L^p-estimates for a class of elliptic operators with unbounded coefficients in $\mathbf{R}^N$

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Abstract

We prove L^p-estimates for second order elliptic operator in $\mathbf{R}^N$ with unbounded, globally Lipschitz coefficients.

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1 Introduction

In this paper we consider second-order elliptic operators in $L^p(\mathbf{R}^N)$ of the following type

$$A = \sum_{i,j=1}^{N} D_i\big(q_{ij} D_j\big) + \sum_{i=1}^{N} (b_i + f_i) D_i. \tag{1.1}$$

under the following assumptions on the coefficients.

(H1) $Q = (q_{ij})$ is a symmetric real matrix, $q_{ij} \in C_b^1(\mathbf{R}^N)$ and there is $\nu > 0$ such that

$$Q(x)\xi \cdot \xi \geq \nu |\xi|^2 \qquad x, \xi \in \mathbf{R}^N.$$

(H2) $B = (b_1, \dots, b_N)$ is a (globally) Lipschitz vector field on $\mathbf{R}^N$, $F = (f_1, \dots, f_N) \in C_b(\mathbf{R}^N, \mathbf{R}^N)$.

Observe that, since $q_{ij} \in C_b^1(\mathbf{R}^N)$ and F is only supposed to be continuous and bounded, the operator A can be written in the divergence form (1.1) or in the non-divergence form

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} \Big(b_i + f_i + \sum_{j=1}^{N} D_j q_{ij}\Big) D_i. \tag{1.2}$$

When endowed with its maximal domain

$$D_{p,max}(A) := \Big\{u \in L^p(\mathbf{R}^N) \cap W^{2,p}_{loc}(\mathbf{R}^N) : \ Au \in L^p(\mathbf{R}^N)\Big\} \tag{1.3}$$

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the operator A is the generator of a strongly continuous semigroup $(P(t))_{t\geq 0}$ in $L^p(\mathbf{R}^N)$ such that $P(t)f$ solves the Cauchy problem

$$\begin{cases} D_t u = Au & \text{in } (0,\infty)\times\mathbf{R}^N \\ u(0)=f & \text{in } \mathbf{R}^N \end{cases} \tag{1.4}$$

for $f \in L^p(\mathbf{R}^N)$. The main results of this paper are maximal regularity estimates for the corresponding resolvent equation

$$\lambda u - Au = f \qquad \text{in } \mathbf{R}^N, \qquad\qquad \lambda > 0, \tag{1.5}$$

that yield a complete description of the domain. In fact, we prove the following result.

Theorem 1 *Assume that (H1) and (H2) are satisfied, and, in addition, that*

(H3) $\sup_{x\in\mathbf{R}^N} |\nabla q_{ij}\cdot B| < \infty$, *for all* $i,j=1,\dots,N$.

Then, the domain $D_{p,max}(A)$ *of the generator of the semigroup* $(P(t))_{t\geq 0}$ *coincides with*

$$D_p := \{u \in W^{2,p}(\mathbf{R}^N):\ B\cdot\nabla u \in L^p(\mathbf{R}^N)\}.$$

The above result can be rephrased by saying that requiring that $u \in D_{p,max}(A)$, i.e. $Au \in L^p(\mathbf{R}^N)$, is equivalent to requiring that the two leading terms in Au, i.e., the diffusion term $\sum_{i,j=1}^N D_i\big(q_{ij}(x)D_j u\big)$ and the drift term $B\cdot\nabla u$ *separately* belong to $L^p(\mathbf{R}^N)$.

We point out that in the special case of the Ornstein-Uhlenbeck operators, that is when the matrix Q is constant and $B(x)=Bx$, where B is a non-zero $N\times N$ real matrix, the above result has been proved in [15].

The approach presented in this paper is more geometric. In fact, it is based upon a change of variables determined by the flow generated by the drift term (see Section 3). This allows us to reduce problem (1.4) to a uniformly parabolic one, and also gives a better understanding of the intrinsic geometry related to the operator A (see also [7], where this point of view is deeply pursued).

The above characterisation of the domain of A follows from regularity results for the solution of the more general problem

$$\begin{cases} D_t u - Au = g & \text{in } (0,T)\times\mathbf{R}^N \\ u(0)=f & \text{in } \mathbf{R}^N. \end{cases}$$

As in [13], we use a suitable change of variables to transform this problem into a non autonomous uniformly parabolic one (i.e., with regular *bounded* coefficients), so that the well-known estimates available for the transformed problem can be recovered in the original setting. Assumption (H3) is crucial for this approach as it guarantees that the coefficients of the transformed operator are uniformly continuous. It could likely be relaxed by requiring that the coefficients (q_{ij}) are only uniformly continuous, but have a small variation along the characteristics anduced by B as in (3.3). Actually, in [7] a Harnack inequality is proved with respect to a geometry determined by the operator. However, we do not know whether Theorem 1 holds if condition (H3) is completely dropped.

Finally, leu us point out that there is a wide literature on domain characterisation of operators with unbounded coefficients. However, in most cases, the operators contain an unbounded potential whose growth is used to balance the growth of the drift term. Such an approach is used e.g. in [1], [2], [16]. We also refer to [11] for the case of Hölder spaces.

