# Analytic semigroups generated in $L^1(\Omega)$ by second order elliptic operators via duality methods

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**Abstract**

Given an open domain (possibly unbounded) $\Omega \subset \mathbf{R}^n$, we prove that uniformly elliptic second order differential operators, under nontangential boundary conditions, generate analytic semigroups in $L^1(\Omega)$. We use a duality method, and, further, give estimates of first order derivatives for the resolvent and the semigroup, through properties of the generator in Sobolev spaces of negative order.

## 1 Introduction

The aim of this paper is to show how a duality method can be used to prove that uniformly elliptic operators endowed with non-tangential boundary conditions generate analytic semigroups in $L^1$. It is well-known that the methods used in $L^p$ spaces, with $1 < p < \infty$, cannot be extended to the cases $p = 1, \infty$ because the classical Agmon-Douglis-Nirenberg estimates do not hold in these cases. In fact, the generation results in $L^\infty$ follow from the $L^p$ case through a clever passage to the limit and Sobolev embedding known as Masuda-Stewart technique, see [18], [19], [12]. The known approaches relative to the $L^1$ case are based either on an integral representation of the semigroup and suitable estimates on the kernel, see e.g. [20], or on duality arguments. In order to deduce $L^1$ results from $L^\infty$, duality arguments have been developed both for the adjoint semigroups, see [2], and their generators in the case of Dirichlet boundary conditions, see [14], [7], [24] and [9], where also elliptic systems are studied. The generation result in $L^1$ has been proved also in [3] under $L^\infty$ conditions on the diffusion coefficients. In this paper Dirichlet, Neumann or mixed boundary conditions are considered, and the fact that the resulting operator is symmetric seems to be essential. In this paper, we apply duality arguments to general (non-tangential) boundary conditions involving first order derivatives. Moreover, we prove gradient estimates for the solution of the resolvent equation and for the semigroup solution of the parabolic initial-boundary value problem. The parabolic estimates are deduced from the elliptic ones, while the elliptic estimates are obtained through a discussion of the sectoriality of the generator in Sobolev spaces of negative order. In this part, we follow ideas from [22], [13].

Let us come to a more technical description of the results. Throughout this paper, $\Omega$ denotes an open (possibly unbounded) domain in $\mathbf{R}^n$ with uniformly $C^3$ boundary $\partial\Omega$ (see

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Remark 2.1). We denote by $\nu(x)$ the outward normal unit vector to $\partial\Omega$ at $x$. We consider the uniformly elliptic second order operator in divergence form:

$$\mathcal{A}(x,D)=\sum_{i,j=1}^{n}D_i(a_{ij}(x)D_j)+\sum_{i=1}^{n}b_i(x)D_i+c(x) \tag{1.1}$$

with real coefficients satisfying the following assumptions

$$a_{ij}=a_{ji}\in W^{1,\infty}(\Omega),\qquad b_i,c\in L^\infty(\Omega) \tag{1.2}$$

and the following uniform ellipticity condition: there exists $\mu\geq 1$ such that for any $x\in\overline{\Omega}$ and $\xi\in\mathbf{R}^n$

$$\mu^{-1}|\xi|^2\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq\mu|\xi|^2. \tag{1.3}$$

We also consider a linear first order differential operator with real coefficients defined for $x\in\partial\Omega$:

$$\mathcal{B}(x,D)=\sum_{i=1}^{n}\beta_i(x)D_i+\gamma(x) \tag{1.4}$$

with $\beta_i,\gamma$ and their first derivatives uniformly continuous and bounded, and assume that the uniform non-tangentiality condition

$$\inf_{x\in\partial\Omega}\left|\sum_{i=1}^{n}\beta_i(x)\nu_i(x)\right|>0 \tag{1.5}$$

holds. We are looking for the solution of the following problem

$$\begin{cases}\partial_t u-\mathcal{A}u=0 & \text{in } (0,\infty)\times\Omega\\ u(0)=u_0 & \text{in } \Omega\\ \mathcal{B}u=0 & \text{in } (0,\infty)\times\partial\Omega\end{cases} \tag{1.6}$$

with initial datum $u_0\in L^1(\Omega)$. In the language of semigroups, this leads us to consider the realization $A_1: D(A_1)\subset L^1(\Omega)\to L^1(\Omega)$ of $\mathcal{A}$ in $L^1(\Omega)$, where the domain $D(A_1)$ takes into account the boundary conditions. We prove that $(A_1,D(A_1))$ is sectorial in $L^1(\Omega)$, hence it is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $L^1(\Omega)$.

The plan of the paper is as follows: in the second section we prove the generation result in $L^1(\Omega)$ and in the third one we prove the estimates on the gradient of the resolvent and of the semigroup. This last section relies on the analysis of the resolvent equation in Sobolev spaces of negative order.

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## 2 Preliminary results in $L^1(\Omega)$

In this section we prove that the realization of $\mathcal{A}$ with boundary conditions $\mathcal{B}u = 0$ is sectorial in $L^1(\Omega)$. Since we would like to solve the problem in $L^1$ by duality from $L^\infty$, we point out that the assumptions (2.2) on the regularity of the coefficients guarantee that the realization in $L^\infty(\Omega)$ of the adjoint $(\mathcal{A}^*, \mathcal{B}^*)$, as defined in (2.3) and (2.4) below, generates an analytic semigroup in $L^\infty(\Omega)$. In this section we could simplify the presentation by considering first $b_i = c = 0$ and then recovering the general case by a perturbation argument. However, we discuss the duality theory for general $b_i,\ c$ because we shall use it in Section 3. Let us start from the elliptic problem

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \end{cases} \tag{2.1}$$

and let us temporarily assume some more regularity on the coefficients $b_i$, namely

$$a_{ij} = a_{ji}, \quad a_{ij},\, b_i \in W^{1,\infty}(\Omega), \qquad c \in L^\infty(\Omega), \tag{2.2}$$

setting

$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{1,\infty}(\Omega)}, \|b_i\|_{W^{1,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}.$$

