A GENERAL FORMULA FOR THE ANISOTROPIC OUTER MINKOWSKI CONTENT OF A SET

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Abstract. We generalize to the anisotropic case some classical and recent results on the $(n-1)$-Minkowski content of rectifiable sets in $\mathbb{R}^n$, and on the outer Minkowski content of subsets of $\mathbb{R}^n$. In particular, a general formula for the anisotropic outer Minkowski content is provided and applies to a wide class of sets, stable under finite unions.

Keywords: Minkowski content, outer Minkowski content, anisotropy, rectifiability.

2010 Mathematics Subject Classification: 28A75, 49Q15, 52A39.

1. Introduction

The notion of Minkowski content has been introduced by H. Minkowski in order to study an intrinsic definition of k-dimensional area of a compact set. Precisely, if $S \subset \mathbb{R}^n$ is a closed set, the $(n-1)$-dimensional Minkowski content of S is defined by

$$\mathcal{M}^{n-1}(S) := \lim_{\varepsilon \to 0} \frac{|\{x \in \mathbb{R}^n : d(x,S) \leq \varepsilon\}|}{2\varepsilon} \tag{1.1}$$

whenever the limit on the right-hand side exists. A natural question arises: is it true that the $(n-1)$-dimensional Minkowski content coincides with the $(n-1)$-dimensional Hausdorff measure? A first result in this direction can be found in [6]: if the set S is compact and $(n-1)$-rectifiable then $\mathcal{M}^{n-1}(S) = \mathcal{H}^{n-1}(S)$. A weaker version of the previous result can be found in [2]; it is suffiicent to have compactness, countable $\mathcal{H}^{n-1}$-rectifiability, and a further condition: $\eta(B_r(x)) \geq \gamma r^{n-1}$ holds true for all $x \in S$ and for all $r \in (0,1)$, for some $\gamma > 0$ and some Radon measure η on $\mathbb{R}^n$ absolutely continuous with respect to $\mathcal{H}^{n-1}$. More recently, in [1], motivated by problems arising from stochastic geometry, the notion of outer Minkowski content of a set has been introduced:

$$\mathcal{SM}(E) := \lim_{\varepsilon \to 0} \frac{|\{x \in \mathbb{R}^n : d(x,E) \leq \varepsilon\} \setminus E|}{\varepsilon}.$$

In [1] the authors investigated general conditions ensuring the existence of $\mathcal{SM}$: in particular they prove that $\mathcal{SM}(E)$ coincides with $\mathcal{P}(E)$, the perimeter of E, whenever E has finite perimeter and $\mathcal{M}^{n-1}(\partial E) = \mathcal{P}(E)$, being ∂E the topological boundary of E. A more general formula for the outer Minkowski content of a set has been investigated in [11]: more precisely, if E^0 denotes the set of points of density 0 for E,

$$\mathcal{SM}(E) = \mathcal{P}(E) + 2\mathcal{H}^{n-1}(E^0 \cap \partial E), \tag{1.2}$$

holds true for a suitable class of sets which is stable under finite unions, and such a stability is a feature particularly relevant in connection with applications to the study of some stochastic processes (see again [11] for further details). Very recently, an anisotropic variant of the outer Minkowski content of a set has been considered in [4], motivated also by the study of anisotropic perimeters arising from discrete ones (see [3]). The idea introduced in [4] is to look at the limit, whenever it exists, given by

$$\lim_{\varepsilon\to 0} \frac{|(E+\varepsilon C)| - |E|}{\varepsilon} =: \mathcal{SM}_C(E),$$

where $C \subset \mathbb{R}^n$ is a convex body. Contrarily with respect to the isotropic case, much less is known about the anisotropic situation. In [4] the authors prove that for sets with finite perimeter and such that $\mathcal{SM}(E) = \mathcal{P}(E)$, the anisotropic outer Minkowski content of E takes the form

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} \tag{1.3}$$

being h_C is the support function of C and $\mathcal{F}E$ the reduced boundary of E.

In this paper, first of all we want to find an integral formula also for the anisotropic Minkowski content of a set, and we are able to prove that under suitable conditions on S similar to what one has to assume for the isotropic case (see Theorems 3.4 and 3.7) the anisotropic Minkowski content of S takes the form

$$\frac{1}{2}\int_S (h_C(\nu_S) + h_C(-\nu_S))\, d\mathcal{H}^{n-1}.$$

Moreover, we will generalize also formula (1.2) to the anisotropic case: precisely (see Theorem 4.3), we will prove that

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} + \int_{E^0\cap\partial E} (h_C(\nu_E) + h_C(-\nu_E))\, d\mathcal{H}^{n-1}$$

holds true for a suitable class of sets stable under finite unions.

2. Notation and preliminaries

2.1. **Notation.** Let $n \geq 1$ be integer. Given a measurable set $A \subset \mathbb{R}^n$, we will denote by $|A|$ its Lebesgue measure. If $k \in \{0, \dots, n\}$, the k-dimensional Hausdorff measure of $S \subset \mathbb{R}^n$ will be denoted by $\mathcal{H}^k(S)$. We will use the notation $x \cdot y$ for the standard scalar product in $\mathbb{R}^n$ between x and y, $B_r(x)$ for the closed ball of radius r centered in x. For each $k \in \mathbb{N}$ with $k \leq n$ we denote by $\mathbf{G}_k$ the set of unoriented k-planes on $\mathbb{R}^n$; for any $\pi \in \mathbf{G}_k$ we denote by $\pi^\perp \in \mathbf{G}_{n-k}$ the $(n-k)$-plane orthogonal to π. Finally, if μ is a positive, real or vector measure on some space X and $f\colon X \to Y$ is measurable, we define the measure $f_\sharp\mu$ on Y as $f_\sharp\mu(F) := \mu(f^{-1}(F))$ for any F measurable in Y.

2.2. **Geometric Measure Theory.** In this paragraph we recall some basic notions of geometric measure theory we will need; for all details we refer the reader to [2], [6] and [10]. Let $n \geq 1$ be integer and let $k \in \mathbb{N}$ with $k \leq n$. The following general property of Radon measures holds true; here ω_k is the volume of the k-dimensional unit ball.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let μ be a positive Radon measure on Ω. Then, for any $t > 0$ and for any B Borel set in Ω the following implications hold:*

$$\limsup_{\rho\to 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} \geq t \quad \forall x \in B \Rightarrow \mu \geq t\mathcal{H}^k \llcorner B,$$

$$\limsup_{\rho\to 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} \leq t \quad \forall x \in B \Rightarrow \mu \leq 2^k t\mathcal{H}^k \llcorner B.$$

A very useful consequence of Theorem 2.1 turns out to be the following fact:

$$B \text{ Borel in } \Omega \text{ with } \mu(B) = 0 \Rightarrow \lim_{\rho\to 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} = 0, \text{ for } \mathcal{H}^k\text{-a.e. } x \in B. \tag{2.1}$$

Let now $S \subset \mathbb{R}^n$. We say that S is k-rectifiable if there exist a bounded set $B \subset \mathbb{R}^k$ and a Lipschitz function $f\colon B \to \mathbb{R}^n$ such that $S = f(B)$. We say that $S \subset \mathbb{R}^n$ is countably $\mathcal{H}^k$-rectifiable if there exist countably many Lipschitz functions $f_h\colon \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k\left(S \setminus \bigcup_{h=0}^{+\infty} f_h(\mathbb{R}^k)\right) = 0.$$