In Section 2 we construct the semigroup generated by $(A, D_{p,max}(A))$. In Section 3 we prove, under slightly stronger hypotheses on the first-order coefficients B, the regularity results for the above problem. Section 4 is devoted to the proof of Theorem 1.

Notation. For $x \in \mathbf{R}^N$, $|x|$ denotes the euclidean norm, and $B_\varrho = \{x \in \mathbf{R}^N : |x| < \varrho\}$ the open ball with radius $\varrho > 0$. As regards function spaces, we write $\| \cdot \|_p$ for the norm of $L^p(\mathbf{R}^N)$. We denote by $C^k(\mathbf{R}^N)$ (resp. $C_b^k(\mathbf{R}^N)$) the space of functions on $\mathbf{R}^N$ with continuous (resp. bounded and continuous) derivatives up to the order k, and write $C_b(\mathbf{R}^N)$ instead of $C_b^0(\mathbf{R}^N)$. $BUC(\mathbf{R}^N)$ is the space of all bounded, uniformly continuous functions on $\mathbf{R}^N$ and $C_0(\Omega) = \{f \in C(\overline{\Omega} : \ f(x) = 0 \ \forall x \in \partial\Omega\}$. $W^{k,p}(\Omega)$ is the Sobolev spaces of the measurable functions in the open set $\Omega \subset \mathbf{R}^N$ which have weak derivatives p-summable in Ω up to order k, endowed with the usual norm $\| \cdot \|_{W^{k,p}(\Omega)}$. Finally, for $T > 0$, we define $Q_T := (0, T) \times \mathbf{R}^N$ and the spaces $\mathcal{W}_p^{1,2}(Q_T)$ of the functions $f : Q_T \to \mathbf{C}$ whose first-order partial derivative with respect to t and partial derivatives with respect to x up to the second order are p-summable in Q_T, endowed with the norm

$$\|f\|_{\mathcal{W}_p^{1,2}(Q_T)} := \Big(\int_{Q_T} |f|^p + |D_t f|^p + \sum_{i=1}^{N} |D_{x_i} f|^p + \sum_{i,j=1}^{N} |D_{x_i x_j} f|^p \, dxdt \Big)^{1/p}.$$

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2 Construction of the semigroup

In this section we construct the strongly continuous semigroup generated by $(A, D_{p,max}(A))$ in L^p. This fact is not completely new. In fact, the existence of a semigroup generated by A is proved e.g. in [12] where the coefficients q_{ij} are only supposed to be in L^∞ or can be deduced from the more general results of [4, Theorem 2.3]. However, these results do not show (directly) that the domain of the generator is $D_{p,max}(A)$. For this reason and for the sake of completeness, we give the construction below. In this section we denote by A_0 the operator

$$A_0 = \sum_{i,j=1}^{N} D_i(q_{ij} D_j).$$

We need the following lemma.

Lemma 2.1 *Let Ω be a bounded domain with a C^2 boundary or $\Omega = \mathbf{R}^N$ and $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Let, moreover, $\eta \in C_b^1(\Omega)$ be nonnegative. Then for $1 < p < \infty$*

$$(p-1) \int_\Omega \eta \sum_{i,j} q_{ij} |u|^{p-2} D_i u D_j u \chi_{\{u \neq 0\}} + \int_\Omega \sum_{i,j} q_{ij} |u|^{p-2} u D_i u D_j \eta \leq - \int_\Omega \eta (A_0 u) u |u|^{p-2}. \tag{2.1}$$

Proof. Let us prove the result for a bounded Ω. The case $\Omega = \mathbf{R}^N$ is even simpler. Observe that if $p \geq 2$, then (2.1) holds with equality. This is readily seen since the function $u|u|^{p-2}$ belongs to $W^{1,p'}(\Omega)$ and therefore integration by parts in the right hand side of (2.1) is allowed.

Let then be $1 < p < 2$, and take first $u \in C^2(\overline{\Omega}) \cap C_0(\Omega)$. For $\delta > 0$ we have

$$
\begin{aligned}
-\int_\Omega (A_0 u)\eta u\Big(u^2+\delta\Big)^{p/2-1} &= \int_\Omega \eta\Big(u^2+\delta\Big)^{p/2-2}\Big((p-1)u^2+\delta\Big)\sum_{i,j} q_{ij}D_iuD_ju \\
&+ \int_\Omega u\Big(|u|^2+\delta\Big)^{p/2-1}\sum_{i,j} q_{ij}D_iuD_j\eta. \qquad (2.2)
\end{aligned}
$$

Letting $\delta \to 0$ and recalling that $\nabla u = 0$ a.e. on the set $\{u = 0\}$, from Fatou's lemma we obtain

$$
(p-1)\int_\Omega \eta\sum_{i,j} q_{ij}D_iuD_ju|u|^{p-2}\chi_{\{u\neq 0\}} \le -\int_\Omega (A_0u)\eta u|u|^{p-2} - \int_\Omega u|u|^{p-2}\sum_{i,j} q_{ij}D_iuD_j\eta.
$$

Therefore, the function $\eta\sum_{i,j} q_{ij}D_iuD_ju|u|^{p-2}\chi_{\{u\neq 0\}}$ belongs to $L^1(\Omega)$ and one obtains (2.1) with equality, letting $\delta \to 0$ in (2.2) and using dominated convergence.

