Approximation problems for curvature varifolds

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Introduction.

Many variational problems involve geometric objects, both problems essentially geometric in nature and problems where a set is one of the unknowns, such as for instance free boundary problems, which are by now classical, and (more recently studied) free discontinuity problems (see [2], [3]). This leads to the introduction of objects more general than the classical differential manifolds, in order to be able to describe various phenomena not allowed in the classical context, or also to get a (suitably defined) weak solution to be hopefully regularized. Examples of such generalized manifolds are the sets of finite perimeter, the varifolds, various classes of currents, whose introduction is mainly motivated by least area type problems. It has an obvious interest to try to compare such classes (see for instance [6]), and a considerable effort has been constantly produced in this direction.

More recently, new classes of geometric objects have been introduced in order to study variational problems involving curvature depending functionals. In particular, we refer to r-dimensional varifolds with second fundamental form in L^p (here denoted by $W^{2,p}_r$) introduced by J.E.Hutchinson in [7] via an integration by parts formula (see also [4]), and Sobolev classes proposed by E.De Giorgi as weak limits (with L^p bounds on the second fundamental form) of functions which are locally the sum of characteristic functions of graphs of regular functions of r variables (denoted F_rC^κ).

In this paper we introduce a different notion of Sobolev type manifolds, closer to De Giorgi's F_rC^κ classes and denoted by $F_rW^{2,p}$ (with $p > r$), whose members are (locally) sums of characteristic functions of graphs of $W^{2,p}$ functions of r variables, and attempt a comparison between some of these classes. Our first motivation is the following: on the one hand, a regularity result of Hutchinson's states that a r-dimensional varifold with second fundamental form in L^p, $p > r$, is locally the sum of graphs of $C^{1,\alpha}$ multiple-valued functions. On the other hand, it can be proved that any varifold in the Sobolev classes belongs to the corresponding Hutchinson's class. The natural questions is: are these two classes equal? We will see that in general the answer is *no* (Example 4.4), because Sobolev classes satisfy a stronger local regularity property.

A related question concerns the closure of $F_rC^\kappa(\Omega)$ and $F_rW^{2,p}(\Omega)$ classes in $W^{2,p}_r(\Omega)$ with respect to the natural topology, *i.e.* weak* convergence of the corresponding measures in Ω with L^p bounds on the second fundamental form. In particular, we address the question whether the closure of $F_rC^\kappa(\Omega)$ is contained in $F_rW^{2,p}(\Omega)$. The answer is *yes* if (Theorems 2.3 and 2.4) and only if (§4) $\kappa = \omega$ (case of analytic regularity) or $r = 1$.

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The outline of the paper is the following: in §1 we recall the main definitions needed in the paper, such as rectifiable sets and integer valued functions, and describe the classes $W^{2,p}_r(\Omega)$, $F_rC^\kappa(\Omega)$, $F_rW^{2,p}(\Omega)$. In §2, after discussing some properties of $W^{2,p}_r(\Omega)$, we present the main results and deduce some consequences of the monotonicity formula (see the Appendix for the statement and a short proof) which will be useful in proving theorems 2.3 and 2.4; such proofs occupy the whole of §3. Finally, §4 is devoted to explain some geometric properties that the various classes may (or may not) share, in order to clarify the precise bounds of our results. The discussion therein depends on the comparison between the notions of *local decomposability* and *global decomposability* of an integer valued function. In particular, we show that, except for the analytic case, local decomposability does not imply global decomposability, even if the domain is simply connected. We also show that for any $p > r > 1$ there exists a function $f \in W^{2,p}_r$ which is approximable by F_rC^∞ functions but does not belong to $F_rW^{2,p}$.

1. Preliminaries.

In this section we introduce some classes of piecewise C^κ manifolds with multiplicities, and we describe how the divergence theorem on manifolds leads to a "distributional" definition of the second fundamental form (see [7], [8]).

Let $n \geq 2$ and $1 \leq r < n$ be integers. We denote by $O(r,n)$ the set of (unoriented) r-dimensional subspaces of $\mathbf{R}^n$ (henceforth r-planes), and use the same notation P for an element of $O(r,n)$, the orthogonal projection $P : \mathbf{R}^n \to \mathbf{R}^n$ which maps $\mathbf{R}^n$ onto P and the associated matrix, whose entries are in turn denoted P_{ij}; the orthogonal complement of P is denoted by $P^\perp$ and the distance $|P - Q|$ is that one induced over $O(r,n)$ as a subset of $\mathbf{R}^{n^2}$. We shall use the notation $B^r_\rho(x)$ for $\{y \in \mathbf{R}^r : |x - y| < \rho\}$, drop x if $x = 0$ and put $\omega_r = \mathcal{H}^r(B^r_1)$, for $\mathcal{H}^r$ the r-dimensional Hausdorff measure. Moreover, $G(\varphi)$ denotes the graph of the function φ, $G(\varphi) = \{(z,y) : y = \varphi(z)\}$ and $\mathrm{char}S(x)$ denotes the characteristic function of the set S evaluated at x. Notice also that when repeated indices are present the sum is understood.

In the following definition we introduce the well known notions of rectifiability, approximate tangent space, and tangential differential operators (see [5], [13]).

Definition 1.1. Let $S \subset \mathbf{R}^n$. We say that S is *countably $\mathcal{H}^r$-rectifiable* if there exists a sequence of C^1 submanifolds Γ_i of dimension r such that

$$\mathcal{H}^r\left(S \setminus \bigcup_{i=1}^\infty \Gamma_i\right) = 0.$$

Let $\Omega \subset \mathbf{R}^n$ be an open set and $f : \Omega \to \mathbf{N}$. We set $S_f = \{x \in \Omega : f(x) \neq 0\}$, and we say that f is $\mathcal{H}^r$-rectifiable in Ω if $f^{-1}(i)$ is countably $\mathcal{H}^r$-rectifiable for any $i \geq 1$ and $\int_\Omega f d\mathcal{H}^r < +\infty$. If f is $\mathcal{H}^r$-rectifiable then for $f\mathcal{H}^r$-a.e. $x \in \Omega$ there is a unique $P \in O(r,n)$ such that

$$(1.1)\quad \lim_{\rho\to 0^+} \rho^{-r} \int_{B^n_\rho(x)} \phi\Big(\frac{y-x}{\rho}\Big) f(y) d\mathcal{H}^r(y) = f(x) \int_P \phi(y)\, d\mathcal{H}^r(y) \qquad \forall \phi \in C^1_0\big(\mathbf{R}^n\big).$$

If (1.1) holds, we say that P is the *approximate tangent space* to f and we denote it by $P_f(x)$. To any $\mathcal{H}^r$-rectifiable function $f : \Omega \to \mathbf{N}$ we associate the *varifold* measure μ_f in $\Omega \times O(r,n)$ by the formula

$$\int_{\Omega\times O(r,n)} \varphi(x,P)\, d\mu_f = \int_\Omega \varphi\big(x, P_f(x)\big)\, f d\mathcal{H}^r \quad \forall \varphi \in C^0_0\big(\Omega \times O(r,n)\big).$$

Finally, if x satisfies (1.1) and ϕ is a C^1 function defined in a neighbourhood of x we define the tangential gradient of ϕ at x

$$\delta^f\phi(x) = \big(\delta^f_1\phi(x), \ldots, \delta^f_n\phi(x)\big)$$

as the projection of $\nabla\phi(x)$ on $P_f(x)$.

We now introduce the classes $F_rC^\kappa\big(\Omega\big)$ whose members are locally the sum of the characteristic functions of a finite number of r-dimensional graphs of C^κ functions (see [3]).

Definition 1.2. Let $\Omega \subset \mathbf{R}^n$ be an open set, $\kappa \in \mathbf{N} \cup \{\infty, \omega\}$, $f : \Omega \to \mathbf{N}$. We say that $f \in F_rC^\kappa\big(\Omega\big)$ if for any $x \in S_f$ we can find a neighbourhood U of x in Ω, a positive integer q and r-dimensional C^κ (analytic if $\kappa = \omega$) manifolds Γ_i (not necessarily distinct) such that

$$f(x) = \sum_{i=1}^q \mathrm{char}\Gamma_i(x) \qquad \forall x \in U.$$

Remark 1.3. In an analogous way, we can define the class $F_rW^{2,p}\big(\Omega\big)$ by requiring that each Γ_i in Definition 1.2 is locally the graph of a Lipschitz continuous function of r variables belonging to the class $W^{2,p}$. Notice that the Sobolev embedding theorem yields

$$F_rW^{2,p}\big(\Omega\big) \subset F_rC^1\big(\Omega\big)$$

for any $p > r$. Since graphs of Lipschitz continuous functions are (locally) rectifiable, the condition $f \in F_rW^{2,p}\big(\Omega\big)$ implies that f is locally $\mathcal{H}^r$-rectifiable in Ω. From now on, we shall be interested only to the case $p > r$.