The hypothesis $b \in W^{1,\infty}$ will be removed in Theorem 3.6 by a perturbation argument. Following the notation in [2], we consider the formally adjoint operators $\mathcal{A}^*$ and $\mathcal{B}^*$ given by

$$\mathcal{A}^* = \sum_{i,j=1}^n D_j(a^*_{ij} D_i) + \sum_{j=1}^n b^*_j D_j + c^*, \tag{2.3}$$

with

$$a^*_{ij} = a_{ij} \quad b^*_i = -b_i \quad c^* = c - \operatorname{div} b,$$

and

$$\mathcal{B}^* = \langle \beta^*, D\rangle + \gamma^*. \tag{2.4}$$

Setting

$$\nu_a := a \cdot \nu = (\sum_{j=1}^n a_{ij}\nu_j)_{i=1\ldots,n} \qquad \rho(x) := \frac{\langle \nu_a(x), \nu(x)\rangle}{\langle \beta(x), \nu(x)\rangle}, \qquad \tau := \nu_a - \rho\beta,$$

the coefficients $\beta^*,\ \gamma^*$ are given by

$$\rho\beta^* := \nu_a + \tau, \qquad \rho\gamma^* := \rho\gamma - \langle b, \nu\rangle + \operatorname{div}_{\partial\Omega}\tau,$$

where $\operatorname{div}_{\partial\Omega}$ denotes the tangential divergence along $\partial\Omega$. Then by the divergence theorem we obtain

$$\int_\Omega v\mathcal{A}u\, dx = \int_\Omega u\mathcal{A}^* v\, dx + \int_{\partial\Omega} \rho(v\mathcal{B}u - u\mathcal{B}^* v)\ d\mathcal{H}^{n-1}$$

for every $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

**Remark 2.1.** The $C^3$ regularity hypothesis on $\partial\Omega$ ensures that the function $\tau$, and then $\beta^*$, is uniformly $C^1$. If $\beta = a \cdot \nu$ and $\gamma = 0$, we get the *conormal* boundary conditions, and in this case the (uniform) $C^2$ regularity of $\partial\Omega$ is sufficient for our purposes because $\tau = 0$. This is the case in [3], [4], [5].

Assumption (2.2) is needed to apply the classical $L^\infty$ theory also to $\mathcal{A}^*$, but (1.2) is sufficient to guarantee generation results for the realization of $(\mathcal{A}, \mathcal{B})$ in $L^p$ with $1 < p \leq \infty$. For every $1 < p < \infty$ the operator $A_p^B : D(A_p^B) \subseteq L^p(\Omega) \to L^p(\Omega)$ defined by

$$\begin{cases} D(A_p^B) = \{u \in W^{2,p}(\Omega),\ \mathcal{B}u = 0 \text{ in } \partial\Omega\} \\ A_p^B u = \mathcal{A}u \end{cases}$$

is the infinitesimal generator of an analytic semigroup in $L^p(\Omega)$, see [12, Chapter 3]. In particular, there exists $\omega \in \mathbf{R}$ depending on $n, p, \mu, \Omega$ such that for each $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda \geq \omega$ and for each $f \in L^p(\Omega)$ the equation

$$(\lambda - \mathcal{A})u = f$$

has a unique solution $u \in D(A_p^B)$ satisfying

$$|\lambda| \|u\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|Du\|_{L^p(\Omega)} + \|D^2 u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \tag{2.5}$$

Moreover, let $A_\infty^B : D(A_\infty^B) \subseteq L^\infty(\Omega) \to L^\infty(\Omega)$ be the operator defined by

$$\begin{cases} D(A_\infty^B) = \{u \in \bigcap_{p \geq 1} W^{2,p}_{loc}(\Omega); \quad u, \mathcal{A}u \in L^\infty(\Omega),\ \mathcal{B}u_{|\partial\Omega} = 0\}, \\ A_\infty^B u = \mathcal{A}u. \end{cases}$$

Let us present the $L^\infty$ results that will be exploited later, see [18], [19], and also [12].

**Theorem 2.2.** *The following hold:*

(*i*) *There exist $\omega_0$, $M_0 \in \mathbf{R}$ and $\varphi_0 \in (\pi/2, \pi)$ such that the sector*

$$\Sigma_{\varphi_0} = \{\lambda \in \mathbf{C} :\ |arg(\lambda - \omega_0)| < \varphi_0\}$$

*belongs to the resolvent set of $A_\infty^B$. Moreover for each $\lambda \in \Sigma_{\varphi_0}$ we have*

$$|\lambda - \omega_0| \|R(\lambda, A_\infty^B)\|_{\mathcal{L}(L^\infty(\Omega))} \leq M_0$$

*where $R(\lambda, A_\infty^B) = (\lambda - A_\infty^B)^{-1}$.*

(*ii*) *There exist $\omega_0' \geq \omega_0$, $M \geq M_0$ and $\varphi \in (\pi/2, \varphi_0)$ such that for each $\lambda$ verifying $|arg(\lambda - \omega_0')| < \varphi$ we have*

$$|\lambda - \omega_0'|^{1/2} \|DR(\lambda, A_\infty^B)\|_{\mathcal{L}(L^\infty(\Omega))} \leq M.$$

In order to deduce a result of generation in $L^1(\Omega)$ we set

$$D_{\mathcal{A}} = \{u \in L^1(\Omega) \cap C^2(\bar{\Omega});\ \mathcal{A}u \in L^1(\Omega), \mathcal{B}u = 0 \text{ in } \partial\Omega\}.$$

**Lemma 2.3.** *$\mathcal{A} : D_{\mathcal{A}} \subset L^1(\Omega) \to L^1(\Omega)$ is closable in $L^1(\Omega)$.*

PROOF. Let $(u_j)$ be a sequence in $D_{\mathcal{A}}$ such that $u_j \to 0$ and $\mathcal{A}u_j \to v$ in $L^1(\Omega)$. Then, integrating by parts,

$$\int_\Omega \varphi\, v\, dx = \lim_{j\to\infty} \int_\Omega \varphi \mathcal{A} u_j\, dx = \lim_{j\to\infty} \int_\Omega u_j \mathcal{A}^* \varphi\, dx = 0$$

for every $\varphi \in C_c^\infty(\Omega)$. Hence $v = 0$, which implies the assertion. □

By Lemma 2.3 we can define the realization of $\mathcal{A}$ in $L^1$ with boundary condition $\mathcal{B}u = 0$, that will be denoted by $(A_1, D(A_1))$ as the closure of $\mathcal{A}_{|D_{\mathcal{A}}}$ in $L^1(\Omega)$, that is, the smallest closed extension of $\mathcal{A}_{|D_{\mathcal{A}}}$ in $L^1(\Omega)$. Then $D(A_1)$ is the closure of $D_{\mathcal{A}}$ with respect to the graph norm in $L^1$. Now we are in a position to prove the following result.