In the general case $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ we find a sequence $(u_n) \in C^2(\overline{\Omega}) \cap C_0(\Omega)$ such that $u_n \to u$ in $W^{2,p}(\Omega)$ and a.e., $\nabla u_n \to \nabla u$ a.e. and obtain (2.1) (with inequality) from the previous case, using again Fatou's lemma. □

Let us define

$$
\lambda_p = \frac{K^2}{4\nu(p-1)} - \frac{1}{p}\inf_{x\in\mathbf{R}^n} \operatorname{div} B(x), \qquad K = \|F\|_\infty. \qquad (2.3)
$$

Theorem 2.2 *Suppose that (H1) and (H2) hold. Then the operator $(A, D_{p,max}(A))$ generates a semigroup $(P(t))_{t\ge 0}$ in $L^p(\mathbf{R}^N)$ which satisfies the estimate*

$$
\|P(t)f\|_p \le e^{\lambda_p t}\|f\|_p \qquad (2.4)
$$

for every $f \in L^p(\mathbf{R}^N)$.

Proof. First of all, notice that the operator $(A, D_{p,max}(A))$ is closed, by local L^p regularity. In order to apply Hille-Yosida's theorem, let us prove that for every $\lambda > \lambda_p$ the operator $(\lambda - A)$ is bijective on $L^p(\mathbf{R}^N)$ and the resolvent estimate

$$
\|u\|_p \le \frac{\|f\|_p}{\lambda - \lambda_p} \qquad (2.5)
$$

holds.

Let then $f \in L^p(\mathbf{R}^N)$ be given, and consider the Dirichlet problem

$$
\begin{cases} \lambda u - Au = f & \text{in } B_\varrho \\ u = 0 & \text{on } \partial B_\varrho. \end{cases} \qquad (2.6)
$$

in $L^p(B_\varrho)$. According to [6, Theorem 9.15], a unique solution u_ϱ exists in $W^{2,p}(B_\varrho) \cap W^{1,p}_0(B_\varrho)$ for large λ. Observe that the dissipativity estimate

$$
(\lambda - \lambda_p)\|u_\varrho\|_p \le \|f\|_p \qquad (2.7)
$$

holds. To show this, we multiply the equation $\lambda u - Au = f$ by $u^* = u|u|^{p-2}$ and integrate over B_ϱ. Then we have, using Lemma 2.1 with $\Omega = B_\varrho$ and $\eta = 1$,

$$-\int_{B_\varrho} u^* A_0 u \geq (p-1)\int_{B_\varrho} |u|^{p-2}\sum_{i,j} q_{ij} D_i u D_j u,$$

$$\int_{B_\varrho} u^* B\cdot\nabla u = -\frac{1}{p}\int_{B_\varrho} |u|^p \mathrm{div}\, B.$$

It follows that

$$\begin{aligned}\int_{B_\varrho}\Big(\lambda + \frac{1}{p}\mathrm{div}\ B\Big)|u|^p + \nu(p-1)\int_{B_\varrho}|\nabla u|^2|u|^{p-2} &\leq \|f\|_p\|u\|_p^{p-1} + K\int_{B_\varrho}|\nabla u||u|^{p-1}\\ &\leq \|f\|_p\|u\|_p^{p-1} + K\Big(\int_{B_\varrho}|\nabla u|^2|u|^{p-2}\Big)^{1/2}\Big(\int_{B\varrho}|u|^p\Big)^{1/2} \quad (2.8)\\ &\leq \|f\|_p\|u\|_p^{p-1} + K\sigma\int_{B_\varrho}|\nabla u|^2|u|^{p-2} + \frac{K}{4\sigma}\int_{B_\varrho}|u|^p.\end{aligned}$$

Choosing $\sigma = \nu(p-1)/K$, (2.7) follows and therefore (2.6) has a (unique) solution satisfying (2.7) for every $\lambda > \lambda_p$.

Let us fix $\varrho_1 < \varrho_2$. Then, for $\varrho > \varrho_2$, u_ϱ belongs to $W^{2,p}(B_{\varrho_2})$ and combining estimate (2.7) with [6, Theorem 9.11] we obtain

$$\|u_\varrho\|_{W^{2,p}(B_{\varrho_1})} \leq C_1\left(\|\lambda u_\varrho - Au_\varrho\|_{L^p(B_{\varrho_2})} + \|u_\varrho\|_{L^p(B_{\varrho_2})}\right) \leq C\|f\|_p \qquad (2.9)$$

for a constant $C := C(p, \varrho_1, \varrho_2, \lambda, A) > 0$. From (2.9) it follows that the u_ϱ are bounded in $W^{2,p}_{loc}(\mathbf{R}^N)$, hence there is a sequence (u_{ϱ_n}) weakly converging to $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ which solves $\lambda u - Au = f$. Moreover, $u \in L^p(\mathbf{R}^N)$ and (2.5) holds. Finally, by difference, $Au \in L^p(\mathbf{R}^N)$ and then $u \in D_{p,max}(A)$.