Let us assume that $\Omega = \Omega' \times \mathbf{R}^{n-r}$ with $\Omega' \subset \mathbf{R}^r$. We now define the C^κ-local decomposability property by requiring that the surfaces Γ_i of Definition 1.2 are graphs of C^κ maps defined on the same r-plane.

Definition 1.4. Let $\kappa \in \mathbf{N}\cup\{\infty, \omega\}$. Let $\Omega' \subset \mathbf{R}^r$ be an open set and let $f : \Omega' \times \mathbf{R}^{n-r} \to \mathbf{N}$. The function f is said to be C^κ-*locally decomposable* if for any $z \in \Omega'$ there exist an open neighbourhood U_z of z in Ω', an integer $q(z) > 0$, and functions $\varphi_z^{(i)} \in C^\kappa(U_z, \mathbf{R}^{n-r})$, $i = 1, \ldots, q(z)$ such that

$$f(x) = \sum_{i=1}^{q(z)} \mathrm{char} G(\varphi_z^{(i)})(x) \qquad \forall x \in U_z \times \mathbf{R}^{n-r}.$$

Remark 1.5. It is clear that any C^κ-locally decomposable function belongs to $F_r C^\kappa\big(\Omega' \times \mathbf{R}^{n-r}\big)$. Conversely, any $f \in F_r C^\kappa\big(\Omega' \times \mathbf{R}^{n-r}\big)$ is C^κ-locally decomposable if the tangent planes to the surfaces Γ_i are nowhere vertical with respect to the r-plane $\mathbf{R}^r \times \{0\}$.
The function $q(z)$ does not depend on the local decomposition. Indeed,

$$q(z) = \sum_{y \in \mathbf{R}^{n-r}} f(z, y) \qquad \forall z \in \Omega'.$$

Furthermore, q is locally constant in Ω' and therefore q is constant if Ω' is connected.

Let us now recall the divergence theorem on curved manifolds (see [13], 7.1). For any r-dimensional manifold without boundary $M \subset \mathbf{R}^n$ of class C^2 and any $\phi \in C_0^1\big(\mathbf{R}^n\big)$, the following integration by parts formula holds:

$$\int_M \delta_i^M \phi \, d\mathcal{H}^r = -\int_M \phi H_i \, d\mathcal{H}^r \qquad i = 1, \ldots, n, \tag{1.2}$$

where H is the mean curvature vector of M. Inserting in (1.2) a test function $\phi(x) = \varphi\big(x, P_M(x)\big)$, where φ (depending both on $x \in \mathbf{R}^n$ and $P \in O(r, n)$) has compact support with respect to x, we get

$$\begin{aligned} \int_M \big[\delta_i^M \varphi\big(x, P_M(x)\big) + D_{jk}\varphi\big(x, P_M(x)\big) A_{ijk}^M(x)\big] \, d\mathcal{H}^r &= \\ = \int_M \varphi\big(x, P_M(x)\big) H_i(x) \, d\mathcal{H}^r&, \end{aligned} \tag{1.3}$$

where

$$\delta_i^M \varphi(x, P) = \delta_i^M \varphi(\cdot, P)(x) \qquad \text{and} \qquad D_{jk}\varphi(x, P) = \frac{\partial \varphi(x, \cdot)}{\partial P_{jk}}(P),$$

and $A_{ijk}^M(x) = \delta_i^M (P_M)_{jk}(x) = \delta_i^M \delta_j^M x_k$. The coefficients A_{ijk}^M and the second fundamental form of M are each expressible in terms of the other and

$$H_i = A_{jji}^M \qquad \forall i = 1, \ldots, n \tag{1.4}$$

(see [7], [8]). These remarks led Hutchinson to give in [7] a definition of sets (with multiplicities) with p-summable weak second fundamental form equivalent (in an obvious sense) to the following one, which is expressed in terms of integer valued functions.

Definition 1.6. Let $\Omega \subset \mathbf{R}^n$ be an open set, let $f : \Omega \to \mathbf{N}$ be a $\mathcal{H}^r$-rectifiable function, and let $p \geq 1$. We say that $f \in W_r^{2,p}(\Omega)$ if and only if for $i, j, k = 1, \ldots, n$ there are Borel functions $A_{ijk}^f : \Omega \to \mathbf{R}$ such that

$$\|A^f\|_p = \Big(\int_\Omega f \sum_{i,j,k} \big|A_{ijk}^f\big|^p \, d\mathcal{H}^r\Big)^{1/p} < +\infty \qquad \text{if } p < \infty,$$

$$\|A^f\|_\infty = \text{ess sup}_x \Big\{ f(x) \sum_{i,j,k} \big|A_{ijk}^f(x)\big| \Big\} < +\infty \quad \text{if } p = \infty,$$

and

$$\begin{aligned} \int_\Omega f(x)\big(\delta_i^f \varphi(x, P_f(x)) + D_{jk}\varphi(x, P_f(x)) A_{ijk}^f(x)\big) \, d\mathcal{H}^r = \\ = -\int_\Omega f(x)\varphi(x, P_f(x)) A_{jji}^f(x) \, d\mathcal{H}^r \end{aligned} \tag{1.5}$$

for any $\varphi \in C_0^1(\Omega \times \mathbf{R}^{n^2})$ and any $i \in \{1, \ldots, n\}$. For any $f \in W_r^{2,p}(\Omega)$ and any $x \in \Omega$ we define

$$\mathrm{Tan}_x(f) = \big\{ P \in O(r,n) : (x, P) \in \mathrm{supp}(\mu_f) \big\}.$$

Remark 1.7. Let $f \in F_r W^{2,p}(\Omega)$. Then, by (1.3), $f \in W_r^{2,p}(B)$ for any open set $B \subset\subset \Omega$, and

$$A_{ijk}^f = \delta_i^f \delta_j^f x_k. \qquad f\mathcal{H}^r\text{−a.e. in } \Omega.$$

Moreover, if $p > r$, $\mathrm{Tan}_x(f)$ consists of the tangent r-planes to the sheets of S_f at x. The converse implication does not hold. Indeed, $F_r W^{2,p}(\Omega)$ is in general strictly contained in $W_r^{2,p}(\Omega)$ (see Example 4.4).

2. The main results.

It is natural to endow the set $W_r^{2,p}(\Omega)$ with the metrizable topology given by the weak* convergence of the Hausdorff measures associated to f, namely f_h converges to f if and only if

$$\lim_{h\to+\infty}\int_\Omega \phi f_h\,d\mathcal{H}^r=\int_\Omega \phi f\,d\mathcal{H}^r$$

for any $\phi\in C^0_0(\Omega)$. In the following theorem we recall useful properties of $F_rC^\kappa(\Omega)$ and $W^{2,p}_r(\Omega)$.