If in adjoint $\mathcal{H}^k(S) < +\infty$ then S is said to be $\mathcal{H}^k$-rectifiable. A classical rectifiability criterium says that a Borel set $S \subset \mathbb{R}^n$ with $\mathcal{H}^k(S) < +\infty$ is $\mathcal{H}^k$-rectifiable if and only

$$\lim_{\rho\to 0} \frac{\mathcal{H}^k(S \cap B_\rho(x))}{\omega_k \rho^k} = 1, \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in S. \tag{2.2}$$

It turns out that if S is countably $\mathcal{H}^k$-rectifiable then for $\mathcal{H}^k$-almost any point $x_0 \in S$ it is well defined the approximate tangent space $\mathrm{Tan}^k(S, x_0) \in \mathbf{G}_k$, that is

$$\lim_{\rho\to 0} \frac{1}{\rho^k} \int_S \phi\left(\frac{x - x_0}{\rho}\right) d\mathcal{H}^k(x) = \int_{\mathrm{Tan}^k(S,x_0)} \phi(y)\, d\mathcal{H}^k(y), \quad \forall \phi \in C_c^\infty(\mathbb{R}^n).$$

In particular, if $k = n-1$ then $\mathrm{Tan}^{n-1}(S, x_0)^\perp$ is generated by some unit vector denoted by ν_S. We recall that by Lipschitz k-graph we mean the graph of a Lipschitz function $\phi\colon \pi \to \pi^\perp$, where $\pi \in \mathbf{G}_k$. Given a countably $\mathcal{H}^k$-rectifiable set S, it is well known that S can be covered, up to a $\mathcal{H}^k$-negligible set, by a countable family of pairwise disjoint compact subsets of S which are contained in some Lipschitz k-graph and with finite k-dimensional Hausdorff measure. We now recall the notion of k-dimensional Jacobian and the area formula. Let $L\colon \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. The k-dimensional Jacobian of L is defined by $\mathbf{J}_k L := \sqrt{\det(L^* \circ L)}$ where

$L^*\colon \mathbb{R}^n \to \mathbb{R}^k$ denotes the transpose of L. The Jacobian is related, as it is well known in the smooth case, to the change of variable formula for multiple integrals: more precisely, if $f\colon \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz then for any measurable set $E \subset \mathbb{R}^k$ the multiplicity function $y \mapsto \mathcal{H}^0(E \cap f^{-1}(\{y\}))$ is measurable and the area formula

$$\int_{\mathbb{R}^n} \sum_{x \in E \cap f^{-1}(\{y\})} g(x)\, d\mathcal{H}^k(y) = \int_E g(x)\, \mathbf{J}_k df_x\, dx \tag{2.3}$$

holds for each $g\colon E \to \mathbb{R}$ Borel. There is another useful formula, which is known as coarea formula: if Ω is open in $\mathbb{R}^n$, $f\colon \Omega \to \mathbb{R}$ is Lipschitz continuous and $g\colon \Omega \to [0, +\infty]$ is Borel then

$$\int_\Omega g(x)|\nabla f(x)|\, dx = \int_{-\infty}^{+\infty} \int_{\Omega \cap \{f = t\}} g(y)\, d\mathcal{H}^{n-1}(y)\, dt. \tag{2.4}$$

Let now $E \subset \mathbb{R}^n$ be a measurable set and $\Omega \subset \mathbb{R}^n$ be an open domain; we denote by χ_E the characteristic function of E. We say that E has finite perimeter in Ω if the distributional derivative of χ_E, denoted by $D\chi_E$, is a $\mathbb{R}^n$-valued Radon measure on Ω with finite total variation; the perimeter of E in Ω is defined by $\mathcal{P}(E;\Omega) := |D\chi_E|(\Omega)$, where $|D\chi_E|$ denotes the total variation of $D\chi_E$; we also let $\mathcal{P}(E) := \mathcal{P}(E;\mathbb{R}^n)$. For sufficiently smooth boundaries the perimeter coincides with the $(n-1)$-dimensional Hausdorff measure of the topological boundary. An interesting situation is the following one: given $f\colon A \to \mathbb{R}$ be a Lipschitz map, with A open and bounded in $\mathbb{R}^{n-1}$, the subgraph of f turns out to be a set with finite perimeter in $A \times \mathbb{R}$ and its perimeter coincides with the $(n-1)$-dimensional Hausdorff measure of the graph of f. The upper and lower n-dimensional densities of E at x are respectively defined by

$$\Theta_n^*(E,x) := \limsup_{\rho \to 0} \frac{|E \cap B_\rho(x)|}{\omega_n \rho^n}, \quad \Theta_{*n}(E,x) := \liminf_{\rho \to 0} \frac{|E \cap B_\rho(x)|}{\omega_n \rho^n}.$$

If $\Theta_n^*(E,x) = \Theta_{*n}(E,x)$ their common value is denoted by $\Theta_n(E,x)$. For every $t \in [0,1]$ we define $E^t := \{x \in \mathbb{R}^n : \Theta_n(E,x) = t\}$. The essential boundary of E is defined as $\partial^* E := \mathbb{R}^n \setminus (E^0 \cup E^1)$. It turns out that if E has finite perimeter in Ω, then $\mathcal{H}^{n-1}(\partial^* E \setminus E^{1/2}) = 0$, and $\mathcal{P}(E;\Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$. Moreover, one can define a subset of $E^{1/2}$ as the set of points x where there exists a unit vector $\nu_E(x)$ such that

$$\frac{E - x}{\rho} \to \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \le 0\}, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } \rho \to 0,$$

and which is referred to as the outer normal to E at x. The set where $\nu_E(x)$ exists is called the reduced boundary and is denoted by $\mathcal{F}E$. One can show that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0$, moreover, one has the decomposition $D\chi_E = (-\nu_E)\mathcal{H}^{n-1} \llcorner \mathcal{F}E$. We also introduce the set $\partial^2 E := \{x \in \partial E \cap E^0 : \exists \operatorname{Tan}^{n-1}(\partial E, x)\}$. Let us collect some elementary properties of sets with countably $\mathcal{H}^{n-1}$-rectifiable boundary and with finite perimeter in Ω; for any $E \subseteq \mathbb{R}^n$ we let $E^c := \mathbb{R}^n \setminus E$.

Lemma 2.2. *Assume that E has finite perimeter in Ω and ∂E is countably $\mathcal{H}^{n-1}$-rectifiable. Then the following holds:*

$$\mathcal{H}^{n-1}(\mathcal{F}E) = \mathcal{H}^{n-1}(\mathcal{F}E^c); \tag{2.5}$$

$$\mathcal{H}^{n-1}(\partial E \setminus \mathcal{F}E) = \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^c); \tag{2.6}$$

$$\nu_{E^c}(x) = -\nu_E(x), \quad \text{for any } x \in \mathcal{F}E; \tag{2.7}$$

$$\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial^2(\partial E)). \tag{2.8}$$

Proof. Properties (2.5), (2.7) and (2.8) are trivial. In order to prove (2.6) we notice that

$$\begin{aligned}\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial E^c) &= \mathcal{H}^{n-1}(\mathcal{F}E^c) + \mathcal{H}^{n-1}(\partial E^c \cap (E^c)^1) + \mathcal{H}^{n-1}(\partial E^c \cap (E^c)^0)\\ &= \mathcal{H}^{n-1}(\mathcal{F}E) + \mathcal{H}^{n-1}(\partial E \cap E^0) + \mathcal{H}^{n-1}(\partial^2 E^c)\\ &= \mathcal{H}^{n-1}(\mathcal{F}E) + \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^c)\end{aligned}$$