It remains to show that this solution is unique in $D_{p,max}(A)$. Assume that $u \in D_{p,max}(A)$ satisfies $\lambda u - Au = 0$. We multiply the equation by $u|u|^{p-2}\eta_n^2$, where $\eta_n(x) = \eta(x/n)$, $\eta \in C_0^\infty$ and $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, and integrate over $\mathbf{R}^N$. Integrating by parts and using Lemma 2.1 with $\Omega = \mathbf{R}^N$ we obtain

$$\begin{aligned}&\int_{\mathbf{R}^N}\Big(\lambda + \frac{1}{p}\mathrm{div}\, B\Big)\eta_n^2|u|^p + (p-1)\int_{\mathbf{R}^N}\eta_n^2|u|^{p-2}\sum_{i,j} q_{ij}D_iuD_ju \qquad (2.10)\\ \leq\ & -2\int_{\mathbf{R}^N}\eta_n u|u|^{p-2}\sum_{i,j}q_{ij}D_juD_i\eta_n - \frac{2}{p}\int_{\mathbf{R}^N}\eta_n|u|^pB\cdot\nabla\eta_n + \int_{\mathbf{R}^N}\eta_n^2u|u|^{p-2}F\cdot\nabla u.\end{aligned}$$

Since $\|q_{ij}\|_\infty \leq C$, $|\nabla\eta_n| \leq C/n$ and $|B||\nabla\eta_n| \leq C$, for some $C > 0$, we deduce

$$\begin{aligned}&\int_{\mathbf{R}^N}\Big(\lambda + \frac{1}{p}\mathrm{div}\ B\Big)\eta_n^2|u|^p + \nu(p-1)\int_{\mathbf{R}^N}\eta_n^2|u|^{p-2}|\nabla u|^2\\ \leq\ & (K + C/n)\int_{\mathbf{R}^N}\eta_n|u|^{p-1}|\nabla u| + \frac{2}{p}\int_{\mathbf{R}^N}\eta_n|u|^p|B||\nabla\eta_n|\\ \leq\ & (K + C/n)\Big(\int_{\mathbf{R}^N}\eta_n^2|u|^{p-2}|\nabla u|^2\Big)^{1/2}\Big(\int_{\mathbf{R}^N}|u|^p\Big)^{1/2} + \frac{2}{p}\int_{\mathbf{R}^N}\eta_n|u|^p|B||\nabla\eta_n|\\ \leq\ & (K\sigma + C/n)\int_{\mathbf{R}^N}\eta_n^2|\nabla u|^2|u|^{p-2} + (K/(4\sigma) + C/n)\int_{\mathbf{R}^N}|u|^p + \frac{2}{p}\int_{\mathbf{R}^N}\eta_n|u|^p|B||\nabla\eta_n|.\end{aligned}$$

Setting $\sigma = \frac{\nu(p-1)}{(1+\varepsilon)K}$, $\alpha_\varepsilon = \frac{\varepsilon K}{4\nu(p-1)}$, $\beta_\varepsilon = \frac{\varepsilon\nu(p-1)}{1+\varepsilon}$ (with $\varepsilon > 0$ to be chosen), we obtain

$$\int_{\mathbf{R}^N} (\lambda - \lambda_p - \alpha_\varepsilon)\eta_n^2 |u|^p + (\beta_\varepsilon - C/n) \int_{\mathbf{R}^N} \eta_n^2 |u|^{p-2} |\nabla u|^2 \le \frac{C}{n} \int_{\mathbf{R}^N} |u|^p + \frac{2C}{p} \int_{n \le |x| \le 2n} |u|^p.$$

Choosing ε such that $\lambda - \lambda_p - \alpha_\varepsilon > 0$ and letting $n \to \infty$ it follows that $u = 0$. $\square$

Notice that for the existence of a C_0-semigroup generated by A the much weaker hypothesis $\operatorname{div} B \ge K$ for some $K \in \mathbf{R}$ suffices, by the proof of the above theorem. The linear growth of B has been used only to prove that the domain of the generator is $D_{p,max}(A)$.

Remark 2.3 Notice that for $1 < p \le 2$, besides (2.5), the gradient estimate

$$\|\nabla u\|_p \le C\|f\|_p$$

follows from Theorem 2.2. In fact, choosing $\sigma < \nu(p-1)/K$ in (2.8) and letting $\varrho \to \infty$, we have

$$\int_{\mathbf{R}^N} |u|^{p-2} |\nabla u|^2 \le C\|f\|_p^p$$

and then using Hölder inequality, we obtain

$$\int_{\mathbf{R}^N} |\nabla u|^p \le \Big(\int_{\mathbf{R}^N} |u|^{p-2} |\nabla u|^2\Big)^{p/2} \Big(\int_{\mathbf{R}^N} |u|^p\Big)^{1-p/2}.$$

3 A special case

In this section we assume that A is given in the non-divergence form

$$A = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i=1}^N b_i D_i = \operatorname{Tr}\big[Q\, D^2\big] + B \cdot \nabla. \tag{3.1}$$

We assume that (H1), (H2) hold with $F = 0$, $\nabla B \in C_b^2(\mathbf{R}^N)$ and that the coefficients q_{ij} and (b_i) satisfy (H3).