Theorem 2.1. *Let $n\ge 2$, $1\le r\le n-1$ be integers, and let $p>1$.*
(1) If f_h converges to f in $W^{2,p}_r(\Omega)$ and

$$\int_\Omega |H^h| f_h\,d\mathcal{H}^r\le\Gamma \qquad \forall h\in\mathbf{N} \tag{2.1}$$

for some $\Gamma\ge 0$ (with $H^h_i=A^{f_h}_{jji}$), then μ_{f_h} weakly converges to μ_f in $\Omega\times O(r,n)$.
(2) For any $\Gamma\ge 0$, the set

$$\left\{f\in W^{2,p}_r(\Omega):\int_\Omega f\,d\mathcal{H}^r+\|A^f\|_p\le\Gamma\right\}$$

is compact.
(3) If $f\in W^{2,p}_r(\Omega)$ for some $p>r$, then for any $x\in\overline{S_f}\cap\Omega$ we can find a finite number of r-planes $P_1,\ldots,P_N\in O(r,n)$ and positive integers $q_1,\ldots,q_N$ such that

$$\mathrm{Tan}_x(f)=\{P_1,\ldots,P_N\}$$

and

$$\lim_{\rho\to 0^+}\rho^{-r}\int_{B^n_\rho(x)\times O(r,n)}\psi(P)\,d\mu_f(y,P)=\omega_r\sum_{i=1}^N q_i\psi(P_i) \tag{2.2}$$

for any continuous function $\psi:O(r,n)\to\mathbf{R}$. Moreover, $\mathrm{Tan}_x(f)$ coincides with the tangent cone to $\overline{S_f}$ at x.
(4) Let $f\in F_rC^\kappa(\Omega)$ for some $\kappa>0$, and let us assume that there is a finite number of pairwise disjoint compact sets $C_1,\ldots,C_N\subset O(r,n)$ such that

$$\mathrm{supp}(\mu_f)\subset\Omega\times\bigcup_{i=1}^N C_i.$$

Then, we can find $f_1,\ldots,f_N$ in $F_rC^\kappa(\Omega)$ such that

$$\mu_{f_i}(B)=\mu_f\big(B\cap(\Omega\times C_i)\big)$$

for any Borel set $B\subset\Omega\times O(r,n)$ and any $i=1,\ldots,N$, and (for $p<+\infty$)

$$\sum_{i=1}^N\|A^{f_i}\|_p^p=\|A^f\|_p^p.$$

Proof. (1) The upper bound (2.1) and the Allard compactness theorem (see *e.g.* [13], Theorem 42.7 and Remark 42.8) yield a locally $\mathcal{H}^r$-rectifiable function $f' : \Omega \to \mathbf{N}$ such that (possibly passing to a subsequence) μ_{f_h} weakly converges to $\mu_{f'}$. In particular, by taking $\varphi(x, P) = \phi(x)$ we infer

$$\int_\Omega \phi f \, d\mathcal{H}^r = \int_\Omega \phi f' \, d\mathcal{H}^r \qquad \forall \phi \in C_0^1(\Omega),$$

hence $f = f'$ $\mathcal{H}^r$-a.e. and P_f, $P_{f'}$ are equal $\mathcal{H}^r$-almost everywhere in Ω.

(2) See [7], Theorem 4.4.2 or [9], Theorem 4.

(3) See [8], Theorem 3.4 and [13], Lemma 17.11.

(4) The construction of f_i is simply obtained by selecting (locally) the graphs whose tangent r-planes belong to C_i.

Remark 2.2. The decomposition property of Theorem 2.1(4) is also valid for $f \in W_r^{2,p}(\Omega)$. By using this property and (2.2), Hutchinson proved in [8] that, for $p > r$, any $f \in W_r^{2,p}(\Omega)$ can be locally described as the graph of a multi-valued $C^{1,\alpha}$ function in the sense of Almgren (see [1]).

The main results of our paper are the following theorems.

Theorem 2.3. *Let $\Omega \subset \mathbf{R}^n$ be an open set, let $r \in [2, n-1]$ be an integer, and let $p > r$. For any $\Gamma > 0$ we define*

$$E = \Big\{ f \in F_r C^\omega(\Omega) : \|A^f\|_p \le \Gamma \Big\}$$

Then, the closure of E is a compact set strictly contained in

$$\Big\{ f \in F_r W^{2,p}(\Omega) : \|A^f\|_p \le \Gamma \Big\},$$

hence strictly contained in $W_r^{2,p}(\Omega)$.

The characterization of the closure of E is still an open problem. We also note that, because of the L^p bound on the second fundamental form, we cannot expect that the functions whose graphs locally represent $f \in \overline{E}$ are better than $W^{2,p}$. Our proof, based on a monotonicity formula, crucially depends on the assumption $p > r$ and the case $p = r$ is, to our knowledge, open. In the case $p = r = 2$ related Lipschitz approximation results have been proved in [11] and [14].
If we replace ω by ∞ in the definition of E then, by Example 4.5, $\overline{E}$ is not contained in $F_r W^{2,p}(\Omega)$, but it is still contained in $W_r^{2,p}(\Omega)$.
The following theorem deals with the one dimensional case.

Theorem 2.4. *Let $\Omega \subset \mathbf{R}^n$ be an open set and let $p > 1$. Then, for any $\Gamma \geq 0$ the closure of the set*

$$\left\{ f \in F_1 C^2(\Omega) : \|A^f\|_p \leq \Gamma \right\}$$

is contained in $F_1 W^{2,p}(\Omega)$.

We now recall some facts which will be useful in the proof of Theorems 2.3, 2.4 presented in §3. Let be $A = B^r_{\rho_1} \times B^{n-r}_{\rho_2}$, $f : A \to \mathbf{N}$ a locally $\mathcal{H}^r$-rectifiable function, let

$$P_0 = \left\{ x \in \mathbf{R}^n : x_{r+1} = \ldots = x_n = 0 \right\},$$

and let

$$m(z) = \sum \left\{ f(x) : x \in A,\ P_0(x) = z \right\}.$$

For any $P \in O(r, n)$ we define

$$\mathbf{J}_r(P, P_0) = \mathcal{H}^r\left(P_0(Q_P)\right)$$

where $Q_P \subset P$ is any unit cube. We remark that $0 \leq \mathbf{J}_r(P, P_0) \leq 1$. Then, the area formula (see *e.g.* [5], 3.2.3) yields

$$\int_{B^r_{\rho_1}} m(z)\, dz = \int_A f(x) \mathbf{J}_r\left(P_f(x), P_0\right) d\mathcal{H}^r.$$

Let us check that for any $\delta < 1$ we can choose $\epsilon > 0$ so small that

$$|P - P_0| < \epsilon \quad \Longrightarrow \quad \mathbf{J}_r(P, P_0) \geq \delta. \tag{2.3}$$

In fact, (2.3) holds if $\epsilon < 1/(r+1)$ and δ are tied by the following relation:

$$\delta = \left[\frac{1 - (r+1)\epsilon}{1 - \epsilon} \right]^r.$$

Indeed, let $(v_1, \ldots, v_r)$ be an orthonormal basis of P, and set $\gamma_i = P_0(v_i)$, and

$$\gamma_1' = \gamma_1, \quad \gamma_2' = \gamma_2 - \langle \gamma_2, \frac{\gamma_1}{|\gamma_1|} \rangle \frac{\gamma_1}{|\gamma_1|}, \ldots$$

Then, $\mathbf{J}_r(P, P_0)$ is the r-dimensional measure of the r-simplex generated by $\gamma_1, \ldots, \gamma_r$, hence $\mathbf{J}_r(P, P_0) = \prod_{i=1}^r |\gamma_i'|$. From the inequality

$$|P_0^\perp v| = |P_0^\perp v - P^\perp v| \leq |P_0^\perp - P^\perp| = |P - P_0| < \epsilon,$$

(which holds for any $v \in P$) it readily follows $|\gamma_i| \geq (1 - \epsilon)$, $|\langle \gamma_i, \gamma_j \rangle| \leq \epsilon^2$ for $i \neq j$, and

$$|\gamma_i'| \geq (1 - \epsilon) - (i - 1) \frac{\epsilon}{1 - \epsilon} \geq \frac{1 - (r+1)\epsilon}{1 - \epsilon},$$

which proves our statement.

The following lemma, whose proof is based on the monotonicity formula (see [8], and also the Appendix below) is fundamental to estimate the oscillation of the tangent planes.