Therefore, $\mathcal{H}^{n-1}(\partial E \setminus \mathcal{F}E) = \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^c)$ which is (2.6). □

Remark 2.3. *Using Lemma 2.2, we may observe that if E is such that its topological boundary ∂E is a set countably $\mathcal{H}^{n-1}$-rectifiable and bounded, then one of the followings holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$:*

(a) *$x \in \mathcal{F}E$, and the outer normal $\nu_E(x)$ to E at x exists;*
(b) *$x \in E^1 \cap \partial E$ (in such a case no outer normal exists);*
(c) *$x \in \partial^2 E$, and two outer normals to E at x exist, say $\nu_E(x)$ and $-\nu_E(x)$.*

This is in accordance with already known results in literature for sets with positive reach (for instance, see [5]*); namely, it can be shown that the topological boundary of a compact subset E of $\mathbb{R}^n$ with positive reach is $(n-1)$-rectifiable, and that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$, either $x \in \mathcal{F}E$ or $x \in \partial^2 E$ (see also* [1]*).*

We recall now the well known Besicovitch's Theorem.

Theorem 2.4. *Let $A \subset \mathbb{R}^n$ be a bounded set and $\rho \colon A \to (0, +\infty)$ be a function. There exists a set $S \subset A$ at most countable such that*

$$A \subset \bigcup_{x \in S} B_{\rho(x)}(x).$$

Moreover, every point of $\mathbb{R}^n$ belongs at most to ξ balls $B_{\rho(x)}(x)$ centred in a point of S, where ξ is a constant depending only on n.

We will need also the following variant of the Vitali covering Theorem concerning Lebesgue measure; if $A \subset \mathbb{R}^n$ we say that $\mathcal{F}$ is a fine cover of A if for each $x \in A$ there exist balls in $\mathcal{F}$ centred in x and with arbitrarily small radii.

Theorem 2.5. *Let $A \subset \mathbb{R}^n$ be a bounded and Borel set and let $\mathcal{F}$ be a fine cover of A. Then for any positive Radon measure μ in $\mathbb{R}^n$ there exists a disjoint family $\mathcal{F}' \subset \mathcal{F}$ such that*

$$\mu\biggl(A \setminus \bigcup_{F \in \mathcal{F}'} F\biggr) = 0.$$

3. The anisotropic Minkowski content

Let $C \subset \mathbb{R}^n$ be a convex body, that is a compact and convex subset of $\mathbb{R}^n$ with 0 in its interior; here and in what follows $a, b \in \mathbb{R}$ are such that $0 < a < b$ and $B_a(0) \subset C \subset B_b(0)$. We denote by h_C its support function, i.e.

$$h_C(v) := \sup_{x \in C} x \cdot v, \quad v \in \mathbb{R}^n.$$

Define, for any $S \subset \mathbb{R}^n$ closed,

$$\mathcal{M}^*_C(S) := \limsup_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}, \quad \mathcal{M}_{*C}(S) := \liminf_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}.$$

If $\mathcal{M}^*_C(S) = \mathcal{M}_{*C}(S)$ their common value is denoted by $\mathcal{M}_C(S)$. As we have seen in the introduction, there exists a relation relation between $\mathcal{M}_{B_1(0)}(\partial E)$ and $\mathcal{SM}(E)$. To be more precise, let

$$\mathcal{SM}(E; \Omega) := \lim_{\varepsilon \to 0} \frac{|\{x \in \Omega : d(x, E) \le \varepsilon\} \setminus E|}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{|(E + \varepsilon B_1(0)) \setminus E|}{\varepsilon}$$

whenever such a limit exists; of course we have $\mathcal{SM}(E) = \mathcal{SM}(E; \mathbb{R}^n)$. Hence the following Theorem holds (see [1]).

Theorem 3.1. *If E has finite perimeter in Ω and $\mathcal{M}_{B_1(0)}(\partial E) = \mathcal{P}(E; \Omega)$ then $\mathcal{SM}(E; \Omega) = \mathcal{P}(E; \Omega)$.*

Moreover, we need to give the precise statement of the main result in [4]; for, let

$$\mathcal{SM}_C(E; \Omega) := \lim_{\varepsilon \to 0} \frac{|(E + \varepsilon C) \cap \Omega| - |E|}{\varepsilon}.$$

Notice that we have also in this case $\mathcal{SM}_C(E) = \mathcal{SM}_C(E; \mathbb{R}^n)$.

Theorem 3.2. *If E has finite perimeter in Ω and $\mathcal{SM}(E; \Omega) = \mathcal{P}(E; \Omega)$ then*

$$\mathcal{SM}_C(E; \Omega) = \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1}.$$

Lemma 3.3. *Let $A \subset \mathbb{R}^{n-1}$ be open and bounded and let $f \colon A \to \mathbb{R}$ be Lipschitz continuous. Let G be the graph of f, that is $G := \{(x, y) \in A \times \mathbb{R} : y = f(x)\}$. Then $\mathcal{M}_C(G)$ exists and*

$$\mathcal{M}_C(G) = \int_G \phi_C(\nu_G)\, d\mathcal{H}^{n-1}$$

being

$$\phi_C(v) := \frac{h_C(v) + h_C(-v)}{2}, \quad \forall v \in \mathbb{R}^n.$$

Proof. Let $E := \{(x, y) \in A \times \mathbb{R} : y < f(x)\}$ be the subgraph of f. Since f is Lipschitz, E has finite perimeter in $A \times \mathbb{R}$ and $\mathcal{M}^{n-1}(\partial E) = \mathcal{M}_{B_1(0)}(\partial E) = \mathcal{P}(E; A \times \mathbb{R})$. Applying Theorem 3.1 we deduce that also $\mathcal{SM}(E; \Omega) = \mathcal{P}(E; \Omega)$. Thus, using (2.7) and Theorem 3.2 we get

$$\begin{aligned}\mathcal{M}_C(G) &= \frac{\mathcal{SM}_C(E;\Omega) + \mathcal{SM}_C(\Omega \setminus E;\Omega)}{2} \\ &= \int_{\mathcal{F}E} \frac{h_C(\nu_E) + h_C(-\nu_E)}{2}\, d\mathcal{H}^{n-1} \\ &= \int_G \phi_C(\nu_G)\, d\mathcal{H}^{n-1}\end{aligned}$$

which yields the conclusion. □

We are ready to prove the first main Theorem of this section, which may be seen as the generalization to the anisotropic case of Theorem 2.104 in [2]. Of course, the next proofs owe a bit to ones contained in [2] for the classical Minkowski content, which are anyway not elementary, but for the convenience of the reader we will explain all details.

