"ENTROPIC" SOLUTIONS TO A THERMODYNAMICALLY CONSISTENT PDE SYSTEM FOR PHASE TRANSITIONS AND DAMAGE

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ABSTRACT. In this paper we analyze a PDE system modelling (non-isothermal) phase transitions and damage phenomena in thermoviscoelastic materials. The model is thermodynamically consistent: in particular, no *small perturbation assumption* is adopted, which results in the presence of quadratic terms on the right-hand side of the temperature equation, only estimated in L^1. The whole system has a highly nonlinear character.

We address the existence of a weak notion of solution, referred to as "entropic", where the temperature equation is formulated with the aid of an entropy inequality, and of a total energy inequality. This solvability concept reflects the basic principles of thermomechanics as well as the thermodynamical consistency of the model. It allows us to obtain *global-in-time* existence theorems without imposing any restriction on the size of the initial data.

We prove our results by passing to the limit in a time discretization scheme, carefully tailored to the nonlinear features of the PDE system (with its "entropic" formulation), and of the a priori estimates performed on it. Our time-discrete analysis could be useful towards the numerical study of this model.

Key words: damage, phase transitions, thermoviscoelasticity, global-in-time weak solutions, time discretization.

AMS (MOS) subject classification: 35D30, 74G25, 93C55, 82B26, 74A45.

1. Introduction

We consider the following PDE system

$$\vartheta_t + \chi_t\vartheta + \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

supplemented with the boundary conditions (here n denotes the outward unit normal to $\partial\Omega$)

$$\mathsf{K}(\vartheta)\nabla\vartheta \cdot n = h, \qquad \mathbf{u} = 0, \qquad \partial_n\chi = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{1.4}$$

Equations (1.1)–(1.3) were derived according to M. FRÉMOND's modeling approach (see [12, 13]), in [28]. There, it was shown that this PDE system describes (non-isothermal) phase transitions, or (non-isothermal) damage, in a material body occupying a reference domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$. We refer to [28] for a quite detailed survey on the literature on phase transition and damage problems in thermoviscoelasticity. In (1.1)–(1.3), the symbols ϑ and $\mathbf{u}$ respectively denote the absolute temperature of the system and the small displacement vector, while χ is an internal parameter: its meaning depends on the phenomenon described by (1.1)–(1.3), which also determines the choices of the coefficients a and b in the momentum equation (1.2), and of the constant $\mu \in \{0,1\}$ in (1.3). More precisely,

- the choices $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$ correspond to the case of *phase transitions* in thermoviscoelastic materials: in this case, χ is the order parameter, standing for the local proportion of one of the two phases. We assume that χ takes values between 0 and 1, choosing 0 and 1 as reference values: in the case of phase transitions, $\chi = 1$ stands for the liquid phase while $\chi = 0$ for the solid one and one has $0 < \chi < 1$ in the so-called *mushy regions*. Unidirectionality, or irreversibility, of the phase transition process may be encompassed in the model by taking $\mu = 1$ in (1.3), which "activates" the term $\partial I_{(-\infty,0]}(\chi_t)$ (i.e. the subdifferential in the sense of convex analysis of the indicator function $I_{(-\infty,0]}$, evaluated at χ_t), yielding the constraint $\chi_t \leq 0$ a.e. in $\Omega \times (0,T)$. The meaning of $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$ in (1.2) is that, in the *purely* solid phase $\chi = 0$ only the elastic energy, in addition to the thermal expansion energy, contributes to the stress $\sigma = a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}$ (where $\mathbb{E}$ and $\mathbb{V}$ are the elasticity and viscosity tensors, respectively). Instead, in the *purely* liquid, or "viscous", phase $\chi = 1$ only the viscosity contribution remains, whereas in mushy regions both elastic and viscous effects are present.
- The choices $a(\chi) = b(\chi) = \chi$ correspond to *damage*. In this case, χ is the damage parameter, assessing the soundness of the material microscopically, around a point in the material domain Ω. In fact, we have $\chi = 0$ in the presence of complete damage, while χ takes the value 1 when the material is fully sound, and $0 < \chi < 1$ describes *partial damage*.

Finally, K in (1.1) is the heat conductivity, W in (1.3) is a mixing energy density, which we assume of the form

$$W = \widehat{\beta} + \widehat{\gamma} \qquad \text{with } \widehat{\beta}: \mathrm{dom}(\widehat{\beta}) \to \mathbb{R} \text{ convex, possibly nonsmooth, and } \widehat{\gamma} \in \mathrm{C}^2(\mathbb{R}),$$

while $\mathbf{f}$ is a given bulk force, and g and h heat sources.

Observe that, in the case when both coefficients $a(\chi)$ and $b(\chi)$ in the momentum equation degenerate to zero (which happens, for instance, with $a(\chi) = b(\chi) = \chi$, when complete damage occurs), the equation for $\mathbf{u}$ loses its elliptic character. This leads to serious troubles as, for instance, no control of the term $b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2}$ on the right-hand side of (1.3) is possible. That is why, in what follows we shall confine our analysis of system (1.1)–(1.3) only to the case case in which the functions $a, b \in \mathrm{C}^1(\mathbb{R})$ are bounded from below away from 0 (cf. (2.16) in Sec. 2). The reader may refer to our previous contribution [28], where we deal with complete damage and elliptic degeneracy of the momentum equation, in a *simplified* case.

In fact, in [28] we analyzed the following *reduced* system

$$\begin{aligned}
&\vartheta_t + \chi_t\vartheta + \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega \times (0,T), \\
&\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \\
&\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega \times (0,T),
\end{aligned} \qquad (1.5)$$

where the quadratic contributions in the velocities on the right-hand side in the internal energy balance (1.1) are neglected by means of the *small perturbation assumption* (cf. [14]).

In this paper, instead, we address the *full* system (1.1)–(1.3). Let us stress that, since we keep the quadratic terms $a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)$ and $|\chi_t|^2$ on the right-hand side of (1.1), the model is *thermodynamically consistent*, as shown in [28]. However,

- the highly nonlinear character of the whole system, with the multivalued term $\partial I_{(-\infty,0]}(\chi_t)$ and the possibly nonsmooth contribution $\widehat{\beta}$ to the energy W;
- the quadratic terms on the right-hand side of (1.1), which make it difficult to get suitable estimates on $(\vartheta, \mathbf{u}, \chi)$,

bring about severe difficulties in the analysis of (1.1)–(1.3). This is the reason why we are going to develop an existence analysis only for a suitable weak solution concept for (1.1)–(1.3), which we illustrate in the following lines.

The "entropic" formulation. We resort to a weak solution notion for (1.1)–(1.3) partially mutuated from [9]. There, a thermodynamically consistent model for phase transitions, consisting of the temperature and

of the phase parameter equations, was analyzed: the temperature equation, featuring quadratic terms on its right-hand side, was weakly formulated in terms of an *entropy inequality* and of a *total energy inequality.* In the present framework, the pointwise internal energy balance (1.1) is thus replaced by this *entropy inequality*

$$\int_s^t \int_\Omega (\log(\vartheta) + \chi)\varphi_t \, dx \, dr + \rho \int_s^t \int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi \, dx \, dr - \int_s^t \int_\Omega \mathsf{K}(\vartheta)\nabla \log(\vartheta) \cdot \nabla\varphi \, dx \, dr$$

$$\leq - \int_s^t \int_\Omega \mathsf{K}(\vartheta)\frac{\varphi}{\vartheta}\nabla \log(\vartheta) \cdot \nabla\vartheta \, dx \, dr - \int_s^t \int_\Omega \left(g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2 \right) \frac{\varphi}{\vartheta} \, dx \, dr \qquad (1.6)$$

$$- \int_s^t \int_{\partial\Omega} h\frac{\varphi}{\vartheta} \, dS \, dr,$$

where φ is a sufficiently regular, *positive* test function (cf. (2.37)), coupled with the following *total energy inequality*

$$\mathscr{E}(\vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t), \chi(t)) \leq \mathscr{E}(\vartheta(s), \mathbf{u}(s), \mathbf{u}_t(s), \chi(s)) + \int_s^t \int_\Omega g \, dx \, dr + \int_0^t \int_{\partial\Omega} h \, dS \, dr + \int_s^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \, dx \, dr, \quad (1.7)$$

where

$$\mathscr{E}(\vartheta, \mathbf{u}, \mathbf{u}_t, \chi) := \int_\Omega \vartheta \, dx + \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2 \, dx + \frac{1}{2}\int_\Omega b(\chi(t))\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t)) \, dx + \frac{1}{p}\int_\Omega |\nabla\chi|^p \, dx + \int_\Omega W(\chi) \, dx. \quad (1.8)$$

Both (1.6) and (1.7) are required to hold for almost all $t \in (0, T]$ and almost all $s \in (0, t)$, and for $s = 0$. This formulation of the heat equation has been first developed in [7, 2] in the framework of heat conduction in fluids, and then applied to a phase transition model, also derived according to FRÉMOND's approach [12], firstly in [9]. Successively, the so-called *entropic* notion of solution has been used to prove global-in-time existence results in models for special materials like liquid crystals (cf. [8], [10], [11]), and more recently in the analysis of models for the evolution of non-isothermal binary incompressible immiscible fluids (cf. [6]).

This notion of solution for the temperature equation corresponds exactly to the physically meaningful requirement that the system should satisfy the second and first principle of Thermodynamics. Indeed, one of the main advantages of this formulation resides in the fact that the thermodynamically consistency of the the model immediately follows from the existence proof. It can be also shown that it is consistent with the standard one, (cf. the discussion in Sec. 2.3, in particular Remark 2.3, and in [9]).

From an analytical viewpoint, observe that the entropy inequality (1.6) has the advantage that all the troublesome quadratic terms on the right-hand side of (1.1) feature as multiplied by a negative test function. This, and the fact that (1.6) is an inequality, allows for upper semicontinuity arguments in the limit passage in a suitable approximation of (1.6)–(1.8).

In addition to (1.6)–(1.8), the *entropic formulation* of system (1.1)–(1.3) also consists of the momentum balance (1.2), given pointwise a.e. in $\Omega \times (0, T)$, and of the internal variable equation (1.3). The latter is required to hold pointwise almost everywhere in the reversible case $\mu = 0$. In the irreversible case $\mu = 1$, we confine the analysis to the case in which $\widehat{\beta}$ is the indicator function $I_{[0, +\infty)}$ of $[0, +\infty)$, hence $W(\chi) = I_{[0, +\infty)}(\chi) + \widehat{\gamma}(\chi)$. For reasons expounded in Sec. 2.3, we have to weakly formulate (1.3) in terms of the requirement $\chi_t \leq 0$ a.e. in $\Omega \times (0, T)$, of the *one-sided variational inequality*

$$\int_\Omega \left(\chi_t - \mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) + \xi + \gamma(\chi) + b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} - \vartheta \right)\psi \, dx \geq 0 \text{ for all } \psi \in W^{1,p}(\Omega) \text{ with } \psi \leq 0, \quad (1.9)$$

almost everywhere in $(0, T)$ (where $\gamma := \widehat{\gamma}'$), and of the *energy inequality*

$$\int_s^t \int_\Omega |\chi_t|^2 \, dx \, dr + \int_\Omega \left(\frac{1}{p}|\nabla\chi(t)|^p + W(\chi(t)) \right) \, dx$$

$$\leq \int_\Omega \left(\frac{1}{p}|\nabla\chi(s)|^p + W(\chi(s)) \right) \, dx + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \right) \, dx \, dr \qquad (1.10)$$

for all $t \in (0,T]$ and almost all $s \in (0,t)$, with ξ a selection in the (convex analysis) subdifferential $\partial \widehat{\beta}(\chi) = \partial I_{[0,+\infty)}(\chi)$ of $I_{[0,+\infty)}$. In [28, Prop. 2.14] (see also [16]), we prove that, under additional regularity properties any weak solution in fact fulfills (1.3) pointwise.

Let us also mention that other approaches to the weak solvability of coupled PDE systems with an L^1-right-hand side are available in the literature: in particular, we refer here to [33] and [30]. In [33], the notion of *renormalized solution* has been used in order to prove a global-in-time existence result for a nonlinear system in thermoviscoelasticity. In [30] the focus is on rate-independent processes coupled with viscosity and inertia in the displacement equation, and with the temperature equation. There the internal variable equation is not of gradient-flow type as (1.3) but instead features a 1-positively homogeneous dissipation potential. For the resulting PDE system, a weak solution concept partially mutuated from the theory of rate-independent processes by A. MIELKE (cf., e.g., [22]) is analyzed. An existence result is proved combining techniques for rate-independent evolution, with Boccardo-Gallouët type estimates of the temperature gradient in the heat equation with L^1-right-hand side.

Our existence results. The main results of this paper, <u>Theorems 2.5</u> and <u>2.8</u>, state the existence of *entropic solutions* for system (1.1–1.4) in the irreversible ($\mu = 1$) and reversible ($\mu = 0$) cases.

More precisely, in the case of unidirectional evolution for χ we can prove the existence of a global-in-time entropic solution (i.e. satisfying the *entropy* (1.6) and the *total energy* (1.7) inequalities, the (pointwise) momentum balance (1.2), the *one-sided* (1.9) and the *energy* (1.10) inequalities for χ). We work under fairly general assumptions on the nonlinear functions in (1.1)–(1.3). More precisely, we require that a and b are sufficiently smooth and bounded from below by a positive constant, b convex, and we standardly assume that $W = I_{[0,+\infty)} + \widehat{\gamma}$, with $\widehat{\gamma}$ smooth and λ-convex. A crucial role is played by the requirement that the heat conductivity function $\mathsf{K} = \mathsf{K}(\vartheta)$ grows at least like ϑ^κ with $\kappa > 1$, and that the exponent p in the gradient regularization of the equation for χ fulfills $p > d$. This ensures that χ is estimated in $W^{1,p}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$. Moreover, under some restriction on κ (i.e. $\kappa \in (1, 5/3)$ for space dimension $d = 3$), we can also obtain an enhanced regularity for ϑ and that conclude that the *total energy inequality* actually holds as an *equality*.

In the reversible case ($\mu = 0$), instead, under the same assumptions above described (but with a general $\widehat{\beta}$), we improve the estimates, hence the regularity, of the internal variable χ. Therefore, we prove the existence of a weak formulation of (1.1)–(1.3), featuring, in addition to (1.6), (1.7), and (1.2), a *pointwise* formulation of equation (1.3). Again, in the case of the aforementioned restriction on κ, we enhance the time-regularity of ϑ. What is more, also exploiting the improved formulation of the equation for χ, we are able to conclude existence for a stronger formulation of the heat equation (1.2), of variational type. Instead, a uniqueness result seems to be out of reach, at the moment, not only in the irreversible but also in the reversible cases (cf. Remarks 2.6 and 2.9). Only for the *isothermal* reversible system a continuous dependence result, yielding uniqueness, can be proved exactly like in [28, Thm.3].

Finally, in the last Section 6 we address the analysis of system (1.1)–(1.3), with $\mu = 1$, in the case the p-Laplacian regularization in (1.3) is replaced by the standard Laplacian operator. We approximate it by adding a p-Laplacian term, modulated by a small parameter δ, on the left-hand side of (1.3), so that Thm. 2.8 guarantees the existence of approximate solutions $(\vartheta_\delta, \mathbf{u}_\delta, \chi_\delta)$. Then, we let δ tend to zero. In this context, the enhanced elliptic regularity estimates on the momentum equation exploited in the proof of Thm. 2.5, and which would here yield some suitable compactness for the quadratic term $a(\chi_\delta)\varepsilon(\partial_t \mathbf{u}_\delta)\mathbb{V}\varepsilon(\partial_t \mathbf{u}_\delta)$ on the right-hand side of (1.1), are no longer available. In fact, they rely on the requirement $p > d$. A crucial step for proving the existence of (a slightly weaker notion of) entropic solutions to system (1.1)–(1.3) (cf. <u>Theorem 6.2</u>), then consists in deriving some suitable strong convergence for $(\partial_t \mathbf{u}_\delta)_\delta$ with an ad hoc technique, strongly relying on the fact that $\mu = 1$, and on the additional assumption that b is non-decreasing.

Our main existence results Thms. 2.5 and 2.8 are proved by passing to the limit in a time-discretization scheme, unique for the reversible and the irreversible cases, carefully tuned to the nonlinear features of the PDE system. In particular, it is devised in such a way as to obtain that the piecewise constant and piecewise linear interpolants of the discrete solutions satisfy the discrete versions of the entropy inequality (1.6), of total energy

inequality (1.8), and of the energy inequality (1.10) in the case $\mu = 1$. Moreover, with delicate calculations we are also able to translate on the time-discrete level a series of a priori estimates on the heat equation, having a nonlinear character. This detailed time-discrete analysis could be interesting in view of further numerical studies of this model.

Plan of the paper. In Section 2 we fix some notation, state some preliminaries that will be used in the rest of the paper, list our assumptions on the data as well as our main global-in-time existence results. In Section 3 we perform a series of *formal* a-priori estimates on our system. We render them rigorously in Section 4, where we set up our time-discrete scheme. Theorems 2.5 and 2.8 are proved by passing to the limit in the approximated entropy and energy inequality, as well as in the discretized versions of (1.2) and (1.3), throughout Sec. 5. Section 6 is then devoted to the proof of Theorem 6.2.

2. Setup and results

After fixing some notation and results which shall be used throughout the paper, in Section 2.2 we collect our working assumptions on the nonlinear functions K, a, b, and W in the PDE system (1.1)–(1.3), and on the data. Then, in Secs. 2.3 and 2.4 we discuss the weak formulations of (the initial-boundary value problem for) (1.1)–(1.3) in the irreversible and reversible cases, respectively corresponding to $\mu = 1$ and $\mu = 0$ in (1.3).

2.1. Preliminaries.

Notation 2.1. Throughout the paper, given a Banach space X we shall denote by $\| \cdot \|_X$ its norm, and use the symbol $\langle \cdot, \cdot \rangle_X$ for the duality pairing between X' and X. Moreover, we shall denote by $\mathrm{BV}([0,T]; X)$ (by $\mathrm{C}^0_{\mathrm{weak}}([0,T]; X)$, respectively), the space of functions from $[0,T]$ with values in X that are defined at every $t \in [0,T]$ and have bounded variation on $[0,T]$ (and are *weakly* continuous on $[0,T]$, resp.).

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $d \in \{2,3\}$. We set $Q := \Omega \times (0,T)$. We identify both $L^2(\Omega)$ and $L^2(\Omega; \mathbb{R}^d)$ with their dual spaces, and denote by $(\cdot, \cdot)$ the scalar product in $\mathbb{R}^d$, by $(\cdot, \cdot)_{L^2(\Omega)}$ both the scalar product in $L^2(\Omega)$, and in $L^2(\Omega; \mathbb{R}^d)$, and by $H^1_0(\Omega; \mathbb{R}^d)$ and $H^2_0(\Omega; \mathbb{R}^d)$ the spaces

$$H^1_0(\Omega; \mathbb{R}^d) := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \,:\, \mathbf{v} = 0 \ \text{ on } \partial\Omega\}, \text{ endowed with the norm } \|\mathbf{v}\|^2_{H^1_0(\Omega; \mathbb{R}^d)} := \int_\Omega \varepsilon(\mathbf{v}) : \varepsilon(\mathbf{v}) \, \mathrm{d}x,$$

$$H^2_0(\Omega; \mathbb{R}^d) := \{\mathbf{v} \in H^2(\Omega; \mathbb{R}^d) \,:\, \mathbf{v} = 0 \ \text{ on } \partial\Omega\}.$$

Note that $\| \cdot \|_{H^1_0(\Omega; \mathbb{R}^d)}$ is a norm equivalent to the standard one on $H^1(\Omega; \mathbb{R}^d)$. We will use the symbol $\mathcal{D}(\overline{Q})$ for the space of the C^∞-functions with compact support on $Q := \Omega \times (0,T)$ and for $q > 1$ we will adopt the notation

$$W^{1,q}_+(\Omega) := \left\{ \zeta \in W^{1,q}(\Omega) \,:\, \zeta(x) \geq 0 \quad \text{for a.a. } x \in \Omega \right\}, \quad \text{and analogously for } W^{1,q}_-(\Omega).$$

We denote by A_p the p-Laplacian operator with zero Neumann boundary conditions, viz.

$$A_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)' \text{ given by } \langle A_p u, v \rangle_{W^{1,p}(\Omega)} := \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x.$$

In the weak formulation of the momentum equation (1.2), besides $\mathcal{V}$ and $\mathcal{E}$ we will also make use of the operator

$$\mathcal{C}_\rho : L^2(\Omega) \to H^{-1}(\Omega; \mathbb{R}^d) \quad \text{defined by} \quad \langle \mathcal{C}_\rho(\theta), \mathbf{v} \rangle_{H^1(\Omega; \mathbb{R}^d)} := -\rho \int_\Omega \theta \, \mathrm{div}(\mathbf{v}) \, \mathrm{d}x. \tag{2.1}$$

Finally, throughout the paper we shall denote by the symbols c, c', C, C' various positive constants depending only on known quantities. Furthermore, the symbols I_i, $i = 0, 1, \ldots$, will be used as place-holders for several integral terms popping in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, the symbol I_1 will occur several times with different meanings.

Recaps of mathematical elasticity. The elasticity and viscosity tensors fulfill

$$\mathbb{E} = (e_{ijkh}), \ \mathbb{V} = (v_{ijkh}) \in \mathrm{C}^1(\Omega; \mathbb{R}^{d \times d \times d \times d}) \tag{2.2}$$

with coefficients satisfying the classical symmetry and ellipticity conditions (with the usual summation convention)

$$e_{ijkh} = e_{jikh} = e_{khij}, \qquad v_{ijkh} = v_{jikh} = v_{khij}$$

$$\exists\,\alpha_0 > 0 : \qquad e_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha_0\xi_{ij}\xi_{ij} \quad \forall\,\xi_{ij} : \xi_{ij} = \xi_{ji} \tag{2.3}$$

$$\exists\,\beta_0 > 0 : \qquad v_{ijkh}\xi_{ij}\xi_{kh} \geq \beta_0\xi_{ij}\xi_{ij} \quad \forall\,\xi_{ij} : \xi_{ij} = \xi_{ji}.$$

Observe that with (2.3) we also encompass in our analysis the case of an anisotropic and inhomogeneous material.

In order to give the variational formulation of the momentum equation, we need to introduce the bilinear forms related to the χ-dependent elliptic operators appearing in (1.2). Hence, given a *non-negative* function $\eta \in L^\infty(\Omega)$ (later, $\eta = a(\chi)$ or $\eta = b(\chi)$), let us consider the bilinear symmetric forms $\mathsf{e}(\eta\cdot,\cdot)$, $\mathsf{v}(\eta\cdot,\cdot) : H_0^1(\Omega;\mathbb{R}^d) \times H_0^1(\Omega;\mathbb{R}^d) \to \mathbb{R}$ defined for all $\mathbf{u}, \mathbf{v} \in H_0^1(\Omega;\mathbb{R}^d)$ by

$$\mathsf{e}(\eta\mathbf{u}, \mathbf{v}) := \langle -\operatorname{div}(\eta\mathbb{E}\varepsilon(\mathbf{u})), \mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)} = \sum_{i,j,k,h=1}^{d} \int_\Omega \eta\, e_{ijkh}\, \varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}),$$

$$\mathsf{v}(\eta\mathbf{u}, \mathbf{v}) := \langle -\operatorname{div}(\eta\mathbb{V}\varepsilon(\mathbf{u})), \mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)} = \sum_{i,j,k,h=1}^{d} \int_\Omega \eta\, v_{ijkh}\, \varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}). \tag{2.4}$$

Thanks to (2.3) and Korn's inequality (see eg [4, Thm. 6.3-3]), the forms $\mathsf{e}(\eta\cdot,\cdot)$ and $\mathsf{v}(\eta\cdot,\cdot)$ fulfill

$$\exists\,C_1 > 0 \;\forall\,\mathbf{u}, \mathbf{v} \in H_0^1(\Omega;\mathbb{R}^d) \;:\; \begin{cases} \mathsf{e}(\eta\mathbf{u}, \mathbf{u}) \geq \inf_{x\in\Omega}(\eta(x))\,C_1\|\mathbf{u}\|_{H^1(\Omega)}^2, \\ \mathsf{v}(\eta\mathbf{u}, \mathbf{u}) \geq \inf_{x\in\Omega}(\eta(x))\,C_1\|\mathbf{u}\|_{H^1(\Omega)}^2. \end{cases} \tag{2.5}$$

It follows from (2.2) that they are also continuous, namely

$$\exists\,C_2 > 0 \;\forall\,\mathbf{u}, \mathbf{v} \in H_0^1(\Omega;\mathbb{R}^d) \;:\; \quad |\mathsf{e}(\eta\mathbf{u}, \mathbf{v})| + |\mathsf{v}(\eta\mathbf{u}, \mathbf{v})| \leq C_2\|\eta\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}. \tag{2.6}$$

We shall denote by $\mathcal{E}(\eta\,\cdot) : H_0^1(\Omega;\mathbb{R}^d) \to H^{-1}(\Omega;\mathbb{R}^d)$ and $\mathcal{V}(\eta\,\cdot) : H_0^1(\Omega;\mathbb{R}^d) \to H^{-1}(\Omega;\mathbb{R}^d)$ the linear operators associated with the forms $\mathsf{e}(\eta\cdot,\cdot)$ and $\mathsf{v}(\eta\cdot,\cdot)$, respectively, that is

$$\langle \mathcal{E}(\eta\mathbf{v}), \mathbf{w}\rangle_{H^1(\Omega;\mathbb{R}^d)} := \mathsf{e}(\eta\mathbf{v}, \mathbf{w}), \quad \langle \mathcal{V}(\eta\mathbf{v}), \mathbf{w}\rangle_{H^1(\Omega;\mathbb{R}^d)} := \mathsf{v}(\eta\mathbf{v}, \mathbf{w}) \qquad \text{for all } \mathbf{v}, \mathbf{w} \in H_0^1(\Omega;\mathbb{R}^d). \tag{2.7}$$

It can be checked via an approximation argument that the following regularity results hold

$$\text{if } \eta \in L^\infty(\Omega) \text{ and } \mathbf{u} \in H_0^1(\Omega;\mathbb{R}^d), \text{ then } \mathcal{E}(\eta\mathbf{u}), \mathcal{V}(\eta\mathbf{u}) \in H^{-1}(\Omega;\mathbb{R}^d), \tag{2.8a}$$

$$\text{if } \eta \in W^{1,d}(\Omega) \text{ and } \mathbf{u} \in H_0^2(\Omega;\mathbb{R}^d), \text{ then } \mathcal{E}(\eta\mathbf{u}), \mathcal{V}(\eta\mathbf{u}) \in L^2(\Omega;\mathbb{R}^d). \tag{2.8b}$$

Finally, let us recall the following elliptic regularity result (see e.g. [23, Lemma 3.2, p. 260])

$$\exists\,C_3, C_4 > 0 \quad \forall\,\mathbf{u} \in H_0^2(\Omega;\mathbb{R}^d) \;:\; \qquad C_3\|\mathbf{u}\|_{H^2(\Omega)} \leq \|\operatorname{div}(\varepsilon(\mathbf{u}))\|_{L^2(\Omega)} \leq C_4\|\mathbf{u}\|_{H^2(\Omega)}. \tag{2.9}$$

Useful inequalities. In order to make the paper as self-contained as possible, we recall here the Gagliardo-Nirenberg inequality (cf. [24, p. 125]) in a particular case: for all $r, q \in [1, +\infty]$, and for all $v \in L^q(\Omega)$ such that $\nabla v \in L^r(\Omega)$, there holds

$$\|v\|_{L^s(\Omega)} \leq C_{\mathrm{GN}}\|v\|_{W^{1,r}(\Omega)}^\theta\|v\|_{L^q(\Omega)}^{1-\theta} \qquad \text{with } \frac{1}{s} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + (1-\theta)\frac{1}{q}, \;\; 0 \leq \theta \leq 1, \tag{2.10}$$

the positive constant C_{GN} depending only on d, r, q, θ. Combining the compact embedding

$$H_0^2(\Omega;\mathbb{R}^d) \Subset W^{1,d^\star - \eta}(\Omega;\mathbb{R}^d), \quad \text{with } d^\star = \begin{cases} \infty & \text{if } d = 2, \\ 6 & \text{if } d = 3, \end{cases} \quad \text{for all } \eta > 0, \tag{2.11}$$

(where for $d = 2$ we mean that $H_0^2(\Omega;\mathbb{R}^d) \Subset W^{1,q}(\Omega;\mathbb{R}^d)$ for all $1 \leq q < \infty$), with [20, Thm. 16.4, p. 102], we have

$$\forall\,\varrho > 0 \;\; \exists\,C_\varrho > 0 \;\; \forall\,\mathbf{u} \in H_0^2(\Omega;\mathbb{R}^d) \;:\; \|\varepsilon(\mathbf{u})\|_{L^{d^\star - \eta}(\Omega)} \leq \varrho\|\mathbf{u}\|_{H^2(\Omega)} + C_\varrho\|\mathbf{u}\|_{L^2(\Omega)}. \tag{2.12}$$

We will also use the following *nonlinear* Poincaré-type inequality (cf. e.g. [15, Lemma 2.2]), with $m(w)$ the mean value of w:

$$\forall q > 0 \quad \exists C_q > 0 \quad \forall w \in H^1(\Omega) : \qquad \||w|^q w\|_{H^1(\Omega)} \le C_q(\|\nabla(|w|^q w)\|_{L^2(\Omega)} + |m(w)|^{q+1}). \tag{2.13}$$

2.2. **Assumptions.** In most of this paper, we shall suppose that

$$\Omega \subset \mathbb{R}^d, \quad d \in \{2, 3\} \text{ is a bounded connected domain, with } C^2\text{-boundary } \partial\Omega. \tag{2.14}$$

This will allow us to apply elliptic regularity results and to conclude $H^2(\Omega; \mathbb{R}^d)$-regularity for $\mathbf{u}$. In Section 6 we will see that this regularity requirement can be dropped, at the price of proving the existence of a weaker notion of solution for the irreversible system (cf. Remark 6.3).

We list below our basic assumptions on the functions K, a, b, and W in system (1.1)–(1.3).

Hypothesis (I). We suppose that

$$\begin{aligned} &\text{the function } \mathsf{K} : [0, +\infty) \to (0, +\infty) \text{ is continuous and} \\ &\exists c_0, c_1 > 0 \quad \kappa > 1 \quad \forall \vartheta \in [0, +\infty) : \quad c_0(1 + \vartheta^\kappa) \le \mathsf{K}(\vartheta) \le c_1(1 + \vartheta^\kappa). \end{aligned} \tag{2.15}$$

We will denote by $\widehat{\mathsf{K}}$ the primitive $\widehat{\mathsf{K}}(x) := \int_0^x \mathsf{K}(r)\, dr$ of K.

Hypothesis (II). We require

$$a \in C^1(\mathbb{R}), \ b \in C^2(\mathbb{R}) \text{ and } \exists c_2 > 0 : \quad a(x), \ b(x) \ge c_2 \text{ for all } x \in \mathbb{R} \tag{2.16}$$

and that the function b is convex. The latter requirement could be weakened to λ-convexity, i.e. that b'' is bounded from below (cf. also (2.19)), see. Remark 4.9 later on.