Lemma 2.5. *Let $n \geq 2$ and $1 \leq r \leq n-1$ be integers, $\Omega \subset \mathbf{R}^n$, and let $p > r$. Let f_h be converging to f in $W^{2,p}_r(\Omega)$ and let us assume that*

$$\|A^{f_h}\|_p \leq \Gamma < +\infty \qquad \forall h \in \mathbf{N}.$$

Then, the following implication holds:

$$P_h \in \mathrm{Tan}_{x_h}(f_h),\ (x_h, P_h) \to (x, P) \qquad \Longrightarrow \qquad P \in \mathrm{Tan}_x(f). \tag{2.4}$$

Proof. Let (x_h, P_h), (x, P) as in (2.4). Let $\psi \in C^1(\mathbf{R}^{n^2})$ be any Lipschitz function such that $\psi(P) = 1$ and $0 < \delta \leq \psi \leq 1$ for some $\delta > 0$. By Theorem 2.1(3) we infer

$$\liminf_{\sigma \to 0^+} \sigma^{-r} \int_{B^n_\sigma(x_h) \times O(r,n)} \psi(Q)\, d\mu_{f_h} \geq \psi(P_h)\omega_r$$

because $P_h \in \mathrm{Tan}_{x_h}(f_h)$. By letting $\sigma \to 0^+$ in the monotonicity formula we get

$$\left[\rho^{-r} \int_{\overline{B}^n_\rho(x_h) \times O(r,n)} \psi(Q)\, d\mu_{f_h}\right]^{1/p} \geq \left[\psi(P_h)\omega_r\right]^{1/p} - \frac{\Gamma}{1 - r/p}\left(1 + \delta^{-1}\|D\psi\|_\infty\right)\rho^{1-r/p},$$

so that, passing to the limit as $h \to +\infty$ we find

$$\rho^{-r} \int_{\overline{B}^n_\rho(x) \times O(r,n)} \psi(Q)\, d\mu_f \geq \left(\omega_r^{1/p} - \frac{\Gamma}{1 - r/p}\left(1 + \delta^{-1}\|D\psi\|_\infty\right)\rho^{1-r/p}\right)^p.$$

By letting $\rho \to 0^+$ and using Theorem 2.1(3) we obtain

$$\sum_{Q_i \in \mathrm{Tan}_x(f)} q_i \psi(Q_i) \geq 1.$$

Since ψ is arbitrary, this inequality can be true only if P is a member of $\mathrm{Tan}_x(f)$. □

3. Proof of Theorems 2.3, 2.4.

The proof of Theorem 2.3 and Theorem 2.4 is achieved in several steps, in which we first control the cardinality of tangent planes, then their oscillation and finally the number of

sheets. After these estimates, the crucial step consists in finding a *global* representation of f_h as a sum of characteristic functions of graphs (Step 5). This can be achieved (see Theorem 4.4) only in the case of analytic regularity, corresponding to Theorem 2.3, and in the one dimensional case, corresponding to Theorem 2.4. Since the proof of the two theorems are essentially the same, we confine ourselves to Theorem 2.3.
Let an integer $r \in [2, n-1]$ be given, and let us assume that there exist an open set $\Omega \subset \mathbf{R}^n$, $p \in]r, +\infty[$ and a sequence $(f_h) \subset F_r C^\omega\big(\Omega\big)$ converging to f and such that

$$\Gamma = \sup_{h \in \mathbf{N}} \Big(\|A^{f_h}\|_p \Big) < +\infty.$$

We have to show that $f \in F_r W^{2,p}\big(\Omega\big)$. By Theorem 2.1(2), $f \in W_r^{2,p}\big(\Omega\big)$. We fix $x_0 \in \Omega \cap \overline{S_f}$ and we construct a finite number of graphs of $W^{2,p}$ functions which represent f in a neighbourhood of x_0. For simplicity we assume $x_0 = 0$.

Step 1. We claim that it is not restrictive to assume that $\mathrm{Tan}_0(f)$ is a singleton. Let $P_1, \ldots, P_N$ be the members of $\mathrm{Tan}_0(f)$ and let

$$\epsilon = \frac{1}{3} \inf \Big\{ |P_i - P_j| : 1 \le i < j \le N \Big\}.$$

By Lemma 2.5, we can find $R > 0$ so small and $h_0 \in \mathbf{N}$ such that $B_R^n \subset \Omega$ and

$$\mathrm{supp}(\mu_{f_h}) \cap B_R^n \times O(r,n) \subset B_r^n \times \bigcup_{i=1}^N B_\epsilon^{n^2}(P_i)$$

for any $h \ge h_0$. We apply the decomposition property of Theorem 2.1(4) to f_h, getting $(f_{h,i}) \in F_r W^{2,p}\big(B_R^n\big)$ converging (up to subsequences) as $h \to +\infty$ to $f_i \in W_r^{2,p}\big(B_R^n\big)$ for any $i = 1, \ldots, N$. Since

$$\mu_{f_h} = \sum_{i=1}^N \mu_{f_{h,i}},$$

we get

$$\mu_f = \sum_{i=1}^N \mu_{f_i}.$$

Hence, we need only to show that $f_i \in F_r W^{2,p}\big(B_R^n\big)$ for any i. By the inclusion

$$\mathrm{supp}(\mu_{f_{h,i}}) \cap B_R^n \times O(r,n) \subset B_R^n \times C_i,$$

and Theorem 2.1(1) we infer

$$\mathrm{supp}(\mu_{f_i}) \cap B_R^n \times O(r,n) \subset B_R^n \times C_i,$$

so that $\mathrm{Tan}_0(f_i) = \{P_i\}$. $\square$

In the following we assume (up to a rotation in $\mathbf{R}^n$) that $\mathrm{Tan}_0(f) = \{P_0\}$, where we recall that P_0 is the projection on the first r coordinates. By Theorem 2.1(3) we get an integer $q \geq 1$ such that

$$\lim_{\rho \to 0^+} \rho^{-r} \int_{B^n_\rho} f d\mathcal{H}^r = q\omega_r. \tag{3.1}$$

Finally, Theorem 2.1(3) implies that we can find a sufficiently small $R > 0$ such that

$$P_0(x) \neq 0 \qquad \forall x \in \overline{S_f} \cap B^n_R \subset \Omega, \ x \neq 0. \tag{3.2}$$

Step 2. (height estimate) We claim that there exist $R_1 \in]0, R[$ and $h_1 \in \mathbf{N}$ such that

$$h \geq h_1, \ x \in \left(B^r_{R_1} \times B^{n-r}_{R/2}\right), \ f_h(x) \neq 0 \quad \Rightarrow \quad |P_0^\perp(x)| < R/4.$$

Let us show by contradiction that the statement holds true for R_1 small enough. Indeed, were it false, it would be possible to find sequences $h_k \to +\infty$ and x_k such that $f_{h_k}(x_k) \neq 0$, $P_0(x_k) \to 0$ and

$$R/2 > |P_0^\perp(x_k)| \geq R/4.$$

We can assume with no loss of generality that x_k converges as $k \to +\infty$ to $x_\infty \in P_0^\perp \cap \overline{B}^n_{R/2} \setminus \{0\}$. By choosing $\rho < R/2$, $\psi \equiv 1$ and letting $\sigma \to 0^+$ in the monotonicity formula (see the Appendix) we get

$$\left[\rho^{-r} \int_{\overline{B}^n_\rho(x_k)} f_{h_k}\, d\mathcal{H}^r\right]^{1/p} \geq \omega_r^{1/p} - \frac{\Gamma}{1 - r/p} \rho^{1-r/p},$$

so that

$$\rho^{-r} \int_{\overline{B}^n_\rho(x_\infty)} f\, d\mathcal{H}^r \geq \left(\omega_r^{1/p} - \frac{\Gamma}{1 - r/p} \rho^{1-r/p}\right)^p.$$

Since $\rho > 0$ is arbitrary, this shows that $x_\infty \in \overline{S}_f$, and this contradicts (3.2). $\square$

Step 3. (continuity of tangent planes) For any $\epsilon > 0$ there exist $h_2 \geq h_1$ and $R_2 \in]0, R_1[$ such that

$$h \geq h_2, \ x \in \left(B^r_{R_2} \times B^{n-r}_{R/2}\right), \ f_h(x) \neq 0, \ P \in \mathrm{Tan}_x(f_h) \quad \Rightarrow \quad |P - P_0| < \epsilon.$$

Indeed, if the statement were not true it would exist $\epsilon > 0$ and sequences h_k, x_k, P_k such that

$$\lim_{k \to +\infty} h_k = +\infty, \ P_k \in \mathrm{Tan}_{x_k}(f_{h_k}), \ \lim_{k \to +\infty} P_0(x_k) = 0$$

and $|P_k - P_0| \geq \epsilon$. By using the same argument of Step 2 and (3.2), we can show that x_k converges to 0. Since $\mathrm{Tan}_0(f) = \{P_0\}$, the contradiction follows by Lemma 2.5. $\square$

Notation: We set $D = B^r_{R_1} \times B^{n-r}_{R/2}$.