Theorem 3.4. *Let $S \subset \mathbb{R}^n$ be a compact and countably $\mathcal{H}^{n-1}$-rectifiable set such that*

$$\eta(B_r(x)) \geq \gamma r^{n-1}$$

holds for all $x \in S$ and for all $r \in (0, 1)$ for some $\gamma > 0$ and some Radon measure η on $\mathbb{R}^n$ absolutely continuous with respect to $\mathcal{H}^{n-1}$. Then $\mathcal{M}_C(S)$ exists and

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1}. \tag{3.1}$$

Proof. Let $\{S_h\}_{h\in\mathbb{N}}$ be a countable family of pairwise disjoint compact subsets of S which covers S, up to a $\mathcal{H}^{n-1}$-negligible set, which are contained in some Lipschitz $(n-1)$-graph and with finite $\mathcal{H}^{n-1}$ measure. Applying Lemma 3.3 and using the subadditivity of the lim inf operator we get, for any $N \in \mathbb{N}$,

$$\begin{aligned}\mathcal{M}_{*C}(S) \geq \liminf_{\varepsilon\to 0} \frac{\left|\bigcup_{h=1}^N (S_h + \varepsilon C)\right|}{2\varepsilon} &\geq \sum_{h=0}^N \liminf_{\varepsilon\to 0} \frac{|S_h + \varepsilon C|}{2\varepsilon} \\ &= \sum_{h=0}^N \mathcal{M}_{*C}(S_h) = \sum_{h=0}^N \int_{S_h} \phi_C(\nu_S)\, d\mathcal{H}^{n-1}.\end{aligned}$$

Passing to the limit as $N \to +\infty$ we find the estimate from below

$$\mathcal{M}_{*C}(S) \geq \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1}. \tag{3.2}$$

The main point of the proof concerns the proof of the estimate from above, that is

$$\mathcal{M}^*_C(S) \leq \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1}. \tag{3.3}$$

The idea is to use, in a suitable way, a covering argument to control what remains outside a region covered by a finite number of subsets of S where we can say that $\mathcal{M}_C$ exists by Lemma

3.3. Fix $\sigma\in(0,1)$. We can find a finite number N of pairwise disjoint compact subsets S_h of S which are contained in some Lipschitz $(n-1)$-graph, with finite $\mathcal{H}^{n-1}$-measure and such that

$$\eta(S)\le \sigma+\sum_{h=1}^N \eta(S_h).$$

Consider the set

$$E:=S\setminus\bigcup_{h=1}^N S_h$$

and, for any $\varepsilon\in(0,1)$,

$$S_{\sigma,\varepsilon}:=\left\{x\in S: d\left(x,\bigcup_{h=1}^N S_h\right)\ge \sigma^{1/n}\varepsilon\right\}.$$

Using Besicovitch's Theorem 2.4 we are able to cover $S_{\sigma,\varepsilon}$ by many balls $\{B_{a\sigma^{1/n}\varepsilon}(x_j)\}_{j\in J}$ (recall that a has been choosen in such a way that $B_a(0)\subset C$) with $x_j\in S_{\sigma,\varepsilon}$ for each $j\in J$, and such that, for ε small enough, using the assumption on η,

$$\sum_{j\in J}\gamma(a\sigma^{1/n}\varepsilon)^{n-1}\le\sum_{j\in J}\eta(B_{a\sigma^{1/n}\varepsilon}(x_j))\le \xi\eta\left((S+\sigma^{1/n}\varepsilon C)\setminus\bigcup_{h=1}^N S_h\right)\le\xi\sigma$$

where ξ is as in Theorem 2.4. As a consequence we get the estimate

$$\sharp J\le\frac{\xi\sigma^{1/n}}{\gamma a^{n-1}\varepsilon^{n-1}}.$$

Therefore, we obtain, recalling that $C\subset B_b(0)$,

$$|S_{\sigma,\varepsilon}+(1+\sigma^{1/n})\varepsilon C|\le\sum_{j\in J}|B_{b(1+2\sigma^{1/n})\varepsilon}(x_j)|\le\frac{\omega_n b^n(1+2\sigma^{1/n})^n\sigma^{1/n}\xi\varepsilon}{\gamma a^{n-1}}\le\frac{\omega_n b^n 3^n\sigma^{1/n}\xi\varepsilon}{\gamma a^{n-1}}.$$

Now, since it holds

$$S+\varepsilon C\subset(E+\varepsilon C)\cup\bigcup_{h=1}^N(S_h+\varepsilon C)\subset(S_{\sigma,\varepsilon}+(1+\sigma^{1/n})\varepsilon C)\cup\bigcup_{h=1}^N(S_h+\varepsilon C)$$

we deduce that, by Lemma 3.3,

$$\begin{aligned}\mathcal{M}^*_C(S)&=\limsup_{\varepsilon\to0}\frac{|S+\varepsilon C|}{2\varepsilon}\\&\le\limsup_{\varepsilon\to0}\frac{|S_{\sigma,\varepsilon}+(1+\sigma^{1/n})\varepsilon C|}{2\varepsilon}+\sum_{h=1}^N\limsup_{\varepsilon\to0}\frac{|S_h+\varepsilon C|}{2\varepsilon}\\&\le\frac{\omega_n b^n3^n\sigma^{1/n}\xi}{2\gamma a^{n-1}}+\int_S\phi_C(\nu_S)\,d\mathcal{H}^{n-1}\end{aligned}$$

and (3.3) follows sending $\sigma\to0$. □

We now move toward the case of the $(n-1)$-rectifiability, as the first existence result for the Minkowski content, which can be found in [6], pg. 275:

Theorem 3.5. *If S is compact and $(n-1)$-rectifiable then $\mathcal{M}_{B_1(0)}(S) = \mathcal{H}^{n-1}(S)$.*

The next Lemma can be found in [2], see Lemma 2.105.

Lemma 3.6. *Let $K \subset \mathbb{R}^{n-1}$ be a compact set and $f\colon K \to \mathbb{R}^n$ be Lipschitz. Assume that $\mathbf{J}_{n-1} df_x = 0$ a.e. $x \in K$. Then $\mathcal{M}_{B_1(0)}(f(K)) = 0$.*

Theorem 3.7. *Let $S \subset \mathbb{R}^n$ be compact and $(n-1)$-rectifiable. Then $\mathcal{M}_C(S)$ exists and*

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1}.$$

Proof. The estimate from below

$$\mathcal{M}_{*C}(S) \geq \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1}$$

can be proved as in Theorem 3.4. Let $f\colon K \to \mathbb{R}^n$ be Lipschitz with $K \subset \mathbb{R}^{n-1}$ compact such that $S = f(K)$ and fix $\sigma \in (0,1)$. Consider the subset of K given by

$$F := \{x \in K : df_x \text{ exists and } \mathbf{J}_{n-1} df_x > 0\}.$$

Let $K' \subset K \setminus F$ be compact and such that $\mathcal{L}^{n-1}(K \setminus (F \cup K')) < \sigma$, being $\mathcal{L}^{n-1}$ the Lebesgue measure of dimension $n-1$. Moreover, let $S_0 := f(K')$. Combining Theorem 3.5 with Lemma 3.6 we get $\mathcal{H}^{n-1}(S_0) = 0$ and thus we obtain

$$\mathcal{M}_C(S_0) \leq \limsup_{\varepsilon \to 0} \frac{|S_0 + \varepsilon B_b(0)|}{2\varepsilon} = b\mathcal{M}^{n-1}(S_0) = b\mathcal{H}^{n-1}(S_0) = 0$$

which means that $\mathcal{M}_C(S_0) = 0$. Now, consider the measure $\eta := f_\sharp(\mathcal{L}^{n-1} \llcorner F)$. By definition, η is concentrated on $f(K) = S$; moreover, if $S' \subset S$ is $\mathcal{H}^{n-1}$-negligible then, by area formula we deduce that

$$\int_{F \cap f^{-1}(S')} \mathbf{J}_{n-1} df_x\, dx = \int_{S'} \mathcal{H}^0(F \cap f^{-1}(\{y\}))\, d\mathcal{H}^{n-1}(y) = 0.$$

