

Quantitative stability results for the Brunn-Minkowski inequality

Alessio Figalli

Abstract. The Brunn-Minkowski inequality gives a lower bound on the Lebesgue measure of a subset in terms of the measures of the individual sets. This inequality plays a crucial role in the theory of convex bodies and has many interactions with isoperimetry and functional analysis. Stability of optimizers of this inequality in one dimension is a consequence of classical results in additive combinatorics. In this note we describe how optimal transportation and analytic tools can be used to obtain quantitative stability results in higher dimension.

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1. Introduction

Geometric and functional inequalities naturally appear in several problems in the calculus of variations, partial differential equations, geometry, etc. Among the most classical inequalities, we recall the isoperimetric **(Isop)** inequality, Sobolev **(Sob)** and Gagliardo-Nirenberg **(GN)** inequalities, and the Brunn-Minkowski **(BM)** inequality.

Although different, all these inequalities are intimately related. Indeed it is well-known that the following chain of implications holds:

$$\mathbf{(BM)} \Rightarrow \mathbf{(Isop)} \Rightarrow \mathbf{(Sob)} \Rightarrow \mathbf{(GN)}. \quad (1.1)$$

Let us introduce briefly all these inequalities and describe this connection.

The Brunn-Minkowski inequality deals with sum of sets: given A, B nonempty subsets of \mathbb{R}^n one defines $A + B := \{a + b : a \in A, b \in B\}$. Then **(BM)** gives a sharp lower bound on the measure of $A + B$ in terms of the measures of A and B : more precisely,

$$\mathbf{(BM)} \quad |A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

The isoperimetric inequality, instead, deals with boundary measure and volume: if $E \subset \mathbb{R}^n$ is a smooth bounded set, its volume $|E|$ is controlled by the perimeter $P(E)$:

$$\mathbf{(Isop)} \quad P(E) \geq C(n)|E|^{(n-1)/n},$$

where $C(n) > 0$ is an explicit dimensional constant.

Sobolev inequalities are a “generalization” of isoperimetric inequalities but they concern functions instead of sets: they say that the gradient of a function in some L^p norm controls some L^q norm of the function itself. More precisely, for any $u \in C_c^\infty(\mathbb{R}^n)$ and $p < n$,

$$\text{(Sob)} \quad \|\nabla u\|_{L^p(\mathbb{R}^n)} \geq C(n, p) \|u\|_{L^q(\mathbb{R}^n)},$$

where $q := \frac{np}{n-p} > p$, and $C(n, p) > 0$ is an explicit constant depending only on n and p .

Finally, Gagliardo-Nirenberg inequalities control a L^q norm of a function with a weaker L^r norm of the function and the L^p norm of its gradient:

$$\text{(GN)} \quad \|\nabla u\|_{L^p(\mathbb{R}^n)}^\theta \|u\|_{L^r(\mathbb{R}^n)}^{1-\theta} \geq C(n, p, r) \|u\|_{L^q(\mathbb{R}^n)},$$

where $\theta = \theta(n, p, r) \in (0, 1)$, $q = q(n, p, r) \in \left(r, \frac{np}{n-p}\right)$, and $C(n, p, r) > 0$ are explicit constants depending only on n , p , and r .

We now explain the chain of implications (1.1).

• **(BM)** \Rightarrow **(Isop)**. We apply **(BM)** to $A = E$ and $B = B_\epsilon(0)$ for some small $\epsilon > 0$. Then

$$|E + B_\epsilon(0)|^{1/n} \geq |E|^{1/n} + |B_\epsilon(0)|^{1/n} = |E|^{1/n} + \epsilon |B_1(0)|^{1/n}. \quad (1.2)$$

On the other hand $E + B_\epsilon(0)$ coincides with the ϵ -neighborhood of E , hence its volume is approximately

$$|E + B_\epsilon(0)| = |E| + \epsilon P(E) + o(\epsilon).$$

Inserting the above expression in (1.2), a Taylor expansion gives

$$|E|^{1/n} + \epsilon \frac{1}{n} \frac{P(E)}{|E|^{(n-1)/n}} + o(\epsilon) = \left(|E| + \epsilon P(E) + o(\epsilon)\right)^{1/n} \geq |E|^{1/n} + \epsilon |B_1(0)|^{1/n},$$

that is

$$\frac{1}{n} \frac{P(E)}{|E|^{(n-1)/n}} + \frac{o(\epsilon)}{\epsilon} \geq |B_1(0)|^{1/n},$$

and letting $\epsilon \rightarrow 0$ we obtain

$$\frac{1}{n} \frac{P(E)}{|E|^{(n-1)/n}} \geq |B_1(0)|^{1/n}$$

as desired.

• **(Isop)** \Rightarrow **(Sob)**. The basic idea is that the perimeter of a set corresponds to the mass of the gradient of the indicator function of a set E : more precisely, if $\mathbf{1}_E$ denotes the indicator function of a set E , that is,

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E, \end{cases}$$

then

$$P(E) = \int_{\mathbb{R}^n} |\nabla \mathbf{1}_E|.$$

This formula is not rigorous since $\nabla \mathbf{1}_E$ is zero a.e. (both inside and outside E) while it gives a “Dirac mass” on the boundary of E , but it can be made precise with the notions of sets of finite perimeter and of BV functions [28].

Now, to relate sets and functions, one applies the layer-cake formula: assuming for simplicity $u \geq 0$, u can be written as

$$u(x) = \int_0^\infty \mathbf{1}_{\{u>t\}}(x) dt, \quad (1.3)$$

from which one deduces, by differentiation,

$$\nabla u = \int_0^\infty \nabla \mathbf{1}_{\{u>t\}} dt.$$

Although not completely obvious it is possible to show that the above identity still holds when taking the moduli on the gradients, that is

$$|\nabla u| = \int_0^\infty |\nabla \mathbf{1}_{\{u>t\}}| dt,$$

so integrating this identity on \mathbb{R}^n we obtain

$$\int_{\mathbb{R}^n} |\nabla u| = \int_0^\infty \left(\int_{\mathbb{R}^n} |\nabla \mathbf{1}_{\{u>t\}}| \right) dt = \int_0^\infty P(\{u > t\}) dt$$

(see [28, Theorem 13.1] for a rigorous proof of the above identity, usually called co-area formula). Applying now **(Isop)** to the sets $\{u > t\}$ one obtains

$$\int_{\mathbb{R}^n} |\nabla u| \geq C(n) \int_0^\infty |\{u > t\}|^{(n-1/n)} dt = C(n) \int_0^\infty \|\mathbf{1}_{\{u>t\}}\|_{L^{(n-1/n)}(\mathbb{R}^n)} dt. \quad (1.4)$$

Recalling now that the norm of the integral is less than the integral of the norm, from (1.3) we get

$$\|u\|_{L^{(n-1/n)}(\mathbb{R}^n)} = \left\| \int_0^\infty \mathbf{1}_{\{u>t\}}(x) dt \right\|_{L^{(n-1/n)}(\mathbb{R}^n)} \leq \int_0^\infty \|\mathbf{1}_{\{u>t\}}\|_{L^{(n-1/n)}(\mathbb{R}^n)} dt,$$

that combined with (1.4) proves **(Sob)** when $p = 1$.