Let $\delta < 1$ be such that $(q-1) < q\delta$ (this choice will be useful in step 4) and let $\epsilon > 0$ satisfy (2.3). By using this step, possibly replacing R_1 by R_2 and h_1 by h_2, it is not restrictive to assume that

$$h \geq h_1,\ x \in D,\ f_h(x) \neq 0,\ P \in \mathrm{Tan}_x(f_h) \quad \Rightarrow \quad |P - P_0| < \epsilon.$$

Since, by (2.3), $J_r(P, P_0) > \delta > 0$ for $x \in D$, $P \in \mathrm{Tan}_x(f_h)$ and $h \geq h_1$, it follows that the tangent planes are not vertical with respect to P_0. By Remark 1.5 and the height estimate the functions $f_h \mathrm{char} D$ are C^ω-locally decomposable in $B^r_{R_1} \times \mathbf{R}^{n-r}$ and the functions

$$q_h(z) = \sum \Big\{ f_h(x) : x \in D,\ P_0(x) = z \Big\}$$

are constant in $B^r_{R_1}$.

Step 4. (estimate on the number of sheets) We now claim that there exists $h_2 \geq h_1$ such that $q_h = q$ for any $h \geq h_2$, where q is given by (3.1).
Indeed, let q_∞ be a limit point of the sequence q_h, let us choose $R_2 \in]0, R_1[$ and let $A = B^r_{R_2} \times B^{n-r}_{R/2}$. Since $\mathcal{H}^r\big(S_f \cap \partial A\big) = 0$, by weak convergence of measures we get

$$\lim_{h \to +\infty} \int_A f_h \, d\mathcal{H}^r = \int_A f \, d\mathcal{H}^r.$$

The area formula yields

$$q_h \omega_r R_2^r = \int_{B^r_{R_2}} q_h \, dz = \int_A \mathbf{J}_r\big(P_{f_h}(x), P_0\big) f_h \, d\mathcal{H}^r,$$

so that, recalling our choice of ϵ,

$$\delta \int_A f_h \, d\mathcal{H}^r \leq q_h \omega_r R_2^r \leq \int_A f_h \, d\mathcal{H}^r$$

and passing to the limit as $h \to +\infty$ we get

$$\delta \int_A f \, d\mathcal{H}^r \leq q_\infty \omega_r R_2^r \leq \int_A f \, d\mathcal{H}^r.$$

Since $(q-1)/\delta < q$ and R_2 can be so small that

$$\frac{q-1}{\delta} < \frac{1}{\omega_r R_2^r} \int_A f \, d\mathcal{H}^r < q + 1,$$

we find that $q_\infty = q$.

Step 5 For $h \geq h_2$ we can find q analytic functions

$$\varphi_h^{(1)}, \ldots, \varphi_h^{(q)} : B^r_{R_1} \to B^{n-r}_{R/2}$$

such that

$$f_h(x) = \sum_{j=1}^{q} \mathrm{char}G\big(\varphi_h^{(j)}\big)(x) \qquad \forall x \in D.$$

Indeed, since $f_h \mathrm{char} D$ are C^ω-locally decomposable for $h \geq h_1$ and $q_h = q$ for $h \geq h_2$, the statement will follow by Theorem 4.4, to be proven below.

Step 6 The functions $\varphi_h^{(j)}$ are equibounded in $W^{2,p}\big(B^r_{R_1}\big)$ for $h \geq h_2$, $j = 1, \ldots, q$.
First, we remark that by choosing ϵ small enough in Step 3 we can assume that the Lipschitz constants of $\varphi_h^{(j)}$ don't exceed 1. Fix h and j, drop these indices, let $g = \mathrm{char}G(\varphi)$ with $\varphi = \big(\varphi_1, \ldots, \varphi_{n-r}\big) \in W^{2,p}\big(B^r_{R_1}\big)$, and let $i, l \in \{1, \ldots, r\}$. For any $k = r+1, \ldots, n$ the equality (P_{il} and A_{ilk} are evaluated at $x = (z, \varphi(z))$ and $P = P_g$)

$$A^g_{ilk} = \delta^g_i \big(\delta^g_l x_k\big) = \sum_{s=1}^{r} \delta^g_i \big(P_{ls} \frac{\partial \varphi_{k-r}}{\partial z_s}\big)$$

$$= \sum_{s=1}^{r} A^g_{ils} \frac{\partial \varphi_{k-r}}{\partial z_s} + \sum_{s,t=1}^{r} P_{it} P_{ls} \frac{\partial^2 \varphi_{k-r}}{\partial z_t \partial z_s}$$

holds almost everywhere in $B^r_{R_1}$. Summing in i, l and using the Lipschitz estimate we get

$$\sum_{k=r+1}^{n} \sum_{s,t=1}^{r} B_{st}(P) \left| \frac{\partial^2 \varphi_{k-r}}{\partial z_t \partial z_s} \right| \leq 2r^2 \|A^g\|_1 \qquad \mathcal{H}^r - \text{a.e. in } B^r_{R_1}, \tag{3.3}$$

where

$$B_{st}(P) = \left| \sum_{i,l=1}^{r} P_{it} P_{ls} \right|.$$

Since the mapping $P \mapsto B_{st}(P)$ is continuous and $B_{st}(P_0) = 1$, by a suitable choice of ϵ in Step 3 we can assume that

$$B_{st}(P) \geq \frac{1}{2} \qquad \forall s, t \in \{1, \ldots, r\}$$

for any tangent r-plane P to the graph of $g = \varphi_h^{(j)}$. Hence, the boundedness of $\varphi_h^{(j)}$ in $W^{2,p}\big(B^r_{R_1}\big)$ follows by integrating (3.3) with $\varphi = \varphi_h^{(j)}$.

Step 7. Conclusion. Possibly passing to a subsequence, we can assume that $\varphi_h^{(j)}$ weakly converges to some function $\varphi^{(j)}$ in $W^{2,p}\big(B_{R_1}\big)$ for any $j = 1, \ldots, q$. It is then easy to see that $f_h^{(j)}$ converges in $W^{2,p}_r\big(D\big)$ to $f^{(j)}$, where $f_h^{(j)} = \mathrm{char}G(\varphi_h^{(j)})$, $f^{(j)} = \mathrm{char}G(\varphi^{(j)})$. Since

$$\int_D \phi(x) f_h \, d\mathcal{H}^r = \sum_{j=1}^q \int_D \phi(x) \, f_h^{(j)} d\mathcal{H}^r \qquad\qquad \forall \phi \in C_0^0\big(D\big),$$

passing to the limit as $h \to +\infty$ we get

$$\int_D \phi(x) \, f d\mathcal{H}^r = \sum_{j=1}^q \int_D \phi(x) \, f^{(j)} d\mathcal{H}^r \qquad\qquad \forall \phi \in C_0^0\big(D\big),$$

so that in D the equality $f = \sum_{j=1}^q f^{(j)} = \sum_{j=1}^q \mathrm{char} G(\varphi^{(j)})$ holds. This shows that

$$\overline{E} \subset \Big\{ f \in F_r W^{2,p}\big(\Omega\big) : \|A^f\|_p \leq \Gamma \Big\}.$$

Equality cannot hold because the set

$$\Big\{ f \in F_r W^{2,p}\big(\Omega\big) : \|A^f\|_p \leq \Gamma \Big\}$$

is not closed (see Example 4.5).

4. Global decomposability and examples.

In this section we discuss the notion of global decomposability of an integer valued function, a crucial step in the proof of Theorems 2.3, 2.4. Let $\Omega \subset \mathbf{R}^r$ be a connected open set, and let $f : \Omega \times \mathbf{R}^m \to \mathbf{N}$ be a C^κ-locally decomposable function according to Definition 1.4 (with $m = n - r$ and Ω in place of Ω').