Since, by the very definition of F, it holds $\mathbf{J}_{n-1} df_x > 0$ on F, we get $\mathcal{L}^{n-1}(F \cap f^{-1}(S')) = 0$, which proves that η is absolutely continuous with respect to $\mathcal{H}^{n-1}$. Now we are ready to use the same covering argument as in the proof of Theorem 3.4 in order to control the "bad" part of S using the properties of the measure η. More precisely, we can find a finite number N of pairwise disjoint compact subsets S_h of S which are contained in some Lipschitz $(n-1)$-graph, with finite $\mathcal{H}^{n-1}$-measure and such that

$$\eta(S) \leq \sigma + \sum_{h=1}^{N} \eta(S_h).$$

Consider the set

$$E := S \setminus \bigcup_{h=0}^{N} S_h.$$

Notice now that $\eta(E) < \sigma$ and $f^{-1}(E) \setminus F \subset K \setminus (F \cup K')$ since $E \cap S_0 = \emptyset$; we deduce that $\mathcal{L}^{n-1}(f^{-1}(E)) < 2\sigma$. If now L denotes the Lipschitz constant of f and we choose $\bar{\varepsilon} > 0$ such that $\mathcal{L}^{n-1}((K + \bar{\varepsilon} L^{-1} C) \setminus K) < \sigma$, we can consider, for any $\varepsilon \in (0, \bar{\varepsilon})$, the set

$$S_{\sigma,\varepsilon} := \left\{ x \in S : d\left(x, \bigcup_{h=0}^{N} S_h\right) \geq \sigma^{1/n}\varepsilon \right\}.$$

Applying Besicovitch's Theorem we are able to cover $S_{\sigma,\varepsilon}$ by many balls $\{B_{a\sigma^{1/n}\varepsilon}(x_j)\}_{j\in J}$ centred at points of $S_{\sigma,\varepsilon}$ and such that, for ε small enough,

$$\begin{aligned}
\sum_{j\in J} \omega_{n-1}(aL^{-1}\sigma^{1/n}\varepsilon)^{n-1} &\leq \sum_{j\in J} \mathcal{L}^{n-1}((K + \varepsilon L^{-1}C) \cap f^{-1}(B_{a\sigma^{1/n}\varepsilon}(x_j))) \\
&\leq \xi \mathcal{L}^{n-1}\left((K + \varepsilon L^{-1}C) \cap f^{-1}\left(\bigcup_{j\in J} B_{a\sigma^{1/n}\varepsilon}(x_j)\right)\right) \\
&\leq \xi(\mathcal{L}^{n-1}(f^{-1}(E)) + \mathcal{L}^{n-1}((K + \bar{\varepsilon}L^{-1}C) \setminus K))) \leq 3\xi\sigma
\end{aligned}$$

being ξ is as in the Besicovitch's Theorem. Therefore

$$\sharp J \leq \frac{3\xi\sigma^{1/n}}{\omega_{n-1}a^{n-1}L^{1-n}\varepsilon^{n-1}}$$

hence

$$|S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C| \leq \sum_{j\in J} |B_{b(1+2\sigma^{1/n})\varepsilon}(x_j)| \leq \frac{\omega_n b^n (1+2\sigma^{1/n})^n \sigma^{1/n} 3\xi\varepsilon}{\omega_{n-1}a^{n-1}L^{1-n}} \leq \frac{\omega_n b^n 3^{n+1} \sigma^{1/n}\xi\varepsilon}{\omega_{n-1}a^{n-1}L^{1-n}}.$$

Using again

$$S + \varepsilon C \subset (E + \varepsilon C) \cup \bigcup_{h=0}^{N} (S_h + \varepsilon C) \subset (S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C) \cup \bigcup_{h=0}^{N} (S_h + \varepsilon C)$$

we deduce that, by Lemma 3.3,

$$\begin{aligned}
\mathcal{M}^*_C(S) &= \limsup_{\varepsilon\to 0} \frac{|S + \varepsilon C|}{2\varepsilon} \\
&\leq \limsup_{\varepsilon\to 0} \frac{|S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C|}{2\varepsilon} + \sum_{h=0}^{N} \limsup_{\varepsilon\to 0} \frac{|S_h + \varepsilon C|}{2\varepsilon} \\
&\leq \frac{\omega_n b^n 3^{n+1}\sigma^{1/n}\xi}{2\omega_{n-1}a^{n-1}L^{1-n}} + \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1}
\end{aligned}$$

and the conclusion follows. □

4. A more general formula for $\mathcal{SM}_C$

In this section we prove the generalization of formula (1.2). First of all, let us introduce the class $\mathcal{O}$ and $\mathcal{O}'_C$.

Definition 4.1. *Let $\mathcal{O}$ be the class of Borel sets E of $\mathbb{R}^n$ such that:*

(a) *∂E is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set;*

(b) *there exist* $\gamma > 0$ *and a probability measure* η *in* $\mathbb{R}^n$ *absolutely continuous with respect to* $\mathcal{H}^{n-1}$ *such that* $\eta(B_r(x)) \geq \gamma r^{n-1}$ *for all* $x \in \partial E$ *and for all* $r \in (0,1)$.

Moreover, let $\mathcal{O}'_C$ *be the class of Borel sets* E *of* $\mathbb{R}^n$ *such that:*

(a') ∂E *is a countably* $\mathcal{H}^{n-1}$*-rectifiable bounded set and*

$$\mathcal{M}_C(\partial E) = \int_{\partial E} \phi_C(\nu_{\partial E})\, d\mathcal{H}^{n-1};$$

(b') *there exist* $\gamma > 0$ *and a probability measure* η *in* $\mathbb{R}^n$ *such that* $\eta(B_r(x)) \geq \gamma r^{n-1}$, *for all* $x \in \partial E$ *and for all* $r \in (0,1)$.

Remark 4.2. *We point out that condition (b'), and therefore also condition (b), implies, since Theorem 2.1, that* $\mathcal{H}^{n-1}(\partial E)$ *is finite; in particular, any set in* $\mathcal{O}$ *or* $\mathcal{O}'_C$ *has finite perimeter.*

Now we are ready to state the main result of this section.

Theorem 4.3. *The class* $\mathcal{O}$ *and the class* $\mathcal{O}'_C$ *are stable under finite unions, and for any* $E \in \mathcal{O}$ *(or* $\mathcal{O}'_C$*) it holds*

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}. \tag{4.1}$$

Remark 4.4. *We notice that If* $E \in \mathcal{O}$ *(or* $E \in \mathcal{O}'_C$*) is such that* $\mathcal{SM}(E) = \mathcal{P}(E)$, *then by* (1.2) *it follows* $\mathcal{H}^{n-1}(E^0 \cap \partial E) = 0$, *and so* $\mathcal{H}^{n-1}(\partial^2 E) = 0$. *As a consequence, from* (4.1) *we get*

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1},$$

in accordance with Theorem 3.2. Moreover, if

$$\frac{\mathcal{SM}(E) + \mathcal{SM}(E^c)}{2} = \mathcal{P}(E),$$

then by (1.2) *it follows* $\mathcal{H}^{n-1}(E^0 \cap \partial E) + \mathcal{H}^{n-1}(E^1 \cap \partial E) = 0$, *and so* $\mathcal{H}^{n-1}(\partial^2 E) = \mathcal{H}^{n-1}(\partial^2 E^c) = 0$. *In this case, again as a consequence of* (4.1), *we get*

$$\begin{aligned}\frac{\mathcal{SM}_C(E) + \mathcal{SM}_C(E^c)}{2} &= \frac{1}{2}\left(\int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\mathcal{F}E^c} h_C(\nu_{E^c})\, d\mathcal{H}^{n-1}\right)\\ &= \int_{\mathcal{F}E} \frac{h_C(\nu_E) + h_C(-\nu_E)}{2}\, d\mathcal{H}^{n-1},\end{aligned}$$

in accordance with Theorem 3.7 in [4].