**Theorem 2.4.** *Under the assumption* (2.2) *there exist $C > 0$ and $\omega_1 \in \mathbf{R}$, depending on $n, \mu, M_1$ and $\Omega$, such that for $\operatorname{Re}\lambda \geq \omega_1$ and $f \in L^1(\Omega)$ the equation*

$$\lambda u - \mathcal{A}u = f \quad \text{in } \Omega \tag{2.6}$$

*has a unique solution $u \in D(A_1)$ and*

$$|\lambda| \|u\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega)}. \tag{2.7}$$

PROOF. First of all we prove that the range of $(\lambda - A_1)$ contains the space of functions $L_c^\infty(\Omega) = \{\psi \in L^\infty(\Omega); \operatorname{supp}\psi \subset\subset \Omega\}$ which is dense in $L^1(\Omega)$.
Indeed, let $\pi \in C^2(\Omega)$ be such that

$$\begin{cases} \sum_{i,j=1}^n |D_{ij}\pi| + \sum_{i=1}^n |D_i\pi|^2 \leq c \\ e^{-\pi} \in L^1(\Omega) \\ \sum_i \beta_i D_i \pi = 0 \qquad \text{in } \partial\Omega \end{cases}$$

If $\Omega$ is bounded we don't need to consider compactly supported functions and we can simply take $\pi = 0$. If $\Omega$ is unbounded, we require that $\lim_{|x|\to\infty, x\in\Omega} \pi(x) = +\infty$. Such a $\pi$ exists, for instance, when $\Omega = \mathbf{R}^n$ one can choose $\pi(x) = \sqrt{1+|x|^2}$. In the general case one can adapt the previous example modifying $\pi$ near the boundary in a suitable way. Starting e.g. from $\pi(x) = \sqrt{1+|x|^2}$ in $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \delta\}$, and fixing $y \in \partial\Omega$, consider the line $\mathbf{R} \ni t \to y + t\beta(y)$. Call $x$ the nearest point belonging to $\partial\Omega_\delta$ that intersects this line (this point exists thanks to the non-tangentiality condition (1.5)) and extend the function $\pi$ on the segmnent $[x, y]$ by assigning the constant value $\pi(x)$. The final step consists in regularizing the function obtained in this way. Once $\pi$ has been constructed, let us define $\Pi(x) = \exp[\pi(x)]$. Then, for every function $\psi \in L_c^\infty(\Omega)$, we get $\Pi\psi \in L_c^\infty(\Omega)$ and

$$\begin{cases} \lambda u - \mathcal{A}u = \psi \in L_c^\infty(\Omega) \\ \mathcal{B}u = 0 \quad \text{in } \partial\Omega \end{cases}$$

if and only if

$$\begin{cases} \lambda \Pi u - \mathcal{A}_\pi(\Pi u) = \Pi\psi \in L_c^\infty(\Omega) \\ \mathcal{B}(\Pi u) = 0 \quad \text{in } \partial\Omega \end{cases} \tag{2.8}$$

where

$$\mathcal{A}_\pi = \mathcal{A} - 2\sum_{i,j=1}^n a_{ij} D_i \pi D_j + \sum_{i,j=1}^n \Big[ D_i(a_{ij} D_j \pi) - a_{ij} D_i \pi D_j \pi \Big] + \sum_{i=1}^n b_i D_i \pi.$$

As it is easily seen, the operator $\mathcal{A}_\pi$ satisfies the assumptions (1.2)-(1.3), therefore, by applying Theorem 2.2 we get that there exists $\Pi u \in D((A_\pi)^B_\infty) \subseteq L^\infty(\Omega)$ solution of (2.8). Hence $u \in \{v \in C^1(\bar{\Omega}) \cap L^1(\Omega); \mathcal{A}v \in L^1(\Omega)\}$, then $\psi$ is in the range of $(\lambda - A_1)$ and consequently (2.6) has a solution for every $f \in L^1(\Omega)$. Now we prove (2.7). Consider a solution $u$ of $\lambda u - \mathcal{A}u = f \in L^1(\Omega)$ and let

$$\mathcal{A}^* = \sum_{i,j=1}^n D_j(a_{ij}D_i) - \sum_{j=1}^n b_j D_j + (c - \operatorname{div} b)$$

Then, from Theorem 2.2, it follows that ${A^*}^{B^*}_\infty$ generates an analytic semigroup in $L^\infty(\Omega)$ and so the elliptic problem

$$\begin{cases} \lambda w - \mathcal{A}^* w = \varphi \in L^\infty(\Omega) \\ \mathcal{B}^* w = 0 \quad \text{ in } \partial\Omega \end{cases} \tag{2.9}$$

for Re $\lambda$ sufficiently large has a unique solution $w \in D({A^*}^{B^*}_\infty)$ satisfying

$$|\lambda| \|w\|_{L^\infty(\Omega)} \leq \tilde{K} \|\varphi\|_{L^\infty(\Omega)}.$$

Now we can apply the method used in [14] to obtain

$$\begin{aligned} \|u\|_{L^1(\Omega)} &= \sup\left\{ \int u(x)\varphi(x)dx;\ \varphi \in L^\infty_c(\Omega),\ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\} \\ &\leq \sup\left\{ \int u(x)(\lambda - \mathcal{A}^*)w_\varphi dx;\ w_\varphi \in L^\infty(\Omega) \text{ solution of (2.9)},\ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\} \\ &\leq \sup\left\{ \int w_\varphi(\lambda - \mathcal{A})u dx;\ w_\varphi \in L^\infty(\Omega) \text{ solution of (2.9)},\ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\} \end{aligned}$$

in particular,

$$\|u\|_{L^1(\Omega)} \leq \tilde{K}|\lambda|^{-1}\|f\|_{L^1(\Omega)}.$$

So, $(\lambda - A_1)$ is an injective operator with closed range in $L^1(\Omega)$ and the proof is complete. □

As a consequence of the previous theorem $(A_1, D(A_1))$ generates an analytic semigroup in $L^1(\Omega)$ if we assume (2.2). The following result is a consequence of [12, Proposition 2.1.11].

**Proposition 2.5.** *Under assumption* (2.2) *there exist* $K, \omega_1 \in \mathbf{R}$ *and* $\theta_1 \in (\pi/2, \pi)$ *such that*

$$\Sigma_{\theta_1,\omega_1} = \{\lambda \in \mathbf{C};\ \lambda \neq \omega_1, |\arg(\lambda - \omega_1)| < \theta_1\} \subset \rho(A_1)$$

*and*

$$\|R(\lambda, A_1)\|_{\mathcal{L}(L^1(\Omega))} \leq \frac{K}{|\lambda - \omega_1|}$$

*holds for each* $\lambda \in \Sigma_{\theta_1,\omega_1}$. *Therefore* $(A_1, D(A_1))$ *generates an analytic semigroup in* $L^1(\Omega)$.