Hypothesis (III). We suppose that the potential W in (1.3) is given by $W = \widehat{\beta} + \widehat{\gamma}$, where

$$\widehat{\beta} : \operatorname{dom}(\widehat{\beta}) \to \mathbb{R} \text{ is proper, l.s.c., convex}, \quad \widehat{\gamma} \in C^2(\mathbb{R}), \tag{2.17}$$

$$\exists c_W, c_W' > 0 : \quad W(r) \ge c_W|r| - c_W' \quad \forall r \in \operatorname{dom}(\widehat{\beta}). \tag{2.18}$$

Moreover, we impose that

$$\exists \lambda > 0 \ \forall r \in \mathbb{R} : \ \widehat{\gamma}''(r) \ge -\lambda. \tag{2.19}$$

Hereafter, we shall use the notation

$$\beta := \partial\widehat{\beta}, \qquad \gamma := \widehat{\gamma}'.$$

Observe that, we have not required that $\operatorname{dom}(\widehat{\beta}) \subset [0, +\infty)$, which would enforce the (physically feasible) positivity of the phase/damage variable χ. In fact, for the analysis of the irreversible case (i.e. with $\mu = 1$), we will have to confine the discussion to the case $\widehat{\beta} = I_{[0, +\infty)}$, cf. Hypothesis (IV) later on. Instead, in the reversible case $\mu = 0$, we will allow for a general $\widehat{\beta}$ (complying with Hypothesis (III)).

Remark 2.2 (A generalization of the p-Laplacian). In fact, our analysis of system (1.1)–(1.3) extends to the case the p-Laplacian operator $-\operatorname{div}(|\nabla\chi|^{p-2}\nabla\chi)$, with $p > d$, is replaced by an elliptic operator $\mathcal{B} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ of the form

$$\langle \mathcal{B}(\chi), v \rangle_{W^{1,p}(\Omega)} := \int_\Omega \nabla_\zeta \phi(x, \nabla\chi(x)) \cdot \nabla v(x)\, dx, \tag{2.20}$$

where $\phi : \Omega \times \mathbb{R}^d \to [0, +\infty)$ is a Carathéodory integrand such that

$$\text{the map } \phi(x, \cdot) : \mathbb{R}^d \to [0, +\infty) \text{ is convex, with } \phi(x, 0) = 0, \text{ and in } C^1(\mathbb{R}^d) \text{ for a.a. } x \in \Omega,$$

$$\exists c_3, c_4, c_5 > 0 \text{ for a.a. } x \in \Omega \ \forall \zeta \in \mathbb{R}^d : \quad \begin{cases} \phi(x, \zeta) \ge c_3|\zeta|^p - c_4, \\ |\nabla_\zeta \phi(x, \zeta)| \le c_5(1 + |\zeta|^{p-1}). \end{cases}$$

This more general framework was analyzed in [28], to which we refer the reader for all details.

Problem and Cauchy data. We suppose that the data $\mathbf{f}$, g, and h fulfill

$$\mathbf{f} \in L^2(0,T; L^2(\Omega; \mathbb{R}^d)), \tag{2.21}$$

$$g \in L^1(0,T; L^1(\Omega)) \cap L^2(0,T; H^1(\Omega)'), \quad g \geq 0 \quad \text{a.e. in } \Omega \times (0,T), \tag{2.22}$$

$$h \in L^1(0,T; L^2(\partial\Omega)), \quad h \geq 0 \quad \text{a.e. in } \partial\Omega \times (0,T), \tag{2.23}$$

and that the initial data comply with

$$\vartheta_0 \in L^1(\Omega), \quad \exists \vartheta_* > 0 : \quad \inf_\Omega \vartheta_0 \geq \vartheta_* > 0, \quad \log \vartheta_0 \in L^1(\Omega), \tag{2.24}$$

$$\mathbf{u}_0 \in H_0^2(\Omega; \mathbb{R}^d), \quad \mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d), \tag{2.25}$$

$$\chi_0 \in W^{1,p}(\Omega), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega). \tag{2.26}$$

2.3. **A global existence result for the irreversible system.** Before stating precisely our notion of weak solution to (the initial-boundary value problem for) system (1.1)–(1.3) in the case of *unidirectional evolution*, let us briefly motivate the weak formulations for the heat balance equation (1.1), and for the phase/damage parameter subdifferential inclusion (1.3) (with $\mu = 1$). They will be coupled with the pointwise (in time and space) formulation of the momentum equation (1.2) (cf. (2.40) later on).

For (1.1), we adopt the weak formulation of proposed in [2, 7, 9]. It consists of a so-called "entropy inequality", and of an "energy conservation" identity. The former is obtained by formally dividing (1.1) by ϑ, and testing it by a smooth test function φ. Integrating over space and time leads to

$$\int_0^T \int_\Omega (\partial_t \log(\vartheta) + \chi_t + \rho \mathrm{div}(\mathbf{u}_t)) \, \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \mathsf{K}(\vartheta) \nabla \log(\vartheta) \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^T \int_\Omega \mathsf{K}(\vartheta) \frac{\varphi}{\vartheta} \nabla \log(\vartheta) \nabla \vartheta \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega (g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2) \frac{\varphi}{\vartheta} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} h \frac{\varphi}{\vartheta} \, \mathrm{d}S \, \mathrm{d}t \tag{2.27}$$

for all $\varphi \in \mathcal{D}(\overline{Q})$. Then, the entropy inequality (2.37) below follows. The total energy identity (2.38) associated with system (1.1)–(1.3) is obtained by testing (1.1) by 1, (1.2) by $\mathbf{u}_t$, and (1.3) by χ_t.

Remark 2.3. Conversely, it can be checked that, when the functions ϑ and χ are *sufficiently smooth*, inequalities (2.37)–(2.38), combined with (1.2) and (1.3), yield the heat equation (1.1).

Indeed, the weak formulation of (1.1) is *equivalent*, for sufficiently smooth solutions, to the (2.37) with identity sign. Hence, let us suppose, by contradiction, that (2.37) holds with strict inequality sign (hence, (1.1) does not hold). Then, using (1.2) and (1.3), we can conclude that the total energy balance (2.38) is not satisfied.

However, at the moment the necessary regularity for ϑ and χ to carry out this argument is out of reach.

Let us emphasize that the entropy inequality (2.37) below has the advantage that all the troublesome quadratic quantities on the right-hand side of (1.1) are tested by the *negative* function $-\varphi$. This will allow for upper semicontinuity arguments in the limit passage for proving the existence of weak solutions, cf. Sec. 5 later on. Let us also mention in advance that, when dropping the unidirectionality constraint (i.e., in the case $\mu = 0$), under an additional condition (cf. Hypothesis (V)), we will be able to get an existence result for an improved formulation of (1.1), cf. Theorem 2.8 below.

A significant difficulty in the analysis of system (1.1)–(1.3) is due to the triply nonlinear character of (1.3), featuring, in addition to the p-Laplacian and to $\beta = \partial\widehat{\beta}$ which contributes to W', the (maximal monotone) operator $\partial I_{(-\infty,0]}$. Since the latter is unbounded, it is not possible to perform comparison estimates in (1.3) and an estimate for the terms $A_p\chi$ and $\beta(\chi)$ (treated as single-valued in the context of this heuristical discussion) could be obtained only by testing (1.3) by $\partial_t(A_p\chi + \beta(\chi))$. However, the related calculations, involving an

integration by parts in time on the right-hand side of (1.3), cannot be carried out in the present case. That is why, we need to resort to a weak formulation of (1.3) which does not feature the term $A_p \chi + \beta(\chi)$. We draw it from [16, 17], and as therein we confine the analysis to the particular case in which

Hypothesis (IV).

$$\widehat{\beta} = I_{[0,+\infty)}. \tag{2.28}$$

This still ensures the constraint

$$\chi \in [0,1] \quad \text{a.e. in } \Omega \times (0,T) \tag{2.29}$$

provided we start from an initial datum $\chi_0 \le 1$ a.e. in Ω, we will obtain by irreversibility that $\chi(t) \le \chi_0 \le 1$ a.e. in Ω, for almost all $t \in (0,T)$.

To motivate the weak formulation of (1.3) from [16, 17], we observe that (1.3) rephrases as

$$\chi_t \le 0 \qquad \text{in } \Omega \times (0,T), \tag{2.30a}$$

$$\left(\chi_t - \mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) + \xi + \gamma(\chi) + b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} - \vartheta \right) \psi \ge 0 \text{ for all } \psi \le 0 \qquad \text{in } \Omega \times (0,T), \tag{2.30b}$$

$$\left(\chi_t - \mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) + \xi + \gamma(\chi) + b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} - \vartheta \right) \chi_t \le 0 \qquad \text{in } \Omega \times (0,T), \tag{2.30c}$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in $\Omega \times (0,T)$. Our weak formulation of (1.3) in fact consists of (2.30a), of the integrated version of (2.30b), with negative test functions from $W^{1,p}(\Omega)$, and of the energy inequality obtained by integrating (2.30c). In [28, Prop. 2.14] (see also [16, Thm. 4.6]), we prove that, under additional regularity properties, any weak solution in the sense of (2.41)–(2.44) in fact fulfills (1.3) pointwise.

We are now in the position to specify our weak solution concept, for which we borrow the terminology from [9].

Definition 2.4 (Entropic solutions to the irreversible system)**.** Let $\mu = 1$. Given initial data $(\vartheta_0, \mathbf{u}_0, \mathbf{v}_0, \chi_0)$ fulfilling (2.24)–(2.26), we call a triple $(\vartheta, \mathbf{u}, \chi)$ an *entropic solution* to the (initial-boundary value problem) for system (1.1)–(1.3), with the boundary conditions (1.4), if

$$\vartheta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \tag{2.31}$$

$$\log(\vartheta) \in L^2(0,T;H^1(\Omega)), \tag{2.32}$$

$$\mathbf{u} \in H^1(0,T;H^2_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{2.33}$$

$$\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{2.34}$$

$(\vartheta, \mathbf{u}, \chi)$ complies with the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad\qquad \text{for a.a. } x \in \Omega, \tag{2.35}$$

$$\chi(0,x) = \chi_0(x) \qquad\qquad \text{for a.a. } x \in \Omega, \tag{2.36}$$

(while the initial condition for ϑ is implicitly formulated in (2.38) below), and with the *entropic formulation* of (1.1)–(1.3), consisting of

- the *entropy* inequality for almost all $t \in (0,T]$ and almost all $s \in (0,t)$, and for $s = 0$:

$$\int_s^t \int_\Omega (\log(\vartheta) + \chi)\varphi_t \, \mathrm{d}x \, \mathrm{d}r - \rho \int_s^t \int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi \, \mathrm{d}x \, \mathrm{d}r - \int_s^t \int_\Omega \mathsf{K}(\vartheta)\nabla\log(\vartheta) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}r$$

$$\le \langle \log(\vartheta(t)), \varphi(t) \rangle_{W^{1,d+\epsilon}(\Omega)} - \langle \log(\vartheta(s)), \varphi(s) \rangle_{W^{1,d+\epsilon}(\Omega)} - \int_s^t \int_\Omega \mathsf{K}(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta) \cdot \nabla\vartheta \, \mathrm{d}x \, \mathrm{d}r \tag{2.37}$$

$$- \int_s^t \int_\Omega \left(g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2 \right)\frac{\varphi}{\vartheta} \, \mathrm{d}x \, \mathrm{d}r - \int_s^t \int_{\partial\Omega} h\frac{\varphi}{\vartheta} \, \mathrm{d}S \, \mathrm{d}r$$

for all φ in $C^0([0,T];W^{1,d+\epsilon}(\Omega))$ for some $\epsilon > 0$, and $\varphi \in H^1(0,T;L^{6/5}(\Omega))$, with $\varphi \ge 0$;

- the *total energy inequality* for almost all $t \in (0,T]$ and almost all $s \in (0,t)$, and for $s = 0$:

$$\mathscr{E}(\vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t), \chi(t)) \leq \mathscr{E}(\vartheta(s), \mathbf{u}(s), \mathbf{u}_t(s), \chi(s)) + \int_s^t \int_\Omega g \, \mathrm{d}x \, \mathrm{d}r + \int_s^t \int_{\partial\Omega} h \, \mathrm{d}S \, \mathrm{d}r + \int_s^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \, \mathrm{d}x \, \mathrm{d}r \,,$$

(2.38)

where for $s = 0$ we read ϑ_0, and

$$\mathscr{E}(\vartheta, \mathbf{u}, \mathbf{u}_t, \chi) := \int_\Omega \vartheta \, \mathrm{d}x + \frac{1}{2} \int_\Omega |\mathbf{u}_t|^2 \, \mathrm{d}x + \frac{1}{2}\mathsf{e}(b(\chi(t))\mathbf{u}(t), \mathbf{u}(t)) + \frac{1}{p} \int_\Omega |\nabla\chi|^p \, \mathrm{d}x + \int_\Omega W(\chi) \, \mathrm{d}x \,;$$

(2.39)

- the momentum equation

$$\mathbf{u}_{tt} + \mathcal{V}(a(\chi)\mathbf{u}_t) + \mathcal{E}(b(\chi)\mathbf{u}) + \mathcal{C}_\rho(\vartheta) = \mathbf{f} \quad \text{a.e. in } \Omega \times (0,T);$$

(2.40)

- the weak formulation of (1.3), viz.

$$\chi_t(x,t) \leq 0 \qquad \text{for a.a. } (x,t) \in \Omega \times (0,T),$$

(2.41)

$$\int_\Omega \left(\chi_t(t)\psi + |\nabla\chi(t)|^{p-2}\nabla\chi(t) \cdot \nabla\psi + \xi(t)\psi + \gamma(\chi(t))\psi + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))}{2}\psi - \vartheta(t)\psi \right) \mathrm{d}x \geq 0$$

(2.42)

$$\text{for all } \psi \in W_-^{1,p}(\Omega), \quad \text{for a.a. } t \in (0,T),$$

where $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the sense that

$$\xi \in L^1(0,T;L^1(\Omega)) \qquad \text{and} \qquad \langle \xi(t), \psi - \chi(t) \rangle_{W^{1,p}(\Omega)} \leq 0 \quad \forall \psi \in W_+^{1,p}(\Omega), \text{ for a.a. } t \in (0,T),$$

(2.43)

as well as and the energy inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \leq t$

$$\int_s^t \int_\Omega |\chi_t|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_\Omega \left(\frac{1}{p}|\nabla\chi(t)|^p + W(\chi(t)) \right) \mathrm{d}x$$

$$\leq \int_\Omega \left(\frac{1}{p}|\nabla\chi(s)|^p + W(\chi(s)) \right) \mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \right) \mathrm{d}x \, \mathrm{d}r.$$

(2.44)

We now state our existence result for system (1.1)–(1.3) in the case $\mu = 1$. As far as the time-regularity of ϑ goes, observe that we will just prove BV-in-time regularity for $\log(\vartheta)$ (cf. (2.46) below). Indeed, we will obtain BV-in-time regularity for ϑ, as well, under an additional restriction on the exponent κ in Hypothesis (I) (note that the range of the admissible values below depends on the space dimension), viz.

Hypothesis (V). The exponent κ in (2.15) satisfies

$$\kappa \in (1,5/3) \quad \text{if } d = 3 \text{ and } \kappa \in (1,2) \quad \text{if } d = 2 \,.$$

(2.45)

Theorem 2.5 (Existence of entropic solutions, $\mu = 1$). *Let $\mu = 1$. Assume **Hypotheses (I)–(III)** and, in addition, **(IV)** (i.e., $\widehat{\beta} = I_{[0,+\infty)}$), as well as conditions (2.21)–(2.26) on the data $\mathbf{f}$, g, h, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0. Then, there exists an entropic solution (in the sense of Definition 2.4) $(\vartheta, \mathbf{u}, \chi)$ to the initial-boundary value problem for system (1.1)–(1.3), such that*

$$\log(\vartheta) \in \mathrm{BV}([0,T];W^{1,d+\epsilon}(\Omega)^*) \qquad \text{for all } \epsilon > 0,$$

(2.46)

and ξ in (2.43) is given by

$$\xi(x,t) = -\mathcal{I}_{\chi=0}(x,t) \left(\gamma(\chi(x,t)) + b'(\chi(x,t))\frac{\varepsilon(\mathbf{u}(x,t))\mathbb{E}(x)\varepsilon(\mathbf{u}(x,t))}{2} - \vartheta(x,t) \right)^+ ,$$

(2.47)

for almost all $(x,t) \in \Omega \times (0,T)$, where $\mathcal{I}_{\chi=0}$ denotes the characteristic function of the set $\{\chi = 0\}$, and such that $\exists \underline{\vartheta} > 0$ such that

$$\vartheta(x,t) \geq \underline{\vartheta} > 0 \quad \text{for a.a. } (x,t) \in \Omega \times (0,T).$$

(2.48)

*Furthermore, if in addition K satisfies **Hypothesis (V)**, there holds*

$$\vartheta \in \mathrm{BV}([0,T];W^{2,d+\epsilon}(\Omega)^*) \qquad \text{for every } \epsilon > 0,$$

(2.49)

and the total energy inequality (2.38) holds for all $t \in [0,T]$, for $s = 0$, and for almost all $s \in (0,t)$.

Observe that (2.49) yields that there exists $D \subset [0,T]$, at most infinitely countable, such that $\vartheta \in C^0([0,T] \setminus D; W^{2,d+\epsilon}(\Omega)^*)$. We will develop the proof in Section 5, by passing to the limit in the time-discretization scheme carefully devised in Section 4.

Remark 2.6 (Uniqueness and extensions). (1) Uniqueness of solutions for the *irreversible* system, even in the isothermal case, is still an open problem. This is mainly due to the doubly nonlinear character of (1.3) (cf. also [5] for non-uniqueness examples for a general doubly nonlinear equation).

(2) Theorem 2.5 could be easily extended to the case in which the indicator function $I_{(-\infty,0]}$ in (1.3) is replaced by

$$\widehat{\alpha} : \mathbb{R} \to [0,+\infty] \quad \text{convex, 1-positively homogeneous, with } \mathrm{dom}(\widehat{\alpha}) \subset (-\infty,0] \text{ and } 0 \in \alpha(0). \tag{2.50}$$

2.4. **A global existence result for the reversible system.** In the case $\mu = 0$, we are able to cope with a weak solvability notion for system (1.1)–(1.3) stronger than the one from Definition 2.4. Indeed, it features a *pointwise* formulation for the internal parameter equation (1.3), while keeping the entropic formulation for the heat equation (1.1). Under the additional Hypothesis (V), we will also improve the weak formulation of the heat equation (cf. (2.54) below). As a byproduct, we will manage to prove the total energy *identity* for all $t \in [0,T]$.

Definition 2.7 (Entropic solutions to the reversible system). Let $\mu = 0$. Given initial data $(\vartheta_0, \mathbf{u}_0, \mathbf{v}_0, \chi_0)$ fulfilling (2.24)–(2.26), we call a triple $(\vartheta, \mathbf{u}, \chi)$ an *entropic solution* to the (initial-boundary value problem) for system (1.1)–(1.3), with the boundary conditions (1.4), if it has the regularity (2.31)–(2.34), it complies with the initial conditions (2.35)–(2.36), and with

- the *entropy* inequality (2.37);
- the *total energy inequality* (2.38) for almost all $t \in (0,T]$, for $s = 0$, and for almost all $s \in (0,t)$;
- the momentum equation (2.40);
- the internal parameter equation

$$\chi_t + A_p \chi + \xi + \gamma(\chi) = -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \qquad \text{a.e. in } \Omega \times (0,T), \tag{2.51}$$

with

$$\xi \in L^2(0,T;L^2(\Omega)) \text{ s.t.} \quad \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.a. } (x,t) \in \Omega \times (0,T). \tag{2.52}$$

Our second main result states the existence of an entropic solution $(\vartheta, \mathbf{u}, \chi)$ (in the sense of Definition 2.7) to the PDE system (1.1)–(1.3). Furthermore, we show that, under the additional Hypothesis (V), the formulation of the heat equation (1.1) improves to a standard variational formulation (cf. (2.54) below), albeit with suitably smooth test functions, and the total energy inequality (2.38) holds as an equality. We shall refer to the solutions thus obtained as *weak*.

Theorem 2.8 (Existence of entropic and weak solutions, $\mu = 0$). *Let $\mu = 0$. Assume **Hypotheses (I)–(III)** and conditions (2.21)–(2.26) on the data $\mathbf{f}$, g, h, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0,. Then, there exists an entropic solution (in the sense of Definition 2.7) $(\vartheta, \mathbf{u}, \chi)$ to the initial-boundary value problem for system (1.1)–(1.3), such that the strict positivity property (2.48) holds for ϑ, and such that χ has the enhanced regularity*

$$\chi \in L^2(0,T;W^{1+\sigma,p}(\Omega)) \qquad \text{for all } 0 < \sigma < \frac{1}{p}. \tag{2.53}$$

Moreover, if K *also complies with* **Hypothesis (V)***, then* ϑ *has the enhanced regularity* (2.49) *and the heat equation* (1.1) *is fulfilled in the following weak form for all* $t \in [0, T]$

$$\int_\Omega \varphi(t)\,\vartheta(t)(\mathrm{d}x) - \int_0^t \int_\Omega \vartheta\varphi_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \chi_t\vartheta\varphi\,\mathrm{d}x\,\mathrm{d}s + \rho\int_0^t \int_\Omega \mathrm{div}(\mathbf{u}_t)\vartheta\varphi\,\mathrm{d}x\,\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \mathsf{K}(\vartheta)\nabla\vartheta\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_\Omega \left(g + \frac{\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)}{2} + |\chi_t|^2 \right)\varphi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} h\varphi\,\mathrm{d}S\,\mathrm{d}s + \int_\Omega \vartheta_0\varphi(0)\,\mathrm{d}x$$

for all $\varphi \in C^0([0,T]; W^{2,d+\epsilon}(\Omega)) \cap H^1(0,T; L^{6/5}(\Omega))$ *for some* $\epsilon > 0$.

$$(2.54)$$

In this case, the triple $(\vartheta, \mathbf{u}, \chi)$ *complies with the* total energy equality

$$\mathscr{E}(\vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t), \chi(t)) = \mathscr{E}(\vartheta(s), \mathbf{u}(s), \mathbf{u}_t(s), \chi(s)) + \int_s^t \int_\Omega g\,\mathrm{d}x\,\mathrm{d}r + \int_0^t \int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}r + \int_s^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r,$$

$$(2.55)$$

for all $0 \le s \le t \le T$.

The *proof* will be given in Section 5, passing to the limit in the time-discretization scheme set up in Sec. 4. We mention in advance that the argument for (2.54) and for the total energy identity (2.55) *for all* $t \in [0, T]$ relies on obtaining, for the sequence $(\mathbf{u}_k, \chi_k)$ of approximate solutions, the *strong* convergences

$$\mathbf{u}_k \to \mathbf{u} \quad \text{in } H^1(0,T; H_0^1(\Omega; \mathbb{R}^d)), \qquad \chi_k \to \chi \quad \text{in } H^1(0,T; L^2(\Omega)). \tag{2.56}$$

This allows us to pass to the limit on the right-hand side of the approximate version of (2.54). In turn, the proof of (2.56) is based on a lim sup-argument, for which it is essential to have preliminarily obtained the *pointwise* formulation (2.51) of the equation for χ. This is the reason why we have not been able to obtain the improved formulation (2.54) in the irreversible case $\mu = 1$.

Remark 2.9 (Uniqueness in the reversible case)**.** As in the irreversible case, a uniqueness result for the full system seems to be out of reach. Instead, for the *isothermal* case in [28, Thm. 3] we have proved uniqueness and continuous dependence of the solutions on the data. This result has been obtained in the case the p-Laplacian operator $-\mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi)$ is replaced by an elliptic operator of the type described in Remark 2.2, fulfilling an additional non-degeneracy condition, cf. Hypothesis (VII) in [28]: for instance, we may consider $-\mathrm{div}((1 + |\nabla\chi|^2)^{(p-2)/2})$.

3. (Formal) A priori estimates

In this section, we perform a series of *formal* estimates on system (1.1)–(1.3). All of these estimates will be rigorously justified on the time-discrete approximation scheme proposed in Section 4. Yet, we believe that, in order to enhance the readability of the paper, it is worthwhile to develop all the significant calculations on the (easier) time-continuous level. This is especially useful for the Second and the Third a priori estimates, which have a non-standard character and are in fact tailored to handle the quadratic terms on the right-hand side of (1.1).

More in detail, we start by showing the strict positivity of the temperature ϑ, via a comparison argument in the same lines as the one for proving positivity in [9, Subsection 4.2.1]. All the ensuing estimates rely on this property, starting from the basic energy estimate (i.e. the one corresponding to the total energy inequality (2.38)). After this, we test (1.1) by $\vartheta^{\alpha-1}$, with $\alpha \in (0, 1)$. This enables us somehow to confine the troublesome quadratic terms to the left-hand side. Carefully using the Gagliardo-Nirenberg inequality, we infer a bound for ϑ^α in $L^2(0,T; H^1(\Omega))$. Ultimately, exploiting the fact that the heat flux K controls ϑ^κ (cf. (2.15)) we conclude an estimate for ϑ in $L^2(0,T; H^1(\Omega))$. This done, we are in the position to perform all the remaining estimates, i.e. subtracting the temperature equation tested by 1 from the total energy inequality (2.38); performing an elliptic regularity estimate on the momentum equation (1.3), and comparison estimates in (1.1) and (1.3).

We mention in advance that, with the exception of the last one, all of the ensuing estimates hold both in the reversible ($\mu = 0$), and in the irreversible ($\mu = 1$) cases.

Positivity of ϑ [$\mu \in \{0, 1\}$]. Scooping all the quadratic terms in (1.1) to the right-hand side, we obtain

$$\vartheta_t - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2 - \chi_t\vartheta - \rho\vartheta\mathrm{div}(\mathbf{u}_t)$$

$$\geq g + c|\varepsilon(\mathbf{u}_t)|^2 + \frac{1}{2} + |\chi_t|^2 - C\vartheta^2 \geq -C\vartheta^2 \quad \text{a.e. in } \Omega \times (0, T),$$

where we have written (1.1) in a formal way, disregarding the (positive) boundary datum h. Indeed, for the first inequality we have used that $\mathbb{V}$ is positive definite, that a is strictly positive, and the fact that

$$|\mathrm{div}(\mathbf{u}_t)| \leq c(d)|\varepsilon(\mathbf{u}_t)| \quad \text{a.e. in } \Omega \times (0, T) \tag{3.1}$$

with $c(d)$ a positive constant only depending on the space dimension d. The second estimate also relies on the fact that $g \geq 0$ a.e. in $\Omega \times (0, T)$. Therefore we conclude that v solving the Cauchy problem

$$v_t = -\frac{1}{2}v^2, \quad v(0) = \vartheta_* > 0$$

is a subsolution of (1.1). Hence, a comparison argument yields

$$\vartheta(\cdot, t) \geq v(t) > \vartheta_* > 0 \quad \text{for all } t \in [0, T]. \tag{3.2}$$

First estimate [$\mu \in \{0, 1\}$]. Test (1.1) by 1, (1.2) by $\mathbf{u}_t$, (1.3) by χ_t and integrate over $(0, t)$, $t \in (0, T]$. Adding the resulting equations and taking into account cancellations, we obtain

$$\int_\Omega \vartheta(t)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2\,\mathrm{d}x + \frac{1}{2}\mathsf{e}(b(\chi(t))\mathbf{u}(t), \mathbf{u}(t)) + \frac{1}{p}\int_\Omega |\nabla\chi(t)|^p\,\mathrm{d}x + \int_\Omega W(\chi(t))\,\mathrm{d}x$$

$$= \int_\Omega \vartheta_0\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}\mathsf{e}(b(\chi_0)\mathbf{u}_0, \mathbf{u}_0) + \frac{1}{p}\int_\Omega |\nabla\chi_0|^p\,\mathrm{d}x + \int_\Omega W(\chi_0)\,\mathrm{d}x \tag{3.3}$$

$$+ \int_0^t \int_\Omega g\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}s + \int_0^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}s,$$

viz. the total energy equality (2.55). For (3.3), we have also used the integration-by-parts formula

$$\int_0^t \mathsf{e}(b(\chi(t))\mathbf{u}(s), \mathbf{u}_t(s))\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_\Omega b'(\chi)\chi_t\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\mathsf{e}(b(\chi(t))\mathbf{u}(t), \mathbf{u}(t)) - \frac{1}{2}\mathsf{e}(b(\chi_0)\mathbf{u}_0, \mathbf{u}_0) \tag{3.4}$$

as well as the fact that $\int_0^t \int_\Omega \partial I_{(-\infty, 0]}(\chi_t)\chi_t\,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_\Omega I_{(-\infty, 0]}(\chi_t)\,\mathrm{d}x\,\mathrm{d}s = 0$ (where we have formally written $\partial I_{(-\infty, 0]}(\chi_t)$ as a single-valued operator). Using (2.21)–(2.26) for the data f, g, h and the initial data $(\vartheta_0, \mathbf{u}_0, \chi_0)$, the positivity of ϑ (cf. (3.2)), and the coercivity (2.18) of W (cf. Hypothesis (III)), also in view of the Poincaré inequality we conclude the following estimate

$$\|\vartheta\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} + \|b(\chi)^{1/2}\varepsilon(\mathbf{u})\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d \times d}))} + \|\chi\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C, \tag{3.5}$$

as well as

$$\|W(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{3.6}$$

Second estimate [$\mu \in \{0, 1\}$]. Let $F(\vartheta) = \vartheta^\alpha/\alpha$, with $\alpha \in (0, 1)$. We test (1.1) by $F'(\vartheta) := \vartheta^{\alpha-1}$, and integrate on $(0, t)$ with $t \in (0, T]$. We thus have

$$\int_\Omega F(\vartheta_0)\,\mathrm{d}x + \int_0^t \int_\Omega gF'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} hF'(\vartheta)\,\mathrm{d}S\,\mathrm{d}s + \int_0^t \int_\Omega a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s$$

$$+ \int_0^t \int_\Omega |\chi_t|^2 F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega F(\vartheta(t))\,\mathrm{d}x + \int_0^t \int_\Omega \chi_t\vartheta F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s + \rho\int_0^t \int_\Omega \vartheta\,\mathrm{div}(\mathbf{u}_t)F'(\vartheta)\,\mathrm{d}x\,\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \mathsf{K}(\vartheta)\nabla\vartheta\nabla(F'(\vartheta))\,\mathrm{d}x\,\mathrm{d}s$$

whence (cf. (2.5) and the positivity (2.22) and (2.23) of g and h)

$$\frac{4(1-\alpha)}{\alpha^2} \int_0^t \int_\Omega \mathsf{K}(\vartheta) |\nabla(\vartheta^{\alpha/2})|^2 \, \mathrm{d}x \, \mathrm{d}s + c_2 \int_0^t \int_\Omega |\varepsilon(\mathbf{u}_t)|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega |\chi_t|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s$$

$$\leq \int_\Omega |F(\vartheta_0)| \, \mathrm{d}x + I_1 + I_2 + I_3,$$

where we have used (2.5) and (2.16). We estimate

$$I_1 = \int_\Omega |F(\vartheta(t))| \, \mathrm{d}x \leq \frac{1}{\alpha} \int_\Omega \max\{\vartheta(t), 1\}^\alpha \, \mathrm{d}x \leq \frac{1}{\alpha} \int_\Omega \max\{\vartheta(t), 1\} \, \mathrm{d}x \leq C$$

since $\alpha < 1$ and taking into account the previously obtained (3.5). Analogously we can estimate $\int_\Omega |F(\vartheta_0)| \, \mathrm{d}x$; moreover,

$$I_2 = \int_0^t \int_\Omega |\chi_t \vartheta F'(\vartheta)| \, \mathrm{d}x \, \mathrm{d}s \leq \frac{1}{4} \int_0^t \int_\Omega |\chi_t|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega F'(\vartheta) \vartheta^2 \, \mathrm{d}x \, \mathrm{d}s.$$

Using (2.16) and inequality (3.1), we have that

$$I_3 = |\rho| \int_0^t \int_\Omega |\vartheta \operatorname{div}(\mathbf{u}_t) F'(\vartheta)| \, \mathrm{d}x \, \mathrm{d}s \leq \frac{1}{4} \int_0^t \int_\Omega |\varepsilon(\mathbf{u}_t)|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s + C \int_0^t \int_\Omega F'(\vartheta) \vartheta^2 \, \mathrm{d}x \, \mathrm{d}s$$

$$\leq \frac{c_2}{4} \int_0^t \int_\Omega |\varepsilon(\mathbf{u}_t)|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s + C \int_0^t \int_\Omega F'(\vartheta) \vartheta^2 \, \mathrm{d}x \, \mathrm{d}s \,,$$

with c_2 from (2.16).