We are now going to prove Theorem 4.3 by proceeding along the same arguments of the proof of Theorem 3.1 in [11]: for the convenience of the reader we will give complete proofs also for the series of auxiliary lemmas that we needed. First of all, it is easy to observe that $E \cap W \in \mathcal{O}$ for any $E \in \mathcal{O}$ and any closed $W \subset \mathbb{R}^n$; the following lemma implies that the same holds for the class $\mathcal{O}'_C$ as well.

Lemma 4.5. *If $S \subset \mathbb{R}^n$ is a countably $\mathcal{H}^{n-1}$-rectifiable compact set such that*

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1},$$

then

$$\mathcal{M}_C(S \cap W) = \int_{S\cap W} \phi_C(\nu_S)\, d\mathcal{H}^{n-1},$$

for all $W \subset \mathbb{R}^n$ closed.

Proof. Being $S \cap W$ countably $\mathcal{H}^{n-1}$-rectifiable and compact, by (3.2) we know that

$$\mathcal{M}_{*C}(S \cap W) \geq \int_{S\cap W} \phi_C(\nu_S)\, d\mathcal{H}^{n-1}.$$

Let us show that the opposite inequality holds for $\mathcal{M}^*_C(S \cap W)$. Consider the sequence $\{W_h\}_{h\in\mathbb{N}}$ of closed sets $W_h := \{x \in W^c : d(x, W) \geq b/h\}$, which implies that

$$\left(x + \frac{1}{h}C\right) \cap W = \emptyset$$

for any $x \in W^c$, since

$$x + \frac{1}{h}C \subseteq B_{b/h}(x).$$

Note that $W_h \nearrow W^c$ as h goes to infinity. Let us observe that

$$S + \varepsilon C \supseteq ((S \cap W) + \varepsilon C) \cup ((S \cap W_h) + \varepsilon C),$$

and

$$((S \cap W) + \varepsilon C) \cap ((S \cap W_h) + \varepsilon C) = \emptyset$$

for all ε sufficiently small. Hence, for all $h \in \mathbb{N}$

$$\mathcal{M}^*_C(S \cap W) \leq \mathcal{M}^*_C(S) - \mathcal{M}_{*C}(S \cap W_h) \leq \int_S \phi_C(\nu_S)\, d\mathcal{H}^{n-1} - \int_{S\cap W_h} \phi_C(\nu_S)\, d\mathcal{H}^{n-1}$$

By taking now the limit for h which goes to infinity, we get

$$\mathcal{M}^*_C(S \cap W) \leq \int_{S\cap W} \phi_C(\nu_S)\, d\mathcal{H}^{n-1}$$

and so the assertion. □

By observing that for each $G \subset \mathbb{R}^n$ Borel set and for each $r, \rho > 0$ it holds

$$|(\partial G + rC) \cap G \cap B_\rho(x)| \subset |(\partial G + rbB_1(0)) \cap G \cap B_\rho(x)|,$$

the following assertion is a direct application of Lemma 2 in [1].

Lemma 4.6. *Let $G \subset \mathbb{R}^n$ be a Borel set and assume that there exist $\gamma > 0$ and a probability measure η on $\mathbb{R}^n$ such that $\eta(B_r(x)) \geq \gamma r^{n-1}$ for all $x \in \partial G$ and for all $r \in (0,1)$. Then*

$$\limsup_{\varepsilon\to 0} \frac{|(\partial G + \varepsilon C) \cap G \cap B_\rho(x)|}{\varepsilon} = o(\rho^{n-1})$$

for $\mathcal{H}^{n-1}$-a.e. $x \in G^0 \cap \partial G$.

For each A Borel subset of $\mathbb{R}^n$ let $\mathcal{SM}_{*C}(E;A)$ and $\mathcal{SM}^*_C(E;A)$ be given by

$$\mathcal{SM}_{*C}(E;A) := \liminf_{\varepsilon\to 0} \frac{|((E+\varepsilon C)\setminus E)\cap A|}{\varepsilon}, \quad \mathcal{SM}^*_C(E;A) := \limsup_{\varepsilon\to 0} \frac{|((E+\varepsilon C)\setminus E)\cap A|}{\varepsilon}.$$

We also let $\mathcal{SM}_{*C}(E) := \mathcal{SM}_{*C}(E;\mathbb{R}^n)$ and $\mathcal{SM}^*_C(E) := \mathcal{SM}^*_C(E;\mathbb{R}^n)$.

Lemma 4.7. *For any $E\in\mathcal{O}$ (or $\mathcal{O}'_C$) it holds*

$$\mathcal{SM}_{*C}(E;B_\rho(x)) = o(\rho^{n-1}), \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x\in E^1\cap\partial E, \tag{4.2}$$

$$\mathcal{SM}_{*C}(E;B_\rho(x)) \geq \int_{\mathcal{F}E\cap \mathrm{int}B_\rho(x)} h_C(\nu_E)\, d\mathcal{H}^{n-1}, \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x\in E^{1/2}, \tag{4.3}$$

$$\mathcal{SM}_{*C}(E;B_\rho(x)) \geq 2\int_{\partial^2 E\cap \mathrm{int}B_\rho(x)} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} + o(\rho^{n-1}), \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x\in \partial^2 E. \tag{4.4}$$

Proof. Equality (4.2) follows directly from Lemma 4.6 with the choice $G := E^c$, and taking into account that

$$|\partial E+\varepsilon C| = |(E+\varepsilon C)\setminus E| + |(E^c+\varepsilon C)\setminus E^c|. \tag{4.5}$$

Equality (4.3) can be found in [4]. It remains to prove (4.4). Since $0\in \mathrm{int}\, C$, for any closed set $W \subset\subset B_\rho(x)$ there exists $\tilde\varepsilon>0$ such that $W+\varepsilon C\subset B_\rho(x)$, $\forall \varepsilon<\tilde\varepsilon$. So, noticing that $\partial E\cap W$ satisfies the assumption of Theorem 3.4 if $E\in\mathcal{O}$ (or (a') in the definition of the class $\mathcal{O}'_C$ if $E\in\mathcal{O}'_C$), we get that

$$\begin{aligned} 2\int_{\partial E\cap W}\phi_C(\nu_E)\, d\mathcal{H}^{n-1} &= \liminf_{\varepsilon\to 0}\frac{|(\partial E\cap W)+\varepsilon C|}{\varepsilon} \\ &\leq \liminf_{\varepsilon\to 0}\frac{|(\partial E+\varepsilon C)\cap (W+\varepsilon C)|}{\varepsilon} \\ &\leq \liminf_{\varepsilon\to 0}\frac{|(\partial E+\varepsilon C)\cap B_\rho(x)|}{\varepsilon}. \end{aligned}$$

Let $\{W_k\}_{k\in\mathbb{N}}$ be an increasing sequence of closed sets with $W_k\subset\subset B_\rho(x)$ and such that $W_k\nearrow \mathrm{int}B_\rho(x)$. By taking the limit as k tends to ∞, we obtain that

$$\liminf_{\varepsilon\to 0}\frac{|(\partial E+\varepsilon C)\cap B_\rho(x)|}{\varepsilon} \geq 2\lim_{k\to\infty}\int_{\partial E\cap W_k}\phi_C(\nu_E)\, d\mathcal{H}^{n-1} = 2\int_{\partial E\cap \mathrm{int}B_\rho(x)}\phi_C(\nu_E)\, d\mathcal{H}^{n-1}. \tag{4.6}$$

