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An Approximation Result for Generalised Functions of Bounded Deformation and Applications to Damage Problems

Ph.D. Thesis

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A Trieste e a chi potrebbe risentirsi

Per me al mondo non v'ha un più caro e fido
luogo di questo.

Né a te dispiaccia, amico mio, se amore
reco pur tanto al luogo ove son *nata*.

Sai che un più vario, un più movimentato
porto di questo è solo il nostro cuore.

(Umberto Saba, *Il molo*)

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Introduction

The present thesis draws its inspiration from considerations of physical nature arising in the field of Damage Mechanics. Damage models for linearly elastic materials describe the worsening of the elastic properties of the material as a consequence of the applied loads. Roughly speaking, when a material is subject to a damage process, the elastic strain in the mainly damaged regions can become very large. Therefore one expects that the body develops some fractures in the regions where damage is concentrated.

One of the main goals of this thesis is the rigorous study, in the static case, of the asymptotic behaviour of certain damage models under different regimes. In particular we aim at identifying the limit model, which exhibits a strong dependence on the regime which is assumed. Certain regimes really lead to a model for fracture, of brittle or cohesive type. Nevertheless, some other regimes lead to a model for diffuse plasticity.

In the detailed exposition of the results we shall see that the rigorous mathematical investigation of the above-mentioned problems relies on the classical notion of Γ-convergence and requires to formulate the problem in a proper functional space. A crucial tool will be a new density theorem which has an independent theoretical interest.

In order to explain in details our results we need some terminology and preliminaries from Brittle Damage Mechanics. In [41, 42] Pham and Marigo describe the foundations of the variational approach to damage as well as the gradient damage model, which this thesis relies on.

(i) The damage state of the material point is characterized by a scalar internal variable v, defined on the reference configuration $\Omega \subset \mathbb{R}^n$ with values in the interval $[0,1]$. The value $v = 1$ corresponds to the original elastic material, while $v = 0$ represents the totally damaged material.

(ii) For a given state $v \in (0,1]$ the behaviour is elastic and described by the elastic potential $\mathscr{Q}(v, e(u))$, where $e(u)$ is the symmetric gradient of the displacement

u and the function $(v,e) \mapsto \mathscr{Q}(v,e)$ is, in the simplest case, increasing in v and quadratic in e. A prototypical example carrying the relevant features is $\mathscr{Q}(v,e) = v|e|^2$, which we shall consider in this section for the sake of the exposition.

(iii) In an isotropic, homogeneous, and linearized setting, the total energy for the damage model at fixed time is given by

$$\int_\Omega v|e(u)|^2 dx + \int_\Omega a\psi(v)dx + \int_\Omega b|\nabla v|^2 dx, \tag{1}$$

where ψ is strictly decreasing and $\psi(1) = 0$, and $a, b < +\infty$ are positive constants. Here the first term represents the stored elastic energy corresponding to the displacement u and to the internal variable v, the second term is the energy dissipated by the damage process, finally the last term, penalizing the spatial variations of v, guarantees some regularity in the distribution of damage. Assuming that $(u,v) \in H^1(\Omega,\mathbb{R}^n) \times H^1(\Omega,[0,1])$, one ensures that the energy (1) is finite. The functional (1) is complemented by suitable boundary conditions and lower order terms due to the action of external forces.

(iv) The quasistatic evolution is governed by the following rules: the damage process is *irreversible*, the system is in *static stable equilibrium* at each time t, and the total energy is *conserved*.

In this thesis we consider a damage model of the type (1) and we assume in addition that the damage is never complete. We focus on the problem of investigating the asymptotic behaviour of a solution of the stationary damage problem as the concentration and the completion of damage are forced, that is when the model requires regions with smaller and smaller volume where the internal variable tends to 0. As we shall see, a variety of difficulties arises already in this static context.

Our results can be applied to study the asymptotic behaviour of the incremental minimum problems used in the standard approximation of the quasistatic evolution (see, for instance, [27] and [33] for existence results of quasistatic evolutions in brittle fracture; see also [16] and [17] for some numerical simulations). With the exception of the classical Ambrosio-Tortorelli regime (7) in the antiplane context (see the work [36] by Giacomini), the extension to the continuous time is still an open problem and it is out of the aims of this thesis.

The mathematical rewording of the above-mentioned convergences under the non-completion of damage assumption entails the introduction of a positive parameter d such that

$$d \le v \le 1 \tag{2}$$

and the study of the limit behaviour of a minimizer of (1) under the constraint (2), with suitable boundary conditions, as $a \to +\infty$, $b \to 0$, and $d \to 0$. The choice of $a\psi(v)$ as the cost of the damage is in fact the simplest possible. As $a \to +\infty$, the internal variable v is compelled to tend to 1 $\mathcal{L}^n$-a.e. in Ω, entailing concentration of damage in regions with vanishing volume. The transition from the damaged to the undamaged regions occurs in a strip with smaller and smaller width due to the requirement $b \to 0$. Finally, the completion of damage is forced as the minimum d of v in the damaged regions tends to 0.

