RESOLVENT EXPANSION AND TIME DECAY OF THE WAVE FUNCTIONS FOR TWO-DIMENSIONAL MAGNETIC SCHRÖDINGER OPERATORS

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ABSTRACT. We consider two-dimensional Schrödinger operators $H(B, V)$ given by equation (1.1) below. We prove that, under certain regularity and decay assumptions on B and V, the character of the expansion for the resolvent $(H(B, V) - \lambda)^{-1}$ as $\lambda \to 0$ is determined by the flux of the magnetic field B through $\mathbb{R}^2$. Subsequently, we derive the leading term of the asymptotic expansion of the unitary group $e^{-itH(B,V)}$ as $t \to \infty$ and show how the magnetic field improves its decay in t with respect to the decay of the unitary group $e^{-itH(0,V)}$.

1. Introduction

The present paper is concerned with the Schrödinger operator associated to a magnetic field B:

$$H(B, V) = (i\nabla + A)^2 + V \quad \text{in} \quad L^2(\mathbb{R}^2), \tag{1.1}$$

where $\operatorname{curl} A = B$ and V is an electric scalar potential. Both B and V are assumed to have a polynomial decay at infinity. We will analyze the asymptotic expansion of the resolvent $(H(B, V) - \lambda)^{-1}$ as $\lambda \to 0$ and the long time behavior of the unitary group $e^{-itH(B,V)}$ generated by $H(B, V)$. It is well-known that the two problems are closely related to each other, see e.g. [JK, Mu, Sch2]. Asymptotic expansions of the resolvent have been well studied in the absence of magnetic filed, i.e. for the operator $H(0, V)$. In the case of dimension two, in particular, it has been shown that if zero is a regular point of $H(0, V)$, which means that zero is neither an eigenvalue nor a resonance of $H(0, V)$, and if V decays fast enough at infinity, then as $\lambda \to 0$

$$(H(0, V) - \lambda)^{-1} = T_0 + T_1 (\log \lambda)^{-1} + o((\log \lambda)^{-1}) \qquad \left(\text{in} \ \ \mathbb{R}^2\right) \tag{1.2}$$

holds in suitable weighted L^2-spaces; see [JN, Mu, Sch1]. In dimension three, still under the condition that zero is a regular point of $H(0, V)$, the term $(\log \lambda)^{-1}$ in the above equation must be replaced by $\lambda^{1/2}$, cf. [JK, JN, Mu].

As far as the long time behavior of the operator $e^{-itH(0,V)}$ is concerned, the classical results say that in dimension three, for sufficiently short range potentials and under the condition that zero is a regular point, one has for $t \to \infty$ the following asymptotic equation in $L^2(\mathbb{R}^3)$:

$$(1 + |x|)^{-s} e^{-itH(0,V)} P_{ac} (1 + |x|)^{-s} = S_3 \, t^{-\frac{3}{2}} + o(t^{-\frac{3}{2}}), \qquad \left(\text{in} \ \ \mathbb{R}^3\right) \tag{1.3}$$

which holds in $L^2(\mathbb{R}^3)$ for s large enough, see [JK, Je, Mu], and with P_{ac} being the projection on the absolutely continuous subspace of $H(0, V)$. Note that is the decay rate $t^{-3/2}$ corresponds to the free evolution in $\mathbb{R}^3$. For higher-dimensional results we refer to [Je2, Mu].

The situation in dimension two is different since in this case by adding a potential V one may improve the decay rate of $e^{-itH(0,V)}$ with respect to the t^{-1} decay rate of the free evolution operator, provided the weight function $(1 + |x|)^s$ grows fast enough. More precisely, it was proved by Murata, see [Mu], that if zero is a regular point then in $L^2(\mathbb{R}^2)$ we have for $t \to \infty$ the asymptotic expansion

$$(1 + |x|)^{-s} e^{-itH(0,V)} P_{ac} (1 + |x|)^{-s} = S_2 \, t^{-1} (\log t)^{-2} + o(t^{-1} (\log t)^{-2}) \qquad \left(\text{in} \ \ \mathbb{R}^2\right) \tag{1.4}$$

with $s > 3$ and $|V(x)| \lesssim (1 + |x|)^{-6-0}$, see section 2.1 for the precise meaning of the latter condition.

The aim of this paper is to show that the presence of a magnetic field in $\mathbb{R}^2$ changes completely the character of the expansions (1.2) and (1.4). In order to describe how the magnetic field affects these asymptotic expansions, we define the normalized flux of B through $\mathbb{R}^2$ by

$$\alpha := \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x)\, dx. \tag{1.5}$$

The main results of this paper show then if B is sufficiently smooth and decays fast enough at infinity, then the behavior of $(H(B,V) - \lambda)^{-1}$ for $\lambda \to 0$ as well as the behavior of $e^{-itH(B,V)}$ for $t \to \infty$ is determined by the distance between α and the set of integers:

$$\mu(\alpha) := \min_{k \in \mathbb{Z}} |k + \alpha|. \tag{1.6}$$

More precisely, if α is finite and non-integer, and if zero is a regular point of $H(B,V)$, then for $\lambda \to 0$ the expansion

$$(H(B,V) - \lambda)^{-1} = F_0 + F_1\, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \alpha \notin \mathbb{Z}, \qquad \left(\text{in } \mathbb{R}^2 \right) \tag{1.7}$$

holds true in certain weighted L^2-spaces with suitably chosen weights, see Theorem 2.2. Accordingly we obtain a faster decay rate of $e^{-itH(B,V)}$ in t with respect to the decay rate of $e^{-itH(0,V)}$. In fact we show in Theorem 2.5 that there exists a bounded operator $\mathcal{K}$ in $L^2(\mathbb{R}^2)$ such that for $t \to \infty$

$$(1 + |x|)^{-s}\, e^{-itH(B,V)}\, P_{ac}\, (1 + |x|)^{-s} = t^{-1-\mu(\alpha)} \mathcal{K} + o(t^{-1-\mu(\alpha)}) \qquad \alpha \notin \mathbb{Z}, \qquad \left(\text{in } \mathbb{R}^2 \right) \tag{1.8}$$

holds in $L^2(\mathbb{R}^2)$ provided $s > 5/2$, $|V(x)| \lesssim (1+|x|)^{-3-0}$, and B satisfies suitable decay and regularity conditions, see assumption 2.4. On the other hand, For integer values of α one has qualitatively the same behavior as in (1.2) and (1.4), see Theorems 2.3 and 2.7. We also give an explicit formulae for the operators F_0, F_1 and $\mathcal{K}$ in the case $\mu(\alpha) < 1/2$. Hence the character of the expansions for $(H(B,V) - \lambda)^{-1}$ as $\lambda \to 0$ and $e^{-itH(B,V)}$ as $t \to \infty$ is completely determined by the flux of B.

To understand what makes the family of magnetic fields with equal fluxes distinguished, we refer to Lemma 4.4 and equation (3.4). The latter implies that a difference between two magnetic Hamiltonians $H(B_1, V) - H(B_2, V)$ is a first order differential operator with sufficiently short range coefficients, provided we choose a suitable gauge, *if and only if* B_1 and B_2 have equal flux through $\mathbb{R}^2$. Therefore, only in this case the coefficients of $H(B_1, V) - H(B_2, V)$ may decay fast enough, depending on the decay of B_1 and B_2, in order to compensate for the growth of the weight function $(1 + |x|)^s$. When the fluxes of B_1 and B_2 are different, then the coefficients of the first order term in $H(B_1, V) - H(B_2, V)$ cannot decay faster than $|x|^{-1}$ even if both B_1 and B_2 have compact support, cf. equation (3.4).

This indicates a natural strategy for proving (1.7), and consequently (1.8); we choose a concrete magnetic field B_0, see equation (4.2), for which it is possible to calculate the resolvent for small values of λ explicitly. Using this fact we first show that the expansion (1.7) holds for $H(B_0, 0)$, see Proposition 4.1, and then we extend the result to all magnetic fields with the same flux (and sufficient decay) by using the perturbation theory in combination with Lemma 4.4. The proof in the case of integer flux follows the same strategy, cf. Proposition 4.2.

For comparison it should be mentioned that in the case of dimension three the effect of a magnetic field on the asymptotic expansion (1.3) is much weaker. Indeed, from [Mu, Thms. 8.11] it follows that, under sufficient regularity and decay assumptions on B, the unitary group $e^{-itH(B,V)}$ satisfies again the asymptotic expansion (1.3) (with different coefficients), see also [KK].

The paper is organized as follows. In section 2 we introduce some necessary notation and formulate our main results. The proofs are given in section 4. The concrete model associated to the operator $H(B_0, 0)$ is treated in sections 5 and 6. In section 7 we give some auxiliary technical results needed for the proofs of Propositions 4.1 and 4.2.

2. **Main results**

2.1. **Notation.** Before we formulate our main results we need to introduce some notation. Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be given by $\rho(x) = \rho(|x|) = 1 + |x|$ and let $s \in \mathbb{R}$. We define

$$L^2(\mathbb{R}^2, s) = \{u : \|\rho^s u\|_{L^2(\mathbb{R}^2)} < \infty\}, \qquad \|u\|_s := \|\rho^s u\|_{L^2(\mathbb{R}^2)}.$$

We will denote by $\mathscr{B}(X, Y)$ the space of bounded linear operators from a Banach space X into a Banach space Y and by $\|\cdot\|_{\mathscr{B}(X,Y)}$ the corresponding operator norm. For the sake of brevity we will make use of the following shorthands: instead of $\mathscr{B}(L^2(\mathbb{R}^2, s), L^2(\mathbb{R}^2, s'))$ we write $\mathscr{B}(s, s')$, and if $X = Y$, then we use the notation $\mathscr{B}(X) = \mathscr{B}(X, X)$. Given $R > 0$ and a point $x \in \mathbb{R}^2$ we denote by $\mathcal{D}(x, R) \subset \mathbb{R}^2$ the open disc with radius R centred in x. The scalar product in a Hilbert space $\mathscr{H}$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathscr{H}}$. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ we will often use the polar coordinates representation

$$x_1 + ix_2 = re^{i\theta}, \quad y_1 + iy_2 = r'e^{i\theta'}, \qquad r, r' \geq 0, \quad \theta, \theta' \in [0, 2\pi). \tag{2.1}$$

Given functions $f, g \in L^\infty(\mathbb{R}^2)$ we will write $f(x) \lesssim g(x)$ if there exists a numerical constant c such that $f(x) \leq c\,g(x)$ for all $x \in \mathbb{R}^2$. The symbol $f(x) \gtrsim g(x)$ is defined analogously. Finally, for any $f \in L^\infty(\mathbb{R}^2)$ and $\alpha \in \mathbb{R}$ we will make use of the notation

$$f(x) \lesssim (1 + |x|)^{-\alpha - 0} \quad \Rightarrow \quad \lim_{|x| \to \infty} (1 + |x|)^\alpha f(x) = 0.$$

Here is the basic assumption on the magnetic field:

Assumption 2.1. Let $B : \mathbb{R}^2 \to \mathbb{R}$ be continuos and assume that for some $\sigma > 4$ we have

$$\sup_{\theta \in (0, 2\pi)} \left(|B(r, \theta)| + |\partial_\theta B(r, \theta)|\right) \lesssim (1 + r)^{-\sigma}. \tag{2.2}$$

Under this condition B obviously belongs to $L^1(\mathbb{R}^2)$ and therefore has finite flux α. To any magnetic field which satisfies (2.2) may be associated a bounded vector field A such that $\operatorname{curl} A = B$, see Lemma 4.4. We then add an electric potential $V \in L^\infty(\mathbb{R}^2)$ and define the operator $H(B, V)$ through the closed quadratic form

$$Q[u] = \int_{\mathbb{R}^2} \left(|(i\nabla + A)\,u|^2 + V|u|^2\right) dx, \qquad u \in W^{1,2}(\mathbb{R}^2).$$

Moreover, we will always assume that $V(x) \to 0$ as $|x| \to \infty$. Hence by standard compactness arguments

$$\sigma_{es}(H(B, V)) = [0, \infty).$$

Consequently, we use the standard definition of a regular point; we say that *zero is a regular point of $H(B, V)$* if there exists $s > 1/2$ such that

$$\limsup_{\lambda \to 0} \|\rho^{-s} (H(B, V) - \lambda - i0)^{-1} \rho^{-s}\|_{\mathscr{B}(L^2(\mathbb{R}^2))} < \infty. \tag{2.3}$$

2.2. **Resolvent expansion at threshold.** Let $A_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector potential given by

$$A_0(x) = \alpha\,(-x_2, x_1) \begin{cases} |x|^{-1} & |x| \leq 1, \\ |x|^{-2} & |x| > 1, \end{cases} \tag{2.4}$$

and let $T(B, V)$ be defined by

$$T(B, V) = 2i(A - A_0) \cdot \nabla + i\nabla \cdot A + (|A|^2 - |A_0|^2) + V. \tag{2.5}$$

Below $G_0, G_1, \mathcal{G}_0$ and $\mathcal{G}_1$ are integral operators in $L^2(\mathbb{R}^2)$ defined in sections 5 and 6, see equations (5.27), (5.29), (6.6) and (6.7) respectively.

Theorem 2.2. *Let $\alpha \notin \mathbb{Z}$. Suppose that assumption 2.1 is satisfied. Suppose moreover that $s > 3/2$ and $|V(x)| \lesssim (1 + |x|)^{-3-0}$. If zero is a regular point of $H(B,V)$, then there exist operators $F_j(B,V) \in \mathscr{B}(s, -s)$, $j = 0, 1$, such that*

$$(H(B,V) - \lambda - i0)^{-1} = F_0(B,V) + F_1(B,V)\,\lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \quad in \quad \mathscr{B}(s, -s) \qquad (2.6)$$

as $\lambda \to 0$. Moreover, for $\mu(\alpha) < 1/2$ it holds

$$F_0(B,V) = (1 + G_0\,T(B,V))^{-1}\,G_0, \quad F_1(B,V) = (1 + G_0\,T(B,V))^{-1}\,G_1\,(1 + T(B,V)\,G_0)^{-1}.$$

It would be possible to provide an explicit formula for the operators $F_0(B,V)$ and $F_1(B,V)$ also in the case $\mu(\alpha) = 1/2$. In order to avoid complicated expressions with heavy notation we prefer not to do so, see section 5 for details. In the case of an integer flux we have

Theorem 2.3. *Let B satisfy assumption 2.1 and assume that $\alpha \in \mathbb{Z}$. Let $s > 3/2$. If $|V(x)| \lesssim (1 + |x|)^{-3-0}$ is such that zero is a regular point of $H(B,V)$, then*

$$(H(B,V) - \lambda - i0)^{-1} = \mathcal{F}_0(B,V) + \mathcal{F}_1(B,V)\,(\log \lambda)^{-1} + o((\log|\lambda|)^{-1}) \quad in \quad \mathscr{B}(s, -s) \qquad (2.7)$$

as $\lambda \to 0$, where

$$\mathcal{F}_0(B,V) = (1 + \mathcal{G}_0\,T(B,V))^{-1}\,\mathcal{G}_0, \quad \mathcal{F}_1(B,V) = (1 + \mathcal{G}_0\,T(B,V))^{-1}\,\mathcal{G}_1\,(1 + T(B,V)\,\mathcal{G}_0)^{-1}.$$

2.3. Time decay.

Our results concerning the time decay of the unitary group generated by $H(B,V)$ require stronger regularity assumptions on B:

Assumption 2.4. Let $s > 4$ and assume that for any multi-index $\beta \in \mathbb{N}_0^2$ it holds

$$|\partial^\beta B(x)| \lesssim (1 + |x|)^{-s - |\beta|}. \qquad (2.8)$$

Obviously, any B which satisfies assumption 2.4 satisfies also assumption 2.1. Let $\mathcal{H}_d$ be the subspace of $L^2(\mathbb{R}^2)$ spanned by normalized eigenfunctions corresponding to discrete eigenvalues of $H(B,V)$. We denote by P_c the projection on the orthogonal complement of $\mathcal{H}_d$ in $L^2(\mathbb{R}^2)$.

Theorem 2.5. *Let assumption 2.4 be satisfied. Assume that $\alpha \notin \mathbb{Z}$ and let $s > 5/2$. If $|V(x)| \lesssim (1 + |x|)^{-3-0}$ is such that zero is a regular point of $H(B,V)$, then there exists $K(B,V) \in \mathscr{B}(s, s^{-1})$ such that as $t \to \infty$*

$$e^{-itH(B,V)}\,P_c = K(B,V)\,t^{-1-\mu(\alpha)} + o(t^{-1-\mu(\alpha)}) \qquad (2.9)$$

in $\mathscr{B}(s, -s)$, where

$$K(B,V) = \frac{i}{\pi}\,\sin(\pi\mu(\alpha))\,e^{i\pi\mu(\alpha)/2}\,\Gamma(1 + \mu(\alpha))\,F_1(B,V).$$

Remark 2.6. The improved decay rate induced by a two-dimensional magnetic field was observed also for the heat semi-group $e^{-tH(B,0)}$, [Ko1, Kr].

Theorem 2.7. *Let assumption 2.4 be satisfied. Assume that $\alpha \in \mathbb{Z}$ and let $s > 5/2$. If $|V(x)| \lesssim (1 + |x|)^{-3-0}$ is such that zero is a regular point of $H(B,V)$, then as $t \to \infty$*

$$e^{-itH(B,V)}\,P_c = -i\,t^{-1}(\log t)^{-2}\,\mathcal{F}_1(B,V) + o\big(t^{-1}(\log t)^{-2}\big) \qquad (2.10)$$

in $\mathscr{B}(s, -s)$.

Remark 2.8. It should be pointed out that, in view of the magnetic Hardy-type inequality (3.1), zero is a regular point of $H(B,V)$ whenever $|V(x)| \leq V_0\,(1 + |x|)^{-2}$ with V_0 small enough.

Remark 2.9. The assumption $B \in C^\infty(\mathbb{R}^2)$ is needed only for the behavior of the resolvent for high energies, [Ro]. It is natural to suppose that the claims of Theorems 2.5 and 2.7 should hold true under weaker regularity assumptions on B.

3. Discussion

3.1. Hardy inequality.
If the magnetic field satisfies assumption 2.1, then by [LW, W, Ko2] there exists a positive constant $C_h = C_h(B)$ such that the inequality

$$H(B) \geq \frac{C_h}{w}, \qquad w(x) = \begin{cases} 1 + |x|^2, & \alpha \notin \mathbb{Z}, \\ 1 + |x|^2 (\log|x|)^2, & \alpha \in \mathbb{Z} \end{cases} \tag{3.1}$$

holds in the sense of quadratic forms on $W^{1,2}(\mathbb{R}^2)$.

3.2. Dispersive estimates.
Since $e^{-itH(B,V)}$ is a unitary operator from $L^2(\mathbb{R}^2)$ onto itself it is obvious that $\|(1+|x|)^{-s} e^{-itH(B,V)} (1+|x|)^{-s}\|_{L^2(\mathbb{R}^2)} \leq 1$ for every $s \geq 0$ and $t > 0$. In combination with Theorem 2.5 we thus get the dispersive estimate

$$\|(1+|x|)^{-s} e^{-itH(B,V)} P_c (1+|x|)^{-s}\|_{L^2(\mathbb{R}^2)} \lesssim t^{-1-\mu(\alpha)} \qquad \forall\, t > 0. \tag{3.2}$$

We note once again that the effect of faster decay with respect to the non-magnetic evolution is absent in dimension three, see [KK] and [EGS1, EGS2]. Three-dimensional dispersive estimates in weighted L^2−spaces in the absence of magnetic fields were first obtained by Rauch, [Ra]. Extensions of these estimates to the $L^1 \to L^\infty$ setting, also in dimensions higher than three, were established in [JSS, GS]. The case of dimension two was treated by Schlag in [Sch1].

An $L^1 \to L^\infty$ dispersive estimate which corresponds to Murata's asymptotic expansion (1.4) has been obtained only recently by Erdogan and Green in [EG]. They showed that

$$\| (\log(2+|x|))^{-2} e^{-itH(0,V)} P_{ac} (\log(2+|x|))^{-2}\|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \lesssim t^{-1} (\log t)^{-2} \qquad t > 2, \tag{3.3}$$

provided zero is a regular point of $H(0,V)$ and $|V(x)| \lesssim (1+|x|)^{-3-0}$. It is interesting to observe that the logarithmic factor on the left hand side of (3.3) appears also in the weight function w for the Hardy inequality (3.1) in the case $\alpha \in \mathbb{Z}$.

3.3. The case of zero flux.
When $\alpha = 0$, then we cannot apply our perturbative approach, since the reference magnetic field B_0 is identically zero in this case, see equations (2.4) and (4.2). Instead, we treat the operator $H(B,V)$ as a perturbation the free Laplacian $-\Delta$ and apply a result of Murata, see [Mu, Thms.8.4&7.5]. This is possible thanks to the fact that for a magnetic field with zero flux we can find a corresponding vector potential with sufficient decay at infinity, cf. Lemma 4.5.

It must be mentioned, however, that this is the only case in which the perturbation with respect to the free Laplacian, i.e. the operator $H(B,V) + \Delta$, has coefficients decaying fast enough in order to compensate for the growth of the weight function $(1 + |x|)^s$ with $s > 1$. Indeed, if $\mathrm{curl}\, A_1 = B_1$ and $\mathrm{curl}\, A_2 = B_2$, then by the Stokes Theorem we have

$$|A_1(x) - A_2(x)| = o(|x|^{-1}) \quad \text{as} \quad |x| \to \infty \quad \Rightarrow \quad \int_{\mathbb{R}^2} B_1(x)\, dx = \int_{\mathbb{R}^2} B_2(x)\, dx, \tag{3.4}$$

see also [HB]. Hence if $B_1 = 0$ and $B_2 = B$ is such that $\int_{\mathbb{R}^2} B \neq 0$, then the coefficients of the perturbation $H(B,V) + \Delta = 2i\, A \cdot \nabla + i \nabla \cdot A + |A|^2 + V$ cannot decay faster than $|x|^{-1}$ irrespectively of the decay rate of B.