All in all, we conclude

$$\frac{4(1-\alpha)}{\alpha^2} \int_0^t \int_\Omega \mathsf{K}(\vartheta) |\nabla(\vartheta^{\alpha/2})|^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{c_2}{4} \int_0^t \int_\Omega |\varepsilon(\mathbf{u}_t)|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_0^t \int_\Omega |\chi_t|^2 F'(\vartheta) \, \mathrm{d}x \, \mathrm{d}s$$

$$\leq C + C \int_0^t \int_\Omega \vartheta^{\alpha+1} \, \mathrm{d}x \, \mathrm{d}s. \tag{3.7}$$

Now, we fix $q \geq 4$ and introduce the auxiliary quantity $\eta := \max\{\vartheta, 1\}$. Observe that η is still in $H^1(\Omega)$, and that, for q sufficiently big (see below) we have

$$\frac{\alpha}{2} \geq \frac{\alpha+1}{q} \quad \text{whence} \quad \eta^{(\alpha+1)/q} \leq \eta^{\alpha/2} \doteq w. \tag{3.8}$$

Therefore, taking into account that

$$\int_0^t \int_\Omega \mathsf{K}(\vartheta) |\nabla(\vartheta^{\alpha/2})|^2 \, \mathrm{d}x \, \mathrm{d}s \geq c_1 \iint_{\{\vartheta \geq 1\}} |\nabla(\vartheta^{\alpha/2})|^2 \, \mathrm{d}x \, \mathrm{d}s = c_1 \int_0^t \int_\Omega |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}s,$$

thanks to (2.15), we infer from (3.7) and (3.8) that

$$\int_0^t \int_\Omega |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}s \leq C + C \int_0^t \|w\|_{L^q(\Omega)}^q \, \mathrm{d}s. \tag{3.9}$$

We now apply the Gagliardo-Nirenberg inequality for $d = 3$ (for $d = 2$ even better estimates hold true), yielding

$$\|w\|_{L^q(\Omega)} \leq c_1 \|\nabla w\|_{L^2(\Omega; \mathbb{R}^d)}^\theta \|w\|_{L^r(\Omega)}^{1-\theta} + c_2 \|w\|_{L^r(\Omega)} \tag{3.10}$$

with $1 \leq r \leq q$ and θ satisfying $1/q = \theta/6 + (1-\theta)/r$. Hence $\theta = 6(q-r)/q(6-r)$. Observe that $\theta \in (0, 1)$ if $q < 6$ and that, by the way, this restriction on q implies that, for (3.8) we need to have $\alpha \in [1/2, 1)$. Plugging the Gagliardo-Nirenberg estimate into (3.9) and using Young's inequality we ultimately conclude

$$\frac{c}{2} \int_0^t \int_\Omega |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}s \leq C + C \int_0^t \|w\|_{L^r(\Omega)}^{2q(1-\theta)/(2-q\theta)} \, \mathrm{d}s + C' \int_0^t \|w\|_{L^r(\Omega)}^q \, \mathrm{d}s. \tag{3.11}$$

Now, choosing $r \leq 2/\alpha$, we have that

$$\|w\|_{L^r(\Omega)} = \left(\int_\Omega \eta^{r\alpha/2} \, \mathrm{d}x \right)^{1/r} \leq \left(\int_\Omega \eta \, \mathrm{d}x \right)^{1/r} \leq C \|\vartheta\|_{L^\infty(0,T; L^1(\Omega))} + |\Omega| \leq C,$$

where the latter inequality is due to estimate (3.5). Combining the above estimate with (3.11) we infer a bound for $w = \eta^{\alpha/2}$ in $L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^r(\Omega))$. Ultimately, also in view of (3.9), we conclude that

$$\|\vartheta^{\alpha/2}\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^r(\Omega))} \leq C. \tag{3.12}$$

Third estimate $[\mu \in \{0, 1\}]$. It follows from (3.7) and (2.15) that

$$
\begin{aligned}
C \geq \int_0^t \int_\Omega \mathsf{K}(\vartheta) |\nabla(\vartheta^{\alpha/2})|^2 \, dx \, ds &\geq c_1 \int_0^t \int_\Omega \vartheta^\kappa |\nabla(\vartheta^{\alpha/2})|^2 \, dx \, ds \\
&= \int_0^t \int_\Omega |\vartheta^{\kappa+\alpha-2}||\nabla\vartheta|^2 \, dx \, ds \\
&= \int_0^t \int_\Omega |\nabla(\vartheta^{(\kappa+\alpha)/2})|^2 \, dx \, ds
\end{aligned}
\tag{3.13}
$$

with $\alpha \in [1/2, 1)$ arbitrary.

From (3.13) and the strict positivity of ϑ (3.2) it follows that

$$\int_0^t \int_\Omega |\nabla\vartheta|^2 \, dx \, ds \leq C,$$

provided that $\kappa + \alpha - 2 \geq 0$. Observe that, since $\kappa > 1$ we can choose $\alpha \in [1/2, 1)$ such that this inequality holds. Hence, taking into account estimate (3.5) and applying Poincaré inequality, we deduce

$$\|\vartheta\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{3.14}$$

By interpolation (cf. (2.10)), we also get

$$\|\vartheta\|_{L^h(\Omega\times(0,T))} \leq C \quad \text{with } h = 8/3 \quad \text{if } d = 3, \quad h = 3 \quad \text{if } d = 2. \tag{3.15}$$

For later use, we also point out that estimates (3.13) and (3.14) yield that $\|\nabla\vartheta^{(\kappa-\alpha)/2}\|_{L^2(0,T;L^2(\Omega))} \leq C$. Combining this with estimate (3.5) and using a nonlinear version of the Poincaré inequality (cf. e.g. (2.13)), we infer

$$\|\vartheta^{(\kappa-\alpha)/2}\|_{L^2(0,T;H^1(\Omega))}, \ \|\vartheta^{(\kappa+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{3.16}$$

Fourth estimate $[\mu \in \{0, 1\}]$. We test (1.1) by 1, integrate over $(0, t)$, and subtract the resulting identity from the total energy balance (3.3). We thus obtain

$$
\frac{1}{2} \int_\Omega |\mathbf{u}_t(t)|^2 \, dx + \int_0^t \mathsf{v}(a(\chi)\mathbf{u}_t, \mathbf{u}_t) \, ds + \frac{1}{2} \mathsf{e}(b(\chi(t))\mathbf{u}(t), \mathbf{u}(t)) + \int_0^t \int_\Omega |\chi_t|^2 \, dx \, ds + \int_\Omega \frac{1}{p} |\nabla\chi(t)|^p + W(\chi(t)) \, dx
$$
$$
= \frac{1}{2} \int_\Omega |\mathbf{u}_0|^2 \, dx + \frac{1}{2} \mathsf{e}(b(\chi_0)\mathbf{u}_0, \mathbf{u}_0) + \int_\Omega \frac{1}{p} |\nabla\chi_0|^p + \int_\Omega W(\chi_0) \, dx + \int_0^t \int_\Omega \vartheta \left(\rho \operatorname{div} \mathbf{u}_t + \chi_t\right) \, dx \, ds \tag{3.17}
$$
$$
+ \int_0^t \int_\Omega \mathbf{f}\, \mathbf{u}_t \, dx \, ds.
$$

Using now (2.25)–(2.26) to estimate the initial data $(\mathbf{u}_0, \chi_0)$, (2.21) on $\mathbf{f}$, Hyp. (III) (which in particular yields that W is bounded from below), and combining estimate (3.14) on ϑ with (3.1), we obtain

$$\|\chi_t\|_{L^2(\Omega\times(0,T))} + \|a(\chi)^{1/2}\varepsilon(\mathbf{u}_t)\|_{L^2(\Omega\times(0,T);\mathbb{R}^{d\times d})} \leq C, \tag{3.18}$$

whence $\|\mathbf{u}_t\|_{L^2(0,T;H_0^1(\Omega;\mathbb{R}^d))} \leq C$, by (2.16).

Fifth estimate $[\mu \in \{0,1\}]$. We use here the crucial assumption that $p > d$. We test (1.2) by $-\mathrm{div}(\varepsilon(\mathbf{u}_t))$ and integrate on time (cf. also [28, Sec. 3]). Using the assumption $p > d$, we can fix $\zeta > 0$ such that $p \geq d + \zeta$, we get

$$- \int_0^t \int_\Omega \mathcal{E}\left(b(\chi)\mathbf{u}\right) \cdot \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s$$

$$= -\int_0^t \int_\Omega \nabla b(\chi) \mathbb{E}\varepsilon(\mathbf{u}) \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s - \int_0^t \int_\Omega b(\chi) \mathrm{div}(\mathbb{E}(\varepsilon(\mathbf{u}))) \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s$$

$$\leq C \int_0^t \|\nabla b(\chi)\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)} \|\varepsilon(\mathbf{u})\|_{L^{d^\star - \zeta}(\Omega;\mathbb{R}^{d \times d})} \|\mathrm{div}(\varepsilon(\mathbf{u}_t))\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}s$$

$$+ C \int_0^t \|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)} \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}s$$

$$\leq \sigma \int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s + C_\sigma \int_0^t \left(\|\chi\|_{W^{1,p}(\Omega)}^2 \|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)}^2 + \|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)}^2 \right) \, \mathrm{d}s \, .$$

Here, $d^\star$ is from (2.11) and we have exploited inequality (2.12) with a constant σ that we will choose later, and some $C_\sigma > 0$. Moreover, we have used that $\|b(\chi)\|_{L^{d+\zeta}(\Omega)} \leq C\|\chi\|_{W^{1,p}(\Omega)}$. Furthermore, relying on the elliptic regularity result in (2.9) and on (2.16), we obtain

$$- \int_0^t \int_\Omega \mathcal{V}\left(a(\chi)\mathbf{u}_t\right) \cdot \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s = \int_0^t \int_\Omega -\nabla a(\chi) \mathbb{V}\varepsilon(\mathbf{u}_t) \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s \tag{3.19}$$

$$- \int_0^t \int_\Omega a(\chi) \mathrm{div}(\mathbb{V}(\varepsilon(\mathbf{u}_t))) \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s$$

$$\geq C \int_0^t \int_\Omega |\mathrm{div}(\varepsilon(\mathbf{u}_t))|^2 \, \mathrm{d}x \, \mathrm{d}s + I_1 \geq c \int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s + I_1,$$

where we get

$$|I_1| = \left| \int_0^t \int_\Omega \nabla a(\chi) \mathbb{V}\varepsilon(\mathbf{u}_t) \mathrm{div}(\varepsilon(\mathbf{u}_t)) \right| \, \mathrm{d}x \, \mathrm{d}s$$

$$\leq C \int_0^t \|\nabla a(\chi)\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)} \|\varepsilon(\mathbf{u}_t)\|_{L^{d^\star - \zeta}(\Omega;\mathbb{R}^{d \times d})} \|\mathrm{div}(\varepsilon(\mathbf{u}_t))\|_{L^2(\Omega;\mathbb{R}^d)} \, \mathrm{d}s$$

$$\leq \delta \int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s + C_\delta \int_0^t \|\nabla a(\chi)\|_{L^{d+\zeta}(\Omega;\mathbb{R}^d)}^2 \|\varepsilon(\mathbf{u}_t)\|_{L^{d^\star - \zeta}(\Omega;\mathbb{R}^{d \times d})}^2 \, \mathrm{d}s$$

$$\leq \delta \int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s + C_\delta \varrho^2 \int_0^t \|\chi\|_{W^{1,p}(\Omega)}^2 \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s + C_\delta C_\varrho \int_0^t \|\chi\|_{W^{1,p}(\Omega)}^2 \|\mathbf{u}_t\|_{L^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s,$$

again exploiting (2.12), for some positive constants δ and ϱ that we will choose later and for some C_δ, $C_\varrho > 0$. Moreover, we also have that

$$\left| \rho \int_0^t \int_\Omega \nabla\vartheta \cdot \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x \, \mathrm{d}s \right| \leq \eta \int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s + C_\eta \int_0^t \|\nabla\vartheta\|_{L^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s \tag{3.20}$$

holds true for some positive constant η to be fixed later and for some $C_\eta > 0$. Collecting (3.19)–(3.20), (3.5) and (3.14), and also using that $\int_\Omega \mathbf{u}_{tt} \mathrm{div}(\varepsilon(\mathbf{u}_t)) \, \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\varepsilon(\mathbf{u}_t)|^2 \, \mathrm{d}x \right)$, we conclude that

$$\frac{1}{2}\int_\Omega |\varepsilon(\mathbf{u}_t(t))|^2 \, \mathrm{d}x + c \int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s \leq \frac{1}{2}\int_\Omega |\varepsilon(\mathbf{v}_0)|^2 \, \mathrm{d}x + C\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}^2 + \frac{c}{2}\int_0^t \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s$$

$$+ C\left(1 + \|\mathbf{u}_0\|_{H^2(\Omega;\mathbb{R}^d)}^2 + \int_0^t \int_0^s \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}r \, \mathrm{d}s \right),$$

where we have used the fact that $\int_0^t \|\mathbf{u}\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}s \leq \|\mathbf{u}_0\|_{H^2(\Omega;\mathbb{R}^d)}^2 + \int_0^t \int_0^s \|\mathbf{u}_t\|_{H^2(\Omega;\mathbb{R}^d)}^2 \, \mathrm{d}r \, \mathrm{d}s$ and chosen σ, δ, ϱ and η sufficiently small. Taking into account condition (2.21) on $\mathbf{f}$, the assumptions on the initial data

(2.25), and using a standard Gronwall lemma, we conclude

$$\|\mathbf{u}_t\|_{L^2(0,t;H_0^2(\Omega;\mathbb{R}^d))\cap L^\infty(0,t;H_0^1(\Omega;\mathbb{R}^d))} \leq C. \tag{3.21}$$

By comparison in (1.2), taking into account the regularity property (2.8b), we also get

$$\|\mathbf{u}_{tt}\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^d))} \leq C. \tag{3.22}$$

Sixth estimate $[\mu \in \{0,1\}]$. We multiply (1.1) by $\frac{w}{\vartheta}$, with w a test function in $W^{1,d}(\Omega)\cap L^\infty(\Omega)$ (in particular, this is true for $w \in W^{1,d+\epsilon}(\Omega)$ with $\epsilon > 0$). We integrate in space, only. We thus obtain (cf. (2.27)) that

$$\left|\int_\Omega \partial_t \log(\vartheta)w\,\mathrm{d}x\right| \leq \left|\int_\Omega Hw\,\mathrm{d}x\right| + \left|\int_\Omega \frac{\mathsf{K}(\vartheta)}{\vartheta}\nabla\vartheta\cdot\nabla w\,\mathrm{d}x\right| + \left|\int_\Omega \frac{\mathsf{K}(\vartheta)}{\vartheta^2}|\nabla\vartheta|^2 w\,\mathrm{d}x\right| + \left|\int_\Omega Jw\,\mathrm{d}x\right| + \left|\int_{\partial\Omega} hw\,\mathrm{d}S\right|$$
$$\doteq I_1 + I_2 + I_3 + I_4 + I_5$$

where we have used the place-holders $H := -\chi_t - \rho\mathrm{div}(\mathbf{u}_t)$ and $J := \frac{1}{\vartheta}(g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2)$. Estimate (3.18) yields that $\|H\|_{L^2(0,T;L^2(\Omega))} \leq C$, therefore $|I_1| \leq \mathcal{H}(t)\|w\|_{L^2(\Omega)}$ with $\mathcal{H}(t) = \|H(\cdot,t)\|_{L^2(\Omega)} \in L^2(0,T)$. Analogously, also in view of (2.22) and of (3.2) we have that

$$|I_4| \leq \frac{1}{\vartheta_*}\mathcal{J}(t)\|w\|_{L^\infty(\Omega)} \qquad \text{with } \mathcal{J}(t) := \|J(\cdot,t)\|_{L^1(\Omega)} \in L^1(0,T). \tag{3.23}$$

Moreover, $|I_5| \leq \|h(t)\|_{L^2(\partial\Omega)}\|w\|_{L^2(\partial\Omega)}$, with $\|h(t)\|_{L^2(\partial\Omega)} \in L^1(0,T)$ thanks to (2.23). Using the growth condition (2.15) for K, we estimate

$$|I_2| \leq C\int_\Omega \vartheta^{\kappa-1}|\nabla\vartheta||\nabla w|\,\mathrm{d}x + C\int_\Omega \frac{1}{\vartheta}|\nabla\vartheta||\nabla w|\,\mathrm{d}x \doteq I_{2,1} + I_{2,2}. \tag{3.24}$$

Thanks to the previously proved positivity (3.2), we have

$$I_{2,2} \leq \frac{C}{\vartheta^*}\mathcal{O}(t)\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)} \qquad \text{with } \mathcal{O}(t) := \|\nabla\vartheta(t)\|_{L^2(\Omega;\mathbb{R}^d)} \in L^2(0,T)$$

by (3.14). We estimate $I_{2,1}$ via the Hölder inequality, taking into account (3.13) and (3.16), whence, for $d \in \{2,3\}$,

$$I_{2,1} \leq C\|\vartheta^{(\kappa+\alpha-2)/2}\nabla\vartheta\|_{L^2(\Omega;\mathbb{R}^d)}\|\vartheta^{(\kappa-\alpha)/2}\|_{L^6(\Omega)}\|\nabla w\|_{L^3(\Omega;\mathbb{R}^d)} \doteq C\mathcal{O}^*(t)\|\nabla w\|_{L^3(\Omega;\mathbb{R}^d)}$$
$$\text{with } \mathcal{O}^*(t) := \|\vartheta(t)^{(\kappa+\alpha-2)/2}\nabla\vartheta(t)\|_{L^2(\Omega;\mathbb{R}^d)}\|\vartheta(t)^{(\kappa-\alpha)/2}\|_{L^6(\Omega)} \in L^1(0,T).$$

Finally, we have

$$|I_3| \leq C\int_\Omega \vartheta^{\kappa-2}|\nabla\vartheta|^2|w|\,\mathrm{d}x + C\int_\Omega \frac{1}{\vartheta^2}|\nabla\vartheta|^2|w|\,\mathrm{d}x \doteq I_{3,1} + I_{3,2}. \tag{3.25}$$

The positivity property (3.2) again guarantees

$$I_{3,2} \leq \frac{C}{\vartheta_*^2}\mathcal{O}(t)^2\|w\|_{L^\infty(0,T)} \qquad \text{with } \mathcal{O}(t)^2 \in L^1(0,T)$$

while, using that $\vartheta^{\kappa-2} \leq c\vartheta^{\kappa+\alpha-2} + c'$, we infer

$$I_{3,2} \leq \|w\|_{L^\infty(\Omega)}\left(c\int_\Omega \vartheta^{\kappa+\alpha-2}|\nabla\vartheta|^2\,\mathrm{d}x + c'\int_\Omega |\nabla\vartheta|^2\,\mathrm{d}x\right) \doteq \|w\|_{L^\infty(\Omega)}\mathcal{O}_*(t)$$
$$\text{with } \mathcal{O}_*(t) = c\int_\Omega \vartheta(t)^{\kappa+\alpha-2}|\nabla\vartheta(t)|^2\,\mathrm{d}x + c'\int_\Omega |\nabla\vartheta(t)|^2\,\mathrm{d}x \in L^1(0,T), \tag{3.26}$$

thanks to (3.13) and (3.14).

Collecting all of the above calculations, we conclude that

$$\|\partial_t \log(\vartheta)\|_{L^1(0,T;(W^{1,d}(\Omega)\cap L^\infty(\Omega))^*)} \leq C. \tag{3.27}$$

Seventh estimate [$\mu \in \{0,1\}$], $\kappa \in (1, 5/3)$ **if** $d = 3$ **and** $\kappa \in (1, 2)$ **if** $d = 2$. Assume in addition **Hypothesis (V)**. We multiply (1.1) by a test function $w \in W^{1,\infty}(\Omega)$ (which e.g. holds if $w \in W^{2,d+\epsilon}(\Omega)$ for $\epsilon > 0$). By comparison we have

$$\left| \int_\Omega \vartheta_t w \, \mathrm{d}x \right| \leq \left| \int_\Omega Lw \, \mathrm{d}x \right| + \left| \int_\Omega \mathsf{K}(\vartheta) \nabla \vartheta \cdot \nabla w \, \mathrm{d}x \right| + \left| \int_{\partial\Omega} hw \, \mathrm{d}S \right| \doteq I_1 + I_2 + I_3,$$

where we have set $L = -\chi_t \vartheta - \rho \vartheta \mathrm{div}(\mathbf{u}_t) + g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2$. Therefore,

$$|I_1| \leq \mathcal{L}(t)\|w\|_{L^\infty(\Omega)} \quad \text{with } \mathcal{L}(t) := \|L(t)\|_{L^1(\Omega)} \in L^1(0,T), \quad |I_3| \leq \|h(t)\|_{L^2(\partial\Omega)}\|w\|_{L^2(\partial\Omega)} \text{ with } h \in L^1(0,T)$$

thanks to (3.14), (3.18) and (2.23), respectively. As for I_2, in view of (2.15), taking into account (3.13) and using the Hölder inequality, we obtain

$$|I_2| \leq C\|\vartheta^{(\kappa-\alpha+2)/2}\|_{L^2(\Omega)}\|\vartheta^{(\kappa+\alpha-2)/2}\nabla\vartheta\|_{L^2(\Omega;\mathbb{R}^d)}\|\nabla w\|_{L^\infty(\Omega;\mathbb{R}^d)} + C\|\nabla\vartheta\|_{L^2(\Omega;\mathbb{R}^d)}\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)}. \tag{3.28}$$

Observe that, since α can be chosen arbitrarily close to 1, in view of estimate (3.15) we have that $\vartheta^{(\kappa-\alpha+2)/2}$ is bounded in $L^2(0,T;L^2(\Omega))$ if and only if $\kappa < \frac{5}{3}$ if $d = 3$, and $\kappa < 2$ if $d = 2$. Under this restriction on κ, we have that $|I_2| \leq C\mathcal{L}^*(t)\|\nabla w\|_{L^\infty(\Omega)}$ for some $\mathcal{L}^* \in L^1(0,T)$. Ultimately, we conclude that

$$\|\vartheta_t\|_{L^1(0,T;W^{1,\infty}(\Omega)^*)} \leq C. \tag{3.29}$$

Eighth estimate [$\mu = 0$]. In view of the previously obtained estimates (3.5), (3.14), (3.18), and (3.21), a comparison in equation (1.3) yields that (recall that ξ is a selection in $\beta(\chi)$ a.e. in $\Omega \times (0,T)$),

$$\|A_p(\chi) + \xi\|_{L^2(0,T;L^2(\Omega))} \leq C$$

whence, by standard elliptic regularity,

$$\|A_p(\chi)\|_{L^2(0,T;L^2(\Omega))} + \|\xi\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{3.30}$$

In view of the regularity results [31, Thm. 2, Rmk. 2.5], we finally infer the enhanced regularity (2.53) for χ. ∎

Remark 3.1 (**The p-Laplacian regularization**). A close perusal at the above calculations shows that the fact that $p > d$ for the p-Laplacian term in the χ-equation (1.3) has been used only for carrying out the calculations in the *Fifth estimate*. All the other estimates do not depend on the condition $p > d$, and would therefore hold if the operator A_p in (1.3) were replaced by the Laplacian.

In turn, the *Fifth estimate* for $\mathbf{u}$ will play a crucial role in the limit passage arguments at the basis of the proofs of Theorems 2.5 and 2.8: it will ensure strong compactness in $H^1(0,T;H_0^2(\Omega;\mathbb{R}^d))$ (cf. Lemma 5.1) for the sequences of approximate solutions constructed in Sec. 4. Relying on this, we will be able to pass to the limit with the quadratic term $|\varepsilon(\mathbf{u}_t)|^2$ on the right-hand side of (1.1).

Nonetheless, in Sec. 6 we will show that, in the case $\mu = 1$ of unidirectional evolution, it is ultimately possible to drop the constraint $p > d$ and in fact we will obtain an existence result for the entropic formulation of system (1.1)–(1.3), in the case (1.3) simply features the Laplacian (i.e. for $p = 2$).

4. Time discretization

In Section 4.1 we set up a *single* time discretization scheme for both the irreversible ($\mu = 1$) and for the reversible ($\mu = 0$) systems. We then show in Section 4.2 that the piecewise constant and piecewise linear interpolants of the discrete solutions satisfy a total energy inequality, and the approximate versions of the entropy inequality and of equations (1.2)–(1.3). Finally, in Section 4.3 we rigorously prove the a priori estimates from Section 3 in the time-discrete context.

Notation 4.1. In what follows, also in view of the extension (2.50) mentioned at the end of Sec. 2.3, we will use $\widehat{\alpha}$ and α as place-holders for $I_{(-\infty,0]}$ and $\partial I_{(-\infty,0]}$.

4.1. **Setup of the time discretization.** We consider an equidistant partition of $[0, T]$, with time-step $\tau > 0$ and nodes $t_\tau^k := k\tau$, $k = 0, \ldots, K_\tau$. In this framework, we approximate the data $\mathbf{f}$, g, and h by local means, i.e. setting for all $k = 1, \ldots, K_\tau$

$$\mathbf{f}_\tau^k := \frac{1}{\tau} \int_{t_\tau^{k-1}}^{t_\tau^k} \mathbf{f}(s) \, ds, \qquad g_\tau^k := \frac{1}{\tau} \int_{t_\tau^{k-1}}^{t_\tau^k} g(s) \, ds, \qquad h_\tau^k := \frac{1}{\tau} \int_{t_\tau^{k-1}}^{t_\tau^k} h(s) \, ds. \tag{4.1}$$

Consider the following initial data

$$\vartheta_\tau^0 := \vartheta_0, \qquad \mathbf{u}_\tau^0 := \mathbf{u}_0, \qquad \mathbf{u}_\tau^{-1} := \mathbf{u}_0 - \tau \mathbf{v}_0, \qquad \chi_\tau^0 := \chi_0. \tag{4.2}$$

We construct discrete solutions to system (1.1)–(1.3) by solving the following elliptic system, featuring the operator $\mathcal{A}^k : X \to H^1(\Omega)^*$, with

$$X = \{\theta \in H^1(\Omega) : \int_\Omega \mathsf{K}(\theta) \nabla \theta \cdot \nabla v \, dx \text{ is well defined for all } v \in H^1(\Omega)\}, \quad \mathcal{A}^k : X \to H^1(\Omega)^* \text{ defined by}$$

$$\langle \mathcal{A}^k(\theta), v \rangle_{H^1(\Omega)} := \int_\Omega \mathsf{K}(\theta) \nabla \theta \cdot \nabla v \, dx - \int_{\partial\Omega} h_\tau^k v \, dS. \tag{4.3}$$

Problem 4.2 (Time discretization of the reversible system, $\mu \in \{0, 1\}$)**.** Starting from $(\mathbf{u}_\tau^0, \mathbf{u}_\tau^{-1}, \chi_\tau^0, \vartheta_\tau^0)$ as in (4.2), find $\{\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k\}_{k=1}^{K_\tau} \subset X \times H_0^2(\Omega; \mathbb{R}^d) \times W^{1,p}(\Omega)$ fulfilling

$$\frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \vartheta_\tau^k + \rho \operatorname{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \vartheta_\tau^k + \mathcal{A}^k(\vartheta_\tau^k) = g_\tau^k \tag{4.4}$$

$$+ a(\chi_\tau^{k-1}) \varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 + \frac{\tau^{1/2}}{2}\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 \quad \text{in } H^1(\Omega)^*,$$

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left(a(\chi_\tau^{k-1})\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) + \mathcal{C}_\rho(\vartheta_\tau^k) = \mathbf{f}_\tau^k \quad \text{a.e. in } \Omega, \tag{4.5}$$

$$\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \sqrt{\tau}\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mu\zeta_\tau^k + A_p(\chi_\tau^k) + \xi_\tau^k + \gamma(\chi_\tau^k) \ni -b'(\chi_\tau^k)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \vartheta_\tau^k \quad \text{a.e. in } \Omega, \tag{4.6}$$

where $\mathbb{I} \in \mathbb{R}^{d \times d \times d \times d}$ denotes the identity tensor and

$$\xi_\tau^k \in \beta(\chi_\tau^k) \qquad\qquad\qquad \text{a.e. in } \Omega, \tag{4.7}$$

$$\zeta_\tau^k \in \alpha\left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right) \qquad\qquad\qquad \text{a.e. in } \Omega. \tag{4.8}$$

Remark 4.3 (Features of the time-discretization scheme)**.** A few observations on Problem 4.2 are in order.

First of all, let us point out that the scheme is fully implicit and, in particular, (4.6) is coupled to the system (4.4)–(4.5) by the implicit term ϑ_τ^k on the right-hand side, in view of proving the strict positivity (4.10) below for the discrete temperature ϑ_τ^k. As we will see, our argument for (4.10) is the discrete version of the comparison argument developed at the beginning of Section 3 and strongly relies on the structure of the discrete temperature equation (4.4). However, in the case of unidirectional evolution, we could have decoupled the discrete equation for χ from (4.4)–(4.5), replacing (4.6) by

$$\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mu\zeta_\tau^k + A_p(\chi_\tau^k) + \xi_\tau^k + \gamma(\chi_\tau^k) \ni -b'(\chi_\tau^k)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \vartheta_\tau^{k-1} \quad \text{a.e. in } \Omega, \tag{4.9}$$

and, accordingly, replacing the coupling term $\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\vartheta_\tau^k$ on the left-hand side of (4.4) by $\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\vartheta_\tau^{k-1}$. In Remark 4.5 below, we will show how it is still possible to prove the strict positivity of the discrete temperature for this partially decoupled scheme.

Second, observe that $\frac{\tau^{1/2}}{2}\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2$ appears on the right-hand side of (4.4) and, accordingly, $\sqrt{\tau}\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}$ features on the left-hand side of (4.6). These terms have been added for technical reasons, related to the proof

of the discrete version of the total energy inequality (2.38), cf. the text above Proposition 4.8. Clearly, they will disappear when passing to the limit with $\tau \downarrow 0$.

Because of the implicit character of system (4.4)–(4.6), for the existence proof (cf. Lemma 4.4 below) we shall have to resort to a fixed-point type result from the theory for elliptic systems featuring pseudo-monotone operators, drawn from [29, Chap. II]. Indeed, we will not apply it directly to system (4.4)–(4.6), but to an approximation of (4.4)–(4.6), i.e. system (4.15)–(4.17) below, obtained in the following way. We will need to

(1) truncate K, along the lines of [15], in such a way as to have a bounded function in the elliptic operator in the temperature equation (4.4). Therefore, the truncated operator K_M, with M a positive parameter, shall be defined on $H^1(\Omega)$ (in place of X), with values in $H^1(\Omega)^*$ (in place of X^*). Accordingly, we shall truncate all occurrences of ϑ in a quadratic term;

(2) following [30], add the higher order terms $-\nu\mathrm{div}(|\varepsilon(\mathbf{u}_\tau^k)|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}_\tau^k))$ and $\nu|\chi_\tau^k|^{\eta-2}\chi_\tau^k$, with $\nu > 0$ and $\eta > 4$, on the left-hand sides of (4.5) and (4.6), respectively. Their role is to compensate the quadratic terms on the right-hand side of (4.4). As a result, both for $d = 2$ and for $d = 3$ the pseudo-monotone operator by means of which we will rephrase system (4.15)–(4.17) will turn out to be coercive, in its ϑ-component, with respect to the $H^1(\Omega)$-norm;

(3) in the case $\mu = 1$, in order to cope with the (possible) unboundedness of the operator α we will have to replace it with its Yosida-regularization α_ν (cf. [3]), with ν the same parameter as above.