We fix $T > 0$ and consider the parabolic problem

$$\begin{cases} D_t u - Au = g & \text{in } Q_T \\ u(0) = f & \text{in } \mathbf{R}^N. \end{cases} \tag{3.2}$$

We prove that a suitable change of variables allows us to find a parabolic problem equivalent to (3.2), but with (regular and) *bounded* coefficients. Let us consider the ordinary Cauchy problem in $\mathbf{R}^N$:

$$\begin{cases} \dfrac{d\xi}{dt} = B(\xi) \qquad t \in \mathbf{R}, \\ \\ \xi(0) = x \end{cases} \tag{3.3}$$

and denote by $\xi(t,x)$ its solution. We shall look at the equation solved by $v(t,x) := u(t, \xi(-t,x))$. The relevant properties of $\xi(t,x)$ are collected in the following lemma, whose proof is in [13, Section 2]. In order to shorten the notation, we shall denote by ξ_x the Jacobian matrix $(\partial\xi_i/\partial x_j)$ of the derivatives of ξ with respect to x and by ξ_x^* its transpose matrix.

Lemma 3.1 *If B is Lipschitz continuous and ∇B belongs to C_b^2, there is a unique global solution $\xi(t,x)$ of (3.3) and the relationship $x = \xi\big(t,\xi(-t,x)\big)$ holds. Moreover, all the following derivatives are bounded in every strip $[-T,T]\times\mathbf{R}^N$:*

$$\xi_x,\quad \xi_{tx},\ \frac{\partial}{\partial t}\xi_x\big(t,\xi(-t,x)\big),\ \frac{\partial}{\partial x_i}\xi_x\big(t,\xi(-t,x)\big),\ \frac{\partial}{\partial t}\xi_{xx}\big(t,\xi(-t,x)\big),\ \frac{\partial}{\partial x_i}\xi_{xx}\big(t,\xi(-t,x)\big)$$

and the matrix ξ_x is invertible, with determinant bounded away from zero in every strip $[-T,T]\times\mathbf{R}^N$.

We are now in a position to write the equivalent Cauchy problem. Setting $v(t,y) = u\big(t,\xi(-t,y)\big)$, by a direct computation we deduce

$$D_t v(t,y) = D_t u\big(t,\xi(-t,y)\big) - \sum_{i=1}^N b_i\big(\xi(-t,y)\big) D_{x_i} u\big(t,\xi(-t,y)\big) \tag{3.4}$$

and also

$$\begin{aligned} D_{x_i} u(t,x) &= \sum_{h=1}^N D_{x_i}\xi_h(t,x) D_{y_h} v\big(t,\xi(t,x)\big) &(3.5)\\ D_{x_i x_j} u(t,x) &= \sum_{h,k=1}^N D_{x_i}\xi_h(t,x) D_{y_h y_k} v\big(t,\xi(t,x)\big) D_{x_j}\xi_k(t,x) &(3.6)\\ &\quad+\sum_{h=1}^N D_{x_i x_j}\xi_h(t,x) D_{y_h} v\big(t,\xi(t,x)\big). \end{aligned}$$

Let us further set $\tilde f(t,y) = f\big(t,\xi(-t,y)\big)$, $\tilde Q = (\tilde q_{ij})$, $\tilde B = (\tilde b_i)$, with

$$\begin{aligned} \tilde Q(t,y) &= \xi_x^*\big(t,\xi(-t,y)\big) Q\big(\xi(-t,y)\big)\xi_x\big(t,\xi(-t,y)\big)\\ \tilde B(t,y) &= \Big(\mathrm{Tr}\big[D^2\xi_1(t,\xi(-t,y))Q(\xi(-t,y))\big],\dots,\mathrm{Tr}\big[D^2\xi_N(t,\xi(-t,y))Q(\xi(-t,y))\big]\Big), \end{aligned}$$

or, more explicitly,

$$\begin{aligned} \tilde q_{ij}(t,y) &= \sum_{h,k=1}^N D_{x_h}\xi_i(t,\xi(-t,y)) q_{hk}(\xi(-t,y)) D_{x_k}\xi_j(t,\xi(-t,y)), &(3.7)\\ \tilde b_i(t,y) &= \sum_{h,k=1}^N D_{x_h x_k}\xi_i(t,\xi(-t,y)) q_{hk}(\xi(-t,y)) &(3.8) \end{aligned}$$

and finally

$$\tilde A = \sum_{i,j=1}^N \tilde q_{ij}(t,y) D_{y_i y_j} + \sum_{i=1}^N \tilde b_i(t,y) D_{y_i} \tag{3.9}$$

The above computations show that u solves (3.2) if only if v solves the Cauchy problem

$$\begin{cases} D_t v(t,y) = \tilde A v(t,y) + g(t,\xi(-t,y)) & \text{in } Q_T\\ v(0) = f & \text{in } \mathbf{R}^N. \end{cases} \tag{3.10}$$

Notice that from Lemma 3.1 it follows that the coefficients $\tilde{q}_{ij}, \tilde{b}_i$ are in $C_b^1(Q_T)$. Moreover, the inequality

$$\sum_{i,j=1}^{N} \tilde{q}_{ij}(t,y)\eta_i\eta_j \geq \tilde{\nu}|\eta|^2$$

holds for every $y, \eta \in \mathbf{R}^N$, with a suitable $\tilde{\nu} > 0$ (see also [13] for further details). We may therefore apply the standard theory of nonautonomous parabolic problems to infer that the operators $\tilde{A}(t)$ generate a parabolic evolution family $G(t,s)$ in $L^p(\mathbf{R}^N)$, see e.g. [9, Corollary 6.1.6].

Finally, for $t \geq 0$ we define maps $S(t) : W^{k,p}(\mathbf{R}^N) \to W^{k,p}(\mathbf{R}^N)$, for $k = 0,1,2$ by $\big(S(t)f\big)(x) = f(\xi(t,x))$.