Definition 4.1. The function f is said to be C^κ-*globally decomposable* in Ω if there exist an integer $q > 0$ and functions $\varphi^{(i)} \in C^\kappa(\Omega, \mathbf{R}^m)$, $i = 1, \ldots, q$ such that

$$f(x) = \sum_{i=1}^q \mathrm{char} G(\varphi^{(i)})(x) \qquad \forall x \in \Omega \times \mathbf{R}^m.$$

Remark 4.2. If $f : \Omega \times \mathbf{R}^m \to \mathbf{N}$ is a C^κ-locally decomposable function with $\kappa \neq 0$, then we can define a function

$$f' : \Omega \times \mathbf{R}^m \times \mathbf{R}^{rm} \to \mathbf{N}$$

in the following way. If $U \subset \Omega$ and $\varphi^{(i)} \in C^\kappa(U, \mathbf{R}^m)$, $i = 1, \ldots, q$, are functions such that

$$f(x) = \sum_{i=1}^q \mathrm{char} G(\varphi^{(i)})(x) \qquad \forall x \in U \times \mathbf{R}^m,$$

then we set

$$f'(x,p) = \sum_{i=1}^{q} \mathrm{char} G(\phi^{(i)})(x,p) \qquad \forall x \in U \times \mathbf{R}^m,\ p \in \mathbf{R}^{rm},$$

where $\phi^{(i)}(z) = \big(\varphi_i(z), D\varphi_i(z)\big) \in \mathbf{R}^m \times \mathbf{R}^{rm}$ for any $z \in \Omega$. This definition does not depend on the choice of a local decomposition. Indeed, it can be easily seen that $f'(x,p)$ is the greatest integer j such that there are C^κ-surfaces $\Gamma_1, \dots, \Gamma_j$ of dimension r (not necessarily distinct) such that

$$f \geq \sum_{i=1}^{j} \mathrm{char}\Gamma_i$$

in a neighbourhood of x and $\mathrm{Tan}_x(\Gamma_i) = \{P_p\}$ for any i, where, if we think of $p \in \mathbf{R}^{rm}$ as a linear operator from $\mathbf{R}^r$ into $\mathbf{R}^m$, the r-plane $P_p \subset \mathbf{R}^{r+m}$ is given by

$$P_p := \big\{(z, p(z)) : z \in \mathbf{R}^r\big\}.$$

We call the function f' the *blow-up* of f. It is immediate from the definition that f' is a $C^{\kappa-1}$-locally decomposable function if $\kappa \in \mathbf{N}$, a C^κ-locally decomposable function if $\kappa \in \{\infty, \omega\}$.

Lemma 4.3. *Let $\kappa \in \mathbf{N} \cup \{\infty\}$, let $f : \Omega \times \mathbf{R}^m \to \mathbf{N}$ be a C^κ-locally decomposable function and let $\{\varphi^{(1)}, \dots, \varphi^{(q)}\}$, $\{\psi^{(1)}, \dots, \psi^{(q)}\}$ be two C^κ-decompositions of f in an open neighourhood U of $z_0 \in \Omega$. Then, there is a permutation σ of $\{1, \dots, q\}$ such that*

$$D^k\varphi^{(i)}(z_0) = D^k\psi^{\sigma(i)}(z_0) \qquad \forall k \leq \kappa,\ \forall i = 1, \dots, q.$$

Proof. We argue by induction on κ. The thesis is trivial if $\kappa = 0$. If $\kappa > 0$ is an integer we consider the blow up f' of f. The function f' is $C^{\kappa-1}$-locally decomposable, and $\Phi^{(i)} := (\varphi^{(i)}, D\varphi^{(i)})$, $\Psi^{(i)} := (\psi^{(i)}, D\psi^{(i)})$ are two local decompositions in U. By the inductive hypothesis there exists a permutation σ of $\{1, \dots, q\}$ such that

$$D^k\Phi^{(i)}(z_0) = D^k\Psi^{\sigma(i)}(z_0) \qquad \forall k \leq \kappa - 1,\ i = 1, \dots, q,$$

but this is equivalent to

$$D^k\varphi^{(i)}(z_0) = D^k\psi^{\sigma(i)}(z_0) \qquad \forall k \leq \kappa,\ i = 1, \dots, q.$$

Finally, if $\kappa = \infty$ we can find for any integer p a permutation σ_p such that

$$D^k\varphi^{(i)}(z_0) = D^k\psi^{\sigma_p(i)}(z_0) \qquad \forall k \leq p,\ i = 1, \dots, q.$$

Since the set of permutations is finite, at least one of them satisfies the above formula for infinitely many p. $\square$

Theorem 4.4. *Let $\Omega \subset \mathbf{R}^r$ be an open set, and let $f : \Omega \times \mathbf{R}^m \to \mathbf{N}$ be C^κ-locally decomposable in Ω. Then, f is C^κ-globally decomposable in Ω if at least one of the following conditions is satisfied:*
(1) $\kappa = \omega$ and Ω is simply connected;
(2) $\kappa = 0$ and $m = 1$;
(3) $r = 1$.

Proof. (1) We argue by induction on the number of sheets q. If $q = 1$, the thesis is trivial. If $q > 1$, we fix a point $z_0 \in \Omega$ and a C^ω local decomposition $\varphi^{(1)}, \ldots, \varphi^{(q)}$ of f in a neighbourhood U of z_0.
First of all, we claim that $\varphi^{(1)}$ can be analitically continued (in the sense of [12]) along any continuous curve $\gamma : [0,1] \to \Omega$ starting at z_0, *i.e.*, there exists a finite number of pairs (D_i, f_i) $(i = 0, \ldots, p)$ such that:
- each D_i is a open ball contained in Ω and $D_0 \subset U$;
- $\mathrm{char}G(f_i) \leq f$ on $D_i \times \mathbf{R}^m$ and $\gamma([0,1]) \subset \bigcup_{i=0}^{p} D_i$;
- $D_i \cap D_{i-1} \neq \emptyset$ for any $i = 1, \ldots, p$;
- $f_0(z) = \varphi^{(1)}(z)$ $\forall z \in D_0$ and $f_i(z) = f_{i-1}(z)$ $\forall z \in D_i \cap D_{i-1},\ i = 1, \ldots, p$.

In order to show that such an extension exists, let us consider a finite cover of $\gamma([0,1])$ by a finite number of balls $D_0, \ldots, D_p$ such that $D_0 \subset U$, $D_i \cap D_{i-1} \neq \emptyset$ for $i = 1, \ldots, p$ and f is decomposable over D_i. Now let $z \in D_1 \cap D_0$ and let $\psi^{(1)}, \ldots, \psi^{(q)}$ be a C^ω decomposition of f in D_1. By Lemma 4.3 we can find a permutation σ such that

$$D^k \varphi^{(1)}(z) = D^k \psi^{\sigma(1)}(z) \qquad \forall k \in \mathbf{N},$$

hence, by analiticity, $\varphi^{(1)} \equiv \psi^{\sigma(1)}$ on $D_1 \cap D_0$. Therefore $(D_1, \psi^{\sigma(1)})$ is an extension of $(D_0, \varphi^{(1)})$. Proceeding in the same way with $D_1 \cap D_2$, $D_2 \cap D_3, \ldots$, in a finite number of steps we have constructed the required extension along γ.
Since Ω is simply connected, by the construction of Theorem 16.15 of [12] the function $\varphi^{(1)}$ can be analytically extended to a function ϕ defined in Ω such that $f \geq \mathrm{char}G(\phi)$. Now, exploiting once more the analiticity of f and ϕ, it is easy to see that $f - \mathrm{char}G(\phi)$ is a C^ω-locally decomposable function with $q-1$ sheets, and so the thesis follows by the inductive hypothesis.

(2) Let us fix a covering U_β of Ω such that f can be decomposed over each U_β. Since we are interested only in C^0 decompositions, we can assume that on each U_β the decomposition is given by functions $\varphi_\beta^{(1)}, \ldots, \varphi_\beta^{(q)}$ such that

$$\varphi_\beta^{(1)}(z) \leq \varphi_\beta^{(2)}(z) \ldots \leq \varphi_\beta^{(q)}(z) \qquad \forall z \in U_\beta.$$

Since the sets

$$\{\varphi_\beta^{(1)}(z), \ldots, \varphi_\beta^{(q)}(z)\}, \qquad \{\varphi_\gamma^{(1)}(z), \ldots, \varphi_\gamma^{(q)}(z)\}$$

are equal for any $z \in U_\beta \cap U_\gamma$, it follows that $\varphi_\beta^{(i)}(z) = \varphi_\gamma^{(i)}(z)$ for any $z \in U_\beta \cap U_\gamma$ and any $i = 1, \ldots, q$. This obviously leads to a global C^0 decomposition.