Our approach is based on Γ-convergence (see [24] and Section 1.8): a variational convergence which guarantees convergence of minimizers (and of minima) of the damage energies to minimizers (and minima) of the limit models.

The first part of the thesis studies the above-mentioned problem in the case of antiplane shear. This case is studied in its full generality, establishing a hierarchy of limit models depending on the asymptotic ratios of the parameters a, b, and d. The extension of some interesting results from the antiplane to the general case is the object of the second part of the thesis. As we will see this will not be a straightforward generalization of the scalar case, requiring the involvement of a new functional space and the proof of suitable density properties.

The antiplane shear is a special state of strain in a 3-dimensional cylindrical body, achieved when the displacements are parallel to the axis and depend only on the projection onto the basis. Under this hypothesis the displacement is described by a scalar function u defined on the cross section $\Omega \subset \mathbb{R}^2$ of the cylinder, so that the gradient ∇u replaces $e(u)$ in (1).

In order to state precisely our results we introduce three sequences $\delta_k, \varepsilon_k, \eta_k > 0$, with $\delta_k \to 0$, $\varepsilon_k \to 0$, $\eta_k \to 0$, playing the role of the vanishing parameters $1/a$, b, and d, respectively. Without loss of generality we assume that

$$\frac{\eta_k}{\delta_k} \to \alpha \quad \text{and} \quad \frac{\delta_k}{\varepsilon_k} \to \beta, \qquad \text{with} \quad 0 \le \alpha, \beta \le +\infty.$$

Since this does not require any additional difficulty, we consider the general case when $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$ and $|\nabla v|^2$ in the total energy is replaced by $|\nabla v|^p$ with $1 < p < +\infty$.

Given a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, for $u \in H^1(\Omega)$ and $v \in W^{1,p}(\Omega)$ with

$$\eta_k \le v \le 1 \quad \mathcal{L}^n\text{-a.e. in } \Omega, \tag{3}$$

we define

$$F_k(u,v) := \int_\Omega \Big(v\,|\nabla u|^2 + \frac{\psi(v)}{\delta_k} + \gamma\,\varepsilon_k^{p-1} |\nabla v|^p \Big) dx, \tag{4}$$

where $0 < \gamma < +\infty$ and $\psi \in C([0,1])$ is strictly decreasing with $\psi(1) = 0$. We set $F_k(u,v) := +\infty$ otherwise in $L^1(\Omega)\times L^1(\Omega)$.

The limit case $p = +\infty$ is also studied, in the sense that the penalization term $\int_\Omega \varepsilon_k^{p-1}|\nabla v|^p dx$ is now replaced by the constraint

$$|\nabla v| \le \frac{1}{\varepsilon_k} \quad \mathcal{L}^n\text{-a.e. in } \Omega.$$

In this case the energy functional is defined by

$$F_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(v\,|\nabla u|^2 + \frac{\psi(v)}{\delta_k}\Big) dx & \text{if } (u,v) \in H^1(\Omega)\times V_k, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$V_k := \Big\{ v \in W^{1,\infty}(\Omega) : \eta_k \le v \le 1,\ |\nabla v| \le \frac{1}{\varepsilon_k}\ \mathcal{L}^n\text{-a.e. in } \Omega \Big\}.$$

In Chapter 2 we determine the Γ-limit in $L^1(\Omega)\times L^1(\Omega)$ of the sequence (F_k) and we find that this limit depends on α and β (see Theorem 2.1). For some values of the parameters the limit functional is related to a fracture problem; this is due to damage concentration along the limit cracks. For some other values the limit is related to perfect plasticity; in this case we see damage diffusion, which leads to plastic strains. The Γ-limit can be described by means of an auxiliary functional $\Phi_{\alpha,\beta}\colon L^1(\Omega) \mapsto [0,+\infty]$, depending on the values of $0 \le \alpha, \beta \le +\infty$. Precisely the following main regimes can be identified.

- For $\alpha, \beta \in (0,+\infty)$ we define

$$\Phi_{\alpha,\beta}(u) := \int_\Omega |\nabla u|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} \tag{5}$$

  for $u \in SBV^2(\Omega)$, and $\Phi_{\alpha,\beta}(u) := +\infty$ if $u \notin SBV^2(\Omega)$. Here ∇u is the density of the absolutely continuous part of the distributional derivative of u, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, J_u is the jump set of u, $[u]$ is the jump of u, and $u \in SBV^2(\Omega)$ means that $u \in SBV(\Omega)$, $\nabla u \in L^2(\Omega,\mathbb{R}^n)$, and $\mathcal{H}^{n-1}(J_u) < +\infty$ (see [7] and Section 1.2 for the definition of these quantities). Setting $1/p+1/q = 1$, the precise definitions of the constants a_β and b_α are

$$a_\beta := 2\Big(\frac{q}{\beta}\Big)^{\frac{1}{q}} (\gamma p)^{\frac{1}{p}} \int_0^1 \psi^{\frac{1}{q}} ds, \quad b_\alpha := 2(\alpha\psi(0))^{\frac12} \qquad \text{if} \quad 1 < p < +\infty, \tag{6}$$

$$a_\beta := \frac{2}{\beta}\int_0^1 \psi\, ds, \quad b_\alpha := 2(\alpha\psi(0))^{\frac12} \qquad \text{if} \quad p = +\infty.$$

