τ, let us prove by induction on i that $\|z_i - s_h(z_i)\|_1 \leq \frac{i\tau}{\mu}\omega(h)$. Indeed, using the expression of z_i obtained above, we have

$$\begin{aligned} \|z_i - s_h(z_i)\|_{L^1} &= \|z_{i-1} - (\frac{\tau}{\mu}a \wedge z_{i-1})^+ - \big(s_h(z_{i-1}) - (\frac{\tau}{\mu}s_h(a) \wedge s_h(z_{i-1}))^+\big)\|_{L^1} \\ &\leq \ \|z_{i-1} - s_h(z_{i-1})\|_{L^1} + \|\frac{\tau}{\mu}a - \frac{\tau}{\mu}s_h(a)\|_{L^1} \\ &\leq \ \frac{(i-1)\tau}{\mu}\omega(h) + \frac{\tau}{\mu}\omega(h) = \frac{i\tau}{\mu}\omega(h), \end{aligned} \tag{3.41}$$

where the first inequality follows by the fact that the function $(x,y) \mapsto x - (x \wedge y)^+$ is 1-Lipschitz in both the two real variables, and the second inequality follows by the inductive hypothesis. Now, recalling that $\tau = \frac{T}{N}$, (3.41) implies that for all τ

and $t \in [0,T]$ it holds $\|z_\tau(t) - s_h(z_\tau(t))\|_1 \leq \frac{T}{\mu}\omega(h)$. Since $z_\tau(t) \in [1,0]$, we have $|z_\tau(t) - s_h(z_\tau(t))| \leq 1$, and then also

$$\|z_\tau(t) - s_h(z_\tau(t))\|_q^q \leq \frac{T}{\mu}\omega(h). \tag{3.42}$$

Using (3.25h), the last formula implies (3.36). □

We are now ready to prove the conditions governing the flow rule.

Proposition 3.8. *Let $u \in L^\infty([0,T], H^1(\Omega))$ and $z \in L^2([0,T], L^\infty(\Gamma))$ be the functions defined in* (3.25c) *and* (3.25h). *Then for a.e. $t \in [0,T]$ it holds*

$$\langle V([u(t)]), \dot z(t)\rangle_\Gamma + \mu\|\dot z(t)\|_{L^2}^2 - \langle \alpha, \dot z(t)\rangle_\Gamma = 0, \tag{3.43}$$

and

$$\langle V([u(t)]), \eta\rangle_{\{z(t)>0\}} + \mu\langle \dot z(t), \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \geq 0, \tag{3.44}$$

for all $\eta \in L^\infty(\Gamma)$, $\eta \leq 0$.

Proof. Let us fix $t \in [0,T]$, and for all τ we decompose the interface Γ as the union of the three sets $\Gamma = A_\tau^t \cup B_\tau^t \cup C_\tau^t$ where, if $t \in [t_i - 1, t_i)$, then $A_\tau^t := \{z_i = 0 < z_{i-1}\}$, $B_\tau^t := \{z_i = z_{i-1}\}$, $C_\tau^t := \{0 < z_i < z_{i-1}\}$. We recognize these three cases as the three options of (3.38), so that it is readily seen that

$$V([u_\tau])\dot z_\tau + \mu|\dot z_\tau|^2 - \alpha \dot z_\tau = 0, \tag{3.45}$$

on B_τ^t and C_τ^t, while on A_τ^t

$$V([u_\tau]) + \mu\dot z_\tau - \alpha \geq 0. \tag{3.46}$$

The latter being true for all $t \in [0,T]$. In particular, for every positive smooth function φ on $[0,T]$, recalling that $\dot z_\tau \leq 0$, we have

$$\int_0^T \big(\langle V([u_\tau]), \dot z_\tau\rangle_\Gamma + \mu\|\dot z_\tau\|_2^2 - \langle \alpha, \dot z_\tau\rangle_\Gamma\big)\varphi dt \leq 0. \tag{3.47}$$

We would like to pass to the limit in (3.47). To this aim, we first observe that from (3.30c) and the definition of V we see that actually $V([u_\tau])(t)$ is converging in $L^2(\Gamma, \mathbb{R})$. Thus we have $V([u_\tau]) \to V([u])$ strongly in $L^2([0,T], L^2(\Gamma))$. This together with (3.16g) and the Fatou lemma implies

$$\int_0^T \big(\langle V([u]), \dot z\rangle_\Gamma + \mu\|\dot z\|_2^2 - \langle \alpha, \dot z\rangle_\Gamma\big)\varphi dt \leq 0. \tag{3.48}$$

Now formula (3.12) provides

$$\int_0^T \big(\langle V([\tilde u_\tau]), \eta\rangle_{\Gamma\cap\{\tilde z_\tau>0\}} + \mu\langle \dot z_\tau, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma\big)\varphi dt \geq 0. \tag{3.49}$$

for all $\eta \leq 0$. We note that, by definitions of z_τ and $\tilde z_\tau$ it holds $\chi_{\{\tilde z_\tau>0\}} = \chi_{\{z_\tau>0\}}$. From Lemma 3.7 we know that $z_\tau \to z$ strongly in $L^1(\Gamma \times [0,T])$, so that we can suppose it converges almost everywhere in $\Gamma \times [0,T]$. As a consequence we entail

$$\limsup \chi_{\{z_\tau>0\}} \geq \chi_{\{z>0\}}.$$

Then, from (3.49), taking into account that $\eta \leq 0$ and that $V([u_\tau]) \to V([u])$ strongly in $L^1(\Gamma \times [0,T])$, we obtain

$$\int_0^T \big(\langle V([u]), \eta\rangle_{\{z>0\}} + \mu\langle \dot z, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma\big)\varphi dt \geq 0, \tag{3.50}$$

for every smooth nonnegative function φ on $[0,T]$, and for all $\eta \leq 0$. From arbitrariness of φ we get (3.44). Now, plugging $\eta = \dot z$ we recover the opposite inequality of (3.48) provided $\dot z = 0$ almost everywhere on the set $\{z = 0\}$. But this is a straightforward consequence of the fact that z is non-negative, then (3.43) follows and the Proposition is proved.

□

Proof of Theorem 3.1. Let us prove that conditions (3.32), (3.43), and (3.44) implies equations (3.2) and (3.3). Equation (3.2b) holds by definition, while (3.2a) is expressed by (2.14), that is exactly (3.32). From arbitrariness of η equation (3.44) readily implies

$$V([u(t)]) + \mu\dot{z}(t) - \alpha \leq 0 \quad \text{ a.e. on } \Gamma \cap \{z(t) > 0\},$$

that is (3.4), while (3.43) implies (3.3a) and (3.3b), keeping into account that z is nonnegative and nonincreasing. To prove (3.5), we use (3.16b), (3.16d), and the fact that $u_\tau(0) = u_0$ and $z_\tau(0) = z_0$ for all τ. It remains to show that $\dot{u}(0) = v_0$. We first note that (3.16e) and (3.16f) imply that

$$v_\tau \rightharpoonup \dot{u} \quad \text{weakly in } H^1([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)),$$

so that we entail $v_\tau(t) \rightharpoonup \dot{u}(t)$ weakly in $H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ for all $t \in [0,T]$. Thesis follows since by definition $v_\tau(0) = v_0$ for all τ. □

When we deal with nonhomogeneous boundary datum the existence theorem is stated as follows:

Theorem 3.9. *Let $\mathcal{L} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$, u_0,$v_0 \in H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$, $z_0 \in \mathcal{Z}$, and let $w \in H^1([0,T], H^1_D(\Omega, \mathbb{R}^d))$ with $\dot{w} \in H^1([0,T], H_D^{-1}(\Omega, \mathbb{R}^d))$ be such that $w(0) = u_0$ and $\dot{w}(0) = v_0$ on $\partial_D\Omega$. Then there exists a triple (u, σ, z) with*

$$u \in L^\infty([0,T], H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.51a}$$

$$\dot{u} \in L^2([0,T], H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)) \cap L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.51b}$$

$$\ddot{u} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.51c}$$

$$\sigma \in L^2([0,T], L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{sym}), \tag{3.51d}$$

$$z \in H^1([0,T], L^2(\Gamma)) \cap L^\infty([0,T], \mathcal{Z}), \tag{3.51e}$$

satisfying

$$\rho\ddot{u}(t) - \mathrm{div}_D\sigma(t) = \mathcal{L}(t) + T(u,z), \tag{3.52a}$$

$$\sigma(t) = \mathbb{C}^0 e(u)(t) + \mu\mathbb{C}^1 e(\dot{u})(t), \tag{3.52b}$$

on Ω for a.e. $t \in [0,T]$, the Dirichlet condition

$$u(t) = w(t) \text{ on } \partial_D\Omega, \tag{3.52c}$$

for a.e. $t \in [0,T]$, the relations

$$\dot{z}(t) \leq 0, \tag{3.53a}$$

$$\dot{z}(t)(V([u(t)]) + \mu\dot{z}(t) - \alpha) = 0, \tag{3.53b}$$

on Γ,

$$V([u(t)]) + \mu\dot{z}(t) - \alpha \leq 0, \tag{3.54}$$

on $\Gamma \cap \{z(t) > 0\}$, for a.e. $t \in [0,T]$, and the initial data

$$u(0) = u_0, \qquad \dot{u}(0) = v_0, \qquad z(0) = z_0. \tag{3.55}$$

The proof is essentially the same of Theorem 3.1, that can be easyly arranged.