Finally, we have that

$$\begin{aligned} \mathcal{SM}_{*C}(E;B_\rho(x)) &= \liminf_{\varepsilon\to 0}\frac{|(\partial E+\varepsilon C)\cap B_\rho(x)| - |(\partial E+\varepsilon C)\cap E\cap B_\rho(x)|}{\varepsilon} \\ &\geq \liminf_{\varepsilon\to 0}\frac{|(\partial E+\varepsilon C)\cap B_\rho(x)|}{\varepsilon} - \limsup_{\varepsilon\to 0}\frac{|(\partial E+\varepsilon C)\cap E\cap B_\rho(x)|}{\varepsilon}. \end{aligned}$$

Thus (4.4) follows from (4.6), Lemma 4.6, and by taking into account that $\mathcal{H}^{n-1}(\partial E\cap E^0) = \mathcal{H}^{n-1}(\partial^2 E)$. □

Now we are ready to prove the main result of this section.

Proof of Theorem 4.3. Let $E \in \mathcal{O}$ (or $E \in \mathcal{O}'_C$). Let us show that the following lower bound for $\mathcal{SM}_{*C}(E)$ holds:

$$\mathcal{SM}_{*C}(E) \geq \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}. \tag{4.7}$$

Let μ be the measure in $\mathbb{R}^n$ so defined

$$\mu(A) := \int_{\mathcal{F}E\cap A} h_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E\cap A} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}, \quad A \subset \mathbb{R}^n \text{ Borel.}$$

By rectifiability we can say that

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{n-1}(\partial E \cap E^0 \cap B_\rho(x))}{\rho^{n-1}} = \omega_{n-1}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E \cap E^0,$$

and 0 for $\mathcal{H}^{n-1}$-a.e. $x \in (\partial E \cap E^0)^c$; the same conclusions hold for the quantities

$$\frac{\mathcal{H}^{n-1}(\partial E \cap E^1 \cap B_\rho(x))}{\rho^{n-1}}, \quad \frac{\mathcal{H}^{n-1}(\mathcal{F}E \cap B_\rho(x))}{\rho^{n-1}}.$$

Taking into account all those informations and using Lemma 4.7 we get that, for any $\varepsilon > 0$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$,

$$\liminf_{\rho\to 0} \frac{\mathcal{SM}_{*C}(E; B_\rho(x)) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap B_\rho(x))}{\mu(\mathrm{int} B_\rho(x))} \geq 1. \tag{4.8}$$

(Note that the term $\varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap B_\rho(x))$ in the above fraction is to avoid an indetermination of type 0/0 at points $x \in E^1 \cap \partial E$.) Since the family of closed balls $B_\rho(x)$ with $\mu(\partial B_\rho(x)) = 0$ is a fine cover of ∂E, by Vitali-Besicovitch covering Theorem 2.5 for any $\delta > 0$ there exist finitely many disjoint closed balls $W_1, \dots, W_N$ with $\mu(\partial W_i) = 0$ such that

$$\mu\left(\partial E \setminus \bigcup_{i=1}^N W_i\right) < \delta.$$

The balls W_i can be chosen with centers in ∂E and such that

$$\mathcal{SM}_{*C}(E; W_i) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap W_i) \overset{(4.8)}{\geq} (1-\delta)\mu(W_i), \qquad i = 1, \dots, N.$$

Then, the following chain of inequalities holds:

$$\begin{aligned}
&\mathcal{SM}_{*C}(E) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E) \\
&\qquad \geq \mathcal{SM}_{*C}\left(E; \bigcup_{i=1}^{N} W_i\right) + \varepsilon\mathcal{H}^{n-1}\left(E^1 \cap \partial E \cap \bigcup_{i=1}^{N} W_i\right) \\
&\qquad \geq \sum_{i=1}^{N}(\mathcal{SM}_{*C}(E; W_i) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap W_i)) \\
&\qquad \geq (1-\delta)\sum_{i=1}^{N}\mu(W_i) \\
&\qquad = (1-\delta)\left(\mu(\mathbb{R}^n) - \mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{N} W_i\right)\right) \\
&\qquad \geq (1-\delta)\left(\int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} - \delta\right).
\end{aligned}$$

By taking now the limit as $\delta \to 0$, and then as $\varepsilon \to 0$, we obtain the inequality (4.7).

Observing now that E^c belongs to $\mathcal{O}$ (resp. $\mathcal{O}'_C$) too, we can also claim that

$$\mathcal{SM}_{*C}(E^c) \geq \int_{\mathcal{F}E^c} h_C(\nu_{E^c})\, d\mathcal{H}^{n-1} + \int_{\partial^2 E^c} \phi_C(\nu_{E^c})\, d\mathcal{H}^{n-1}. \tag{4.9}$$

Let us now define, for any $\varepsilon > 0$,

$$a_\varepsilon := \frac{|(E + \varepsilon C) \setminus E|}{\varepsilon}, \qquad b_\varepsilon := \frac{|(E^c + \varepsilon C) \setminus E^c|}{\varepsilon}.$$

Observe that by taking into account (2.6) for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E \setminus \mathcal{F}E$

$$\int_{\partial E \setminus \mathcal{F}E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} = \int_{\partial^2 E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\partial^2 E^c} \phi_C(\nu_{E^c})\, d\mathcal{H}^{n-1}. \tag{4.10}$$

Moreover, it holds

$$\liminf_{\varepsilon \to 0} a_\varepsilon = \mathcal{SM}_{*C}(E) \overset{(4.7)}{\geq} \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} =: a,$$

and, using also (2.5) and (2.7),

$$\liminf_{\varepsilon \to 0} b_\varepsilon = \mathcal{SM}_{*C}(E^c) \overset{(4.9)}{\geq} \int_{\mathcal{F}E} h_C(-\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E^c} \phi_C(\nu_{E^c})\, d\mathcal{H}^{n-1} =: b.$$

By (4.5) and by (3.1) if $E \in \mathcal{O}$ (resp. by (a') in the definition of the class $\mathcal{O}'_C$ if $E \in \mathcal{O}'_C$) it follows that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}(a_\varepsilon + b_\varepsilon)\\
&= \limsup_{\varepsilon\to 0} \frac{|\partial E + \varepsilon C|}{\varepsilon}\\
&= 2\mathcal{M}_C(\partial E)\\
&= 2\int_{\partial E} \phi_C(\nu_{\partial E})\, d\mathcal{H}^{n-1}\\
&= 2\int_{\partial E\cap \mathcal{F}E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial E\setminus \mathcal{F}E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}\\
&\overset{(4.10)}{=} \int_{\mathcal{F}E} (h_C(\nu_E) + h_C(-\nu_E))\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} + 2\int_{\partial^2 E^c} \phi_C(\nu_{E^c})\, d\mathcal{H}^{n-1}\\
&\quad = a + b.
\end{aligned}$$

Since

$$\limsup_{\varepsilon\to 0}(a_\varepsilon + b_\varepsilon) \le a + b, \quad \liminf_{\varepsilon\to 0} a_\varepsilon \ge a \in \mathbb{R}, \quad \liminf_{\varepsilon\to 0} b_\varepsilon \ge b \in \mathbb{R}$$

implies

$$\lim_{\varepsilon\to 0} a_\varepsilon = a, \quad \lim_{\varepsilon\to 0} b_\varepsilon = b,$$

equation (4.1) follows.