Then, in the proof of Lemma 4.4 we will

(1) prove the existence of solutions to the approximate discrete system (4.15)–(4.17);

(2) pass to the limit in (4.15)–(4.17) first as the truncation parameter $M \to \infty$ and conclude an existence result for an approximation of system (4.4)–(4.6), still depending on the parameter $\nu > 0$;

(3) pass to the limit in this approximate system as $\nu \to 0$ and conclude the existence of solutions to (4.4)–(4.6).

We postpone to Remark 4.6 some comments on the reason why we need to keep the two limit passages as $M \to \infty$ and $\nu \to 0$ distinct.

Our existence result for Problem 4.2 reads

Lemma 4.4 (Existence for the time-discrete Problem 4.2, $\mu \in \{0,1\}$). *Assume **Hypotheses (I)–(III)**, and assumptions (2.21)–(2.26) on the data $\mathbf{f}$, g, h, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0. Then, there exists $\bar{\tau} > 0$ such that for all $0 < \tau \leq \bar{\tau}$ Problem 4.2, admits at least one solution $\{(\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)\}_{k=1}^{K_\tau}$.*

Furthermore, any solution $\{(\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)\}_{k=1}^{K_\tau}$ of Problem 4.2 fulfills

$$\vartheta_\tau^k(x) \geq \underline{\vartheta} > 0 \quad \text{for a.a. } x \in \Omega \tag{4.10}$$

for some $\underline{\vartheta} = \underline{\vartheta}(T)$.

Proof. We split the proof in some steps.

Step 1: approximation. As already mentioned, we construct our approximation of system (4.4)–(4.6) by truncating K in (4.4) and the quadratic terms in ϑ, replacing α with its Yosida approximation α_ν, and adding higher order terms to (4.5) and (4.6). Namely, let

$$\mathsf{K}_M(r) := \begin{cases} \mathsf{K}(-M) & \text{if } r < -M, \\ \mathsf{K}(r) & \text{if } |r| \leq M, \\ \mathsf{K}(M) & \text{if } r > M \end{cases} \tag{4.11}$$

and accordingly introduce the operator

$$\mathcal{A}_M^k : H^1(\Omega) \to H^1(\Omega)^* \text{ defined by } \langle \mathcal{A}_M^k(\theta), v \rangle_{H^1(\Omega)} := \int_\Omega \mathsf{K}_M(\theta)\nabla\theta \cdot \nabla v \, dx - \int_{\partial\Omega} h_\tau^k v \, dS. \tag{4.12}$$

Observe that, thanks to (2.15) there still holds $\mathsf{K}_M(r) \geq c_0$ for all $r \in \mathbb{R}$, and therefore

$$\langle \mathcal{A}_M^k(\theta), \theta \rangle_{H^1(\Omega)} \geq c_0 \int_\Omega |\nabla\theta|^2 \, dx \qquad \text{for all } \theta \in H^1(\Omega). \tag{4.13}$$

We also introduce the truncation operator $\mathcal{T}_M : \mathbb{R} \to \mathbb{R}$

$$\mathcal{T}_M(r) := \begin{cases} -M & \text{if } r < -M, \\ r & \text{if } |r| \le M, \\ M & \text{if } r > M. \end{cases} \tag{4.14}$$

Furthermore, for a given $\nu > 0$ we denote by α_ν the Yosida approximation of α with parameter ν.

Then, we consider following approximation of system (4.4)–(4.6):

$$\frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \mathcal{T}_M(\vartheta_\tau^k) + \rho \operatorname{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathcal{T}_M(\vartheta_\tau^k) + \mathcal{A}_M^k(\vartheta_\tau^k) = g_\tau^k \tag{4.15}$$

$$+ a(\chi_\tau^{k-1})\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 + \frac{\tau^{1/2}}{2}\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 \quad \text{in } H^1(\Omega)^*,$$

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left(a(\chi_\tau^{k-1})\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) + \mathcal{C}_\rho(\mathcal{T}_M(\vartheta_\tau^k)) - \nu\operatorname{div}(|\varepsilon(\mathbf{u}_\tau^k)|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}_\tau^k)) = \mathbf{f}_\tau^k$$

$$\text{in } W^{1,\eta}(\Omega;\mathbb{R}^d)^*, \tag{4.16}$$

$$\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \sqrt{\tau}\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mu\alpha_\nu\left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right) + A_p(\chi_\tau^k) + \xi_\tau^k + \gamma(\chi_\tau^k) + \nu|\chi_\tau^k|^{\eta-2}\chi_\tau^k$$

$$= -b'(\chi_\tau^k)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \mathcal{T}_M(\vartheta_\tau^k) \qquad \text{a.e. in } \Omega, \tag{4.17}$$

with $\xi_\tau^k \in \beta(\chi_\tau^k)$ a.e. in Ω.

Step 2: existence of solutions for the approximate system. Observe that system (4.15)–(4.17) can be recast as

$$\vartheta_\tau^k + \left(\chi_\tau^k - \chi_\tau^{k-1}\right)\mathcal{T}_M(\vartheta_\tau^k) + \rho\operatorname{div}\left(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}\right)\mathcal{T}_M(\vartheta_\tau^k) + \tau\mathcal{A}_M^k(\vartheta_\tau^k)$$

$$- \tau a(\chi_\tau^{k-1})\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) - \tau\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 - \frac{\tau^{3/2}}{2}\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 \tag{4.18}$$

$$= \vartheta_\tau^{k-1} + \tau g_\tau^k \quad \text{in } H^1(\Omega)^*,$$

$$\mathbf{u}_\tau^k + \tau\mathcal{V}\left(a(\chi_\tau^{k-1})(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1})\right) + \tau^2\mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) + \tau^2\mathcal{C}_\rho(\mathcal{T}_M(\vartheta_\tau^k)) - \nu\tau^2\operatorname{div}(|\varepsilon(\mathbf{u}_\tau^k)|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}_\tau^k))$$

$$= 2\mathbf{u}_\tau^{k-1} - \mathbf{u}_\tau^{k-2} + \tau^2\mathbf{f}_\tau^k \quad \text{in } W^{1,\eta}(\Omega;\mathbb{R}^d)^*, \tag{4.19}$$

$$\chi_\tau^k + \sqrt{\tau}\chi_\tau^k + \mu\tau\alpha_\nu\left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right) + \tau A_p(\chi_\tau^k) + \tau\xi_\tau^k + \tau\gamma(\chi_\tau^k) + \nu\tau|\chi_\tau^k|^{\eta-2}\chi_\tau^k - \tau\mathcal{T}_M(\vartheta_\tau^k)$$

$$= \chi_\tau^{k-1} + \sqrt{\tau}\chi_\tau^{k-1} - \tau b'(\chi_\tau^k)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} \quad \text{a.e. in } \Omega. \tag{4.20}$$

Denoting by $\mathcal{R}_{k-1}$ the operator acting on the unknown $(\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)$ and by H_{k-1} the vector of the terms on the r.h.s. of the above equations, we can reformulate system (4.18)–(4.20) in the abstract form

$$\mathcal{R}_{k-1}(\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k) = H_{k-1}. \tag{4.21}$$

It can be checked that $\mathcal{R}_{k-1}$ is a pseudo-monotone operator (according to [29, Chap. II, Def. 2.1]) on $H^1(\Omega) \times W_0^{1,\eta}(\Omega;\mathbb{R}^d) \times H^1(\Omega)$. In order to check that $\mathcal{R}_{k-1}$ is coercive on that space, it is sufficient to test (4.18) by ϑ_τ^k, (4.19) by $\mathbf{u}_\tau^k$, (4.20) by χ_τ^k and add the resulting equations. To obtain a bound for $\|\vartheta_\tau^k\|_{H^1(\Omega)}$ we use that $\mathcal{A}_M^k$ is coercive (cf. (4.13)). The additional terms $-\nu\operatorname{div}(|\varepsilon(\mathbf{u}_\tau^k)|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}_\tau^k))$ and $\nu|\chi_\tau^k|^{\eta-2}\chi_\tau^k$ in (4.19) and (4.20) enable us to control the quadratic terms on the right-hand side of (4.18). More in detail, the test of

(4.18) by ϑ_τ^k gives rise, e.g., to the term $I_1 := \int_\Omega a(\chi_\tau^{k-1})\varepsilon(\mathbf{u}_\tau^k)\mathbb{V}\varepsilon(\mathbf{u}_\tau^k)\vartheta_\tau^k\,\mathrm{d}x$, which can be estimated as follows

$$
\begin{aligned}
|I_1| &\leq C\|a(\chi_\tau^{k-1})\|_{L^\infty(\Omega)}\|\varepsilon(\mathbf{u}_\tau^k)\|_{L^4(\Omega;\mathbb{R}^{d\times d})}^2\|\vartheta_\tau^k\|_{L^2(\Omega)} \\
&\leq \tfrac{1}{4}\|\vartheta_\tau^k\|_{L^2(\Omega)}^2 + C\|\varepsilon(\mathbf{u}_\tau^k)\|_{L^4(\Omega;\mathbb{R}^{d\times d})}^4 \\
&\leq \tfrac{1}{4}\|\vartheta_\tau^k\|_{L^2(\Omega)}^2 + \tfrac{\nu\tau^2}{4}\|\varepsilon(\mathbf{u}_\tau^k)\|_{L^\eta(\Omega;\mathbb{R}^{d\times d})}^\eta + C,
\end{aligned}
$$

where the first estimate follows from the Hölder inequality, the second one from the fact that $\|a(\chi_\tau^{k-1})\|_{L^\infty(\Omega)} \leq C$ since $\chi_\tau^{k-1} \in W^{1,p}(\Omega)$ and $a \in \mathrm{C}^0(\mathbb{R})$, and the last one relies on $\eta > 4$. Therefore, the right-hand side terms can be absorbed by the left-hand side ones, also resulting from the test of (4.19) by $\mathbf{u}_\tau^k$. With analogous calculations we estimate $I_2 := \int_\Omega(|\chi_\tau^k|^2 + \tfrac{\tau^{1/2}}{2}|\chi_\tau^k|^2)\vartheta_\tau^k\,\mathrm{d}x$, exploiting the term $\nu\tau|\chi_\tau^k|^{\eta-2}\chi_\tau^k$ on the left-hand side of (4.20).

Therefore, the Leray-Lions type existence result of [29, Chap. II, Thm. 2.6] applies, yielding the existence of a solution $(\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)$ (whose dependence on the parameters M and ν is not highlighted, for simplicity) to (4.15)–(4.17).

Step 3: proof of the strict positivity (4.10)**.** Observe first that, for ϑ_τ^k solving (4.15)–(4.17) the strict positivity (4.10) holds for $k = 0$ with $\underline{\vartheta} := \vartheta_*$ due to (2.24). In order to prove that $\vartheta_\tau^k \geq \underline{\vartheta} > 0$ a.e. in Ω, for every $k \geq 1$, we proceed in the same spirit of the proof of the strict positivity of ϑ in Sec. 3 (cf. also [19, Sec. 5.2]). Namely, we start by deducing from (4.4) that

$$
\int_\Omega \frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\tau}w\,\mathrm{d}x + \int_\Omega \mathsf{K}_M(\vartheta_\tau^k)\nabla\vartheta_\tau^k\nabla w\,\mathrm{d}x \geq -C\int_\Omega(\vartheta_\tau^k)^2 w\,\mathrm{d}x \quad \text{for every } w \in W_+^{1,2}(\Omega), \tag{4.22}
$$

where C is independent of k. We now consider the decreasing sequence $\{v_k\} \subseteq \mathbb{R}$ defined recursively as

$$
\frac{v_k - v_{k-1}}{\tau} = -Cv_k^2, \quad v_0 = \vartheta_* > 0, \tag{4.23}
$$

where C is the same constant of (4.22). We write now (4.23), adding the term $-\operatorname{div}(\mathsf{K}_M(\vartheta_\tau^k)\nabla v_k) = 0$, in the form

$$
\frac{1}{\tau}\int_\Omega(v_k - v_{k-1})w\,\mathrm{d}x + \int_\Omega \mathsf{K}_M(\vartheta_\tau^k)\nabla v_k \cdot \nabla w\,\mathrm{d}x = -C\int_\Omega v_k^2 w\,\mathrm{d}x \quad \text{for every } ww \in W_+^{1,2}(\Omega).
$$

Subtracting (4.22) from (4.23) and testing the difference by $w = H_\varepsilon(v_k - \vartheta_k)$, where

$$
H_\varepsilon(v) = \begin{cases} 0 & \text{if } v \leq 0 \\ v/\varepsilon & \text{if } v \in (0,\varepsilon) \\ 1 & \text{if } v \geq \varepsilon \end{cases}
$$

we obtain, since $v_k < v_{k-1}$ that

$$
\int_\Omega \left((v_k - v_{k-1}) - (\vartheta_\tau^k - \vartheta_\tau^{k-1})\right)H_\varepsilon(v_k - \vartheta_\tau^k)\,\mathrm{d}x \leq 0. \tag{4.24}
$$

Assume now that $\vartheta_\tau^{k-1} \geq v_{k-1}$ a.e. in Ω (which is true for $k = 1$). Taking $\varepsilon \searrow 0$, (4.24) yields $\vartheta_\tau^k \geq v_k$ a.e. in Ω, and, by induction, $\vartheta_\tau^k \geq v_k > v_{K_\tau}$ a.e. in Ω for every $k = 1, \ldots, K_\tau$. We now prove that there exists $\underline{\vartheta} > 0$ such that $v_{K_\tau} \geq \underline{\vartheta}$ a.e. in Ω. To this aim, observe that v_{K_τ} rewrites as $v_{K_\tau} = G^{-1}(G(v_{K_\tau}))$, where $G(z) := -\int_z^{v_0}\frac{1}{s^2}\,\mathrm{d}s$ is monotonlcally increasing on $(0, v_0]$, $G(0+) = -\infty$, $G(v_0) = 0$, hence, by the mean value theorem, for every $k = 1, \ldots, K_\tau$ there exists $s_k \in [v_k, v_{k-1}]$ such that

$$
\frac{G(v_k) - G(v_{k-1})}{v_k - v_{k-1}} = G'(s_k) = \frac{1}{s_k^2} \leq \frac{1}{v_k^2},
$$

from which we deduce, using (4.23),

$$
\frac{G(v_k) - G(v_{k-1})}{-C\tau v_k^2} \leq \frac{1}{v_k^2} \implies G(v_{K_\tau}) \geq -C\tau K_\tau.
$$

Hence, we get

$$
\vartheta_\tau^k > v_{K_\tau} = G^{-1}(G(v_{K_\tau})) \geq G^{-1}(-C\tau K_\tau) = G^{-1}(-CT) =: \underline{\vartheta}(T). \tag{4.25}
$$

Thus, we conclude (4.10) with $\underline{\vartheta} = G^{-1}(-CT)$.

Step 4: passage to the limit as $M \to \infty$. We now pass to the limit in (4.15)–(4.17) as $M \to \infty$, for $\nu > 0$ fixed. In this framework, we will denote by $(\vartheta_M, \mathbf{u}_M, \chi_M)$ the solutions of (4.15)–(4.17), with $(\vartheta_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \chi_\tau^{k-1})$ given and $\nu > 0$ fixed. First of all, we derive a bunch of estimates for $(\vartheta_M, \mathbf{u}_M, \chi_M)_M$, holding for constants independent of $M > 0$ (but possibly depending on $\tau > 0$, as well as on norms of $(\vartheta_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \chi_\tau^{k-1})$).

We test (4.15) by 1, (4.16) by $\mathbf{u}_M - \mathbf{u}_\tau^{k-1}$, (4.17) by $\chi_M - \chi_\tau^{k-1}$, and add the resulting relations. Taking into account all cancellations, conditions (2.21)–(2.26), as well as the fact that the Yosida approximation $\widehat{\alpha}_\nu$ of $\widehat{\alpha} = I_{(\infty,0]}$ is a positive function, we obtain that

$$\exists C > 0 \ \forall M > 0 : \quad \|\vartheta_M\|_{L^1(\Omega)} + \|\mathbf{u}_M\|_{H^1(\Omega;\mathbb{R}^d)} + \nu^{1/\eta}\|\varepsilon(\mathbf{u}_M)\|_{L^\eta(\Omega;\mathbb{R}^{d\times d})} + \|\chi_M\|_{W^{1,p}(\Omega)} \leq C. \tag{4.26}$$

We now test (4.15) by $\mathcal{T}_M(\vartheta_M)$. Observing that

$$\left.\begin{array}{l} \mathsf{K}_M(\vartheta_M)\nabla\vartheta_M\nabla(\mathcal{T}_M(\vartheta_M)) = \mathsf{K}(\mathcal{T}_M(\vartheta_M))|\nabla(\mathcal{T}_M(\vartheta_M)))|^2 \\[4pt] \vartheta_M\mathcal{T}_M(\vartheta_M) \geq |\mathcal{T}_M(\vartheta_M)|^2 \end{array}\right\} \quad \text{a.e. in } \Omega,$$

we get

$$\frac{1}{\tau}\int_\Omega |\mathcal{T}_M(\vartheta_M)|^2\,\mathrm{d}x + \int_\Omega \mathsf{K}(\mathcal{T}_M(\vartheta_M))|\nabla(\mathcal{T}_M(\vartheta_M)))|^2\,\mathrm{d}x \leq \int_\Omega |g_\tau^k + \vartheta_\tau^{k-1}||\mathcal{T}_M(\vartheta_M)|\,\mathrm{d}x + \int_{\partial\Omega} h_\tau^k|\mathcal{T}_M(\vartheta_M)|\,\mathrm{d}S$$
$$+ \int_\Omega |\ell_{\tau,M}^k||\mathcal{T}_M(\vartheta_M)|^2\,\mathrm{d}x + \int_\Omega |j_{\tau,M}^k||\mathcal{T}_M(\vartheta_M)|\,\mathrm{d}x \tag{4.27}$$

with the place-holders

$$\ell_{\tau,M}^k := -\frac{\chi_M - \chi_\tau^{k-1}}{\tau} - \rho\,\mathrm{div}\left(\frac{\mathbf{u}_M - \mathbf{u}_\tau^{k-1}}{\tau}\right),$$

$$j_{\tau,M}^k := a(\chi_\tau^{k-1})\varepsilon\left(\frac{\mathbf{u}_M - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_M - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \left|\frac{\chi_M - \chi_\tau^{k-1}}{\tau}\right|^2 + \frac{\tau^{1/2}}{2}\left|\frac{\chi_M - \chi_\tau^{k-1}}{\tau}\right|^2.$$

Arguing in the same way as in the proof of [28, Thm. 2] (see also [28, Rmk. 2.10] and [15]), combining the growth condition (2.15) on K with the Poincaré inequality (2.13), and taking into account estimate (4.26), we deduce that

$$\exists c, C > 0 \ \ \forall M > 0 : \quad \int_\Omega \mathsf{K}(\mathcal{T}_M(\vartheta_M))|\nabla(\mathcal{T}_M(\vartheta_M)))|^2\,\mathrm{d}x \geq c\|\nabla(\mathcal{T}_M(\vartheta_M)))\|_{L^2(\Omega;\mathbb{R}^d)}^2 + \|\mathcal{T}_M(\vartheta_M)\|_{L^{3\kappa+6}(\Omega)}^{\kappa+2} - C.$$

On the other hand,

$$\int_\Omega |\ell_{\tau,M}^k||\mathcal{T}_M(\vartheta_M)|^2\,\mathrm{d}x \leq \|\ell_{\tau,M}^k\|_{L^2(\Omega)}\|\mathcal{T}_M(\vartheta_M)\|_{L^3(\Omega)}\|\mathcal{T}_M(\vartheta_M)\|_{L^6(\Omega)}$$
$$\leq \frac{c}{4}\|\nabla(\mathcal{T}_M(\vartheta_M)))\|_{L^2(\Omega;\mathbb{R}^d)}^2 + C\|\mathcal{T}_M(\vartheta_M)\|_{L^3(\Omega)}^2$$
$$\leq \frac{c}{2}\|\nabla(\mathcal{T}_M(\vartheta_M)))\|_{L^2(\Omega;\mathbb{R}^d)}^2 + C\|\mathcal{T}_M(\vartheta_M)\|_{L^1(\Omega)}^2,$$

where we have used that $\sup_M \|\ell_{\tau,M}^k\|_{L^2(\Omega)} \leq C$ thanks to (4.26). The last inequality follows from the fact that $H^1(\Omega) \Subset L^3(\Omega) \subset L^1(\Omega)$, yielding that for all $\rho > 0$ there exists $C_\rho > 0$ such that $\|\mathcal{T}_M(\vartheta_M)\|_{L^3(\Omega)} \leq \rho\|\mathcal{T}_M(\vartheta_M)\|_{H^1(\Omega)} + C_\rho\|\mathcal{T}_M(\vartheta_M)\|_{L^1(\Omega)}$. In the same way, estimate (4.26) ensures that

$$\int_\Omega |j_{\tau,M}^k||\mathcal{T}_M(\vartheta_M)|\,\mathrm{d}x \leq C\|\mathcal{T}_M(\vartheta_M)\|_{L^2(\Omega)}.$$

All in all, from (4.27), taking into account (4.26) and conditions (2.22) and (2.23) on g and h, we deduce that

$$\exists C > 0 \ \ \forall M > 0 : \quad \|\mathcal{T}_M(\vartheta_M)\|_{H^1(\Omega)} + \|\mathcal{T}_M(\vartheta_M)\|_{L^{3\kappa+6}(\Omega)} \leq C. \tag{4.28}$$

We now introduce the notation

$$\mathcal{S}_M := \{x \in \Omega : \vartheta_M(x) \leq M\}, \qquad \mathcal{O}_M := \Omega \setminus \mathcal{S}_M.$$

In view of estimate (4.28) we have that

$$M^{3\kappa+6} \int_{\mathcal{O}_M} 1 \, \mathrm{d}x \leq \int_{\mathcal{O}_M} |\mathcal{T}_M(\vartheta_M)|^{3\kappa+6} \, \mathrm{d}x \leq C \quad \text{whence} \quad |\mathcal{O}_M| \leq \frac{C}{M^{3\kappa+6}} \to 0 \text{ as } M \to \infty. \tag{4.29}$$

Let us finally test (4.15) by ϑ_M. Relying on the coercivity (4.13) of $\mathcal{A}_M^k$ and again arguing as in the proof of [28, Thm. 2] we find

$$\sup_{M>0} \left(\|\vartheta_M\|_{H^1(\Omega)} + \|\vartheta_M\|_{L^{3\kappa+6}(\mathcal{S}_M)} \right) \leq C. \tag{4.30}$$

Here, we have essentially used the same arguments as for treating (4.27) and estimated the terms involving $\ell_{\tau,M}^k$ and $j_{\tau,M}^k$ by means of (4.26).

In the end, it remains to estimate the terms $\alpha_\nu((\chi_M - \chi_\tau^{k-1})/\tau)$, $A_p(\chi_M)$ and ξ_M in (4.6). First of all, we may suppose that the terms $A_p(\chi_\tau^{k-1})$, $\xi_\tau^{k-1} \in \beta(\chi_\tau^{k-1})$ from the previous step are bounded in $L^2(\Omega)$ by a constant independent of M. Then, we test (4.6) by $(A_p(\chi_M) - A_p(\chi_\tau^{k-1}) + (\xi_M - \xi_\tau^{k-1}))$, thus obtaining

$$\int_\Omega \lambda_M (A_p(\chi_M) - A_p(\chi_\tau^{k-1}) + \xi_M - \xi_\tau^{k-1}) \, \mathrm{d}x + \|A_p(\chi_M) + \xi_M\|_{L^2(\Omega)}^2$$
$$= \int_\Omega (A_p(\chi_M) + \xi_M)(A_p(\chi_\tau^{k-1}) + \xi_\tau^{k-1}) \, \mathrm{d}x + \int_\Omega \mu_M (A_p(\chi_M) - A_p(\chi_\tau^{k-1}) + \xi_M - \xi_\tau^{k-1}) \, \mathrm{d}x \doteq I_1 + I_2.$$

Here, we have used the place-holders $\lambda_M := (\chi_M - \chi_\tau^{k-1})/\tau + \sqrt{\tau}(\chi_M - \chi_\tau^{k-1})/\tau + \alpha_\nu((\chi_M - \chi_\tau^{k-1})/\tau)$ and $\mu_M := \vartheta_M - b'(\chi_M)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} - \gamma(\chi_M) - \nu(\chi_M)^{\eta-2}\eta$. With monotonicity arguments, we see that the first integral on the left-hand side is positive. We estimate

$$I_1 \leq \frac{1}{2}\|A_p(\chi_M) + \xi_M\|_{L^2(\Omega)}^2 + \frac{1}{2}\|A_p(\chi_\tau^{k-1}) + \xi_\tau^{k-1}\|_{L^2(\Omega)}^2.$$

It follows from the estimates on $\mathbf{u}_\tau^{k-1}$, χ_τ^{k-1}, and from (4.26) for χ_M and from (4.30) for ϑ_M that $\|\mu_M\|_{L^2(\Omega)} \leq C$ for a constant independent of $M > 0$. Therefore we have

$$I_2 \leq \frac{1}{4}\|A_p(\chi_M) + \xi_M\|_{L^2(\Omega)}^2 + \frac{1}{4}\|A_p(\chi_\tau^{k-1}) + \xi_\tau^{k-1}\|_{L^2(\Omega)}^2 + C.$$

With this, we conclude that $\|A_p(\chi_M) + \xi_M\|_{L^2(\Omega)} \leq C$ for a constant independent of M. By the monotonicity of the operator β (cf., e.g., [1, Lemma 3.3]), we find $\|A_p(\chi_M)\|_{L^2(\Omega)} \leq C$ and $\|\xi_M\|_{L^2(\Omega)} \leq C$. Then, a comparison argument in (4.6) yields

$$\mu \left\| \alpha_\nu \left(\frac{\chi_M - \chi_\tau^{k-1}}{\tau} \right) \right\|_{L^2(\Omega)} + \|A_p(\chi_M)\|_{L^2(\Omega)} + \|\xi_M\|_{L^2(\Omega)} \leq C. \tag{4.31}$$

Standard compactness arguments together with (4.30) imply that there exists $\vartheta \in H^1(\Omega)$ such that, up to a (not relabeled) subsequence,

$$\vartheta_M \rightharpoonup \vartheta \text{ in } H^1(\Omega), \qquad \vartheta_M \to \vartheta \text{ in } L^q(\Omega) \text{ for all } q < \begin{cases} \infty & \text{if } d = 2, \\ 6 & \text{if } d = 3. \end{cases} \tag{4.32}$$

In particular, $\vartheta_M \to \vartheta$ in measure. Combining this with (4.29) we infer that $\mathcal{T}_M(\vartheta_M) \to \vartheta$ in measure. Therefore, in view of estimate (4.28) and of the Egorov theorem we ultimately have that

$$\vartheta \in L^{3\kappa+6}(\Omega), \quad \mathcal{T}_M(\vartheta_M) \rightharpoonup \vartheta \text{ in } H^1(\Omega) \cap L^{3\kappa+6}(\Omega), \quad \mathcal{T}_M(\vartheta_M) \to \vartheta \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 3\kappa + 6. \tag{4.33}$$

Therefore, taking into account the growth condition (2.15) for K, we have

$$\mathsf{K}_M(\vartheta_M) = \mathsf{K}(\mathcal{T}_M(\vartheta_M)) \to \mathsf{K}(\vartheta) \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 3 + \frac{6}{\kappa}.$$

Combining this with the fact that $\nabla \vartheta_M \rightharpoonup \nabla \vartheta$ in $L^2(\Omega; \mathbb{R}^d)$, we infer on the one hand that $\mathcal{A}_M^k(\vartheta_M)$ weakly converges in the space $W^{1,s}(\Omega)^*$ to the operator $\widetilde{\mathcal{A}}^k(\vartheta)$ defined by $\langle \widetilde{\mathcal{A}}^k(\vartheta), v \rangle_{W^{1,s}(\Omega)} := \int_\Omega \mathsf{K}(\vartheta)\nabla\vartheta\nabla v \, \mathrm{d}x - \int_{\partial\Omega} h_\tau^k v \, \mathrm{d}x$ for all $v \in W^{1,s}(\Omega)$, for some sufficiently big $s > 0$. On the other hand, a comparison in (4.15)

shows that $(\mathcal{A}_M^k(\vartheta_M))_M$ is bounded in $H^1(\Omega)^*$. Therefore, it is not difficult to infer that the operator $\widetilde{\mathcal{A}}^k(\vartheta)$ extends to $H^1(\Omega)$ and coincides with the operator $\mathcal{A}^k$ from (4.3), and that

$$\mathcal{A}_M^k(\vartheta_M) \rightharpoonup \mathcal{A}^k(\vartheta) \quad \text{in } H^1(\Omega)^* \quad \text{as } M \to \infty. \tag{4.34}$$

From estimates (4.26) and (4.31) we also deduce that there exist $\mathbf{u}$, χ, ξ and, if $\mu = 1$, ζ such that, up to a subsequence, $\mathbf{u}_M \rightharpoonup \mathbf{u}$ in $W_0^{1,\eta}(\Omega;\mathbb{R}^d)$, $\chi_M \to \chi$ in $W^{1,p}(\Omega)$ (this follows from the fact that $(\chi_M)_M$ is bounded in $W^{1+\sigma,p}(\Omega)$ for all $0 < \sigma < \frac{1}{p}$ by [31, Thm. 2, Rmk. 2.5]), $\xi_M \rightharpoonup \xi$ in $L^2(\Omega)$, and, if $\mu = 1$, $\alpha_\nu((\chi_M - \chi_\tau^{k-1})/\tau) \rightharpoonup \zeta$ in $L^2(\Omega)$. By the strong-weak closedness in the sense of graphs of α_ν (viewed as a maximal monotone graph in $L^2(\Omega) \times L^2(\Omega)$), we infer, in the case $\mu = 1$, that $\zeta = \alpha_\nu((\chi_\tau^k - \chi_\tau^{k-1})/\tau)$ a.e. in Ω. Analogously, the strong-weak closedness property of β yields that $\xi \in \beta(\chi)$. Combining this convergences with (4.33)–(4.34) we conclude that the functions $\vartheta, \mathbf{u}, \chi, \xi, \zeta$ fulfill a.e. in Ω

$$\frac{\chi - \chi_\tau^{k-1}}{\tau} + \sqrt{\tau}\frac{\chi - \chi_\tau^{k-1}}{\tau} + \mu\alpha_\nu((\chi - \chi_\tau^{k-1})/\tau) + A_p(\chi) + \xi + \gamma(\chi) + \nu|\chi|^{\eta-2}\chi = -b'(\chi)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \vartheta$$

as well as

$$\frac{\vartheta - \vartheta_\tau^{k-1}}{\tau} + \frac{\chi - \chi_\tau^{k-1}}{\tau}\vartheta + \rho\operatorname{div}\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right)\vartheta + \mathcal{A}^k(\vartheta) \tag{4.35}$$

$$= g_\tau^k + a(\chi_\tau^{k-1})\Lambda_k + \left|\frac{\chi - \chi_\tau^{k-1}}{\tau}\right|^2 + \frac{\tau^{1/2}}{2}\left|\frac{\chi - \chi_\tau^{k-1}}{\tau}\right|^2 \quad \text{in } H^1(\Omega)^*, \tag{4.36}$$

$$\frac{\mathbf{u} - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left(a(\chi_\tau^{k-1})\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi)\mathbf{u}\right) + \mathcal{C}_\rho(\vartheta) - \nu\operatorname{div}(\Gamma_k) = \mathbf{f}_\tau^k \quad \text{in } W^{1,\eta}(\Omega;\mathbb{R}^d)^*, \tag{4.37}$$

where Λ_k denotes the weak limit of $\varepsilon\left(\frac{\mathbf{u}_M - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_M - \mathbf{u}_\tau^{k-1}}{\tau}\right)$ in $L^2(\Omega)$, and Γ_k stands for the weak limit of $|\varepsilon(\mathbf{u}_M)|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}_M)$ in $L^{\eta/(\eta-1)}(\Omega;\mathbb{R}^d)$. In order to identify them, it is sufficient to test (4.16) by $\mathbf{u}_M$ and show that

$$\limsup_{M \to \infty} \langle -\operatorname{div}(|\varepsilon(\mathbf{u}_M)|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}_M)), \mathbf{u}_M\rangle_{W^{1,\eta}(\Omega;\mathbb{R}^d)} = \limsup_{M \to \infty} \int_\Omega |\varepsilon(\mathbf{u}_M)|^\eta \, dx \leq \langle -\operatorname{div}(\Gamma_k), \mathbf{u}\rangle_{W^{1,\eta}(\Omega;\mathbb{R}^d)},$$

which we can do, exploiting that $\mathbf{u}$ solves (4.37). This enables us to conclude that $\Gamma_k = -\operatorname{div}(|\varepsilon(\mathbf{u})|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u}))$ and that $\mathbf{u}_M \to \mathbf{u}$ *strongly* in $W^{1,\eta}(\Omega;\mathbb{R}^d)$. The latter convergence clearly allows us to conclude that $\Lambda_k = \varepsilon\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right)$. All in all, $(\vartheta, \mathbf{u}, \chi)$ solves the system

$$\frac{\vartheta - \vartheta_\tau^{k-1}}{\tau} + \frac{\chi - \chi_\tau^{k-1}}{\tau}\vartheta + \rho\operatorname{div}\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right)\vartheta + \mathcal{A}^k(\vartheta) = g_\tau^k \tag{4.38}$$

$$+ a(\chi_\tau^{k-1})\varepsilon\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \left(1 + \frac{\tau^{1/2}}{2}\right)\left|\frac{\chi - \chi_\tau^{k-1}}{\tau}\right|^2 \quad \text{in } H^1(\Omega)^*,$$

$$\frac{\mathbf{u} - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left(a(\chi_\tau^{k-1})\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi)\mathbf{u}\right) + \mathcal{C}_\rho(\vartheta) - \nu\operatorname{div}(|\varepsilon(\mathbf{u})|^{\eta-2}\mathbb{I}\varepsilon(\mathbf{u})) = \mathbf{f}_\tau^k$$

$$\text{in } W^{1,\eta}(\Omega;\mathbb{R}^d)^*, \tag{4.39}$$

$$(1 + \sqrt{\tau})\frac{\chi - \chi_\tau^{k-1}}{\tau} + \mu\alpha_\nu\left(\frac{\chi - \chi_\tau^{k-1}}{\tau}\right) + A_p(\chi) + \xi_\tau^k + \gamma(\chi) + \nu|\chi|^{\eta-2}\chi \ni -b'(\chi)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \vartheta$$

$$\text{a.e. in } \Omega, \tag{4.40}$$

with $\xi_\tau^k \in \beta(\chi_\tau^k)$ a.e. in Ω. It follows from Step 3 and convergences (4.32) that ϑ also fulfills the strict positivity property (4.10).