3.4. Long range magnetic fields.
One might expect that the main results of this paper should remain valid also if B has a slower decay than the one required by assumption 2.1, as long as the flux α remains finite. The asymptotic expansion of the resolvent for decaying magnetic fields with infinite flux is an open question.

3.5. $L^1 \to L^\infty$ **estimates.** It would be interesting to extend the dispersive estimate (3.2) to an $L^1 \to L^\infty$ setting. So far this is known only for the Aharonov-Bohm magnetic field B_{ab} with flux α, see [FFFP, GK]. Such a magnetic field is generated by the vector potential

$$A_{ab}(x) = \frac{\alpha}{|x|^2}(-x_2, x_1).$$

Denote by $H(B_{ab}, 0)$ the Friedrichs extension of the operator $(i\nabla + A_{ab})^2$ defined on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. In [FFFP, Thm.1.9] it is proved that

$$\| e^{-itH(B_{ab},0)} \|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \lesssim t^{-1}, \qquad t > 0. \tag{3.5}$$

A related weighted estimate with an improved decay rate was obtained in [GK, Cor.3.3]:

$$\| (1+|x|)^{-\mu(\alpha)} e^{-itH(B_{ab},0)} (1+|x|)^{-\mu(\alpha)} \|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \lesssim t^{-1-\mu(\alpha)} \qquad t > 0. \tag{3.6}$$

Note that (3.6) gives the decay rate t^{-1} when $\alpha \in \mathbb{Z}$. This is not surprising since the Aharonov-Bohm operator with an integer flux is unitarily equivalent to the free Laplacian $-\Delta$ in $\mathbb{R}^2$.

4. **Proofs of the main results**

In this section we assume throughout that B satisfies assumption 2.1 and that V satisfies the hypothesis of Theorems 2.2 and 2.3. Under these conditions the operator $H(B, V)$ has no positive eigenvalues, see [KT]. Then by [IS]

$$\sup_{a \leq \lambda \leq b} \| \rho^{-s} (H(B, V) - \lambda - i0)^{-1} \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}^2))} < \infty \tag{4.1}$$

for $s \geq 1$ and any $0 < a < b < \infty$. Consequently, the positive part of the spectrum of $H(B, V)$ is purely absolutely continuous. The negative part of the spectrum is either empty or consists of a finite number of eigenvalues each having finite multiplicity, see [Ko2, Thm.3.1].

4.1. **Expansion at threshold.** If zero is a regular point of $H(B, V)$, then the estimate (4.1) can be extended to $a = 0$. Moreover, from the finiteness of the discrete spectrum of $H(B, V)$ it follows that $H(B, V)$ has no spectrum in the left neighborhood of zero. Hence

$$\sup_{-\delta \leq \lambda \leq \delta} \| \rho^{-1} (H(B, V) - \lambda - i0)^{-1} \rho^{-1} \|_{\mathscr{B}(L^2(\mathbb{R}^2))} < \infty$$

for $s \geq 1$ and $\delta > 0$ small enough.

To prove Theorems 2.2 and 2.3 (for $\alpha \neq 0$) we will employ the perturbation procedure mentioned in the introduction. First we establish the asymptotic expansions of the type (2.6) and (2.7) for the resolvent of an operator $H(B_0) = H(B_0, 0)$, where B_0 is by equation (4.2), see Propositions 4.1, 4.2. Then we show by the perturbative technique that any other magnetic field with the same flux as B_0 gives rise to an operator with (qualitatively) the same asymptotic expansion of the resolvent. Adding a bounded electric potential V with a fast enough decay at infinity then won't change the character of the obtained expansion. In the case $\alpha = 0$, which concerns Theorem 2.3 only, we repeat the same procedure with $H(B_0)$ replaced by $-\Delta$.

The reference operator. The reference operator $H(B_0)$, which will play the role of the free Hamiltonian when $\alpha \neq 0$, is associated to the radial magnetic field B_0 given by

$$B_0(x) = B_0(|x|) = \frac{\alpha}{|x|} \quad \text{if} \quad |x| < 1, \qquad B_0(x) = 0 \quad \text{otherwise.} \tag{4.2}$$

It is easily seen that $B_0 = \text{curl}\, A_0$ and that the flux of B_0 through $\mathbb{R}^2$ is equal to α. Let

$$R_0(\lambda + i0) = (H(B_0) - \lambda - i0)^{-1}. \tag{4.3}$$

We have

Proposition 4.1. *Let $\alpha \notin \mathbb{Z}$ and let $s > 3/2$. Then there exists $\widetilde{G}_1 \in \mathscr{B}(s, -s)$ such that*

$$R_0(\lambda + i0) = G_0 + \lambda^{\mu(\alpha)} G_1 + G_2(\lambda) \quad in \quad \mathscr{B}(s, -s), \tag{4.4}$$

where

$$\| G_2(\lambda) \|_{\mathscr{B}(s,-s)} = o\big(|\lambda|^{\mu(\alpha)}\big), \qquad \||\nabla G_2(\lambda)|\|_{\mathscr{B}(s,-s)} = o\big(|\lambda|^{\mu(\alpha)}\big) \qquad \lambda \to 0.$$

Moreover, if $\mu(\alpha) < 1/2$, then $\widetilde{G}_1 = G_1$.

Proposition 4.2. *Let $\alpha \in \mathbb{Z}$, $\alpha \neq 0$ and let $s > 3/2$. Then*

$$R_0(\lambda + i0) = G_0 + (\log \lambda)^{-1} \mathcal{G}_1 + \mathcal{G}_2(\lambda) \quad in \quad \mathscr{B}(s, -s), \tag{4.5}$$

where

$$\| \mathcal{G}_2(\lambda) \|_{\mathscr{B}(s,-s)} = o\big(\log |\lambda| \big)^{-1}\big), \qquad \||\nabla \mathcal{G}_2(\lambda)|\|_{\mathscr{B}(s,-s)} = o\big(\log |\lambda| \big)^{-1}\big) \qquad \lambda \to 0.$$

Proofs of Propositions 4.1 and 4.2 are given in sections 5 and 6 respectively.

Remark 4.3. Note that the field B_0 does not satisfy assumption 2.1.

The following Lemma plays a crucial role in our approach, for it allows us to extend the results of Propositions 4.1 and 4.2 to all magnetic fields satisfying assumption 2.1.

Lemma 4.4. *Let B satisfy assumption 2.1 and let α be given by (1.5). Suppose that $\alpha \neq 0$. Then there exists a differentiable vector field $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{curl} A = B$ and such that*

$$|\nabla \cdot A(x)| \lesssim (1 + |x|)^{-3-0}, \qquad |A(x) - A_0(x)| \lesssim (1 + |x|)^{-3-0} \tag{4.6}$$

where A_0 is given by (2.4).

Proof. Let $\widehat{A}$ be the vector potential associated to B by the Poincaré gauge:

$$\widehat{A}(x) = (-x_2, x_1) \int_0^1 B(t\, x_1, t\, x_2)\, t\, dt, \tag{4.7}$$

see e.g. [Th, Eq. (8.154)]. A direct calculation then shows that $\operatorname{curl} \hat{A} = B$. Passing to the polar coordinates we get

$$\widehat{A}(r, \theta) = r(-\sin\theta, \cos\theta) \int_0^1 B(t\, r, \theta)\, t dt = \frac{(-\sin\theta, \cos\theta)}{r} \int_0^r B(z, \theta)\, z\, dz \tag{4.8}$$

Hence with the notation

$$\psi(\theta) = \int_0^\infty B(z, \theta)\, z\, dz \tag{4.9}$$

we obtain the decomposition

$$\widehat{A}(r, \theta) = F_1(r, \theta) + F_2(r, \theta),$$

where

$$F_1(r, \theta) = \frac{(-\sin\theta, \cos\theta)}{r}\, \psi(\theta), \qquad F_2(r, \theta) = \frac{(\sin\theta, -\cos\theta)}{r} \int_r^\infty B(z, \theta)\, z\, dz. \tag{4.10}$$

Note that by equation (1.5)

$$\int_0^{2\pi} \psi(\theta)\, d\theta = 2\pi\alpha. \tag{4.11}$$

Now fix an $R > 1$ and let $\gamma \subset \mathbb{R}^2 \setminus D(0, R)$ be a piece-wise regular simple closed curve. We denote by $\Omega_\gamma \subset \mathbb{R}^2$ the region enclosed by γ. Note that $\operatorname{curl} F_1 = \operatorname{curl} A_0 = 0$ in $\mathbb{R}^2 \setminus D(0, R)$. Hence if Ω_γ does not intersect $D(0, R)$, then $\Omega_\gamma \setminus D(0, R)$ is simply connected and

$$\oint_\gamma F_1 = \oint_\gamma A_0 = 0.$$

On the other hand, if Ω_γ does intersect $D(0, R)$, then it contains $D(0, R)$ as a proper subset, for $\gamma \cap D(0, R) = \emptyset$. In this case, in view of (4.10) and (4.11), it turns out that

$$\oint_\gamma F_1 = 2\pi\alpha.$$

By the Stokes Theorem we then have

$$\oint_\gamma F_1 = 2\pi\alpha = \int_{D(0,2)} B_0(x)\, dx = \oint_\gamma A_0.$$

So, in either case it holds

$$\oint_\gamma (F_1 - A_0) = 0$$

This means that the vector field $F_1 - A_0$ is conservative in $\mathbb{R}^2 \setminus \overline{D(0, R)}$, since the latter is an open connected subset of $\mathbb{R}^2$. Moreover, by definition of F_1 and A_0 and by the hypothesis on B it follows that $F_1, A_0 \in C^1(\mathbb{R}^2 \setminus \overline{D(0, R)})$. Therefore there exists a scalar field $\varphi \in C^2(\mathbb{R}^2 \setminus \overline{D(0, R)})$ such that

$$F_1 + \nabla\varphi = A_0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \overline{D(0, R)} \qquad R > 1. \tag{4.12}$$

We now put $R \in (1, 2)$. Then $\varphi \in C^2(\mathbb{R}^2 \setminus D(0, 2))$. Since $\mathbb{R}^2 \setminus D(0, 2)$ is closed, we can extend φ into a function $\widehat{\varphi} \in C^2(\mathbb{R}^2)$ in such a way that $\widehat{\varphi} = \varphi$ on $\mathbb{R}^2 \setminus D(0, 2)$, see e.g. [Wh, Thm.1]. It now remains to define

$$A = \widehat{A} + \nabla\widehat{\varphi} = F_1 + F_2 + \nabla\widehat{\varphi} \qquad \text{in} \qquad \mathbb{R}^2.$$

Then $\operatorname{curl} A = \operatorname{curl} \widehat{A} = B$ and $\nabla \cdot A = \nabla \cdot \widehat{A} + \Delta\widehat{\varphi}$. However, $\nabla \cdot \widehat{A} \in L^\infty(\mathbb{R}^2)$ by equation (4.8) and hypothesis on B, and $\Delta\widehat{\varphi} \in L^\infty(R^2)$ by construction of $\widehat{\varphi}$. Therefore $\nabla \cdot A \in L^\infty(\mathbb{R}^2)$. Finally, from (2.2) we easily verify that

$$|\nabla \cdot F_2(x)| \lesssim (1 + |x|)^{-3-0}, \qquad |F_2(x)| \lesssim (1 + |x|)^{-3-0}$$

Since $A - A_0 = F_2$ on $\mathbb{R}^2 \setminus D(0, 2)$ by (4.12) and since $\nabla \cdot A_0 = 0$, this implies (4.6). $\qquad \square$

Lemma 4.5. *Let B satisfy assumption 2.1 and let $\alpha = 0$. Then there exists a differentiable vector field $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{curl} A = B$ and such that*

$$|\nabla \cdot A(x)| \lesssim (1 + |x|)^{-3-0}, \qquad |A(x)| \lesssim (1 + |x|)^{-3-0} \tag{4.13}$$

Proof. In this case we replace B_0 by a continuous radial field $\widetilde{B}_0$ of compact support and with zero flux:

$$\int_{\mathbb{R}^2} \widetilde{B}_0(x)\, dx = \int_{\mathbb{R}^2} \widetilde{B}_0(|x|)\, dx = 0. \tag{4.14}$$

Accordingly, we define

$$\widetilde{A}_0(x) = (-x_2, x_1) \int_0^1 \widetilde{B}_0(t\, x_1, t\, x_2)\, t\, dt = (-x_2, x_1) \int_0^1 \widetilde{B}_0(t\, |x|)\, t\, dt = \frac{(-x_2, x_1)}{|x|} \int_0^{|x|} \widetilde{B}_0(s)\, s\, ds.$$

Then $\operatorname{curl} \widetilde{A}_0 = \widetilde{B}_0$. Equation (4.14) and the fact that the support of $\widetilde{B}_0$ is compact then imply $\widetilde{A}_0(x) = 0$ for $|x|$ large enough. Now it remains to follow the proof of Lemma 4.4 with A_0 replaced by $\widetilde{A}_0$ and with $\alpha = 0$. $\qquad \square$

From now on we will associate to any B satisfying (2.2) a vector potential A given by Lemma 4.4 when $\alpha \neq 0$, or by Lemma 4.5 when $\alpha = 0$.

Remark 4.6. The fact that we use a particular vector potential generating the magnetic field B represents no restriction, since all our statements are gauge invariant. Note also that A is not uniquely defined by Lemmata 4.4 respectively 4.5.

Since $\nabla \cdot A_0 = 0$, see (2.4), from the definition of $T(B,V)$ it follows that

$$T(B,V) = H(B,V) - H(B_0). \tag{4.15}$$

Note also that $T(B,V)$ is symmetric on $W^{1,2}(\mathbb{R}^2)$, see (2.5).

Lemma 4.7. *If B satisfies assumption 2.1, then there exists $s_0 > 3/2$ such that the operator $1 + T(B,V)\,G_0$ is invertible in $\mathscr{B}(s,s)$ for all $3/2 < s \le s_0$.*

Proof. By Lemma 4.4 we can find $s_0 > 3/2$ such that the functions $\rho^{s_0}\,(\nabla \cdot A)\,\rho^{s_0}$ and $\rho^{s_0}\,|A - A_0|\,\rho^{s_0}$ are bounded. From Lemma 5.2 it then follows that the operator $T(B,V)\,G_0$ is compact from $L^2(\mathbb{R}^2, s)$ to $L^2(\mathbb{R}^2, s)$ for any $s \in (3/2, s_0)$. Assume that there exists $u \in L^2(\mathbb{R}^2, s)$ such that

$$u + T(B,V)G_0\,u = 0. \tag{4.16}$$

The resolvent equation in combination with (4.15) says that for every $\varepsilon > 0$

$$R_0(i\varepsilon) = (H(B,V) + i\varepsilon)^{-1}\,(1 + T(B,V)\,R_0(i\varepsilon))$$

holds on $L^2(\mathbb{R}^2, s)$. Hence using equation (2.3), Proposition 4.1 and passing to the limit $\varepsilon \to 0$ we arrive at

$$G_0\,u = H(B,V)^{-1}\,(1 + T(B,V)\,G_0)\,u = 0,$$

since $(1 + T(B,V)\,G_0)\,u$ by (4.16). But then $u = 0$ again in view of (4.16). This means that $\mathrm{Ker}(1 + T(B,V)\,G_0) = \{0\}$ and by the Fredholm alternative $1 + T(B,V)\,G_0$ is invertible. $\qquad\square$

Lemma 4.8. *If B satisfies assumption 2.1, then there exists $s_0 > 3/2$ such that the operator $1 + G_0\,T(B,V)$ is invertible in $\mathscr{B}(-s,-s)$ for all $3/2 < s \le s_0$.*

Proof. The claim follows by duality from Lemma 4.7. $\qquad\square$

Lemma 4.9. *Assume (2.1) and let $\alpha \notin \mathbb{Z}$. Then for $\lambda \to 0$ we have*

$$(1 + R_0(\lambda + i0)\,T(B,V))^{-1} = (1 + G_0\,T(B,V))^{-1} - \tag{4.17}$$
$$- (1 + G_0\,T(B,V))^{-1}\,G_1\,T(B,V)\,(1 + G_0\,T(B,V))^{-1}\,\lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)})$$

in $\mathscr{B}(-s,-s)$ for some $s > 3/2$.

Proof. From Lemma 4.4 and Proposition 4.1 it follows that $\|T(B,V)\,G_2(\lambda)\|_{\mathscr{B}(s,s)} = o(|\lambda|^{\mu(\alpha)})$ as $\lambda \to 0$. Hence by (4.4) and duality we have

$$1 + R_0(\lambda + i0)\,T(B,V) = 1 + G_0\,T(B,V) + G_1\,T(B,V)\,\lambda^{\mu(\alpha)} + o(|\lambda|^{\mu(\alpha)}) \tag{4.18}$$

in $\mathscr{B}(-s,-s)$ for $\lambda \to 0$. The operator $1 + G_0\,T(B,V)$ is invertible in $\mathscr{B}(-s,-s)$ in view of Lemma 4.8. Hence for $|\lambda|$ small enough the operator $1 + R_0(\lambda + i0)\,T(B,V)$ is invertible too and with the help of the Neumann series we arrive at (4.17). $\qquad\square$

Proof of Theorem 2.2. It suffices to prove the statement for $s \le s_0$ with s_0 given by Lemma 4.7. Similarly as for the free resolvent we introduce the notation

$$R(\lambda + i0) = (H(B,V) - \lambda - i0)^{-1}. \tag{4.19}$$

Since $1 + R_0(\lambda + i0)\,T(B,V)$ is invertible in $\mathscr{B}(-s,-s)$ for λ small enough, the resolvent equation yields

$$R(\lambda + i0) = (1 + R_0(\lambda + i0)\,T(B,V))^{-1}\,R_0(\lambda + i0), \tag{4.20}$$

which in combination with (4.4) and (4.17) gives

$$R(\lambda + i0) = (1 + G_0\,T(B,V))^{-1}\,G_0 + F_1(B,V)\,\lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \tag{4.21}$$

as $\lambda \to 0$, where

$$F_1(B,V) = (1 + G_0\,T(B,V))^{-1}\,G_1 - (1 + G_0\,T(B,V))^{-1}\,G_1\,T(B,V)\,(1 + G_0\,T(B,V))^{-1}\,G_0$$
$$= (1 + G_0\,T(B,V))^{-1}\,G_1(1 + T(B,V)\,G_0)^{-1}\left[1 + T(B,V)\,G_0 - T(B,V)\,(1 + G_0\,T(B,V))^{-1}\,G_0\right]$$

in $\mathscr{B}(s,-s)$. Since

$$T(B,V)\,(1 + G_0\,T(B,V))^{-1}\,G_0 = (1 + T(B,V)\,G_0)^{-1}\,T(B,V)\,G_0 \tag{4.22}$$

holds on $L^2(\mathbb{R}^2, s)$, the claim follows from (4.21). $\qquad\square$

Proof of Theorem 2.3. For $\alpha \neq 0$ the result follows from Proposition 4.2 and Lemma 4.4 in the same way as in the case $\alpha \notin \mathbb{Z}$; one only needs to replace the operators G_0 and G_1 by $\mathcal{G}_0$ and $\mathcal{G}_1$ respectively, and use Lemma 6.3 instead of Lemma 5.2.

When $\alpha = 0$, then we replace the reference operator $H(B_0)$ by the Laplacian $-\Delta$. Consequently, we write

$$H(B,V) = -\Delta + 2i\,A\cdot\nabla + i\,\nabla\cdot A + |A|^2 + V.$$

The statement now follows by Lemma 4.5 and Theorems 8.4 and 7.5.(iii) of [Mu]. $\qquad\square$

4.2. **Time decay.** We will use the formula

$$e^{-itH(B,V)} = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-it\lambda}\,R(\lambda + i0)\,d\lambda, \qquad t > 0. \tag{4.23}$$

To prove Theorems 2.5 and 2.7 we have to estimate the behavior of $R(\lambda + i0)$ for $|\lambda| \to \infty$.