- When $\alpha = 0$ and $\beta \in (0, +\infty)$ we define

$$\Phi_{0,\beta}(u) := \int_\Omega |\nabla u|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) \quad \text{for } u \in GSBV^2(\Omega) \cap L^1(\Omega) \tag{7}$$

and $\Phi_{0,\beta}(u) := +\infty$ otherwise (see Section 1.2 for the definition of $GSBV^2(\Omega)$).

- If $\alpha = +\infty$ or $\beta = 0$ we set

$$\Phi_{\alpha,\beta}(u) := \int_\Omega |\nabla u|^2 dx \quad \text{for } u \in H^1(\Omega)$$

and $\Phi_{\alpha,\beta}(u) := +\infty$ if $u \notin H^1(\Omega)$.

- If $\alpha = 0$ and $\beta = +\infty$ we set $\Phi_{0,\infty}(u) := 0$ for $u \in L^1(\Omega)$.

- Finally for $\alpha \in (0, +\infty)$ and $\beta = +\infty$ we define

$$\Phi_{\alpha,\infty}(u) := \int_\Omega f_\alpha(|\nabla u|)dx + b_\alpha |D^s u|(\Omega) \quad \text{for } u \in BV(\Omega) \tag{8}$$

and $\Phi_{\alpha,\infty}(u) := +\infty$ if $u \notin BV(\Omega)$. Here $f_\alpha(t) := t^2$ for $0 \le t < b_\alpha/2$, $f_\alpha(t) := b_\alpha(t - b_\alpha/4)$ for $t \ge b_\alpha/2$, and $D^s u$ is the singular part of the distributional derivative of u.

We prove the following theorem (see Theorem 2.1).

Theorem 1. *The* Γ*-limit of* (F_k) *in* $L^1(\Omega) \times L^1(\Omega)$ *is the functional*

$$F_{\alpha,\beta}(u,v) := \begin{cases} \Phi_{\alpha,\beta}(u) & \text{if } v = 1 \ \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

The previous theorem, combined with standard properties of Γ-convergence, allows us to establish the following result about the limit behaviour of minima and minimizers (see Theorem 2.7).

Theorem 2. *Let* $r > 1$*, let* (δ_k)*,* (ε_k)*, and* (η_k) *be infinitesimal sequences of positive numbers, and let* $g \in L^r(\Omega)$*. For every* k*, let* (u_k, v_k) *be a minimizer of the functional*

$$F_k(u,v) + \int_\Omega |u - g|^r dx, \quad (u,v) \in L^1(\Omega) \times L^1(\Omega). \tag{9}$$

Then $v_k \to 1$ *strongly in* $L^1(\Omega)$ *and a subsequence of* (u_k) *converges strongly in* $L^r(\Omega)$ *to a minimizer* u *of the limit functional*

$$\Phi_{\alpha,\beta}(u) + \int_\Omega |u - g|^r dx, \quad u \in L^1(\Omega). \tag{10}$$

Moreover for every α and β the minimum values of (9) tend to the minimum value of the limit problem.

A few comments on the features of the limit problem are in order. The functional $F_{0,\beta}$ with $0 < \beta < +\infty$ has been originally introduced by Mumford and Shah in [40] for a variational approach to image segmentation and it has been subsequently used to determine stationary solutions in some brittle fracture models (see [18]). Under the latter interpretation the first integral represents the elastic energy stored in the nonfractured regions of the material, whereas the second term is the amount of energy paid to create the fracture surface.

Under this regime our convergence result recovers the work by Ambrosio and Tortorelli [10], where the approximating functionals are of the form

$$\int_\Omega (v^2 + \eta_k)|\nabla u|^2 dx + \varepsilon_k \int_\Omega |\nabla v|^2 dx + \frac{1}{4\varepsilon_k} \int_\Omega (v-1)^2 dx, \tag{11}$$

with $u, v \in H^1(\Omega)$, $0 \le v \le 1$, and $\eta_k/\varepsilon_k \to 0$. To our knowledge, no convergence result has been proved for (11) in the other regimes.

The first new result which inspired the study developed within this thesis corresponds to the regime $0 < \alpha < +\infty$ and $0 < \beta < +\infty$. With respect to the Mumford-Shah functional, the energy $F_{\alpha,\beta}$ now exhibits a further surface term depending on the amplitude of the jump $[u]$. While the first term in (5) again represents the stored elastic energy, the second term plays this time the role of energetic barrier that has to be overcome to unpin certain surfaces. A first interpretation for the last integral in (5) can be given using the terminology of fracture mechanics. A constant force acts between the lips of the crack J_u, whose displacements are u^+ and u^-; therefore the energy for unit area spent to create the crack is proportional to its opening $|[u]|$. This interpretation is not properly covered by the classical Barenblatt's cohesive crack model [12], due to the presence of an activation energy $\mathcal{H}^{n-1}(J_u)$ and to the fact that the cohesive force bridging the crack lips is not decreasing with respect to the crack opening and does not vanish for large values of the opening itself.