# 3 The generation result and gradient estimates in $L^1(\Omega)$

In this section, assuming (2.2), we prove gradient estimates for the solution of the resolvent equation (2.1) and the semigroup solution of problem (1.6). Finally, using these estimates and a perturbation result, we show the generation result in $L^1(\Omega)$ under the weaker assumption (1.2). Following, with significant modifications, ideas from [22], [23], [13], the first step is to get suitable estimates for the weak solution of elliptic boundary value problems in some negative Sobolev spaces. For $1 < p \le \infty$, we denote by $p'$ the conjugate exponent of $p$ (we set $\infty' = 1$) and consider the Banach space $W_*^{-1,p}(\Omega)$, defined as the dual of $W^{1,p'}(\Omega)$. This space can be defined as the completion of $C^\infty(\overline{\Omega}) \cap L^p(\Omega)$ with respect to the norm

$$\|u\|_{W_*^{-1,p}(\Omega)} = \sup\Big\{ \int_\Omega uvdx : \ v \in C^\infty(\overline{\Omega}) \cap W^{1,p'}(\Omega), \ \|v\|_{W^{1,p'}(\Omega)} \le 1 \Big\}.$$

The action of an element $f \in W_*^{-1,p}(\Omega)$ on $u \in W^{1,p'}(\Omega)$ can be written as

$$\langle f, u\rangle = \int_\Omega f_0 u\, dx + \sum_{i=1}^n \int_\Omega f_i D_i u\, dx \qquad u \in W^{1,p'}(\Omega) \tag{3.1}$$

where $f_i \in L^p(\Omega)$, $i = 0, \dots, n$ and $f = f_0 + \sum_{i=1}^n D_i f_i$ in a distributional sense.
The norm can be expressed also as follows:

$$\|f\|_{W_*^{-1,p}(\Omega)} = \inf\left\{ \sum_{i=0}^n \|f_i\|_{L^p(\Omega)} : \ f_0, \dots, f_n \text{ as in (3.1)} \right\}.$$

In the following lemma we prove a property of this function space that extends analogous estimates proved in [23] for the norm of the dual space of $W_0^{1,p'}(\Omega)$.

**Lemma 3.1.** *For each $p > n$ there exist two constants $c_1, c_2$ such that for each $x_0 \in \overline{\Omega}$, $r > 0$ and $u \in L^p(\Omega)$ with support in $\Omega \cap B(x_0, r)$,*

$$\|u\|_{W_*^{-1,p}(\Omega)} \le c_1 r \|u\|_{L^p(\Omega)} \tag{3.2}$$

$$\|u\|_{W_*^{-1,\infty}(\Omega)} \le c_2 r^{1-n/p} \|u\|_{L^p(\Omega)} \tag{3.3}$$

Proof. Let $\varphi \in W^{1,p'}(\Omega)$ be such that $\|\varphi\|_{W^{1,p'}(\Omega)} \le 1$. Then by Sobolev embedding $\varphi \in L^q(\Omega)$ with $q = (np')/(n-p')$ and $\|\varphi\|_{L^q(\Omega)} \le c$ where $c$ depends only on $\Omega$. Hence

$$\|u\|_{W_*^{-1,p}(\Omega)} = \sup\left\{ \int_\Omega u\varphi\, dx\, ; \ \varphi \in W^{1,p'}(\Omega), \ \|\varphi\|_{W^{1,p'}(\Omega)} \le 1 \right\}$$

and, using Hölder's inequality, the following estimate holds

$$\int_\Omega u\varphi\, dx \le \|u\|_{L^{q'}(\Omega \cap B(x_0,r))} \|\varphi\|_{L^q(\Omega)} \le c\, r \|u\|_{L^p(\Omega)}$$

and (3.2) is proved. In a similar way one can prove (3.3). □

Now we consider the realization of $\mathcal{A}$ with homogeneous boundary condition given by $\mathcal{B}$ as in (1.4) in the Banach space $W_*^{-1,p}$, $1 < p < \infty$, so defined

$$E_p : D(E_p) \subset W_*^{-1,p}(\Omega) \to W_*^{-1,p}(\Omega)$$

$$E_p u := \mathcal{A} u \qquad u \in D(E_p).$$

where by $D(E_p)$ we mean the completion of the set $\{u \in W^{2,p}(\Omega) : \mathcal{B}u = 0\}$ with respect to the topology induced by the norm $\|u\| = \|u\|_{W_*^{-1,p}(\Omega)} + \|\mathcal{A}u\|_{W_*^{-1,p}(\Omega)}$. Analogously one could define $E_{p'}$ the realization of $\mathcal{A}^*$ with homogeneous boundary conditions given by $\mathcal{B}^*$ in $W_*^{-1,p'}$. We start with two technical results involving $L^p$ estimates that are true both for $E_p$ and $E_{p'}$ and that for simplicity are stated only in the first case.

**Theorem 3.2.** *For every $1 < p < \infty$ the operator $E_p$ is sectorial in $W_*^{-1,p}(\Omega)$. In particular there is a constant $\omega_p \in \mathbf{R}$ depending on $n, \mu, M_1, \Omega$ such that for each $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda > \omega_p$ and for each $f \in W_*^{-1,p}(\Omega)$ the solution $u \in D(E_p)$ of the equation $(\lambda - \mathcal{A})u = f$ satisfies*

$$|\lambda| \|u\|_{W_*^{-1,p}(\Omega)} + |\lambda|^{1/2} \|u\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \leq K_1 \|f\|_{W_*^{-1,p}(\Omega)} \tag{3.4}$$

*where $K_1 > 0$ is a constant independent of $\lambda$ and $f$.*

Proof. Let $f \in W_*^{-1,p}(\Omega)$. By [15, Corollary 2.2] we deduce that there is a weak solution $u \in L^p(\Omega)$ of the problem

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{in } \partial\Omega\,. \end{cases} \tag{3.5}$$