Proof. We set $w_{-1}^n := w(0) - \tau\dot{w}(0)$, $w_i^n = w(t_i^n)$, $\omega_i^n := \frac{w_i^n - w_{i-1}^n}{\tau}$ for $i = 0, \dots, n$, then we define the piecewise affine functions

$$w_\tau = w_i^n + (t - t_i^n)\frac{w_{i+1}^n - w_i^n}{\tau} \qquad \text{for } t \in [t_i^n, t_{i+1}^n), \tag{3.56a}$$

$$\omega_\tau = v_i^n + (t - t_i^n)\frac{\omega_{i+1}^n - \omega_i^n}{\tau} \qquad \text{for } t \in [t_i^n, t_{i+1}^n), \tag{3.56b}$$

for $i = 0, \dots, n-1$. The fact that

$$w_\tau \to w \quad \text{strongly in } H^1([0,T], H^1(\Omega, \mathbb{R}^d)), \tag{3.57a}$$

$$\omega_\tau \to \dot{w} \quad \text{strongly in } H^1([0,T], H_D^{-1}(\Omega, \mathbb{R}^d)), \tag{3.57b}$$

is standard and easily checked. We also define the piecewise affine function $l_\tau : [0,T] \to H_D^{-1}(\Omega, \mathbb{R}^d))$ by setting

$$l_\tau := \rho\dot{\omega}_\tau - \mathrm{div}_D(\mathbb{C}^0 e(w_\tau) + \mu\mathbb{C}^1 e(\dot{w}_\tau)), \tag{3.58}$$

so that property (2.5), the continuity of div_D, and (3.57) imply that

$$l_\tau \to l \quad \text{strongly in } L^2([0,T], H_D^{-1}(\Omega, \mathbb{R}^d)), \tag{3.59}$$

where $l := \rho\ddot{w} - \mathrm{div}_D(\mathbb{C}^0 e(w) + \mu\mathbb{C}^1 e(\dot{w}))$. Arguing as in the proof of Theorem 3.1 we solve the minimum problems (3.9) and (3.10) with $\mathcal{L}_i^n - l(t_i^n)$ in place of $\mathcal{L}_i^n$ and denote by u_i^n and z_i^n their minimizers. Standard arguments taking into account relation (3.59) ensure one that the same estimates (3.23) hold for the functions u'_τ, z_τ, v'_τ defined as in (3.14). So that we found functions $u' \in H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ with $\dot{u}' \in H^1([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ and $z \in L^\infty([0,T], L^2(\Gamma)) \cap H^1([0,T], \mathcal{Z})$ such that

$$u'_\tau \rightharpoonup u' \quad \text{weakly in } H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.60a}$$

$$u'_\tau(t) \rightharpoonup u'(t) \quad \text{weakly in } H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d), \text{ for every } t \in [0,T], \tag{3.60b}$$

$$z_\tau \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{3.60c}$$

$$z_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t \in [0,T],, \tag{3.60d}$$

$$v'_\tau \rightharpoonup \dot{u}' \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.60e}$$

$$\dot{v}'_\tau \rightharpoonup \ddot{u}' \quad \text{weakly in } L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{3.60f}$$

$$\dot{z}_\tau \rightharpoonup \dot{z} \quad \text{weakly in } L^2([0,T], L^2(\Gamma)). \tag{3.60g}$$

Moreover we also get (3.12), (3.13), while (3.11) is replaced by the following

$$\begin{aligned} &\rho\langle\dot{v}'_\tau, \varphi\rangle + \langle\mathbb{C}^0 e(\tilde{u}'_\tau) + \mu\mathbb{C}^1 e(\dot{u}'_\tau), e(\varphi)\rangle + \langle\nabla V([\tilde{u}'_\tau]) \cdot [\varphi], \tilde{z}_\tau\rangle_\Gamma \\ &= \langle\tilde{\mathcal{L}}_\tau - \tilde{l}_\tau, \varphi\rangle, \end{aligned} \tag{3.61}$$

for all $\varphi \in H_D^1$ and for a.e. $t \in [0,T]$. Arguing as in Proposition 3.5 we see that (3.61) passes to the limit as $\tau \to 0$ and leads one to

$$\begin{aligned} &\rho\langle\ddot{u}', \varphi\rangle + \langle\mathbb{C}^0 e(\tilde{u}'_\tau) + \mu\mathbb{C}^1 e(\dot{u}'_\tau), e(\varphi)\rangle + \langle\nabla V([u']) \cdot [\varphi], z\rangle_\Gamma \\ &= \langle\mathcal{L} - l, \varphi\rangle, \end{aligned} \tag{3.62}$$

for all $\varphi \in H_D^1$ and for a.e. $t \in [0,T]$. If we define $u := u' + w$, observing that, since $w \in H^1(\Omega, \mathbb{R}^d)$, $[w] = 0$ on Γ, then (3.62) reads

$$\rho\langle\ddot{u}, \varphi\rangle + \langle\mathbb{C}^0 e(u) + \mu\mathbb{C}^1 e(\dot{u}), e(\varphi)\rangle + \langle\nabla V([u]) \cdot [\varphi], z\rangle_\Gamma = \langle\mathcal{L}, \varphi\rangle. \tag{3.63}$$

At the same time (3.12) and (3.13) pass to the limit like in the case of homogeneous boundary datum, and give rise to the same equations (3.43) and (3.44). The conclusion easily follows. □

The following Proposition provides the energy balance of the system.

Proposition 3.10. *Let u be the solution of Theorem 3.9. Then for all $0 \leq t_1 < t_2 \leq T$, the following energy balance holds*

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t_2)-\dot{w}(t_2)\|_{L^2}^2+\mathcal{Q}_0(e(u(t_2)))+\langle V([u(t_2)]),z(t_2)\rangle_\Gamma+\mu\int_{t_1}^{t_2}\mathcal{Q}_1(e(\dot{u}))ds\\
&+\mu\int_{t_1}^{t_2}\|\dot{z}\|_{L^2}^2ds-\langle\alpha,z(t_2)\rangle_\Gamma=\frac{\rho}{2}\|\dot{u}(t_1)-\dot{w}(t_1)\|_{L^2}^2+\mathcal{Q}_0(e(u(t_1)))-\langle\alpha,z(t_1)\rangle_\Gamma\\
&+\langle V([u(t_1)]),z(t_1)\rangle_\Gamma+\int_{t_1}^{t_2}\langle\sigma,e(\dot{w})\rangle ds+\int_{t_1}^{t_2}\langle\mathcal{L}-\rho\ddot{w},\dot{u}-\dot{w}\rangle ds, \qquad (3.64)
\end{aligned}$$

where $\sigma=\mathbb{C}^0e(u)+\mu\mathbb{C}^1e(\dot{u})$.

Proof. We put $\varphi=\dot{u}-\dot{w}$ in (3.62) and sum this expression with (3.43). Integrating in time on $[t_1,t_2]$ we get (3.64). $\square$

3.1. **Processes in Mode II.** In order to prove the existence of solution of the problem in Theorem 3.9 which also satisfy constrains as in (2.20), we use a standard argument dealing with a penalization term.

Let $D\subset\mathbb{R}^d$ be the convex and closed cone defined in (2.19b). Let $\Phi:\mathbb{R}\to\mathbb{R}$ be a smooth nonnegative and convex map such that

(i) $\Phi(0)=0$ and $\Phi(x)>0$ if $x\neq 0$.
(ii) The derivative Φ' of Φ is Lipschitz with constant $L>0$.
(iii) There exists $1\leq\delta<q^*$ and $C>0$ such that $|\Phi(x)|\leq C(|x|+1)^\delta$ for all $x\in\mathbb{R}$.

Here $q^*=+\infty$ for $d\leq 2$ and $q^*=\frac{2(d-1)}{d-2}$ for $d>2$. As for V, property (ii) has the following consequence

(iv) For all $x\in\mathbb{R}$ it holds $|\Phi'(x)|\leq L|x|$.

Now we define $\bar{V}:\mathbb{R}^d\times\Gamma\to\mathbb{R}$ the function $\bar{V}(y,x):=\Phi(\mathrm{dist}(y,D(x))$. We then define $\tilde{V}:L^1(\Gamma)\to L^1(\Gamma)$ as $\tilde{V}([u(x)]):=\bar{V}([u(x)],x)$ when $[u]\in L^1(\Gamma)$. Finally, for all positive integers $h>0$, we set $\tilde{V}_h:=h\tilde{V}$.

Let us remind the constraint conditions on the jump of $[u]$ that we want to satisfy. They read

$$[u(t)]\succeq 0, \qquad (3.65a)$$

$$\sigma(t)\nu+T(u(t),z(t))\succeq^* 0, \qquad (3.65b)$$

$$(\sigma(t)\nu+T(u(t),z(t)))\cdot[u(t)]=0. \qquad (3.65c)$$

for a.e. $t\in[0,T]$. Since $\sigma(t)$ is not in general an element of $L^1(\Gamma,\mathbb{R}^d)$, we prove a theorem where the solutions satisfy (3.65) in a weak form.

Theorem 3.11. *Let D be the cone in* (2.19b) *and let $\mathcal{L}$, u_0, v_0, z_0, and w be as in Theorem 3.9. Then there exists a couple (u,z) satisfying* (3.51), (3.52c), (3.55), *and such that, for a.e. $t\in[0,T]$, it satisfies conditions* (3.43), (3.44), *and*

$$u(t)\in D, \qquad (3.66a)$$

$$\langle\rho\ddot{u},\varphi\rangle+\langle\mu\mathbb{C}^1e(\dot{u})+\mathbb{C}^0e(u),e(\varphi)\rangle+\langle\nabla V([u])\cdot[\varphi],z\rangle_\Gamma=\langle\mathcal{L},\varphi\rangle, \qquad (3.66b)$$

for all $\varphi\in H^1_D$ with $[\varphi]\cdot\nu=0$.

We will give a sketch of the proof, being it very similar to the one of Theorem 3.1. Moreover, for simplicity, we will only treat the case with homogeneous boundary datum.

Proof. Let u_i^n be the minimun of the potential

$$U_i^n(u) := \frac{\rho}{2}\|\frac{u-u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2 + \mathcal{Q}(e(u)) + \langle V([u]), z_{i-1}^n\rangle_\Gamma \\ + \frac{\mu}{2}\langle \mathbb{C}e(u-u_{i-1}^n), e(u-u_{i-1}^n)\rangle - \langle \mathcal{L}_i^n, u\rangle + \|\tilde{V}_h([u]\cdot\nu)\|_{L^1(\Gamma)}, \quad (3.67)$$

and z_i^n the minimum of (3.10). The discrete Euler condition then is

$$\frac{\rho}{\tau}\langle\frac{u-u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}, \varphi\rangle + \langle \mathbb{C}^0 e(u_i^n), e(\varphi)\rangle + \langle \tilde{V}_h'([u_i^n]\cdot\nu), [\varphi]\cdot\nu\rangle_\Gamma \\ + \frac{\mu}{\tau}\langle \mathbb{C}^1(e(u_i^n) - e(u_{i-1}^n)), e(\varphi)\rangle - \langle \mathcal{L}_i^n, \varphi\rangle + \langle \nabla V([u_i^n])\cdot[\varphi], z_i^n\rangle_\Gamma = 0, \quad (3.68)$$

for all $\varphi \in H_D^1(\Omega_1\cup\Omega_2, \mathbb{R}^d)$. Arguing in the same way as in proof of Proposition 3.3 we obtain the same bounds and convergences (3.16) and the further information

$$\|\tilde{V}([u_\tau(t_j^n)]\cdot\nu)\|_{L^1(\Gamma)} \le C. \quad (3.69)$$

Passing to the limit as $\tau \to 0$ we obtain that the functions $u_h \in L^\infty([0,T], H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ and $z_h \in H^1([0,T], L^2(\Gamma))$ satisfies (3.43), (3.44), and, in place of (3.32),

$$\langle \rho\ddot{u}_h, \varphi\rangle + \langle \mathbb{C}^0 e(u_h) + \mu\mathbb{C}^1 e(\dot{u}_h), e(\varphi)\rangle + \langle \nabla V([u_h])\cdot[\varphi], z_h\rangle_\Gamma \\ = \langle \mathcal{L}, \varphi\rangle - \langle \tilde{V}_h'([u_h]\cdot\nu), [\varphi]\cdot\nu\rangle_\Gamma, \quad (3.70)$$

for all $\varphi \in H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ and for a.e. $t \in [0,T]$.