To prove the general case $p \in [1, n)$, it suffices to apply first **(Sob)** with $p = 1$ to the function $v := |u|^\gamma$ with $\gamma := \frac{p(n-1)}{n-p}$, and then use Hölder inequality: indeed, recalling that $q = \frac{np}{n-p}$, setting $s := \frac{np-p}{n-p}$ we get

$$\begin{aligned} \|u\|_{L^q(\mathbb{R}^n)} &= \| |u|^s \|_{L^{n/(n-1)}(\mathbb{R}^n)}^{1/s} \leq C(n)^{1/s} \| |\nabla |u|^s \|_{L^1(\mathbb{R}^n)}^{1/s} \\ &\leq (s C(n))^{1/s} \| |\nabla u| |u|^{s-1} \|_{L^1(\mathbb{R}^n)}^{1/s} \\ &\leq (s C(n))^{1/s} \| \nabla u \|_{L^p(\mathbb{R}^n)}^{1/s} \| u \|_{L^q(\mathbb{R}^n)}^{1-1/s}, \end{aligned}$$

as desired.

• **(Sob)** \Rightarrow **(GN)**. As in the last step of the previous argument, also this implication is a consequence of Hölder inequality: more precisely, given any choice of numbers $r < q < s$ one applies Hölder inequality to get

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \|u\|_{L^s(\mathbb{R}^n)}^\theta \|u\|_{L^r(\mathbb{R}^n)}^{1-\theta}, \quad \text{for some } \theta = \theta(r, q, s) \in (0, 1),$$

and then chooses $s := \frac{np}{n-p}$ to control $\|u\|_{L^s(\mathbb{R}^n)}$ with $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ using **(Sob)**.

The discussion above shows how it is possible to derive some inequalities from others, and that Brunn-Minkowski is at the basis of all of them. However, it is interesting to point out that, although one inequality may imply another one, the constants we obtained from the proofs may not be sharp. More precisely, in the discussion above **(BM)** implied **(Isop)** with the sharp constant, and **(Isop)** implied **(Sob)** for $p = 1$ with sharp constant again, but all the other implications (based on Hölder inequality) are non-sharp.

The issue of the sharpness of a constant, as well as the characterization of minimizers, is a classical and important question which is by now well understood (at least for the class of inequalities we are considering). More recently, a lot of attention has been given to the stability issue:

Suppose that a function almost attains the equality in one of the previous inequalities. Can we prove, if possible in some quantitative way, that such a function is close (in some suitable sense) to one of the minimizers?

In the latest years several results have been obtained in this direction, showing stability for isoperimetric inequalities [27, 22, 12, 17, 13], the Brunn-Minkowski inequality on convex sets [23], Sobolev [11, 24, 15] and Gagliardo-Nirenberg inequalities [3, 15]. We notice that, apart from their own interest, this kind of results have applications in the study of geometric problems (see for instance [20, 21, 9]) and can be used to obtain quantitative rates of convergence for diffusion equations (see for instance [3]).

The aim of this note is to describe recent results on the stability of the Brunn-Minkowski inequality [23, 18, 19]. As we shall see, the study of this problem involves an interplay between linear structure, analysis, and affine-invariant geometry of Euclidean spaces.

2. Sumsets and the Brunn-Minkowski inequality

Let $A \subset \mathbb{R}^n$ be a Borel set with $|A| > 0$. We define its semi-sum as

$$\frac{A+A}{2} := \left\{ \frac{a+a'}{2} : a, a' \in A \right\}.$$

Obviously $\frac{A+A}{2} \supset A$, hence

$$\left| \frac{A+A}{2} \right| \geq |A|.$$

In addition it is not difficult to show that equality holds if and only if “ A is convex”: more precisely, a necessary and sufficient condition for equality is

$$|\text{co}(A) \setminus A| = 0,$$

where $\text{co}(A)$ denote the convex hull of A .

This result is a particular case of a more general inequality: the *Brunn-Minkowski inequality*. Although we already introduced it in the previous section we restate it here since, for convenience, we will write it in an equivalent form with a semisum instead of a sum of sets.

Given $A, B \subset \mathbb{R}^n$ Borel sets, with $|A|, |B| > 0$, we define

$$\frac{A+B}{2} := \left\{ \frac{a+b}{2} : a \in A, b \in B \right\}.$$

As we already mentioned in the introduction, the Brunn-Minkowski inequality gives a control from below on the measure of $A+B$ in terms of the measures of A and B :

$$\left| \frac{A+B}{2} \right|^{1/n} \geq \frac{|A|^{1/n} + |B|^{1/n}}{2}. \quad (2.1)$$

In addition, equality holds if and only if “ A and B are homothetic convex sets”, that is, there exist $\alpha, \beta > 0$, $v, w \in \mathbb{R}^n$, and K convex, such that

$$\begin{aligned} A &\subset \alpha K + v, & |(\alpha K + v) \setminus A| &= 0, \\ B &\subset \beta K + w, & |(\beta K + w) \setminus B| &= 0. \end{aligned}$$

The main question we are interested in is the following: *Are these results stable?*

For instance, assume that

$$\left| \frac{A+A}{2} \right| = |A| + \epsilon$$

with $\epsilon \ll |A|$. Is it true that A is close to its convex hull? Moreover, can we quantify the closeness in terms of ϵ ?

This same kind of question can also be asked for the Brunn-Minkowski inequality: Assume that (2.1) is almost an equality. Is it true that both A and B are almost convex, and that actually they are close to the same convex set?

Let us notice that the latter question has two statements in it: we are wondering if:

- The error in the Brunn-Minkowski inequality controls how far A and B are from their convex hulls (**Convexity**).
- The error in the Brunn-Minkowski inequality controls the difference between the shapes of A and B (**Homothety**).

The aim of this note is to address the questions raised above. We will proceed by steps as follows: in Section 3 we will focus only on the (**Homothety**) issue.

More precisely, we assume that A and B are already convex and we prove that, if equality almost holds in (2.1), then A and B have almost the same shape. Then, in Section 4 we will focus on the **(Convexity)** issue in the simpler case $A = B$, and we shall prove that A is close to its convex hull. Finally, in Section 5 we will deal with the general case.

3. Stability on convex sets

Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$, and

$$\delta(A, B) := \left| \frac{|A+B|}{2} \right|^{1/n} - \frac{|A|^{1/n} + |B|^{1/n}}{2}.$$

It follows from (2.1) that $\delta(A, B) \geq 0$, and we would like to show that $\delta(A, B)$ controls some kind of “distance” between the shape of A and the one of B .