Lemma 3.2 *Let S be as above, let G be the evolution family generated by $\tilde{A}(t)$ and let us define $\Gamma(t) = S(t)G(t,0)$. Then, $(\Gamma(t))_{t\geq 0}$ is a strongly continuous semigroup in $L^p(\mathbf{R}^N)$.*

PROOF. We first check that the semigroup law $\Gamma(t+s) = \Gamma(t)\Gamma(s)$ holds. For, let us compute

$$\begin{aligned}
\Gamma(t+s) &= S(t+s)G(t+s,0) = S(t)S(s)G(t+s,s)G(s,0),\\
\Gamma(t)\Gamma(s) &= S(t)G(t,0)S(s)G(s,0),
\end{aligned}$$

hence it suffices to show that $G(t,0)S(s) = S(s)G(t+s,s)$. This can be done by proving that, as functions of t, both sides solve the same Cauchy problem for every $s \geq 0$. We then compute the derivatives

$$\begin{aligned}
&\frac{d}{dt}\big(G(t,0)S(s)\big) = \tilde{A}(t)G(t,0)S(s)\\
&\frac{d}{dt}\big(S(s)G(t+s,s)\big) = S(s)\tilde{A}(t+s)G(t+s,s)
\end{aligned}$$

and notice that the thesis follows from the equality

$$S(s)\tilde{A}(t+s) = \tilde{A}(t)S(s). \tag{3.11}$$

Let us write, for a smooth function u,

$$\big(S(s)\tilde{A}(t+s)u\big)(x) = \mathrm{Tr}\big[P\, D^2u(\xi(s,x))\big] + C\cdot\nabla u(\xi(s,x)),$$

with the matrix R and the vector field $C = (c_i)$ given by

$$\begin{aligned}
R(t,s,x) &= \xi_x^*\big(t+s,\xi(s,\xi(-t-s,x))\big)Q\big(\xi(s,\xi(-t,x))\big)\xi_x\big(t+s,\xi(s,\xi(-t-s,x))\big)\\
c_i(t,s,x) &= \mathrm{Tr}\big[D^2\xi_i(t+s,\xi(s,\xi(-t-s,x)))Q(\xi(s,\xi(-t-s,x)))\big].
\end{aligned}$$

From the semigroup property of the flow ξ and the equalities

$$\begin{aligned}
\xi_x(t+s,x) &= \xi_x\big(s,\xi(t,x)\big)\xi_x(t,x),\\
D^2_{x_hx_k}\xi_i(t+s,x) &= \sum_{j,\ell=1}^{N} D^2_{x_jx_\ell}\xi_i\big(s,\xi(t,x)\big)D_{x_h}\xi_j(t,x))D_{x_k}\xi_\ell(t,x))\\
&\quad+\sum_{j=1}^{N} D_{x_j}\xi_i\big(s,\xi(t,x)\big)D_{x_hx_k}\xi_i(t,x))
\end{aligned}$$

we deduce

$$
\begin{aligned}
\xi\big(s,\xi(-t-s,x)\big) &= \xi(-t,x),\\
D_{x_h}\xi_i\big(t+s,\xi(-t,x)\big) &= \sum_{k=1}^N D_{x_k}\xi_i\big(t,\xi(-t,x)\big)D_{x_h}\xi_k(s,x)),\\
D^2_{x_hx_k}\xi_i\big(t+s,\xi(-t,x)\big) &= \sum_{j,\ell=1}^N D^2_{x_jx_\ell}\xi_i(s,x)D_{x_h}\xi_j\big(t,\xi(-t,x)\big)D_{x_k}\xi_\ell\big(t,\xi(-t,x)\big)\\
&\quad+\sum_{j=1}^N D_{x_j}\xi_i(s,x)D^2_{x_hx_k}\xi_j\big(t,\xi(-t,x)\big).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
R(t,s,x) &= \xi_x^*\big(t+s,\xi(-t,x)\big)Q\big(\xi(-t,x)\big)\xi_x\big(t+s,\xi(-t,x)\big)\\
&= \xi_x^*\big(t,\xi(-t,x)\big)\xi_x^*(s,x)Q\big(\xi(-t,x)\big)\xi_x(s,x)\xi_x\big(t,\xi(-t,x)\big)\\
c_i(t,s,x) &= \mathrm{Tr}\big[\xi_x^*\big(t,\xi(-t,x)\big)D^2\xi_i(s,x)\xi_x\big(t,\xi(-t,x)\big)Q\big(\xi(-t,x)\big)\big]\\
&\quad+\sum_{j,h,k=1}^N D_{x_j}\xi_i(s,x)D^2_{x_hx_k}\xi_j\big(t,\xi(-t,x)\big)q_{kh}\big(\xi(-t,x)\big)\\
&= \mathrm{Tr}\big[\xi_x^*\big(t,\xi(-t,x)\big)D^2\xi_i(s,x)\xi_x\big(t,\xi(-t,x)\big)Q\big(\xi(-t,x)\big)\big]\\
&\quad+\sum_{j=1}^N D_{x_j}\xi_i(s,x)\mathrm{Tr}\big[D^2\xi_j\big(t,\xi(-t,x)\big)Q\big(\xi(-t,x)\big)\big].
\end{aligned}
$$