Another interpretation for the functional (5) has been recently given in [8]. The unpinned surfaces after the overcoming of the energy barrier are now seen in terms of sliding surfaces in a strain localization plastic process. Therefore $|[u]|$ here represents the surface plastic energy, that is the work per unit area that must be expended in order to produce plastic slip, supposed to occur at constant yielding shear stress. The model neglects the final failure stage eventually leading to fracture, so that infinite energy would be necessary to produce a complete separation of the body.

From the mathematical point of view, in [8] a different approximation of the energy (5) is also proposed, involving the elliptic functionals

$$\int_\Omega (v^2+\eta_k)|\nabla u|^2 dx + \varepsilon_k \int_\Omega |\nabla v|^2 dx + \frac{1}{4\varepsilon_k}\int_\Omega (v-1)^2 dx + \int_\Omega (v-1)^2 |\nabla u| dx,$$

with $u, v \in H^1(\Omega)$, $0 \leq v \leq 1$, and $\eta_k/\varepsilon_k \to 0$.

Our last interesting result in the case of antiplane shear is obtained when $0 < \alpha < +\infty$ and $\beta = +\infty$. The functional $F_{\alpha,\infty}$ is now related to the Hencky's diffuse plasticity model (see [11] and [44]), so that we are able to simulate a plastic material by means of damaged elastic materials. To our knowledge, in this case no other approximation result with phase field models is available in the literature. When $\alpha = +\infty$ or $\beta = 0$ the limit functional corresponds to an elasticity problem without cracks.

The Γ-convergence method consists in proving two inequalities: a liminf inequality, which provides a lower bound for the limit functional, together with some compactness properties for sequences with equibounded energies, and a limsup inequality, based on the construction of a recovery sequence, which guarantees that the lower bound is indeed optimal. In our framework, to prove compactness of displacements with equibounded energies, a key tool is a characterization proved in [1] which relates L^1-compactness of sequences with L^1-compactness of slices (see Theorem 1.9). Crucial ingredients in the construction of the recovery sequences are the density result for SBV established in [23] (see Theorem 1.13) and the relaxation result contained in [15].

To conclude the discussion about the case of antiplane shear, let us stress that the variational approximation via families of elliptic functionals has also turned out to be an efficient analytical tool and numerical strategy in order to analyze the behaviour of those energies and of their minimizers, being the approximating functionals easier to handle with respect to their limit counterpart (see for instance [18] and [8]).

For completeness we also recall that some variants of the Ambrosio-Tortorelli approximation have been introduced by other authors to solve different problems: for the purpose of approximating energies arising in the theory of nematic liquid crystals [9], the Blake and Zisserman second order model in computer vision [6], to provide a common framework for curve evolution and image segmentation [43, 2, 3], for general free discontinuity functionals defined over vector-valued fields [29, 30], and finally for functionals defined over bounded fields and corresponding to models for brittle linearly elastic materials [20, 21], which will be also discussed in the next part.

In the second part of this thesis we are concerned with studying the convergence problem for (1) in the general case of linearized elasticity in dimension n, where several additional difficulties arise.

Let us consider first the counterpart of the minimum problem for (10) in the regime $\alpha = 0,\ 0 < \beta < +\infty$:

$$\min_u \Big(\int_{\Omega \setminus J_u} |e(u)|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) + \int_\Omega |u - g|^2 dx \Big), \tag{12}$$

where $e(u)$ is the symmetric part of the gradient of u and a_β is defined as in (6). This represents a prototype of the minimum problems occurring in the mathematical formulation of some variational models in Linearly Elastic Fracture Mechanics (see, *e.g.*, [34, 35], [18]).

Drawing inspiration from the scalar-valued case, numerical computations concerning (12) and similar problems are performed, *e.g.*, in [17, 18], and [16] using a phase-field approximation, which leads to the minimization of Ambrosio-Tortorelli type functionals

$$\min_{(u,v)} \int_\Omega \Big(v|e(u)|^2 + \frac{\psi(v)}{\varepsilon_k} + \gamma \varepsilon_k |\nabla v|^2 + |u - g|^2 \Big) dx, \tag{13}$$

where η_k, ε_k belong to $(0, +\infty)$, $\eta_k/\varepsilon_k \to 0$, and (u, v) runs in $H^1(\Omega, \mathbb{R}^n) \times H^1(\Omega)$ with $\eta_k \le v \le 1$.