Lemma 4.10. *Let $s > 5/2$. Suppose that B satisfies assumption 2.4 and that $|V(x)| \lesssim (1 + |x|)^{-\beta}$, $\beta > 3$. Let $R^{(j)}(\lambda + i0)$ denote the jth derivative of $R(\lambda + i0)$ with respect to λ in $\mathscr{B}(s,-s)$. Then*

$$\|R^{(2)}(\lambda + i0)\|_{\mathscr{B}(s,-s)} = \mathcal{O}(\lambda^{-3/2}) \qquad \lambda \to \infty, \tag{4.24}$$

$$\|R^{(2)}(\lambda + i0)\|_{\mathscr{B}(s,-s)} = \mathcal{O}(|\lambda|^{-3}) \qquad \lambda \to -\infty. \tag{4.25}$$

Proof. We will apply a perturbative argument. This time the unperturbed operator will be $H(B,0) = H(B)$, so that $H(B,V) = H(B) + V$. For convenience we denote

$$R_B(\lambda + i0) = (H(B) - \lambda - i0)^{-1}. \tag{4.26}$$

By assumption 2.4 and [Ro, Thm.5.1] we have

$$\|R_B^{(j)}(\lambda + i0)\|_{\mathscr{B}(s',-s')} = \mathcal{O}(\lambda^{-\frac{j+1}{2}}) \qquad j = 0,1,2, \qquad \lambda \to \infty. \tag{4.27}$$

for any $s' > j + \frac{1}{2}$. Now $R_B(\lambda + i0) \in \mathscr{B}(s', s' - \beta)$ for all $1/2 < s' < \beta - 1/2$ by (4.27) and (4.1). Since $V \in \mathscr{B}(s' - \beta, s')$ by assumption, it follows from (4.27) that

$$\|V\,R_B(\lambda + i0)\|_{\mathscr{B}(s',s')} \to 0 \qquad \forall\, s' \in (1/2,\, \beta - 1/2). \tag{4.28}$$

Hence $\|(1 + V\,R_B(\lambda + i0))^{-1}\|_{\mathscr{B}(s',s')} = \mathcal{O}(1)$ for all $1/2 < s' < \beta - 1/2$ as $\lambda \to \infty$. By duality the same holds for $\|(1 + R_B(\lambda + i0)\,V)^{-1}\|_{\mathscr{B}(-s',-s')}$. This in combination with (4.26) and the resolvent equation implies that the identities

$$R^{(1)}(\lambda + i0) = (1 + R_B(\lambda + i0)\,V)^{-1}\,R_B^{(1)}(\lambda + i0)\,(1 + V\,R_B(\lambda + i0))^{-1}$$

$$R^{(2)}(\lambda + i0) = (1 + R_B(\lambda + i0)\,V)^{-1}\,R_B^{(2)}(\lambda + i0)\,(1 + V\,R_B(\lambda + i0))^{-1}$$
$$- 2R^{(1)}(\lambda + i0)\,V\,R_B^{(1)}(\lambda + i0)(1 + V\,R_B(\lambda + i0))^{-1},$$

hold in $B(s,-s)$ for all $5/2 < s < \beta - 1/2$. The first equation shows that $\|R^{(1)}(\lambda + i0)\|_{\mathscr{B}(s',-s')} = \mathcal{O}(\lambda^{-1})$. By inserting this together with (4.27) into the second equation we obtain (4.24).

As for the negative values of λ, we note that $R(\lambda) = (H(B,V) - \lambda)^{-1}$ is analytic in $\mathscr{B}(L^2(\mathbb{R}^2))$ for $|\lambda|$ large enough. This is a consequence of the fact the $H(B,V)$ has finitely many negative eigenvalues, [Ko2, Thm.3.1]. Hence

$$\|R^{(2)}(\lambda + i0)\|_{\mathscr{B}(L^2(\mathbb{R}^2))} = \|(H(B,V) - \lambda)^{-3}\|_{\mathscr{B}(L^2(\mathbb{R}^2))} = \mathcal{O}(|\lambda|^{-3}) \qquad \lambda \to -\infty,$$

where we have used the fact that $\|(H(B,V) - \lambda)^{-1}\|_{\mathscr{B}(L^2(\mathbb{R}^2))} = (\mathrm{dist}(\sigma(H(B,V)), \lambda)^{-1}$. This implies equation (4.25). $\qquad\square$

Before we come to the proof of Theorems 2.5 and 2.7, we recall [JK, Lem.10.1], from which it follows that if $F : \mathbb{R} \to \mathscr{B}(s,-s)$ is such that $F(\lambda) = 0$ in a vicinity of zero and $F^{(2)} \in L^1(\mathbb{R}; \mathscr{B}(s,-s))$, then

$$\int_{\mathbb{R}} e^{-it\lambda} F(\lambda) \, d\lambda = o(t^{-2}) \qquad t \to \infty \tag{4.29}$$

in $\mathscr{B}(s,-s)$.

Proof of Theorems 2.5 and 2.7. As usual we will split the integral (4.23) into two parts relative to small and large energies. To this end we introduce a function $\phi \in C_0^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$ and $\phi = 1$ in a vicinity of 0. By the resolvent equation $R(\lambda + i0) = (1 + V R_B(\lambda + i0))^{-1} V R_B(\lambda + i0)$ and equations (4.27), (4.28) we have $\|R(\lambda + i0)\|_{\mathscr{B}(s,-s)} = \mathcal{O}(\lambda^{-\frac{1}{2}})$ as $\lambda \to \infty$ for $1/2 < s$. In view of (4.1) Theorems 2.2, 2.3 it thus follows that $R(\lambda + i0)$ is uniformly bounded on $(0,\infty)$ in $\mathscr{B}(s,-s)$ for $1/2 < s$. On the other hand for $\lambda < 0$ the operator $R(\lambda + i0) P_c$ is analytic in λ with respect to the norm $\|\cdot\|_{\mathscr{B}(L^2(\mathbb{R}^2))}$. Hence Lemma 4.10 in combination with equation (4.29) gives

$$\int_{\mathbb{R}} e^{-it\lambda} (1 - \phi(\lambda)) R(\lambda + i0) P_c \, d\lambda = o(t^{-2}) \qquad t \to \infty \tag{4.30}$$

in $\mathscr{B}(s,-s)$ for all $s > 5/2$. To estimate the contribution to (4.23) from small values, we recall two results on Fourier transform:

$$\frac{1}{2\pi i} \int_{\mathbb{R}} e^{-it\lambda} (\lambda + i0)^\nu \, d\lambda = \frac{i \sin(\pi\nu)}{\pi} e^{i\pi\nu/2} \Gamma(1 + \nu) \, t^{-1-\nu} \qquad \nu \in \mathbb{R}, \tag{4.31}$$

and

$$\frac{1}{2\pi i} \int_{\mathbb{R}} e^{-it\lambda} (\log(\lambda + i0))^{-k} \, d\lambda = i \sum_{j=k}^{2} (-1)^k \, k \, t^{-1} (\log t)^{-k-1} + \mathcal{O}(t^{-1}(\log t)^{-4}) \tag{4.32}$$

for $k = 1, 2$ as $t \to \infty$, see e.g. [Mu, Lems.6.6 -6.7]. The last two equations in combination with (4.23), (4.29) and Theorems 2.2 and 2.3 then imply that as $t \to \infty$

$$e^{-itH(B,V)} P_c = i \, F_1(B,V) \, \frac{\sin(\pi\mu(\alpha))}{\pi} \, e^{i\frac{\pi\mu(\alpha)}{2}} \, \Gamma(1 + \mu(\alpha)) \, t^{-1-\mu(\alpha)} + o(t^{-1-\mu(\alpha)}) \quad \alpha \notin \mathbb{Z} \tag{4.33}$$

and

$$e^{-itH(B,V)} P_c = -i \, \mathcal{F}_1(B,V) \, t^{-1} (\log t)^{-2} + o(t^{-1} (\log t)^{-2}) \qquad\qquad \alpha \in \mathbb{Z} \tag{4.34}$$

in $\mathscr{B}(s,-s)$ for all $s > 5/2$. $\qquad\square$

Hence our main results are established provided we can prove auxiliary Propositions 4.1 and 4.2. This will be done in the following two sections. However, the analysis of the resolvent of the operator $H(B_0)$ leads to rather lengthly calculations. Therefore, in order to keep the exposition as smooth as possible, we will often make use of auxiliary technical results presented in section 7 and of selected properties of certain special functions which are described in Appendices A, B and C.

5. Operator $H(B_0)$: non-integrer flux

We are going to study the resolvent $(H(B_0) - \lambda - i0)^{-1}$ separately for positive and negative values of λ. We first derive an explicit expression for the integral kernel, and then we will discuss the behavior of $(H(B_0) - \lambda - i0)^{-1}$ in the vicinity of zero in a suitable operator norm. For the sake of brevity we will suppose that $\mu(\alpha) < 1/2$, which means that

$$\exists!\, k(\alpha): \quad \mu(\alpha) = |k(\alpha) + \alpha|. \tag{5.1}$$

The case $\mu(\alpha) = 1/2$ when the minimum in (1.6) is attained for two different values of $k \in \mathbb{Z}$ can be treated in a completely analogous way.

5.1. The case $\lambda > 0$. We have

Lemma 5.1. *Assume that $\alpha \notin \mathbb{Z}$. For any $x, y \in \mathbb{R}^2$ it holds*

$$R_0(\lambda; x, y) = G_0(x, y) + G_1(x, y)\,\lambda^{\mu(\alpha)} + G_2^+(\lambda; x, y), \tag{5.2}$$

where $G_0(x, y), G_1(x, y)$ are given by equations (5.27), (5.29), and $G_2^+(\lambda; x, y) = o(\lambda^{\mu(\alpha)})$ as $\lambda \to 0+$.

Proof. Without loss of generality we may assume that $\alpha > 0$. To calculate $R_0(\lambda; x, y)$ we write the vector potential A_0 associated to the field B_0 through (2.4) in polar coordinates: $A_0(r, \theta) = a_0(r)\,(-\sin\theta, \cos\theta)$, where

$$a_0(r) = \alpha \quad \text{if} \quad r < 1, \qquad a(r) = \frac{\alpha}{r} \quad \text{if} \quad r \geq 1. \tag{5.3}$$

The quadratic form associated to $H(B_0)$ now reads

$$\int_0^\infty \int_0^{2\pi} \left(|\partial_r u|^2 + |i\, r^{-1} \partial_\theta u + a_0(r)\, u|^2 \right) r\, dr d\theta. \tag{5.4}$$

By expanding a given test function $u \in L^2(\mathbb{R}_+ \times (0, 2\pi))$ into a Fourier series with respect to the basis $\{e^{im\theta}\}_{m \in \mathbb{Z}}$ of $L^2((0, 2\pi))$, we obtain the decomposition

$$H(B_0) = \sum_{m \in \mathbb{Z}} \oplus (h_m \otimes \mathrm{id})\, \Pi_m, \tag{5.5}$$

where h_m are the operators in $L^2(\mathbb{R}_+, r dr)$ acting on thier domain as

$$h_m f = -f'' - \frac{1}{r} f' + \left(\frac{m}{r} + a_0(r) \right)^2 f, \tag{5.6}$$

and Π_m is given by

$$(\Pi_m u)(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{im(\theta - \theta')}\, u(r, \theta')\, d\theta'.$$

The integral kernel of $R_0(\alpha; \lambda)$ then splits accordingly:

$$R_0(\lambda; x, y) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} R_0^m(\lambda; r, r')\, e^{im(\theta - \theta')}. \tag{5.7}$$

Here $R_0^m(\lambda; r, r')$ denotes the integral kernel of $(h_m - \lambda - i0)^{-1}$ in $L^2(\mathbb{R}_+, r dr)$. Now consider the operators

$$\mathfrak{H}_m = \mathcal{U}\, h_m\, \mathcal{U}^{-1} \qquad \text{in} \qquad L^2(\mathbb{R}_+, dr), \tag{5.8}$$

where $\mathcal{U} : L^2(\mathbb{R}_+, r\, dr) \to L^2(\mathbb{R}_+, dr)$ is a unitary mapping defined by $(\mathcal{U}f)(r) = r^{1/2} f(r)$. Note that $\mathfrak{H}_m$ is subject to Dirichlet boundary condition at 0 and that it acts, on its domain, as

$$\mathfrak{H}_m f = -f'' - \frac{1}{4r^2} f + \left(\frac{m}{r} + a_0(r) \right)^2 f. \tag{5.9}$$

Let $\mathfrak{R}_0^m(\lambda; r, r')$ denote the integral kernel of $(\mathfrak{H}_m - \lambda - i0)^{-1}$ in $L^2(\mathbb{R}_+, dr)$. By (5.8)

$$R_0^m(\lambda; r, r') = \frac{1}{\sqrt{rr'}}\, \mathfrak{R}_0^m(\lambda; r, r'). \tag{5.10}$$

Hence it suffices to calculate $\mathfrak{R}_0^m(\lambda; r, r')$. To do so we will find two solutions, $f_{m,\lambda}$ and $\phi_{m,\lambda}$ to the generalized eigenvalue equation

$$-f'' - \frac{f}{4r^2} + \left(\frac{m^2}{r^2} + \frac{2\,m\,a_0(r)}{r} + a_0^2(r)\right) f = \lambda\,f, \tag{5.11}$$

such that $f_{m,\lambda+i\varepsilon} \in H_0^1((0,1), r\,dr)$ and $\phi_{m,\lambda+i\varepsilon} \in H^1((1,\infty), r\,dr)$ for $\varepsilon > 0$. The Sturm-Liouville theory then gives

$$\mathfrak{R}_0^m(\lambda; r, r') = \frac{1}{W_m(\lambda)} \begin{cases} f_{m,\lambda}(r)\,\phi_{m,\lambda}(r'), & r \le r', \\[2mm] f_{m,\lambda}(r')\,\phi_{m,\lambda}(r), & r' < r, \end{cases} \tag{5.12}$$

where

$$W_m(\lambda) = \phi_{m,\lambda}\,f'_{m,\lambda} - \phi'_{m,\lambda}\,f_{m,\lambda}$$

is the Wronskian of $\phi_{m,\lambda}$ and $f_{m,\lambda}$. In view of (5.9) it follows that $W\{\phi_{m,\lambda}, f_{m,\lambda}\}$ is constant. In order to find $f_{m,\lambda}$ and $\phi_{m,\lambda}$ we have to solve equation (5.11) separately for $r \le 1$ and $r > 1$ and match the solutions smoothly. Let

$$\kappa = \sqrt{\alpha^2 - \lambda}\,, \tag{5.13}$$

To simplify the notation in the sequel we define the functions $v_m(\lambda, \cdot), u_m(\lambda, \cdot) : (0,1) \to \mathbb{R}$ by

$$v_m(\lambda, r) = e^{-\kappa r}\,(2\kappa\,r)^{|m|}\,M\left(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa}, 1 + 2|m|, 2\kappa r\right) \tag{5.14}$$

$$u_m(\lambda, r) = e^{-\kappa r}\,(2\kappa\,r)^{|m|}\,U\left(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa}, 1 + 2|m|, 2\kappa r\right)\,, \tag{5.15}$$

where $M(a, b, z)$ and $U(a, b, z)$ are the Kummer's confluent hypergeometric functions, see [AS, Sec.13.1]. Using equations (5.3), (5.9) and a suitable change of variables we find they (5.11) lead to a Whittaker's equation for $r \le 1$ and to the Bessel equation for $r > 1$, see [AS, Chaps.9&13]. We thus obtain

$$f_{m,\lambda}(r) = \begin{cases} \sqrt{r}\,v_m(\lambda, r), & r \le 1, \\[2mm] \sqrt{r}\left(A_m(\lambda)\,J_{|\alpha+m|}(\sqrt{\lambda}\,r) + B_m(\lambda)\,Y_{|\alpha+m|}(\sqrt{\lambda}\,r)\right), & 1 < r, \end{cases} \tag{5.16}$$

where $A_m(\lambda), B_m(\lambda)$ are numerical coefficients whose values will be determined later. Similarly,

$$\phi_{m,\lambda}(r) = \begin{cases} \sqrt{r}\left(C_m(\lambda)\,v_m(\lambda, r) + D_m(\lambda)\,u_m(\lambda, r)\right), & r \le 1, \\[2mm] \sqrt{r}\left(J_{|\alpha+m|}(\sqrt{\lambda}\,r) + i\,Y_{|\alpha+m|}(\sqrt{\lambda}\,r)\right), & 1 < r, \end{cases} \tag{5.17}$$

In order to find the coefficients $A_m(\lambda)$ and $B_m(\lambda)$ we impose the differentiability condition at $r = 1$ on the function $\frac{1}{\sqrt{r}}\,f_{m,\lambda}(r)$ which is equivalent to the differentiability (at $r = 1$) of $f_{m,\lambda}$. With the help of (B.3) we get

$$A_m(\lambda) = \frac{\pi}{2}\left((v_m(\lambda, 1)\,|\alpha + m| - v'_m(\lambda, 1))\,Y_{|\alpha+m|}(\sqrt{\lambda}) - \sqrt{\lambda}\,v_m(\lambda, 1)\,Y_{|\alpha+m|+1}(\sqrt{\lambda})\right) \tag{5.18}$$

$$B_m(\lambda) = \frac{\pi}{2}\left((v'_m(\lambda, 1) - |\alpha + m|\,v_m(\lambda, 1))\,J_{|\alpha+m|}(\sqrt{\lambda}) + \sqrt{\lambda}\,v_m(\lambda, 1)\,J_{|\alpha+m|+1}(\sqrt{\lambda})\right). \tag{5.19}$$

Here $v'_m(\lambda, 1)$ and $u'_m(\lambda, 1)$ denote the derivatives of v_m and u_m with respect to r. Similarly, the matching conditions for $\frac{1}{\sqrt{r}} \phi_{m,\lambda}$ yield

$$C_m(\lambda) = \frac{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}{\Gamma(1 + 2|m|)} \left((u'_m(\lambda, 1) - u_m(\lambda, 1) |\alpha + m|) J_{|\alpha+m|}(\sqrt{\lambda}) + \sqrt{\lambda}\, u_m(\lambda, 1) J_{|\alpha+m|+1}(\sqrt{\lambda}) \right.$$
$$\left. + i \left[u'_m(\lambda, 1) - (u_m(\lambda, 1) |\alpha + m|) Y_{|\alpha+m|}(\sqrt{\lambda}) + \sqrt{\lambda}\, u_m(\lambda, 1) Y_{|\alpha+m|+1}(\sqrt{\lambda}) \right] \right)$$
(5.20)

$$D_m(\lambda) = \frac{2\,\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}{\pi\,\Gamma(1 + 2|m|)} \left(iA_m(\lambda) - B_m(\lambda) \right), \tag{5.21}$$

where we have used the identity

$$v'_m(\lambda, 1)\, u_m(\lambda, 1) - v_m(\lambda, 1)\, u'_m(\lambda, 1) = \frac{\Gamma(1 + 2|m|)}{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}\,, \tag{5.22}$$

see [AS, Eq.13.1.22]. From (B.3) and [AS, Eq.13.1.22] we then calculate the Wronskian

$$W_m(\lambda) = D_m(\lambda)\,\frac{\Gamma(1 + 2\,|m|)}{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})} = \frac{2}{\pi}(B_m(\lambda) - iA_m(\lambda)). \tag{5.23}$$

These formulas in combination with (5.12) provide the expression for $\mathfrak{R}_0^m(\lambda; r, r')$ and consequently for $R_0(\lambda; x, y)$, via (5.7) and (5.10). To analyse the asymptotic behavior of $R_0(\alpha, \lambda; x, y)$ as $\lambda \to 0+$ we introduce the following shorthands:

$$a_m = v_m(0, 1), \qquad a'_m = v'_m(0, 1), \qquad b_m = u_m(0, 1), \qquad b'_m = u'_m(0, 1)\,.$$

Then, in view of (B.4) and (5.18)-(5.21) as $\lambda \to 0+$ we have:

$$A_m(\lambda) = \frac{\Gamma(|m + \alpha|)}{2} \left(a'_m + |m + \alpha| a_m \right) \left(\frac{1}{2}\sqrt{\lambda} \right)^{-|\alpha+m|} \left(1 + \mathcal{O}(\lambda) \right)$$
$$+ \frac{i\,\pi\,\cot(|m + \alpha|\,\pi)}{2\,\Gamma(|m + \alpha| + 1)} (a'_m - |m + \alpha| a_m) \left(\frac{1}{2}\sqrt{\lambda} \right)^{|\alpha+m|} \left(1 + \mathcal{O}(\lambda) \right) \tag{5.24}$$

$$B_m(\lambda) = \frac{\pi}{2\,\Gamma(|m + \alpha| + 1)} (a'_m - |m + \alpha| a_m) \left(\frac{1}{2}\sqrt{\lambda} \right)^{|\alpha+m|} \left(1 + \mathcal{O}(\lambda) \right), \tag{5.25}$$

where the error terms are uniform in m. Similarly we find

$$C_m(\lambda) = \frac{\Gamma(\frac{1}{2} + m + |m|)\,(1 + i\cot(|m + \alpha|\,\pi))}{\Gamma(1 + 2|m|)\,\Gamma(|m + \alpha| + 1)} (b'_m - |m + \alpha|\, b_m) \left(\frac{1}{2}\sqrt{\lambda} \right)^{|\alpha+m|} (1 + \mathcal{O}(\lambda))$$
$$- i\,\frac{\Gamma(\frac{1}{2} + m + |m|)\,\Gamma(|m + \alpha|)}{\pi\Gamma(1 + 2|m|)} (b'_m + |m + \alpha|\, b_m) \left(\frac{1}{2}\sqrt{\lambda} \right)^{-|\alpha+m|} (1 + \mathcal{O}(\lambda)) \tag{5.26}$$

Hence from equations (5.7), (5.10), (5.12) and (5.16)-(5.23), after elementary but somewhat lengthly calculations, we obtain

$$\lim_{\lambda \to 0+} R_0(\lambda; x, y) =: G_0(x, y) = \sum_{m \in \mathbb{Z}} G_{m,0}(r, r')\, e^{im(\theta - \theta')}\,, \tag{5.27}$$

where

$$G_{m,0}(r, r') = \frac{\Gamma(\frac{1}{2} + m + |m|)}{\Gamma(1 + 2|m|)} \, v_m(0, r) \left(u_m(0, r') - \frac{b'_m + |m + \alpha| \, b_m}{a'_m + |m + \alpha| \, a_m} \, v_m(0, r') \right) \qquad r < r' \leq 1,$$

$$G_{m,0}(r, r') = \frac{v_m(0, r)}{a'_m + |m + \alpha| \, a_m} \, (r')^{-|m+\alpha|} \qquad\qquad r \leq 1 < r',$$

$$(5.28)$$

$$G_{m,0}(r, r') = \frac{1}{2|m + \alpha|} \left[\left(\frac{r}{r'}\right)^{|m+\alpha|} - \frac{a'_m - |m + \alpha| \, a_m}{a'_m + |m + \alpha| \, a_m} \, (rr')^{-|m+\alpha|} \right] \qquad 1 < r < r'$$

Note that $a'_m + |m + \alpha| \, a_m > 0$ for all $m \in \mathbb{Z}$, see (5.31). Consider now the remainder term in (5.27). Since $\alpha > 0$ by assumption, we have $k(\alpha) \leq 0$. To simplify the notation we will write

$$k(\alpha) = k, \qquad \mu(\alpha) = \mu = |k + \alpha|.$$

Then, using again (5.7), (5.10), (5.12) and (5.16)-(5.23) we find that

$$\lim_{\lambda \to 0+} \lambda^{-\mu} \left(R_0(\lambda; x, y) - G_0(x, y) \right) = G_1(x, y) = g_1(r, r') \, e^{ik(\theta - \theta')}, \qquad (5.29)$$

where

$$g_1(r, s) = \frac{2\pi \, v_k(0, r) \, v_k(0, s)}{4^\mu \, \Gamma^2(\mu) \, (a'_k + \mu \, a_k)^2} \, (i - \cot(\mu\pi)) \qquad\qquad r < s \leq 1,$$

$$g_1(r, s) = \frac{\pi \, v_k(0, r) \, (i - \cot(\mu\pi))}{\mu \, 4^\mu \, \Gamma^2(\mu) \, (a'_k + \mu \, a_k)} \left(s^\mu - \frac{a'_k - \mu \, a_k}{a'_k + \mu \, a_k} \, s^{-\mu} \right) \qquad\qquad r \leq 1 < s, \qquad (5.30)$$

$$g_1(r, s) = \frac{\pi \, (i - \cot(\mu\pi))}{4^\mu \, 2 \, \mu^2 \, \Gamma^2(\mu)} \left(r^\mu - \frac{a'_k - \mu \, a_k}{a'_k + \mu \, a_k} \, r^{-\mu} \right) \left(s^\mu - \frac{a'_k - \mu \, a_k}{a'_k + \mu \, a_k} \, s^{-\mu} \right) \qquad 1 \leq r < s.$$

This implies (5.2). □

In the sequel we denote by G_0 and G_1 the operators on $L^2(\mathbb{R}^2)$ with kernels $G_0(x, y)$ and $G_1(x, y)$. The following Lemma shows that these operators have the properties needed for the proof of Proposition 4.1.

