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Kirszbraun's extension theorem fails for Almgren's multiple valued functions

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ABSTRACT. We show that there is no analog of Kirszbraun's extension theorem for Almgren's multiple valued functions.

KEYWORDS: Kirszbraun's extension theorem, multiple valued functions, geometric measure theory.

MSC (2010): 54C20, 49Q20.

1. INTRODUCTION

Almgren's multiple valued functions play a key role in geometric measure theory since they are employed in the analysis of the branching behaviour of minimal surfaces in codimension larger than or equal to 2 (see [1] and [3]).

We recall basic definitions for multiple valued functions. Let Q be a positive integer, then

$$\mathcal{A}_Q(\mathbb{R}^n) = \left\{ \sum_{i=1}^{Q} [\![P_i]\!] : P_i \in \mathbb{R}^n,\ 1 \le i \le Q \right\},$$

where $[\![P]\!]$ denotes the Dirac measure at P. This space is endowed with the L^2-Wasserstein distance: for $T_1 = \sum_{i=1}^{Q} [\![P_i]\!]$ and $T_2 = \sum_{i=1}^{Q} [\![S_i]\!]$ we define

$$\mathcal{G}(T_1, T_2) = \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_{i=1}^{Q} |P_i - S_{\sigma(i)}|^2},$$

where $\mathcal{P}_Q$ denotes the group of permutations of $\{1, \dots, Q\}$.

One of the main ingredients in the theory of multiple valued funtions is the following extension theorem (see Theorem 1.7 in [3]).

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1.1. Theorem. *Let $B \subset \mathbb{R}^m$ be a measurable set and let $f : B \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then there exists a constant $C = C(m, Q) > 0$ and an extension $\bar{f} : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ of f such that*

$$\mathrm{Lip}(\bar{f}) \le C \mathrm{Lip}(f).$$

In the Euclidean case, the classical Kirszbraun's extension theorem (see Theorem 2.10.43 in [2]) states that an analogous result holds with $C = 1$. More precisely, Kirszbraun's theorem states that Lipschitz functions defined on a subset of $\mathbb{R}^m$ with values in $\mathbb{R}^n$ (both endowed with the Euclidean distance) can be extended to all of $\mathbb{R}^m$ without increasing the Lipschitz constant. The conclusion may fail as soon as $\mathbb{R}^m$ or $\mathbb{R}^n$ is remetrized by a metric which is not induced by an inner product, as shown in 2.10.44 in [2].

In §2 we prove that the conclusion also fails in the setting of multiple valued functions, by exhibiting a $\sqrt{2/3}$-Lipschitz function f defined on a subset of $\mathbb{R}^2$ with values in $\mathcal{A}_2(\mathbb{R}^2)$ with the property that any Lipschitz extension $\bar{f}$ to $\mathbb{R}^2$ has Lipschitz constant at least 1.

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2. Construction of the counterexample

Let $A = (0, 1), B = (-\sqrt{3}/2, -1/2), C = (\sqrt{3}/2, -1/2)$ and let $P_1, \dots, P_6$ be the vertices of a regular hexagon centered at 0, with side length 1: $P_1 = (0, 1)$, $P_2 = (\sqrt{3}/2, 1/2)$, $P_3 = (\sqrt{3}/2, -1/2)$, $P_4 = (0, 1)$, $P_5 = (-\sqrt{3}/2, -1/2)$ and $P_6 = (-\sqrt{3}/2, 1/2)$.

Consider the map $f : \{A, B, C\} \subset \mathbb{R}^2 \to \mathcal{A}_2(\mathbb{R}^2)$ given by

$$\begin{aligned} f(A) &= [\![P_1]\!] + [\![P_4]\!], \\ f(B) &= [\![P_2]\!] + [\![P_5]\!], \\ f(C) &= [\![P_3]\!] + [\![P_6]\!]. \end{aligned}$$

The Lipschitz constant of f is $\sqrt{2/3}$. In fact, $|A-B| = |A-C| = |B-C| = \sqrt{3}$ and

$$\mathcal{G}(f(A), f(B)) = \mathcal{G}(f(A), f(C)) = \mathcal{G}(f(B), f(C)) = \sqrt{2}.$$