Nevertheless, so far in the literature there is no complete rigorous proof of the convergence of these minimum problems to problem (12) in the vector-valued case. An important contribution in this direction has been given by Chambolle in [20, 21], where the problem (12) is set in the space $SBD(\Omega)$ (we refer to [44] and to Section 1.3 for its definition) and the convergence result is proved under the assumption of an *a priori* bound on the L^∞-norm of the function u. Actually, even the existence of solutions in $SBD(\Omega)$ to the problem (12) is guaranteed only if an *a priori* L^∞-bound for minimizing sequences is assumed (see [14, Theorem 3.1]).

In Section 4.2 we provide the first complete proof of the convergence of the solutions to (13) toward a solution to (12), formulating these problems in a more convenient framework. Precisely, if (u_k, v_k) is a sequence of minimizers of the problem (13), we prove (see Corollary 4.2) that $v_k \to 1$ in $L^1(\Omega)$ and a subsequence of u_k converges in $L^2(\Omega, \mathbb{R}^n)$ to a minimizer u of the problem (12) in the space $GSBD(\Omega)$ of Generalized Special Functions of Bounded Deformation.

This space has been recently introduced by Dal Maso in [25] to solve minimum problems of the form (12) without L^∞-bounds on the minimizing sequences. For every $u \in GSBD(\Omega)$ it is possible to define the approximate symmetric gradient $e(u) \in L^1(\Omega, \mathbb{M}^{n\times n}_{sym})$, the approximate one-sided limits $u^\pm$ on regular submanifolds, and the approximate jump set J_u, which turns out to be $(\mathcal{H}^{n-1}, n-1)$-rectifiable (see Section 1.5 for a summary of these fine properties of $GSBD$-functions). Therefore the functional occurring in (12) makes sense in this more general context and a solution in $GSBD(\Omega)$ to the minimum problem is ensured by the compactness and semicontinuity result proved in [25, Theorem 11.3] (see also Theorem 1.12).

The strategy leading to the proof of the convergence of (13) to (12) is close in spirit to the one devised by Chambolle in [20, 21] and consists of three fundamental steps. The first and crucial step allows us (see Chapter 3) to approximate a function $u \in GSBD(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$, for which $e(u)$ is square integrable and $\mathcal{H}^{n-1}(J_u)$ is finite, with a sequence $(u_k) \subset SBV(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ of piecewise continuous functions in a way that

$$\begin{gathered} ||u_k - u||_{L^2(\Omega,\mathbb{R}^n)} \to 0, \\ ||e(u_k) - e(u)||_{L^2(\Omega,\mathbb{M}^{n\times n}_{sym})} \to 0, \\ \mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \to 0, \\ \int_{J_{u_k}\cup J_u} |u_k^\pm - u^\pm| \wedge 1 \, d\mathcal{H}^{n-1} \to 0, \end{gathered}$$

where $\triangle$ denotes the symmetric difference and $a \wedge b := \min\{a, b\}$.

The second step concerns the Γ-convergence of the functionals occurring in (13) to the one occurring in (12) (see Theorem 4.1). In particular the liminf inequality is obtained through a slicing technique. The Density Theorem 3.1 is involved in the proof of the Γ-$\limsup$ inequality, allowing us to construct a recovery sequence starting from more regular functions.

The third step is the proof of the compactness of the minimizing sequences of (13). This is established in Proposition 4.5 using again [1, Theorem 6.6] on the L^1-compactness of slices and its adaptation to the $GSBD$-context [25, Lemma 10.7] (see Section 1.6).

The last issue we face within this thesis is the extension to the n-dimensional case of the convergence result for (5) (see Section 4.3). To this aim we define

$$F_k(u, v) := \int_\Omega \Big(v|e(u)|^2 + \frac{\psi(v)}{\varepsilon_k} + \gamma\varepsilon_k|\nabla v|^2 \Big) dx \tag{14}$$

if $(u, v) \in H^1(\Omega, \mathbb{R}^n) \times V_{\varepsilon_k}$, where $V_{\varepsilon_k} = \big\{ v \in H^1(\Omega) : \varepsilon_k \le v \le 1 \ \mathcal{L}^n\text{-a.e. in } \Omega \big\}$, $F_k(u, v) := +\infty$ otherwise in $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$. We prove that the asymptotic behaviour of the sequence (F_k) is described by the cohesive type energy

$$\int_\Omega |e(u)|^2 \, dx + a_1 \, \mathcal{H}^{n-1}(J_u) + b_1 \int_{J_u} |[u] \odot \nu_u| d\mathcal{H}^{n-1}, \quad \text{for } u \in SBD(\Omega), \tag{15}$$

where a_1 and b_1 are defined as in (6) and the symbol $\odot$ denotes the symmetrized tensor product between vectors.

Let us stress that the previous results also hold if we replace the term $|e(u)|^2$ in the functionals by a more general quadratic form $\mathscr{Q}(e(u))$; consequently, the term $|[u]\odot\nu|$ in (15) is replaced by $\mathscr{Q}^{1/2}([u]\odot\nu)$.