Lemma 5.2. *Let $\alpha \notin \mathbb{Z}$ and let $s > 1$. Then $\rho^{-s} \, G_j \, \rho^{-s}$ and $\rho^{-s} \, \nabla \, G_j \, \rho^{-s}$ with $j = 0, 1$ are compact operators from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2, \mathbb{C}^2)$ respectively.*

Proof. Consider first the operator G_0. We denote by $G_{m,0}$ the integral operator in $L^2(\mathbb{R}_+, rdr)$ with the kernel $G_{m,0}(r, r')$. For $m = 0$ it is easily seen from (5.27), using equations (A.3) and (A.4), that the kernels $G_{0,0}(r, r')$ and $\partial_{r'} G_{0,0}(r, r')$ generate Hilbert-Schmidt operators in $L^2(\mathbb{R}_+, rdr)$. Hence we may suppose in the rest of the proof that $m \neq 0$.

To continue we note that with the help of equations (A.3), (A.5), (A.6) and (A.7) it is straightforward to verify that

$$|b'_m + |m + \alpha| \, b_m| \lesssim \frac{|2\alpha|^{-|m|} \, \Gamma(2|m|)}{\Gamma(\frac{1}{2} + 2|m|)}, \qquad |a'_m + |m + \alpha| \, a_m| \gtrsim (1 + |m|) \, |2\alpha|^{|m|} \qquad \forall \, m \in \mathbb{Z}. \quad (5.31)$$

On the other hand, equations (5.14), (5.15) in combination with (A.3) and (A.6) imply

$$|v_m(0, r)| \leq e^{\kappa r} \, (2\kappa r)^{|m|}, \qquad |u_m(0, r')| \leq C_\alpha \, e^{\kappa r'} \, (2\kappa r')^{-|m|} \, \Gamma(2|m|), \qquad (5.32)$$

with a constant C_α independent of m. From (5.28) and (5.32) we thus obtain the following estimates:

$$|G_{m,0}(r,r')| \lesssim \frac{1}{|m+\alpha|} \begin{cases} (r/r')^{|m|} & r < r' \leq 1, \\[2mm] (r/r')^{|m+\alpha|} & 1 \leq r < r' \\[2mm] r^{|m|}\,(r')^{-|m+\alpha|} & r < 1 < r'. \end{cases} \tag{5.33}$$

and

$$\big|\frac{m}{r'}\,G_{m,0}(r,r')\big| + \big|\partial_{r'}G_{m,0}(r,r')\big| \lesssim \frac{1}{r'}\,|m+\alpha|\,|G_{m,0}(r,r')|, \tag{5.34}$$

where the roles of r and r' have to be interchanged if $r' < r$. We thus find that

$$\|\,\rho^{-s}\,G_{m,0}\,\rho^{-s}\|_{HS(\mathbb{R}_+,rdr)} \lesssim |m+\alpha|^{-3/2} \qquad \forall\, m \in \mathbb{Z}. \tag{5.35}$$

Here $\|\cdot\|_{HS(\mathbb{R}_+,rdr)}$ denotes the Hilbert-Schmidt norm on $L^2(\mathbb{R}_+,rdr)$. This shows that $\rho^{-s}\,G_{m,0}\,\rho^{-s}$ is compact on $L^2(\mathbb{R}_+,rdr)$ for every m and that $\|\rho^{-s}\,G_{m,0}\,\rho^{-s}\|_{L^2(\mathbb{R}_+,rdr)}$ converges to zero as $|m| \to \infty$. Hence $\rho^{-s}\,G_0\,\rho^{-s}$ is compact on $L^2(\mathbb{R}^2)$ in view of (5.27). Moreover, from (5.33) and (5.34) we infer that

$$\Big\|\rho^{-s}\,\frac{m}{r'}\,G_{m,0}\,\rho^{-s}\Big\|_{HS(\mathbb{R}_+,rdr)} + \|\rho^{-s}\,\partial_{r'}G_{m,0}\,\rho^{-s}\|_{HS(\mathbb{R}_+,rdr)} \lesssim |m+\alpha|^{-1/2}.$$

This implies that the operators $\rho^{-s}\,\frac{m}{r'}\,G_{m,0}\,\rho^{-s}$ and $\rho^{-s}\,\partial_{r'}G_{m,0}\,\rho^{-s}$ are compact on $L^2(\mathbb{R}_+,rdr)$ for every m, and that their operator norm tends to zero as $|m| \to \infty$. Since the integral kernel of the operator ∇G_0 is given by

$$\nabla G_0(r,r',\theta,\theta') = \sum_{m\in\mathbb{Z}} e^{im(\theta-\theta')}\,\Big(\partial_{r'}G_{m,0}(r,r')\,,\,\frac{m}{r'}\,G_{m,0}(r,r')\Big),$$

this proves the compactness of the operator $\rho^{-s}\,\nabla G_0\,\rho^{-s}$ on $L^2(\mathbb{R}^2)$.

The analysis of the operators $\rho^{-s}\,G_1\,\rho^{-s}$ and $\rho^{-s}\,\nabla G_1\,\rho^{-s}$ is easier, since we have a contribution only from $m = k(\alpha)$. Indeed, from the explicit expression for $g_1(r,r')$ it follows that

$$\rho^{-s}(r')\,g_1(r,r')\,\rho^{-s}(r), \qquad \rho^{-s}(r')\,\partial_{r'}g_1(r,r')\,\rho^{-s}(r), \qquad \rho^{-s}(r')\,\frac{1}{r'}\,g_1(r,r')\,\rho^{-s}(r),$$

are Hilbert-Schmidt kernels on $L^2(\mathbb{R}_+,rdr)$. $\qquad\square$

Next we are going to study the behavior of the remainder term in (5.2). Let us denote by $G_2^+(\lambda)$ the integral operator in $L^2(\mathbb{R}^2)$ with kernel $G_2^+(\lambda;x,y)$ given by (5.2) and (5.7).

Lemma 5.3. *Let $\alpha \notin \mathbb{Z}$ and assume that $3/2 < s < 3/2 + \mu(\alpha)$. Then, as $\lambda \to 0+$ we have $G_2^+(\lambda) = o(\lambda^{\mu(\alpha)})$ in $\mathscr{B}(s,-s)$.*

Proof. From Lemma 5.1 it follows that

$$\|G_2^+(\lambda)\|_{\mathscr{B}(s,-s)} = \|\rho^{-s}\,G_2^+(\lambda)\,\rho^{-s}\|_{\mathscr{B}(L^2(\mathbb{R}^2))} = \sup_{m\in\mathbb{Z}} \|\rho^{-s}\,G_{m,2}^+(\lambda)\,\rho^{-s}\|_{\mathscr{B}(L^2(\mathbb{R}_+,rdr))}, \tag{5.36}$$

where $G_{m,2}^+(\lambda)$ is the integral operator with kernel

$$G_{m,2}^+(\lambda,r,r') = R_0^m(\lambda,r,r') - G_{m,0}(r,r') - \lambda^{\mu(\alpha)}\,\delta_{m,k(\alpha)}\,g_1(r,r'), \tag{5.37}$$

and δ_{jk} denotes the Kronecker delta. Let us denote

$$M_{(0,1)\times(0,1)}^1[G_{m,2}^+(\lambda)] = \sup_{0\leq r\leq 1}\int_0^1 |\rho^{-s}(r)\,G_{m,2}^+(\lambda,r,r')\,\rho^{-s}(r')|\,r'\,dr'$$

$$M_{(0,1)\times(0,1)}^2[G_{m,2}^+(\lambda)] = \sup_{0\leq r'\leq 1}\int_0^1 |\rho^{-s}(r)\,G_{m,2}^+(\lambda,r,r')\,\rho^{-s}(r')|\,r\,dr$$

We will estimate the norm of $\rho^{-s} G_{m,2}^+(\lambda)\,\rho^{-s}$ in $L^2(\mathbb{R}_+, rdr)$ as follows:

$$\|\rho^{-s} G_{m,2}^+(\lambda)\,\rho^{-s}\|^2_{\mathscr{B}(L^2(\mathbb{R}_+, rdr))} \leq M^1_{(0,1)\times(0,1)}[G_{m,2}^+(\lambda)]\,M^2_{(0,1)\times(0,1)}[G_{m,2}^+(\lambda)] \tag{5.38}$$

$$+ 2\int_0^1 \rho^{-2s}(r)\int_1^\infty |G_{m,2}^+(\lambda, r, r')|^2\,\rho^{-2s}(r')\,r'\,dr'\,r\,dr$$

$$+ \int_1^\infty\int_1^\infty \rho^{-2s}(r)\,|G_{m,2}^+(\lambda, r, r')|^2\,\rho^{-2s}(r')\,rr'\,dr\,dr$$

Here we use the Schur-Holmgren bound for the part of the operator relative to the region $(0,1)\times(0,1)$ and the Hilbert-Schmidt norm on the rest of $\mathbb{R}_+ \times \mathbb{R}_+$.

Let us first estimate the last term on the right hand side of (5.38). We recall the formula for the resolvent kernel on $(1,\infty)\times(1,\infty)$:

$$R_0^m(\lambda, r, r') = \frac{1}{W_m(\lambda)}\Big[A_m(\lambda)\,J_{|m+\alpha|}(\sqrt{\lambda}\,r)\,J_{|m+\alpha|}(\sqrt{\lambda}\,r') + i\,A_m(\lambda)\,J_{|m+\alpha|}(\sqrt{\lambda}\,r)\,Y_{|m+\alpha|}(\sqrt{\lambda}\,r')$$

$$+ B_m(\lambda)\,Y_{|m+\alpha|}(\sqrt{\lambda}\,r)\,J_{|m+\alpha|}(\sqrt{\lambda}\,r') + i\,B_m(\lambda)\,Y_{|m+\alpha|}(\sqrt{\lambda}\,r)\,Y_{|m+\alpha|}(\sqrt{\lambda}\,r')\Big], \tag{5.39}$$

see equations (5.12), (5.16) and (5.17).

We now use identity (B.1), keeping in mind that $\alpha\notin\mathbb{Z}$ to write $Y_{|m+\alpha|}$ in terms of $J_{|m+\alpha|}$ and $J_{-|m+\alpha|}$, and estimate each term in the above sum separately. To this end we are going to use integral operators $T_\pm^m$ on $L^2((1,\infty), rdr)$ with kernels

$$T_+^m(r, r') = \frac{4^{-|m+\alpha|}\,(rr')^{|m+\alpha|}}{\Gamma^2(|m+\alpha|+1)}\,, \qquad T_-^m(r, r') = \left(\frac{rr'}{4}\right)^{-|m+\alpha|}\Gamma^2(|m+\alpha|)\,\frac{\sin^2(|m+\alpha|\pi)}{\pi^2}$$

$$T_\pm^m(r, r') = -(r/r')^{|m+\alpha|}\,\frac{\sin(|m+\alpha|\,\pi)}{\pi\,|m+\alpha|}\,, \quad T_\mp^m(r, r') = T_\pm^m(r', r). \tag{5.40}$$

In view of Lemma 7.1, equation (7.1) of Lemma 7.2, and equation (5.1) we then have

$$J_{|m+\alpha|}(\sqrt{\lambda}\,r)\,J_{|m+\alpha|}(\sqrt{\lambda}\,r') = \begin{cases} \lambda^{\mu(\alpha)}\,T_+^{k(\alpha)}(r, r')(1+\mathcal{O}(\lambda^\varepsilon)) & m = k(\alpha), \\ (1+m^2)^{-1}\,\mathcal{O}(\lambda^{\frac{1}{2}+\varepsilon}) & m\neq k(\alpha) \end{cases} \tag{5.41}$$

as $\lambda\to 0+$ with respect to the Hilbert-Schmidt norm on $L^2((1,\infty), rdr)$ and with the error terms uniform in m. Similarly, using equations (7.2) and (7.3) of Lemma 7.2, we find that

$$\lambda^{|m+\alpha|}\,J_{-|m+\alpha|}(\sqrt{\lambda}\,r)\,J_{-|m+\alpha|}(\sqrt{\lambda}\,r') = T_-^m(r, r')(1+\mathcal{O}(\lambda))$$

$$J_{|m+\alpha|}(\sqrt{\lambda}\,r)\,J_{-|m+\alpha|}(\sqrt{\lambda}\,r') = T_\pm^m(r, r') + (1+|m|)^{-1}\,\mathcal{O}(\lambda^{\frac{1}{2}+\varepsilon})$$

$$J_{-|m+\alpha|}(\sqrt{\lambda}\,r)\,J_{|m+\alpha|}(\sqrt{\lambda}\,r') = T_\mp^m(r, r') + (1+|m|)^{-1}\,\mathcal{O}(\lambda^{\frac{1}{2}+\varepsilon})$$

as $\lambda\to 0+$ with respect to the Hilbert-Schmidt norm on $L^2((1,\infty), rdr)$ and error terms uniform in m. On the other hand, from equations (5.24) and (5.25) it follows that

$$\frac{B_m(\lambda)}{W_m(\lambda)} = \frac{i\,\pi^2\,4^{-|m+\alpha|}}{2\,|m+\alpha|\,\Gamma^2(|m+\alpha|)}\,\left(\frac{a'_m - |m+\alpha|\,a_m}{a'_m + |m+\alpha|\,a_m}\right)\lambda^{|m+\alpha|}(1+\mathcal{O}(\lambda))$$

$$+ \frac{\pi^3\,(1+i\cot(|m+\alpha|\,\pi))}{4^{2|m+\alpha|}\,2\,|m+\alpha|^2\,\Gamma^4(|m+\alpha|)}\,\left(\frac{a'_m - |m+\alpha|\,a_m}{a'_m + |m+\alpha|\,a_m}\right)^2\lambda^{2|m+\alpha|}(1+\mathcal{O}(\lambda)) \tag{5.42}$$

and

$$\frac{A_m(\lambda)}{W_m(\lambda)} = \frac{\pi}{2}\,\left(i + \frac{\pi\cot(|m+\alpha|\,\pi)\,\lambda^{|m+\alpha|}}{4^{|m+\alpha|}\,|m+\alpha|\,\Gamma^2(|m+\alpha|)}\,\left(\frac{a'_m - |m+\alpha|\,a_m}{a'_m + |m+\alpha|\,a_m}\right)\right)(1+\mathcal{O}(\lambda)), \tag{5.43}$$

with the respective error terms uniform in m. We now write (5.39) with $Y_{|m+\alpha|}$ replaced by the right hand side of (B.1) and insert in the resulting equation for $R_0^m(\lambda, r, r')$ the asymptotic expansions (5.40)-(5.43). With the help of (5.31) this yields

$$\int_1^\infty \int_1^\infty \rho^{-2s}(r) \, |G_{m,2}^+(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, rr' \, dr \, dr' \; \lesssim \; \frac{\lambda^{2\mu(\alpha)}}{1+|m|} \tag{5.44}$$

for λ small enough.

Next we are going to estimate the first term on the right hand side of (5.38). In view of (5.12), (5.16) and (5.17) we have

$$R_0^m(\lambda, r, r') = \frac{v_m(\lambda, r)}{W_m(\lambda)} \; (C_m(\lambda) \, v_m(\lambda, r') + D_m(\lambda) \, u_m(\lambda, r')), \qquad r \le r' \le 1. \tag{5.45}$$

As a first step we will show that the second term on the right hand side of the above equation is differentiable with respect to λ in the norm given by the right hand side of (5.38) and that the derivative is uniformly bounded in m. By (5.23)

$$\frac{D_m(\lambda)}{W_m(\lambda)} = \frac{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}{\Gamma(1+2|m|)}.$$

Therefore keeping in mind the definition of u_m, see equations (5.15) and (5.13), and using Lemmata A.1 and A.12 we find that for λ small enough it holds

$$\left| \partial_\lambda \left(\frac{D_m(\lambda)}{W_m(\lambda)} \, u_m(\lambda, r') \right) \right| \; \lesssim \; (2\kappa \, r')^{-|m|} + \frac{(4\kappa \, r')^{|m|}}{\Gamma(2|m|)}.$$

This in combination with (5.14) and Lemma A.2 gives

$$\left| v_m(\lambda, r) \, \partial_\lambda \left(\frac{D_m(\lambda)}{W_m(\lambda)} \, u_m(\lambda, r') \right) \right| \; \lesssim \; \left(\frac{r}{r'} \right)^{|m|} + \frac{(8\kappa)^{|m|} \, (rr')^{|m|}}{\Gamma(2|m|)}. \tag{5.46}$$

On the other hand, combining Lemma A.2 with Lemma A.3 we obtain that for λ small enough

$$|\partial_\lambda v_m(\lambda, r)| \; \lesssim \; |m| \, (2\kappa \, r)^{|m|} \qquad \forall \, r \in (0, 1). \tag{5.47}$$

Since

$$\left| \frac{D_m(\lambda)}{W_m(\lambda)} \, u_m(\lambda, r') \right| = \left| \frac{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}{\Gamma(1+2|m|)} \, u_m(\lambda, r') \right| \; \lesssim \; \frac{(2\kappa \, r')^{-|m|}}{|m|} + \frac{(4\kappa \, r')^{|m|}}{\Gamma(1+2|m|)},$$

in view of Lemma A.1, by putting the above estimates together we arrive at

$$\left| \partial_\lambda \left(\frac{D_m(\lambda)}{W_m(\lambda)} \, u_m(\lambda, r') \, v_m(\lambda, r) \right) \right| \; \lesssim \; \left(\frac{r}{r'} \right)^{|m|} + (rr')^{|m|}, \qquad 0 < r \le r' \le 1,$$

where we have used the fact that $(8\kappa)^{|m|}/\Gamma(2|m|)$ is bounded in $m \in \mathbb{Z} \setminus \{0\}$. As usual, r and r' have to be interchanged if $r' < r$. Since

$$\sup_{0<r<1} \int_r^1 \left| \left(\frac{r}{r'} \right)^{|m|} + (rr')^{|m|} \right| r' \, dr' + \sup_{0<r'<1} \int_0^{r'} \left| \left(\frac{r}{r'} \right)^{|m|} + (rr')^{|m|} \right| r \, dr \; \lesssim \; \frac{1}{1+|m|}$$

we find that

$$M_{(0,1)\times(0,1)}^j \left[\frac{D_m(\lambda)}{W_m(\lambda)} \, u_m(\lambda, \cdot) \, v_m(\lambda, \cdot) - \frac{D_m(0)}{W_m(0)} \, u_m(0, \cdot) \, v_m(0, \cdot) \right] \; \lesssim \; \frac{\lambda}{1+|m|}, \qquad j = 1, 2 \tag{5.48}$$

holds for all $m \in \mathbb{Z}$ and λ small enough. Here $\frac{D_m(0)}{W_m(0)}$ is to be understood as the limiting value of $\frac{D_m(\lambda)}{W_m(\lambda)}$. Consider now the first term on the right hand side of (5.45). From (5.47) and Lemma A.2 we obtain the bound

$$M_{(0,1)\times(0,1)}^j \left[\partial_\lambda \left((v_m(\lambda, r) \, v_m(\lambda, r')) \right) \right] \; \lesssim \; (4\kappa^2)^{|m|} \; \le \; (4\alpha^2)^{|m|} \tag{5.49}$$