Step 5**: passage to the limit as** $\nu \to 0$. We now pass to the limit in (4.38)–(4.40) as $\nu \to 0$. We denote by by $(\vartheta_\nu, \mathbf{u}_\nu, \chi_\nu)$ the solutions of (4.38)–(4.40) and, as before, obtain a series of estimates independent of the parameter ν.

First, we test (4.38) by 1, (4.39) by $\mathbf{u}_\nu - \mathbf{u}_\tau^{k-1}$, (4.40) by $\chi_\nu - \chi_\tau^{k-1}$, and add the resulting relations. We thus conclude that

$$\exists\, C > 0 \;\forall \nu > 0: \quad \|\vartheta_\nu\|_{L^1(\Omega)} + \|\mathbf{u}_\nu\|_{H^1(\Omega;\mathbb{R}^d)} + \nu^{1/\eta}\|\varepsilon(\mathbf{u}_\nu)\|_{L^\eta(\Omega;\mathbb{R}^d)} + \|\chi_\nu\|_{W^{1,p}(\Omega)} \leq C. \qquad (4.41)$$

Second, we test (4.38) by $\vartheta_\nu^{\alpha-1}$, with $\alpha \in (0,1)$. With the very same calculations as for the *Second a priori estimates*, cf. also the proof of Prop. 4.10 ahead, we conclude that (cf. (3.7)) that

$$c\int_\Omega \mathsf{K}(\vartheta_\nu)|\nabla\vartheta_\nu^{\alpha/2}|^2\,\mathrm{d}x + c\int_\Omega \left|\varepsilon\left(\frac{\mathbf{u}_\nu - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right|^2 \vartheta_\nu^{\alpha-1}\,\mathrm{d}x + c\int_\Omega \left|\frac{\chi_\nu - \chi_\tau^{k-1}}{\tau}\right|^2 \vartheta_\nu^{\alpha-1}\,\mathrm{d}x \leq C + C\int_\Omega \vartheta_\nu^{\alpha+1}\,\mathrm{d}x$$

whence, with the same arguments as throughout (3.8)–(3.13), we arrive at $\int_\Omega |\nabla\vartheta_\nu^{(\kappa+\alpha)/2}|^2\,\mathrm{d}x \leq C$ for a constant independent of ν. Then, choosing $\alpha \in [1/2, 1)$ such that $\kappa + \alpha \geq 2$, we conclude that

$$\|\vartheta_\nu\|_{H^1(\Omega)} \leq C \qquad (4.42)$$

and, again arguing via the nonlinear Poincaré inequality, we also have that

$$\|\vartheta_\nu^{(\kappa+\alpha)/2}\|_{H^1(\Omega)} \leq C\,. \qquad (4.43)$$

We then test (4.40) by $(A_p(\chi_\nu) - A_p(\chi_\tau^{k-1}) + \xi_\nu - \xi_\tau^{k-1})$ and, arguing in the very same way as in Step 4, conclude that

$$\mu\left\|\alpha_\nu\left(\frac{\chi_\nu - \chi_\tau^{k-1}}{\tau}\right)\right\|_{L^2(\Omega)} + \|A_p(\chi_\nu)\|_{L^2(\Omega)} + \|\xi_\nu\|_{L^2(\Omega)} \leq C. \qquad (4.44)$$

We can now pass to the limit in system (4.38)–(4.40) as $\nu \downarrow 0$. It follows from the previously proved a priori estimates that, along a (not relabeled) subsequence, $\mathbf{u}_\nu \rightharpoonup \mathbf{u}$ in $H_0^1(\Omega;\mathbb{R}^d)$, $\chi_\nu \to \chi$ in $W^{1,p}(\Omega)$, and $\vartheta_\nu \rightharpoonup \vartheta$ in $H^1(\Omega)$. Using these convergences, it is not difficult to pass to the limit in (4.39) and conclude that $\mathbf{u}$ fulfills (4.5). With the same argument as in Step 4, testing (4.39) by $\mathbf{u}_\nu$ we conclude that

$$\limsup_{\nu\to 0} \int_\Omega \varepsilon(\mathbf{u}_\nu)\mathbb{E}\varepsilon(\mathbf{u}_\nu)\,\mathrm{d}x \leq \int_\Omega \varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})\,\mathrm{d}x,$$

yielding that $\mathbf{u}_\nu \to \mathbf{u}$ *strongly* in $H^1(\Omega;\mathbb{R}^d)$. Therefore,

$$a(\chi_\tau^{k-1})\varepsilon\left(\frac{\mathbf{u}_\nu - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\nu - \mathbf{u}_\tau^{k-1}}{\tau}\right) \to a(\chi_\tau^{k-1})\varepsilon\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u} - \mathbf{u}_\tau^{k-1}}{\tau}\right) \qquad \text{in } L^1(\Omega).$$
$$(4.45)$$

We use this information to pass to the limit in (4.38). Moreover, estimate (4.43) allows us to conclude that, up to a subsequence, $\vartheta_\nu^{(\kappa+\alpha)/2} \rightharpoonup \vartheta^{(\kappa+\alpha)/2}$ in $H^1(\Omega)$, hence $\vartheta_\nu^{(\kappa+\alpha)/2} \to \vartheta^{(\kappa+\alpha)/2}$ in $L^{6-\epsilon}(\Omega)$ for all $\epsilon > 0$, whence, taking into account the growth condition on K, that

$$\mathsf{K}(\vartheta_\nu) \to \mathsf{K}(\vartheta) \qquad \text{in } L^{3+\alpha/\kappa-\epsilon}(\Omega) \qquad \text{for all } \epsilon > 0.$$

This allows us to pass to the limit in the term $\mathsf{K}(\vartheta_\nu)\nabla\vartheta_\nu$, tested against $v \in W^{1,s}(\Omega)$ for some sufficiently big $s > 0$. All in all, we infer that $(\vartheta, \mathbf{u}, \chi)$ satisfies (4.4) in some dual space $W^{1,s}(\Omega)^*$, such that, also, $W^{1,s}(\Omega) \subset L^\infty$ in accord with the L^1-convergence (4.45). Finally, we pass to the limit in (4.40). Due to estimate (4.44), we have that there exist $\xi \in L^2(\Omega)$ and, if $\mu = 1$, $\zeta \in L^2(\Omega)$ such that

$$\alpha_\nu\left(\frac{\chi_\nu - \chi_\tau^{k-1}}{\tau}\right) \rightharpoonup \zeta, \qquad \xi_\nu \rightharpoonup \xi \qquad \text{in } L^2(\Omega).$$

The strong-weak closedness of β yields that $\xi \in \beta(\chi)$ a.e. in Ω. In order to show that, in the case $\mu = 1$, $\zeta \in \alpha((\chi - \chi_\tau^{k-1})/\tau)$ a.e. in Ω, we show that

$$\limsup_{\nu\downarrow 0} \int_\Omega \alpha_\nu\left(\frac{\chi_\nu - \chi_\tau^{k-1}}{\tau}\right)\left(\frac{\chi_\nu - \chi_\tau^{k-1}}{\tau}\right)\mathrm{d}x \leq \int_\Omega \zeta\left(\frac{\chi - \chi_\tau^{k-1}}{\tau}\right)\mathrm{d}x$$

and invoke well-knows results from the theory of maximal monotone operators.

All in all, we conclude that $(\vartheta, \mathbf{u}, \chi)$ solves system (4.4)–(4.6), where (4.4) is to be understood in $W^{1,s}(\Omega)^*$.

Step 6: $H^2(\Omega; \mathbb{R}^d)$-regularity for $\mathbf{u}_\tau^k$ and conclusion. A comparison argument in (4.5) yields that

$$\mathcal{V}\left(a(\chi_\tau^k)\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \quad \text{and } \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) \quad \text{are in } L^2(\Omega; \mathbb{R}^d).$$

From the latter information we now deduce that $\mathbf{u}_\tau^k \in H^2(\Omega; \mathbb{R}^d)$. Indeed, multiplying $\mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) \doteq \mathbf{h}_\tau^k \in L^2(\Omega; \mathbb{R}^d)$ by $-\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k))$, we get

$$\int_\Omega b(\chi_\tau^k)\mathbb{E}|\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k))|^2 \, \mathrm{d}x \leq \left|\int_\Omega \nabla b(\chi_\tau^k)\mathbb{E}\varepsilon(\mathbf{u}_\tau^k)\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k)) \, \mathrm{d}x\right| + \left|\int_\Omega \mathbf{h}_\tau^k \operatorname{div}(\varepsilon(\mathbf{u}_\tau^k)) \, \mathrm{d}x\right|$$

$$\leq C_\delta + \delta \int_\Omega \left|\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k))\right|^2 \, \mathrm{d}x$$

where in the latter estimate $\delta > 0$ is sufficiently small, and we have relied on the fact that $\|\mathbf{u}_\tau^k\|_{H^1(\Omega; \mathbb{R}^d)} + \|\chi_\tau^k\|_{W^{1,p}(\Omega)} \leq C$, combined with assumption (2.16) on b. Also using (2.16) and choosing $0 < \delta < C_1 c_2$ (cf. (2.5) and (2.16)), we then infer that

$$\int_\Omega |\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k))|^2 \, \mathrm{d}x \leq C.$$

Then, a standard regularity result for elliptic systems with constant coefficients (cf. (2.9)), yields that $\mathbf{u}_\tau^k \in H^2(\Omega; \mathbb{R}^d)$.

In the end, exploiting that that $\mathbf{u}_\tau^k \in H^2(\Omega; \mathbb{R}^d)$, a comparison argument in the heat equation allows us to conclude that $\int_\Omega \mathsf{K}(\vartheta)\nabla\vartheta \cdot \nabla v \, \mathrm{d}x$ is well defined for all test functions $v \in H^1(\Omega)$, hence (4.4) is solved in $H^1(\Omega)^*$. $\qquad\square$

Remark 4.5. In the case $\mu = 1$, as mentioned in Remark 4.3, the discrete χ-equation could be decoupled from the discrete equations for ϑ and $\mathbf{u}$, cf. (4.9). This would lead to having the term $\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\vartheta_\tau^{k-1}$. The argument for the strict positivity of ϑ_τ^k in Step 3 in this case would not go through. Nonetheless, it would be possible to prove that $\vartheta_\tau^k \geq 0$ a.e. in Ω, by testing the discrete heat equation by $-(\vartheta_\tau^k)^-$, and using that $\int_\Omega \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\vartheta_\tau^{k-1}(-(\vartheta_\tau^k)^-) \, \mathrm{d}x \geq 0$ since $\chi_\tau^k \leq \chi_\tau^{k-1}$ a.e. in Ω.

Remark 4.6. We briefly comment on the reason why we need to perform two distinct passages to the limit in the proof of Lemma 4.4. As the above proof shows, in the passage to limit as $\nu \to 0$ we lose the information that the right-hand side of the equation for ϑ is estimated in $L^2(\Omega)$. Hence, we need to carry out refined estimates on the ϑ-equation (i.e., testing it by $\vartheta^{\alpha-1}$), where we fully exploit the growth of K to carry out the related calculations. Clearly, to do so we first have to pass to the limit with the truncation parameter.

4.2. **Approximate entropy and total energy inequalities.** Preliminarily, we establish the

Notation 4.7 (Interpolants and discrete integration-by-parts formula). Hereafter, for a given Banach space X and a K_τ-tuple $(\mathfrak{h}_\tau^k)_{k=1}^{K_\tau} \subset X$, we shall use the short-hand notation

$$\mathrm{D}_{\tau,k}(\mathfrak{h}) := \frac{\mathfrak{h}_\tau^k - \mathfrak{h}_\tau^{k-1}}{\tau}, \qquad \mathrm{D}_{\tau,k}^2(\mathfrak{h}) := \mathrm{D}_{\tau,k}(\mathrm{D}_{\tau,k}(\mathfrak{h})) = \frac{\mathfrak{h}_\tau^k - 2\mathfrak{h}_\tau^{k-1} + \mathfrak{h}_\tau^{k-2}}{\tau^2}.$$

We recall the well-known *discrete by-part integration* formula

$$\sum_{k=1}^{K_\tau} \tau \mathrm{D}_{\tau,k}(\mathfrak{h})v_\tau^k = \mathfrak{h}_\tau^{K_\tau}v_\tau^{K_\tau} - \mathfrak{h}_\tau^0 v_\tau^1 - \sum_{k=2}^{K_\tau} \tau\mathfrak{h}_\tau^{k-1}\mathrm{D}_{\tau,k}(v) \quad \text{for all } \{\mathfrak{h}_\tau^k\}_{k=1}^{K_\tau}, \{v_\tau^k\}_{k=1}^{K_\tau} \subset X. \qquad (4.46)$$

We introduce the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants of the values $\{\mathfrak{h}_\tau^k\}_{k=1}^{K_\tau}$ by

$$
\left.
\begin{aligned}
\overline{\mathfrak{h}}_\tau &: (0,T) \to X \quad \text{defined by} \quad \overline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^k, \\
\underline{\mathfrak{h}}_\tau &: (0,T) \to X \quad \text{defined by} \quad \underline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^{k-1}, \\
\mathfrak{h}_\tau &: (0,T) \to X \quad \text{defined by} \quad \mathfrak{h}_\tau(t) := \frac{t-t_\tau^{k-1}}{\tau}\mathfrak{h}_\tau^k + \frac{t_\tau^k - t}{\tau}\mathfrak{h}_\tau^{k-1}
\end{aligned}
\right\} \quad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].
$$

We also introduce the piecewise linear interpolant of the values $\{(\mathfrak{h}_\tau^k - \mathfrak{h}_\tau^{k-1})/\tau\}_{k=1}^{K_\tau}$ (namely, the values taken by the -piecewise constant- function $\mathfrak{h}_\tau'$), viz.

$$
\widehat{\mathfrak{h}}_\tau : (0,T) \to X \qquad \widehat{\mathfrak{h}}_\tau(t) := \frac{t-t_\tau^{k-1}}{\tau}\frac{\mathfrak{h}_\tau^k - \mathfrak{h}_\tau^{k-1}}{\tau} + \frac{t_\tau^k - t}{\tau}\frac{\mathfrak{h}_\tau^{k-1} - \mathfrak{h}_\tau^{k-2}}{\tau} \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].
$$

Note that $\widehat{\mathfrak{h}}_\tau'(t) = \mathrm{D}_{\tau,k}^2(\mathfrak{h})$ for $t \in (t_\tau^{k-1}, t_\tau^k]$.

Furthermore, we denote by $\bar{\mathsf{t}}_\tau$ and by $\underline{\mathsf{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition, i.e. $\bar{\mathsf{t}}_\tau(t) := t_\tau^k$ if $t_\tau^{k-1} < t \le t_\tau^k$ and $\underline{\mathsf{t}}_\tau(t) := t_\tau^{k-1}$ if $t_\tau^{k-1} \le t < t_\tau^k$. Clearly, for every $t \in [0,T]$ we have $\bar{\mathsf{t}}_\tau(t) \downarrow t$ and $\underline{\mathsf{t}}_\tau(t) \uparrow t$ as $\tau \to 0$.

In view of (2.21), (2.22), and (2.23), it is easy to check that the piecewise constant interpolants $(\bar{\mathbf{f}}_\tau)_{k=1}^{K_\tau}$, $(\bar{g}_\tau)_{k=1}^{K_\tau}$, $(\bar{h}_\tau)_{k=1}^{K_\tau}$ of the values $\mathbf{f}_\tau^k$, g_τ^k, h_τ^k (4.1) fulfill as $\tau \downarrow 0$

$$
\bar{\mathbf{f}}_\tau \to \mathbf{f} \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{4.47}
$$

$$
\bar{g}_\tau \to g \text{ in } L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)'). \tag{4.48}
$$

$$
\bar{h}_\tau \to h \text{ in } L^1(0,T;L^2(\partial\Omega)). \tag{4.49}
$$

We now rewrite the discrete equations (4.4)–(4.6) in terms of the interpolants $\bar{\vartheta}_\tau$, ϑ_τ, $\bar{\mathbf{u}}_\tau$, $\underline{\mathbf{u}}_\tau$, $\mathbf{u}_\tau$, $\widehat{\mathbf{u}}_\tau$, $\bar{X}_\tau$, $\underline{X}_\tau$, X_τ, $\bar{\xi}_\tau$, and $\bar{\zeta}_\tau$ of the elements $(\vartheta_\tau^k, \mathbf{u}_\tau^k, X_\tau^k, \xi_\tau^k, \zeta_\tau^k)_{k=1}^{K_\tau}$. Indeed, we have for almost all $t \in (0,T)$

$$
\begin{aligned}
\partial_t \vartheta_\tau(t) + \partial_t X_\tau(t)\bar{\vartheta}_\tau(t) + \rho\,\mathrm{div}(\partial_t \mathbf{u}_\tau(t))\bar{\vartheta}_\tau(t) + \mathcal{A}^{\frac{\bar{\mathsf{t}}_\tau(t)}{\tau}}(\bar{\vartheta}_\tau(t)) = \bar{g}_\tau(t) + \\
+ a(\underline{X}_\tau(t))\varepsilon(\partial_t\mathbf{u}_\tau(t))\mathbb{V}\varepsilon(\partial_t\mathbf{u}_\tau(t)) + (1+\tau^{1/2})|\partial_t X_\tau(t)|^2 \quad \text{in } X^*,
\end{aligned} \tag{4.50}
$$

$$
\partial_t \widehat{\mathbf{u}}_\tau(t) + \mathcal{V}(a(\underline{X}_\tau(t))\partial_t\mathbf{u}_\tau(t)) + \mathcal{E}(b(\bar{X}_\tau(t))\bar{\mathbf{u}}_\tau(t)) + \mathcal{C}_\rho(\bar{\vartheta}_\tau) = \bar{\mathbf{f}}_\tau(t) \tag{4.51}
$$
$$
\text{a.e. in } \Omega,
$$

$$
(1+\sqrt{\tau})\partial_t X_\tau(t) + \mu\bar{\zeta}_\tau(t) + A_p\bar{X}_\tau(t) + \bar{\xi}_\tau(t) + \gamma(\bar{X}_\tau(t))
$$
$$
= -b'(\bar{X}_\tau(t))\frac{\varepsilon(\underline{\mathbf{u}}_\tau(t))\mathbb{E}\varepsilon(\underline{\mathbf{u}}_\tau(t))}{2} + \bar{\vartheta}_\tau(t) \tag{4.52}
$$
$$
\text{a.e. in } \Omega,
$$

with $\bar{\xi}_\tau \in \beta(\bar{X}_\tau)$ and $\bar{\zeta}_\tau \in \partial I_{(-\infty,0]}(\partial_t X_\tau)$ a.e. in $\Omega \times (0,T)$.

Our next result states that the interpolants of suitable discrete solutions to system (4.4)–(4.6) also satisfy the approximate versions of the entropy inequality (2.37) and of the total enegy inequality (2.38).

For stating the discrete entropy inequality (4.55) below, we need to introduce *discrete* test functions. Namely, with every test function $\varphi \in C^0([0,T];W^{1,d+\epsilon}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ we associate

$$
\text{for } k = 1,\ldots,K_\tau \quad \varphi_\tau^k := \varphi(t_\tau^k) \tag{4.53}
$$

and consider the piecewise constant and linear interpolants $\bar{\varphi}_\tau$ and φ_τ of the values $(\varphi_\tau^k)_{k=1}^{K_\tau}$. It can be shown that the following convergences hold as $\tau \to 0$

$$
\bar{\varphi}_\tau \to \varphi \quad \text{in } L^\infty(0,T;W^{1,d+\epsilon}(\Omega)) \text{ and } \quad \partial_t\varphi_\tau \to \partial_t\varphi \quad \text{in } L^2(0,T;L^{6/5}(\Omega)). \tag{4.54}
$$

Then, (4.55) is obtained by testing (4.4) by $\varphi_\tau^k/\vartheta_\tau^k$, for $k = 1,\ldots,K_\tau$.

As for the total energy inequality (4.56) below, let us mention that it results from our carefully designed time-discretization scheme, observing in addition that (4.6) is indeed the Euler-Lagrange equation for a suitable minimum problem, cf. (4.58) below, where the additional term

$$\frac{\tau^{3/2}}{2} \int_\Omega \left| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \right|^2 \mathrm{d}x$$

has the role to "compensate" for the possible non-convexity of $\int_\Omega \widehat{\gamma}(\chi)\,\mathrm{d}x$. Accordingly, to get the discrete total energy inequality (4.56) we need to add the term $\frac{\tau^{1/2}}{2} \left| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \right|^2$ to the right-hand side of (4.4). This will lead to the necessary cancellations, cf. (4.66) below.

Proposition 4.8 (Discrete entropy and total energy inequalities, $\mu \in \{0,1\}$). *Under **Hypotheses (I)−(III)**, for $\tau > 0$ sufficiently small, the discrete solutions $(\vartheta_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)_{k=1}^{K_\tau}$ to Problem 4.2 fulfill*
    - *the* discrete *entropy inequality*

$$
\int_{\underline{t}_\tau(s)}^{\underline{t}_\tau(t)} \int_\Omega (\log(\overline{\vartheta}_\tau(r)) + \overline{\chi}_\tau(r)) \partial_t \varphi_\tau(r+\tau)\,\mathrm{d}x\,\mathrm{d}r + \rho \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega \mathrm{div}(\partial_t \mathbf{u}_\tau(r)) \overline{\varphi}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r
$$
$$
- \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega \mathsf{K}(\overline{\vartheta}_\tau(r)) \nabla \log(\overline{\vartheta}_\tau(r)) \cdot \nabla \overline{\varphi}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r
$$
$$
\leq \int_\Omega (\log(\overline{\vartheta}_\tau(\overline{t}_\tau(t))) + \overline{\chi}_\tau(\overline{t}_\tau(t))) \varphi(\overline{t}_\tau(t))\,\mathrm{d}x - \int_\Omega (\log(\overline{\vartheta}_\tau(\underline{t}_\tau(s))) + \overline{\chi}_\tau(\underline{t}_\tau(s))) \varphi(\overline{t}_\tau(s))\,\mathrm{d}x
$$
$$
- \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega \mathsf{K}(\overline{\vartheta}_\tau(r)) \frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)} \nabla \log(\overline{\vartheta}_\tau(r)) \cdot \nabla \overline{\vartheta}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r
$$
$$
- \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega \left(\overline{g}_\tau(r) + a(\underline{\chi}_\tau(r)) \varepsilon(\partial_t \mathbf{u}_\tau(r)) \mathbb{V} \varepsilon(\partial_t \mathbf{u}_\tau(r)) + |\partial_t \chi_\tau(r)|^2 + \frac{\tau^{3/2}}{2} |\partial_t \chi_\tau(r)|^2 \right) \frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\,\mathrm{d}x\,\mathrm{d}r
$$
$$
- \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_{\partial\Omega} \overline{h}_\tau(r) \frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\,\mathrm{d}S\,\mathrm{d}r
$$
(4.55)

    for all $0 \leq s \leq t \leq T$ and for all $\varphi \in \mathrm{C}^0([0,T]; W^{1,d+\epsilon}(\Omega)) \cap H^1(0,T; L^{6/5}(\Omega))$ with $\varphi \geq 0$;
    - *the* discrete *total energy inequality for all $0 \leq s \leq t \leq T$, viz.*

$$
\mathscr{E}(\overline{\vartheta}_\tau(t), \overline{\mathbf{u}}_\tau(t), \partial_t \mathbf{u}_\tau(t), \overline{\chi}_\tau(t)) \leq \mathscr{E}(\overline{\vartheta}_\tau(s), \overline{\mathbf{u}}_\tau(s), \partial_t \mathbf{u}_\tau(s), \overline{\chi}_\tau(s)) + \int_{\overline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega (\overline{g}_\tau + \overline{\mathbf{f}}_\tau \cdot \partial_t \mathbf{u}_\tau)\,\mathrm{d}x\,\mathrm{d}r
$$
$$
+ \int_{\overline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_{\partial\Omega} \overline{h}_\tau\,\mathrm{d}S\,\mathrm{d}r ,
$$
(4.56)

    with $\mathscr{E}$ from (2.39).

For the proof of the discrete entropy inequality, we will rely on a crucial inequality satisfied by any <u>concave</u> function $\psi : \mathrm{dom}(\psi) \to \mathbb{R}$, i.e.

$$\psi(x) - \psi(y) \leq \psi'(y)(x - y) \qquad \text{for all } x, y \in \mathrm{dom}(\psi).$$
(4.57)

Proof. We split the proof in two steps.
Step 1: proof of the total energy inequality. Let us consider the minimum problem

$$
\min_{\chi \in W^{1,p}(\Omega)} \left\{ \int_\Omega \left(\frac{\tau^{3/2}}{2} \left| \frac{\chi - \chi_\tau^{k-1}}{\tau} \right|^2 + \left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \right) \chi + \mu \widehat{\alpha} \left(\frac{\chi - \chi_\tau^{k-1}}{\tau} \right) + \frac{|\nabla\chi|^p}{p} + \widehat{\beta}(\chi) \right. \right.
$$
$$
\left. \left. + \widehat{\gamma}(\chi) + b(\chi) \frac{\varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E} \varepsilon(\mathbf{u}_\tau^{k-1})}{2} - \vartheta_\tau^k \chi \right) \mathrm{d}x \right\}
$$
(4.58)

where χ_τ^k is the discrete solution from Lemma 4.4, and let $\lambda > 0$ such that $\widehat{\gamma}'' \geq -\lambda$ as in (2.19). Then, the function

$$r \mapsto \widehat{\gamma}(r) + \lambda |r|^2 \qquad \text{is strictly convex.} \tag{4.59}$$

Let $\bar{\tau} > 0$ such that $1/(2\tau) > \lambda$ for all $0 < \tau \leq \bar{\tau}$. We may rewrite the minimum problem (4.58) as

$$\min_{\chi \in W^{1,p}(\Omega)} \Big\{ \int_\Omega \Big(\Big(\frac{1}{2\sqrt{\tau}} - \lambda \Big) |\chi - \chi_\tau^{k-1}|^2 + \Big(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \Big) \chi + \mu \widehat{\alpha} \Big(\frac{\chi - \chi_\tau^{k-1}}{\tau} \Big) + \frac{|\nabla \chi|^p}{p} + \widehat{\beta}(\chi) + \widehat{\gamma}(\chi)$$
$$+ \lambda |\chi|^2 + b(\chi) \frac{\varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} - \vartheta_\tau^k \chi + \lambda |\chi_\tau^{k-1}|^2 + 2\lambda \chi \chi_\tau^{k-1} \Big) \, dx \Big\}. \tag{4.60}$$

Observe that the Euler-Lagrange equation for (4.60) is exactly (4.6). Using the convexity of $\widehat{\alpha}$, $\widehat{\beta}$, b, and the λ-convexity of $\widehat{\gamma}$ (whence (4.59)), it is not difficult to check that (4.6) has a unique solution. We may thus conclude that the minimum problem (4.60) has a unique solution, which coincides with the discrete solution χ_τ^k from Lemma 4.4.