In order to compare A and B , we first want them to have the same volume. Hence, we renormalize A so that it has the same measure of B : if $\gamma := \frac{|B|^{1/n}}{|A|^{1/n}}$ then

$$|\gamma A| = |B|.$$

We then define a “distance”¹ between A and B as follows:

$$d(A, B) := \min_{x \in \mathbb{R}^n} |B \Delta (x + \gamma A)|,$$

where

$$E \Delta F := (E \setminus F) \cup (F \setminus E).$$

The following result has been obtained in [22, Section 4] (see also [23]):

Theorem 3.1. *Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$. There exists $C = C(n, \lambda, \Lambda)$ such that*

$$d(A, B) \leq C \delta(A, B)^{1/2}.$$

As observed in [22, Section 4], the exponent $1/2$ is optimal and the constant C is explicit.

Sketch of the proof. Notice that Theorem 3.1 contains as a corollary the validity of the Brunn-Minkowski inequality on convex sets, as it implies in particular that $\delta(A, B) \geq 0$. Hence, as a general principle, in order to hope for a stability estimate to hold, we should at least be able to prove the easier inequality $\delta \geq 0$.

¹Notice that d is not properly a distance since it is not symmetric. Still, it is a natural geometric quantity which measures, up to translations, the L^1 -closeness between γA and B : indeed, observe that an equivalent formulation for d is

$$d(A, B) := \min_{x \in \mathbb{R}^n} \frac{1}{2} \|\mathbf{1}_B - \mathbf{1}_{x+\gamma A}\|_{L^1(\mathbb{R}^n)}.$$

Thus, we will first show how optimal transportation can be used to prove the Brunn-Minkowski inequality, and then we will explain how the same proof can be exploited to obtain the desired stability result. We notice that a proof of (2.1) using optimal transportation was first given in [29]. Here we follow the argument given in [23] since (as we shall see in Step 2 below) it is particularly suitable to be used to obtain a stability estimate.

Step 1: A proof of (2.1) via optimal transportation. We notice that this part of the proof does not require A and B to be convex. Hence, we consider $A, B \subset \mathbb{R}^n$ Borel with $|A|, |B| > 0$, and we define the probability measures

$$\mu := \frac{\mathbf{1}_A(x)}{|A|} dx, \quad \nu := \frac{\mathbf{1}_B(y)}{|B|} dy.$$

Since μ is absolutely continuous with respect to the Lebesgue measure, Brenier's Theorem [1] ensures the existence of a convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ whose gradient $T := \nabla\varphi$ sends μ onto ν :

$$T_{\#}\mu = \nu, \quad \text{i.e., } \mu(T^{-1}(E)) = \nu(E) \text{ for all } E \text{ Borel.}$$

It is easy to check that T satisfies (at least formally) the following properties:

- (i) $T(A) = B$;
- (ii) $\det(\nabla T) = |B|/|A| = \gamma^n$.

Indeed, (i) is a consequence of the fact that μ lives on A and ν on B .

For (ii) we observe that if $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes a test function, the condition $T_{\#}\mu = \nu$ gives

$$\int_{\mathbb{R}^n} \chi(T(x)) \frac{\mathbf{1}_A(x)}{|A|} dx = \int_{\mathbb{R}^n} \chi(y) \frac{\mathbf{1}_B(y)}{|B|} dy.$$

Now, assuming in addition that T is a diffeomorphism, we can set $y = T(x)$ and use the change of variable formula to obtain that the second integral is equal to

$$\int_{\mathbb{R}^n} \chi(T(x)) \frac{\mathbf{1}_B(T(x))}{|B|} |\det(\nabla T(x))| dx.$$

Hence, since $T(A) = B$, setting $\hat{\chi} := \chi \circ T$ we obtain

$$\frac{1}{|A|} \int_A \hat{\chi}(x) dx = \frac{1}{|B|} \int_B \hat{\chi}(x) |\det(\nabla T(x))| dx,$$

so (ii) follows by the arbitrariness of $\hat{\chi}$ (or equivalently of χ).

We now define

$$\frac{Id + T}{2}(A) := \left\{ \frac{a + T(a)}{2} : a \in A \right\}.$$

Then it follows by (i) that

$$\frac{Id + T}{2}(A) \subset \frac{A + B}{2},$$

from which we deduce that

$$\begin{aligned} \left| \frac{A + B}{2} \right| &\geq \left| \frac{Id + T}{2}(A) \right| = \int_A \det \left(\frac{Id + \nabla T}{2} \right) \\ &= \frac{1}{2^n} \int_A \det(Id + \nabla T). \end{aligned} \tag{3.1}$$

Up to now we never used that T is the gradient of a convex function. We now exploit this: since $T = \nabla\varphi$ with φ convex, the eigenvalues $\lambda_1, \dots, \lambda_n$ of $D^2\varphi = \nabla T$ are nonnegative. Thus, using the inequality between AM-GM (arithmetic mean and geometric mean), one can prove that

$$\begin{aligned} \det(Id + \nabla T) &= \prod_{i=1}^n (1 + \lambda_i) \geq \left(1 + \left(\prod_{i=1}^n \lambda_i \right)^{1/n} \right)^n \\ &\stackrel{(2)}{=} \left(1 + (|B|/|A|)^{1/n} \right)^n. \end{aligned}$$

Combining this inequality with (3.1) we finally obtain

$$\left| \frac{A + B}{2} \right| \geq \frac{1}{2^n} \int_A \left(1 + (|B|/|A|)^{1/n} \right)^n = \left(\frac{|A|^{1/n} + |B|^{1/n}}{2} \right)^n,$$

as desired. Notice that this argument is formal since a priori the transport map is not smooth, but there are two possible ways of fixing this: if A and B are general Borel set, then the above proof can be made rigorous by using some fine results on BV functions and sets of finite perimeter, see [22]; if instead one only wants to prove the result when A and B are convex, then one can rely on [2] to say that the map T is actually smooth.

Step 2: The quantitative estimate. In the previous step we proved that $\delta(A, B) \geq 0$ for any Borel sets A, B . We now want to control $d(A, B)$ with $\delta(A, B)$ when A and B are convex. The first observation is that, by the proof above (using the same notation), we have

$$2^n \delta \geq \int_A \left[\prod_{i=1}^n (1 + \lambda_i) - \left(1 + \left(\prod_{i=1}^n \lambda_i \right)^{1/n} \right)^n \right].$$

Notice that before we used AM-GM to say that the integrand in the right hand side was nonnegative. Also, recall that equality holds in AM-GM if and only if all numbers are equal. Hence, by an improved version of AM-GM which quantifies the closeness of the numbers in terms of the error (see [22, Lemma 2.5]), we obtain a precise quantitative form of the following rough statement:

$$\delta \ll 1 \quad \Rightarrow \quad \lambda_i(x) \simeq \lambda_j(x) \quad \forall i, j, \text{ for most } x \in A.$$