We are going to combine (5.49) with the asymptotic expansion of $\frac{C_m(\lambda)}{W_m(\lambda)}$:

$$\frac{C_m(\lambda)}{W_m(\lambda)} = -\frac{\Gamma(\frac{1}{2} + m + |m|)}{\Gamma(1 + 2|m|)} \frac{b'_m + |m + \alpha| \, b_m}{a'_m + |m + \alpha| \, a_m} \, (1 + \mathcal{O}(\lambda))$$
$$+ \lambda^{|m+\alpha|} \frac{2\pi \, (i - \cot(|m + \alpha| \, \pi))}{\Gamma^2(|m + \alpha|) \, (a'_m + |m + \alpha| \, a_m)^2} \, (1 + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^{|m+\alpha|}))$$

as $\lambda \to 0+$ with the error terms uniform in m, see (5.23) and (5.26). In the calculation of the coefficient of $\lambda^{|m+\alpha|}$ in the above equation we have used the fact that $a'_m b_m - a_m b'_m = \Gamma(1 + |m|)/\Gamma(\frac{1}{2} + m + |m|)$, cf. (5.22). Taking into account (5.31) we conclude that for $j = 1, 2$ and λ small enough

$$M^j_{(0,1)\times(0,1)} \left[\frac{C_m(\lambda)}{W_m(\lambda)} \, v_m(\lambda, r) \, v_m(\lambda, r') - \frac{C_m(0)}{W_m(0)} \, v_m(0, r) \, v_m(0, r') \right] \lesssim \frac{\lambda}{1 + |m|}$$

when $m \neq k(\alpha)$, and

$$M^j_{(0,1)\times(0,1)} \left[\frac{C_m(\lambda)}{W_m(\lambda)} \, v_m(\lambda, r) \, v_m(\lambda, r') - \frac{C_m(0)}{W_m(0)} \, v_m(0, r) \, v_m(0, r') - \lambda^{\mu(\alpha)} \, g_1(r, r') \right] \lesssim \frac{\lambda^{2\mu(\alpha)}}{1 + |m|},$$

when $m = k(\alpha)$. This implies that for λ small enough

$$M^j_{(0,1)\times(0,1)}[G^+_{m,2}(\lambda)] \lesssim \frac{\lambda^{2\mu(\alpha)}}{1 + |m|}, \qquad j = 1, 2, \tag{5.50}$$

It remains to bound the cross term on the right hand side of (5.38). By (B.1) for $r \leq 1 < r'$ it holds

$$R^m_0(\lambda, r, r') = \frac{v_m(\lambda, r)}{W_m(\lambda)} \, \left(J_{|m+\alpha|}(\sqrt{\lambda} \, r') + i \, Y_{|m+\alpha|}(\sqrt{\lambda} \, r') \right) =$$
$$= \frac{v_m(\lambda, r)}{W_m(\lambda)} \, \left(J_{|m+\alpha|}(\sqrt{\lambda} \, r') \, (1 + i \cot(|m + \alpha| \, \pi)) - \frac{i}{\sin(|m + \alpha| \, \pi)} \, J_{-|m+\alpha|}(\sqrt{\lambda} \, r') \right), \tag{5.51}$$

We now insert the asymptotic expansion of the inverse Wronskian

$$\frac{1}{W_m(\lambda)} = \frac{i \, \pi \, \lambda^{\frac{|m+\alpha|}{2}}}{\Gamma(|m + \alpha|) \, (a'_m + |m + \alpha| \, a_m)} \, (1 + \mathcal{O}(\lambda))$$
$$+ \lambda^{|m+\alpha|} \frac{\pi^2 (a'_m - |m + \alpha| \, a_m) \, (1 + i \cot(|m + \alpha| \, \pi))}{|m + \alpha| \, \Gamma^3(|m + \alpha|) \, (a'_m + |m + \alpha| \, a_m)^2} \, (1 + \mathcal{O}(\lambda) + \mathcal{O}(\lambda^{2|m+\alpha|}))$$

into (5.51). With the help of the integral estimates on the derivative of $\lambda^{|m+\alpha|/2} \, J_{-|m+\alpha|}(\sqrt{\lambda} \, \cdot)$ with respect to λ, see inequality (7.11) of Lemma 7.4, and equations (5.31), (5.47), (B.4) we then obtain

$$\frac{-v_m(\lambda, r)}{\sin(|m + \alpha| \, \pi) \, W_m(\lambda)} \, J_{-|m+\alpha|}(\sqrt{\lambda} \, r') = \frac{-v_m(0, r) \, \lambda^{-\frac{|m+\alpha|}{2}}}{\sin(|m + \alpha| \, \pi) \, W_m(\lambda)} \, \lambda^{\frac{|m+\alpha|}{2}} \, J_{-|m+\alpha|}(\sqrt{\lambda} \, r')$$
$$= G_{m,0}(r, r') + \frac{\mathcal{O}(\lambda)}{1 + |m|} \, , \qquad \lambda \to 0+$$

where the convergence is taken with respect to the Hilbert-Schmidt norm on $(1, \infty) \times (1, \infty)$. Similarly, using inequality (7.12) of Lemma 7.4 and the above expansion of $W_m(\lambda)$ together with equations (5.31), (5.47), (B.4) we get

$$\frac{v_m(\lambda, r)}{W_m(\lambda)} \, J_{|m+\alpha|}(\sqrt{\lambda} \, r')(1 + i \cot(|m + \alpha| \, \pi)) = g_1(r, r') \, \delta_{m,k(\alpha)} + \frac{\mathcal{O}(\lambda^{\frac{|m+\alpha|+1}{2}})}{1 + |m|} \, , \qquad \lambda \to 0+$$

with respect to the Hilbert-Schmidt norm on $(0, 1) \times (1, \infty)$. In view of (5.51) this implies that

$$\int_0^1 \rho^{-2s}(r) \int_1^\infty |G^+_{m,2}(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r' \, dr' \, r \, dr = \frac{o(\lambda^{2\mu(\alpha)})}{1 + m^2} \, , \qquad \lambda \to 0+ \, . \tag{5.52}$$

By inserting (5.44), (5.50) and (5.52) into (5.38) we arrive at

$$\sup_{m \neq k(\alpha)} \| \rho^{-s} G_{m,2}^{+}(\lambda) \, \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, r dr))}^2 = \mathcal{O}(\lambda) \tag{5.53}$$

and

$$\| \rho^{-s} G_{k(\alpha),2}(\lambda) \, \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, r dr))}^2 = o(\sqrt{\lambda}) \tag{5.54}$$

The claim thus follows in view of (5.36).   □

Finally, we have to estimate also the operator $\nabla G_2^+(\lambda)$ in $\mathscr{B}(s, -s)$ for λ small enough.

Lemma 5.4. *Let* $\alpha \notin \mathbb{Z}$. *Assume that* $3/2 < s < 3/2 + \mu(\alpha)$. *Then for* $\lambda \to 0+$ *we have* $|\nabla G_2^+(\lambda)| = o(\lambda^{\mu(\alpha)})$ *in* $\mathscr{B}(s, -s)$.

Proof. The integral kernel of $\nabla G_2^+(\lambda)$ for $r \leq r'$ reads as

$$\nabla G_2^+(\lambda, r, r', \theta, \theta') = \sum_{m \in \mathbb{Z}} e^{im(\theta - \theta')} \left(\partial_{r'} G_{m,2}^+(r, r') \, , \, \frac{m}{r'} G_{m,2}^+(r, r') \right).$$

Hence the claim will follow if we show that

$$\sup_{m \in \mathbb{Z}} \| \rho^{-s} \widetilde{G}_{m,2}^+(\lambda) \, \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, r dr))} = o(\lambda^{\mu(\alpha)}), \qquad \lambda \to 0+, \tag{5.55}$$

where $\widetilde{G}_{m,2}^+(\lambda)$ is the integral operator with the kernel given by

$$\left(\widetilde{G}_{m,2}^+(\lambda, r, r') \right)^2 = \left(\partial_{r'} G_{m,2}^+(\lambda, r, r') \right)^2 + \frac{m^2}{r'^2} \left(G_{m,2}^+(\lambda, r, r') \right)^2. \tag{5.56}$$

As usual, r and r' in the above formula have to be interchanged when $r' < r$. We use again the upper bound

$$\| \rho^{-s} \widetilde{G}_{m,2}^+(\lambda) \, \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, r dr))}^2 \leq M_{(0,1) \times (0,1)}^1 [G_{m,2}^+(\lambda)] \, M_{(0,1) \times (0,1)}^2 [\widetilde{G}_{m,2}^+(\lambda)] \tag{5.57}$$

$$+ 2 \int_0^1 \rho^{-2s}(r) \int_1^\infty |\widetilde{G}_{m,2}^+(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r' \, dr' \, r \, dr$$

$$+ \int_1^\infty \int_1^\infty \rho^{-2s}(r) \, |\widetilde{G}_{m,2}^+(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r r' \, dr \, dr.$$

Let us consider the last term on the right hand side. Using inequality (7.8) of Lemma 7.3 instead of (7.1) we mimic the proof of the upper bound (5.44) and obtain

$$\int_1^\infty \int_1^\infty \rho^{-2s}(r) \left[\left(\partial_{r'} G_{m,2}^+(r, r') \right)^2 + \frac{(m + \alpha)^2}{r'^2} G_{m,2}^2(r, r') \right] \rho^{-2s}(r') \, r r' \, dr \, dr' \lesssim \frac{\lambda^{2\mu(\alpha)}}{1 + |m|}$$

for λ small enough. This shows that

$$\sup_{m \in \mathbb{Z}} \int_1^\infty \int_1^\infty \rho^{-2s}(r) \, |\widetilde{G}_{m,2}^+(r, r')|^2 \, \rho^{-2s}(r') \, r r' \, dr \, dr' \lesssim \lambda^{2\mu(\alpha)} \tag{5.58}$$

as $\lambda \to 0+$. As for the remaining terms on the right hand side of (5.57) we note that in view of (5.56)

$$M_{(0,1) \times (0,1)}^j [\widetilde{G}_{m,2}^+(\lambda)] \lesssim (1 + |m|) \, M_{(0,1) \times (0,1)}^j [G_{m,2}^+(\lambda)], \qquad j = 1, 2,$$

and

$$\int_0^1 \rho^{-2s}(r) \int_1^\infty |\widetilde{G}_{m,2}^+(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r' \, dr' \, r \, dr \lesssim$$

$$\lesssim (1 + m^2) \int_0^1 \rho^{-2s}(r) \int_1^\infty |G_{m,2}^+(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r' \, dr' \, r \, dr.$$

A combination of these estimates with (5.50) and (5.52) gives

$$\sup_{m \in \mathbb{Z}} M_{(0,1) \times (0,1)}^j [\widetilde{G}_{m,2}^+(\lambda)] = \mathcal{O}(\lambda^{2\mu(\alpha)}), \qquad j = 1, 2,$$

and

$$\sup_{m \in \mathbb{Z}} \int_0^1 \rho^{-2s}(r) \int_1^\infty |\widetilde{G}_{m,2}^+(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r' \, dr' \, r \, dr = o(\lambda^{2\mu(\alpha)})$$

as $\lambda \to 0+$. In view of (5.57) and (5.58) this proves (5.55) and the claim follows. $\qquad\square$

5.2. **The case $\lambda < 0$.** For negative values of λ we repeat the same procedure as in the case $\lambda > 0$. The calculations are identical to those made in section 5.1. We therefore omit some details. Recall that $k(\alpha)$ is defined (5.1).

Lemma 5.5. *Let $\alpha \notin \mathbb{Z}$. Then for any $x, y \in \mathbb{R}^2$ and $|\lambda|$ small enough we have*

$$R_0(\lambda; x, y) = G_0(x, y) + \lambda^{\mu(\alpha)} \, G_1(x, y) + G_2^-(\lambda; x, y),$$

where $G_0(x, y)$ and $G_1(x, y)$ are given by (5.27) and (5.29), and $G_2^-(\lambda; x, y)) = o(|\lambda|^{\mu(\alpha)})$ as $\lambda \to 0-$.

Proof. We calculate the integral kernel of $R_0(\lambda)$ in the same way as above. Inside the unit disc the generalized eigenvalue equation (5.11) has the same solutions, i.e. u_m and v_m, as for $\lambda > 0$. Outside of the unit disc we have to replace the Bessel functions J_ν and Y_ν by a suitable linear combination of the modified Bessel functions I_ν and K_ν, see Appendix C for details. We then find that $R_0(\lambda; x, y)$ is given by (5.7) with $R_0^m(\lambda; r, r')$ replaced by

$$\widetilde{R}_0^m(\lambda; r, r') = \left(\widetilde{W}_m(\lambda)\right)^{-1} \begin{cases} \tilde{f}_{m,\lambda}(r) \, \tilde{\phi}_{m,\lambda}(r'), & r \leq r', \\[2mm] \tilde{f}_{m,\lambda}(r') \, \tilde{\phi}_{m,\lambda}(r), & r' < r, \end{cases} \tag{5.59}$$

where

$$\tilde{f}_{m,\lambda}(r) = v_m(\lambda, r), \qquad \tilde{\phi}_{m,\lambda}(r) = \widetilde{C}_m(\lambda) \, v_m(\lambda, r) + \widetilde{D}_m(\lambda) \, u_m(\lambda, r) \qquad\qquad r \leq 1$$

$$\tilde{f}_{m,\lambda}(r) = \widetilde{A}_m(\lambda) \, I_{|\alpha+m|}(\sqrt{|\lambda|}\, r) + \widetilde{B}_m(\lambda) \, K_{|\alpha+m|}(\sqrt{|\lambda|}\, r), \qquad \tilde{\phi}_{m,\lambda}(r) = K_{|m+\alpha|}(\sqrt{|\lambda|}\, r), \qquad 1 < r.$$

and $\widetilde{W}_m(\lambda) = \tilde{\phi}_{m,\lambda} \, \tilde{f}'_{m,\lambda} - \tilde{\phi}'_{m,\lambda} \, \tilde{f}_{m,\lambda}$ is the corresponding Wronskian. From the matching conditions at $r = 1$ and the properties of modified Bessel functions, see equations (C.5), (C.6), we then find

$$\widetilde{A}_m(\lambda) = (v'_m(\lambda, 1) - |\alpha + m| \, v_m(\lambda, 1)) \, K_{|\alpha+m|}(\sqrt{|\lambda|}) + \sqrt{|\lambda|} \, v_m(\lambda, 1) \, K_{|\alpha+m|+1}(\sqrt{|\lambda|})$$

$$\widetilde{B}_m(\lambda) = (|\alpha + m| \, v_m(\lambda, 1) - v'_m(\lambda, 1)) \, I_{|\alpha+m|}(\sqrt{|\lambda|}) + \sqrt{|\lambda|} \, v_m(\lambda, 1) \, I_{|\alpha+m|+1}(\sqrt{|\lambda|})$$

and

$$\widetilde{C}_m(\lambda) = \frac{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}{\Gamma(1 + 2|m|)} \left((|\alpha + m| \, u_m(\lambda, 1) - u'_m(\lambda, 1)) \, I_{|\alpha+m|}(\sqrt{|\lambda|}) + \sqrt{|\lambda|} \, u_m(\lambda, 1) \, I_{|\alpha+m|+1}(\sqrt{|\lambda|}) \right.$$

$$\left. + i \left[(|\alpha + m| \, u_m(\lambda, 1) - u'_m(\lambda, 1)) \, K_{|\alpha+m|}(\sqrt{|\lambda|}) + \sqrt{|\lambda|} \, u_m(\lambda, 1) \, K_{|\alpha+m|+1}(\sqrt{|\lambda|}) \right] \right)$$

$$\widetilde{D}_m(\lambda) = \frac{\Gamma(\frac{1}{2} + |m| + \frac{m\alpha}{\kappa})}{\Gamma(1 + 2|m|)} \, \widetilde{A}_m(\lambda).$$

Moreover, with the help of (C.2) we obtain

$$\widetilde{W}_m(\lambda) = \widetilde{A}_m(\lambda). \tag{5.60}$$

Using the behavior of functions I_ν and K_ν for small arguments, see equation (C.3), we then get the asymptotic expansions

$$\widetilde{A}_m(\lambda) = \frac{\Gamma(|m + \alpha|)}{2} \, (a'_m + |m + \alpha| \, a_m) \left(\frac{1}{2}\sqrt{|\lambda|}\right)^{-|\alpha+m|} \left(1 + \mathcal{O}(\lambda)\right)$$

$$- \frac{a'_m - |m + \alpha| \, a_m}{\sin(|m + \alpha| \, \pi) \, \Gamma(|m + \alpha| + 1)} \left(\frac{1}{2}\sqrt{|\lambda|}\right)^{|\alpha+m|} \left(1 + \mathcal{O}(\lambda)\right),$$

$$\widetilde{B}_m(\lambda) = \frac{a'_m - |m + \alpha| \, a_m}{\Gamma(|m + \alpha| + 1)} \left(\frac{1}{2}\sqrt{|\lambda|}\right)^{|\alpha+m|} \left(1 + \mathcal{O}(\lambda)\right)$$

where the error terms are uniform in m. Similarly we find that the expansion of $\widetilde{C}_m(\lambda)$ is given by the right hand side of (5.26) with the factor $i - \cot(\mu(\alpha)\pi)$ replaced by $-\frac{1}{\sin(\mu(\alpha)\pi)}$. Hence when we insert the above equations into (5.59) and use (C.3), after a bit lengthly calculations, we arrive at

$$\widetilde{R}_0^m(\lambda; r, r') = G_{m,0}(r, r') - \delta_{m,k(\alpha)} \frac{|\lambda|^{\mu(\alpha)} g_1(r, r')}{\sin(\mu(\alpha)\pi)(i - \cot(\mu(\alpha)\pi))} + o(|\lambda|^{\mu(\alpha)}) \qquad \lambda \to 0-. \quad (5.61)$$

Recall that $g_1(\cdot, \cdot)$ is defined by (5.30). Since $|\lambda|^{\mu(\alpha)} = \lambda^{\mu(\alpha)}(\cos(\mu(\alpha)\pi) - i\sin(\mu(\alpha)\pi)$, this implies

$$R_0(\lambda; x, y) = G_0(x, y) + \lambda^{\mu(\alpha)} G_1(x, y) + o(|\lambda|^{\mu(\alpha)}) \qquad \lambda \to 0-, \quad x, y \in \mathbb{R}^2.$$

$$\square$$

As in the case $\lambda > 0$ we need an estimate on $G_2^-(\lambda)$ and $\nabla G_2^-(\lambda)$ in $\mathscr{B}(s, -s)$.

Lemma 5.6. *Let $G_2^-(\lambda)$ be the integral operator with the kernel $G_2^-(\lambda; x, y)$ defined in Lemma 5.5. Assume that $3/2 < s < 3/2 + \mu(\alpha)$. Then*

$$\| G_2^-(\lambda)\|_{\mathscr{B}(s,-s)} = o(|\lambda|^{\mu(\alpha)}), \qquad \||\nabla G_2^-(\lambda)|\|_{\mathscr{B}(s,-s)} = o(|\lambda|^{\mu(\alpha)}) \qquad \lambda \to 0-. \quad (5.62)$$

Proof. To simplify the notation we write below μ instead of $\mu(\alpha)$. Similarly as in the case $\lambda > 0$, see Lemma 5.3, we note that

$$\|G_2^-(\lambda)\|_{\mathscr{B}(s,-s)} = \sup_{m \in \mathbb{Z}} \| \rho^{-s} G_{m,2}^-(\lambda) \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, r dr))}, \quad (5.63)$$

where

$$G_{m,2}^-(\lambda, r, r') = R_0^m(\lambda, r, r') - G_{m,0}(r, r') - \lambda^\mu \delta_{m,k(\alpha)} g_1(r, r').$$

As in (5.38) we have

$$\| \rho^{-s} G_{m,2}^-(\lambda) \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, r dr))}^2 \leq M_{(0,1)\times(0,1)}^1[G_{m,2}^-(\lambda)] \, M_{(0,1)\times(0,1)}^2[G_{m,2}^-(\lambda)] \quad (5.64)$$

$$+ 2 \int_0^1 \rho^{-2s}(r) \int_1^\infty |G_{m,2}^-(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r' \, dr' \, r \, dr$$

$$+ \int_1^\infty \int_1^\infty \rho^{-2s}(r) \, |G_{m,2}^-(\lambda, r, r')|^2 \, \rho^{-2s}(r') \, r r' \, dr' \, dr$$

For $r < r' \leq 1$ is the kernel $R_0^m(\lambda, r, r')$ given by the same solutions, v_m and u_m, as in the case $\lambda > 0$. Hence from the proof of Lemma 5.3, namely from the proof of upper bound (5.50), we deduce that

$$M_{(0,1)\times(0,1)}^j[G_{m,2}^-(\lambda)] \lesssim \frac{|\lambda|^{2\mu}}{1+|m|}, \qquad j = 1, 2, \quad (5.65)$$

holds for all m and $|\lambda|$ small enough. Let us now consider the last term on the right hand side of (5.64). From (5.59), taking into account (5.60), we deduce that for $1 \leq r < r'$

$$R_0^m(\lambda, r, r') = I_{|\alpha+m|}(\sqrt{|\lambda|}\, r) K_{|m+\alpha|}(\sqrt{|\lambda|}\, r') + \frac{\widetilde{B}_m(\lambda)}{\widetilde{A}_m(\lambda)} K_{|\alpha+m|}(\sqrt{|\lambda|}\, r) K_{|m+\alpha|}(\sqrt{|\lambda|}\, r'). \quad (5.66)$$