Actually more regularity for the solution $u$ can be deduced, in fact by [16, Theorem 3.1] we get that it belongs to $W^{1,p}(\Omega)$ and

$$\|u\|_{W^{1,p}(\Omega)} \leq C \|f\|_{W_*^{-1,p}(\Omega)}$$

for some positive constant $C$. Now, in order to deduce (3.4) we denote by $A_p^B$ the realization of $\mathcal{A}$ in $L^p$ with homogeneous boundary conditions $\mathcal{B}u = 0$ and analogously $A^*{}_{p'}^{B^*}$ the realization of $\mathcal{A}^*$ in $L^{p'}$ with homogeneous boundary conditions $\mathcal{B}^*u = 0$. We know that $D(A_p^B) = \{u \in W^{2,p}(\Omega) : \mathcal{B}u = 0 \text{ in } \partial\Omega\}$. Then for each $u \in D(A^*{}_{p'}^{B^*})$ and $v \in L^p(\Omega)$, we have $\langle A^*{}_{p'}^{B^*} u, v\rangle = \langle u, (A^*{}_{p'}^{B^*})^* v\rangle$. Here $(A^*{}_{p'}^{B^*})^*$ is the adjoint of $A^*{}_{p'}^{B^*}$ and belongs to $\mathcal{L}(L^p(\Omega), (D(A^*{}_{p'}^{B^*}))')$, where $(D(A^*{}_{p'}^{B^*}))'$ is the dual space of $D(A^*{}_{p'}^{B^*})$. Note that the restriction of $(A^*{}_{p'}^{B^*})^*$ to $D(A_p^B)$ coincides with $A_p^B$. Therefore, from the complex interpolation theory, we have that $(A^*{}_{p'}^{B^*})^*$ is a bounded linear operator from $[L^p(\Omega), D(A_p^B)]_{1/2}$ to $[(D(A^*{}_{p'}^{B^*}))', L^p(\Omega)]_{1/2}$ where $[\cdot,\cdot]_{1/2}$ is the complex interpolation space of order 1/2, (see [21] for the relevant definitions and results). Using [17, Theorem 4.1], which holds for domains with uniformly smooth boundary, we can characterize the complex interpolation spaces in the following way:

$$\begin{gathered} [L^p(\Omega), D(A_p^B)]_{1/2} = W^{1,p}(\Omega) \\ [(D(A^*{}_{p'}^{B^*}))', L^p(\Omega)]_{1/2} = \big([L^{p'}(\Omega), D(A^*{}_{p'}^{B^*})]_{1/2}\big)' = (W^{1,p'}(\Omega))' = W_*^{-1,p}(\Omega) \end{gathered} \tag{3.6}$$

where in the second line in (3.6) we have used that under our assumptions $[X, Y]'_\theta = [Y', X']_{1-\theta}$, see [21, section 1.11.3]. Therefore the restriction of $(A^*{}_{p'}^{B^*})^*$ to the space $D(E_p)$ is a bounded linear operator from $D(E_p)$ to $W_*^{-1,p}(\Omega)$ and coincides with $E_p$. Now, since $A_p^B$ and $A^*{}_{p'}^{B^*}$ are sectorial operators, there exist $\lambda_1, \lambda_2 \in \mathbf{R}$ and $k_1, k_2 > 0$ such that

$$\|(\lambda - A_p^B)^{-1}\|_{\mathcal{L}(L^p, D(A_p^B))} \leq k_1 \quad \text{for } \mathrm{Re}\,\lambda > \lambda_1 \tag{3.7}$$

and analogously

$$\|(\lambda - A^{*B^*}_{p'})^{-1}\|_{\mathcal{L}(L^{p'},D(A^{*B^*}_{p'}))} \le k_2 \quad \text{for } \operatorname{Re}\lambda > \lambda_2. \tag{3.8}$$

Using (3.8) we get that

$$[(\lambda - A^{*B^*}_{p'})^{-1}]^* = [(\lambda - A^{*B^*}_{p'})^*]^{-1} \in \mathcal{L}((D(A^{*B^*}_{p'}))', L^p),$$

hence an argument similar to the previous one yields that the operator $[(\lambda - A^{*B^*}_{p'})^{-1}]^*$ belongs to $\mathcal{L}(W^{-1,p}_*(\Omega), D(E_p))$ and coincides with $(\lambda - E_p)^{-1}$.

Set $K = k_1 + k_2$ and $\omega_p > \max\{\lambda_1, \lambda_2\}$; then, for every $\lambda$ with $\operatorname{Re}\lambda > \omega_p$ and for every $f \in W^{-1,p}_*(\Omega)$ we have that $\|u\|_{W^{1,p}(\Omega)} \le K\|f\|_{W^{-1,p}_*(\Omega)}$ where $u = (\lambda - E_p)^{-1}f$. Then, for every $v \in W^{1,p'}(\Omega)$,

$$\langle f, v\rangle = \lambda\langle u, v\rangle - \langle E_p u, v\rangle.$$

Thus

$$\begin{aligned}
|\langle u, v\rangle| &\le |\lambda|^{-1}\left(|\langle E_p u, v\rangle| + |\langle f, v\rangle|\right)\\
&\le c|\lambda|^{-1}\left(\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)} + \|f\|_{W^{-1,p}_*(\Omega)}\|v\|_{W^{1,p'}(\Omega)}\right)\\
&\le c|\lambda|^{-1}\left(\|f\|_{W^{-1,p}_*(\Omega)}\|v\|_{W^{1,p'}(\Omega)}\right)
\end{aligned}$$

where we have used that $\|E_p u\|_{W^{-1,p}_*(\Omega)} \le c\|u\|_{W^{1,p}(\Omega)}$ since $(A^{*B^*}_{p'})^*$ is a bounded linear operator from $W^{1,p}(\Omega)$ to $W^{-1,p}_*(\Omega)$ and its restriction to $D(E_p)$ coincides with $E_p$.
Hence we have proved that

$$|\lambda|\|u\|_{W^{-1,p}_*(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \le c\|f\|_{W^{-1,p}_*(\Omega)}. \tag{3.9}$$

Therefore, (3.4) is consequence of (3.9), of the equality

$$(W^{-1,p}(\Omega), W^{1,p}(\Omega))_{1/2,p} = L^p(\Omega)$$

for $1 < p < \infty$ where $W^{-1,p}(\Omega)$ is the dual space of $W^{1,p'}_0(\Omega)$ (see [21, Section 2.4.2, Theorem 1; Section 4.2.1, Definition 1]) and of the continuous embedding $W^{-1,p}_*(\Omega) \hookrightarrow W^{-1,p}(\Omega)$. □

**Lemma 3.3.** *Let $p > n$ and $f \in W^{-1,p}_*(\Omega)$ with $f_i \in L^p(\Omega)$ as in* (3.1)*; then for each $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_p$, for each $x_0 \in \overline{\Omega}$ and for each $r > 0$, the solution $u \in D(E_p)$ of the equation $\lambda u - \mathcal{A}u = f$ satisfies the following estimate*

$$\|u\|_{W^{1,p}(B_r)} \le K_2\Big(\sum_{i=0}^{n}\|f_i\|_{L^p(B_{2r})} + r^{-1}\|u\|_{L^p(B_{2r})}\Big) \tag{3.10}$$