The same argument for Proposition 3.10 gives the following energy balance

$$\frac{\rho}{2}\|\dot{u}_h(t)\|_{L^2}^2 + \mathcal{Q}_0(e(u_h)(t)) + \langle V([u_h(t)]), z_h(t)\rangle_\Gamma + \mu\int_0^t \mathcal{Q}_1(e(\dot{u}_h))ds \\ + \mu\int_0^t \|\dot{z}_h\|_{L^2}^2 ds - \int_0^t \langle \alpha, \dot{z}_h\rangle_\Gamma + \|\tilde{V}_h([u_h(t)]\cdot\nu\|_{L^1(\Gamma)} \\ = \frac{\rho}{2}\|v_0\|_{L^2}^2 + \mathcal{Q}_0(e(u_0)) + \langle V([u_0]), z_0\rangle_\Gamma - \int_0^t \langle \mathcal{L}, \dot{u}_h\rangle ds, \quad (3.71)$$

for all $t \in [0,T]$ We write

$$\int_0^t \langle \mathcal{L}, \dot{u}_h\rangle ds \le \frac{1}{2\lambda}\int_0^t \|\mathcal{L}\|_{H_D^{-1}}^2 ds + \frac{\beta\lambda}{2}\int_0^t \|e(\dot{u}_h)\|_2^2 ds,$$

where $\lambda = \frac{\mu\alpha_1}{2\beta}$, so that, plugging this into the energy balance (3.71) and using (2.5) we obtain that there is a positive constant C independent of h such that

$$frac\rho 2\|\dot{u}_h(t)\|_{L^2}^2 + \frac{\alpha_0}{2}\|e(u_h)(t)\|_2^2 + \langle V([u_h(t)]), z(t)\rangle_\Gamma + \frac{\mu\alpha_1}{4}\int_0^t \|e(\dot{u}_h)\|_2^2 ds \\ + \mu\int_0^t \|\dot{z}_h\|_2^2 ds - \int_0^t \langle \alpha, \dot{z}_h\rangle_\Gamma + \|\tilde{V}_h([u_h(t)]\cdot\nu)\|_{L^1(\Gamma)} \le C. \quad (3.72)$$

Thanks to this apriori estimate we have that there exists $u \in H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d))$ and $z \in H^1([0,T], \mathcal{Z})$ such that, up to a subsequence,

$$u_h \rightharpoonup u \quad \text{weakly in } H^1([0,T], H_D^1(\Omega_1\cup\Omega_2, \mathbb{R}^d)), \quad (3.73a)$$

$$u_h(t) \rightharpoonup u(t) \quad \text{weakly in } H_D^1(\Omega_1\cup\Omega_2, \mathbb{R}^d), \text{ for every } t \in [0,T], \quad (3.73b)$$

$$z_h \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \quad (3.73c)$$

$$z_h(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t \in [0,T], \quad (3.73d)$$

$$\dot{z}_\tau \rightharpoonup \dot{z} \quad \text{weakly in } L^2([0,T], L^2(\Gamma)). \quad (3.73e)$$

as $h \to +\infty$. The proof of this fact is identical to the proof of Proposition 3.3. Moreover, since the Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^q(\Gamma)$ is compact for all $1 \leq q < q^*$, (3.73b) implies

$$[u_h(t)] \to [u(t)] \quad \text{strongly in } L^q(\Gamma), \text{ for every } t \in [0,T], \tag{3.73f}$$

for all $1 \leq q < q^*$ as $h \to +\infty$. By definition of $\tilde{V}_h$, one has $\|\tilde{V}_h([u_h(t)] \cdot \nu)\|_{L^1(\Gamma)} = h\|\tilde{V}([u_h(t)] \cdot \nu)\|_{L^1(\Gamma)}$, so that (3.72) implies

$$\tilde{V}([u_h(t)] \cdot \nu) \to 0 \quad \text{strongly in } L^1(\Gamma), \text{ for every } t \in [0,T], \tag{3.73g}$$

as $h \to +\infty$, and in particular we get that $\chi_{D^c}([u_h(t)] \cdot \nu)[u_h(t)] \cdot \nu \to 0$ almost everywhere on Γ for all $t \in [0,T]$. This implies the important condition

$$[u(t)] \in D. \tag{3.74}$$

Thanks to convergences (3.73) it is now easy to pass to the limit as $h \to +\infty$ in (3.43) and (3.44). Indeed, passing to the limit in the first one, we get the inequality (3.48), thanks to (3.73d) and (3.73f). To get (3.44) we argue as in the proof of Proposition 3.8, getting also equality in (3.48), and then (3.43). Instead (3.70) passes to the limit in the case that $[\varphi] \cdot \nu = 0$ providing condition (3.66b). This concludes the proof. $\square$

Corollary 3.12. *Let (u, z) be a solution of* (3.1)*,* (3.43)*, and* (3.66)*. Then the energy balance* (3.64) *holds.*

Proof. The proof is the same as Proposition 3.10, since $\dot{u}$ satisfies the constraint $[\dot{u}] \cdot \nu = 0$ and we can employ (3.66b) with $\varphi = \dot{u} - \dot{w}$. $\square$

4. Limit of solutions in rescaled time

In this section we study the asymptotic behaviour of dynamic evolutions when the rate of the external loads and the boundary conditions become slower and slower. This can be done throught a suitable rescaling of the data. If we start with an external load $\mathcal{L}$ and a datum w on $[0,T]$, we set $\mathcal{L}^\epsilon(t) := \mathcal{L}(t/\epsilon)$ and $w^\epsilon(t) := w(t/\epsilon)$ so that $\mathcal{L}^\epsilon$ and w^ϵ are defined on $[0, T/\epsilon]$. If (u^ϵ, z^ϵ) is the solution given by Theorem 3.9 with these data, we are interested into studying their behaviour as $\epsilon \to 0$. To handle with this, another rescaling is required. We define $(u_\epsilon(t), z_\epsilon(t)) := (u^\epsilon(\epsilon t), z^\epsilon(\epsilon t))$, in such a way that the functions (u_ϵ, z_ϵ) are now defined on the same interval $[0,T]$. A straightforward change of variables shows that (u_ϵ, z_ϵ) solves the same equations of (u, z), with a scalar ϵ appearing besides all the terms with one time derivative, and ϵ^2 appearing beside the second derivative. In other words, this rescaling provides that (u_ϵ, z_ϵ) are the solutions of the beginning delamination problem with a density mass equal to $\rho\epsilon^2$ and a viscosity parameter equal to $\mu\epsilon$. For simplicity in what follows we simply replace ρ by ϵ^2 and μ by ϵ.

Now we are ready to compute the analysis of (u_ϵ, z_ϵ) as the parameter ϵ vanishes. We will restrict to the dimension $d \leq 3$, and we will assume that the potential $V([u])$ has the form

$$V([u]) := \frac{1}{2}\mathbb{K}[u] \cdot [u],$$

where $\mathbb{K}$ is called the elastic coefficient of the adhesive, and is constant on Γ. We assume also that $\mathbb{K}$ is positive definite, that is $\langle \mathbb{K}[u] \cdot [u] \rangle_\Gamma$ is a an equivalent norm on $L^2(\Gamma, \mathbb{R}^d)$. Such hypothesis are classical in literature. Moreover we will need to assume more regularity on the data. In particular we suppose that $w \in H^2([0,T], H^1_D(\Omega, \mathbb{R}^d))$ and $\mathcal{L} \in H^1([0,T], H^{-1}_D(\Omega, \mathbb{R}^d))$.

We first state the Theorem in the case of homogeneous boundary datum.

Theorem 4.1. *Let $\mathcal{L} \in H^1([0,T], H_D^{-1}(\Omega, \mathbb{R}^d))$ and u_0, v_0, z_0 as in Theorem 3.1. Let (u_ϵ, z_ϵ) be a solution of the problem in Theorem 3.1, then there exist $u \in L^\infty([0,T], H_D^1(\Omega_1 \cup \Omega_2))$ and $z \in L^2([0,T], \mathcal{Z})$ such that, up to a subsequece,*

$$u_\epsilon \to u \text{ strongly in } L^2([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{4.1a}$$

$$z_\epsilon \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], \mathcal{Z}), \tag{4.1b}$$

$$z_\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{for all} t \in [0,T], \tag{4.1c}$$

as $\epsilon \to 0$. There also exist two nonnegative Borel measures $\mu_z \in \mathcal{M}([0,T] \times \Gamma)$ and $\mu_b \in \mathcal{M}([0,T] \times \Omega)$ such that, for the same subsequence

$$\epsilon \dot{z}_\epsilon^2 \rightharpoonup \mu_z \text{ weakly* in } \mathcal{M}([0,T] \times \Gamma), \tag{4.1d}$$

$$\epsilon \mathbb{C}^1 e(\dot{u}_\epsilon) \cdot e(\dot{u}_\epsilon) \rightharpoonup \mu_b \text{ weakly* in } \mathcal{M}([0,T] \times \Omega). \tag{4.1e}$$

as $\epsilon \to 0$. Moreover (u, z) satisfies for a.e. $t \in [0,T]$ the semistability condition

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi) \rangle + \langle \mathbb{K}[u(t)] \cdot [\varphi], z(t) \rangle_\Gamma = \langle \mathcal{L}(t), \varphi \rangle, \tag{4.2}$$

for all $\varphi \in H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$, and the energy equality

$$\begin{aligned}
&\mathcal{Q}_0(e(u)(t_2)) + \langle \frac{1}{2}\mathbb{K}[u(t_2)] \cdot [u(t_2)], z(t_2) \rangle_\Gamma - \langle \alpha, z(t_2) \rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) \rangle \\
&= \mathcal{Q}_0(e(u(t_1))) + \langle \frac{1}{2}\mathbb{K}[u(t_1)] \cdot [u(t_1)], z(t_1) \rangle_\Gamma - \langle \alpha, z(t_1) \rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) \rangle \\
&\quad + \mu_z(]t_1, t_2] \times \Gamma) + \mu_b(]t_1, t_2] \times \Omega) + \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}, u \rangle ds,
\end{aligned} \tag{4.3}$$

for a.e. $0 \le t_1 < t_2 \le T$.