Now consider a map $\bar{f} : \{A, B, C\} \cup \{0\} \to \mathcal{A}_2(\mathbb{R}^2)$. We will prove that if $\bar{f}$ is an extension of f, then the Lipschitz constant of $\bar{f}$ is at least 1. Indeed, let $\bar{f}(0) = [\![S_1]\!] + [\![S_2]\!]$. Assume by contradiction $\mathrm{Lip}(\bar{f}) < 1$, then S_1 and S_2 should lie on different sides of the perpendicular bisector of the line segment $\overline{P_1P_4}$ (see Figure 1). In fact, if for example S_1 and S_2 both lie in the half plane $\{y \leq 0\}$ then $|P_1 - S_i| \geq 1$ for $i = 1, 2$ which implies $\mathcal{G}(\bar{f}(0), f(A)) \geq 1$. The latter contradicts the assumption since $|A| = 1$.

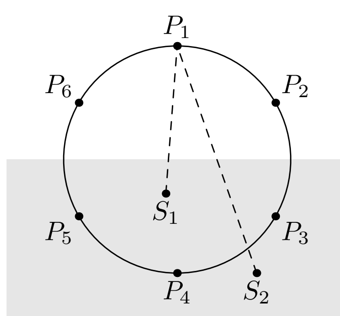

FIGURE 1. S_1 and S_2 must lie on different sides of $y = 0$

Arguing analogously for $\overline{P_2P_5}$ and $\overline{P_3P_6}$ we deduce that S_1 and S_2 must lie on opposite sectors among the six determined by the three perpendicular bisectors.

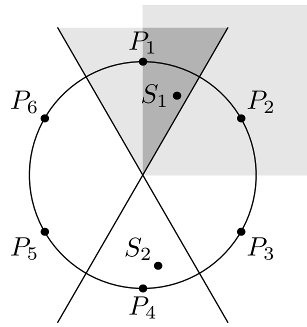

FIGURE 2.

Without loss of generality we can assume that S_1 belongs to the intersection of the sector containing P_1 and the first orthant (see Figure 2).

Since $|S_1 - P_6| \le |S_1 - P_3|$ and $|S_2 - P_6| \ge |S_2 - P_3|$, we can estimate the distance between $\bar{f}(0)$ and $f(C)$ and get

$$\mathcal{G}(\bar{f}(0), f(C))^2 = |S_1 - P_6|^2 + |S_2 - P_3|^2 \ge \frac{3}{4} + \frac{1}{4} = 1,$$

which contradicts our assumption since $|C| = 1$.

2.1. Remark. Following the proof of Theorem 1.1 in [3] one can explicitly determine the growth of the constant C depending on m and Q. It would be desirable to understand if the sharp constant has the same growth (or at least if $C(m, Q)$ goes to infinity as either m or Q goes to infinity). Clearly, just considering one-point extensions cannot lead to an answer to this question as the following general argument shows.

Let (M, d_M) and (N, d_N) be two complete metric spaces, A a subset of M and $f : A \to N$ be Lipschitz continuous. Then for every $P \in M \setminus A$ there exists a Lipschitz extension $\bar{f} : A \cup \{P\} \to N$ such that

$$\mathrm{Lip}(\bar{f}) \le 2\mathrm{Lip}(f).$$

In fact, let $S \in \bar{A}$ be a point realizing the distance between P and $\bar{A}$. Let $\bar{f}(P)$ be the value at S of the unique continuous extension of f to $\bar{A}$, denoted by $f(S)$. Then for every $y \in A \setminus \{S\}$ we get

$$\frac{d_N(\bar{f}(P), f(y))}{d_M(P, y)} = \frac{d_N(f(S), f(y))}{d_M(S, y)} \frac{d_M(S, y)}{d_M(P, y)} \le 2\mathrm{Lip}(f),$$

because $d_M(S, y) \le d_M(S, P) + d_M(P, y)$ and $d_M(S, P) \le d_M(P, y)$ by the definition of S.

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