*where we have set $B_\varrho = B(x_0, \varrho) \cap \Omega$ and $K_2$ is a constant independent of $\lambda$ and $f$.*

Proof. For $x_0 \in \overline{\Omega}$ and $r < 1$ we consider $\theta \in C^2(\mathbf{R}^n)$ with $\theta(x) = 1$ for $|x - x_0| \le r$, $\theta(x) = 0$ for $|x - x_0| \ge 2r$, $|D\theta| \le cr^{-1}$, $|D^2\theta| \le cr^{-2}$ and $\sum_i \beta_i D_i\theta = 0$ on $\partial\Omega$. In this way the function $w := \theta u$ satisfies the problem

$$\begin{cases} \lambda w - \mathcal{A}w = g \\ \mathcal{B}w = 0 \quad \text{in } \partial\Omega \end{cases} \tag{3.11}$$

where

$$(3.12)\qquad g = \theta f - \sum_{i,j=1}^{n} u[D_i a_{ij} D_j\theta + a_{ij} D_{ij}\theta] - u\sum_{i=1}^{n} b_i D_i\theta - 2\sum_{i,j=1}^{n} a_{ij} D_j u D_i\theta.$$

By Theorem 3.2 applied to the function $\theta u$, we get

$$\begin{aligned}\|u\|_{W^{1,p}(B_r)} \le &\|\theta u\|_{W^{1,p}(B_{2r})} \le K_1 \|g\|_{W_*^{-1,p}(B_{2r})} \\ \le &K_1\Big[\sum_{i=0}^{n}\|f_i\|_{L^p(B_{2r})} + r^{-1}\Big(\sum_{i,j=1}^{n}\|a_{ij}\|_{W^{1,\infty}} + \sum_{i=1}^{n}\|b_i\|_{L^\infty}\Big)\|u\|_{L^p(B_{2r})} \\ &+ \sum_{i,j=1}^{n}\|a_{ij}D_j u D_i\theta\|_{W_*^{-1,p}(B_{2r})} + \sum_{i,j=1}^{n}\|u a_{ij} D_{ij}\theta\|_{W_*^{-1,p}(B_{2r})}\Big].\end{aligned}$$

By Lemma 3.1, we get

$$(3.13)\qquad \begin{aligned}\|a_{ij}D_j u D_i\theta\|_{W_*^{-1,p}(B_{2r})} &\le cr\|a_{ij}D_j u D_i\theta\|_{L^p(B_{2r})} \\ &\le c\|a_{ij}\|_\infty \|Du\|_{L^p(B_{2r})} \le c\|a_{ij}\|_\infty \sum_{i=0}^{n}\|f_i\|_{L^p(B_{2r})},\end{aligned}$$

$$(3.14)\qquad \|u a_{ij} D_{ij}\theta\|_{W_*^{-1,p}(B_{2r})} \le cr\|u a_{ij} D_{ij}\theta\|_{L^p(B_{2r})} \le cr^{-1}\|a_{ij}\|_\infty\|u\|_{L^p(B_{2r})},$$

where $c$ depends on $n, p, \Omega$ and may change from a line to the other. Summing up we find

$$\|\theta u\|_{W^{1,p}(B_{2r})} \le K_2\Big(\sum_{i=0}^{n}\|f_i\|_{L^p(B_{2r})} + r^{-1}\|u\|_{L^p(B_{2r}))}\Big).$$

Since $\theta u = u$ on $B_r$ we get the statement. □

The following estimate is proved by using a modification of Stewart's technique (see [18], [19]). It will be useful in order to obtain the estimate of the gradient of the solution of (2.6) in $L^1(\Omega)$.

**Theorem 3.4.** *Let $p > n$, $f \in W_*^{-1,\infty}(\Omega) \cap W_*^{-1,p}(\Omega)$; then, there exists $\omega_\infty > \omega_p$ such that for every $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega_\infty$ the solution $u \in D(E_p)$ of $\lambda u - \mathcal{A}u = f$ belongs to $W^{1,p}(\Omega)$ and satisfies*

$$(3.15)\qquad |\lambda|^{1/2}\|u\|_{L^\infty(\Omega)} \le K_3\|f\|_{W_*^{-1,\infty}(\Omega)},$$

*where $K_3$ is a constant independent of $\lambda, u$ and $f$.*

Proof. Let $x_0 \in \overline{\Omega}$, $r > 0$ to be fixed later, and let $\theta$ be a cut-off function as that considered in proof of Lemma 3.3: $\theta \in C^2(\mathbf{R}^n)$, $\theta(x) = 1$ on $B(x_0, r) \cap \Omega$, $\theta(x) = 0$ outside $B(x_0, 2r)$, $\sum_i \beta_i D_i\theta = 0$ on $\partial\Omega$ and with $\|D^\alpha\theta\|_{L^\infty(\Omega)} \le cr^{-|\alpha|}$ for each $|\alpha| \le 2$. As $f$ belongs to $W_*^{-1,\infty}(\Omega)$, it admits a distributional representation as in (3.1), with $f_i \in L^\infty(\Omega)$ for each $i = 0, 1, \ldots, n$ and

$$(3.16)\qquad \|f\|_{W_*^{-1,\infty}(\Omega)} \le \sum_{i=0}^{n}\|f_i\|_{L^\infty(\Omega)} \le 2\|f\|_{W_*^{-1,\infty}(\Omega)}.$$

Note that $u \in W^{1,p}(\Omega)$ for $p > n$ by Theorem 3.2, therefore $w = \theta u \in W^{1,p}(\Omega) \cap D(E_p)$ and solves

$$\lambda w - \mathcal{A} w = g,$$

where $g$ is defined in (3.12). By (3.13) and (3.14) we get (we confine to considering $r \le 1$ and set $B_\varrho = B(x_0, \varrho) \cap \Omega$)

$$\begin{aligned} \|g\|_{W_*^{-1,p}(\Omega)} &\le K_4 \left\{ \sum_{i=0}^{n} \|f_i\|_{L^p(B_{4r})} + r^{-1} \|u\|_{L^p(B_{4r})} \right\} \\ &\le K_5 r^{n/p} \left\{ \sum_{i=0}^{n} \|f_i\|_{L^\infty(\Omega)} + r^{-1} \|u\|_{L^\infty(\Omega)} \right\}, \end{aligned} \tag{3.17}$$

where $K_4$ and $K_5$ are constants independent of $r, \lambda, f$ and $u$. Since

$$W^{1,p}(B_{2r}) \hookrightarrow C^0(\overline{B}_{2r}) \hookrightarrow L^p(B_{2r})$$

for $p > n$ and the first injection is compact, then for each $\varepsilon > 0$ we get

$$\|\theta u\|_{L^\infty(\Omega)} \le \varepsilon r^{1-n/p} \|\theta u\|_{W^{1,p}(\Omega)} + c(\varepsilon) r^{-n/p} \|\theta u\|_{L^p(\Omega)}, \tag{3.18}$$

where $c(\varepsilon)$ is independent of $r$, $\lambda$, $u$ and $f$ (see Lemma 5.1 of [11]). Moreover, (3.3) and the Hölder inequality imply

$$\|\theta u\|_{W_*^{-1,\infty}(B_{2r})} \le c_2 r^{1-n/p} \|\theta u\|_{L^p(B_{2r})} \le c_2 r \|\theta u\|_{L^\infty(\Omega)}. \tag{3.19}$$