The proof of the theorem is essentially the same of [28, Proposition 3.2], with the only difference that we have the addition of viscosity in the adhesive. We summarize some important steps and emphasize some differences, and then refer to [28] for a complete discussion.

Proof. **Step 1: apriori bounds**. We recall the energy balance for the solution (u_ϵ, z_ϵ), that is

$$\begin{aligned}
&\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \mathcal{Q}_0(e(u_\epsilon)(t)) + \langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)] \cdot [u_\epsilon(t)], z_\epsilon(t) \rangle_\Gamma + \epsilon \int_0^t \mathcal{Q}_1(e(\dot{u}_\epsilon)) ds \\
&+ \epsilon \int_0^t \|\dot{z}_\epsilon\|_{L^2}^2 ds - \int_0^t \langle \alpha, \dot{z}_\epsilon \rangle_\Gamma \\
&= \epsilon^2 \|u_0\|_{L^2}^2 + \mathcal{Q}_0(e(u_0)) + \langle \frac{1}{2}\mathbb{K}[u_0] \cdot [u_0], z_0 \rangle_\Gamma + \int_0^t \langle \mathcal{L}, \dot{u}_\epsilon \rangle ds.
\end{aligned} \tag{4.4}$$

Integrating by parts in time and then using the Cauchy and the Korn inequalities, we see that the right-hand side of (4.4) is bounded by the quantity

$$\frac{C_0}{\lambda} + \frac{\beta\lambda}{2}\|e(u_\epsilon)(t)\|_2^2 + C_1 \int_0^t \|e(u_\epsilon)\|_2^2 ds,$$

for some constants $C_0,\ C_1 > 0$ depending on the data of the problem but independent of ϵ, and for an arbitrary constant $\lambda > 0$. Setting $\lambda = \frac{\alpha_0}{2\beta}$, from (4.4) we obtain

$$\begin{aligned}
&\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)\|_{L^2}^2 + \frac{\alpha_0}{4}\|e(u_\epsilon)(t)\|_2^2 + \langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)] \cdot [u_\epsilon(t)], z_\epsilon(t) \rangle_\Gamma + \epsilon\alpha_1 \int_0^t \|e(\dot{u_\epsilon})\|_2^2 ds \\
&+ \epsilon \int_0^t \|\dot{z}_\epsilon\|_{L^2}^2 ds - \int_0^t \langle \alpha, \dot{z}_\epsilon \rangle_\Gamma \le \frac{2\beta C_0}{\alpha_0} + C_1 \int_0^t \|e(u_\epsilon)\|_2^2 ds,
\end{aligned} \tag{4.5}$$

and in particular, since all the term in the left-hand side are non-negative, we entail

$$\|e(u_\epsilon)(t)\|_2^2 \leq C + C\int_0^t \|e(u_\epsilon)\|_2^2 ds, \tag{4.6}$$

for some constant $C > 0$ independent of ϵ. The Gronwall Lemma then implies that the right-hand side of (4.5) is bounded by a constant. This provides the following estimates: there exists a constant $C > 0$ such that

$$\|e(u_\epsilon)(t)\|_2^2 \leq C \quad \text{ for all } t \in [0,T], \tag{4.7a}$$

$$\langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)] \cdot [u_\epsilon(t)], z_\epsilon(t))\rangle_\Gamma \leq C \quad \text{ for all } t \in [0,T], \tag{4.7b}$$

$$\epsilon\|\dot{u}_\epsilon(t)\|_2 \leq C \quad \text{ for all } t \in [0,T], \tag{4.7c}$$

$$\int_0^T \epsilon\|e(\dot{u}_\epsilon)\|_2^2 ds \leq C, \tag{4.7d}$$

$$\int_0^T \epsilon\|\dot{z}_\epsilon\|_2^2 ds \leq C. \tag{4.7e}$$

and arguing as in [28, Proposition 3.2] we find $z \in L^\infty([0,T], \mathcal{Z})$ such that

$$z_\epsilon(t) \rightharpoonup z(t) \quad \text{ weakly* in } L^\infty(\Gamma), \tag{4.8}$$

for all $t \in [0,T]$. The boundness

$$\|u_\epsilon(t)\|_{H^1} \leq C \quad \text{ for all } t \in [0,T], \tag{4.9}$$

implies that there exists $u \in L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ such that, up to a subsequence,

$$u_\epsilon \rightharpoonup u \text{ weakly* in } L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)), \tag{4.10a}$$

$$[u]_\epsilon \rightharpoonup [u] \text{ weakly* in } L^\infty([0,T], H^{\frac{1}{2}}(\Gamma, \mathbb{R}^d)), \tag{4.10b}$$

as $\epsilon \to 0$. Finally, the bounds (4.7d) and (4.7e) show that the functions $\epsilon\dot{z}_\epsilon^2$ and $\epsilon\mathbb{C}^1 e(\dot{u}_\epsilon) \cdot e(\dot{u}_\epsilon)$ are uniformly bounded in $L^1([0,T] \times \Gamma)$ and $L^1([0,T] \times \Omega)$ respectively, so that there exist two nonnegative Borel measures μ_z and μ_b such that, up to a subsequence,

$$\epsilon\dot{z}_\epsilon^2 \rightharpoonup \mu_z \text{ weakly* in } \mathcal{M}([0,T] \times \Gamma), \tag{4.10c}$$

$$\epsilon\mathbb{C}^1 e(\dot{u}_\epsilon) \cdot e(\dot{u}_\epsilon) \rightharpoonup \mu_b \text{ weakly* in } \mathcal{M}([0,T] \times \Omega). \tag{4.10d}$$

Step 2. The two following key lemma is proved in [28, Proposition 3.2].

Lemma 4.2. *For all $\varphi \in H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ and all ψ compactly supported real smooth function on $[0,T]$, it holds*

$$\lim_{\epsilon \to 0} \int_0^T \langle \mathbb{K}[u_\epsilon(s)]\psi(s) \cdot [\varphi], z_\epsilon(s)\rangle_\Gamma = \int_0^T \langle \mathbb{K}[u(s)]\psi(s) \cdot [\varphi], z(s)\rangle_\Gamma. \tag{4.11}$$

Lemma 4.3. *It holds*

$$\int_0^t \langle \mathbb{K}[u(s)] \cdot [u(s)], z(s)\rangle_\Gamma ds \leq \liminf_{\epsilon \to 0} \int_0^t \langle \mathbb{K}[u_\epsilon(s)] \cdot [u_\epsilon(s)], z_\epsilon(s)\rangle_\Gamma ds. \tag{4.12}$$

Step 3. Let ψ be a smooth and compactly supported positive function on $[0,T]$. Multiplying equation (3.63) by ψ and integrating in time on $[0,T]$ we obtain

$$\begin{aligned} &\int_0^T \big(\langle C^0 e(u_\epsilon) + \epsilon\mathbb{C}^1 e(\dot{u}_\epsilon), e(\varphi)\rangle + \langle \mathbb{K}[u_\epsilon] \cdot [\varphi], z_\epsilon\rangle_\Gamma\big)\psi ds \\ &= \int_0^T \langle \epsilon^2 \dot{u}_\epsilon, \varphi\rangle\dot{\psi} + \langle \mathcal{L}, \varphi\rangle\psi ds. \end{aligned} \tag{4.13}$$

Lemma 4.2 allows us to pass to the limit obtaining, thanks to (4.7c), (4.7d), (4.7e), (4.8), (4.10), and the arbitrariness of ψ,

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle \mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t), \varphi\rangle, \tag{4.14}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ and a.e. $t \in [0,T]$. The

Taking $\varphi = u_\epsilon$ in (3.62) and then integrating in time on $[0,t]$ we obtain

$$\begin{aligned} &\epsilon^2\langle \dot u_\epsilon(t), u_\epsilon(t)\rangle + \frac{\epsilon}{2}\mathcal{Q}_1(e(u_\epsilon(t))) + \int_0^t \epsilon^2\|\dot u\|^2 + \mathcal{Q}_0(e(u_\epsilon))ds \\ &= \epsilon^2\langle v_0, u_0\rangle + \frac{\epsilon}{2}\mathcal{Q}_1(e(u_0))) - \int_0^t \langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon\rangle_\Gamma + \langle \mathcal{L}, u_\epsilon\rangle ds, \end{aligned} \tag{4.15}$$

and taking into account the bounds (4.7c), (4.7d), and (4.7e), letting $\epsilon \to 0$, we entail

$$\lim_{\epsilon\to 0}\int_0^t \mathcal{Q}_0(e(u_\epsilon)) + \langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon\rangle_\Gamma ds = \int_0^t \langle \mathcal{L}, u\rangle ds. \tag{4.16}$$

From (4.14) with $\varphi = u$, the right-hand side equals $\int_0^t \mathcal{Q}_0(e(u)) + \langle \mathbb{K}[u]\cdot[u], z\rangle_\Gamma ds$. Now,

$$\int_0^t \mathcal{Q}_0(e(u)) \le \liminf_{\epsilon\to 0}\int_0^t \mathcal{Q}_0(e(u_\epsilon))ds,$$

and, from Lemma 4.3,

$$\int_0^t \langle \mathbb{K}[u]\cdot[u], z\rangle_\Gamma ds \le \liminf_{\epsilon\to 0}\int_0^t \langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon\rangle_\Gamma ds,$$

so that by (4.16) we entail that equalities hold, and hence

$$u_\epsilon \to u \text{ strongly in } L^2([0,T], H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^d)). \tag{4.17}$$

In particular this gives that for a.e. $t \in [0,T]$ one has

$$u_\epsilon(t) \to u(t) \text{ strongly in } H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^d), \tag{4.18}$$

$$[u_\epsilon](t) \to [u](t) \text{ strongly in } H^{\frac12}(\Gamma,\mathbb{R}^d), \tag{4.19}$$

so that, thanks to (4.8), we also have

$$\langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon(t)\rangle_\Gamma \to \langle \mathbb{K}[u]\cdot[u], z(t)\rangle_\Gamma, \tag{4.20}$$

for a.e. $t \in [0,T]$. This allows us to pass to the limit as $\epsilon \to 0$ in (3.64), getting (4.25).