On the other hand, using (3.6), (3.7) and (3.7), (3.8), we have

$$
\begin{aligned}
\big(\tilde A(t)S(s)u\big)(x) &= \mathrm{Tr}\big[\xi_x^*\big(t,\xi(-t,x)\big)Q\big(\xi(-t,x)\big)\xi_x\big(t,\xi(-t,x)\xi_x^*(s,x)D^2u(\xi(s,x))\xi_x(s,x)\big]\\
&\quad+\sum_{i=1}^N D_iu\big(\xi(s,x)\big)\cdot\Big\{\sum_{j,k=1}^N D^2_{x_jx_k}\xi_i(s,x)\tilde q_{jk}(t,x)\\
&\quad+\sum_{j=1}^N D_{x_j}\xi_j(s,x)\mathrm{Tr}\big[D^2\xi_i\big(t,\xi(-t,x)\big)Q\big(\xi(-t,x)\big)\big]\Big\}\\
&= \mathrm{Tr}\big[R\,D^2u\big(\xi(s,x)\big)\big]+C\cdot\nabla u(\xi(s,x)).
\end{aligned}
$$

Since the strong continuity of $\Gamma(t)$ follows easily from the strong continuity of $S(t)$ and $G(t,0)$, the proof is complete. $\square$

Theorem 3.3 *Assume that A, given by (3.1) satisfies (H1), (H2) and (H3) and also that $\nabla B\in C_b^2(\mathbf{R}^N)$. For every $f\in L^p(\mathbf{R}^N)$ and $T>0$, the function $P(\cdot)f$ belongs to $C\big(]0,T];W^{2,p}(\mathbf{R}^N)\big)\cap C^1\big(]0,T];L^p_{loc}(\mathbf{R}^N)\big)$ and satisfies the estimates*

$$
\|D^2P(t)f\|_p\le\frac{C_T}{t}\|f\|_p,\qquad \|\nabla P(t)f\|_p\le\frac{C_T}{\sqrt t}\|f\|_p. \tag{3.12}
$$

Proof. Let $v(t) = G(t,0)f$ be the solution of the problem

$$\begin{cases} D_t v(t) = \tilde{A}(t)v(t) & \text{in } Q_T \\ v(0) = f & \text{in } \mathbf{R}^N. \end{cases} \tag{3.13}$$

From [9, Corollary 6.1.6], it follows that $v \in C\big(]0,T]; W^2_p(\mathbf{R}^N)\big) \cap C\big([0,T]; L^p(\mathbf{R}^N)\big) \cap C^1\big(]0,T]; L^p(\mathbf{R}^N)\big)$ and that

$$\|D^2 v(t,\cdot)\|_p \le \frac{C_T}{t}\|f\|_p, \qquad \|\nabla v(t,\cdot)\|_p \le \frac{C_T}{\sqrt{t}}\|f\|_p.$$

for $0 < t \le T$. The function $u(t,x) := v\big(t, \xi(-t,x)\big)$ then belongs to $C\big(]0,T]; W^2_p(\mathbf{R}^N)\big) \cap C\big([0,T]; L^p(\mathbf{R}^N)\big) \cap C^1\big(]0,T]; L^p_{loc}(\mathbf{R}^N)\big)$, and satisfies a similar estimate for $0 < t \le T$:

$$\|D^2 u(t,\cdot)\|_p \le \frac{C'_T}{t}\|f\|_p, \qquad \|\nabla u(t,\cdot)\|_p \le \frac{C'_T}{\sqrt{t}}\|f\|_p.$$

We have only to show that $u(t,\cdot) = P(t)f$.

Since $u(t,\cdot)$ is nothing but $\Gamma(t)f$, we have to show that the semigroups $(\Gamma(t))_{t\ge0}$ and $(P(t))_{t\ge0}$ coincide. Let $(C, D(C))$ be the generator of $(\Gamma(t))_{t\ge0}$ in $L^p(\mathbf{R}^N)$ and observe that $u = \Gamma(\cdot)f$ belongs to $C^1\big(]0,T]; L^p_{loc}(\mathbf{R}^N)\big)$ and satisfies $u_t = Au$ in $]0,T] \times \mathbf{R}^N$. Therefore, if $f \in D(C)$, then $Cf = Af \in L^p(\mathbf{R}^N)$. This shows that $(C, D(C))$ is a restriction of $(A, D_{p,max}(A))$, hence coincides with this last since both operators are generators of semigroups. Therefore, $P(t) = \Gamma(t)$ and this concludes the proof. □

We study now the regularity of the mild solution of problem (3.2) with $f = 0$ and $g \in L^p(Q_T)$. Defining $g_s(x) = g(s,x)$, we may identify $L^p(Q_T)$ with $L^p((0,T); L^p(\mathbf{R}^N))$. The mild solution is then given by $u(t) = \int_0^t P(t-s)g_s\,ds$.