Since $\prod_{i=1}^n \lambda_i(x) = |B|/|A|$ (by (ii)), we also deduce that

$$\lambda_i(x) \simeq \gamma \quad \forall i, \text{ for most } x \in A,$$

where $\gamma = (|B|/|A|)^{1/n}$. Using now that $\nabla T = D^2\varphi$ is a symmetric matrix, from the fact that all its eigenvalues are close to γ we deduce that

$$\nabla T(x) \simeq \gamma \text{Id} \quad \text{for most } x \in A,$$

To be more precise, by carefully performing the above estimates, one can prove that

$$\int_A |\nabla T - \gamma \text{Id}| \leq C \delta(A, B)^{1/2}.$$

We now want to use the estimate above on $\nabla(T - \gamma x)$ to obtain a bound on $T - \gamma x$. For this, we wish to apply a trace inequality of the form

$$C \int_A |\nabla f| \geq \inf_{c \in \mathbb{R}} \int_{\partial A} |f - c| d\sigma \quad \forall f \in C^\infty(\mathbb{R}^n),$$

where σ denotes the surface measure on ∂A . Since A is convex its boundary is Lipschitz, so the above trace inequality holds and we can show that, up to a translation,

$$C \delta(A, B)^{1/2} \geq \int_{\partial A} |T(x) - \gamma x| d\sigma.$$

In particular, since $T(x) \in \overline{B}$ for $x \in \overline{A}$, we deduce that

$$C \delta(A, B)^{1/2} \geq \int_{\partial A} \text{dist}(x, B/\gamma) d\sigma. \quad (3.2)$$

A simple geometric argument then shows that

$$C \int_{\partial A} \text{dist}(x, B/\gamma) d\sigma \geq |A \setminus (B/\gamma)| \quad (3.3)$$

(see [23, Proof of Theorem 1, Step 4]). Since $|A| = |B/\gamma|$ one observes that

$$|A \setminus (B/\gamma)| = |(B/\gamma) \setminus A| = \frac{1}{2} |A \Delta (B/\gamma)|,$$

so the desired result follows from (3.2) and (3.3). \square

4. Stability when $A = B$

As we already mentioned before, the quantitative estimate in the proof of Theorem 3.1 works only if A and B are convex. There are several technical reasons why we need this assumption, but there is also a simple way to see why one cannot hope to use the above proof to solve the **(Convexity)** issue that we raised at the end of Section 2.

Indeed, assume that $A = B$. In that case the map T in the proof of Theorem 3.1 is simply the identity map, that is $T(x) = x$, and the argument given in the first part of the proof is completely “empty”, in the sense that it does not introduce any new information. In particular, the proof by optimal transportation does not help in showing that $\delta(A, A)$ controls the distance between A and its convex hull. Hence a completely new strategy is needed to address this issue.

4.1. The case $n = 1$. Already in the one dimensional case the problem is far from being trivial. Up to rescale A we can always assume that $|A| = 1$. Define

$$\delta_1(A) := |A + A| - 2|A|.$$

It is easy to see that $\delta_1(A)$ cannot control in general $|\text{co}(A) \setminus A|$: indeed take

$$A := [0, 1/2] \cup [L, L + 1/2]$$

with $L \gg 1$. Then

$$A + A = [0, 1] \cup [L, L + 1] \cup [2L, 2L + 1],$$

which implies that $\delta_1(A) = 1 (= |A|)$ while $|\text{co}(A) \setminus A| = L - 1/2$ is arbitrarily large. Luckily, as shown by the following theorem, this is essentially the only thing that can go wrong.

Theorem 4.1. *Let $A \subset \mathbb{R}$ be a measurable set with $|A| = 1$, and denote by $\text{co}(A)$ its convex hull. If $\delta_1(A) < 1$ then*

$$|\text{co}(A) \setminus A| \leq \delta_1(A).$$

This theorem can be obtained as a corollary of a result of G. Freiman [25] about the structure of additive subsets of \mathbb{Z} . (See [26] or [31, Theorem 5.11] for a statement and a proof.) However, it turns out that to prove of Theorem 4.1 one only needs weaker results. For convenience of the reader, instead of relying on deep and intricate combinatorial results, we will give an elementary proof of Theorem 4.1. Our proof is based on the simple observation that a subset of \mathbb{R} can be discretized to a subset of \mathbb{Z} starting at 0 and ending at a prime number p . This may look strange from an analytic point of view, but it considerably simplifies the combinatorial aspects.

Sketch of the proof of Theorem 4.1. The proof consists of three steps:

- first, one proves a Brunn-Minkowski type inequality in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$;
- then, we show a simple case of the so-called Freiman’s $3k - 3$ Theorem;
- finally, an approximation argument proves the theorem on \mathbb{R} .

We give here a sketch of the proof, referring to [18, Section 2] for more details. Note that, in the discrete setting, $|A|$ will always denote the cardinality of the set A .

Step 1: Cauchy-Davenport inequality. *If $\emptyset \neq A, B \subset \mathbb{Z}_p$ with p prime, then $|A + B| \geq \min\{|A| + |B| - 1, p\}$.*

The proof is by induction on the size of $|B|$, the case $|B| = 1$ being trivial. To perform the induction, it is useful to define the e -transform of A and B : given $e \in A - B$, define

$$A_{(e)} := A \cup (B + e), \quad B_{(e)} := B \cap (A - e).$$

Notice that

$$A_{(e)} + B_{(e)} \subset A + B, \quad |A_{(e)}| + |B_{(e)}| = |A| + |B| \quad (4.1)$$

We now consider two cases:

Case 1: there exists $e \in A - B$ such that $|B_e| < |B|$. Then by the inductive step

$$|A_{(e)} + B_{(e)}| \geq \min\{|A_{(e)}| + |B_{(e)}| - 1, p\},$$

and we conclude by (4.1).

Case 2: $|B_e| = |B|$ for any $e \in A - B$. This means that $B_{(e)} = B$ for any $e \in A - B$, which implies that $B + e \subset A$ for any $e \in A - B$, that is $A + B - B \subset A$. Thus $B - B$ is contained inside to the subgroup $Sym_1(A) := \{h \in \mathbb{Z}_p : A + h = A\}$. Since $|B| > 1$ and the only subgroups of \mathbb{Z}_p are $\{0\}$ and \mathbb{Z}_p , this means that $Sym_1(A) = \mathbb{Z}_p$. This implies that $A = \mathbb{Z}_p$, so the result is trivially true since $|A + B| \geq |A| = p$.

Step 2: Freiman's $3k - 3$ Theorem. *Let A be a finite nonempty subset of \mathbb{Z} with $\min(A) = 0$ and $\max(A) = p$, with p prime. Assume that $|A + A| - 2|A| < |A| - 3$. Then $|\{0, \dots, p\} \setminus A| \leq |A + A| - 2|A| + 1$.*

Before proving this result, we notice that in Step 3 we will apply it to sets with very high cardinality. Hence, if we forget about the terms -3 and $+1$, the above statement says the following: if $\delta_1(A) < |A|$ then $|\{0, \dots, p\} \setminus A| \leq \delta_1(A)$. Notice that this statement is exactly what we wanted, if one thinks that $\{0, \dots, p\}$ is the ‘‘convex hull’’ of A in this discrete setting.