Step 4. The same argument of [28, Proposition 3.2] applies to prove (4.3).

□

Theorem 4.1 easily generalizes to the case of nonhomogeneous boundary datum. Let us remark that in this case $u \in H^1$ and no longer in H^1_D, so convergences (4.1) hold with this dirrerence.

Theorem 4.4. *Let $\mathcal{L} \in H^1([0,T], H^{-1}_D(\Omega,\mathbb{R}^d))$, $w \in H^2([0,T], H^1_D(\Omega,\mathbb{R}^d))$, and u_0, v_0, z_0 as in Theorem 3.1. Let (u_ϵ, z_ϵ) be the solution given by Theorem 3.1, then there exist $u \in L^\infty([0,T], H^1(\Omega_1\cup\Omega_2))$ with $u(t) = w(t)$ on $\partial_D\Omega$, and $z \in L^2([0,T],\mathcal{Z})$ such that for a subsequece (4.1) hold as $\epsilon \to 0$ and for a.e. $t \in [0,T]$ the semistability condition holds*

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle \mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t), \varphi\rangle, \tag{4.21}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$*. Moreover the energy equality*

$$\mathcal{Q}_0(e(u)(t_2)) + \langle \frac{1}{2}\mathbb{K}[u(t_2)] \cdot [u(t_2)], z(t_2)\rangle_\Gamma - \langle \alpha, z(t_2)\rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) - w(t_2)\rangle$$
$$= \mathcal{Q}_0(e(u(t_1))) + \langle \frac{1}{2}\mathbb{K}[u(t_1)]\cdot[u(t_1)], z(t_1)\rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) - w(t_1)\rangle - \langle \alpha, z(t_1)\rangle_\Gamma$$
$$+ \mu_z(]t_1, t_2] \times \Gamma) + \mu_b(]t_1, t_2] \times \Omega) - \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}, u - w\rangle ds + \int_{t_1}^{t_2} \langle \sigma, e(\dot{w})\rangle ds, \quad (4.22)$$

is true for a.e. $0 \le t_1 < t_2 \le T$*, where* $\sigma := \mathbb{C}^0 e(u)$*.*

Proof. The following energy balance holds

$$\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t) - \dot{w}(t)\|^2_{L^2} + \mathcal{Q}_0(e(u_\epsilon(t))) + \langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)] \cdot [u_\epsilon(t)], z_\epsilon(t)\rangle_\Gamma + \epsilon \int_0^t \mathcal{Q}_1(e(\dot{u}_\epsilon))ds$$
$$+ \epsilon \int_0^t \|\dot{z}_\epsilon\|^2_{L^2} ds - \langle \alpha, z_\epsilon(t)\rangle_\Gamma = \epsilon^2\|v_0 - \dot{w}_0\|^2_{L^2} + \mathcal{Q}_0(e(u_0)) - \langle \alpha, z(0)\rangle_\Gamma$$
$$+ \langle \frac{1}{2}\mathbb{K}[u_0] \cdot [u_0], z_0\rangle_\Gamma + \int_0^t \langle \sigma_\epsilon, e(\dot{w})\rangle ds + \int_0^t \langle \mathcal{L} - \epsilon^2 \ddot{w}, \dot{u}_\epsilon - \dot{w}\rangle ds, \quad (4.23)$$

where $\sigma_\epsilon = \mathbb{C}^0 e(u_\epsilon) + \epsilon \mathbb{C}^1 e(\dot{u}_\epsilon)$. We then write

$$\int_0^t \langle \mathcal{L}, \dot{u}_\epsilon - \dot{w}\rangle ds \le |\langle \mathcal{L}(t), u_\epsilon(t) - w(t)\rangle| + \int_0^t \|\dot{\mathcal{L}}\|_{H^{-1}} \|u_\epsilon - w\|_{H^1} ds + C$$
$$\le C\|\mathcal{L}(t)\|_{H^{-1}} \|e(u_\epsilon(t)) - e(w(t))\|_2 + C\int_0^t \|\dot{\mathcal{L}}\|_{H^{-1}} \|e(u_\epsilon) - e(w(t))\|_2 ds + C$$
$$\le C + C\|e(u_\epsilon(t))\|_2 + C\int_0^t \|e(u_\epsilon)\|_2 ds$$
$$\le C + \frac{\alpha_0}{4}\|e(u_\epsilon(t))\|_2^2 + C\int_0^t \|e(u_\epsilon)\|_2^2 ds,$$

for some constant $C > 0$ possibly different from line to line. Moreover

$$\int_0^T \langle \sigma_\epsilon, e(\dot{w})\rangle ds \le C + C\int_0^T \|e(u_\epsilon)\|_2^2 ds + \frac{\epsilon\alpha_1}{2}\int_0^T \|e(\dot{u}_\epsilon)\|_2^2 ds,$$

and

$$|\int_0^t \langle \epsilon^2 \ddot{w}, \dot{u}_\epsilon - \dot{w}\rangle ds| \le C + \epsilon^2 \int_0^T \|\ddot{w}\|_{H^{-1}} \|e(\dot{u})\|_2 ds \le C + \epsilon^2 \int_0^T \|e(\dot{u})\|_2^2 ds.$$

Hence the right-hand side of (4.23) is bounded by

$$C + \frac{\alpha_0}{4}\|e(u_\epsilon(t))\|_2^2 + C\int_0^t \|e(u_\epsilon)\|_2^2 ds + \frac{\epsilon\alpha_1}{2}\int_0^T \|e(\dot{u}_\epsilon)\|_2^2 ds + \epsilon^2\int_0^T \|e(\dot{u})\|_2^2 ds,$$

and we are lead to

$$\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)\|^2_{L^2} + \frac{\alpha_0}{4}\|e(u_\epsilon)(t)\|_2^2 + \langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)] \cdot [u_\epsilon(t)], z_\epsilon(t)\rangle_\Gamma + \epsilon\int_0^t \|\dot{z}_\epsilon\|^2_{L^2} ds$$
$$+ \frac{\epsilon\alpha_1 - 2\epsilon^2}{2}\int_0^t \|e(\dot{u}_\epsilon)\|_2^2 ds - \int_0^t \langle \alpha, \dot{z}_\epsilon\rangle_\Gamma \le C + C\int_0^t \|e(u_\epsilon)\|_2^2 ds, \quad (4.24)$$

for some $C > 0$. This again implies (4.6) and the apriori bounds (4.7). The proof now is very similar to the previous and can be arranged straightforwardly. $\square$

An immediate consequence of (4.3) is the following:

Corollary 4.5. *Let (u, z) be the evolution obtained in the previous theorem. Then*

$$\begin{aligned}&\mathcal{Q}_0(e(u)(t_2)) + \langle\frac{1}{2}\mathbb{K}[u(t_2)]\cdot[u(t_2)], z(t_2)\rangle_\Gamma - \langle\alpha, z(t_2)\rangle_\Gamma - \langle\mathcal{L}(t_2), u(t_2) - w(t_2)\rangle\\ &\le \mathcal{Q}_0(e(u(t_1))) + \langle\frac{1}{2}\mathbb{K}[u(t_1)]\cdot[u(t_1)], z(t_1)\rangle_\Gamma - \langle\mathcal{L}(t_1), u(t_1) - w(t_1)\rangle\\ &- \langle\alpha, z(t_1)\rangle_\Gamma - \int_{t_1}^{t_2}\langle\dot{\mathcal{L}}, u - w\rangle ds + \int_{t_1}^{t_2}\langle\sigma, e(\dot{w})\rangle ds,\end{aligned} \tag{4.25}$$

for a.e. $0 \le t_1 < t_2 \le T$.

Remark 4.6 (Limit of processes in mode II). The limit of evolution with constrains as provided by Theorem 3.11 is straightforwardly arranged. The limit (u, z) will satisfy for a.e. $t \in [0, T]$ the property

$$u(t) \in D, \tag{4.26}$$

while the semistability condition (4.21) is replaced by

$$\langle\mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle\mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle\mathcal{L}, \varphi\rangle, \tag{4.27}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ with $[\varphi]\cdot\nu = 0$.

We are now in position to discuss the flow rule of the limit evolution (u, z). The presence of the viscosity term $\dot{z}$ in the flow rules (3.43) and (3.44), in contrast to [28] where the flow rule is rate-independent, makes the following analysis necessary.

Lemma 4.7. *For a.e.* $(x, t) \in \Gamma \times [0, T]$ *it holds*

$$\frac{1}{2}\mathbb{K}[u(x,t)]\cdot[u(x,t)] - \alpha(x) \le 0 \text{ or } z(x,t) = 0. \tag{4.28}$$

Proof. By (3.4), for all $\epsilon > 0$ it holds

$$(\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] - \epsilon\dot{z}_\epsilon - \alpha)\chi_{\{z_\epsilon>0\}} \le 0.$$

Up to a subsequence we have that $\chi_{\{z_\epsilon>0\}} \rightharpoonup \zeta$ weakly* in $L^\infty([0,T]\times\Gamma)$ for some $\zeta \in L^\infty([0,T]\times\Gamma)$. Thanks to (4.7e) we know that $\epsilon\dot{z}_\epsilon \to 0$ strongly in $L^2([0,T], L^2(\Gamma))$, while thanks to (4.9) and (4.19) we know that $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] \to \frac{1}{2}\mathbb{K}[u]\cdot[u]$ strongly in $L^1([0,T], L^1(\Gamma))$, so that at the limit as $\epsilon \to 0$ the previous relation gives rise to

$$(\frac{1}{2}\mathbb{K}[u]\cdot[u] - \alpha)\zeta \le 0, \tag{4.29}$$

almost everywhere on $[0,T]\times\Gamma$. Now the thesis follows if we prove that $\zeta > 0$ on the set $\{z > 0\}$. Let $A := \{(t,x) \in [0,T]\times\Gamma : 0 = \zeta(t,x) < z(x,t)\}$, and let us prove that $|A| = 0$. Then suppose $|A| > 0$. From the fact that $z_\epsilon(t) \rightharpoonup z(t)$ weakly* in $L^\infty(\Gamma)$ for all $t \in [0,T]$, the Fubini Theorem and the Dominated Convergence Theorem implies that

$$0 < \int_A z = \lim_{\epsilon\to 0}\int_A z_\epsilon,$$

but, on the other side we see that the right-hand side must be zero. Indeed we claim that $z_\epsilon \to 0$ strongly in $L^1(A)$. Since $z_\epsilon \le 1$, the claim follows if we prove that $|\{z_\epsilon > 0\}\cap A| \to 0$. But this is true since $|\{z_\epsilon > 0\}\cap A| = \int_A \chi_{\{z_\epsilon>0\}} \to \int_A \zeta = 0$ by hypothesis, and the lemma is proved.