Now, choosing χ_τ^{k-1} as a competitor for χ_τ^k in the minimum problem (4.58) yields

$$\tau \int_\Omega \Big| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \Big|^2 \, dx + \int_\Omega \frac{\tau^{3/2}}{2} \Big| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \Big|^2 \, dx + \mu \int_\Omega \widehat{\alpha} \Big(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \Big) \, dx + \int_\Omega \frac{|\nabla \chi_\tau^k|^p}{p} \, dx + \int_\Omega \widehat{\beta}(\chi_\tau^k) \, dx$$
$$+ \int_\Omega \widehat{\gamma}(\chi_\tau^k) \, dx + \int_\Omega b(\chi_\tau^k) \frac{\varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} \, dx - \int_\Omega \vartheta_\tau^k \chi_\tau^k \, dx$$
$$\leq \int_\Omega \frac{|\nabla \chi_\tau^{k-1}|^p}{p} \, dx + \int_\Omega \widehat{\beta}(\chi_\tau^{k-1}) \, dx + \int_\Omega \widehat{\gamma}(\chi_\tau^{k-1}) \, dx$$
$$+ \int_\Omega b(\chi_\tau^{k-1}) \frac{\varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} \, dx - \int_\Omega \vartheta_\tau^k \chi_\tau^{k-1} \, dx. \tag{4.61}$$

Hence, we test (4.5) by $\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}$ and observe that , for all $k = 1, \ldots, K_\tau$,

$$\tau \int_\Omega \mathrm{D}_{\tau,k}^2(\mathbf{u}) \cdot \mathrm{D}_{\tau,k}(\mathbf{u}) \, dx \geq \frac{1}{2} \|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\mathrm{D}_{\tau,k-1}(\mathbf{u})\|_{L^2(\Omega;\mathbb{R}^d)}^2 \tag{4.62}$$

$$\langle \mathcal{V}\big(a(\chi_\tau^{k-1}) \mathrm{D}_{\tau,k}(\mathbf{u}) \big), \mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1} \rangle_{H^1(\Omega)} = \tau \int_\Omega a(\chi_\tau^{k-1}) \varepsilon\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \Big) \mathbb{V}\varepsilon\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \Big) \, dx. \tag{4.63}$$

Furthermore, we have

$$\langle \mathcal{E}\big(b(\chi_\tau^k) \mathbf{u}_\tau^k \big), \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \rangle_{H^1(\Omega;\mathbb{R}^d)} \geq \frac{1}{2} \int_\Omega b(\chi_\tau^k) \varepsilon(\mathbf{u}_\tau^k) \mathbb{E}\varepsilon(\mathbf{u}_\tau^k) \, dx - \frac{1}{2} \int_\Omega b(\chi_\tau^k) \varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1}) \, dx$$
$$= \frac{1}{2} \int_\Omega b(\chi_\tau^k) \varepsilon(\mathbf{u}_\tau^k) \mathbb{E}\varepsilon(\mathbf{u}_\tau^k) \, dx - \frac{1}{2} \int_\Omega b(\chi_\tau^{k-1}) \varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1}) \, dx \tag{4.64}$$
$$- \frac{1}{2} \int_\Omega (b(\chi_\tau^k) - b(\chi_\tau^{k-1})) \varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1}) \, dx.$$

Finally,

$$\tau \langle \mathcal{C}_\rho(\vartheta_\tau^k), \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \rangle_{H^1(\Omega;\mathbb{R}^d)} = -\rho \int_\Omega \vartheta_\tau^k \mathrm{div}\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \Big) \, dx. \tag{4.65}$$

Next, we multiply (4.4) by τ and integrate over Ω. We add the resulting relation to the equation obtained testing (4.16) by $\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}$ and to (4.61). The terms

$$\tau \int_\Omega \mathrm{D}_{\tau,k}(\chi) \vartheta_\tau^k \, dx, \qquad\qquad\qquad \rho\tau \int_\Omega \vartheta_\tau^k \, \mathrm{div}(\mathrm{D}_{\tau,k}(\mathbf{u})) \, dx,$$
$$\tau \int_\Omega a(\chi_\tau^{k-1}) \varepsilon\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \Big) \mathbb{V}\varepsilon\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} \Big) \, dx, \quad \tau \int_\Omega \Big| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \Big|^2 \, dx \tag{4.66}$$
$$\frac{\tau^{3/2}}{2} \int_\Omega \Big| \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \Big|^2 \, dx, \qquad\qquad \frac{1}{2} \int_\Omega (b(\chi_\tau^k) - b(\chi_\tau^{k-1})) \varepsilon(\mathbf{u}_\tau^{k-1}) \mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1}) \, dx$$

cancel out.

We sum over the index $k = m, \ldots, j$, for any couple of indexes $1 \leq m < j \leq K_\tau$. Taking into account (4.61)–(4.65), we ultimately obtain

$$\int_\Omega \left(\vartheta_\tau^j + \frac{1}{2}|\mathrm{D}_{\tau,j}(\mathbf{u})|^2 + \frac{1}{2}b(\chi_\tau^j)\varepsilon(\mathbf{u}_\tau^j)\mathbb{E}\varepsilon(\mathbf{u}_\tau^j) + \frac{|\nabla\chi_\tau^j|^p}{p} + \widehat{\beta}(\chi_\tau^j) + \widehat{\gamma}(\chi_\tau^j) \right) \,\mathrm{d}x$$

$$\leq \int_\Omega \left(\vartheta_\tau^m + \frac{1}{2}|\mathrm{D}_{\tau,m}(\mathbf{u})|^2 + \frac{1}{2}b(\chi_\tau^m)\varepsilon(\mathbf{u}_\tau^m)\mathbb{E}\varepsilon(\mathbf{u}_\tau^m) + \frac{|\nabla\chi_\tau^m|^p}{p} + \widehat{\beta}(\chi_\tau^m) + \widehat{\gamma}(\chi_\tau^m) \right) \,\mathrm{d}x \qquad (4.67)$$

$$+ \sum_{k=m}^{j} \tau \left(\int_\Omega \left(g_\tau^k + \mathbf{f}_\tau^k \cdot \mathrm{D}_{\tau,k}(\mathbf{u}) \right) \,\mathrm{d}x + \int_{\partial\Omega} h_\tau^k \,\mathrm{d}S \right),$$

which yields (4.56).

Step 2: proof of the entropy inequality. Let us fix an arbitrary *posititive* test function

$$\varphi \in \mathrm{C}^0([0,T]; W^{1,d+\epsilon}(\Omega)) \cap H^1(0,T; L^{6/5}(\Omega))$$

with $(\varphi_\tau^k)_{k=1}^{K_\tau}$ defined by (4.53). We multiply (4.4) by $\frac{\varphi_\tau^k}{\vartheta_\tau^k} \in H^1(\Omega)$ (hence, an admissible test function for (4.4)) and integrate over Ω. We obtain

$$\int_\Omega \left(g_\tau^k + a(\chi_\tau^k)\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 + \frac{\tau^{1/2}}{2}\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 \right) \frac{\varphi_\tau^k}{\vartheta_\tau^k} \,\mathrm{d}x$$

$$+ \int_{\partial\Omega} h_\tau^k \frac{\varphi_\tau^k}{\vartheta_\tau^k} \,\mathrm{d}S$$

$$= \int_\Omega \left(\frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\vartheta_\tau^k + \rho\,\mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\vartheta_\tau^k \right) \varphi_\tau^k \,\mathrm{d}x + \int_\Omega \mathsf{K}(\vartheta_\tau^k)\nabla\vartheta_\tau^k \cdot \nabla\left(\frac{\varphi_\tau^k}{\vartheta_\tau^k}\right) \,\mathrm{d}x \qquad (4.68)$$

$$\leq \int_\Omega \left(\frac{\log(\vartheta_\tau^k) - \log(\vartheta_\tau^{k-1})}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \rho\,\mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) \right) \varphi_\tau^k \,\mathrm{d}x$$

$$+ \int_\Omega \left(\frac{\mathsf{K}(\vartheta_\tau^k)}{\vartheta_\tau^k}\nabla\vartheta_\tau^k \cdot \nabla\varphi_\tau^k - \frac{\mathsf{K}(\vartheta_\tau^k)}{|\vartheta_\tau^k|^2}|\nabla\vartheta_\tau^k|^2\varphi_\tau^k \right) \,\mathrm{d}x$$

where we have used that (cf. (4.57))

$$\frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\vartheta_\tau^k} \leq \log(\vartheta_\tau^k) - \log(\vartheta_\tau^{k-1}) \qquad \text{a.e. in } \Omega.$$

Note that this inequality is preserved by the positivity of the discrete test function φ_τ^k. We now sum (4.68), multiplied by τ, over $k = m, \ldots, j$, for any couple of indexes $1 \leq m < j \leq K_\tau$. We use the discrete integration by parts formula (4.46), yielding

$$\sum_{k=m}^{j} \tau \int_\Omega \mathrm{D}_{\tau,k}(\log(\vartheta_\tau^k))\varphi_\tau^k \,\mathrm{d}x = \int_\Omega \log(\vartheta_\tau^j)\varphi_\tau^j \,\mathrm{d}x - \int_\Omega \log(\vartheta_\tau^m)\varphi_\tau^{m+1} \,\mathrm{d}x - \sum_{k=m}^{j-1} \tau \int_\Omega \log(\vartheta_\tau^k)\mathrm{D}_{\tau,k+1}(\varphi) \,\mathrm{d}x$$

$$\sum_{k=m}^{j} \tau \int_\Omega \mathrm{D}_{\tau,k}(\chi_\tau^k)\varphi_\tau^k \,\mathrm{d}x = \int_\Omega \chi_\tau^j\varphi_\tau^j \,\mathrm{d}x - \int_\Omega \chi_\tau^m\varphi_\tau^{m+1} \,\mathrm{d}x - \sum_{k=m}^{j-1} \tau \int_\Omega \chi_\tau^k\mathrm{D}_{\tau,k+1}(\varphi) \,\mathrm{d}x.$$

Inserting the two above inequalities in (4.68) (summed up over $k = m, \ldots, j$), rearranging terms, we conclude (4.55). $\qquad\square$

Remark 4.9. A close perusal of the proof of Proposition 4.8 reveals that, b is only λ-convex, in place of convex, it is still possible to prove that the discrete equation for χ (4.6) admits a unique solution, and therefore conclude that χ_τ^k is the unique minimizer for (4.58). This, provided we replace the p-Laplacian operator in (4.6) with its *non-degenerate* version, cf. Remark 2.9.

4.3. **A priori estimates revisited.** The following result collects all the a priori estimates for the approximate solutions constructed via time discretization. In particular, the proof renders on the discrete level the *Second* and *Sixth* estimates, which have a *nonlinear* character and thus translate with some difficulty within the frame of the discrete system (4.4)–(4.6). In particular, the Sixth estimate (cf. (4.69h) below) is deduced with careful calculations from the discrete entropy inequality (4.55).

Proposition 4.10. *Assume **Hypotheses (I)–(III)** and* (2.21)–(2.26). *Let* $\mu \in \{0, 1\}$. *Then, there exists a constant* $S > 0$ *such that for all* $\tau > 0$ *the following estimates*

$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H_0^2(\Omega;\mathbb{R}^d))} \leq S, \tag{4.69a}$$

$$\|\mathbf{u}_\tau\|_{H^1(0,T;H_0^2(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d))} \leq S, \tag{4.69b}$$

$$\|\widehat{\mathbf{u}}_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S, \tag{4.69c}$$

$$\|\overline{\chi}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq S, \tag{4.69d}$$

$$\|\chi_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq S, \tag{4.69e}$$

$$\|\log(\overline{\vartheta}_\tau)\|_{L^2(0,T;H^1(\Omega))} \leq S, \tag{4.69f}$$

$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^1(\Omega))} \leq S, \tag{4.69g}$$

$$\|\log(\overline{\vartheta}_\tau)\|_{BV([0,T];W^{1,d+\varepsilon}(\Omega)^*)} \leq S \quad \text{for all } \epsilon > 0 \tag{4.69h}$$

*hold. Furthermore, under **Hypothesis (V)** (i.e. if* $1 < \kappa < 5/3$), *we have in addition*

$$\sup_{\tau>0} \|\vartheta_\tau\|_{BV([0,T];W^{2,d+\epsilon}(\Omega)^*)} \leq S \quad \text{for all } \epsilon > 0. \tag{4.69i}$$

Finally, if $\mu = 0$ *we also have*

$$\sup_{\tau>0} \left(\|\overline{\chi}_\tau\|_{L^2(0,T;W^{1+\sigma,p}(\Omega))} + \|\overline{\xi}_\tau\|_{L^2(0,T;L^2(\Omega))} \right) \leq S \quad \text{for all } 1 \leq \sigma < \frac{1}{p}. \tag{4.69j}$$

We now sketch the proof, showing how the formal a priori estimates in Section 3 can be translated in the framework of the time discretization scheme; we shall only detail the argument for the discrete version of the *Sixth estimate*.

Proof. From the discrete total energy inequality (4.56), arguing in the very same way as for the **First a priori estimate**, we deduce

$$\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}_\tau\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} + \|\overline{\chi}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C, \tag{4.70}$$

whence (4.69d). We also infer that $\|b(\overline{\chi}_\tau)^{1/2}\varepsilon(\overline{\mathbf{u}}_\tau)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \leq C$ which gives, via (2.16) and Korn's inequality, that

$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H_0^1(\Omega;\mathbb{R}^d))} \leq C.$$

Next, along the lines of the **Second a priori estimate**, we test (4.4) by $F'(\vartheta_\tau^k) = (\vartheta_\tau^k)^{\alpha-1}$, with $\alpha \in (0,1)$. Since $F(\vartheta) = \vartheta^\alpha/\alpha$ is concave, by (4.57) we have

$$(\vartheta_\tau^k - \vartheta_\tau^{k-1})F'(\vartheta_\tau^k) \leq F(\vartheta_\tau^k) - F(\vartheta_\tau^{k-1}) \qquad \text{a.e. in } \Omega,$$

therefore we obtain

$$\int_\Omega \left(g_\tau^k + a(\chi_\tau^k)\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \left(1 + \frac{\tau^{1/2}}{2}\right)\left|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right|^2 \right) F'(\vartheta_\tau^k)\, \mathrm{d}x + \int_{\partial\Omega} h_\tau^k F'(\vartheta_\tau^k)\, \mathrm{d}S$$

$$\leq \int_\Omega \left(\frac{F(\vartheta_\tau^k) - F(\vartheta_\tau^{k-1})}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\vartheta_\tau^k F'(\vartheta_\tau^k) + \rho\, \mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\vartheta_\tau^k F'(\vartheta_\tau^k) + \mathsf{K}(\vartheta_\tau^k)\nabla\vartheta_\tau^k\nabla(F'(\vartheta_\tau^k)) \right) \mathrm{d}x.$$
$$\tag{4.71}$$

Then, we multiply (4.71) by τ. Summing over the index k and recalling that $g \geq 0$ and $h \geq 0$, we obtain for all $t \in (0, T]$

$$\frac{4(1-\alpha)}{\alpha^2} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \mathsf{K}(\overline\vartheta_\tau) |\nabla((\overline\vartheta_\tau)^{\alpha/2})|^2 \,\mathrm{d}x\,\mathrm{d}s$$

$$+ \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \left(c_2 |\varepsilon(\partial_t \mathbf{u}_\tau)|^2 F'(\overline\vartheta_\tau) + \left(1 + \frac{\tau^{1/2}}{2}\right) |\partial_t \chi_\tau|^2 F'(\overline\vartheta_\tau) \right) \,\mathrm{d}x\,\mathrm{d}s$$

$$\leq \int_\Omega F(\overline\vartheta_\tau(t)) \,\mathrm{d}x - \int_\Omega F(\vartheta_0) \,\mathrm{d}x + \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \left(\partial_t \chi_\tau \overline\vartheta_\tau F'(\overline\vartheta_\tau) + \rho \operatorname{div}(\partial_t \mathbf{u}_\tau) \overline\vartheta_\tau F'(\overline\vartheta_\tau) \right) \,\mathrm{d}x\,\mathrm{d}s \,.$$

Starting from this inequality, we develop calculations completely analogous to the ones in Section 3 **for the Second a priori estimate**. In particular, we conclude that

$$\int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \mathsf{K}(\overline\vartheta_\tau) |\nabla((\overline\vartheta_\tau)^{\alpha/2})|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C \,. \tag{4.72}$$

The same calculations as for the **Third estimate** allow us then to deduce from (4.72) and (4.70) estimate (4.69g). As a byproduct of these calculations, we again have for all $\alpha \in [1/2, 1)$

$$\|(\overline\vartheta_\tau)^{(\kappa-\alpha)/2}\|_{L^2(0,T;H^1(\Omega))}, \ \|(\overline\vartheta_\tau)^{(\kappa+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} \leq C \,. \tag{4.73}$$

Moreover, since

$$\overline\vartheta_\tau(t) \geq \underline\vartheta \qquad \text{a.e. in } \Omega \quad \text{for all } t \in [0, T], \tag{4.74}$$

(with $\underline\vartheta$ from (4.25)), we also have (4.69f).

As for the **Fourth estimate**, we subtract from the discrete total energy inequality (4.56) the discrete heat equation (4.4) multiplied by τ and summed over the index k. Therefore, we obtain for all $t \in [0, T]$

$$\frac{1}{2} \int_\Omega |\partial_t \mathbf{u}_\tau(\bar{\mathsf{t}}_\tau(t))|^2 \,\mathrm{d}x + \int_0^{\bar{\mathsf{t}}_\tau(t)} \mathsf{v}(a(\underline\chi_\tau)\partial_t \mathbf{u}_\tau, \partial_t \mathbf{u}_\tau) \,\mathrm{d}s + \frac{1}{2} \mathsf{e}(b(\overline{X}_\tau(\bar{\mathsf{t}}_\tau(t)))\overline{\mathbf{u}}_\tau(\bar{\mathsf{t}}_\tau(t)), \overline{\mathbf{u}}_\tau(\bar{\mathsf{t}}_\tau(t)))$$

$$+ \left(1 + \frac{\tau^{1/2}}{2}\right) \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\partial_t \chi_\tau|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \frac{1}{p} |\nabla \overline{X}_\tau(\bar{\mathsf{t}}_\tau(t))|^p + W(\overline{X}_\tau(\bar{\mathsf{t}}_\tau(t))) \,\mathrm{d}x$$

$$= I_0 + \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \overline\vartheta_\tau \left(\rho \operatorname{div}(\partial_t \mathbf{u}_\tau) + \partial_t \chi_\tau\right) \,\mathrm{d}x\,\mathrm{d}s + \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \overline{\mathbf{f}}_\tau \cdot \partial_t \mathbf{u}_\tau \,\mathrm{d}x\,\mathrm{d}s \,,$$

where we have used the place-holder $I_0 = \frac{1}{2}\mathsf{e}(b(\chi_0)\mathbf{u}_0, \mathbf{u}_0) + \int_\Omega(\frac{1}{2}|\mathbf{v}_0|^2 + \frac{1}{p}|\nabla \chi_0|^p + W(\chi_0)) \,\mathrm{d}x$. Exploiting (2.21) and estimate (4.69g), we control the second term on the right-hand side with $\int_0^t \int_\Omega |\partial_t \chi_\tau|^2 \,\mathrm{d}x\,\mathrm{d}s$ and the second term on the left-hand side, which bounds $\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\partial_t \mathbf{u}_\tau\|^2_{H^1(\Omega;\mathbb{R}^d)} \,\mathrm{d}s$ thanks to (2.5). Therefore, we conclude that $\|\partial_t \mathbf{u}_\tau\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))} \leq C$, as well as estimate (4.69e).

The **Fifth estimate** is performed on the time-discretization scheme by testing (4.5) by $-\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}))$. For all the calculations, we refer to [28, (3.61)–(3.67)]: therein, the equation for $\mathbf{u}$ was the same as our own (1.2), but the elasticity and viscosity tensors $\mathbb{E}$ and $\mathbb{V}$ were assumed to be independent of the space variable x. Nonetheless, the computations from [28] carry over to the present setting, cf. also the formal calculations for the **Fourth a priori estimate** in Sec. 3. Therefore, we conclude estimates (4.69a) and (4.69b). A comparison argument in (4.5), joint with (2.8b), yields (4.69c).

In order to render the **Sixth estimate** in the time discrete setting, let us fix a partition $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_J = T$ of the interval $[0, T]$. Preliminarily, from the discrete entropy inequality (4.55), written on the interval $[\sigma_{i-1}, \sigma_i]$ and for a constant-in-time test function $\varphi \in W^{1,d+\epsilon}(\Omega)$ for some $\epsilon > 0$, we deduce that

$$\int_\Omega (\ell_i - \ell_{i-1})\varphi \,\mathrm{d}x + \Lambda_i(\varphi) \geq 0 \qquad \text{for all } \varphi \in W^{1,d+\epsilon}_+(\Omega), \tag{4.75}$$

$$\int_\Omega (\ell_{i-1} - \ell_i)\varphi \,\mathrm{d}x - \Lambda_i(\varphi) \geq 0 \qquad \text{for all } \varphi \in W^{1,d+\epsilon}_-(\Omega), \tag{4.76}$$

where we have used the place-holders

$$\ell_i = \log(\overline{\vartheta}_\tau(\sigma_i)) + \overline{X}_\tau(\sigma_i),$$

$$\Lambda_i(\varphi) = \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_\Omega \mathsf{K}(\overline{\vartheta}_\tau) \nabla \log(\overline{\vartheta}_\tau) \cdot \nabla \varphi \, dx \, dr - \rho \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_\Omega \mathrm{div}(\partial_t \mathbf{u}_\tau) \varphi \, dx \, dr$$

$$- \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_\Omega \mathsf{K}(\overline{\vartheta}_\tau) \frac{\varphi}{\overline{\vartheta}_\tau} \nabla(\log(\overline{\vartheta}_\tau)) \nabla \overline{\vartheta}_\tau \, dx \, dr - \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_{\partial\Omega} \overline{h}_\tau \frac{\varphi}{\overline{\vartheta}_\tau} \, dS \, dr$$

$$- \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_\Omega \left(\overline{g}_\tau + a(\underline{X}_\tau)\varepsilon(\partial_t \mathbf{u}_\tau)\mathbb{V}\varepsilon(\partial_t \mathbf{u}_\tau) + \left(1 + \frac{\tau^{1/2}}{2}\right) |\partial_t X_\tau|^2 \right) \frac{\varphi}{\overline{\vartheta}_\tau} \, dx \, dr.$$

For later use, we also introduce the place-holder

$$\mathcal{R}_\tau := \rho \, \mathrm{div}(\partial_t \mathbf{u}_\tau) + \mathsf{K}(\overline{\vartheta}_\tau) |\nabla(\log(\overline{\vartheta}_\tau))|^2 + \left(\overline{g}_\tau + a(\underline{X}_\tau)\varepsilon(\partial_t \mathbf{u}_\tau)\mathbb{V}\varepsilon(\partial_t \mathbf{u}_\tau) + \left(1 + \frac{\tau^{1/2}}{2}\right) |\partial_t X_\tau|^2 \right) \frac{1}{\overline{\vartheta}_\tau},$$

such that

$$\Lambda_i(\varphi) = \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_\Omega \left(\mathsf{K}(\overline{\vartheta}_\tau) \nabla \log(\overline{\vartheta}_\tau) \cdot \nabla \varphi - \mathcal{R}_\tau \varphi \right) dx \, dr - \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_{\partial\Omega} \overline{h}_\tau \frac{\varphi}{\overline{\vartheta}_\tau} \, dS \, dr. \qquad (4.77)$$

We also deduce from (4.75) with $\varphi \equiv 1$ that

$$\int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_\Omega \left(\frac{\ell_i - \ell_{i-1}}{\overline{t}_\tau(\sigma_i) - \overline{t}_\tau(\sigma_{i-1})} - \mathcal{R}_\tau \right) dx \, dr \geq \int_{\overline{t}_\tau(\sigma_{i-1})}^{\overline{t}_\tau(\sigma_i)} \int_{\partial\Omega} \overline{h}_\tau \frac{1}{\overline{\vartheta}_\tau} \, dS \, dr \geq 0. \qquad (4.78)$$

We now estimate the total variation of $(\log(\overline{\vartheta}_\tau) + \overline{X}_\tau)$ with values in $W^{1,d+\epsilon}(\Omega)^*$, i.e.

$$\mathrm{Var}_{W^{1,d+\epsilon}(\Omega)^*}(\log(\overline{\vartheta}_\tau) + \overline{X}_\tau; [0, T])$$

$$= \sup_{0=\sigma_0 < \sigma_1 < ... < \sigma_J = T} \sum_{i=1}^J \|(\log(\overline{\vartheta}_\tau(\sigma_i)) + \overline{X}_\tau(\sigma_i)) - (\log(\overline{\vartheta}_\tau(\sigma_{i-1})) + \overline{X}_\tau(\sigma_{i-1}))\|_{W^{1,d+\epsilon}(\Omega)^*}$$

by proceeding as follows. We observe that for every fixed $\varphi \in W^{1,d+\epsilon}(\Omega)$ with $\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1$, there holds

$$\left| \langle \ell_i - \ell_{i-1}, \varphi \rangle_{W^{1,d+\epsilon}(\Omega)} \right| \leq \left| \int_\Omega (\ell_i - \ell_{i-1})\varphi^+ \, dx + \Lambda_i(\varphi^+) \right| + |\Lambda_i(\varphi^+)| + \left| \int_\Omega (\ell_{i-1} - \ell_i)(-\varphi^-) \, dx - \Lambda_i(-\varphi^-) \right|$$

$$+ |\Lambda_i(\varphi^-)|$$

$$= \int_\Omega (\ell_i - \ell_{i-1})|\varphi| \, dx + \Lambda_i(|\varphi|) + |\Lambda_i(\varphi^+)| + |\Lambda_i(\varphi^-)|,$$

$$(4.79)$$

where φ^+ (φ^-, resp.) denotes the positive (negative) part of φ. The last equality ensues from (4.75)–(4.76), allowing us to remove the absolute values, and from the linearity of the map $\varphi \mapsto \Lambda_i(\varphi)$, yielding $\Lambda_i(\varphi^+) -$

$\Lambda_i(-\varphi^-) = \Lambda_i(|\varphi|)$. Therefore,

$$
\sum_{i=1}^{J} \|\ell_i - \ell_{i-1}\|_{W^{1,d+\epsilon}(\Omega)^*}
$$

$$
= \sum_{i=1}^{J} \sup_{\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1} \left| \langle \ell_i - \ell_{i-1}, \varphi \rangle_{W^{1,d+\epsilon}(\Omega)} \right|
$$

$$
\overset{(1)}{\leq} \sum_{i=1}^{J} \sup_{\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1} \int_{\Omega} (\ell_i - \ell_{i-1}) |\varphi| \, \mathrm{d}x + \Lambda_i(|\varphi|) + |\Lambda_i(\varphi^+)| + |\Lambda_i(\varphi^-)| \tag{4.80}
$$

$$
\overset{(2)}{\leq} \sum_{i=1}^{J} \sup_{\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1} \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \left(\frac{\ell_i - \ell_{i-1}}{\bar{\mathfrak{t}}_\tau(\sigma_i) - \bar{\mathfrak{t}}_\tau(\sigma_{i-1})} - \mathcal{R}_\tau \right) |\varphi| \, \mathrm{d}x \, \mathrm{d}r
$$

$$
+ \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \mathsf{K}(\overline{\vartheta}_\tau) \nabla \log(\overline{\vartheta}_\tau) \cdot \nabla(|\varphi|) \, \mathrm{d}x \, \mathrm{d}r - \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\partial\Omega} \overline{h}_\tau \frac{|\varphi|}{\overline{\vartheta}_\tau} \, \mathrm{d}S \, \mathrm{d}r + |\Lambda_i(\varphi^+)| + |\Lambda_i(\varphi^-)|,
$$

where (1) follows from (4.79) and (2) follows from rewriting $\Lambda_i(|\varphi|)$ by means of (4.77). Then, we continue the above chain of inequalities by observing that

$$
\int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \left(\frac{\ell_i - \ell_{i-1}}{\bar{\mathfrak{t}}_\tau(\sigma_i) - \bar{\mathfrak{t}}_\tau(\sigma_{i-1})} - \mathcal{R}_\tau \right) |\varphi| \, \mathrm{d}x \, \mathrm{d}r
$$

$$
\leq \|\varphi\|_{L^\infty(\Omega)} \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \left(\frac{\ell_i - \ell_{i-1}}{\bar{\mathfrak{t}}_\tau(\sigma_i) - \bar{\mathfrak{t}}_\tau(\sigma_{i-1})} - \mathcal{R}_\tau \right) \, \mathrm{d}x \, \mathrm{d}r \tag{4.81}
$$

$$
\leq C \left(\int_{\Omega} (\ell_i - \ell_{i-1}) \, \mathrm{d}x - \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \mathcal{R}_\tau \, \mathrm{d}x \, \mathrm{d}r \right) \quad \text{for all } \varphi \in W^{1,d+\epsilon}(\Omega) \text{ with } \|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1
$$

where the first inequality follows from (4.78), and the second one from the continuous embedding $W^{1,d+\epsilon}(\Omega) \subset L^\infty(\Omega)$. We then estimate (cf. in particular (3.26))

$$
\sum_{i=1}^{J} \left| \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \mathcal{R}_\tau \, \mathrm{d}x \, \mathrm{d}r \right| \leq C \int_0^T \left(\|\partial_t \mathbf{u}_\tau\|_{H^1(\Omega;\mathbb{R}^d)} + \int_{\Omega} |\overline{\vartheta}_\tau|^{\kappa+\alpha-2} |\nabla\overline{\vartheta}_\tau|^2 \, \mathrm{d}x + \int_{\Omega} |\nabla\overline{\vartheta}_\tau|^2 \, \mathrm{d}x \right.
$$

$$
\left. + \|\overline{g}_\tau\|_{L^1(\Omega)} + \|\varepsilon(\partial_t\mathbf{u}_\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}^2 + \|\partial_t\chi_\tau\|_{L^2(\Omega)}^2 \right) \mathrm{d}s, \tag{4.82}
$$

where we have used the fact that $|1/\overline{\vartheta}_\tau| \leq C$ a.e. in $\Omega \times (0,T)$ by (4.74). We also estimate

$$
\sum_{i=1}^{J} \sup_{\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1} \left| \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\Omega} \mathsf{K}(\overline{\vartheta}_\tau) \nabla \log(\overline{\vartheta}_\tau) \cdot \nabla(|\varphi|) \, \mathrm{d}x \, \mathrm{d}r \right|
$$

$$
\leq \sum_{i=1}^{J} \sup_{\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \leq 1} \|\varphi\|_{W^{1,3}(\Omega)} \int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \|(\overline{\vartheta}_\tau)^{(\kappa+\alpha-2)/2} \nabla\overline{\vartheta}_\tau\|_{L^2(\Omega;\mathbb{R}^d)} \|(\overline{\vartheta}_\tau)^{(\kappa-\alpha)/2}\|_{L^6(\Omega)} \, \mathrm{d}s \tag{4.83}
$$

$$
\leq C \int_0^T \|(\overline{\vartheta}_\tau)^{(\kappa+\alpha-2)/2} \nabla\overline{\vartheta}_\tau\|_{L^2(\Omega;\mathbb{R}^d)} \|(\overline{\vartheta}_\tau)^{(\kappa-\alpha)/2}\|_{L^6(\Omega)} \, \mathrm{d}s
$$

and proceed analogously for the term $\int_{\bar{\mathfrak{t}}_\tau(\sigma_{i-1})}^{\bar{\mathfrak{t}}_\tau(\sigma_i)} \int_{\partial\Omega} \overline{h}_\tau \frac{|\varphi|}{\overline{\vartheta}_\tau} \, \mathrm{d}S \, \mathrm{d}r$, relying on (4.49) and again on the fact that $|1/\overline{\vartheta}_\tau| \leq C$ a.e. in $\Omega \times (0,T)$ by (4.74). With the same calculations as throughout (4.82)–(4.83) we also estimate the terms $|\Lambda_i(\varphi^+)|$ and $|\Lambda_i(\varphi^-)|$, exploiting (4.77). Inserting (4.81)–(4.83) into (4.80), we then get

$$
\sum_{i=1}^{J} \|\ell_i - \ell_{i-1}\|_{W^{1,d+\epsilon}(\Omega)^*} \overset{(1)}{\leq} C \int_{\Omega} \left(\log(\overline{\vartheta}_\tau(T)) + \overline{\chi}_\tau(T) - \log(\vartheta_0) - \chi_0 \right) \mathrm{d}x + C \overset{(2)}{\leq} C,
$$

where (1) follows from the previously proved estimates (4.69b), (4.69d), (4.69e), (4.69g), (4.72), and (4.49). Finally, (2) is due to (4.69d) and to the fact that $|\log(\overline{\vartheta}_\tau(t))| \leq C\left(|\overline{\vartheta}_\tau(t)| + \frac{1}{|\overline{\vartheta}_\tau(t)|}\right) \leq C\left(|\overline{\vartheta}_\tau(t)| + \frac{1}{\underline{\vartheta}(T)}\right)$ a.e. in Ω for all $t \in [0,T]$ thanks to (4.74). Also view of (4.69f), we ultimately conclude that

$$\|\log(\overline{\vartheta}_\tau) + \overline{\chi}_\tau\|_{\mathrm{BV}([0,T];W^{1,d+\epsilon}(\Omega)^*)} \leq C$$

for all $\epsilon > 0$. Therefore, (4.69h) follows, on account of (4.69e).