In order to estimate the right hand side we cannot use the modified splitting formula (C.1), as we did in the with the equation (B.1) in case $\lambda > 0$, since both functions $I_{-\nu}$ and I_ν are exponentially increasing at infinity. Instead we proceed as follows; we consider first the contribution to $G_{m,2}^-(\lambda, r, r')$ relative to the product $I_{|m+\alpha|}(\sqrt{|\lambda|}\, r) K_{|m+\alpha|}(\sqrt{|\lambda|}\, r')$. This means that we have to estimate the L^2-norm, restricted to $(1, \infty) \times (1, \infty)$ of the kernel $\rho^{-s}(r) w_m(\lambda, r, r') \rho^{-s}(r')$, where

$$w_m(\lambda, r, r') := I_{|m+\alpha|}(\sqrt{|\lambda|}\, r) K_{|m+\alpha|}(\sqrt{|\lambda|}\, r') - \frac{(r/r')^{|m+\alpha|}}{2|m+\alpha|} + \frac{\delta_{m,k(\alpha)} |\lambda|^\mu \pi \, 4^{-\mu} (rr')^\mu}{2\sin(\mu\pi)\mu^2(\alpha)\Gamma^2(\mu)}, \quad (5.67)$$

see (5.30) and (5.61). Without loss of generality we may assume that $|\lambda| < 1$. We have

$$\int_1^\infty \int_r^\infty \rho^{-2s}(r)\,|w_m(\lambda,r,r')|^2\,\rho^{-2s}(r')\,r'\,dr'\,r\,dr = \int_{|\lambda|^{-\frac{1}{2}}}^\infty \int_r^\infty |w_m(\lambda,r,r')|^2\,(rr')^{-2-2\varepsilon}\,dr'\,dr$$

$$+ \int_1^{|\lambda|^{-\frac{1}{2}}} \int_{|\lambda|^{-\frac{1}{2}}}^\infty |w_m(\lambda,r,r')|^2\,(rr')^{-2-2\varepsilon}\,dr'dr + \int_1^{|\lambda|^{-\frac{1}{2}}} \int_r^{|\lambda|^{-\frac{1}{2}}} |w_m(\lambda,r,r')|^2\,(rr')^{-2-2\varepsilon}\,dr'dr$$

$$=: X_1(m,\lambda) + X_2(m,\lambda) + X_3(m,\lambda). \tag{5.68}$$

In $X_1(m,\lambda)$ and $X_2(m,\lambda)$ we bound each term on the right hand side of (5.67) separately; note that by monotonicity of $K_{|m+\alpha|}$ and (C.7)

$$\int_{|\lambda|^{-\frac{1}{2}}}^\infty \int_r^\infty I_{|m+\alpha|}^2(\sqrt{|\lambda|}\,r)\,K_{|m+\alpha|}^2(\sqrt{|\lambda|}\,r')\,(rr')^{-2-2\varepsilon}dr'dr \leq \frac{1}{|m+\alpha|^2}\int_{|\lambda|^{-\frac{1}{2}}}^\infty r^{-3-4\varepsilon}\,dr \lesssim \frac{|\lambda|^{1+2\varepsilon}}{|m+\alpha|^2}\,.$$

Similarly we obtain

$$\int_1^{|\lambda|^{-\frac{1}{2}}} \int_{|\lambda|^{-\frac{1}{2}}}^\infty I_{|m+\alpha|}^2(\sqrt{|\lambda|}\,r)\,K_{|m+\alpha|}^2(\sqrt{|\lambda|}\,r')\,(rr')^{-2-2\varepsilon}dr'dr$$

$$\leq |\lambda|^{\frac{1}{2}+\varepsilon}\,K_{|m+\alpha|}^2(1)\int_1^{|\lambda|^{-\frac{1}{2}}} I_{|m+\alpha|}^2(\sqrt{|\lambda|}\,r)\,r^{-3-4\varepsilon}\,dr \lesssim \frac{|\lambda|^{|m+\alpha|+\frac{1}{2}+\varepsilon}}{|m+\alpha|^2},$$

where we have used equations (C.3) to estimate $K_{|m+\alpha|}^2(1)$ and $I_{|m+\alpha|}^2(\sqrt{|\lambda|}\,r)$. Elementary calculations now show that the contributions from the remaining terms on the right hand side of (5.67) to $X_1(m,\lambda)$ and $X_2(m,\lambda)$ are of the same order in $|\lambda|$ and m. Since $\mu \leq \frac{1}{2}$, this implies that

$$X_1(m,\lambda) + X_2(m,\lambda) \lesssim \frac{|\lambda|^{|m+\alpha|+\frac{1}{2}+\varepsilon} + |\lambda|^{1+2\varepsilon}}{|m+\alpha|^2} \leq \frac{|\lambda|^{2\mu+\varepsilon}}{|m+\alpha|^2}\,.$$

It remains to estimate $X_3(m,\lambda)$. Here we use the fact that $\sqrt{|\lambda|}\,r \leq \sqrt{|\lambda|}\,r' \leq 1$, see (5.68). Hence equations (C.3) we then obtain a pointwise estimate on $w_m(\lambda,r,r')$ which yields

$$X_3(m,\lambda) \lesssim \frac{|\lambda|^{2\mu+\varepsilon}}{|m+\alpha|^2}\,.$$

The remaining part of $|G_{m,2}^-(\lambda,r,r')|^2$ on $(1,\infty) \times (1,\infty)$ is estimated in the same way using the asymptotic behavior of the coefficients $\widetilde{A}_m(\lambda)$ and $\widetilde{B}_m(\lambda)$ established above. We thus conclude that

$$\int_1^\infty \int_1^\infty \rho^{-2s}(r)\,|G_{m,2}^-(\lambda,r,r')|^2\,\rho^{-2s}(r')\,rr'\,dr'\,dr \lesssim \frac{|\lambda|^{2\mu+\varepsilon}}{|m+\alpha|^2}. \tag{5.69}$$

Finally, we consider the mixed term in (5.64). We have

$$R_0^m(\lambda;r,r') = \frac{v_m(\lambda,r)}{\widetilde{A}_m(\lambda)}\,K_{|m+\alpha|}(\sqrt{|\lambda|}\,r') \qquad r \leq 1 < r'.$$

Consequently, by (5.28), (5.30) and (5.61)

$$G_{m,2}^-(\lambda,r,r') = \frac{v_m(\lambda,r)}{\widetilde{A}_m(\lambda)}\,K_{|m+\alpha|}(\sqrt{|\lambda|}\,r') - \frac{v_m(0,r)\,(r')^{-|m+\alpha|}}{a_m' + |m+\alpha|\,a_m}$$

$$+ |\lambda|^\mu\,\frac{\delta_{m,k(\alpha)}\,\pi\,v_k(0,r)}{\mu\,\sin(\mu\pi)\,4^\mu\,\Gamma^2(\mu)\,(a_k' + \mu\,a_k)}\left(r'^\mu - \frac{a_k' - \mu\,a_k}{a_k' + \mu\,a_k}\,r'^{-\mu}\right) \tag{5.70}$$

We proceed similarly as above and for $r' \geq |\lambda|^{-\frac{1}{2}}$ estimate the contribution of each term to the integral in (5.64) separately. By (5.31), Lemma A.2 and the asymptotic expansion of $\widetilde{A}_m(\lambda)$ it follows that

$$\int_0^1 \frac{v_m^2(\lambda,r)}{\widetilde{A}_m^2(\lambda)}\,\rho^{-2s}(r)\,r\,dr \lesssim \frac{4^{-|m+\alpha|}\,|\lambda|^{|m+\alpha|}}{(1+|m|)^3\,\Gamma^2(|m+\alpha|)}$$

Taking into account the fact that $K_{|m+\alpha|}$ is decreasing and using (C.3), we thus get

$$\int_0^1 \rho^{-2s}(r)\, r\, dr \int_{|\lambda|^{-\frac{1}{2}}}^\infty |G_{m,2}^-(\lambda,r,r')|^2 \, \rho^{-2s}(r')\, r'\, dr' \;\lesssim\; \frac{|\lambda|^{\mu+\frac{1}{2}+\varepsilon}}{(|m|+1)^3}.$$

In the region where $r' \leq |\lambda|^{-\frac{1}{2}}$ we use the two-term asymptotic expansion of the function $K_\nu(z)$ for $0 < z \leq 1$, cf. (C.3). This gives

$$\int_0^1 \rho^{-2s}(r)\, r\, dr \int_r^{|\lambda|^{-\frac{1}{2}}} |G_{m,2}^-(\lambda,r,r')|^2 \, \rho^{-2s}(r')\, r'\, dr' \;\lesssim\; \frac{|\lambda|^{\mu+\frac{1}{2}+\varepsilon}}{(|m|+1)^3}.$$

Inserting the above estimates together with (5.69) and (5.65) into (5.64) then yields

$$\| \rho^{-s}\, G_{m,2}^-(\lambda)\, \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+,\,rdr))} \;\lesssim\; \frac{|\lambda|^{\mu+\varepsilon}}{|m|+1}. \tag{5.71}$$

This proves the first part or (5.62). As in the proof of Lemma 5.4, to prove the second part of (5.62) it suffices to show that

$$\sup_{m\in\mathbb{Z}} \int_0^\infty \int_0^\infty \rho^{-2s}(r) \left[\left(\partial_{r'} G_{m,2}^-(\lambda,r,r')\right)^2 + \frac{m^2}{r'^2}\, (G_{m,2}^-(\lambda,r,r'))^2 \right] \rho^{-2s}(r')\, r\, r'\, dr dr' = o(|\lambda|^{2\mu})$$

as $\lambda \to 0-$, see the proof of Lemma 5.4. However, this follows from the explicit expression for $G_{m,2}^-(\lambda,r,r')$ and from (5.71). $\qquad\square$

5.3. **Proof of Proposition 4.1.** The claim of the Proposition follows by combining Lemmata 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6.

6. The operator $H(B_0)$: integer flux

For integer values of α we can use the results of the previous sections for the contributions to the asymptotic expansion for $R_0(\lambda)$ from the channels $m \neq -\alpha$. Hence it remains to analyze the contribution from $m = -\alpha$ which.

6.1. **The case $\lambda > 0$.**

Lemma 6.1. *Assume that $\alpha \in \mathbb{Z}$, $\alpha \neq 0$. Then for every $x, y \in \mathbb{R}^2$ it holds*

$$R_0(\lambda; x, y) = \mathcal{G}_0(x,y) + \ (\log\lambda)^{-1}\, \mathcal{G}_1(x,y) + \mathcal{G}_2^+(\lambda; x, y), \tag{6.1}$$

where $\mathcal{G}_0(x,y)$ and $\mathcal{G}_1(x,y)$ are given by equations (6.6) and (6.7) and $\mathcal{G}_2^+(\lambda; x, y) = o\big((\log\lambda)^{-1}\big)$ as $\lambda \to 0+$.

Proof. Without loss of generality we may assume that $\alpha > 0$. We proceed in the same way as above using the functions $f_{m,\lambda}$ and $\phi_{m,\lambda}$ defined by (5.16) and (5.17). For $m \neq -\alpha$ we calculate $R_0^m(\lambda; r, r')$ from the formulas (5.10) and (5.12)-(5.23). If $m = -\alpha < 0$, then

$$A_{-\alpha}(\lambda) = -\frac{\pi}{2} \left(v'_{-\alpha}(\lambda, 1)\, Y_0(\sqrt{\lambda}) + \sqrt{\lambda}\, v_{-\alpha}(\lambda, 1)\, Y_1(\sqrt{\lambda}) \right) \tag{6.2}$$

$$B_{-\alpha}(\lambda) = \frac{\pi}{2} \left(v'_{-\alpha}(\lambda, 1)\, J_0(\sqrt{\lambda}) + \sqrt{\lambda}\, v_{-\alpha}(\lambda, 1)\, J_1(\sqrt{\lambda}) \right), \tag{6.3}$$

and

$$C_{-\alpha}(\lambda) = \frac{\Gamma(\frac{1}{2}+2\alpha)}{\Gamma(1+2\alpha)} \left(u'_{-\alpha}(\lambda, 1)\, (J_0(\sqrt{\lambda}) - i\, Y_0(\sqrt{\lambda})) + \sqrt{\lambda}\, u_{-\alpha}(\lambda, 1)\, (J_1(\sqrt{\lambda}) - i\, Y_1(\sqrt{\lambda})) \right) \tag{6.4}$$

$$D_{-\alpha}(\lambda) = \frac{2\,\Gamma(\frac{1}{2}+2\alpha)}{\pi\,\Gamma(1+2\alpha)} \left(i\, A_{-\alpha}(\lambda) - B_{-\alpha}(\lambda) \right). \tag{6.5}$$

The rest of the calculation proceeds in the same way as is the case $\alpha \notin \mathbb{Z}$. To simplify the notation we introduce the following shorthands:

$$\mathfrak{a}_\alpha = v_{-\alpha}(0,1), \qquad \mathfrak{a}'_\alpha = v'_{-\alpha}(0,1), \qquad \mathfrak{b}_\alpha = u_{-\alpha}(0,1), \qquad \mathfrak{b}'_\alpha = u'_{-\alpha}(0,1).$$

With the help of (5.7), (5.12), and properties of Bessel functions, see equations (B.4) and (B.5), we then obtain

$$\lim_{\lambda \to 0+} R_0(\lambda; x, y) = \mathcal{G}_0(x, y) = \sum_{m \neq -\alpha} G_{m,0}(r, r')\, e^{im(\theta - \theta')} + \mathcal{G}_{\alpha,0}(r, r')\, e^{i\alpha(\theta' - \theta)}, \qquad (6.6)$$

with

$$\mathcal{G}_{\alpha,0}(r, r') = \frac{\Gamma(\frac{1}{2} + 2\alpha)}{\Gamma(1 + 2\alpha)}\, v_{-\alpha}(0, r) \left(u_{-\alpha}(0, r') - \frac{\mathfrak{b}'_\alpha}{\mathfrak{a}'_\alpha}\, v_{-\alpha}(0, r') \right) \qquad r < r' \leq 1,$$

$$\mathcal{G}_{\alpha,0}(r, r') = \frac{v_{-\alpha}(0, r)}{\mathfrak{a}'_\alpha} \qquad\qquad r \leq 1 < r',$$

$$\mathcal{G}_{\alpha,0}(r, r') = \frac{\mathfrak{a}_\alpha}{\mathfrak{a}'_\alpha} + \log r \qquad\qquad 1 < r < r'.$$

Similarly, we find that

$$\lim_{\lambda \to 0+} \log \lambda\, \Big(R_0(\lambda; x, y) - \mathcal{G}_0(x, y) \Big) =: \mathcal{G}_1(x, y) = k_1(\alpha; r, r')\, e^{i\alpha(\theta - \theta')}, \qquad (6.7)$$

with

$$k_1(\alpha; r, r') = \frac{2\, v_{-\alpha}(0, r)\, v_{-\alpha}(0, r')}{(\mathfrak{a}'_\alpha)^2} \qquad\qquad r < r' \leq 1,$$

$$k_1(\alpha; r, r') = \frac{2\, v_{-\alpha}(0, r)}{\mathfrak{a}'_\alpha} \left(\frac{\mathfrak{a}_\alpha}{\mathfrak{a}'_\alpha} + \log r' \right) \qquad\qquad r < 1 < r',$$

$$k_1(\alpha; r, r') = 2 \left(\frac{\mathfrak{a}_\alpha}{\mathfrak{a}'_\alpha} + \log r \right) \left(\frac{\mathfrak{a}_\alpha}{\mathfrak{a}'_\alpha} + \log r' \right) \qquad\qquad 1 \leq r < r'.$$

$\square$

Remark 6.2. Note that by a direct calculation using equations (5.14), (A.1) and (A.5) we have

$$\mathfrak{a}'_\alpha = |\alpha|\, e^{-|\alpha|}\, |2\alpha|^{|\alpha|} \left[|\alpha|\, M\left(\frac{1}{2}, 1 + 2|\alpha|, 2|\alpha|\right) + \frac{1}{2 + 4|\alpha|}\, M\left(\frac{3}{2}, 2 + 2|\alpha|, 2|\alpha|\right) \right] > 0.$$

Lemma 6.3. *Let $s > 1$. Then $\rho^{-s}\, \mathcal{G}_j\, \rho^{-s}$ and $\rho^{-s}\, \nabla \mathcal{G}_j\, \rho^{-s}$ with $j = 0, 1$ are compact operators from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2, \mathbb{C}^2)$ respectively.*

Proof. By Lemma 5.2 the operators with kernels

$$\rho^{-s} \sum_{m \neq -\alpha} G_{m,0}(r, r')\, e^{im(\theta - \theta')}\, \rho^{-s} \qquad \text{and} \qquad \rho^{-s} \left(\nabla \sum_{m \neq -\alpha} G_{m,0}(r, r')\, e^{im(\theta - \theta')} \right) \rho^{-s}$$

are compact from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2, \mathbb{C}^2)$. On the other hand, from the explicit equation for $\mathcal{G}_{\alpha,0}(r, r')$ and from Lemmata A.2 and A.3 it easily follows that the kernels

$$\rho^{-s}\, \mathcal{G}_{\alpha,0}(r, r')\, e^{i\alpha(\theta - \theta')}\, \rho^{-s} \qquad \text{and} \qquad \rho^{-s}\, |\nabla \mathcal{G}_{\alpha,0}(r, r')\, e^{i\alpha(\theta - \theta')}|\, \rho^{-s}$$

are Hilbert-Schmidt in $L^2(\mathbb{R}^2)$. $\square$

Lemma 6.4. *Let $\mathcal{G}_2^+(\lambda)$ denote the integral operator with the kernel $\mathcal{G}_2^+(\lambda; x, y)$ defined by (6.1). Assume that $s > 3/2 + \varepsilon$, $0 < \varepsilon < 1$. Then, as $\lambda \to 0+$ we have $\mathcal{G}_2^+(\lambda) = o\big((\log \lambda)^{-1}\big)$ and $|\nabla \mathcal{G}_2^+(\lambda)| = o\big((\log \lambda)^{-1}\big)$ in $\mathscr{B}(s, -s)$.*

Proof. We have

$$\mathcal{G}_2^+(\lambda; x, y) = \sum_{m \in \mathbb{Z}} \mathcal{G}_{m,2}^+(r, r') \, e^{im(\theta - \theta')} , \tag{6.8}$$

where

$$\mathcal{G}_{m,2}^+(\lambda, r, r') = \begin{cases} R_0^m(\lambda, r, r') - G_{m,0}(r, r') & m \neq -\alpha, \\[2mm] R_0^{-\alpha}(\lambda, r, r') - \mathcal{G}_{\alpha,0}(r, r') - (\log \lambda)^{-1} k_1(r, r'), & m = -\alpha. \end{cases} \tag{6.9}$$

Let us first consider the case $m \neq -\alpha$. We are going to use the results of Lemma 5.3. Note that all the indexes of the Bessel function included in $G_{m,0}(r, r')$ are integers. Consequently, we define $Y_{|m+\alpha|}$ by its integral representation (B.8) instead of (B.1). By using Lemmata 7.2, 7.3 and 7.4 with the function $J_{-\nu}$ replaced by Y_ν it follows that the results of Lemma 5.3 remain valid. In particular, from (5.53) we infer that

$$\sup_{m \neq -\alpha} \| \rho^{-s} (R_0^m(\lambda) - G_{m,0}) \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, rdr))} = \mathcal{O}(\lambda), \qquad \lambda \to 0+, \tag{6.10}$$

where $R_0^m(\lambda)$ is the operator in $L^2(\mathbb{R}_+, rdr)$ generated by the kernel $R_0^m(\lambda, r, r')$. In order to treat the case $m = -\alpha$, we will need the following asymptotic expansions:

$$\frac{A_{-\alpha}(\lambda)}{W_{-\alpha}(\lambda)} = \frac{i\pi}{2} \left(1 + i\pi (\log \lambda)^{-1} - 2\pi \left(\frac{\pi}{2} + i \frac{c_\alpha}{\mathfrak{a}_\alpha'} \right) (\log \lambda)^{-2} + o((\log \lambda)^{-2}) \right), \tag{6.11}$$

$$\frac{B_{-\alpha}(\lambda)}{W_{-\alpha}(\lambda)} = -\frac{i\pi^2}{2} (\log \lambda)^{-1} + \pi^2 \left(\frac{\pi}{2} - \frac{c_\alpha}{\mathfrak{a}_\alpha'} \right) (\log \lambda)^{-2} - 2i\pi^2 \left(\frac{c_\alpha^2}{\mathfrak{a}_\alpha'^2} - \frac{\pi^2}{4} \right) (\log \lambda)^{-3} + o((\log \lambda)^{-3}), \tag{6.12}$$

$$\frac{C_{-\alpha}(\lambda)}{W_{-\alpha}(\lambda)} = \frac{2\,\Gamma(\frac{1}{2} + \alpha)\,\mathfrak{b}_\alpha'}{i\,\Gamma(1 + 2\alpha)\,\mathfrak{a}_\alpha'} \left(1 + i\pi (\log \lambda)^{-1} - 2\pi \left(\frac{\pi}{2} + i \frac{c_\alpha}{\mathfrak{a}_\alpha'} \right) (\log \lambda)^{-2} + o((\log \lambda)^{-2}) \right), \tag{6.13}$$

as $\lambda \to 0+$, where $c_\alpha = \mathfrak{a}_\alpha - \mathfrak{a}_\alpha'(\gamma - \log 2)$. These expansions can be derived directly from equations (6.2)-(6.4) and asymptotics (B.4), (B.5). By inserting the asymptotic equations (6.11)-(6.13) into the expression for $R_0^{-\alpha}(\lambda, r, r')$ and using inequalities (B.18) of Lemma B.2 we find that

$$\| \rho^{-s} (R_0^{-\alpha}(\lambda, r, r') - \mathcal{G}_{\alpha,0}(r, r') - (\log \lambda)^{-1} k_1(r, r')) \rho^{-s} \|_{HS(\mathbb{R}_+, rdr)} = o((\log \lambda)^{-1}), \quad \lambda \to 0+ .$$

This in combination with (6.10) and (6.9) shows that $\mathcal{G}_2^+(\lambda) = o((\log \lambda)^{-1})$ in $\mathscr{B}(s, -s)$. To prove the remaining claim we proceed in the similar way. From Lemma 5.4 we conclude that

$$\sup_{m \neq -\alpha} \| \rho^{-s} \left(\nabla (R_0^m(\lambda) - G_{m,0}) \right) \rho^{-s} \|_{\mathscr{B}(L^2(\mathbb{R}_+, rdr))} = \mathcal{O}(\lambda), \qquad \lambda \to 0+ .$$

On the other hand, with the help of (6.11)-(6.13) and inequalities (B.19) of Lemma B.2 we obtain

$$\int_0^\infty \int_0^\infty \rho^{-2s}(r') \big| \partial_{r'} \left(R_0^{-\alpha}(\lambda, r, r') - \mathcal{G}_{\alpha,0}(r, r') - (\log \lambda)^{-1} k_1(r, r') \right) \big|^2 \rho^{-2s}(r)\, rr'\, dr\, dr' = o((\log \lambda)^{-2})$$

as $\lambda \to 0+$. The statement now follows again by (6.10) and (6.9). $\square$

6.2. The case $\lambda < 0$.

Lemma 6.5. *Let $\alpha \in \mathbb{Z}, \alpha \neq 0$. Assume that $s > 3/2 + \varepsilon$, $0 < \varepsilon < 1$. Then for $\lambda < 0$ and $|\lambda|$ small enough we have*

$$R_0(\lambda + i0) = \mathcal{G}_0 + (\log \lambda)^{-1} \mathcal{G}_1 + \mathcal{G}_2^-(\lambda) \qquad in \qquad \mathscr{B}(s, -s),$$

where

$$\| \mathcal{G}_2^-(\lambda) \|_{\mathscr{B}(s,-s)} = o\big(\log |\lambda| \big)^{-1}), \qquad \| \nabla \mathcal{G}_2^-(\lambda) \|_{\mathscr{B}(s,-s)} = o\big(\log |\lambda| \big)^{-1}) \qquad \lambda \to 0- .$$

Proof. Taking into account the asymptotic equation (C.4), the claim follows in the same way as in the case $\lambda > 0$, cf. Lemma 6.4. $\square$

6.3. **Proof of Proposition 4.2.** The statement of the Proposition follows from Lemmata 6.1, 6.3, 6.4 and 6.5.