□

Now we prove that there is a representant $\bar{z} : [0, T] \times \Gamma \to [0, 1]$ in the class of $z \in L^1([0,T]\times\Gamma)$ such that for a.e. $(t, x) \in [0, T]\times\Gamma$ there exists the time derivative $\frac{d}{dt}\bar{z}(t, x) \in \mathbb{R}$. Let us define

$$\bar{z}(t, x) := \liminf_{\delta\to 0} \int_{B_{x,\delta}} z(t, y)dy, \tag{4.30}$$

where $B_{x,\delta}$ is the ball in Γ centered at x and with radius $\delta > 0$. It turns out that such limit exists and coincides with $z(t, x)$ for a.e. $(t, x) \in [0, T] \times \Gamma$. Moreover for all x and all $0 \le t_1 < t_2 \le T$ it holds $\bar{z}(t_1, x) \le \bar{z}(t_2, x)$, since this inequality holds for z_ϵ and we have $\int_{B_{x,\delta}} z(t, y)dy = \lim_{\epsilon\to 0} \int_{B_{x,\delta}} z_\epsilon(t, y)dy$ for all $\delta > 0$ by (4.8). In particular for all fixed $x \in \Gamma$ the function $t \to \bar{z}(t, x)$ is nonincreasing so that it is differentiable almost everywhere on $[0, T]$. Note also that with such definition for all $t \in [0, T]$ the function $\bar{z}(t, \cdot)$ coincides with $z(t, \cdot)$ almost everywhere on Γ, that is $\bar{z}(t)$ is a particular representant of $z(t)$ in $L^\infty(\Gamma)$.

For $\bar{z}$ the following is true.

Lemma 4.8. *For a.e. $(t, x) \in [0, T] \times \Gamma$ it holds*

$$(\frac{1}{2}\mathbb{K}[u(t, x)] \cdot [u(t, x)] - \alpha(x))\dot{\bar{z}}(t, x) = 0. \tag{4.31}$$

Proof. For all real numbers $0 \le a < b \le T$ and all open set $A \subset \Gamma$ we can define the total variation of z_ϵ on $[a, b] \times A$ as

$$\mathrm{Var}(z_\epsilon, [a, b] \times A) := \langle \chi_A, z_\epsilon(a) - z_\epsilon(b)\rangle_\Gamma, \tag{4.32}$$

that defines a nonnegative measure on the Borel subsets of $[0, T] \times \Gamma$. Defining similarly the total variation of z we see that $\mathrm{Var}(z_\epsilon, \cdot) \rightharpoonup \mathrm{Var}(z, \cdot)$ weakly* in the space of nonnegative Radon measures $\mathcal{M}_b([0, T] \times \Gamma)$. Writing $z_\epsilon(a) - z_\epsilon(b) = -\int_a^b \dot{z}_\epsilon(s)ds$ and similarly $z(a)-z(b) = -\int_a^b D_t\bar{z}(s)ds$ where D_t is the distributional derivative in time, we also obtain that for all Borel set $B \subset [0, T] \times \Gamma$,

$$-\int_B \dot{\bar{z}} \le \mathrm{Var}(\bar{z}, B) \le \mathrm{Var}(z_\epsilon, B) = -\int_B \dot{z}_\epsilon, \tag{4.33}$$

where the first inequality is due to the fact that $-\dot{\bar{z}}$ is only the part of $-D_t\bar{z}$ that is absolutely continuous with respect to the Lebesgue measure, while the second one follows by the lower semocontinuity of the mass.

Now from the fact that $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] \to \frac{1}{2}\mathbb{K}[u]\cdot[u]$ strongly in $L^1([0, T], L^1(\Gamma))$ we have that $\frac{1}{2}\mathbb{K}[u_\epsilon(t, x)] \cdot [u_\epsilon(t, x)] \to \frac{1}{2}\mathbb{K}[u(t, x)] \cdot [u(t, x)]$ for a.e. $(t, x) \in [0, T] \times \Gamma$. Let us define $C := \{(t, x) \in [0, T]\times\Gamma : \dot{\bar{z}}(t, x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t, x)]\cdot[u(t, x)]-\alpha(x) \neq 0\}$. From the fact that $\bar{z}$ is nonnegative and nonincreasing it is straightforward that $\dot{\bar{z}} = 0$ on the set $\bar{z} = 0$, so that condition (4.28) tells us that $|C\Delta C'| = 0$, with $C' := \{(t, x) \in [0, T]\times\Gamma : \dot{\bar{z}}(t, x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t, x)]\cdot[u(t, x)]-\alpha(x) < 0\}$. Let us then prove that $|C'| = 0$. Suppose it is not the case, so that for some $n > 0$ it holds that $|C_n| > 0$, with $C_n := \{(t, x) \in [0, T] \times \Gamma : \dot{\bar{z}}(t, x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t, x)] \cdot [u(t, x)] - \alpha(x) < -\frac{1}{n}\}$. Thanks to the pointwise convergence of $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon]$ to $\frac{1}{2}\mathbb{K}[u]\cdot[u]$ we can find a subset $B \subset C_n$ with positive measure and a number ϵ_0 such that for all $\epsilon < \epsilon_0$ and all $(t, x) \in B$ it holds $\mathbb{K}[u_\epsilon(t, x)]\cdot[u_\epsilon(t, x)]-\alpha(x) < 0$. This means that, thanks to (3.3b), $\dot{z}_\epsilon(t, x) = 0$ for all $\epsilon < \epsilon_0$ and all $(t, x) \in B$. So that

$$0 = -\lim_{\epsilon\to 0} \int_B \dot{z}_\epsilon \ge -\int_B \dot{\bar{z}},$$

where we have used (4.33). But since $-\dot{\bar{z}}$ is nonnegative we find $\dot{\bar{z}} = 0$ almost everywhere on B, contradicting the hypothesis. ∎

Let us define $\mathcal{E} : [0,T] \to \mathbb{R}$ the energy of the limit evolution (u,z) obtained in Theorem 4.1 as

$$\begin{aligned}\mathcal{E}(t) :=& \mathcal{Q}_0(e(u)(t)) + \langle \frac{1}{2}\mathbb{K}[u(t)]\cdot[u(t)], z(t)\rangle_\Gamma \\ &- \langle \alpha, z(t)\rangle_\Gamma - \langle \mathcal{L}(t), u(t)\rangle + \int_0^t \langle \dot{\mathcal{L}}, u\rangle ds, \end{aligned} \tag{4.34}$$

for all $t \in [0,T]$. Inequality (4.25) says exactly that $\mathcal{E}$ is an essentially nonincreasing function. Essentially means that there exists a negligible set $N \subset [0,T]$ such that $\mathcal{E}$ is nonincreasing on $[0,T] \setminus N$. We can then always extend it to a (unique) left-continuous nonincreasing function on the whole $[0,T]$. As a consequence the new $\mathcal{E}$ is discontinuous on an at most countable set $J_E \subset [0,T]$, and this set does not depend on the value of $\mathcal{E}$ on N. We will also denote by J_z the subset of $[0,T]$ where the function z is discontinuous with respect to the strong topology of $L^1(\Gamma)$. Since z is a nonincreasing function with values in $[0,1]$, we see that J_z is at most countable as well.

Theorem (4.1) shows that the evolution (u,z) limit of (u_ϵ, z_ϵ) satisfies the stability condition almost everywhere on $[0,T]$. The next Lemma gives a more precise description of the set of times where stability holds, and at the same time tells us that we can change the map $u \in L^\infty([0,T], L^2(\Omega,\mathbb{R}^d))$ on the negligible set N in such a way that the energy $\mathcal{E}$ is globally nonincreasing.

Lemma 4.9. *Suppose $\bar{t} \in [0,T] \setminus (J_E \cup N)$ is such that z is continuous at $\bar{t}$ with respect to the strong topology of $L^1(\Gamma)$, i.e. $\bar{t} \notin J_z$. Then the stability condition* (4.2) *holds at such $\bar{t}$.*

Moreover there exists a representant of $u \in L^\infty([0,T], L^2(\Omega,\mathbb{R}^d))$, still denoted by u, such that the stability condition (4.2) *holds at all $t \in [0,T] \setminus J_z$ and the corresponding energy* (4.34) *is nonincreasing and continuous at all $t \in [0,T] \setminus J_z$.*

Proof. Condition (4.2) tells us that $u(t)$ is the (unique) minimizer in $H^1_\Gamma(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ of the potential

$$W_t(u) := \mathcal{Q}_0(e(u)) + \langle \frac{1}{2}\mathbb{K}[u]\cdot[u], z(t)\rangle_\Gamma - \langle \mathcal{L}(t), u\rangle. \tag{4.35}$$

Let us denote by $M(t) := \min W_t$. The fact that z is continuous at $\bar{t}$ entails that also M is continuous at $\bar{t}$. Let us choose a sequence t_n such that $t_n \notin N$ and $u(t_n)$ satisfies the stability condition (4.2) for all $n > 0$, then we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{E}(t_n) &= \lim_{n\to\infty} \big(M(t_n) + \langle \alpha, z(t_n)\rangle - \int_0^{t_n} \langle \dot{\mathcal{L}}, u\rangle ds\big) \\ &= M(\bar{t}) + \langle \alpha, z(\bar{t})\rangle - \int_0^{\bar{t}} \langle \dot{\mathcal{L}}, u\rangle ds = \mathcal{E}(\bar{t}), \end{aligned} \tag{4.36}$$

where the last equality follows from the continuity of $\mathcal{E}$. This says that $W_{\bar{t}}(u(\bar{t})) = M(\bar{t})$, which, thanks to the uniqueness of the minimizer of $W_{\bar{t}}$, entails that $u(\bar{t})$ is such minimizer, so that it also satisfies (4.2), and the first part of the statement is proved.