Theorem 3.4 *Assume that A, given by (3.1) satisfies (H1), (H2) and (H3) and that $\nabla B \in C^1_b(\mathbf{R}^N)$. Let $T > 0$ and $g \in L^p(Q_T)$ be given, and consider the mild solution u of the Cauchy problem (3.2) with $f = 0$. Then, u belongs to $\mathcal{W}^{1,2}_{p,loc}(Q_T)$ and satisfies*

$$u,\ D_t u - B\cdot\nabla u,\ D_{x_i}u,\ D_{x_i x_j}u \in L^p(Q_T). \tag{3.14}$$

Proof. Since $P(t) = S(t)\circ G(t,0)$ and $G(t-s,0)S(s) = S(s)G(t,s)$, u is given by

$$u(t) = \int_0^t S(t-s)\circ G(t-s,0)g_s\,ds = S(t)\int_0^t G(t,s)S(-s)g_s\,ds.$$

Let $h_s = S(-s)g_s \in L^p((0,T); L^p(\mathbf{R}^N))$ and $v(t) = \int_0^t G(t,s)h_s\,ds$. Then $u(t) = S(t)v(t)$, i.e. $u(t,x) = v\big((t,\xi(t,x)\big)$ and conditions $u \in \mathcal{W}^{1,2}_{p,loc}(Q_T)$ and (3.14) translate into $v \in \mathcal{W}^{1,2}_{p,loc}(Q_T)$ (see (3.4), (3.6), (3.7)). Let us show that v belongs to $\mathcal{W}^{1,2}_p(Q_T)$.

Let $(h^{(n)}) \in C_0^\infty(Q_T)$ be convergent to h in $L^p(Q_T)$ and define $v_n(t) = \int_0^t G(t,s)h^{(n)}_s\,ds$. Using [9, Proposition 6.1.3] we deduce that $v_n \in C([0,T]; W^{2,p}(\mathbf{R}^N)) \cap C^1([0,T]; L^p(\mathbf{R}^N))$ is a classical solution of the problem

$$\begin{cases} D_t w - \tilde{A}(t)w(t) = h^{(n)}(t) & \text{in } Q_T \\ w(0) = 0 & \text{in } \mathbf{R}^N. \end{cases}$$

Theorem IV.9.1 of [8] yields

$$\|v_n\|_{\mathcal{W}^{1,2}_p(Q_T)} \le C_T\|h^{(n)}\|_{L^p(Q_T)},$$

for a suitable constant C_T, and the thesis follows letting $n \to \infty$. □

Remark 3.5 Notice that the above Theorem does not say that the time derivative of the solution u of (3.2) belongs to $L^p(Q_T)$: only the derivative along the characteristic curves defined by system (3.3), namely $D_t u - B \cdot \nabla u$, is p-summable on the whole of Q_T.

4 Proof of Theorem 1

The inclusion $D_p \subset D_{p,max}(A)$ being trivial, we have only to prove the opposite one, i.e., the following implication:

$$u \in L^p(\mathbf{R}^N), \quad Au \in L^p(\mathbf{R}^N) \qquad \Longrightarrow \qquad u \in W^{2,p}(\mathbf{R}^N). \tag{4.1}$$

For clarity reasons, we split the proof in two steps.

Step 1. Assume $\nabla B \in C_b^2, F = 0$. Let $u \in D_{p,max}(A)$ be given and set $f = Au$. Then

$$u = P(t)u - \int_0^t P(t-s)f\, ds$$

and Theorem 3.3 shows that, for any $t > 0$, $P(t)u \in W^{2,p}(\mathbf{R}^N)$. Moreover, Theorem 3.4 implies that the function w defined by $w(t) = \int_0^t P(t-s)f\, ds$ belongs to $L^p((0,T); W^{2,p}(\mathbf{R}^N))$, hence $w(t) \in W^{2,p}(\mathbf{R}^N)$ for almost every t. Considering such a $\bar{t}$ we deduce that $u = P(\bar{t})u - w(\bar{t}) \in W^{2,p}(\mathbf{R}^N)$.

Step 2. The general case Let $0 \le \eta \in C_0^\infty(\mathbf{R}^N)$, $\int_{\mathbf{R}^N} \eta = 1$ and define $\hat{B} = B * \eta$. Set moreover

$$\hat{A} = \sum_{i,j=1}^N D_i\big(q_{ij} D_j\big) + \hat{B} \cdot \nabla.$$

From Step 1, we know that $D_{p,max}(\hat{A}) = D_p := \{u \in W^{2,p}(\mathbf{R}^N) : \ \hat{B} \cdot \nabla u \in L^p(\mathbf{R}^N)\}$. Since B is globally Lipschitz continuous, $B - \hat{B}$ is bounded and therefore

$$\|Au - \hat{A}u\|_p = \|(B - \hat{B} + F) \cdot \nabla u\|_p \le C\|u\|_{W^{1,p}(\mathbf{R}^N)}$$

for $u \in D_p$. Moreover $D_p = \{u \in W^{2,p}(\mathbf{R}^N) : \ B \cdot \nabla u \in L^p(\mathbf{R}^N)\}$. Let $(\hat{P}(t)_{t \ge 0}$ be the semigroup generated by $(\hat{A}, D_p)$. Combining the above estimate with (3.12) it follows that

$$\|(A - \hat{A})\hat{P}(t)f\|_p \le \frac{C}{\sqrt{t}}\|f\|_p$$

for every $f \in D_p$ and the Miyadera-Voigt perturbation Theorem (see e.g. [5, Corollary 3.16]) shows that (A, D_p) is a generator. Since $D_p \subset D_{p,max}(A)$ and $(A, D_{p,max}(A))$ is also a generator by Theorem 2.2 we conclude that $D_p = D_{p,max}(A)$. $\square$

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