The natural compactness for the problem and the identification of the domain of the possible limits are two main issues. Solving the former fixes the topology to be the strong L^p one for all $p\in[1,1^*)$, while the latter is given by the space $SBD^2(\Omega)$, an appropriate subset of $SBD(\Omega)$. To prove such assertions we establish first the equi-coercivity in the space BD of the energies F_k in (14) (see (4.52)). Given this, we use a global technique introduced by Ambrosio in [4] (see also [29, 30]) to gain coercivity in the space SBD. To this aim we construct a new sequence of displacements, with SBV regularity, by cutting around suitable sublevel sets of v in order to decrease the elastic contribution of the energy at the expense of introducing a surface term that can be kept controlled (see (4.58)). Thus, the SBD compactness result leads to the identification of the domain of the Γ-limit, and it provides the necessary convergences to prove the lower bound inequality for the volume term in (15) simply by applying a classical lower semicontinuity result due to De Giorgi and Ioffe (see estimate (4.49)).

From a technical point of view, the preliminary BD-compactness step is instrumental in order to fulfill the assumptions of the compactness theorem in SBD without imposing L^∞ bounds on the relevant sequences as it typically happens in problems of this kind. Therefore, our proof is completely developed within the theory of the space SBD, without making use of its extension $GSBD$.

The two $(n-1)$-dimensional terms in the target functional in (15) are the result of different contributions: the $\mathcal{H}^{n-1}$ measure of the jump set is detected as in the standard case by the Modica-Mortola type term in (14) and it quantifies the energy paid by the function v, being forced to make a transition from values close to 1 to values close to ε_k (see (4.50)); the cohesive term, instead, is associated to the size of the zone where v takes the minimal value ε_k, and, in the general case, it is related to the behaviour close to 0 of the family of quadratic forms in (4.35) (see assumption (H4)). A refinement of the arguments developed in establishing the compactness properties referred to above and the blow-up technique by Fonseca and Müller are then used to infer the needed estimate (cp. with (4.51)). All these issues are dealt with in the proof of Theorem 4.8 below.

Technical problems of different nature arise when we want to show that the lower bound that we have established is matched. Recovery sequences in Γ-convergence problems are built typically for classes of fields that are dense in energy and having more regular members. In our setting the density result for $GSBD$ established in Chapter 3 enables us to prove the sharpness of the estimate from below only for bounded fields in $SBD^2(\Omega)$ (see Theorem 4.9). Actually, we can extend it also to *all* fields in $SBV^2(\Omega,\mathbb{R}^n)$ by means of classical density theorems (see Remark 4.10 for more details). Clearly, these are strong hints that the lower bound we have derived is optimal, and that we cannot draw the conclusion in the general case for difficulties probably only of technical nature.

Eventually, let us recall briefly the structure of the thesis: Chapter 1 is devoted to fixing the notations and recalling some of the prerequisites needed in what follows.

In Chapter 2 we study the asymptotic behaviour of certain damage models in the case of antiplane shear, as some relevant parameters tend to 0. Chapter 3 is devoted to state and prove the Density Theorem 3.1 for $GSBD$. In Chapter 4 we show the applications of the density theorem to the Ambrosio-Tortorelli approximation of (12) (Section 4.2), and (15) (Section 4.3).

The results of Chapter 2 have been published in [26] and in [39], the first being in collaboration with Gianni Dal Maso and based on [37]. Precisely [26] contains the results stated in Subsection 2.2.2 in the particular case $\beta = 1$. The generalization to the case $\beta \neq 1$, the removal of a technical hypothesis (see (2.44)), and the involvement of a different penalization condition on the spatial variations of the damage variable are obtained in [39] and discussed in Subsection 2.2.1. The results of Subsection 1.5.1, of Chapter 3, and of Section 4.2 will appear in [38]. The content of Section 4.3 corresponds to a joint work with Matteo Focardi [31].

Chapter 1

Preliminary results

In this chapter we collect some notation and preliminary results that will be useful in the sequel. We start fixing the Measure Theory notation in Section 1.1. The main definitions and properties for the functional spaces BV, BD, and GBD are recalled respectively in Sections 1.2, 1.3, and 1.5. In Section 1.4 we fix the notation concerning the slicing method, while in Section 1.6 we recall some compactness properties descending from compactness of slices. Section 1.7 is devoted to a significant density result for the space SBV.

Some fine properties about GBD-functions discussed in Section 1.5 are contained in [38].

1.1 Notation

Let $n \geq 1$ be a fixed integer. The Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$, respectively.

The unit sphere of $\mathbb{R}^n$ is indicated by $\mathbb{S}^{n-1}$, while the open ball of $\mathbb{R}^n$ with centre x and radius r is indicated by $B(x,r)$ or $B_r(x)$; if $x = 0$, we write also B_r in place of $B_r(0)$. The Lebesgue measure of the unit ball of $\mathbb{R}^n$ is denoted by ω_n. Moreover let $d(x,E)$ be the Euclidean distance of the point x from the set $E \subset \mathbb{R}^n$, let $\mathrm{diam}(E)$ be the diameter of E, and let $E \triangle F$ be the symmetric difference of E and F. The symbols $\vee$ and $\wedge$ denote the maximum and the minimum operators respectively.