Let us now fix $t \in [0,T] \setminus J_z$, if we choose t_n such that $t_n \to t$ and $u(t_n)$ satisfies the stability condition (4.2), formula (4.36) still holds with $\bar{t}$ replaced by t thanks to the continuity of z and proves that we can redefine u at all points $t \in N \setminus J_z$ as the minimizer of W_t. We see that the new u coincides with the old one almost everywhere and satisfies (4.2) at all $t \in N \setminus J_z$ by definition. This concludes the proof, noting that the new $\mathcal{E}$ corresponding to the new u is continuous on $[0,T] \setminus J_z$. □

Remark 4.10. A consequence of Lemma 4.9 is that the set of times $t \in [0,T]$ such that the new $u(t)$ does not satisfy the stability condition (4.2) is an at most countable set. Let us denote it by S_u. Lemma 4.9 then reads

$$(S_u \cup J_E) \subset J_z.$$

Another consequence of this fact is that at any time where z is continuous, also u is continuous with respect to the strong topology of $H^1_\Gamma(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$. If we denote by J_u the set of times where u is discontinuous, then J_u is at most countable and $J_u \subset J_z$.

Another consequence of Lemma (4.9) is that the definition of the new u implies that for all $t \in [0,T] \setminus J_z$ relation (4.28) holds true for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma_1$.

Let us finally remark that, with the new definition of $\mathcal{E}$, the energy inequality (4.25) holds for all $t_1, t_2 \in [0,T] \setminus J_z$.

Theorem 4.11. *Suppose that there exists $0 < s \leq T$ such that $z(t,x) > 0$ at a.e. $x \in \Gamma$ for all $0 \leq t \leq s$. Then the energy $\mathcal{E}$ is constant on $[0,s] \setminus J_z$, i.e. $\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \in [0,s] \setminus J_z$. In particular $\mu_z = 0$ on $[0,s] \times \Gamma$ and $\mu_b = 0$ on $[0,s] \times \Omega$.*

Proof. Taking into account (4.25), it suffices to show that $\mathcal{E}(0) \leq \mathcal{E}(s)$. To prove this, for all integers $n > 0$ let us choose a sequence of times $0 = t_0 < t_1 < \cdots < t_n = s$ such that $t_i \in [0,T] \setminus S_u$ for all $i \leq n$ and such that $\max_{i<n} |t_{i+1} - t_i| \to 0$ as $n \to \infty$. The minimality of W_{t_i} at $u(t_i)$ implies $W_{t_i}(u(t_i)) \leq W_{t_i}(u(t_{i+1}))$ for all $0 \leq i < n$. This is equivalent to

$$\begin{aligned}
&\mathcal{Q}_0(e(u(t_i))) - \mathcal{Q}_0(e(u(t_{i+1}))) - \langle \mathcal{L}(t_i), u(t_i)\rangle + \langle \mathcal{L}(t_{i+1}), u(t_{i+1})\rangle \\
&+ \langle \frac{1}{2}\mathbb{K}[u(t_i)] \cdot [u(t_i)], z(t_i)\rangle_\Gamma - \langle \frac{1}{2}\mathbb{K}[u(t_{i+1})] \cdot [u(t_{i+1})], z(t_{i+1})\rangle_\Gamma \\
&\leq \langle \frac{1}{2}\mathbb{K}[u(t_{i+1})] \cdot [u(t_{i+1})], z(t_i) - z(t_{i+1})\rangle_\Gamma + \langle \mathcal{L}(t_{i+1}), u(t_{i+1}) - u(t_i)\rangle \\
&\leq \langle \alpha, z(t_i) - z(t_{i+1})\rangle_\Gamma + \langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1})\rangle,
\end{aligned} \tag{4.37}$$

where in the last inequality we have used (4.28) with Remark 4.10. Summing this expression for $i = 0, \dots, n-1$ we obtain

$$\begin{aligned}
&\mathcal{Q}_0(e(u(0))) - \mathcal{Q}_0(e(u(s))) - \langle \mathcal{L}(0), u(0)\rangle + \langle \mathcal{L}(s), u(s)\rangle \\
&+ \langle \frac{1}{2}\mathbb{K}[u(0)] \cdot [u(0)], z(0)\rangle_\Gamma - \langle \frac{1}{2}\mathbb{K}[u(s)] \cdot [u(s)], z(s)\rangle_\Gamma \\
&\leq \langle \alpha, z(0)\rangle_\Gamma - \langle \alpha, z(s)\rangle_\Gamma + \sum_{i=0}^{n-1} \langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1})\rangle,
\end{aligned} \tag{4.38}$$

but the last term tends to $\int_0^s \langle \dot{\mathcal{L}}, u\rangle ds$ as $n \to \infty$ thanks to the regularity of $\mathcal{L}$ and the fact that J_u is at most countable. So that the inequality above implies exactly $\mathcal{E}(0) \leq \mathcal{E}(s)$, and the thesis follows. □

Remark 4.12. If we do not redefine the functions $\mathcal{E}$ and u as in Lemma 4.9, Theorem 4.11 still holds, with the only difference that the equality $\mathcal{E}(t) = \mathcal{E}(0)$ holds only for a.e. $t \in [0,s] \setminus (N \cup J_z)$. To see this it sufficies to apply the same proof with the only difference that we have to choose the times t_i in the set where (4.21) holds for the original u.

4.1. **The one-dimensional case.** In this section we consider the case $d = 1$. Without lose of generality we set $\Omega_1 :=]0,1[$, $\Omega_2 :=]-1,0[$, $\Gamma := \{0\}$ and $\partial_D\Omega := \{-1,1\}$ and assume that $\mathbb{C}^0 = 1$ and $\mathbb{K} = 1$. We denote by u the displacement, and we want to study an evolution with Dirichlet conditions $u(t,1) = a_1(t)$ and $u(t,-1) = a_{-1}(t)$ for all $t \in [0,]$, and external forces $\mathcal{L}(t,x)$. This arises imponing

$w(t,x) := a_{-1}(t) + \frac{x+1}{2}(a_1(t) - a_{-1}(t))$. We assume that at the initial time we have $z_0 = 1$.

Let us first state the following preliminary fact:

Lemma 4.13. $\mathcal{L} \in H_D^{-1}(]-1,0[\cup]0,1[,\mathbb{R})$ *if and only if there exists* $F \in L^2(]-1,0[\cup]0,1[)$ *such that* $\langle \mathcal{L}, \varphi\rangle = -\langle F, \varphi_x\rangle$*, for all* $\varphi \in H_D^1(]-1,0[\cup]0,1[,\mathbb{R})$.

Proof. We can write

$$\langle \mathcal{L}, \varphi\rangle \le C_1\|\varphi\|_{H^1} \le C_2\|\varphi_x\|_2,$$

thanks to the Poincaré inequality. In particular, since the linear map $A : H_D^{-1} \to L^2(]-1,0[\cup]0,1[)$ given by $A(\varphi) = \varphi_x$ is bijective, we see that the map $\mathcal{L}\circ A^{-1}$ belongs to the dual of $L^2(]-1,0[\cup]0,1[)$, and then there exists $F \in L^2(]-1,0[\cup]0,1[)$ such that $\mathcal{L}\circ A^{-1}(\psi) = -\langle F, \psi\rangle$ for all $\psi \in L^2(]-1,0[\cup]0,1[)$. The claim follows by writing $\psi = \varphi_x$. □

Lemma 4.9 guarantees that (u,z) satisfies (4.21) and (4.25) everywhere on $[0,T]\setminus J_z$. Now we prove that, up to suitably change the function $t \to (u(t),z(t))$ on a negligible set, we can assume that such conditions are satisfied for all $t \in [0,T[$. In the one-dimensional case $z(t)$ is just a real number, and convergence (4.1c) ensures that z is nonincreasing, and then coincides with $\bar{z}$ defined in Lemma 4.8. We define

$$\tilde{z}(t) := \lim_{s\to t^-} z(s).$$

In particular $\tilde{z}$ is left-continuous. Let $S_u \subset [0,T]$ be the set of all t at which (4.21) does not hold. Then for all $t \in S_u$ we define $u'(t)$ as the (unique) solution of problem (4.21) with $z(t)$ replaced by $\tilde{z}(t)$ and boundary datum $w(t)$. Then we set

$$\tilde{u}(t) := \begin{cases} u'(t) & \text{if } t \in L \\ u(t) & \text{otherwise.}\end{cases}$$

Not to weight up the notation since now on we will still denote $(\tilde{u},\tilde{z})$ by (u,z). Let us remark that, thanks to Lemma 4.9, the fact that z is left-continuous at all $t \in [0,t_1]$, it is easily seen that the energy (4.34) turns out to be globally nonincreasing, i.e. it is a nonincreasing function on the whole interval $[0,t_1]$.

In other words we have first redefined z in order that it is left-continuous, and then we have redefine u as in Lemma 4.9. Thanks to the left-continuity of z we see that the proof of Lemma 4.9 provides that the new u satisfies (4.21) on the whole $[0,t_1]$.

When (t,z) are fixed, (4.21) is equivalent to the fact that u is the minimizer of the functional

$$u \to \frac{1}{2}\langle u_x, u_x\rangle + \frac{1}{2}[u]^2 z - \langle \mathcal{L}, u\rangle,$$

among all the functions $u \in H^1(]-1,0[\cup]0,1[)$ with $u(1) = w(t,-1)$ and $u(-1) = w(t,1)$. Equivalently, this is expressed by the following system of equations

$$\begin{cases} -u_{xx}(t,x) = \mathcal{L}(t,x) & \text{on } \ ]-1,0[\ \cup\ ]0,1[, \\ u_x(t,0) = [u(t,0)]z(t) & \\ u(1) = w(t,-1) & \\ u(-1) = w(t,1). & \end{cases} \tag{4.39}$$

It is not difficult to compute explicitly the solutions of such system. Let $F \in H^1([0,T], L^2(]-1,0[\cup]0,1[))$ be the function, provided by Lemma 4.13, such that $\langle \mathcal{L}(t), \varphi\rangle = -\langle F(t), \varphi_x\rangle$ and set $G(t,x) := \int_0^x F(t)(y)dy$ for all $x \in]-1,0[\cup]0,1[$, the solution $u = u(t,x)$ of (4.39) takes the form

$$u(t,x) = \begin{cases} G(t,1) - G(t,x) + g(t)\frac{z(t)}{1+2z(t)}(x-1) + w(t,1) & \text{if } x > 0 \\ G(t,-1) - G(t,x) + g(t)\frac{z(t)}{1+2z(t)}(x+1) + w(t,-1) & \text{if } x < 0, \end{cases} \tag{4.40}$$

where $g(t) := G(t,1) - G(t,-1) + w(t,1) - w(t,-1)$. We can compute

$$[u(t)] := \frac{g(t)}{1+2z(t)}. \tag{4.41}$$

Let us define

$$\begin{aligned} t_0 &:= \inf_t \{\frac{1}{2}[u(t)]^2 - \alpha \geq 0\}, \\ t_1 &:= \inf_t \{\frac{1}{2}[u(t)]^2 - \alpha > 0\}, \end{aligned} \tag{4.42}$$

and let these values be T if the corresponding infima are computed on empty sets. Obviously we have $t_0 \leq t_1$. We see that the times t_0 and t_1 depend only on g and the value of z, in particular

$$\begin{aligned} t_0 &:= \inf_t \{z(t) \leq \frac{g(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}}\}, \\ t_1 &:= \inf_t \{z(t) < \frac{g(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}}\}. \end{aligned} \tag{4.43}$$