7. Auxiliary integral estimates

In this section we prove several integral estimates on Bessel functions and their derivatives. These results will be used in the proof of Lemmata 5.3 and 5.4 .

Lemma 7.1. *Let* $\nu \geq 0$. *Assume that* $s > \frac{3}{2} + \varepsilon$, $0 < \varepsilon < 1$. *Then for all* $\lambda \in (0,1)$ *we have*

$$\int_1^\infty J_\nu^2(\sqrt{\lambda}\, r)\, \rho^{-2s}(r)\, r\, dr \;\lesssim\; (\lambda^\nu + \lambda^{\frac{1}{2}+\varepsilon})\, (1+\nu^2)^{-1}.$$

Proof. For $\nu \leq 2$ use (B.11) and (B.4):

$$\int_1^\infty J_\nu^2(\sqrt{\lambda}\, r)\, \rho^{-2s}(r)\, r\, dr \leq \lambda^{\frac{1}{2}+\varepsilon} \int_{\sqrt{\lambda}}^\infty J_\nu^2(t)\, t^{-2-2\varepsilon}\, dt \;\lesssim\; \lambda^{\frac{1}{2}+\varepsilon} \left(\int_{\sqrt{\lambda}}^1 t^{2\nu-2-2\varepsilon}\, dt + \int_1^\infty t^{-2-2\varepsilon}\, dt \right)$$

$$\lesssim\; \lambda^\nu + \lambda^{\frac{1}{2}+\varepsilon}.$$

For $\nu > 2$ the estimate follows from (B.13) and the identity $\Gamma(\nu + \frac{3}{2}) = (\nu^2 - \frac{1}{4})\, \Gamma(\nu - \frac{1}{2})$. $\qquad\square$

Lemma 7.2. *Let* $\nu > 0$. *Assume that* $s > \frac{3}{2} + \varepsilon$, $0 < \varepsilon < 1$. *Then for all* $\lambda \in (0,1)$ *it holds*

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda \Big(\lambda^{-\nu} J_\nu(\sqrt{\lambda}\, r)\, J_\nu(\sqrt{\lambda}\, r') \Big) \right|^2 \rho^{-2s}(r')\, \rho^{-2s}(r)\, r\, r'\, dr dr' \;\lesssim\; \lambda^{2\varepsilon-1-2\nu}\, (1+\nu)^{-2} \quad (7.1)$$

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda \Big(\lambda^{\nu} J_{-\nu}(\sqrt{\lambda}\, r)\, J_{-\nu}(\sqrt{\lambda}\, r') \Big) \right|^2 \rho^{-2s}(r')\, \rho^{-2s}(r)\, r\, r'\, dr dr' \;\lesssim\; 2^{4\nu}\, \Gamma^4(\nu) \qquad\qquad (7.2)$$

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda \Big(J_{\pm\nu}(\sqrt{\lambda}\, r)\, J_{\mp\nu}(\sqrt{\lambda}\, r') \Big) \right|^2 \rho^{-2s}(r')\, \rho^{-2s}(r)\, r\, r'\, dr dr' \;\lesssim\; \lambda^{\varepsilon-1}\, (1+\nu)^{-1}. \qquad (7.3)$$

Moreover, the function $J_{-\nu}$ *in the above estimates can be replaced by* Y_ν *throughout without changing the right hand side.*

Proof. Assume first that $\nu > 2$ and consider the bound (7.1). From (B.10) we find that

$$\partial_\lambda \Big(\lambda^{-\nu} J_\nu(\sqrt{\lambda}\, r)\, J_\nu(\sqrt{\lambda}\, r') \Big) = -\frac{\lambda^{-\nu-\frac{1}{2}}}{2}\, \Big(r\, J_{\nu+1}(\sqrt{\lambda}\, r)\, J_\nu(\sqrt{\lambda}\, r') + r'\, J_{\nu+1}(\sqrt{\lambda}\, r')\, J_\nu(\sqrt{\lambda}\, r) \Big).$$
$$\qquad\qquad (7.4)$$

With the help of (B.13) we estimate the first term as follows

$$\int_1^\infty \int_r^\infty J_{\nu+1}^2(\sqrt{\lambda}\, r)\, J_\nu^2(\sqrt{\lambda}\, r')\, r^3\, r'\, \rho^{-2s}(r')\, \rho^{-2s}(r)\, dr dr'$$

$$\leq \int_1^\infty J_{\nu+1}^2(\sqrt{\lambda}\, r)\, r^{-2\varepsilon}\, dr \int_1^\infty J_\nu^2(\sqrt{\lambda}\, r')\, r'^{-2-2\varepsilon} dr' \;\lesssim\; \lambda^{\varepsilon-\frac{1}{2}}\, \lambda^{\varepsilon+\frac{1}{2}}\, (1+\nu)^{-2}.$$

The second term in (7.4) is estimated in the same way. This proves (7.1) for $\nu > 2$. To prove we use again (B.9) and calculate

$$\partial_\lambda \Big(\lambda^{\nu} J_{-\nu}(\sqrt{\lambda}\, r)\, J_{-\nu}(\sqrt{\lambda}\, r') \Big) = -\frac{\lambda^{\nu-\frac{1}{2}}}{2}\, \Big(r\, J_{1-\nu}(\sqrt{\lambda}\, r)\, J_{-\nu}(\sqrt{\lambda}\, r') + r'\, J_{1-\nu}(\sqrt{\lambda}\, r')\, J_{-\nu}(\sqrt{\lambda}\, r) \Big).$$
$$\qquad\qquad (7.5)$$

To control the first term on the right hand side we recall (B.2) and (B.6) (with $j = 0$). This gives

$$\int_1^\infty \int_r^\infty J_{1-\nu}^2(\sqrt{\lambda}\,r)\, J_{-\nu}^2(\sqrt{\lambda}\,r')\, r^3\, r'\, \rho^{-2s}(r')\, \rho^{-2s}(r)\, dr dr' \leq$$

$$\leq \int_1^\infty J_{1-\nu}^2(\sqrt{\lambda}\,r) \int_r^\infty \left(J_\nu^2(\sqrt{\lambda}\,r') + Y_\nu^2(\sqrt{\lambda}\,r') \right) r'^2 \rho^{-2s}(r')\, dr'\, \rho^{-2s}(r)\, r^2\, dr$$

$$\lesssim \int_1^\infty J_{1-\nu}^2(\sqrt{\lambda}\,r) \left(J_\nu^2(\sqrt{\lambda}\,r) + Y_\nu^2(\sqrt{\lambda}\,r) \right) \rho^{-2s}(r)\, r^2\, dr$$

$$\leq \int_1^\infty \left(J_{\nu-1}^2(\sqrt{\lambda}\,r) + Y_{\nu-1}^2(\sqrt{\lambda}\,r) \right) \left(J_\nu^2(\sqrt{\lambda}\,r) + Y_\nu^2(\sqrt{\lambda}\,r) \right) \rho^{-2s}(r)\, r^2\, dr$$

$$\lesssim \left(J_{\nu-1}^2(\sqrt{\lambda}) + Y_{\nu-1}^2(\sqrt{\lambda}) \right) \left(J_\nu^2(\sqrt{\lambda}) + Y_\nu^2(\sqrt{\lambda}) \right) \lesssim \lambda^{1-2\nu}\, 2^{4\nu}\, \Gamma^4(\nu).$$

In the last step we have used (B.1) and (B.4). The second term in (7.5) can be estimated with the help of (B.6) applied with $j = 1$. Indeed, proceeding as above we find

$$\int_1^\infty \int_r^\infty J_{1-\nu}^2(\sqrt{\lambda}\,r')\, J_{-\nu}^2(\sqrt{\lambda}\,r)\, r'^3\, r\, \rho^{-2s}(r')\, \rho^{-2s}(r)\, dr dr' \leq$$

$$\leq \int_1^\infty J_{-\nu}^2(\sqrt{\lambda}\,r) \int_r^\infty \left(J_{\nu-1}^2(\sqrt{\lambda}\,r') + Y_{\nu-1}^2(\sqrt{\lambda}\,r') \right) r'^3 \rho^{-2s}(r')\, dr'\, \rho^{-2s}(r)\, r^2\, dr$$

$$\lesssim \int_1^\infty J_{-\nu}^2(\sqrt{\lambda}\,r) \left(J_{\nu-1}^2(\sqrt{\lambda}\,r) + Y_{\nu-1}^2(\sqrt{\lambda}\,r) \right) \rho^{-2s}(r)\, r^2\, dr$$

$$\lesssim \left(J_{\nu-1}^2(\sqrt{\lambda}) + Y_{\nu-1}^2(\sqrt{\lambda}) \right) \left(J_\nu^2(\sqrt{\lambda}) + Y_\nu^2(\sqrt{\lambda}) \right) \lesssim \lambda^{1-2\nu}\, 2^{4\nu}\, \Gamma^4(\nu).$$

As for (7.3), we note that (B.10) implies

$$\partial_\lambda \left(J_\nu(\sqrt{\lambda}\,r)\, J_{-\nu}(\sqrt{\lambda}\,r) \right) = -\frac{1}{2\sqrt{\lambda}} \left(r\, J_{\nu+1}(\sqrt{\lambda}\,r)\, J_{-\nu}(\sqrt{\lambda}\,r') + r'\, J_{1-\nu}(\sqrt{\lambda}\,r')\, J_\nu(\sqrt{\lambda}\,r) \right). \quad (7.6)$$

We now recall again (B.6) with $j = 0$ and estimate the first term on the right hand side of (7.6) as follows:

$$\int_1^\infty \int_r^\infty J_{\nu+1}^2(\sqrt{\lambda}\,r)\, J_{-\nu}^2(\sqrt{\lambda}\,r')\, r^3\, r'\, \rho^{-2s}(r')\, \rho^{-2s}(r)\, dr dr' \leq$$

$$\leq \int_1^\infty J_{\nu+1}^2(\sqrt{\lambda}\,r) \int_r^\infty \left(J_\nu^2(\sqrt{\lambda}\,r') + Y_\nu^2(\sqrt{\lambda}\,r') \right) r'^2 \rho^{-2s}(r')\, dr'\, \rho^{-2s}(r)\, r^2\, dr$$

$$\lesssim \int_1^\infty J_{\nu+1}^2(\sqrt{\lambda}\,r) \left(J_\nu^2(\sqrt{\lambda}\,r) + Y_\nu^2(\sqrt{\lambda}\,r) \right) r^{-1-4\varepsilon}\, dr$$

$$= \lambda^{2\varepsilon} \int_{\sqrt{\lambda}}^\infty J_{\nu+1}^2(t) \left(J_\nu^2(t) + Y_\nu^2(t) \right) t^{-1-4\varepsilon}\, dt. \quad (7.7)$$

To proceed we split the integration in (7.7) with respect to t in three parts as follows:

$$\int_{\sqrt{\lambda}}^1 J_{\nu+1}^2(t) \left(J_\nu^2(t) + Y_\nu^2(t) \right) t^{-1-4\varepsilon}\, dt \lesssim \nu^{-2},$$

where we have used (B.4),

$$\int_1^{\nu+1} J_{\nu+1}^2(t) \left(J_\nu^2(t) + Y_\nu^2(t) \right) t^{-1-4\varepsilon}\, dt \lesssim \nu^{-4/3} \int_1^{\nu+1} t^{-1-4\varepsilon}\, dt \lesssim \nu^{-4/3},$$

where we have used (B.14), and

$$\int_{\nu+1}^\infty J_{\nu+1}^2(t) \left(J_\nu^2(t) + Y_\nu^2(t) \right) t^{-1-4\varepsilon}\, dt \lesssim \int_{\nu+1}^\infty \frac{t^{-1-4\varepsilon}}{\sqrt{(t^2 - (\nu+1)^2)\,(t^2 - \nu^2)}}\, dt \lesssim \nu^{-1},$$

in view of (B.12). The second term on the right hand side of (7.6) is treated in the analogous way; using (B.6) with $j = 1$ and following the estimates used in (7.7) we get

$$\int_1^\infty \int_r^\infty J_\nu^2(\sqrt{\lambda}\,r)\, J_{1-\nu}^2(\sqrt{\lambda}\,r')\, r^3\, r'\rho^{-2s}(r')\, \rho^{-2s}(r)\, drdr'$$

$$\leq \int_1^\infty J_\nu^2(\sqrt{\lambda}\,r) \int_r^\infty \left(J_{\nu-1}^2(\sqrt{\lambda}\,r') + Y_{\nu-1}^2(\sqrt{\lambda}\,r')\right) r'^3 \rho^{-2s}(r')\, dr'\rho^{-2s}(r)\, r\, dr$$

$$\lesssim \int_1^\infty J_\nu^2(\sqrt{\lambda}\,r) \left(J_{\nu-1}^2(\sqrt{\lambda}\,r) + Y_{\nu-1}^2(\sqrt{\lambda}\,r)\right) r^{-1-4\varepsilon}\, dr$$

$$= \lambda^{2\varepsilon} \int_{\sqrt{\lambda}}^\infty J_\nu^2(t)\left(J_{\nu-1}^2(t) + Y_{\nu-1}^2(t)\right) t^{-1-4\varepsilon}\, dt.$$

By splitting the last integral in three parts in the same way as above and taking into account (7.6) we arrive at

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda\Big(J_\nu(\sqrt{\lambda}\,r)\, J_{-\nu}(\sqrt{\lambda}\,r')\Big)\right|^2 \rho^{-2s}(r')\,\rho^{-2s}(r)\, r\, r'\, drdr' \lesssim \lambda^{\varepsilon-1}\,(1+\nu)^{-1}, \quad \nu > 2.$$

The term $J_{-\nu}(\sqrt{\lambda}\,r)\, J_\nu(\sqrt{\lambda}\,r')$ in (7.3) is estimated in the same way. This completes the proof of the Lemma in the case $\nu > 2$. If $\nu \leq 2$, then the bounds (7.1)-(7.3) follow directly from (7.4), (7.5) and (7.6) by (B.12).

When $J_{-\nu}$ is replaced by Y_ν on the left hand side of (7.1)- (7.3), then we proceed in the same way as above using the obvious inequality $Y_\nu^2(z) \leq Y_\nu^2(z) + J_\nu^2(z)$ instead of (B.2). $\qquad\square$

Lemma 7.3. *Let $\nu > 0$. Assume that $s > \frac{3}{2} + \varepsilon$, $0 < \varepsilon < 1$. Denote*

$$J_\nu^{(1)}(\sqrt{\lambda}\,r) = \partial_r J_\nu(\sqrt{\lambda}\,r), \qquad J_\nu^{(2)}(\sqrt{\lambda}\,r) = \frac{\nu}{r}\, J_\nu(\sqrt{\lambda}\,r).$$

Then for all $\lambda \in (0,1)$ and $n = 1,2$ it holds

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda\Big(\lambda^{-\nu}\, J_\nu(\sqrt{\lambda}\,r)\, J_\nu^{(n)}(\sqrt{\lambda}\,r')\Big)\right|^2 \rho^{-2s}(r')\,\rho^{-2s}(r)\, r\, r'\, drdr' \lesssim \lambda^{\varepsilon-1-2\nu}\,(1+\nu)^{-2} \quad (7.8)$$

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda\Big(\lambda^\nu\, J_{-\nu}(\sqrt{\lambda}\,r)\, J_{-\nu}^{(n)}(\sqrt{\lambda}\,r')\Big)\right|^2 \rho^{-2s}(r')\,\rho^{-2s}(r)\, r\, r'\, drdr' \lesssim 2^{4\nu}\,\Gamma^4(\nu) \qquad (7.9)$$

$$\int_1^\infty \int_r^\infty \left| \partial_\lambda\Big(J_{\pm\nu}(\sqrt{\lambda}\,r)\, J_{\mp\nu}^{(n)}(\sqrt{\lambda}\,r')\Big)\right|^2 \rho^{-2s}(r')\,\rho^{-2s}(r)\, r\, r'\, drdr' \lesssim \lambda^{\varepsilon-1}\,(1+\nu)^{-1} \qquad (7.10)$$

Proof. From (B.9) and (B.10) we obtain

$$J_\nu^{(1)}(\sqrt{\lambda}\,r) = \frac{\sqrt{\lambda}}{2}\left[J_{\nu-1}(\sqrt{\lambda}\,r) - J_{\nu+1}(\sqrt{\lambda}\,r)\right], \quad J_\nu^{(2)}(\sqrt{\lambda}\,r) = \frac{\sqrt{\lambda}}{2}\left[J_{\nu-1}(\sqrt{\lambda}\,r) + J_{\nu+1}(\sqrt{\lambda}\,r)\right].$$

Hence the result follows in the same way as the proof of Lemma 7.2. $\qquad\square$

Lemma 7.4. *Let $\nu > 1$. Assume that $s > \frac{3}{2} + \varepsilon$, $0 < \varepsilon < 1$. Then for all $\lambda \in (0,1)$ it holds*

$$\int_1^\infty \left| \partial_\lambda\big(\lambda^{\frac{\nu}{2}}\, J_{-\nu}(\sqrt{\lambda}\,r)\big)\right|^2 \rho^{-2s}(r)\, r\, dr \lesssim \lambda^\varepsilon\, 4^\nu\, \Gamma^2(\nu-1), \qquad (7.11)$$

$$\int_1^\infty \left| \partial_\lambda\big(\lambda^{\frac{\nu}{2}}\, J_\nu(\sqrt{\lambda}\,r)\big)\right|^2 \rho^{-2s}(r)\, r\, dr \lesssim \lambda^{\nu-\frac{3}{2}+\varepsilon}. \qquad (7.12)$$

Moreover, the function $J_{-\nu}$ in the first bound can be replaced by Y_ν without changing the right hand side.