In order to conclude the proof it remains to show that the class $\mathcal{O}'_C$ is stable under finite unions, since the stability of the class $\mathcal{O}$ under finite unions has been already proved in [11]. Let $E_1, E_2 \in \mathcal{O}'_C$ and let $E := E_1 \cup E_2$. Being $\partial E \subseteq \partial E_1 \cup \partial E_2$, it is clear that ∂E is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set, and that (b') in the definition of the class $\mathcal{O}'_C$ is fulfilled. We know that, from (3.2),

$$\mathcal{M}_{*C}(\partial E) \ge \int_{\partial E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}. \tag{4.11}$$

Next, we have to prove that

$$\mathcal{M}^*_C(\partial E) \le \int_{\partial E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}. \tag{4.12}$$

We first localize the lower and upper anisotropic Minkowski content: if S is compact and $\mathcal{H}^{n-1}$-rectifiable, $A \subset \mathbb{R}^n$ is closed and $B \subset \mathbb{R}^n$ is open, let

$$\mathcal{M}^*_C(S; A) := \limsup_{\varepsilon\to 0} \frac{|(S + \varepsilon C) \cap A|}{2\varepsilon}, \quad \mathcal{M}_{*C}(S; B) := \liminf_{\varepsilon\to 0} \frac{|(S + \varepsilon C) \cap B|}{2\varepsilon}.$$

Of course, we have $\mathcal{M}^*_C(S; \mathbb{R}^n) = \mathcal{M}^*_C(S)$ and $\mathcal{M}_{*C}(S; \mathbb{R}^n) = \mathcal{M}_{*C}(S)$; furthermore, following the same proof of (3.2), we can realize that for any $B \subset \mathbb{R}^n$ open it holds

$$\mathcal{M}_{*C}(S; B) \ge \int_{S\cap B} \phi_C(\nu_S)\, d\mathcal{H}^{n-1}. \tag{4.13}$$

By observing that

$$\chi_{\partial E+\varepsilon C} + \chi_{(\partial E_1\cap\partial E_2)+\varepsilon C} \le \chi_{\partial E_1+\varepsilon C} + \chi_{\partial E_2+\varepsilon C},$$

we get, for any $x \in \mathbb{R}^n$ and for $\mathcal{H}^1$-a.e. $\rho > 0$, using also (4.13),

$$\begin{aligned}
&\mathcal{M}^*_C(\partial E; B_\rho(x)) \\
&\qquad \le \int_{\partial E_1\cap B_\rho(x)} \phi_C(\nu_{E_1})\, d\mathcal{H}^{n-1} + \int_{\partial E_2\cap B_\rho(x)} \phi_C(\nu_{E_2})\, d\mathcal{H}^{n-1} \\
&\qquad\qquad - \int_{\partial E_1\cap\partial E_2\cap \mathrm{int} B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\, d\mathcal{H}^{n-1} \\
&\qquad = \int_{(\partial E_1\cup\partial E_2)\cap B_\rho(x)} \phi_C(\nu_{E_1\cup E_2})\, d\mathcal{H}^{n-1} + \int_{\partial E_1\cap\partial E_2\cap B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\, d\mathcal{H}^{n-1}
\end{aligned}$$

and thus

$$\begin{aligned}
\mathcal{M}^*_C(\partial E; B_\rho(x)) \le &\int_{\partial E\cap B_\rho(x)} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} \\
&+ \int_{(\partial E_1\cup\partial E_2)\cap(\partial E)^c\cap B_\rho(x)} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} \\
&+ \int_{\partial E_1\cap\partial E_2\cap\partial B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\, d\mathcal{H}^{n-1}. \qquad (4.14)
\end{aligned}$$

We notice now that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$,

$$\int_{(\partial E_1\cup\partial E_2)\cap(\partial E)^c\cap B_\rho(x)} \phi_C(\nu_{E_1\cup E_2})\, d\mathcal{H}^{n-1} = o(\rho^{n-1})$$

since we can apply (2.1) to the Radon measure η given by

$$\eta(D) := \int_{(\partial E_1\cup\partial E_2)\cap(\partial E)^c\cap D} \phi_C(\nu_{E_1\cup E_2})\, d\mathcal{H}^{n-1}, \quad D \text{ Borel in } \mathbb{R}^n.$$

Moreover, observe that $\mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x)) = 0$ for $\mathcal{H}^1$-a.e. $\rho > 0$. Indeed, by contradiction if for any $\rho \in A$ with $\mathcal{L}^1(A) > 0$ we had $\mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x)) > 0$, by coarea formula

$$\mathcal{L}^n(\partial E_1\cap\partial E_2) = \int_0^{+\infty} \mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x))\, d\rho \ge \int_A \mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x))\, d\rho$$

and therefeore $\mathcal{L}^n(\partial E_1\cap\partial E_2) > 0$ which implies that $\mathcal{H}^{n-1}(\partial E_1\cap\partial E_2) = +\infty$, which is a contradiction (see Remark (4.2)). Thus,

$$\int_{\partial E_1\cap\partial E_2\cap\partial B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\, d\mathcal{H}^{n-1} = 0, \quad \mathcal{H}^1\text{-a.e. } \rho > 0.$$

from which we obtain, by (4.14), the key estimate

$$\mathcal{M}^*_C(\partial E; B_\rho(x)) \le \int_{\partial E\cap B_\rho(x)} \phi_C(\nu_E)\, d\mathcal{H}^{n-1} + o(\rho^{n-1}), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E. \qquad (4.15)$$

Now the assertion follows applying Theorem 2.5. For any $D \subset \mathbb{R}^n$ Borel, let

$$\sigma(D) := \int_{\partial E \cap D} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}.$$

As a consequence of (4.15) and (2.2) we may claim that

$$\limsup_{\rho\to 0} \frac{\mathcal{M}^*_C(\partial E; B_\rho(x))}{\sigma(B_\rho(x))} \le 1, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E.$$

Since ∂E is bounded, for any $\delta > 0$ there exists a finite covering $B_1, \dots, B_N$ of ∂E, being B_i disjoint closed balls in $\mathbb{R}^n$ with

$$\sigma\left(\partial E \setminus \bigcup_{i=1}^N B_i\right) < \delta;$$

notice that the balls B_i can be assumed to have centers in ∂E, and such that $\sigma(\partial B_i) = 0$, and

$$\frac{\mathcal{M}^*_C(\partial E; B_i)}{\sigma(B_i)} \le 1 + \delta.$$

Finally, let $B := \mathbb{R}^n \setminus \bigcup_{i=1}^N \operatorname{int} B_i$. Notice that $\mathcal{M}^*_C(\partial E; B) = 0$ and thus

$$\begin{aligned}
\mathcal{M}^*_C(\partial E) &\le \mathcal{M}^*_C(\partial E, B) + \mathcal{M}^*_C\left(\partial E, \bigcup_{i=1}^N B_i\right) \\
&\le \sum_{i=1}^N \mathcal{M}^*_C(\partial E, B_i) \\
&\le (1+\delta) \sum_{i=1}^N \sigma(B_i) \\
&= (1+\delta)\sigma\left(\bigcup_{i=1}^N B_i\right) \\
&\le (1+\delta) \int_{\partial E} \phi_C(\nu_E)\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Inequality (4.12) follows sending $\delta \to 0$, and this completes the proof.

Acknowledgments

The authors wish to thank Luigi Ambrosio (SNS, Pisa) for useful discussions and suggestions on this subject.

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