Under the additional Hypothesis (V), the same comparison argument in (4.4) as for the **Eighth estimate** yields (4.69i).

For the **Ninth estimate**, in the case $\mu = 0$ we perform a comparison in (4.6). Based on (4.69a), (4.69d), (4.69e), and (4.69g) we conclude

$$\sup_{\tau>0}\left(\|A_p(\overline{\chi}_\tau)\|_{L^2(0,T;L^2(\Omega))} + \|\overline{\xi}_\tau\|_{L^2(0,T;L^2(\Omega))}\right) \leq C$$

whence (4.69j) by the aforementioned regularity results from [31]. □

5. Passage to the limit

Let $(\overline{\vartheta}_\tau, \vartheta_\tau, \overline{\mathbf{u}}_\tau, \underline{\mathbf{u}}_\tau, \mathbf{u}_\tau, \widehat{\mathbf{u}}_\tau, \overline{\chi}_\tau, \underline{\chi}_\tau, \chi_\tau)_\tau$ be a family of approximate solutions, fulfilling the discrete entropy inequality (4.55) and the discrete total energy inequality (4.56): its existence is ensured by Proposition 4.8. We derive a preliminary compactness result, relying on the a priori estimates from Prop. 4.10.

Lemma 5.1 (Compactness, $\mu \in \{0,1\}$). *Under **Hypotheses (I)–(III)** and conditions (2.21)–(2.26) on the data $\mathbf{f}, g, h, \vartheta_0, \mathbf{u}_0, \mathbf{v}_0, \chi_0$, for any sequence $(\tau_k)_k \subset (0,+\infty)$ with $\tau_k \downarrow 0$ as $k \to \infty$, there exist a (not relabeled) subsequence, and a triple $(\vartheta, \mathbf{u}, \chi)$ such that the following convergences hold*

$$\mathbf{u}_{\tau_k} \overset{*}{\rightharpoonup} \mathbf{u} \qquad \text{in } H^1(0,T;H_0^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d)), \tag{5.1}$$

$$\overline{\mathbf{u}}_{\tau_k}, \underline{\mathbf{u}}_{\tau_k} \to \mathbf{u} \qquad \text{in } L^\infty(0,T;H^{2-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{5.2}$$

$$\mathbf{u}_{\tau_k} \to \mathbf{u} \qquad \text{in } \mathrm{C}^0([0,T];H^{2-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{5.3}$$

$$\partial_t \widehat{\mathbf{u}}_{\tau_k} \rightharpoonup \mathbf{u}_{tt} \qquad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{5.4}$$

$$\partial_t \mathbf{u}_{\tau_k} \to \mathbf{u}_t \qquad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), \tag{5.5}$$

$$\overline{\chi}_{\tau_k}, \underline{\chi}_{\tau_k}, \chi_{\tau_k} \rightharpoonup \chi \qquad \text{in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{5.6}$$

$$\chi_{\tau_k} \to \chi \qquad \text{in } \mathrm{C}^0([0,T];X) \text{ for all } X \text{ such that } W^{1,p}(\Omega) \Subset X \subset L^2(\Omega), \tag{5.7}$$

$$\overline{\vartheta}_{\tau_k} \rightharpoonup \vartheta \qquad \text{in } L^2(0,T;H^1(\Omega)), \tag{5.8}$$

$$\log(\overline{\vartheta}_{\tau_k}) \rightharpoonup \log(\vartheta) \qquad \text{in } L^2(0,T;H^1(\Omega)), \tag{5.9}$$

$$\log(\overline{\vartheta}_{\tau_k}) \to \log(\vartheta) \qquad \text{in } L^2(0,T;L^s(\Omega)) \text{ for all } s \in [1,6) \text{ if } d=3, \text{ and all } s \in [1,\infty) \text{ if } d=2, \tag{5.10}$$

$$\log(\overline{\vartheta}_{\tau_k}(t)) \rightharpoonup \log(\vartheta(t)) \qquad \text{in } W^{1,d+\epsilon}(\Omega)^* \text{ for all } \epsilon > 0 \text{ and for all } t \in [0,T], \tag{5.11}$$

$$\overline{\vartheta}_{\tau_k} \to \vartheta \qquad \text{in } L^h(\Omega \times (0,T)) \text{ for all } h \in [1,8/3) \text{ for } d=3 \text{ and all } h \in [1,3) \text{ if } d=2, \tag{5.12}$$

and ϑ also fulfills

$$\vartheta \in L^\infty(0,T;L^1(\Omega)), \quad \log\vartheta \in \mathrm{BV}([0,T];W^{1,d+\epsilon}(\Omega)^*) \text{ for all } \epsilon > 0, \quad \vartheta \geq \underline{\vartheta} \text{ a.e. in } \Omega \times (0,T) \tag{5.13}$$

(with $\underline{\vartheta}$ from (4.10)).

*Under the additional **Hypothesis (V)**, we also have $\vartheta \in \mathrm{BV}([0,T];W^{2,d+\epsilon}(\Omega)^*)$ for all $\epsilon > 0$, and*

$$\overline{\vartheta}_{\tau_k} \to \vartheta \qquad \text{in } L^2(0,T;Y) \text{ for all } Y \text{ such that } H^1(\Omega) \Subset Y \subset W^{2,d+\epsilon}(\Omega)^*, \tag{5.14}$$

$$\overline{\vartheta}_{\tau_k}(t) \to \vartheta(t) \qquad \text{in } W^{2,d+\epsilon}(\Omega)^* \text{ for all } t \in [0,T]. \tag{5.15}$$

Proof. Due to due to estimates (4.69b) and (4.69c), there holds

$$\begin{aligned}
\|\mathbf{u}_\tau - \overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H_0^2(\Omega;\mathbb{R}^d))} &\leq \tau^{1/2}\|\partial_t\mathbf{u}_\tau\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^d))} \leq S\tau^{1/2}, \\
\|\widehat{\mathbf{u}}_\tau - \partial_t\mathbf{u}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} &\leq \tau^{1/2}\|\partial_t\widehat{\mathbf{u}}_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S\tau^{1/2}.
\end{aligned} \tag{5.16}$$

Taking into account (4.69a), (4.69b), (4.69c), and applying well-known weak and strong compactness results (for the latter, cf. e.g. [32]), we conclude convergences (5.1)–(5.5). The same kind of arguments yields (5.6)–(5.7) on account of estimates (4.69d) and (4.69e). The bound (4.69g) gives the weak convergence (5.8). Since the family $(\log(\overline{\vartheta}_\tau))_\tau$ is bounded in $L^2(0,T;H^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,d+\epsilon}(\Omega)^*)$ for all $\epsilon > 0$, an Aubin-Lions type compactness result for BV-functions (see, for instance, [32, Cor.4] or [29, Chap. 7, Cor. 4.9]) ensures that, up to a subsequence, $\log(\overline{\vartheta}_{\tau_k})$ converges to some λ in $L^2(0,T;Z)$ for every Banach space Z such that $H^1(\Omega) \Subset Z \subset W^{1,d+\epsilon}(\Omega)^*$. Therefore, $\log(\overline{\vartheta}_{\tau_k})$ converges to λ pointwise almost everywhere in $\Omega \times (0,T)$ and accordingly $\overline{\vartheta}_{\tau_k}$ converges to e^λ. Then, in view of (5.8), $\lambda = \log(\vartheta)$, and convergences (5.9) and (5.10) ensue. The BV-compactness result [22, Thm. 6.1] also ensures that $\log(\vartheta) \in \mathrm{BV}([0,T];W^{1,d+\epsilon}(\Omega)^*)$, and the additional weak convergence (5.11). With a lower semicontinuity argument one also has that $\vartheta \in L^\infty(0,T;L^1(\Omega))$, and convergence (5.12) follows from an interpolation argument (cf. (3.15)). Relying on this and on the approximate positivity property (4.74), we also conclude the last of (5.13).

Under the additional Hypothesis (V), we also dispose of the BV-estimate (4.69i) for $\overline{\vartheta}_\tau$. Combining this with (4.69g) and applying the aforementioned compactness results from [32] and [22], we conclude (5.14)–(5.15). $\quad\square$

We are now in the position to develop the **<u>Proof of Theorem 2.5</u>**, by passing to the limit in the time-discrete scheme set up in Sec. 4. Let $(\tau_k)_k$ be a vanishing sequence of time-steps, and let

$$(\overline{\vartheta}_{\tau_k}, \vartheta_{\tau_k}, \overline{\mathbf{u}}_{\tau_k}, \underline{\mathbf{u}}_{\tau_k}, \mathbf{u}_{\tau_k}, \widehat{\mathbf{u}}_{\tau_k}, \overline{\chi}_{\tau_k}, \underline{\chi}_{\tau_k}, \chi_{\tau_k})_k$$

be a sequence of approximate solutions. We can exploit the compactness results from Lemma 5.1. We split the limit passage in the following steps.

Ad the weak momentum equation (2.40). Relying on convergences (5.1), (5.4)–(5.5), (5.7) and (5.8), as well as (4.47) for $(\overline{\mathbf{f}}_{\tau_k})_k$, we pass to the limit in (4.51) and conclude that the triple $(\vartheta, \mathbf{u}, \chi)$ fulfills (2.40).

Ad the weak formulation (2.41)–(2.44) **of the equation for χ, $\mu = 1$.** The argument for obtaining (2.41)–(2.44) in the limit follows exactly the same lines as the proof of [16, Thms. 4.4, 4.6] (see also [28, Thm. 3]). Therefore we only recapitulate it, referring to the latter papers for all details.

First of all, as we have pointed out in the proof of Proposition 4.8, (4.6) can be interpreted as the Euler-Lagrange equation for the minimum problem (4.58), i.e. (recall that here $\mu = 1$ and that $\widehat{\alpha} = I_{(-\infty,0]}$ and $\widehat{\beta} = I_{[0,+\infty)}$)

$$\begin{aligned}
\min_{\chi \in W^{1,p}(\Omega)} \Bigg\{ \int_\Omega \Bigg(&\frac{\tau^{3/2}}{2}\left|\frac{\chi - \chi_\tau^{k-1}}{\tau}\right|^2 + \left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right)\chi + I_{(-\infty,0]}\left(\frac{\chi - \chi_\tau^{k-1}}{\tau}\right) + \frac{|\nabla\chi|^p}{p} + I_{[0,+\infty)}(\chi) \\
&+ \widehat{\gamma}(\chi) + b(\chi)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathbb{E}\varepsilon(\mathbf{u}_\tau^{k-1})}{2} - \vartheta_\tau^k\chi \Bigg)\,\mathrm{d}x \Bigg\}
\end{aligned} \tag{5.17}$$

Writing necessary optimality conditions for the minimum problem (5.17), with the very same calculations as for [28, Thm. 3], we arrive at

$$\int_\Omega \Big(\partial_t\chi_\tau(t)\psi + \sqrt{\tau}\partial_t\chi_\tau(t)\psi + |\nabla\overline{\chi}_\tau(t)|^{p-2}\nabla\overline{\chi}_\tau(t)\cdot\nabla\psi + \gamma(\overline{\chi}_\tau(t))\psi + \overline{j}_\tau(t)\psi\Big)\,\mathrm{d}x \geq 0 \tag{5.18}$$

for all $t \in [0,T]$ and all $\psi \in W^{1,p}(\Omega)$ s.t. there exists $\nu > 0$ with $0 \leq \nu\psi + \overline{\chi}_\tau(t) \leq \underline{\chi}_\tau(t)$ a.e. in Ω,

where where we have used the place-holder

$$\overline{j}_\tau := b'(\overline{\chi}_\tau)\frac{\varepsilon(\underline{\mathbf{u}}_\tau)\mathbb{E}\varepsilon(\underline{\mathbf{u}}_\tau)}{2} - \overline{\vartheta}_\tau. \tag{5.19}$$

Choosing $\psi = -\partial_t \mathcal{X}_\tau(t)$ in (5.18) and and summing over the index k we deduce the *discrete version* of the energy inequality (2.44) for all $0 \le s \le t \le T$, viz.

$$
\int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega (1 + \tau^{1/2}) |\partial_t \mathcal{X}_\tau|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_\Omega \left(\frac{1}{p} |\nabla \overline{\mathcal{X}}_\tau(\bar{\mathfrak{t}}_\tau(t)))|^p + W(\mathcal{X}_\tau(\mathfrak{t}_\tau(t))) \right) \, \mathrm{d}x
$$
$$
\le \int_\Omega \left(\frac{1}{p} |\nabla \overline{\mathcal{X}}_\tau(\bar{\mathfrak{t}}_\tau(s))|^p + W(\mathcal{X}_\tau(\mathfrak{t}_\tau(s))) \right) \, \mathrm{d}x \tag{5.20}
$$
$$
+ \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \partial_t \mathcal{X}_\tau \left(-b'(\overline{\mathcal{X}}_\tau) \frac{\varepsilon(\mathbf{u}_\tau)\mathbb{E}\varepsilon(\mathbf{u}_\tau)}{2} + \overline{\vartheta}_\tau \right) \, \mathrm{d}x \, \mathrm{d}r + C\tau \|\partial_t \mathcal{X}_\tau\|^2_{L^2(0,T;L^2(\Omega))} ,
$$

where we have used that

$$
\int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \gamma(\overline{\mathcal{X}}_\tau) \partial_t \mathcal{X}_\tau \, \mathrm{d}x \, \mathrm{d}r = \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \gamma(\mathcal{X}_\tau) \partial_t \mathcal{X}_\tau \, \mathrm{d}x \, \mathrm{d}r + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \left(\gamma(\overline{\mathcal{X}}_\tau) - \gamma(\mathcal{X}_\tau) \right) \partial_t \mathcal{X}_\tau \, \mathrm{d}x \, \mathrm{d}r \doteq I_1 + I_2
$$

and that, by the chain rule,

$$
I_1 = \int_\Omega \widehat{\gamma}(\mathcal{X}_\tau(\bar{\mathfrak{t}}_\tau(t))) \, \mathrm{d}x - \int_\Omega \widehat{\gamma}(\mathcal{X}_\tau(\bar{\mathfrak{t}}_\tau(s))) \, \mathrm{d}x = \int_\Omega W(\mathcal{X}_\tau(\bar{\mathfrak{t}}_\tau(t))) \, \mathrm{d}x - \int_\Omega W(\mathcal{X}_\tau \bar{\mathfrak{t}}_\tau(s)) \, \mathrm{d}x
$$

(due to $\widehat{\beta} = I_{[0,+\infty)}$), while

$$
I_2 \le \|\partial_t \mathcal{X}_\tau\|_{L^2(0,T;L^2(\Omega))} \|\gamma(\mathcal{X}_\tau) - \gamma(\overline{\mathcal{X}}_\tau)\|_{L^2(0,T;L^2(\Omega))} \le C\tau \|\partial_t \mathcal{X}_\tau\|^2_{L^2(0,T;L^2(\Omega))}
$$

thanks to the Lipschitz continuity of γ.

Second, repeating the "recovery sequence" argument from [16, proof of Thm. 4.4], we improve the weak convergence (5.6) to

$$
\overline{\mathcal{X}}_{\tau_k} \to \mathcal{X} \quad \text{in } L^p(0,T;W^{1,p}(\Omega)). \tag{5.21}
$$

We refer to [16] and [28] for all the related calculations.

We are now in the position to taking the limit as $\tau_k \downarrow 0$ in the approximate energy inequality (5.20). We pass to the limit on the left-hand side by lower semicontinuity, relying on convergences (5.6) and (5.21). For the right-hand side, we exploit the latter strong convergence as well as (5.7), yielding that $\overline{\mathcal{X}}_{\tau_k}(s) \to \mathcal{X}(s)$ in $W^{1,p}(\Omega)$, whence $\overline{\mathcal{X}}_{\tau_k}(s) \to \mathcal{X}(s)$ in $C^0(\overline{\Omega})$, for almost all $s \in (0,T)$. It follows from $\widehat{\gamma} \in C^2(\mathbb{R})$ that $\widehat{\gamma}$ has at most quadratic growth on bounded subsets of $\mathbb{R}$. We combine this with the uniform convergence of $(\overline{\mathcal{X}}_{\tau_k}(s))_k$ to conclude that $\int_\Omega \widehat{\gamma}(\overline{\mathcal{X}}_{\tau_k}(s)) \, \mathrm{d}x \to \int_\Omega \widehat{\gamma}(\mathcal{X}(s)) \, \mathrm{d}x$ for almost all $s \in (0,T)$. Since $\widehat{\beta} = I_{[0,+\infty)}$, we have $\int_\Omega W(\overline{\mathcal{X}}_{\tau_k}(s)) \, \mathrm{d}x \to \int_\Omega W(\mathcal{X}(s)) \, \mathrm{d}x$ for almost all $s \in (0,T)$. Since $(\mathcal{X}_\tau)_\tau$ is bounded in $H^1(0,T;L^2(\Omega))$, we also have

$$
\sqrt{\tau} \partial_t \mathcal{X}_{\tau_k} \to 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{5.22}
$$

Combining the weak convergence (5.6) with the strong ones (5.2), (5.7), and (5.12), we also pass to the limit in the second integral term on the right-hand side of (5.20). The last summand obviously tends to zero. Therefore, we conclude the energy inequality (2.44).

Clearly, convergence (5.6) and the fact that $\partial_t \mathcal{X}_\tau \le 0$ a.e. in $\Omega \times (0,T)$ ensure that $\mathcal{X}_t \le 0$ .e. in $\Omega \times (0,T)$, i.e. (2.41). To obtain the variational inequality (2.42), together with (2.43), we proceed exaclty as in [16, 28]. The main steps are as follows: passing to the limit in (5.18) as $\tau_k \downarrow 0$ with suitable test functions from [16, Lemma 5.2], also relying on (5.22), we prove that for almost all $t \in (0,T)$

$$
\int_\Omega \left(\mathcal{X}_t(t)\tilde{\psi} + |\nabla \mathcal{X}(t)|^{p-2} \nabla \mathcal{X}(t) \cdot \nabla \tilde{\psi} + \gamma(\mathcal{X}(t))\tilde{\varphi} + b'(\mathcal{X}(t)) \frac{\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))}{2} \tilde{\psi} - \vartheta(t)\tilde{\psi} \right) \, \mathrm{d}x \ge 0
$$
$$
\text{for all } \tilde{\psi} \in W^{1,p}_-(\Omega) \text{ with } \{\tilde{\psi} = 0\} \supset \{\mathcal{X}(t) = 0\}.
$$

From this, arguing as in the proof of [16, Thm. 4.4] we deduce that for almost all $t \in (0, T)$

$$\int_\Omega \left(\chi_t(t)\psi + |\nabla\chi(t))|^{p-2}\nabla\chi(t) \cdot \nabla\psi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))}{2}\psi - \vartheta(t)\psi \right) \mathrm{d}x$$

$$\geq \int_{\{\chi(t)=0\}} \left(\gamma(\chi(t)) + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))}{2} - \vartheta(t) \right)^+ \psi \, \mathrm{d}x \quad \text{for all } \psi \in W^{1,p}_-(\Omega). \tag{5.23}$$

Relying on (5.23), it is possible to check that the function ξ from (2.47) complies with (2.42) and (2.43).

Ad the entropy inequality (2.37). Let us fix a test function $\varphi \in C^0([0, T]; W^{1,d+\epsilon}(\Omega)) \cap H^1(0, T; L^{6/5}(\Omega))$ (for some $\epsilon > 0$), for the entropy inequality (2.37). We pass to the limit as $\tau_k \downarrow 0$ in the discrete entropy inequality (4.55), with the discrete test functions constructed from φ in (4.53). In order to pass to the limit in the first two integral terms on the left-hand side of (4.55), we combine convergences (5.1), (5.7), and (5.10), with the convergence (4.54) for the test functions. In order to deal with the last integral on the left-hand side, we observe that the family

$$(\mathsf{K}(\overline{\vartheta}_\tau)\nabla\log(\overline{\vartheta}_\tau))_\tau \text{ is bounded in } L^{1+\delta}(Q; \mathbb{R}^d) \text{ for some } \delta > 0. \tag{5.24}$$

Indeed, the growth condition (2.15) implies that

$$|\mathsf{K}(\overline{\vartheta}_\tau)\nabla\log(\overline{\vartheta}_\tau)| \leq C\left(|\overline{\vartheta}_\tau|^{\kappa-1} + \frac{1}{\overline{\vartheta}_\tau} \right)|\nabla\overline{\vartheta}_\tau| \leq | \leq C\left(|\overline{\vartheta}_\tau|^{\kappa-1} + \frac{1}{\underline{\vartheta}(T)} \right)|\nabla\overline{\vartheta}_\tau| \qquad \text{a.e. in } \Omega \times (0, T)$$

(also due to (4.74)). Thus, it remains to bound the term $|\overline{\vartheta}_\tau|^{\kappa-1}|\nabla\overline{\vartheta}_\tau|$. To do so, we observe

$$\iint_Q \left(|\overline{\vartheta}_\tau|^{\kappa-1}|\nabla\overline{\vartheta}_\tau| \right)^r \mathrm{d}x \, \mathrm{d}t \leq \|(|\overline{\vartheta}_\tau|^{(\kappa-\alpha)/2})^r\|_{L^{2/(2-r)}(Q)} \|(|\overline{\vartheta}_\tau|^{(\kappa+\alpha-2)/2}|\nabla\overline{\vartheta}_{\tau_k}|^r\|_{L^{2/r}(Q;\mathbb{R}^d)}$$

$$\leq C\|(|\overline{\vartheta}_\tau|^{(\kappa-\alpha)/2})^r\|_{L^{2/(2-r)}(Q)} \tag{5.25}$$

for some $r > 0$ (to be chosen below), where we exploited that $(|\overline{\vartheta}_\tau|^{(\kappa+\alpha-2)/2}\nabla\overline{\vartheta}_\tau)_\tau$ is bounded in $L^2(Q; \mathbb{R}^d)$ thanks to (3.16) (cf. also (3.13)). Indeed the latter estimate yields that $((\overline{\vartheta}_\tau)^{(\kappa+\alpha)/2})_\tau$ is bounded in $L^2(Q)$, hence that $((\overline{\vartheta}_\tau)^{(\kappa-\alpha)/2})_\tau$ is bounded in $L^{2(\kappa+\alpha)/(\kappa-\alpha)}(Q)$. Therefore, it is sufficient to choose in (5.25) r such that $2r/(2-r) = 2(\kappa+\alpha)/(\kappa-\alpha)$, i.e. $r = (\kappa+\alpha)/\kappa$, which is strictly bigger than 1. Therefore, up to some subsequence $\mathsf{K}(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k})$ weakly converges to some η in $L^{1+\delta}(Q; \mathbb{R}^d)$. In order to identify η as $\mathsf{K}(\vartheta)\nabla\log(\vartheta)$, we use these facts. We first show that

$$|\overline{\vartheta}_{\tau_k}|^{(\kappa+\alpha-2)/2}\nabla\overline{\vartheta}_{\tau_k} \rightharpoonup |\vartheta|^{(\kappa+\alpha-2)/2}\nabla\vartheta \qquad \text{in } L^2(Q; \mathbb{R}^d). \tag{5.26}$$

Indeed, on the one hand, (5.8) gives $\nabla\overline{\vartheta}_{\tau_k} \rightharpoonup \nabla\vartheta$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^d))$. On the other hand, the pointwise convergence $\overline{\vartheta}_{\tau_k} \to \vartheta$ a.e. in $\Omega \times (0, T)$ combined with the fact that $(\overline{\vartheta}_{\tau_k})_k$ is bounded in $L^{\kappa+\alpha}(\Omega)$ yields that $\overline{\vartheta}_{\tau_k} \to \vartheta$ in $L^{\kappa+\alpha-\epsilon}(\Omega)$ for all $\epsilon > 0$. Therefore $|\overline{\vartheta}_\tau|^{(\kappa+\alpha-2)/2} \to |\vartheta|^{(\kappa+\alpha-2)/2}$ in $L^{2(\kappa+\alpha)/(\kappa+\alpha-2)-\epsilon}(\Omega)$ for all $\epsilon > 0$. Since $(|\overline{\vartheta}_{\tau_k}|^{(\kappa+\alpha-2)/2}\nabla\overline{\vartheta}_{\tau_k})_k$ is bounded in $L^2(Q; \mathbb{R}^d)$, (5.26) follows. Second, we have that

$$|\overline{\vartheta}_{\tau_k}|^{(\kappa-\alpha)/2} \to \vartheta^{(\kappa-\alpha)/2} \text{ in } L^{2(\kappa+\alpha)/(\kappa-\alpha)-\epsilon}(\Omega) \quad \text{for all } \epsilon > 0, \tag{5.27}$$

again due to the pointwise convergence of $\overline{\vartheta}_{\tau_k}$ and to the fact $(\overline{\vartheta}_{\tau_k})_k$ is bounded in $L^{\kappa+\alpha}(\Omega)$. It follows from (5.26), (5.27) , and the growth condition on K, that

$$\mathsf{K}(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k}) \rightharpoonup \mathsf{K}(\vartheta)\nabla\log(\vartheta) \qquad \text{in } L^{1+\delta}(Q; \mathbb{R}^d). \tag{5.28}$$

This and convergence (4.54) enables us to take the limit in third term on the left-hand side of (4.55). The passage to the limit in the first two integrals on the right-hand side results from convergences (5.7), (5.11), and again (4.54). For the third term, we use that

$$\liminf_{k\to\infty} \int_{\underline{\mathfrak{t}}_{\tau_k}(s)}^{\overline{\mathfrak{t}}_{\tau_k}(t)} \int_\Omega \mathsf{K}(\overline{\vartheta}_{\tau_k}(r))\overline{\varphi}_{\tau_k}(r) \left| \nabla\log(\overline{\vartheta}_{\tau_k}(r)) \right|^2 \mathrm{d}x \, \mathrm{d}r \geq \int_s^t \int_\Omega \mathsf{K}(\vartheta(r))\varphi(r) \left| \nabla\log(\vartheta(r)) \right|^2 \mathrm{d}x \, \mathrm{d}r$$

which results from the weak convergence (5.9), combined with the pointwise convergence $\overline{\vartheta}_{\tau_k} \to \vartheta$ a.e. in $\Omega \times (0,T)$, (4.54) for the discrete test functions, applying the Ioffe theorem [18]. With analogous arguments we pass to the limit in the last two integrals on the right-hand side of (4.55), and therefore conclude (2.37).

Ad the total energy inequality (2.38). It follows from passing to the limit as $\tau_k \downarrow 0$ in the discrete total energy inequality (4.56), based on convergences (4.47)–(4.49) for $\overline{\mathbf{f}}_{\tau_k}$, $\overline{g}_{\tau_k}$, $\overline{h}_{\tau_k}$, and on (5.2), (5.5), (5.7), and (5.12). Observe that convergences (5.2), (5.5), and (5.7) are sufficient to pass to the limit on the left-hand side of (4.56), by lower semicontinuity, *for all* $t \in [0,T]$. However, (5.12) only guarantees that $\overline{\vartheta}_{\tau_k}(t) \to \vartheta(t)$ in $L^1(\Omega)$ *for almost all* $t \in (0,T)$.

Enhanced regularity and improved total energy inequality under Hypothesis (V). If in addition Hyp. (V) holds, in view of Lemma 5.1 ϑ is in $\mathrm{BV}([0,T]; W^{2,d+\epsilon}(\Omega)^*)$ for every $\epsilon > 0$, and the enhanced convergences (5.14) and (5.15) hold. The latter pointwise convergence allows us to pass to the limit on the left-hand side of (4.56) *for all* $t \in [0,T]$. This concludes the proof. ∎

We conclude this section with the **<u>Proof of Theorem 2.8:</u>** Let $(\tau_k)_k$ be a vanishing sequence of time-steps, and $(\overline{\vartheta}_{\tau_k}, \vartheta_{\tau_k}, \overline{\mathbf{u}}_{\tau_k}, \underline{\mathbf{u}}_{\tau_k}, \mathbf{u}_{\tau_k}, \widehat{\mathbf{u}}_{\tau_k}, \overline{X}_{\tau_k}, \underline{X}_{\tau_k}, X_{\tau_k})_k$ be a sequence of approximate solutions; let $(\overline{\xi}_{\tau_k})_k$ be a sequence of selections in $\beta(\overline{X}_{\tau_k})$, such that $(\overline{X}_{\tau_k}, \overline{\xi}_{\tau_k})$ satisfy for all $k \in \mathbb{N}$ the approximate equation (4.52).

In the case $\mu = 0$, in addition to convergences (5.1)–(5.15), estimates (4.69j) yield, up to a subsequence, the further convergences

$$\overline{X}_{\tau_k} \rightharpoonup X \quad \text{in } L^2(0,T; W^{1+\sigma,p}(\Omega)) \text{ for all } 1 \leq \sigma < \frac{1}{p}, \qquad \overline{X}_{\tau_k} \rightharpoonup X \quad \text{in } L^q(0,T; W^{1,p}(\Omega)) \text{ for all } 1 \leq q < \infty. \tag{5.29}$$

Furthermore, there exists $\xi \in L^2(0,T; L^2(\Omega))$ such that

$$\overline{\xi}_{\tau_k} \rightharpoonup \xi \quad \text{in } L^2(0,T; L^2(\Omega)). \tag{5.30}$$

The strong convergence (5.29) and the strong-weak closedness of β (as a maximal monotone operator from $L^2(\Omega)$ to $L^2(\Omega)$) immediately yield that $\xi \in \beta(X)$ a.e. in $\Omega \times (0,T)$.

Therefore, also exploiting convergences (5.1)–(5.8) we pass to the limit in the discrete equation for X (4.52) and immediately conclude that the quadruple $(\vartheta, \mathbf{u}, X, \xi)$ fulfills the pointwise formulation (2.51)–(2.52) of the internal parameter equation (1.3). The proof of the entropy inequality, of the total energy inequality, and of the momentum equation is clearly the same as for Theorem 2.5.

Under the additional Hypothesis (V), as previously seen ϑ is in $\mathrm{BV}([0,T]; W^{2,d+\varepsilon}(\Omega)^*)$. We prove the weak form (2.54) of the heat equation by passing to the limit as $\tau_k \downarrow 0$ in the approximate heat equation (4.50), tested by an arbitrary $\varphi \in \mathrm{C}^0([0,T]; W^{2,d+\epsilon}(\Omega)) \cap H^1(0,T; L^{6/5}(\Omega))$. The passage to the limit in the first three terms on the left-hand side, and on the first two terms on the right-hand side, results from convergences (4.48), (4.49) for $(\overline{g}_{\tau_k})_k$ and $(\overline{h}_{\tau_k})_k$, and from (5.1)–(5.2), (5.5)–(5.8): in particular, we exploit that $\varepsilon(\partial_t \mathbf{u}_{\tau_k}) \mathbb{E} \varepsilon(\partial_t \mathbf{u}_{\tau_k}) \to \varepsilon(\mathbf{u}_t) \mathbb{E} \varepsilon(\mathbf{u}_t)$ *strongly* in $L^1(Q)$ thanks to the strong convergence (5.5).