To prove this step, let $\phi_p : \mathbb{Z} \rightarrow \mathbb{Z}_p$ denote the canonical quotient map. We have

$$\begin{cases} A + A \supset A \cup (A + p) \\ \phi_p(A) = \phi_p(A + p) \end{cases} \Rightarrow |\phi_p(A + A)| \leq |A + A| - |A|,$$

hence

$$|\phi_p(A + A)| < 2|A| - 3 = 2|\phi_p(A)| - 1$$

(observe that $|\phi_p(A)| = |A| - 1$). Thus, by Step 1,

$$|\phi_p(A + A)| = p,$$

which implies

$$p \leq |A + A| - |A|,$$

as desired.

Step 3: The discretization argument. We now prove the theorem. We first notice that, by approximation, it suffices to prove the result when A is compact.

Without loss of generality assume $\text{co}(A) = [0, M]$. We then approximate A with sets A_k of the form

$$A_k := \bigcup_{j: I_{k,j} \cap A \neq \emptyset} I_{k,j}, \quad I_{k,j} := \left[\frac{jM}{p_k + 1}, \frac{(j+1)M}{p_k + 1} \right],$$

where p_k is a sequence of prime numbers with $p_k \rightarrow +\infty$. Then, we define

$$B_k := \{j \in \mathbb{Z} : I_{k,j} \subset A_k\} \subset \mathbb{Z}.$$

Thanks to the assumption $\delta_1(A) < |A|$ it is easy to check that B_k satisfies the assumptions of Step 2 for k large enough. Hence, it follows by Step 2 that

$$|\{0, \dots, p_k\} \setminus B_k| \leq |B_k + B_k| - 2|B_k| + 1,$$

that rewritten in terms of A_k gives

$$|[0, M] \setminus A_k| \leq \delta_1(A_k) + \frac{1}{p_k + 1}.$$

Letting $k \rightarrow \infty$ concludes the proof. \square

4.2. The case $n \geq 2$. Let us define the *deficit* of A as

$$\delta(A) := \frac{|\frac{1}{2}(A + A)|}{|A|} - 1 = \frac{|A + A|}{|2A|} - 1.$$

In the previous section we showed how to obtain a precise stability result in one dimension by translating it into a problem on \mathbb{Z} . The main result in [18] is a quantitative stability result in arbitrary dimension, showing that a power of $\delta(A)$ dominates the measure of the difference between A and its convex hull $\text{co}(A)$.

Theorem 4.2. *Let $n \geq 2$. There exist computable dimensional constants $\delta_n, c_n > 0$ such that if $A \subset \mathbb{R}^n$ is a measurable set of positive measure with $\delta(A) \leq \delta_n$, then*

$$\delta(A)^{\alpha_n} \geq c_n \frac{|\text{co}(A) \setminus A|}{|A|}, \quad \alpha_n := \frac{1}{8 \cdot 16^{n-2} n! (n-1)!}.$$

Sketch of the proof. Let \mathcal{H}^k denote the k -dimensional Hausdorff measure on \mathbb{R}^n , denote by $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in \mathbb{R}^n , and let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the canonical projection $\pi(y, t) := y$. Given $E \subset \mathbb{R}^n$ and $y \in \mathbb{R}^{n-1}$, we use the notation

$$E_y := E \cap \pi^{-1}(y).$$

We say that E is *t-convex* if E_y is a segment for every $y \in \pi(E)$.

Our proof is by induction on n , the case $n = 1$ being true by Theorem 4.1. As a preliminary observation we notice that if $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine transformation with $\det L = 1$, then $\delta(A) = \delta(L(A))$ and $|\text{co}(A) \setminus A| = |\text{co}(L(A)) \setminus L(A)|$. Hence it is enough to prove the theorem for $L(A)$. This simple remark will be extremely

useful, as it will allow us to reduce to the case when A is bounded (see Step 3 below).

Step 1. The first argument consists in combining Theorem 4.1 with a Fubini's type argument to show that, for most $y \in \pi(A)$, the set $A_y \subset \{y\} \times \mathbb{R}$ is close to its convex hull. Since this part is elementary and also it will be useful to explain one of the main differences with the case $A \neq B$, we detail it.

By Fubini

$$\delta(A) = \left| \frac{1}{2}(A + A) \right| - |A| = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\left(\left(\frac{1}{2}(A + A)\right)_y\right) - \mathcal{H}^1(A_y) dy,$$

hence, since $\frac{1}{2}(A_y + A_y) \subset \left(\frac{1}{2}(A + A)\right)_y$, we deduce

$$\delta(A) \geq \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\left(\frac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y) dy.$$

We now distinguish between two cases, depending whether we can apply Theorem 4.1 or not:

- y is *good* if $\mathcal{H}^1\left(\frac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y) \leq \mathcal{H}^1(A_y)/2$.

- y is *bad* if $\mathcal{H}^1\left(\frac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y) \geq \mathcal{H}^1(A_y)/2$.

Notice that if y is “good” we can apply Theorem 4.1 to A_y , while in the “bad” case $\mathcal{H}^1\left(\frac{1}{2}(A_y + A_y)\right) - \mathcal{H}^1(A_y)$ trivially controls $\mathcal{H}^1(A_y)$, therefore

$$\begin{aligned} \delta(A) &\geq \int_{y \text{ good}} \mathcal{H}^1(\text{co}(A_y) \setminus A_y) dy \\ &\quad + \int_{y \text{ bad}} \mathcal{H}^1(A_y)/2 dy. \end{aligned}$$

From this estimate we deduce that A is (quantitatively) close to the t -convex set

$$A' := \bigcup_{y \text{ good}} \text{co}(A_y).$$

Now, applying the inductive hypothesis with $n - 1$, an argument similar to the one above shows that

$$E_{s_0} := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A_y) > s_0\}$$

is close to its convex hull for some small $s_0 > 0$. Using this fact we prove that, for s_0 small enough, A is close to the t -convex set

$$A^* := \bigcup_{y \in E_{s_0}} \text{co}(A_y).$$

Step 2. Define $\text{co}(A_y) = \{y\} \times [a(y), b(y)]$. A careful analysis based on the assumption that $\delta(A)$ is small shows that the midpoint $c(y) := (a(y) + b(y))/2$ of A_y^* have bounded second differences as a function of y : more precisely,

$$|c(y') + c(y'') - 2c(y)| \leq 6 \quad \forall y, y', y'' \in E_{s_0}, y = \frac{y' + y''}{2}.$$

Step 3. As mentioned before, it is enough to prove the result for $L(A)$ instead of A , where $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a measure preserving affine transformation. We show here that we can find a map L such that $L(A^*)$ is bounded.

Using the above bound for c , we prove that c is at bounded distance from an affine function ℓ . In addition, since $\pi(A^*) = E_{s_0}$ and E_{s_0} is close to its convex hull (by Step 1), a classical result in convex geometry (called John's Lemma) states that we can find a measure preserving affine transformation $T : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ such that $T(\text{co}(E_{s_0}))$ is bounded. Hence, up to applying the affine measure-preserving transformation

$$L(y, t) := (Ty, t - \ell(Ty)),$$

A^* is bounded.