For every set A the characteristic function χ_A is defined by $\chi_A(x) := 1$ if $x \in A$ and by $\chi_A(x) := 0$ if $x \notin A$. Throughout the thesis Ω is assumed to be a bounded open subset of $\mathbb{R}^n$. Moreover c will denote a constant which may vary from line to line.

For every $j \in \mathbb{N} \cup \{\infty\}$, we will denote by $C^j_0(\Omega;\mathbb{R}^m)$ and $C^j_c(\Omega;\mathbb{R}^m)$ respectively the standard spaces of C^j functions vanishing on $\partial\Omega$ and with compact support in Ω. When $m = 1$ we omit the second argument $\mathbb{R}$.

Let us denote by $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ the set of all bounded vector Radon measures in Ω and by $\mathcal{M}_b^+(\Omega)$ the set of scalar nonnegative ones. Given $\mu_k, \mu \in \mathcal{M}_b(\Omega, \mathbb{R}^m)$, we say that $\mu_k \rightharpoonup \mu$ weakly* in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ if

$$\int_\Omega \varphi \, d\mu_k \to \int_\Omega \varphi \, d\mu \quad \text{for every } \varphi \in C_0^0(\Omega, \mathbb{R}^m).$$

1.2 BV-functions

For the general theory of BV-functions we refer to [7]; here we just recall the essential notation. For every $u \in BV(\Omega, \mathbb{R}^m)$ the distributional gradient Du is a bounded Radon measure. One can define the one-sided approximate limits u^+ and u^- on regular submanifold, the approximate differential ∇u, and the jump set J_u (see [7, Sections 3.1, 3.6]). The jump function $u^+ - u^-$ is denoted by $[u]$. The jump set J_u is $(\mathcal{H}^{n-1}, n-1)$-rectifiable according to [7, Definition 2.57] and a measure theoretic normal ν can be defined on J_u.

The strong convergence in $BV(\Omega, \mathbb{R}^m)$ is intended with respect to the norm $||u||_{BV(\Omega,\mathbb{R}^m)} := ||u||_{L^1(\Omega,\mathbb{R}^m)} + |Du|(\Omega)$, whereas the weakly* convergence of u_k to u in $BV(\Omega, \mathbb{R}^m)$ is intended as the strong convergence $u_k \to u$ in $L^1(\Omega, \mathbb{R}^m)$ joined with the weakly* convergence of the measures Du_k to the measure Du.

If $u \in BV(\Omega, \mathbb{R}^m)$ then the distributional derivative can be decomposed as $Du = D^a u + D^j u + D^c u$, where $D^a u$ is absolutely continuous and $D^s u = D^j u + D^c u$ is singular with respect to the Lebesgue measure. In particular $D^j u$ denotes the jump derivative of u and $D^j u = [u] \otimes \nu \mathcal{H}^{n-1} \lfloor J_u$, where $\otimes$ denotes the tensor product, whereas $D^c u$ is the Cantor part of the derivative of u (see [7, Section 3.9]). The approximate differential ∇u coincides $\mathcal{L}^n$-a.e. in Ω with the density of $D^a u$.

The spaces $SBV(\Omega, \mathbb{R}^m)$, $GBV(\Omega, \mathbb{R}^m)$, $GSBV(\Omega, \mathbb{R}^m)$ are defined as in [7]. We recall that a GBV-function is weakly approximately differentiable $\mathcal{L}^n$-a.e. in Ω (see [7, Definition 4.31, Theorem 4.34]). Since an approximately differentiable function u is also weakly approximately differentiable and the approximate differential coincides $\mathcal{L}^n$-a.e. in Ω with the weak approximate differential $\mathcal{L}^n$-a.e. in Ω, we also denote the weak approximate differential by ∇u.

For $p \in (1, +\infty)$ let us define

$$SBV^p(\Omega, \mathbb{R}^m) := \{u \in SBV : \nabla u \in L^p(\Omega, \mathbb{M}^{n\times m}),\ \mathcal{H}^{n-1}(J_u) < +\infty\},$$
$$GSBV^p(\Omega, \mathbb{R}^m) := \{u \in GSBV : \nabla u \in L^p(\Omega, \mathbb{M}^{n\times m}),\ \mathcal{H}^{n-1}(J_u) < +\infty\},$$

being $\mathbb{M}^{n\times m}$ the space of all $n\times m$ matrices.

Let us point out that for $n = m = 1$ one has that $u \in SBV^2(\Omega)$ entails $u \in H^1(\Omega \setminus J_u)$. Conversely, if $\Omega \subset \mathbb{R}$ and there exists a finite set F such that $u \in H^1(\Omega \setminus F)$, then $u \in SBV^2(\Omega)$ and $J_u \subset F$. By a truncation argument one deduces that in the one-dimensional case $GSBV^2(\Omega) \cap L^1(\Omega) = SBV^2(\Omega)$.