The energy (4.34) reads

$$\begin{aligned} \mathcal{E}(t) =& \frac{1}{2}\langle u_x(t,x), u_x(t,x)\rangle + \frac{1}{2}[u(t)]^2 z(t) - \alpha z(t) \\ &+ \langle F(t), u_x(t) - w_x(t)\rangle - \int_0^t \langle \dot{F}(s), u_x(s) - w_x(s)\rangle ds \\ &- \int_0^t \langle u_x(s), \dot{w}_x(s)\rangle ds, \end{aligned}$$

and plugging the formulae found above in this expression we obtain

$$\begin{aligned} \mathcal{E}(t) =& \frac{1}{2}\frac{g(t)^2 z(t)}{1+2z(t)} - \alpha z(t) - \frac{(G(0,1) - G(0,-1))(w(0,1) - w(0,-1))}{2} \\ &- \int_0^t \frac{g(s)\dot{g}(s)}{1+2z(s)} z(s). \end{aligned}$$

We will now employ a standard formula providing the expression of the distributional derivative of the composition of a smooth function with a function with bounded variation (see, e.g., [32], or [2]). If $z : [0,T] \to \mathbb{R}$ is a BV function and $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth, such formula applied to the function $t \to f(t, z(t))$ reads

$$\begin{aligned} & D_t f(\cdot, z(\cdot)) = \\ & \quad f_1(\cdot, z(\cdot))\mathcal{L}^1 + f_2(\cdot, \bar{z}(\cdot)) D_t z \llcorner_{C_z} + \sum_{s \in \mathbb{R}^+} [f(s, z(s^+)) - f(s, z(s^-))]\delta_s, \end{aligned} \tag{4.44}$$

where f_i is the derivative of f with respect to the i-th variable, $\mathcal{L}^1$ is the Lebesgue measure on $\mathbb{R}$, $\bar{z}$ is the continuous representant of z on the set C_z, the set where z is continuous, $z(s^+)$ (resp. $z(s^-)$) is the limit from the right (resp. left) of z at $s \in \mathbb{R}$, and δ_s is the Dirac delta at $s \in \mathbb{R}$. We use this formula to compute the distributional derivative of $\mathcal{E}$. Let us recall that the function z itself is continuous at every t except at the jump times. Therefore we find

$$\begin{aligned} D_t\mathcal{E}(t) =& (\frac{1}{2}\frac{g(t)^2}{(1+2z(t))^2} - \alpha)(\dot{z} + \dot{z}^c) \\ &+ \sum_{s \in [0,T]} \Big(\frac{1}{2}\frac{g(s^+)z(s^+)}{(1+2z(s^+))} - \frac{1}{2}\frac{g(s^-)z(s^-)}{(1+2z(s^-))} - \alpha z(s^+) + \alpha z(s^-)\Big)\delta_s, \end{aligned} \tag{4.45}$$

where $\dot{z}$ and $\dot{z}^c$ are the absolutely continuous part of $D_t z \llcorner_{C_z}$ with respect to $\mathcal{L}^1$ and the Cantor part respectively. We can write the jumps of (4.45) in the following equivalent way

$$-\sum_{s\in[0,T]} \Big(\int_{z(t^+)}^{z(t^-)} \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha dr \Big)\delta_s. \tag{4.46}$$

From the energy inequality we know that the energy is a nonincreasing function, so that its total derivative (4.45) must be a nonpositive measure on $[0,T]$. Since the absolutely continuous, the Cantor and the jump part of this measure are mutually singular, they must all be nonpositive. This applied to the jumps implies that the integrals appearing in the sum (4.46) are all nonnegative. On the other hand we have

$$\int_{z(t^+)}^{z(t^-)} \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha dr \leq \int_{z(t^+)}^{z(t^-)} \frac{1}{2}\frac{g(s)^2}{(1+2z(t^+))^2} - \alpha ds \leq 0,$$

where the first inequality follows from the fact that $r \to \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha$ is nonincreasing, and the second inequality follows untill $t \in [0,t_1[$. Moreover, the first inequality is strict if $g(s) \neq 0$, since $r \to \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha$ is strictly decreasing in this case, while if $g(s) = 0$ the second inequality is strict since $\alpha > 0$. In particular we find out that no jump can occur in the interval $[0,t_1[$.

We claim that, if there is a jump of z, than such jump is unique and takes place at $t = t_1$. Moreover $z(t) = 0$ for $t > t_1$. Without lose of generality suppose $t_1 < T$. Since z is left-continuous, the function $\frac{1}{2}\frac{g(t)^2}{(1+2z(t))^2} - \alpha$ is left-continuous, so that by definition of t_1 it holds $\frac{1}{2}\frac{g(t_1)^2}{(1+2z(t_1))^2} - \alpha \leq 0$, and there is a sequence $t_k \searrow t_1$ such that $f(t_k) > 0$ for all k. Again, since f is left-continuous we obtain that for all $\delta > 0$ the set of all t such that $f(t) > 0$ has positive Lebesgue measure on $[t_1, t_1+\delta]$. This, thanks to (4.28), implies that $z(t) = 0$ for $t > t_1$, getting the claim.

Let us now consider the Cantor and absolutely continuous part of (4.45). We see that $\dot{z}$ and $\dot{z}^c$ might concentrate only on the set $A := \{t \in [0,t_1] : \frac{1}{2}\frac{g(t)^2}{(1+2z(t))^2} - \alpha = 0\} = \{t \in [0,t_1] : z(t) = \frac{g(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}\}$. This is the set where the continuous function $g(t)$ coincides with $f(t) := \sqrt{2\alpha}(1+2z(t))$. We claim that the distributional derivatives of the BV functions g and f coincide on A. It is a particular case of a more general fact provided by [19, Theorem A.1]. As a consequence we get

$$\dot{g} = 2\sqrt{2\alpha}(\dot{z} + \dot{z}^c),$$

which implies that $\dot{z}^c = 0$ since the right-hand side is absolutely continuous with respect to the Lebesgue measure. Moreover we find out that $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{g}$.

We can summarize our discussion with the following results, which holds in the 1-dimensional case:

Theorem 4.14 (1-dimensional case). *Let (u,z) be the limit of dynamic processes given by Theorem 4.4. Then there is a representant of z that is left-continuous. Let t_0, t_1 be as in (4.43). Then there is a representant of u such that $u(t)$ is the solution of (4.39) for all $t \in [0,t_1]$. For these representants, still denoted by (u,z), it holds that z is constant on the interval $[0,t_0]$ and it is such that $z(t) \equiv 0$ for $t > t_1$. Moreover z can jump only at $t = t_1$, $\dot{z}^c \equiv 0$ on $[0,T]$, and $\dot{z}$ is concentrated on the set*

$$A := \{t \in [t_0,t_1] : z(t) = \frac{g(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}\}, \tag{4.47}$$

where it also holds $\dot z = \frac{1}{2\sqrt{2\alpha}}\dot g$, *with* $g(t) := G(t,1) - G(t,-1) + w(t,1) - w(t,-1)$. *In formula*

$$\dot z = \frac{1}{2\sqrt{2\alpha}}\dot g \chi_A.$$

In terms of the data of the problem we can state the following:

Theorem 4.15. *Let* (u,z) *be the limit of dynamic processes given by Theorem 4.4 with initial condition* $z(0) = z_0 > 0$ *and suppose* z *is left-continuous. Let*

$$\tilde t_0 := \inf_{t\in[0,T]}\{g(t) \geq (1+2z_0)\sqrt{2\alpha}\}, \qquad \tilde t_1 := \inf_{t\in[0,T]}\{g(t) > (1+2z_0)\sqrt{2\alpha}\},$$

then it holds $z(t) = z_0$ *if* $t \leq \tilde t_0$, $z(t) = 0$ *if* $t > \tilde t_1$, $\dot z = \frac{1}{2\sqrt{2\alpha}}\dot g\chi_A$, *and* z *can jump only at* $t = t_1$.

Corollary 4.16. *If* $g(t)$ *is strictly increasing and is such that* $g(0) < (1+2z_0)\sqrt{2\alpha}$, *then there is only one solution* $\bar t > 0$ *of* (4.47) *and* $z(t) = z_0$ *for* $t \leq \bar t$, *while* $z(t) = 0$ *for* $t > \bar t$.

Proof. In such a case $t_0 = t_1 = \bar t$. Note that hypothesis $g(0) < (1+2z_0)\sqrt{2\alpha}$ prevents that $\bar t = 0$. □

The last statement proves that the function (u,z) given by an external load and boundary condition as in the example of [28, Section 4] coincides with the couple of such example. We emphasize that Theorem 4.14 refers to a couple (u,z) which evolves without constrains on the jump. However, if the jump remains positive, as in the example of [28, Section 4], the evolution itself satisfies the constraint of mode I.

We conclude the section with the following remark, that show that the conditions we have obtained by the analysis of the limit (u,z) is not sufficient to conclude whether jumps occur or not.

Remark 4.17. Suppose that the function $g \in \mathcal{C}^\infty(\mathbb{R})$ be such that $g(0) = 0$, $g(1) = 3\sqrt{2\alpha}$, $g(2) = \sqrt{2\alpha}$, and g is strictly monotone in the intervals $[0,1]$ and $[1,2]$. Let then $z = 1$ on $[0,1]$, $z(t) = \frac{g(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}$ for $t \in [1,2]$, and $z(t) = 0$ for $t > 2$. Then let $u(t)$ be the solution of (4.39), i.e. the function in (4.40). For such (u,z) we see that (4.21) holds by definition while (4.45) shows that (4.22) holds true with $\mu_b = \mu_z = 0$. This is an example of an evolution satisfying the conditions of the limit of dynamic processes with initial condition $z_0 = 1$, and which does not show any jump, actually being smooth in time. However it is still not clear if there exists some dynamic process whose limit is such function. In particular it is not clear if the measures μ_b and μ_z must be strictly positive, as in the case of Corollary 4.16, or may vanish.

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