(3) If $\kappa = \omega$ the thesis follows by (1). If $\kappa \in \mathbf{N} \cup \{\infty\}$, let us consider the maximal open interval $I \subset \Omega$ where f can be decomposed, and let $\varphi^{(1)}, \ldots, \varphi^{(q)}$ be such a decomposition.

Let us assume by contradiction that $I \neq \Omega$, *e.g.* $z_0 := \sup I < \sup \Omega$. Let $\psi^{(1)}, \ldots, \psi^{(q)}$ be a local decomposition in an open neighbourhood U_0 of z_0, and let $z \in U_0 \cap I$. By Lemma 4.3, up to a permutation of the functions $\psi^{(i)}$, we can assume that

$$D^k \varphi^{(i)}(z) = D^k \psi^{(i)}(z) \qquad \forall k \leq \kappa,\ \forall i = 1, \ldots, q.$$

This equality easily leads to a decomposition of f in $I \cup U_0$, and this contradicts the maximality of I. If $\inf I > \inf \Omega$ we follow a similar argument. $\square$

Now we show that if all the conditions (1), (2), (3) of Theorem 4.4 fail to be true, then there exist functions f which are locally but not globally decomposable.

Example 4.1. In this example we exhibit a function $f : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{N}$ that is C^0-locally decomposable but not C^0-globally decomposable. This example can be easily generalized to C^κ functions, with $\kappa \in \mathbf{N} \cup \{\infty\}$, defined on $\mathbf{R}^r \times \mathbf{R}^m$ with r, $m \geq 2$.
Let $\mathbf{R}^4 = \mathbf{C}^2$, and let

$$\Gamma_1 := \big\{(z, 0) \in \mathbf{C}^2 : |z| \leq 1\big\}, \qquad \Gamma_2 := \big\{(\psi(w), w) \in \mathbf{C}^2 : w \neq 0\big\},$$

where $\psi : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ is defined in polar coordinates by

$$\psi(\rho, \theta) = (\rho + 1, 2\theta), \qquad \rho > 0,\ \theta \in \mathbf{R}.$$

Let $f = 2\mathrm{char}\Gamma_1 + \mathrm{char}\Gamma_2$; we claim that f is locally decomposable with respect to the plane $\mathbf{C} \times \{0\}$. Indeed, if $z \in B_1(0)$ then we can take $U_z = B_1(0)$ and $\varphi_z^{(1)} = \varphi_z^{(2)} = 0$. If $z \notin B_1(0)$, then z belongs to an open sector $S_{\alpha\beta}$ of the form

$$S_{\alpha\beta} := \big\{(\rho, \theta) \in \mathbf{C} : \rho > 0,\ \alpha < \theta < \beta\big\}$$

with $|\alpha - \beta| < 2\pi$. We set $U_z = S_{\alpha\beta}$ and

$$\varphi_1(\rho, \theta) = \begin{cases} 0 & \text{if } \rho \leq 1; \\ (\rho - 1, \theta/2) & \text{if } \rho > 1; \end{cases}$$

and

$$\varphi_2(\rho, \theta) = \begin{cases} 0 & \text{if } \rho \leq 1; \\ (\rho - 1, \pi + \theta/2) & \text{if } \rho > 1; \end{cases}$$

for any $(\rho, \theta) \in S_{\alpha\beta}$. Let us check that f is the sum of the graphs of φ_1 and φ_2 in $S_{\alpha\beta} \times \mathbf{C}$. Indeed, let $z \in S_{\alpha\beta}$ and assume that $(z, w) \in \Gamma_1 \cup \Gamma_2$ for some $w \in \mathbf{C}$. If $(z, w) \in \Gamma_1$, then $w = 0$ and $|z| \leq 1$, hence

$$(z, w) = (z, \varphi_1(z)) = (z, \varphi_2(z)).$$

If $(z,w)\in\Gamma_2$, then $z=\psi(w)$, with $w\neq 0$. Setting $z=(\rho_z,\theta_z)$ and $w=(\rho_w,\theta_w)$ we have

$$(\rho_z,\theta_z)=\psi(w)=(\rho_w+1,2\theta_w),$$

so that $\rho_w=\rho_z+1$ and either $\theta_w=\theta_z/2$ or $\theta_w=\theta_z/2+\pi$, *i.e.*, either $w=\varphi_1(z)$ or $w=\varphi_2(z)$.
Now we claim that f is not globally C^0-decomposable. Indeed, let us assume that there exists $\varphi:\mathbf{C}\to\mathbf{C}$ such that $G(\varphi)\subset\Gamma_1\cup\Gamma_2$ and let $D=\{z\in\mathbf{C}:|z|>1\}$; since $(z,\varphi(z))\in\Gamma_2$ for any $z\in D$ we get $\psi(\varphi(z))=z$ in D. This is impossible because $\psi:\mathbf{C}\setminus\{0\}\to D$ is a covering space of degree two (see [10], Theorem 5.1).

Remark 4.5 In order to have C^∞ regularity in Example 4.1, it is enough to substitute ψ with

$$\bar\psi(\rho,\theta)=\big(|\log\rho|^{-\frac{1}{2}}+1,2\theta\big)$$

and $\rho-1$ with $\exp\{(\rho-1)^{-2}\}$ in the definition of φ_1 and φ_2. In higher dimensions, we can consider the function $g=f\circ p$, where

$$p(x_1,\ldots,x_r;y_1,\ldots,y_m)=(x_1,x_2;y_1,y_2).$$

Example 4.2. Let us consider the restriction of the function f of Example 4.1 to $\Omega\times\mathbf{C}$, where $\Omega:=\{z\in\mathbf{C}:|z|>1\}$. In this case Ω is not simply connected, f is a C^ω-locally decomposable function, and the same argument of Example 4.1 shows that f is not even C^0-globally decomposable.

Example 4.3. Let us consider the function $g:\mathbf{R}^2\times\mathbf{R}\to\mathbf{N}$ defined by

$$g(x,y,z)=f(x,y,z,0),$$

where f is the locally C^∞-decomposable function of Remark 4.5. It is easy to see that also g is locally C^∞-decomposable. Furthermore, g is C^0-globally decomposable because of Theorem 4.4, but not C^1-globally decomposable. Indeed, arguing as in the construction of the blow up, f can be recovered from g by adding the θ derivative, hence C^1-decomposability of g implies C^0-decomposability of f. As in Remark 4.5, this example can be extended to higher dimensions.

Example 4.4. In this example we show that if $n\geq 3$, and $r\in[2,n-1]$ are integers and $\Omega\subset\mathbf{R}^n$ is an open set, then the class $F_rW^{2,p}\big(\Omega\big)$ is strictly contained in $W^{2,p}_r\big(\Omega\big)$ for any $p>r$. It is not restrictive to take $\Omega=B^n_1$. We first consider the case $n=3$ and $r=2$. We define

$$\Gamma:=\big\{(\rho\cos 2\theta,\rho\sin 2\theta,e^{-\frac{1}{\rho^2}}\cos\theta):\rho>0,\ \theta\in\mathbf{R}\big\},$$

and the function $f=\mathrm{char}\Gamma$. We claim that $f\in W^{2,\infty}_2\big(B^3_1\big)$. Indeed, a direct calculation shows that $A^f_{ijk}=\delta^f_i\delta^f_j x_k$ are bounded in B^3_1. Since $f\in F_2C^\infty\big(B^3_1\setminus\{0\}\big)$, by (1.3) and

(1.4) we infer that (1.5) holds for any $\varphi(x,P)$ whose support does not touch $\{0\}\times\mathbf{R}^9$. Taking into account that

$$\lim_{\rho\to 0^+}\frac{\mathcal{H}^2\big(B^3_\rho\cap\Gamma\big)}{\pi\rho^2}=2,$$

a standard approximation argument shows that (1.5) remains valid for any $\varphi\in C^1_0\big(B^3_1\times\mathbf{R}^9\big)$.
On the other hand, Γ cannot be described in any neighbourhood of the origin as the union of the graphs of two C^1 functions, and therefore $f\notin F_2W^{2,p}\big(B^3_1\big)$ for any $p>2$. In the general case ($n\geq 4$, $r\in[2,n-1]$) the argument above may be applied to $f=\mathrm{char}T$, where T is the cylinder $T=\Gamma\times\mathbf{R}^{r-2}\times\big\{0_{\mathbf{R}^{n-r-1}}\big\}\subset\mathbf{R}^n$.