Step 4. We want to show that A^* is close to a convex set. For this, we need to prove a statement of the following form (the exact form of the statement proved in [18] is more involved):

Assume that

$$\frac{f(y') + f(y'')}{2} \leq f(y) + \gamma \quad \forall y, y', y'' \in E, y = \frac{y' + y''}{2},$$

$$|f| \leq 1, \quad |\text{co}(E) \setminus E| \leq \gamma$$

for some $E \subset \mathbb{R}^{n-1}$, $\gamma \ll 1$. Then there exist $C, \alpha > 0$ such that

$$\int_E |f - F| \leq C\gamma^\alpha, \quad \text{for some function } F : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \text{ concave.}$$

Recalling that $\text{co}(A_y) = \{y\} \times [a(y), b(y)]$, the above statement applied to $f = b$ and $f = -a$ shows that A^* (hence also A) is quantitatively close to a convex set.

Step 5. By a simple geometric argument we prove that the convex set obtained in Step 4 can be assumed to be the convex hull of A , concluding the proof. \square

5. Stability when $A \neq B$

As in the case $A = B$, when $n = 1$ a sharp stability result holds as a consequence of classical theorems in additive combinatorics (an elementary proof of this result can be given using Kemperman's theorem [7, 8]):

Theorem 5.1. *Let $A, B \subset \mathbb{R}$ be measurable sets. If $|A + B| < |A| + |B| + \delta$ for some $\delta \leq \min\{|A|, |B|\}$, then there exist two intervals $I, J \subset \mathbb{R}$ such that $A \subset I$, $B \subset J$, $|I \setminus A| \leq \delta$, and $|J \setminus B| \leq \delta$.*

Concerning the higher dimensional case, in [5, 6] M. Christ proved a *qualitative* stability result for (2.1), namely, if $|A + B|^{1/n}$ is close to $|A|^{1/n} + |B|^{1/n}$ then A and B are close to homothetic convex sets.

The main result in [19] is a quantitative version of Christ's result. After dilating A and B appropriately, we can assume $|A| = |B| = 1$ while replacing the semisum $(A + B)/2$ by a convex combination $S := tA + (1 - t)B$ with $t \in (0, 1)$. It follows by (2.1) that $|S| = 1 + \delta$ for some $\delta \geq 0$. Since our proof is by induction on the dimension, it will be convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of δ . The main result of [19] shows that the measure of the difference between the sets A and B and their convex hull is bounded by a power δ^ε , confirming a conjecture of Christ [5].

Theorem 5.2. *Let A, B be measurable subsets of \mathbb{R}^n with $n \geq 2$, and define $S := tA + (1 - t)B$ for some $t \in [\tau, 1 - \tau]$, $0 < \tau \leq 1/2$. There are computable dimensional constants N_n and computable functions $M_n(\tau), \varepsilon_n(\tau) > 0$ such that if*

$$||A| - 1| + ||B| - 1| + ||S| - 1| \leq \delta \quad (5.1)$$

for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset \mathcal{K} \quad \text{and} \quad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq \tau^{-N_n} \delta^{\varepsilon_n(\tau)}.$$

Explicitly, we may take

$$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n}}{\tau^{3^n}}, \quad \varepsilon_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}}.$$

Sketch of the proof. For convenience we reintroduce some of the notation, although identical to the one in the proof of Theorem 4.2.

Let \mathcal{H}^k denote k -dimensional Hausdorff measure on \mathbb{R}^n , denote by $x = (y, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in \mathbb{R}^n , and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ and $\bar{\pi} : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the canonical projections, i.e.,

$$\pi(y, s) := y \quad \text{and} \quad \bar{\pi}(y, s) := s.$$

Given a compact set $E \subset \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, and $\lambda > 0$, we use the notation

$$E_y := E \cap \pi^{-1}(y) \subset \{y\} \times \mathbb{R}, \quad E(s) := E \cap \bar{\pi}^{-1}(s) \subset \mathbb{R}^{n-1} \times \{s\},$$

$$\mathcal{E}(\lambda) := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_y) > \lambda\}.$$

Following Christ [6], we consider different symmetrizations:

We define the *Schwarz symmetrization* E^* of E as follows. For each $t \in \mathbb{R}$,

- If $\mathcal{H}^{n-1}(E(s)) > 0$, then $E^*(s)$ is the closed disk centered at $0 \in \mathbb{R}^{n-1}$ with the same measure.

- If $\mathcal{H}^{n-1}(E(s)) = 0$, then $E^*(s)$ is empty.

We define the *Steiner symmetrization* E^* of E so that for each $y \in \mathbb{R}^{n-1}$, the set E_y^* is empty if $\mathcal{H}^1(E_y) = 0$; otherwise it is the closed interval of length $\mathcal{H}^1(E_y)$ centered at $0 \in \mathbb{R}$. Finally, we define $E^\natural := (E^*)^*$.

The proof of Theorem 5.2 is very elaborate, combining the techniques of M. Christ with those developed by the present authors in [18] (where we proved Theorem 5.2 in the special case $A = B$ and $t = 1/2$), as well as several new ideas. Before describing the proof, we begin by showing one of the differences with respect to the case $A = B$.

Let us try to repeat Step 1 in the proof of Theorem 4.2: arguing in the very same way as we did there, one would obtain

$$\begin{aligned} \left| \frac{1}{2}(A+B) \right| - \frac{|A|+|B|}{2} &= \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\left(\left(\frac{1}{2}(A+B)\right)_y\right) - \frac{\mathcal{H}^1(A_y)+\mathcal{H}^1(B_y)}{2} dy \\ &\geq \int_{\mathbb{R}^{n-1}} \mathcal{H}^1\left(\frac{1}{2}(A_y+B_y)\right) - \frac{\mathcal{H}^1(A_y)+\mathcal{H}^1(B_y)}{2} dy \\ &\geq 0. \end{aligned}$$

However the above inequality is *false* when $n \geq 2$, as one can immediately check by taking $A = B_1(0)$ and $B = \{0\}$, so that $\frac{1}{2}(A+B) = B_{1/2}(0)$. The mistake in the above argument is the following: in the last inequality we applied the one dimensional Brunn-Minkowski inequality

$$\mathcal{H}^1\left(\frac{1}{2}(A_y+B_y)\right) - \frac{\mathcal{H}^1(A_y)+\mathcal{H}^1(B_y)}{2} \geq 0,$$

but the latter is true only when both A_y and B_y are *nonempty* (since the semisum of any set with the empty set is the empty set).

As we shall see, this is just the first of several new issues that arise in the stability proof when $A \neq B$. We now give a detailed description of the proof of the theorem.