Proof. By (B.10) it holds

$$\partial_\lambda\big(\lambda^{\frac{\nu}{2}}\, J_{-\nu}(\sqrt{\lambda}\,r)\big) = -\lambda^{\frac{\nu-1}{2}}\, \frac{r}{2}\, J_{1-\nu}(\sqrt{\lambda}\,r). \qquad (7.13)$$

Now we use (B.2) and (B.6) with $j = 1$ to get

$$\int_1^\infty J_{1-\nu}^2(\sqrt{\lambda}\, r))\, \rho^{-2s}(r)\, r^3\, dr \;\le\; \int_1^\infty \left(J_{\nu-1}^2(\sqrt{\lambda}\, r) + Y_{\nu-1}^2(\sqrt{\lambda}\, r)\right) r^{-2\varepsilon}\, dr \tag{7.14}$$

$$= \lambda^{\varepsilon - \frac{1}{2}} \int_{\sqrt{\lambda}}^\infty t\left(J_{\nu-1}^2(t) + Y_{\nu-1}^2(t)\right) t^{-1-2\varepsilon}\, dt$$

$$\lesssim \lambda^\varepsilon \left(J_{\nu-1}^2(\sqrt{\lambda}) + Y_{\nu-1}^2(\sqrt{\lambda})\right) \lesssim \lambda^{\varepsilon+1-\nu}\, 4^\nu\, \Gamma^2(\nu - 1),$$

where we have applied (B.4) in the last step. This in combination with (7.13) proves (7.11). When $J_{-\nu}$ is replaced by Y_ν in (7.11), then we obtain the same estimate as can be seen from the first line of (7.14). The second estimate of the Lemma is proven in a similar way; by (B.9) we have

$$\partial_\lambda\left(\lambda^{\frac{\nu}{2}} J_\nu(\sqrt{\lambda}\, r)\right) = \lambda^{\frac{\nu-1}{2}}\, \frac{r}{2}\, J_{\nu-1}(\sqrt{\lambda}\, r).$$

Hence

$$\int_1^\infty \left|\partial_\lambda\left(\lambda^{\frac{\nu}{2}} J_\nu(\sqrt{\lambda}\, r)\right)\right|^2 \rho^{-2s}(r)\, r\, dr \;\lesssim\; \lambda^{\nu-1} \int_1^\infty J_\nu^2(\sqrt{\lambda}\, r)\, r^{-2\varepsilon}\, dr$$

and (7.12) follows from Lemma 7.1. $\square$

APPENDIX A. CONFLUENT HYPERGEOMETRIC FUNCTIONS

Recall first the definition of the Kummer's hypergeometric series;

$$M(a, b, z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\, \frac{z^n}{n!}, \tag{A.1}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (b)_n = b(b+1)\cdots(b+n-1), \qquad a_0 = b_0 = 1. \tag{A.2}$$

Note that by (A.1)

$$1 \;\le\; \left|M\left(\frac{1}{2} + m + |m|, 1 + 2|m|, z\right)\right| \;\le\; e^{|z|} \qquad \forall\, z \in \mathbb{R}, \;\; \forall\, m \in \mathbb{Z}. \tag{A.3}$$

For the function U, for $\operatorname{Re} a, \, z > 0$ we will often use its integral representation

$$\Gamma(a)\, U(a, b, z) = \int_0^\infty e^{-zt}\, t^{a-1}\, (1+t)^{b-a-1}\, dt, \tag{A.4}$$

see [AS, 13.2.5]. By [AS, Sect.13.4] the functions M and U are related to their derivatives in the following way:

$$\frac{d}{dz}\, M(a, b, z) = \frac{a}{b}\, M(a+1, b+1, z), \qquad \frac{d}{dz}\, U(a, b, z) = -a\, U(a+1, b+1, z). \tag{A.5}$$

Lemma A.1. *Let $z > 0$ and suppose that a and $b > 1$. Then*

$$\Gamma(a)\, |U(a, b, z)| \;\le\; \begin{cases} e^z\, z^{1-b}\, \Gamma(b-1) & a \ge 1, \\[2mm] a^{-1}\, 2^{b-a-1} + 2^{1-a}\, e^z\, z^{1-b}\, \Gamma(b-1) & a < 1 \end{cases} \tag{A.6}$$

Moreover, for any $a > 0, b > 0$ we have

$$\Gamma(a)\left|\frac{d}{dz}\, U(a, b, z)\right| \;\le\; \Gamma(b)\, e^z\, z^{-b} \tag{A.7}$$

Proof. For $a \geq 1$ equation (A.4) gives

$$\Gamma(a) \, |U(a,b,z)| \leq \int_0^\infty e^{-zt} \, (t+1)^{b-2} \, dt \; \leq \; e^z \int_0^\infty e^{-zt} \, t^{b-2} \, dt = \Gamma(b-1) \, e^z \, z^{1-b}.$$

This implies the first bound in (A.6). Assume now that $a < 1$. Then (A.4) implies

$$\Gamma(a) \, |U(a,b,z)| = \int_0^1 e^{-zt} \, t^{a-1} \, (t+1)^{b-a-1} \, dt + \int_1^\infty e^{-zt} \, t^{a-1} \, (t+1)^{b-a-1} \, dt$$

$$\leq \frac{1}{a} \, 2^{b-a-1} + \int_1^\infty e^{-zt} \left(\frac{1+t}{t}\right)^{1-a} (t+1)^{b-2} \, dt$$

$$\leq \frac{1}{a} \, 2^{b-a-1} + 2^{1-a} \, e^z \, z^{1-b} \, \Gamma(b-1).$$

The bound (A.7) follows from (A.5), the first part of the proof and the identity $\Gamma(a+1) = a \, \Gamma(a)$. $\square$

Lemma A.2. *Fix $z > 0$ and assume that $\lambda \leq \frac{3}{4} \alpha^2$. Then*

$$\left| M\left(\frac{1}{2} + j + |m| + m\frac{\alpha}{\kappa}, 1 + j + 2|m|, z\right) \right| \leq e^{2z} \qquad \forall \, m \in \mathbb{Z}, \quad j = 0, 1. \tag{A.8}$$

Moreover, there exists $\lambda_c \leq \frac{3}{4} \alpha^2$ and $m_c \in \mathbb{N}$, independent of λ_c, such that for all $\lambda \in (0, \lambda_c)$ and all $m \in \mathbb{Z}$ with $|m| \geq m_c$ we have

$$M\left(\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}, 1 + 2|m|, z\right) \geq \frac{1}{2}. \tag{A.9}$$

Proof. Let $j = 0$. Since $|\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}| < 2(1 + 2|m|)$ for all $m \in \mathbb{Z}$ by assumption, with the notation introduced in (A.2) we find that

$$\left(\left|\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}\right|\right)_n \leq 2^n \, \left(1 + 2|m|\right)_n.$$

Since $|M(a,b,z)| \leq M(|a|,b,z)$ if $b > 0$, this in combination with (A.1) implies (A.8). The proof for $j = 1$ follows in the same way.

Next we note that if $m \geq 0$ then (A.9) follows immediately from (A.1). We may thus assume that $m < 0$. Note that

$$\lim_{m \to -\infty} \frac{|\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}|}{1 + 2|m|} = \frac{\alpha - \sqrt{\alpha^2 - \lambda}}{2\sqrt{\alpha^2 - \lambda}} =: q(\lambda) \tag{A.10}$$

uniformly with respect to $\lambda \in (0, \frac{3}{4}\alpha^2)$. Hence there exists $m_c \in \mathbb{N}$ such that for all $m \leq -m_c$ and all $\lambda \leq \frac{3}{4}\alpha^2$ we have

$$\frac{|\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}|}{1 + 2|m|} \leq 2 \, q(\lambda).$$

Since m is negative and $2 \, q(\lambda) \leq 1$, by assumption, in view of (A.2) we conclude that

$$\left(\left|\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}\right|\right)_n \leq 2 \, q(\lambda) \, \left(1 + 2|m|\right)_n \qquad \forall \, n \geq 1 \quad \forall \, m \leq -m_c.$$

By (A.1) this implies

$$M\left(\frac{1}{2} + |m| + m\frac{\alpha}{\kappa}, 1 + 2|m|, z\right) \geq 1 - 2 \, q(\lambda) \sum_{n=1}^\infty \frac{z^n}{n!}.$$

To finish the proof it suffices to take λ_c small enough, cf. (A.10), so that

$$2 \, q(\lambda) \sum_{n=1}^\infty \frac{z^n}{n!} \leq \frac{1}{2} \qquad \forall \, \lambda \leq \lambda_c.$$

$\square$

Lemma A.3. *Let* $b > 0$ *and* $z > 0$. *Then*

$$\left| \frac{d}{da} M(a,b,z) \right| \lesssim \frac{M(|a|,b,2z)}{1+|a|} .$$ (A.11)

Proof. From the definition of $(a)_n$, see (A.2), we find that

$$\left| \frac{d}{da} (a)_n \right| \lesssim \frac{n\,(|a|)_n}{1+|a|} \qquad \forall\, n \in \mathbb{N}.$$

Hence

$$\left| \frac{d}{da} M(a,b,z) \right| \lesssim \frac{1}{1+|a|} \sum_{n=1}^{\infty} \frac{(|a|)_n}{(b)_n} \frac{n\,z^n}{n!} \leq \frac{1}{1+|a|} \sum_{n=1}^{\infty} \frac{(|a|)_n}{(b)_n} \frac{(2z)^n}{n!}$$

$$\leq \frac{1}{1+|a|}\, M(|a|,b,2z).$$

$\square$

Lemma A.4. *Let* $a,b > 0$ *and* $z > 0$. *Then*

$$\left| \frac{d}{da} \big(\Gamma(a)\, U(a,b,z) \big) \right| \lesssim 2^{b-a-1} + e^z\, z^{1-b}\, \Gamma(b-1).$$ (A.12)

Proof. By (A.4) we have

$$\frac{d}{da}(\Gamma(a)\,U(a,b,z)) = \int_0^\infty e^{-zt} \log\left(\frac{t+1}{t}\right) t^{a-1}\,(1+t)^{b-a-1}\,dt \leq 2^{b-a-1} \int_0^1 \log\left(\frac{t+1}{t}\right) t^{a-1}\,dt$$

$$+ \int_1^\infty e^{-zt}\, t^{a-1}\,(1+t)^{b-a-1}\,dt \lesssim 2^{b-a-1} + e^z \int_1^\infty e^{-z(t+1)}\,(t+1)^{b-2}\,dt$$

$$\leq 2^{b-a-1} + e^z\, z^{1-b}\, \Gamma(b-1).$$

$\square$

Appendix B. Bessel functions J and Y

The functions $J_\nu(z)$ and $Y_\nu(z)$ are related through the identity [AS, Eq.9.1.2]

$$Y_\nu(z) = \frac{J_\nu(z)\,\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} .$$ (B.1)

The right hand side is to be replaced by a limiting value when $\nu \in \mathbb{Z}$. The above formula implies

$$J_{-\nu}^2(z) \leq J_\nu^2(z) + Y_\nu^2(z).$$ (B.2)

By [AS, Eq.9.1.16] their Wronskian is independent of ν and reads as follows:

$$J_\nu'(z)\,Y_\nu(z) - J_\nu(z)\,Y_\nu'(z) = J_{\nu+1}(z)\,Y_\nu(z) - Y_{\nu+1}(z)\,J_\nu(z) = \frac{2}{\pi z}.$$ (B.3)

For $z \to 0$ we have

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)}(1+\mathcal{O}(z^2)), \qquad\qquad\qquad \nu \geq 0$$

$$Y_\nu(z) = -\frac{\Gamma(\nu)\,(z/2)^{-\nu}}{\pi}(1+\mathcal{O}(z^2)) + \frac{\cot(\nu\pi)\,(z/2)^\nu}{\Gamma(1+\nu)}(1+\mathcal{O}(z^2)), \quad \nu > 0,$$ (B.4)

where the error terms are uniform with respect to ν. When $\nu = 0$, then

$$Y_0(z) = \frac{2}{\pi}(\log z + \gamma - \log 2) + o(z), \qquad z \to 0,$$ (B.5)

where $\gamma \simeq 0,577$ is the Euler gamma constant, see [AS, Eqs.9.1.7-9]. From [Wa, pp.446] we learn that if $\nu > 1/2$, then

$$\frac{d}{dz}\left[z^j\,(Y_\nu^2(z) + J_\nu^2(z)) \right] \leq 0, \qquad \forall\, z > 0, \quad j = 0,1.$$ (B.6)

We will also need a representations of $J_\nu(z)$ in terms of a power series in z:

$$J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{4^k k! \, \Gamma(\nu + k + 1)}, \qquad \nu \neq -1, -2, \dots \tag{B.7}$$

and integral representation of Y_ν for $\nu > 0$:

$$Y_\nu(z) = \frac{2 \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \, \Gamma(\nu + \frac{1}{2})} \left[\int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \sin(zt) \, dt \right) - \int_0^\infty e^{-zt} (1 + t^2)^{\nu - \frac{1}{2}} \, dt \right]. \tag{B.8}$$

We refer to [AS, Eqs.9.1.10] and [Wa, Sect.6.1, Eq.(4)] respectively for the above formulas. Let us also recall the well-known relations between the Bessel functions and their derivatives: with the above notation we have

$$\frac{d}{dz} \mathcal{L}_\nu(z) = \mathcal{L}_{\nu-1}(z) - \frac{\nu}{z} \mathcal{L}_\nu(z) \tag{B.9}$$

$$\frac{d}{dz} \mathcal{L}_\nu(z) = -\mathcal{L}_{\nu+1}(z) + \frac{\nu}{z} \mathcal{L}_\nu(z). \tag{B.10}$$

Next we recall several pointwise bounds on J_ν and Y_ν. By [AS, Eq.9.1.60]

$$|J_\nu(z)| \leq 1 \qquad \forall \, \nu \geq 0, \quad \forall \, z > 0. \tag{B.11}$$

For large values of z and any fixed $\nu > 0$ it holds

$$Y_\nu^2(z) + J_\nu^2(z) \leq \frac{2}{\pi \sqrt{z^2 - \nu^2}}, \qquad \forall \, z \geq \nu, \tag{B.12}$$

see [Wa, p.447, eq.(1)]. Finally we mention an integral identity due to [Wa, p.403, eq.(2)]: for any $a > 0$ we have

$$\int_0^\infty J_\nu^2(a \, t) \, t^{-\beta} \, dt = \frac{a^{\beta-1} \Gamma(\beta) \Gamma(\nu + \frac{1-\beta}{2})}{2^\beta \Gamma^2(\frac{1+\beta}{2}) \Gamma(\nu + \frac{1+\beta}{2})}, \qquad 2\nu + 1 > \beta > 0. \tag{B.13}$$

Lemma B.1. *Let* $\nu \geq 1$. *Then*

$$\sup_{1 \leq t \leq \nu+j} \left(J_\nu^2(t) + |J_{\nu+j}(t) \, Y_\nu(t)| + |J_\nu(t) \, Y_{\nu+j}(t)| \right) \lesssim \nu^{-2/3}, \qquad j = 0, 1, 2. \tag{B.14}$$

Proof. We use the substitution $t = z\nu$. Suppose first that $1 \leq t \leq \nu$. In this case $z \in [\nu^{-1}, 1]$. By [AS, Eq.9.3.6, Eq.9.3.38] we have

$$J_\nu(t) = J_\nu(\nu z) = \left(\frac{4 \xi^{2/3}}{1 - z^2} \right)^{1/4} \left[\frac{\text{Ai}\big((\nu\xi)^{2/3}\big)}{\nu^{1/3}} + \frac{e^{-\nu\xi}}{1 + |\nu\,\xi|^{1/6}} \, \mathcal{O}(\nu^{-4/3}) \right] \tag{B.15}$$

$$Y_\nu(t) = Y_\nu(\nu z) = -\left(\frac{4 \xi^{2/3}}{1 - z^2} \right)^{1/4} \left[\frac{\text{Bi}\big((\nu\xi)^{2/3}\big)}{\nu^{1/3}} + \frac{e^{\nu\xi}}{1 + |\nu\,\xi|^{1/6}} \, \mathcal{O}(\nu^{-4/3}) \right], \tag{B.16}$$

where

$$\xi = \xi(z) = \frac{3}{2} \int_z^1 \frac{\sqrt{1 - t^2}}{t} \, dt, \qquad z < 1,$$

and Ai, Bi are Airy functions of the first and second kind. From their asymptotic behavior it follows that

$$|\text{Ai}(x)| \lesssim \begin{cases} x^{-\frac{1}{4}} \exp(-\frac{2}{3} x^{3/2}) & 1 \leq x \\ 1 & 0 < x < 1, \end{cases} \qquad |\text{Bi}(x)| \lesssim \begin{cases} x^{-\frac{1}{4}} \exp(\frac{2}{3} x^{3/2}) & 1 \leq x \\ 1 & 0 < x < 1, \end{cases}$$

Since $\xi(z)^{2/3}(1 - z^2)^{-1}$ remains bounded as $z \to 1$, which follows from the definition of $\xi(z)$, the above bounds on the Airy functions together with (B.15) and (B.16) imply

$$|J_\nu(t)| = |J_\nu(\nu z)| \lesssim \nu^{-1/3} e^{-\nu \xi(z)}, \qquad |Y_\nu(t)| = |Y_\nu(\nu z)| \lesssim \nu^{-1/3} e^{\nu \xi(z)}, \qquad \frac{1}{\nu} \leq z \leq 1.$$

This proves (B.14) for $j = 0$. Now if $\nu \le t \le \nu + j$, then we estimate $|J_{\nu+j}(t)|$ and $|Y_{\nu+j}(t)|$ in the same way as in the last equation with ν replaced by $\nu + j$. It remains to estimate $J_\nu(t)$ and $Y_\nu(t)$ on $[\nu, \nu + j]$. To this end we note that (B.15) and (B.16) still hold with $1 \le z \le (\nu + j)/\nu$ and

$$\xi(z) = -\frac{3}{2} \int_1^z \frac{\sqrt{t^2 - 1}}{t} \, dt, \qquad z > 1, \tag{B.17}$$

see [AS, Eq.9.3.39]. Therefore, following the above procedure we find that

$$|J_\nu(t)| = |J_\nu(\nu z)| \lesssim \nu^{-1/3} e^{\nu |\xi(z)|}, \quad |Y_\nu(t)| = |Y_\nu(\nu z)| \lesssim \nu^{-1/3} e^{\nu |\xi(z)|}, \qquad 1 \le z \le \frac{\nu + j}{\nu}.$$

Since $\nu |\xi(z)|$ is uniformly bounded with respect to ν on $[1, \frac{\nu+j}{\nu}]$, see (B.17), this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma B.2. *Let $z > 0$. Then*

$$|J_0(z) - 1| \lesssim \min\{z^2, 1\} \qquad \left|Y_0(z) - \frac{2}{\pi}\left(\log\frac{z}{2} + \gamma\right)\right| \lesssim (|\log z| + 1) \min\{z^2, 1\}, \tag{B.18}$$

and

$$|J_0'(z)| \lesssim \min\{z, \sqrt{z}\}, \qquad \left|Y_0'(z) - \frac{2}{\pi z}\right| \lesssim \min\{z, \sqrt{z}\} \tag{B.19}$$

Proof. The upper bounds in (B.18) follow from the integral representaions

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \cos t) \, dt, \qquad Y_0(z) = \frac{4}{\pi^2} \int_0^{\pi/2} \cos(z \cos t)(\gamma + \log(2z \sin^2 t)) \, dt,$$

see [AS, Sec.9.1]. To prove (B.19) we use the fact that $J_0'(z) = -J_1(z)$ and $Y_0'(z) = -Y_1(z)$, see (B.10). With the help of (B.4) and (B.12) we then arrive at (B.19). $\qquad\qquad\square$

Appendix C. Modified Bessel functions I and K

The modified Bessel functions I_ν and K_ν satisfy the relation

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu \pi)}, \tag{C.1}$$

see [AS, Eq.9.6.2], where as usual the right hand side is replaced by its limiting value if $\nu \in \mathbb{Z}$. From [AS, Eq.9.6.15] we learn that the Wronskian of I_ν and K_ν reads as follows:

$$K_\nu'(z) I_\nu(z) - K_\nu(z) I_\nu'(z) = K_{\nu+1}(z) I_\nu(z) + K_\nu(z) I_{\nu+1}(z) = \frac{1}{z}. \tag{C.2}$$

As $z \to 0$, then

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1 + \nu)}(1 + \mathcal{O}(z^2)), \qquad \nu \ge 0,$$

$$K_\nu(z) = \frac{\Gamma(\nu)(z/2)^{-\nu}}{2}(1 + \mathcal{O}(z^2)) - \frac{\pi (z/2)^\nu}{2 \sin(\nu\pi) \Gamma(\nu + 1)}(1 + \mathcal{O}(z^2)), \qquad \nu > 0, \tag{C.3}$$

with the errors terms uniform in ν, see [AS, Eq.9.6.10] and (C.1). For $\nu = 0$ we have by [AS, Eq.9.6.13]

$$K_0(z) = -\log z - \gamma + \log 2 + o(z) \qquad z \to 0. \tag{C.4}$$

Finally, we recall the relations between the functions I_ν and K_ν and their derivatives:

$$I_\nu'(z) = I_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z) \tag{C.5}$$

$$K_\nu'(z) = -K_{\nu\pm1} \pm \frac{\nu}{z} K_\nu(z), \tag{C.6}$$

see [AS, Eq.9.6.26].

Lemma C.1. *Let $\nu > 0$. Then for every $z > 0$ it holds*

$$I_{\nu+j}(z)\,K_\nu(z) \,\le\, \frac{1}{2\nu} \qquad j = 0, 1. \tag{C.7}$$

Proof. From equation (C.2) and the positivity of $I_\nu(z), K_\nu(z)$ (for $z > 0$) it follows that

$$I_{\nu+1}(z)\,K_\nu(z) \,\le\, \frac{1}{z}, \qquad I_\nu(z)\,K_{\nu+1}(z) \,\le\, \frac{1}{z}. \tag{C.8}$$

On the other hand, by [AS, Eq.9.6.26]

$$\frac{2\nu}{z}\,K_\nu(z) = -K_{\nu-1}(z) + K_{\nu+1}(z) \tag{C.9}$$

Hence

$$\frac{2\nu}{z}\,I_\nu(z)\,K_\nu(z) = K_{\nu+1}(z)\,I_\nu(z) - K_{\nu-1}(z)\,I_\nu(z) \le K_{\nu+1}(z)\,I_\nu(z) \le \frac{1}{z},$$

where we have used (C.8) and the positivity of $I_\nu(z)$ and $K_\nu(z)$. This proves (C.7) for $j = 0$. Similarly we obtain from (C.9) the upper bound

$$\frac{2\nu}{z}\,K_\nu(z)\,I_{\nu+1}(z) = I_{\nu+1}(z)\,K_{\nu+1}(z) - I_{\nu+1}(z)\,K_{\nu-1}(z) \,\le\, I_{\nu+1}(z)\,K_{\nu+1}(z) \le \frac{1}{\nu+1},$$

where we applied (C.7) with $j = 0$. It follows that $K_\nu(z)\,I_{\nu+1}(z) \le \frac{z}{\nu(\nu+1)}$. This in combination with (C.8) proves (C.7) for $j = 1$. $\qquad\square$

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