Therefore, from (3.18) and (3.19) we get

$$r^{-2} \|\theta u\|_{W_*^{-1,\infty}(\Omega)} + r^{-1} \|\theta u\|_{L^\infty(\Omega)} \le \varepsilon r^{-n/p} \|\theta u\|_{W^{1,p}(\Omega)} + c(\varepsilon) r^{-1-n/p} \|\theta u\|_{L^p(\Omega)}. \tag{3.20}$$

On the other hand, from Theorem 3.2

$$|\lambda| \|\theta u\|_{W_*^{-1,p}(\Omega)} + |\lambda|^{1/2} \|\theta u\|_{L^p(\Omega)} + \|\theta u\|_{W^{1,p}(\Omega)} \le K_1 \|g\|_{W_*^{-1,p}(\Omega)}. \tag{3.21}$$

Therefore, by (3.20), (3.21) and (3.17) we deduce

$$\begin{aligned} & r^{-2} \|\theta u\|_{W_*^{-1,\infty}(\Omega)} + r^{-1} \|\theta u\|_{L^\infty(\Omega)} \\ & \quad \le K_1 K_5 \big( \varepsilon + c(\varepsilon) r^{-1} |\lambda|^{-1/2} \big) \Big( r^{-1} \|u\|_{L^\infty(\Omega)} + \sum_{i=0}^{n} \|f_i\|_{L^\infty(\Omega)} \Big). \end{aligned}$$

Set $K_6 = 4K_1K_5$ and choose $\omega_\infty \ge \omega_p$ and $\varepsilon = K_6^{-1}$, $r = K_6 c(K_6^{-1}) |\lambda|^{-1/2} = K_7 |\lambda|^{-1/2}$. Then, if $x_0$ is a maximum point for the function $|u|$ (which exists because $u \in W^{1,p}(\Omega)$ with $p > n$ implies that $u$ is continuous and vanishes at infinity by the Sobolev embedding if $\Omega$ is unbounded) using (3.16) we obtain

$$K_7^{-2} |\lambda| \|\theta u\|_{W_*^{-1,\infty}(\Omega)} + \frac{1}{2} K_7^{-1} |\lambda|^{1/2} \|u\|_{L^\infty(\Omega)} \le \frac{1}{2} \sum_{i=0}^{n} \|f_i\|_{L^\infty(\Omega)} \le \|f\|_{W_*^{-1,\infty}(\Omega)}.$$

Thus (3.15) is proved. □

**Theorem 3.5.** *Under the assumptions of Theorem* 2.4*, there exist* $\omega_1' \geq \omega_1$*, depending on* $n, \mu, M_1$ *and* $\Omega$ *such that for every* $\lambda$ *with* $\mathrm{Re}\,\lambda > \omega_1'$ *the solution* $u \in D(A_1)$ *of equation* (2.6) *satisfies*

$$|\lambda|^{1/2}\|Du\|_{L^1(\Omega)} \leq K_3\|f\|_{L^1(\Omega)},$$

*where* $K_3$ *is the constant in Theorem* 3.4.

Proof. Let $\phi = \mathrm{div}\psi \in W_*^{-1,\infty}(\Omega) \cap W_*^{-1,p}(\Omega)$ for some $p > n$ as the datum $f$ in Theorem 3.4. By the estimate (3.15) we know that for $\lambda$ with $\mathrm{Re}\,\lambda > \omega_\infty$, the solution of the following problem

$$\begin{cases} \lambda v - \mathcal{A}^* v = \phi \\ \mathcal{B}^* v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.22}$$

satisfies

$$|\lambda|^{1/2}\|v\|_{L^\infty(\Omega)} \leq K_3\|\phi\|_{W_*^{-1,\infty}(\Omega)}. \tag{3.23}$$

We notice that

$$\|\phi\|_{W_*^{-1,\infty}} = \|\mathrm{div}\psi\|_{W_*^{-1,\infty}} = \sup\{\langle \mathrm{div}\psi, \varphi\rangle :\ \varphi \in W^{1,1}(\Omega),\ \|\varphi\|_{W^{1,1}(\Omega)} \leq 1\} \leq \|\psi\|_{L^\infty}.$$

Now, if $u \in D(A_1)$ is the solution of (2.6), we get

$$\begin{aligned}
\|Du\|_{L^1(\Omega)} &= \sup\left\{\int_\Omega \langle Du(x), \psi(x)\rangle dx :\ \psi \in C_c^\infty(\Omega;\mathbf{R}^n),\ \|\psi\|_{L^\infty(\Omega)} \leq 1\right\}\\
&= \sup\left\{\int_\Omega u(x)\,\mathrm{div}\psi(x) dx :\ \psi \in C_c^\infty(\Omega;\mathbf{R}^n),\ \|\psi\|_{L^\infty(\Omega)} \leq 1\right\}\\
&\leq \sup\left\{\int_\Omega u(x)\,\mathrm{div}\psi(x) dx : \psi \in C_c^\infty(\Omega;\mathbf{R}^n),\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\}\\
&= \sup\left\{\int_\Omega u\,(\lambda - \mathcal{A}^*)\,v_\psi\, dx :\ v_\psi \text{ solution of (3.22)},\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\}\\
&= \sup\left\{\int_\Omega [(\lambda - \mathcal{A})u]\,v_\psi\, dx :\ v_\psi \text{ solution of (3.22)},\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\}\\
&\leq \sup\left\{\|f\|_{L^1(\Omega)}\|v_\psi\|_{L^\infty(\Omega)} :\ v_\psi \text{ solution of (3.22)},\ \|\mathrm{div}\psi\|_{W_*^{-1,\infty}(\Omega)} \leq 1\right\}. \qquad (3.24)
\end{aligned}$$