Example 4.5. In this example we show that in general the closure of $F_rC^\infty\big(\Omega\big)$ is not contained in $F_rW^{2,p}\big(\Omega\big)$. Let $\Gamma^{(\infty)}\subset\mathbf{C}^2$ be defined, in polar coordinates, by

$$\Gamma^{(\infty)}:=\Big\{\big((|\log\rho|^{-\frac{1}{2}},2\theta),(\rho,3\theta)\big):\rho>0,\ \theta\in\mathbf{R}\Big\}\cup\{0\}.$$

Let

$$\Gamma^{(n)}_1:=\big\{(z,0)\in\mathbf{C}^2:|z|\leq 1/n\big\},\qquad \Gamma^{(n)}_2:=\big\{(\varphi_n(z),\psi(z))\in\mathbf{C}^2:z\neq 0\big\},$$

where φ_n, ψ are defined in $\mathbf{C}\setminus\{0\}$ by

$$\varphi_n(\rho,\theta)=(|\log\rho|^{-\frac{1}{2}}+\frac{1}{n},2\theta),\qquad \psi(\rho,\theta)=(\rho,3\theta).$$

Finally, we set

$$f_n:=2\mathrm{char}\Gamma^{(n)}_1+\mathrm{char}\Gamma^{(n)}_2,\qquad f_\infty=\mathrm{char}\Gamma^{(\infty)}.$$

It is easy to see that the same argument of Example 4.1 shows that $f_n\in F_2C^\infty(\mathbf{C}^2)$. Furthermore, a direct computation shows that $A^{f_n}_{ijk}$ are equi-bounded in L^∞ on compact sets, and $f_n\mathcal{H}^2$ weakly converges to $f_\infty\mathcal{H}^2$. However, $f_\infty\notin F_2C^0(\mathbf{C}^2)$.

Appendix. The monotonicity formula.

In this appendix we state and prove the monotonicity formula for curvature varifolds which we have exploited in the paper. Notice that here all the balls are n-dimensional, hence we omit the dimension index.

Theorem. Let $\Omega\subset\mathbf{R}^n$ open, $p>r$, $f\in W^{2,p}_r\big(\Omega\big)$, and let

$$\Gamma=\bigg(\int_\Omega\sum_{i,j,k}|A^f_{ijk}|^p f\,d\mathcal{H}^r\bigg)^{1/p}.$$

Then, we have

$$(A.1)\qquad \begin{aligned}&\left[\rho^{-r}\int_{B_\rho(x_0)}\psi\big(P_f(x)\big)\,f d\mathcal{H}^r\right]^{1/p}-\left[\sigma^{-r}\int_{B_\sigma(x_0)}\psi\big(P_f(x)\big)\,f d\mathcal{H}^r\right]^{1/p}\geq\\ &\geq\frac{\Gamma\big(1+\lambda\big)}{p-r}\big[\sigma^{1-r/p}-\rho^{1-r/p}\big],\end{aligned}$$

whenever $\psi\in C^1\big(\mathbf{R}^{n^2}\big)$, $0\leq\psi\leq 1$, $|D\psi|\leq\lambda\psi$ and

$$B_\sigma(x_0)\subset B_\rho(x_0)\subset\Omega.$$

Proof. Let $x_0\in\Omega$, $R=\operatorname{dist}(x_0,\partial\Omega)$, $S=\{x\in\Omega: f(x)\neq 0\}$, $\delta=\delta^f$, let ψ as in the statement of the theorem and let

$$E=\big\{s\in]0,R[:\mathcal{H}^r\big(S\cap\partial B_s(x_0)\big)>0\big\}.$$

For any $0<\sigma<\rho<R$ we set

$$\varphi_{\sigma,\rho}(x)=(x-x_0)\big[\inf\{\sigma^{-r},|x-x_0|^{-r}\}-\rho^{-r}\big]^+,$$

and

$$\varphi(x,P)=\varphi_{\sigma,\rho}(x)\psi(P).$$

Since $\varphi:\mathbf{R}^n\times\mathbf{R}^{n^2}\to\mathbf{R}^n$ is continuously differentiable outside the set

$$\big(\partial B_\sigma(x_0)\cup\partial B_\rho(x_0)\big)\times\mathbf{R}^{n^2},$$

and globally Lipschitz continuous, a standard approximation argument shows that the integration by parts formula is valid for each component of φ, provided neither σ nor ρ belong to E.
We remark that E is at most countable and

$$\operatorname{supp}(\varphi_{\sigma,\rho})\subset\overline{B}_\rho(x_0),\qquad \varphi_{\sigma,\rho}(x)=(x-x_0)\big(\sigma^{-r}-\rho^{-r}\big)\quad\forall x\in B_\sigma(x_0).$$

Taking into account the identity

$$\sum_{i,j=1}^n(x_i-x_{0i})\delta_i(x_j-x_{0j})^2=2|\pi_x(x-x_0)|^2$$

where π_x denotes the orthogonal projection on $P_f(x)$, we find

$$\begin{aligned}\operatorname{div}^f\varphi_{\sigma,\rho}(x,P)&=\sum_{i,j=1}^n(\delta_i x_i)\big(|x-x_0|^{-r}-\rho^{-r}\big)-r\frac{(x_i-x_{0i})\delta_i(x_j-x_{0j})^2}{2|x-x_0|^{r+2}}=\\ &=-r\rho^{-r}+r|x-x_0|^{-r}+r\frac{|\pi_x(x-x_0)|^2}{|x-x_0|^{r+2}}\end{aligned}$$

for any $x \in B_\rho(x_0) \setminus \overline{B}_\sigma(x_0)$ where $P_f(x)$ is defined. Similarly

$$\mathrm{div}^f \varphi_{\sigma,\rho}(x) = r\big(\sigma^{-r} - \rho^{-r}\big)$$

for any $x \in B_\sigma(x_0)$ where $P_f(x)$ is defined. Inserting $\varphi = \varphi_{\sigma,\rho}\psi$ in the integration by parts formula, we get

$$\begin{aligned} & r\left[\rho^{-r}\int_{B_\rho(x_0)} \psi\big(P(x)\big)\, f d\mathcal{H}^r\right] - r\left[\sigma^{-r}\int_{B_\sigma(x_0)} \psi\big(P(x)\big)\, f d\mathcal{H}^r\right] \geq \\ & \geq -\int_{B_\rho(x_0)} \sum_{i,j,k} \big(A^f_{ijk}(\varphi_{\sigma,\rho})_i D_{jk}\psi + A^f_{jji}(\varphi_{\sigma,\rho})_i \psi\big)\, f d\mathcal{H}^r. \end{aligned}$$

By using our assumptions on ψ, the Hölder inequality and the estimate

$$|\varphi_{\sigma,\rho}| \leq \rho(\sigma^{-r} - \rho^{-r})$$

we obtain

$$\begin{aligned} & r\left[\rho^{-r}\int_{B_\rho(x_0)} \psi\big(P(x)\big)\, f d\mathcal{H}^r\right] - r\left[\sigma^{-r}\int_{B_\sigma(x_0)} \psi\big(P(x)\big)\, f d\mathcal{H}^r\right] \geq \\ & \geq -\Gamma(1+\lambda)\rho\big(\sigma^{-r} - \rho^{-r}\big)\left[\int_{B_\rho(x_0)} \psi\big(P(x)\big)\, f d\mathcal{H}^r\right]^{1-1/p}. \end{aligned} \tag{A.2}$$

By approximation, the same inequality is true for any $0 < \sigma < \rho < R$. Denoting by $\gamma :]0, R[\to \mathbf{R}$ the function

$$\gamma(s) = \left[s^{-r}\int_{B_s(x_0)} \psi\big(P(x)\big)\, f d\mathcal{H}^r\right],$$

dividing both sides by $\rho - \sigma$ in (A.2) and letting $\sigma \uparrow \rho$ we get

$$\gamma'(\rho) \geq -\Gamma(1+\lambda)\rho^{-r}\big[\gamma(\rho)\rho^r\big]^{1-1/p},$$

which is equivalent to

$$p\big[\gamma^{1/p}\big]'(\rho) \geq -\Gamma(1+\lambda)\rho^{-r/p}.$$

Taking into account that the negative part of the distributional derivative of γ is absolutely continuous, (A.1) follows by integration between σ and ρ. $\square$

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