Case 1: $A = A^\natural$ and $B = B^\natural$. First we prove the theorem in the special case $A = A^\natural$ and $B = B^\natural$. In this case we have that

$$A_y = \{y\} \times [-a(y), a(y)] \quad \text{and} \quad B_y = \{y\} \times [-b(y), b(y)],$$

for some functions $a, b : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^+$, and it is easy to show that a and b satisfy the “3-point concavity inequality”

$$ta(y') + (1-t)b(y'') \leq [ta + (1-t)b](y) + \delta^{1/4} \quad (5.2)$$

whenever y', y'' , and $y := ty' + (1-t)y''$ belong to a large subset F of $\pi(A) \cap \pi(B)$. From this 3-point inequality and an elementary argument we show that a satisfies the “4-point concavity inequality”

$$a(y_1) + a(y_2) \leq a(y'_{12}) + a(y''_{12}) + \frac{2}{t}\delta^{1/4} \quad (5.3)$$

with $y'_{12} := t'y_1 + (1-t')y_2$, $y''_{12} := t''y_1 + (1-t'')y_2$, $t' := \frac{1}{2-t}$, $t'' := 1-t'$, provided all four points belong to F . (The analogous inequality for b involves a different set of four points.)

Using this inequality and a variant of the argument in Step 4 of the proof of Theorem 4.2, we deduce that a is quantitatively close in L^1 to a concave function.

Once we know that a (and analogously b) is L^1 -close to a concave function, we deduce that both A and B are L^1 -close to convex sets K_A and K_B respectively, and we would like to say that these convex sets are nearly the same. This is demonstrated by first showing that S is close to $tK_A + (1-t)K_B$, then applying Theorem 3.1 to deduce that K_A and K_B are almost homothetic, and then constructing a convex set \mathcal{K} close to A and B and containing both of them.

This concludes the proof of Theorem 4.2 in the case $A = A^\natural$ and $B = B^\natural$.

Case 2: The general case. We now consider the general case, which we prove in several steps, culminating in induction on dimension.

Step 1. This first step is very close to the argument used by M. Christ in [6], although our analysis is more elaborate since we have to quantify every estimate.

Given A , B , and S , as in the theorem, we consider their symmetrizations A^\natural , B^\natural , and S^\natural , and apply Case 1 above to deduce that A^\natural and B^\natural are close to the same convex set. This information combined with a lemma of Christ allows us to deduce that functions $y \mapsto \mathcal{H}^1(A_y)$ and $y \mapsto \mathcal{H}^1(B_y)$ are almost equipartitioned (that is, the measure of their level sets $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are very close). This fact combined with a Fubini argument yields that, for most levels λ , $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are almost optimal for the $(n-1)$ -dimensional Brunn-Minkowski inequality. Thus, by the inductive step, we can find a level $\bar{\lambda} \sim \delta^\zeta$ ($\zeta > 0$) such that we can apply the inductive hypothesis to $\mathcal{A}(\bar{\lambda})$ and $\mathcal{B}(\bar{\lambda})$. Consequently, after removing sets of small measure both from A and B and translating in y , we deduce that $\pi(A), \pi(B) \subset \mathbb{R}^{n-1}$ are close to the same convex set.

Step 2. This step is the analogue of Step 1 in the proof of Theorem 4.2: we apply a Fubini argument and Theorem 5.1 to most of the sets A_y and B_y for $y \in \mathcal{A}(\bar{\lambda}) \cap \mathcal{B}(\bar{\lambda})$ to deduce that they are close to their convex hulls. Note, however, that to apply Fubini and Theorem 5.1 it is crucial that, thanks to Step 1, we found a set in \mathbb{R}^{n-1} onto which both A and B project almost fully. Indeed, as we already mentioned at the beginning of the proof, to say that $\mathcal{H}^1(A_y + B_y) \geq \mathcal{H}^1(A_y) + \mathcal{H}^1(B_y)$ it is necessary to know that both A_y and B_y are nonempty, as otherwise the inequality would be false.

Step 3. To understand the properties of the barycenter of A_y and B_y (in analogy with Step 2 in the proof of Theorem 4.2), we consider the ‘‘upper’’ (resp. ‘‘lower’’) profile of A and B , that is the functions $a^+(y) := \max\{t \in \mathbb{R} : t \in A_y\}$ (resp. $a^-(y) := \min\{t \in \mathbb{R} : t \in A_y\}$) and $b^+(y) := \max\{t \in \mathbb{R} : t \in B_y\}$ (resp. $b^-(y) := \min\{t \in \mathbb{R} : t \in B_y\}$). With this notation we obtain a 3-point concavity inequality as in (5.2) for a^+ and b^+ (and the analogous one but with opposite signs

for a^- and b^-). This inequality allows us to say that the barycenter of A_y satisfies the 4-point inequality (5.3) both from above and from below, and from this information we can deduce that, as a function of y , the barycenter of A_y (resp. B_y) is at bounded distance from a linear function. It follows that the barycenters of \tilde{S}_y are at bounded distance from a linear function for a set \tilde{S} which is almost of full measure inside S . Then a variation of Step 3 in the proof of Theorem 4.2 allows us to show that, after an affine measure preserving transformation, \tilde{S} is universally bounded, that is, bounded in diameter by a constant of the form $C_n \tau^{-M_n}$ where C_n and M_n are dimensional constants.

Step 4. By a relatively easy argument we find sets A^\sim and B^\sim of the form

$$A^\sim = \bigcup_{y \in F} \{y\} \times [a^A(y), b^A(y)] \quad B^\sim = \bigcup_{y \in F} \{y\} \times [a^B(y), b^B(y)]$$

which are close to A and B , respectively, and are universally bounded.

Step 5. This is a crucial step: we want to show that A^\sim and B^\sim are close to convex sets. As in the case $A = A^\natural$ and $B = B^\natural$, we would like to deduce that b^A and b^B (resp. a^A and a^B) are L^1 -close to concave (resp. convex) functions.

The main issue for proving this is to show first that the level sets of b^A and b^B are close to their convex hulls. To deduce this we wish to prove that most slices of A^\sim and B^\sim are nearly optimal in the Brunn-Minkowski inequality in dimension $n - 1$ and invoke the inductive hypothesis. We achieve this by introducing a new inductive proof of the Brunn-Minkowski inequality, based on combining the validity of Brunn-Minkowski in dimension $n - 1$ with 1-dimensional optimal transport.

An examination of this new proof of the Brunn-Minkowski inequality in the situation near equality shows that if A and B are almost optimal for the Brunn-Minkowski inequality in dimension n , then for most levels s , the slices $A(s)$ and $B(T(s))$ have comparable $(n - 1)$ -measure, where T is the 1-dimensional optimal transport map, and this pair of sets is almost optimal for the Brunn-Minkowski inequality in dimension $n - 1$. In particular, we can apply the inductive hypothesis to deduce that most $(n - 1)$ -dimensional slices are close to their convex hulls.

In this way, we end up proving that A^\sim and B^\sim are close to convex sets, as desired.

Step 6. Since A^\sim and B^\sim are close to A and B respectively, as in the case $A = A^\natural$ and $B = B^\natural$ we find a convex set \mathcal{K} close to A and B and containing both of them.