Now, taking into account (3.23), we get

$$\|Du\|_{L^1(\Omega)} \leq K_3|\lambda|^{-1/2}\|f\|_{L^1(\Omega)},$$

for $\mathrm{Re}\,\lambda > \max\{\omega_1, \omega_\infty\}$. □

**Theorem 3.6.** *Under the hypotheses* (1.2)*, the operator* $(A_1, D(A_1))$ *generates an analytic semigroup in* $L^1(\Omega)$*.*

Proof. Suppose $\omega_1 = 0$, otherwise consider $A_1 - \omega_1$. Assume first that $b_i = 0$, so that (2.2) holds. Consider the first order perturbation $\mathcal{C} = \sum_{i=1}^n b_i D_i$ with $b_i \in L^\infty(\Omega)$. Let $C_1$ be the realization of $\mathcal{C}$ in $L^1(\Omega)$ with domain $D(C_1) = W^{1,1}(\Omega)$. The operator $C_1$ is $A_1$-bounded with $A_1$ bound 0 (we refer to [8, Chapter III] for the relevant definitions), i.e., for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\|C_1 u\|_{L^1(\Omega)} \leq \varepsilon\|A_1 u\|_{L^1(\Omega)} + c(\varepsilon)\|u\|_{L^1(\Omega)}$$

holds for every $u \in D(A_1)$. Indeed let $u \in D(A_1)$, then $u = R(\lambda, A_1)f$ for every $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda > 0$ and $f \in L^1(\Omega)$. Thus, for $\lambda > 0$ and using Theorem 3.5, we get

$$\begin{aligned} \|DR(\lambda, A_1)f\|_{L^1(\Omega)} &\le c\frac{1}{\sqrt{\lambda}}\|f\|_{L^1(\Omega)} = c\frac{1}{\sqrt{\lambda}}\|\lambda u - A_1 u\|_{L^1(\Omega)} \\ &\le c\Big[\sqrt{\lambda}\|u\|_{L^1(\Omega)} + \frac{1}{\sqrt{\lambda}}\|A_1 u\|_{L^1(\Omega)}\Big]. \end{aligned}$$

This implies that $D(A_1) \hookrightarrow W^{1,1}(\Omega)$. By minimizing over $\lambda > 0$, we get $\|Du\|_{L^1(\Omega)} \le c\|u\|^{1/2}_{L^1(\Omega)}\|A_1 u\|^{1/2}_{L^1(\Omega)}$ from which we derive

$$\|Du\|_{L^1(\Omega)} \le \varepsilon\|A_1 u\|_{L^1(\Omega)} + \frac{c}{\varepsilon}\|u\|_{L^1(\Omega)} \tag{3.25}$$

and by [8, Theorem III.2.10] we conclude. Moreover from the classical theory of semigroups, there exist $c_i = c_i(\Omega, \mu, M_1)$, $i = 0, 1$ such that

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_0, \quad \text{and} \quad t\|A_1 T(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_1, \qquad t > 0. \tag{3.26}$$

In the general case estimates (3.26) become

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_0 e^{\omega_1 t}, \quad \text{and} \quad t\|A_1 T(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_1 e^{\omega_1 t}, \qquad t > 0. \tag{3.27}$$

Finally, since $D(A_1)$ is dense in $L^1(\Omega)$ by construction, $T(t)$ is strongly continuous in $L^1(\Omega)$. □

The following estimate for the gradient of the semigroup follows by a standard argument, we present the proof for completeness.

**Proposition 3.7.** *Let $T(t)$ be the semigroup generated by $(A_1, D(A_1))$ and assume (1.2). Then, there exists $c_2$ depending on $\Omega, \mu, M_1$ such that for $t > 0$,*

$$t^{1/2}\|DT(t)\|_{\mathcal{L}(L^1(\Omega))} \le c_2 e^{\omega_1 t}. \tag{3.28}$$

PROOF. Suppose first $\omega_1 = 0$, and let $S(t)$ be the semigroup generated by $\tilde{A}_1 = A_1 - C_1$ in $L^1(\Omega)$ where $\mathcal{C} = \sum_{i=1}^n b_i D_i$. Using (3.25) with $S(t)u$ in place of $u$ and $\varepsilon = \sqrt{t}$, and (3.26), we get

$$\|DS(t)u\|_{L^1(\Omega)} \le \sqrt{t}\|\tilde{A}_1 S(t)u\|_{L^1(\Omega)} + \frac{c}{\sqrt{t}}\|S(t)u\|_{L^1(\Omega)} \le \frac{c_2}{\sqrt{t}}\|u\|_{L^1(\Omega)}$$

for every $u \in L^1(\Omega)$. Now, let $T(t)$ be the semigroup generated by $A_1$ in $L^1(\Omega)$ and take $u \in L^1(\Omega)$. For every $\varepsilon > 0$, set $u_\varepsilon = T(\varepsilon)u \in W^{1,1}(\Omega)$, so that

$$T(t)u_\varepsilon = S(t)u_\varepsilon + \int_\varepsilon^t S(t-s)C_1 T(s)u_\varepsilon ds.$$

Therefore, we may differentiate under the integral sign and get

$$DT(t)u_\varepsilon = DS(t)u_\varepsilon + \int_\varepsilon^t DS(t-s)C_1 T(s)u_\varepsilon ds,$$

whence

$$\|DT(t)u_\varepsilon\|_{L^1(\Omega)} \leq \left(\frac{c_2}{\sqrt{t}}\|u_\varepsilon\|_{L^1(\Omega)} + \|b\|_{L^\infty(\Omega)}\int_\varepsilon^t \frac{1}{\sqrt{t-s}}\|DT(s)u_\varepsilon\|_{L^1(\Omega)}ds\right).$$

By using Gronwall's generalized inequality (see for instance [10, Lemma 7.1.1]) we deduce

$$\|DT(t)u_\varepsilon\|_{L^1(\Omega)} \leq c\frac{1}{\sqrt{t}}\|u_\varepsilon\|_{L^1(\Omega)}$$

and then

$$\|DT(t)u\|_{L^1(\Omega)} \leq \liminf_{\varepsilon\to 0}\|DT(t)u_\varepsilon\|_{L^1(\Omega)} \leq \liminf_{\varepsilon\to 0} c\frac{1}{\sqrt{t}}\|u_\varepsilon\|_{L^1(\Omega)} = c\frac{1}{\sqrt{t}}\|u\|_{L^1(\Omega)}.$$

Finally, if $\omega_1 \neq 0$ (3.28) follows by applying estimates (3.27) instead of (3.26). □

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