In order to pass to the limit with the fourth term on the left-hand side of (4.50), we need to derive a finer estimate for $(\mathsf{K}(\overline{\vartheta}_{\tau_k}) \nabla \overline{\vartheta}_{\tau_k})_k$. Arguing as for (3.28) we use that

$$|\mathsf{K}(\overline{\vartheta}_{\tau_k}) \nabla \overline{\vartheta}_{\tau_k}| \leq C |\overline{\vartheta}_{\tau_k}|^{(\kappa-\alpha+2)/2} |\overline{\vartheta}_{\tau_k}|^{(\kappa+\alpha-2)/2} |\nabla \overline{\vartheta}_{\tau_k}| + C |\nabla \overline{\vartheta}_{\tau_k}|.$$

Now, $(\overline{\vartheta}_{\tau_k})^{(\kappa+\alpha-2)/2} \nabla \overline{\vartheta}_{\tau_k}$ is bounded in $L^2(0,T; L^2(\Omega; \mathbb{R}^d))$ (thanks to (4.72)). On the other hand, $(\overline{\vartheta}_{\tau_k})_k$ is bounded in $L^p(Q)$ for all $1 \leq p < 8/3$, in the case $d = 3$ (to which we confine this discussion). Therefore, choosing $\alpha \in [1/2, 1)$ such that $\alpha > \kappa - \frac{2}{3}$ (this can be done since $\kappa < 5/3$ by assumption), we conclude that $((\overline{\vartheta}_{\tau_k})^{(\kappa-\alpha+2)/2})_k$ is bounded in $L^{2+\delta}(Q)$ for some $\delta > 0$. Ultimately, we conclude that $(\mathsf{K}(\overline{\vartheta}_{\tau_k}) \nabla \overline{\vartheta}_{\tau_k})_k$ is bounded in $L^{1+\bar{\delta}}(0,T; L^{1+\bar{\delta}}(\Omega))$ for some $\bar{\delta} > 0$, hence

$$\exists \eta \in L^{1+\bar{\delta}}(0,T; L^{1+\bar{\delta}}(\Omega)) : \qquad \mathsf{K}(\overline{\vartheta}_{\tau_k}) \nabla \overline{\vartheta}_{\tau_k} \rightharpoonup \eta \text{ in } L^{1+\bar{\delta}}(0,T; L^{1+\bar{\delta}}(\Omega)). \tag{5.31}$$

In order to identify the weak limit η, it is sufficient to observe that (cf. [21]) $\mathsf{K}(\overline{\vartheta}_{\tau_k}) \nabla \overline{\vartheta}_{\tau_k} = \nabla \widehat{\mathsf{K}}(\overline{\vartheta}_{\tau_k})$ a.e. in $\Omega \times (0,T)$. Combining the growth property (2.15) of K (where $1 \leq \kappa < 5/3$), with the strong convergence

(5.12) of $\overline{\vartheta}_{\tau_k}$ in $L^p(Q)$ for all $1 \leq p < 8/3$, we ultimately conclude that $(\widehat{\mathsf{K}}(\overline{\vartheta}_{\tau_k}))_k$ strongly converges to $\widehat{\mathsf{K}}(\vartheta)$ in $L^{1+\tilde{\delta}}(Q)$ for some $\tilde{\delta} > 0$. A standard argument then yields

$$\eta = \nabla \widehat{\mathsf{K}}(\vartheta) = \mathsf{K}(\vartheta)\nabla\vartheta \qquad \text{a.e. in } \Omega \times (0,T). \tag{5.32}$$

Combining (5.31) and (5.32) leads to

$$\int_0^T \int_\Omega \mathsf{K}(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k}) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \mathsf{K}(\vartheta)\nabla\vartheta \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t$$

for every test function $\varphi \in \mathrm{C}^0([0,T]; W^{2,d+\epsilon}(\Omega))$.

To complete the passage to the limit on the right-hand side of (4.50), it remains to show that

$$\partial_t \chi_{\tau_k} \to \chi_t \qquad \text{in } L^2(0,T; L^2(\Omega)). \tag{5.33}$$

This follows from testing the discrete equation for χ (4.52) by $\partial_t \chi_{\tau_k}$, integrating in time, and passing to the limit as $k \to \infty$. Indeed, exploiting convergences (5.2) and (5.6)–(5.8) we deduce that

$$\limsup_{k\to\infty} \int_0^T \int_\Omega |\partial_t \chi_{\tau_k}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \int_0^T \int_\Omega |\chi_t|^2 \, \mathrm{d}x \, \mathrm{d}t,$$

whence (5.33). This concludes the proof of (2.54).

The total energy *equality* (2.55) then ensues from testing (2.54) by $\varphi = 1$, the momentum balance (2.40) by $\mathbf{u}_t$, and the (pointwise) χ-equation (2.51) by χ_t, adding the resulting relations, and integrating in time. ∎

6. From the p-Laplacian to the Laplacian

In this Section we prove a global-in-time existence result for a suitable *entropic formulation* of the initial-boundary value problem for system (1.1)–(1.3), in the case the p-Laplacian operator $-\operatorname{div}(|\nabla\chi|^{p-1}\nabla\chi)$ is replaced by the Laplacian $-\Delta\chi$, i.e. for $p = 2$, keeping the evolution *unidirectional* (i.e., $\mu = 1$). Hence, (1.3) rewrites as

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) - \Delta\chi + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega \times (0,T). \tag{6.1}$$

We restrict, apparently for technical reasons (which however we cannot bypass), to the irreversible case $\mu = 1$. The main idea of the technique consists in passing to the limit as $\delta \searrow 0$ in the following approximation of (6.1)

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) - \Delta\chi - \delta\operatorname{div}(|\nabla\chi|^{p-1}\nabla\chi) + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega \times (0,T). \tag{6.2}$$

Indeed, we can apply Thm. 2.5 to the initial-boundary value problem for system (1.1)–(1.2), (6.2), with $p > d$ (supplemented with the boundary conditions (1.4)), and conclude the existence of global-in-time entropic solutions. In this entropic formulation we will pass to the limit as $\delta \searrow 0$, recovering an existence result for the case $p = 2$. Let us now state the notion of entropic solution for the limit system as $\delta \to 0$. We mention in advance that the solution concept introduced below is weaker than the one we have obtained in the case $p > d$ (cf. Definition (2.4)). In fact, the total energy inequality holds true only on $(0,t)$ (cf. (6.7) below), and not on a generic interval (s,t), and so does the energy inequality in the weak formulation of the equation for χ. Moreover, the momentum equation is no longer formulated pointwise a.e. in $\Omega \times (0,T)$, but in $H^{-1}(\Omega; \mathbb{R}^d)$, a.e. in time, only. Let us also anticipate that we will confine to initial data $\chi_0 \in H^1(\Omega)$ such that $\chi_0 \geq 0$ a.e. in Ω (which gives $\widehat{\beta}(\chi_0) \in L^1(\Omega)$ as in (2.26)) and, at the same time, $\chi_0 \leq 1$ a.e. in Ω. This and the irreversible character of the evolution will ensure that $\chi \in [0,1]$ a.e. in $\Omega \times (0,T)$, in accord with its physical meaning.

Definition 6.1 (Entropic solutions to the irreversible system with $p = 2$). Given initial data $(\vartheta_0, \mathbf{u}_0, \mathbf{v}_0$ fulfilling (2.24)–(2.25), and χ_0 such that

$$\chi_0 \in H^1(\Omega), \quad 0 \leq \chi_0 \leq 1 \text{ a.e. in } \Omega, \tag{6.3}$$

we call a triple $(\vartheta, \mathbf{u}, \chi)$ an *entropic solution* to the (initial-boundary value problem) for system (1.1)–(1.2), (6.1) with the boundary conditions (1.4), if

$$\vartheta \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^1(\Omega)), \tag{6.4}$$

$$\mathbf{u} \in H^1(0, T; H_0^1(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)) \cap H^2(0, T; H^{-1}(\Omega; \mathbb{R}^d)), \tag{6.5}$$

$$\chi \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \tag{6.6}$$

$(\vartheta, \mathbf{u}, \chi)$ complies with the initial conditions (2.35)–(2.36), and with the *entropic formulation* of (1.1)–(1.2), (6.1) consisting of

- the *entropy* inequality (2.37);
- the *total energy inequality* for almost all $t \in (0, T]$:

$$\mathscr{E}(\vartheta(t), \mathbf{u}(t), \mathbf{u}_t(t), \chi(t)) \leq \mathscr{E}(\vartheta_0, \mathbf{u}_0, \mathbf{v}_0, \chi_0) + \int_0^t \int_\Omega g \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_{\partial\Omega} h \, \mathrm{d}S \, \mathrm{d}r + \int_0^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \, \mathrm{d}x \, \mathrm{d}r, \tag{6.7}$$

where

$$\mathscr{E}(\vartheta, \mathbf{u}, \mathbf{u}_t, \chi) := \int_\Omega \vartheta \, \mathrm{d}x + \frac{1}{2} \int_\Omega |\mathbf{u}_t|^2 \, \mathrm{d}x + \frac{1}{2} \mathsf{e}(b(\chi(t))\mathbf{u}(t), \mathbf{u}(t)) + \frac{1}{2} \int_\Omega |\nabla\chi|^2 \, \mathrm{d}x + \int_\Omega W(\chi) \, \mathrm{d}x; \tag{6.8}$$

- the momentum equation

$$\mathbf{u}_{tt} + \mathcal{V}(a(\chi)\mathbf{u}_t) + \mathcal{E}(b(\chi)\mathbf{u}) + \mathcal{C}_\rho(\vartheta) = \mathbf{f} \quad \text{in } H^{-1}(\Omega; \mathbb{R}^d); \tag{6.9}$$

- the weak formulation of (6.1), viz.

$$\chi_t(x, t) \leq 0 \qquad \text{for a.a. } (x, t) \in \Omega \times (0, T), \tag{6.10}$$

$$\int_\Omega \left(\chi_t(t)\psi + \nabla\chi(t) \cdot \nabla\psi + \xi(t)\psi + \gamma(\chi(t))\psi + b'(\chi(t)) \frac{\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))}{2}\psi - \vartheta(t)\psi \right) \mathrm{d}x \geq 0$$
$$\text{for all } \psi \in W_-^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \text{for a.a. } t \in (0, T), \tag{6.11}$$

where $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the sense that

$$\xi \in L^1(0, T; L^1(\Omega)) \qquad \text{and} \qquad \langle \xi(t), \psi - \chi(t) \rangle_{W^{1,2}(\Omega)} \leq 0 \quad \forall \psi \in W_+^{1,2}(\Omega) \cap L^\infty(\Omega), \text{ for a.a. } t \in (0, T), \tag{6.12}$$

as well as the energy inequality for all $t \in (0, T]$:

$$\int_0^t \int_\Omega |\chi_t|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_\Omega \left(\frac{1}{2} |\nabla\chi(t)|^2 + W(\chi(t)) \right) \mathrm{d}x$$
$$\leq \int_\Omega \left(\frac{1}{2} |\nabla\chi_0|^2 + W(\chi_0) \right) \mathrm{d}x + \int_0^t \int_\Omega \chi_t \left(-b'(\chi) \frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \right) \mathrm{d}x \, \mathrm{d}r. \tag{6.13}$$

We are in the position now to state the main existence result of this section.

Theorem 6.2 (Existence of entropic solutions, $\mu = 1$ and $p = 2$). *Let Ω be a bounded connected domain with Lipschitz boundary. Assume* **Hypotheses (I)–(III)** *with*

$$b'(x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \tag{6.14}$$

and, in addition, **Hypothesis (IV)** *(i.e., $\widehat{\beta} = I_{[0,+\infty)}$), as well as conditions (2.21)–(2.25) on the data $\mathbf{f}$, g, h, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, and (6.3) on χ_0. Then, there exists an entropic solution (in the sense of Definition 6.1) $(\vartheta, \mathbf{u}, \chi)$ to the initial-boundary value problem for system (1.1)–(1.2), (6.1), such that ξ in (6.12) is given by (2.47) and ϑ satisfies (2.48).*

Remark 6.3. Let us note that in Thm. 6.2 we are able to deal with the case of a Lipschitz domain Ω and we do not need C^2-regularity of Ω (2.14). The latter condition was exploited in the previous sections in order to perform the elliptic regularity estimate on $\mathbf{u}$ (cf. the *Fifth estimate* (3.21)), which is not carried out here. Indeed the regularity requirement (6.5) on $\mathbf{u}$ we ask for in Definition 6.1 is weaker than the one prescribed in Section 2

(cf., e.g., (2.33)). Moreover, for the same reason, in this case we could also consider more general boundary conditions on $\mathbf{u}$ than the homogeneous Dirichlet (1.4): for example mixed Dirichlet-Neumann conditions could be taken into account.

Proof. Let $(\vartheta_\delta, \mathbf{u}_\delta, \chi_\delta)$ be a suitable family of *entropic* solutions to the initial-boundary value problem for (1.1)–(1.2), supplemented with initial data $(\vartheta_0, \mathbf{u}_0, \mathbf{v}_0)$ fulfilling (2.24)–(2.25), and with a sequence of data $(\chi_0^\delta)_\delta$ such that

$$(\chi_0^\delta)_\delta \subset W^{1,p}(\Omega), \quad 0 \le \chi_0^\delta(x) \le 1 \text{ for all } x \in \Omega \text{ for all } \delta > 0, \quad \chi_0^\delta \to \chi_0 \text{ in } H^1(\Omega). \tag{6.15}$$

Observe that we cannot rigorously perform on the entropic formulation of (1.1)–(1.2) the a priori estimates in Section 3. Therefore we need to confine the discussion only to the entropic solutions which arise from the time-discretization scheme set up in Sec. 4. In the present framework (i.e. with $p = 2$ and $\mu = 1$, and no upper bound on κ, cf. Hypothesis (V)), the a priori estimates for the time-discrete solutions in Prop. 4.10 are inherited in the time-continuous limit by the entropic solutions, with the exception of those corresponding to the *Fifth*, the *Seventh*, and the *Eighth a priori estimates* in Sec. 3, cf. also Remark 3.1.

The convergences from Lemma 5.1 combined with lower semicontinuity arguments indeed ensure that the strict positivity of ϑ_δ (cf. (3.2)), as well as estimates (3.5), (3.14), (3.16), (3.18), (3.27), hold with constants uniform w.r.t. δ. Moreover, combining the fact that $\widehat{\beta} = I_{[0,+\infty)}$ with the unidirectional character of the evolution and with the fact that $\chi_\delta(0) = \chi_0^\delta \in [0,1]$ on Ω, we infer that

$$\exists C > 0 \ \forall \delta > 0 : \qquad \|\chi_\delta\|_{L^\infty(Q)} \le C. \tag{6.16}$$

Therefore, repeating the compactness arguments in the proof of Lemma 5.1, based on the compactness results in [32], for every vanishing sequence $\delta_k \downarrow 0$ as $k \to \infty$ there exist a not relabeled subsequence and a triple $(\vartheta, \mathbf{u}, \chi)$ along which there holds as $k \to \infty$:

$$\vartheta_{\delta_k} \rightharpoonup^* \vartheta \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \tag{6.17}$$

$$\mathbf{u}_{\delta_k} \rightharpoonup^* \mathbf{u} \quad \text{in } H^2(0,T;H^{-1}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^1(\Omega;\mathbb{R}^d)), \tag{6.18}$$

$$\partial_t \mathbf{u}_{\delta_k} \to \partial_t \mathbf{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{6.19}$$

$$\chi_{\delta_k} \rightharpoonup^* \chi \quad \text{in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{6.20}$$

$$\chi_{\delta_k} \to \chi \quad \text{in } L^h(\Omega \times (0,T)) \quad \text{for all } h \in [1,+\infty), \tag{6.21}$$

$$\log(\vartheta_{\delta_k}) \to \log(\vartheta) \quad \text{in } L^2(0,T;L^s(\Omega)) \quad \text{for all } s \in (1,6) \text{ for } d = 3 \text{ and for all } s \in (1,+\infty) \text{ for } d = 2, \tag{6.22}$$

$$\vartheta_{\delta_k} \to \vartheta \quad \text{in } L^h(\Omega \times (0,T)), \text{ for every } h \in [1,8/3) \text{ for } d = 3 \text{ and } h \in [1,3) \text{ if } d = 2. \tag{6.23}$$

Now, in order to pass to the limit as $\delta \searrow 0$ we need to prove the following further convergence. Observe that, in the case of the p-Laplacian regularization for χ, we were able to prove an additional the strong convergence for $\partial_t \mathbf{u}$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^d))$. Our argument resulted from compactness arguments, relying on the *Fifth a priori estimate* (i.e. the elliptic regularity estimate on $\mathbf{u}$). The latter is no longer at our disposal, now.

Strong convergence of $\partial_t \mathbf{u}_\delta$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^d))$. This argument is strongly based on the irreversible character of our system. Let us test the weak formulation (2.40) of momentum equation fulfilled by the approximate solutions $(\vartheta_{\delta_k}, \mathbf{u}_{\delta_k}, \chi_{\delta_k})_k$, by $\partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})$, where $\mathbf{u}$ is the limit of $(\mathbf{u}_{\delta_k})_k$ as in (6.18)–(6.19). We get

$$0 = \int_0^t \int_\Omega \partial_{tt}^2 \mathbf{u}_{\delta_k} \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \, dx \, ds + \int_0^t \mathsf{v}(a(\chi_{\delta_k})\partial_t \mathbf{u}_{\delta_k}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds$$

$$+ \int_0^t \mathsf{e}(b(\chi_{\delta_k})\mathbf{u}_{\delta_k}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds - \rho \int_0^t \int_\Omega \vartheta_{\delta_k} \operatorname{div}(\partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, dx \, ds - \int_0^t \int_\Omega \mathbf{f} \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \, dx \, ds =: \sum_{i=1}^5 I_i \,.$$

Let us now deal separately with the single integrals:

$$I_1 := \int_0^t \int_\Omega \partial_{tt}^2 \mathbf{u}_{\delta_k} \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \, dx \, ds = \int_0^t \int_\Omega \partial_{tt}^2(\mathbf{u}_{\delta_k} - \mathbf{u}) \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \, dx \, ds + \int_0^t \int_\Omega \partial_{tt}^2 \mathbf{u} \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \, dx \, ds$$

$$= \frac{1}{2} \|\partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})(t)\|_{L^2(\Omega;\mathbb{R}^d)}^2 - \frac{1}{2} \|\partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})(0)\|_{L^2(\Omega;\mathbb{R}^d)}^2 + \int_0^t \langle \partial_{tt}^2 \mathbf{u}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \rangle_{H^1(\Omega;\mathbb{R}^d)} \, ds,$$

and the third integral tends to 0 when $\delta_k \searrow 0$ due to (6.18). Moreover,

$$I_2 := \int_0^t \mathsf{v}(a(\chi_{\delta_k}) \partial_t \mathbf{u}_{\delta_k}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds$$

$$= \int_0^t \mathsf{v}(a(\chi_{\delta_k}) \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}), \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds + \int_0^t \mathsf{v}(a(\chi_{\delta_k}) \partial_t \mathbf{u}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds.$$

Now, observe that

$$a(\chi_{\delta_k}) \partial_t \mathbf{u} \to a(\chi) \partial_t \mathbf{u} \qquad \text{in } L^2(0, T; H^1(\Omega; \mathbb{R}^d)). \tag{6.24}$$

This follows from the fact that $a(\chi_{\delta_k}) \mathbf{u}_t \to a(\chi) \mathbf{u}_t$ and $a(\chi_{\delta_k}) \varepsilon(\mathbf{u}_t) \to a(\chi) \varepsilon(\mathbf{u}_t)$ a.e. in $\Omega \times (0, T)$, in view of convergence (6.21) and of the continuity of a. Moreover, also due to (6.16), we have that $\|a(\chi_{\delta_k}) \mathbf{u}_t\|_{H^1(\Omega;\mathbb{R}^d)} \leq C \|\mathbf{u}_t\|_{H^1(\Omega;\mathbb{R}^d)}$ for a constant independent of $k \in \mathbb{N}$. Therefore, using the Lebesgue theorem (6.24) ensues. This implies that $\int_0^t \langle \partial_{tt}^2 \mathbf{u}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \rangle_{H^1(\Omega;\mathbb{R}^d)} \, ds$ tends to 0 when $\delta_k \searrow 0$, due to (6.18). Integrating by parts in time, we get

$$I_3 := \int_0^t \mathsf{e}(b(\chi_{\delta_k}) \mathbf{u}_{\delta_k}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds$$

$$= \int_0^t \mathsf{e}(b(\chi_{\delta_k})(\mathbf{u}_{\delta_k} - \mathbf{u}), \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds + \int_0^t \mathsf{e}(b(\chi_{\delta_k}) \mathbf{u}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds$$

$$= -\int_0^t \int_\Omega b'(\chi_{\delta_k}) \partial_t \chi_{\delta_k} \frac{\varepsilon(\mathbf{u}_{\delta_k} - \mathbf{u}) \mathbb{E} \varepsilon(\mathbf{u}_{\delta_k} - \mathbf{u})}{2} \, dx \, ds + \frac{1}{2} \mathsf{e}(b(\chi_{\delta_k}(t)(\mathbf{u}_{\delta_k} - \mathbf{u})(t), (\mathbf{u}_{\delta_k} - \mathbf{u})(t))$$

$$- \frac{1}{2} \mathsf{e}(b(\chi_{\delta_k}(0))(\mathbf{u}_{\delta_k} - \mathbf{u})(0), (\mathbf{u}_{\delta_k} - \mathbf{u})(0)) + \int_0^t \mathsf{e}(b(\chi_{\delta_k}) \mathbf{u}, \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds,$$

where the last integral tends to 0 (this can be shown arguing in the same way as for I_2), while the first integral is non-negative due to the fact that $\partial_t \chi_{\delta_k} \leq 0$ a.e. on $\Omega \times (0, T)$ and that $b' \geq 0$. This is the point where we exploit the unidirectional character of the system (i.e. $\mu = 1$). Finally,

$$I_4 := -\int_0^t \int_\Omega \vartheta_{\delta_k} \varepsilon(\partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, dx \, ds \to 0, \quad I_5 := -\int_0^t \int_\Omega \mathbf{f} \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}) \, dx \, ds \to 0,$$

as $\delta_k \searrow 0$, due to the convergences (6.18), (6.23), as well as assumption (2.21) on $\mathbf{f}$. Ultimately, we get

$$\|\partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})(t)\|_{L^2(\Omega;\mathbb{R}^d)}^2 + \int_0^t \mathsf{v}(a(\chi_{\delta_k}) \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u}), \partial_t(\mathbf{u}_{\delta_k} - \mathbf{u})) \, ds + \mathsf{e}(b(\chi_{\delta_k}(t)(\mathbf{u}_{\delta_k} - \mathbf{u})(t), (\mathbf{u}_{\delta_k} - \mathbf{u})(t)) \to 0$$

as $\delta_k \searrow 0$, which entails

$$\mathbf{u}_{\delta_k} \to \mathbf{u} \quad \text{strongly in } W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)) \cap H^1(0, T; H^1(\Omega; \mathbb{R}^d)). \tag{6.25}$$

Conclusion of the proof. Using this strong convergence, we can now pass to the limit as $k \to \infty$ in the energy inequality (2.42) in the weak formulation of the equation for χ_{δ_k} as follows. We have to identify the weak limit of

$$\xi_{\delta_k}(x, t) = -\mathfrak{I}_{\chi_{\delta_k}=0}(x, t) \left(\gamma(\chi_{\delta_k}(x, t)) + b'(\chi_{\delta_k}(x, t)) \frac{\varepsilon(\mathbf{u}_{\delta_k}(x, t)) \mathbb{E}(x) \varepsilon(\mathbf{u}_{\delta_k}(x, t))}{2} - \vartheta_{\delta_k}(x, t) \right)^+. \tag{6.26}$$

First of all note that $(\mathfrak{I}_{\chi_{\delta_k}=0})_k$ is bounded in $L^\infty(Q)$ independently of $k \in \mathbb{N}$. Hence, we can select a subsequence $(\mathfrak{I}_{\chi_{\delta_k}=0})_k$ weakly star converging in $L^\infty(Q)$ to some $\mathfrak{I}$. Observe that we cannot establish that $\mathfrak{I} = \mathfrak{I}_{\chi=0}$. On

the other hand, it follows from the previously proved convergences that $(\gamma(\chi_{\delta_k}) + b'(\chi_{\delta_k})\frac{\varepsilon(\mathbf{u}_{\delta_k})\mathbb{E}\varepsilon(\mathbf{u}_{\delta_k})}{2} - \vartheta_{\delta_k})^+$ strongly converges in $L^1(Q)$ to $(\gamma(\chi) + b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} - \vartheta)^+$. Hence we identify

$$\xi = -\mathfrak{I}(x,t)(\gamma(\chi(x,t)) + b'(\chi(x,t))\frac{\varepsilon(\mathbf{u}(x,t))\mathbb{E}\varepsilon(\mathbf{u}(x,t))}{2} - \vartheta(x,t))^+ \tag{6.27}$$

and observe that $\xi_{\delta_k} \rightharpoonup \xi$ in $L^1(Q)$. Then, integrating $(2.42)_{\delta_k}$ from 0 to T and passing to the limit as $k \to \infty$, using the fact that or all $\psi \in L^p(0,T;W_-^{1,p}(\Omega)) \cap L^\infty(Q)$

$$\left| \int_0^T \int_\Omega \delta_k |\nabla\chi_{\delta_k}|^{p-2}\nabla\chi_{\delta_k} \cdot \nabla\psi \, dx \, dt \right| \le \delta_k \|\nabla\chi_{\delta_k}\|^{p-1}_{L^{p-1}(Q;\mathbb{R}^d)} \|\nabla\psi\|_{L^p(Q;\mathbb{R}^d)} \to 0,$$

we get

$$\int_0^T \int_\Omega \left(\chi_t(t)\psi + \nabla\chi(t) \cdot \nabla\psi + \gamma(\chi(t))\psi + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))}{2}\psi - \vartheta(t)\psi \right) dx \, dt \ge - \int_0^T \int_\Omega \xi(t)\psi \, dx dt, \tag{6.28}$$

for all $\psi \in L^p(0,T;W_-^{1,p}(\Omega)) \cap L^\infty(Q)$, where ξ is defined in (6.27). From (6.28), we get (6.11).

It remains to show that χ complies with the variational inequality (6.12). To do so, we have to pass to the limit in $(2.43)_{\delta_k}$, whence

$$\int_0^T \left(\int_\Omega \xi_{\delta_k}(\psi - \chi_{\delta_k}(t)) \, dx \right) \zeta(t) \, dt \ge 0 \quad \text{for all } \psi \in W_+^{1,p}(\Omega) L^\infty(\Omega) \text{ and all } \zeta \in L^\infty(0,T) \text{ with } \psi, \zeta \ge 0.$$

Observe that the two weak convergences $\chi_{\delta_k} \rightharpoonup^* \chi$ in $L^\infty(Q)$ and $\xi_{\delta_k} \rightharpoonup \xi$ in $L^1(Q)$ do not allow for a direct limit passage in the term $\iint_Q \xi_{\delta_k}\chi_{\delta_k}\zeta \, dx \, dt$, which equals zero for all $k \in \mathbb{N}$ due to (6.26). Indeed, we need to argue in a more refined way. It follows from (6.21) that χ_{δ_k} converges almost uniformly to χ in Q, i.e. for every $\epsilon > 0$ there exists $Q_\epsilon \subset Q$ such that $|Q \setminus Q_\epsilon| < \epsilon$ and $\chi_{\delta_k} \to \chi$ uniformly on Q_ϵ. The latter property implies that

$$\mathfrak{I} \equiv 0 \quad \text{on } \quad Q_\epsilon \cap \{\mathfrak{I}_{\chi=0} \equiv 0\}. \tag{6.29}$$

Indeed, $\mathfrak{I}_{\chi=0}(x,t) = 0$ implies $\chi(x,t) \ne 0$. Since χ_{δ_k} converges to χ uniformly on Q_ϵ, there exists an index $\bar{k}$, independent of (x,t), such that for all $k \ge \bar{k}$, $\chi_{\delta_k}(x,t) \ne 0$, hence $\mathfrak{I}_{\chi_{\delta_k}=0}(x,t) = 0$. With this argument we conclude that that $\mathfrak{I}_{\chi_{\delta_k}=0} \equiv 0$ on $Q_\epsilon \cap \{\mathfrak{I}_{\chi=0} \equiv 0\}$, whence (6.29). It follows from (6.29) and (6.27) that

$$\xi(x,t)\chi(x,t) = 0 \quad \text{for a.a. } (x,t) \in Q_\epsilon, \quad \text{whence} \quad \iint_{Q_\epsilon} \xi(x,t)\chi(x,t)\zeta(t) \, dx \, dt = 0.$$

On the other hand, using the properties of the Lebesgue integral we have that

$$\forall \eta > 0 \ \exists \epsilon = \epsilon_\eta > 0 \ : \ |Q \setminus Q_\epsilon| < \epsilon \ \Rightarrow \ \iint_{Q \setminus Q_\epsilon} |\xi(x,t)\chi(x,t)\zeta(t)| \, dx \, dt < \eta.$$

Therefore we conclude that

$$\forall \eta > 0 \qquad \left| \iint_Q \xi(x,t)\chi(x,t)\zeta(t) \, dx \, dt \right| < \eta,$$

i.e.

$$\iint_Q \xi(x,t)\chi(x,t)\zeta(t) \, dx \, dt = 0 = \lim_{k\to\infty} \iint_Q \xi_{\delta_k}\chi_{\delta_k}\zeta \, dx \, dt$$

Hence

$$0 \le \iint_Q \xi_{\delta_k}(\psi - \chi_{\delta_k})\zeta \, dx \, dt \to \iint_Q \xi(\psi - \chi)\zeta \, dx \, dt = \int_0^T \left(\int_\Omega \xi(\psi - \chi(t)) \, dx \right) \zeta(t) \, dt,$$

which implies

$$\int_\Omega \xi(t)(\psi - \chi(t)) \, dx \ge 0 \quad \text{for a.e. } t \in (0,T) \qquad \text{for all } \psi \in W_+^{1,p}(\Omega) \cap L^\infty(\Omega).$$

With a density argument we get (6.12) for all $\psi \in W_+^{1,2}(\Omega)$.

Convergences (6.17)–(6.23) also guarantee the passage to the limit in the momentum equation, whence (6.9).

Finally, we pass to the limit in the entropy inequality (2.37) and in the total energy inequality (2.38) by the very same compactness/lower semicontinuity arguments as in the proof of Theorem 2.5, thus deducing (2.37) and the total energy inequality (6.7) on the generic interval $(0, t)$.   ∎

Remark 6.4. Notice that, we have been able to obtain the energy inequalities (6.13) and (6.8) only on intervals of the type $(0, t)$, and not on the generic interval $(s, t) \subset (0, T)$, due to the weak convergence of $(\nabla \mathcal{X}_{\delta_k})$ in $L^2(Q; \mathbb{R}^d)$, which does not yield the pointwise-in-time convergence required to take the limit of the right-hand sides of (2.44) and (2.38). It is an open poblem to improve the convergence of $(\nabla \mathcal{X}_{\delta_k})$ to a strong one.

This limit passage also reveals that the notion of entropic solution enjoys stability properties. It is clearly the right one in the present framework, and, seemingly, the entropy inequality cannot be improved to an equality, at least with these techniques. Indeed, due to a lack of elliptic regularity estimates on the displacement which were previously made possible by the p-Laplacian regularization, in the limit as $\delta \downarrow 0$ the right-hand side of the heat equation is only estimated in $L^1(Q)$.

□

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