Step 7. Tracking down the exponents in the proof, we provide an explicit lower (resp. upper) bound on $\varepsilon_n(\tau)$ (resp. $M_n(\tau)$), which concludes the proof of the theorem. \square

6. Concluding remarks

In this note we have seen three substantially different methods to obtain stability results: the proof of Theorem 3.1 relies on optimal transportation techniques, the one of Theorem 4.1 is based on additive combinatorics' arguments, while Theorems 4.2 and 5.2 involve an interplay between measure theory, analysis, and affine-invariant geometry. While Theorem 4.1 is sharp, Theorems 3.1, 4.2, and 5.2 still leave space for improvements.

First of all, in the statement of Theorem 3.1 it would be interesting to find a sharp dependence on the constant C with respect to the parameters n , λ , and Λ : if for instance λ and Λ are comparable (that is, A and B have comparable volumes) then the proof in [23] provides the estimate of the form $C(n) \approx A^n$ for some universal constant $A > 1$, while [22] improves it to a polynomial bound $C(n) \approx n^{17/2}$. As later shown in [30], a careful examination of the methods presented in [22, 23] permits to improve the constant and get a (still non-sharp) bound $C(n) \approx n^{7/2}$.

Concerning Theorems 4.2 and 5.2, there are even more fundamental questions. For instance, notice that the exponents α_n and $\beta_n(\tau)$ depend on the dimension, and it looks very plausible to us that they are both non-sharp. An important question in this direction would be to improve our exponents and, if possible, understand what the sharp exponents should be.

Let us conclude by pointing out that improving the exponent in a stability estimate is not a merely academic question. For instance, the exponent in the stability estimate for the Gagliardo-Nirenberg inequality used in [3] is related to the rate of convergence to the stationary states for solutions of the critical Keller-Segel equation, so a better exponent would give a faster rate. Even more surprisingly, the results in [9] rely in a crucial way on the sharpness of the exponent in the stability estimate for the isoperimetric inequality found in [27, 22].

It is our belief that this line of research will continue growing in the next years, producing new and powerful stability results.

References

- [1] Brenier, Y., Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44** (1991), no. 4, 375–417.
- [2] Caffarelli, L. A., The regularity of mappings with a convex potential. *J. Amer. Math. Soc.* **5** (1992), no. 1, 99–104.
- [3] Carlen, E. A., Figalli, A., Stability for a GN inequality and the Log-HLS inequality, with application to the critical mass Keller-Segel equation. *Duke Math. J.* **162** (2013), no. 3, 579–625.
- [4] Carlen, E. A., Loss, M., Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on \mathbb{S}^n . *Geom. Funct. Anal.* **2** (1992), 90–104.
- [5] Christ, M., Near equality in the two-dimensional Brunn-Minkowski inequality. Preprint, 2012. Available online at <http://arxiv.org/abs/1206.1965>

- [6] Christ, M., Near equality in the Brunn-Minkowski inequality. Preprint, 2012. Available online at <http://arxiv.org/abs/1207.5062>
- [7] Christ, M., An approximate inverse Riesz-Sobolev inequality. Preprint, 2012. Available online at <http://arxiv.org/abs/1112.3715>
- [8] Christ, M., Personal communication.
- [9] Cicalese, M., Spadaro, E., Droplet minimizers of an isoperimetric problem with long-range interactions. *Comm. Pure Appl. Math.* **66** (2013), no. 8, 1298–1333.
- [10] Cordero-Erausquin, D., Nazaret, B., Villani, C., A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.* **182** (2004), no. 2, 307–332.
- [11] Cianchi, A., Fusco, N., F. Maggi and Pratelli, A., The sharp Sobolev inequality in quantitative form. *J. Eur. Math. Soc. (JEMS)* **11** (2009), no. 5, 1105–1139.
- [12] Cianchi, A., Fusco, N., F. Maggi and Pratelli, A., A. On the isoperimetric deficit in Gauss space. *Amer. J. Math.* **133** (2011), no. 1, 131–186.
- [13] Cicalese, M., Leonardi, G. P., A Selection Principle for the Sharp Quantitative Isoperimetric Inequality. *Arch. Ration. Mech. Anal.* **206** (2012), no. 2, 617–643.
- [14] Del Pino, M., Dolbeault, J., Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.* **81** (2002), no. 9, 847–875.
- [15] Dolbeault, J., Toscani, G., Improved interpolation inequalities, relative entropy and fast diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** (2013), no. 5, 917–934.
- [16] Esposito, L., Fusco, N., Trombetti, C., A quantitative version of the isoperimetric inequality: the anisotropic case. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4** (2005) no. 4, 619–651.
- [17] Figalli, A., Indrei, E., A sharp stability result for the relative isoperimetric inequality inside convex cones. *J. Geom. Anal.* **23** (2013), no. 2, 938–969.
- [18] Figalli, A., Jerison, D., Quantitative stability for sumsets in \mathbb{R}^n . Preprint, 2012.
- [19] Figalli, A., Jerison, D., Quantitative stability for the Brunn-Minkowski inequality Preprint, 2013.
- [20] Figalli, A., Maggi, F., On the shape of liquid drops and crystals in the small mass regime. *Arch. Ration. Mech. Anal.* **201** (2011), no. 1, 143–207.
- [21] Figalli, A., Maggi, F., On the isoperimetric problem for radial log-convex densities. *Calc. Var. Partial Differential Equations* **48** (2013), no. 3-4, 447–489.
- [22] Figalli, A., Maggi, F., Pratelli, A., A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.* **182** (2010), no. 1, 167–211.
- [23] Figalli, A., Maggi, F., Pratelli, A., A refined Brunn-Minkowski inequality for convex sets. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), no. 6, 2511–2519.
- [24] Figalli, A., Maggi, F., Pratelli, A., Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation. *Adv. Math.* **242** (2013), 80–101.
- [25] Freiman, G. A., The addition of finite sets. I. (Russian) *Izv. Vyss. Ucebn. Zaved. Matematika* **13** (1959), no. 6, 202–213.

- [26] Freiman, G. A., *Foundations of a structural theory of set addition*. Translated from the Russian. Translations of Mathematical Monographs, Vol 37. American Mathematical Society, Providence, R. I., 1973.
- [27] Fusco, N., Maggi, F., Pratelli, A., The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)* **168** (2008), 941–980.
- [28] Maggi, F., *Sets of finite perimeter and geometric variational problems*. An introduction to geometric measure theory. Cambridge Studies in Advanced Mathematics, 135. Cambridge University Press, Cambridge, 2012.
- [29] McCann, R. J., A convexity principle for interacting gases. *Adv. Math.* **128** (1997), no. 1, 153–179.
- [30] Segal, A., Remark on stability of Brunn-Minkowski and isoperimetric inequalities for convex bodies. *Geometric aspects of functional analysis*, 381–391, Lecture Notes in Math., 2050, Springer, Heidelberg, 2012.
- [31] Tao, T., Vu, V., *Additive combinatorics*. Cambridge Studies in Advanced Mathematics, 105. Cambridge University Press, Cambridge, 2006.

The University of Texas at Austin, Mathematics Dept. RLM 8.100, 2515 Speedway
Stop C1200, Austin, Texas 78712-1202, USA
E-mail: figalli@math.utexas.edu