1.3 BD-functions

We recall briefly some notions related to the space $BD(\Omega)$ and to its subspace $SBD(\Omega)$. For complete results we refer to [45], [44], [13], [5], [14], and [28].

The symmetrized distributional derivative Eu of a function $u \in BD(\Omega)$ is by definition a finite Radon measure on Ω. Its density with respect to the Lebesgue measure on Ω is represented by the approximate symmetric gradient $e(u)$, the approximate jump set J_u is a $(\mathcal{H}^{n-1}, n-1)$ rectifiable set on which a measure theoretic normal ν and approximate one-sided limits $u^{\pm}$ can be defined $\mathcal{H}^{n-1}$-a.e.. Furthermore, we denote by $[u] := u^+ - u^-$ the related jump function.

For $u_k, u \in BD(\Omega)$, we say that $u_k \rightharpoonup u$ weakly* in $BD(\Omega)$ if $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly* in $\mathcal{M}_b(\Omega, \mathbb{M}^{n\times n}_{sym})$, where $\mathbb{M}^{n\times n}_{sym}$ is the space of all $n\times n$ symmetric matrices.

We point out that if Ω has Lipschitz boundary and $u \in L^1(\Omega, \mathbb{R}^n)$ satisfies $Eu \in L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$, where $\mathbb{M}^{n\times n}_{sym}$ is the set of all $n\times n$ symmetric matrices, then u actually belongs to $H^1(\Omega, \mathbb{R}^n)$. A key instrument to prove this result is the Korn's inequality [44, Proposition 1.1].

We define $SBD^p(\Omega)$, $1 < p < +\infty$, by

$$SBD^p(\Omega) := \big\{u \in SBD(\Omega) : e(u) \in L^p(\Omega, \mathbb{M}^{n\times n}_{sym}) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\big\}. \quad (1.1)$$

1.4 Slices

Fixed $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$, let π_ξ be the orthogonal projection onto the hyperplane $\Pi^\xi := \big\{y \in \mathbb{R}^n : y \cdot \xi = 0\big\}$, and for every subset $A \subset \mathbb{R}^n$ set

$$A^\xi_y := \big\{t \in \mathbb{R} : y + t\xi \in A\big\} \quad \text{for } y \in \Pi^\xi.$$

For $v \colon \Omega \to \mathbb{R}$, $u \colon \Omega \to \mathbb{R}^n$, and $e \colon \Omega \to \mathbb{M}^{n\times n}_{sym}$ we define the slices $v^\xi_y, u^\xi_y, e^\xi_y \colon \Omega^\xi_y \to \mathbb{R}$ by

$$v^\xi_y(t) := v(y+t\xi), \qquad u^\xi_y(t) := u(y+t\xi)\cdot\xi, \qquad \text{and} \qquad e^\xi_y(t) := e(y+t\xi)\xi\cdot\xi. \quad (1.2)$$

If $u_k, u \in L^1(\Omega, \mathbb{R}^n)$ and $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$, then for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence (u_j) of (u_k) such that

$$(u_j)^\xi_y \to u^\xi_y \text{ in } L^1(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi .$$

If $u \in BV(\Omega)$, then for every $\xi \in S^{n-1}$ the following properties hold:

$$\int_{J_u} |\nu_u \cdot \xi| d\mathcal{H}^{n-1}(y) = \int_{\Pi^\xi} \mathcal{H}^0((J_u)^\xi_y) d\mathcal{H}^{n-1}(y), \tag{1.3}$$

$$\int_{J_u} |\nu_u \cdot \xi| |[u]| d\mathcal{H}^{n-1}(y) = \int_{\Pi^\xi} \left[\int_{(J_u)^\xi_y} |[u]^\xi_y| d\mathcal{H}^0(t) \right] d\mathcal{H}^{n-1}(y), \tag{1.4}$$

$$\text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \text{ we have} \quad |\nabla(u^\xi_y)| = |(\nabla u)^\xi_y \cdot \xi| \leq |(\nabla u)^\xi_y| \quad \mathcal{L}^1\text{-a.e. on } \Omega^\xi_y. \tag{1.5}$$

Moreover for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$(J_u)^\xi_y = J_{u^\xi_y} \quad \text{and} \quad |[u]^\xi_y| = |[u^\xi_y]| \quad \text{on } \Omega^\xi_y. \tag{1.6}$$

We also make use of the fine properties of GBV-functions collected in [7, Theorem 4.34].

We recall next the slicing theorem in SBD (see [5]).

Theorem 1.1. *Let $u \in L^1(\Omega, \mathbb{R}^n)$ and let $\{\xi_1, ..., \xi_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then the following two conditions are equivalent:*

(i) *For every $\xi = \xi_i + \xi_j$, $1 \leq i, j \leq n$, the slice u^ξ_y belongs to $SBV(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and*

$$\int_{\Pi^\xi} \left| Du^\xi_y \right| (\Omega^\xi_y) \, d\mathcal{H}^{n-1}(y) < \infty;$$

(ii) *$u \in SBD(\Omega)$.*

Moreover, if $u \in SBD(\Omega)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ the following properties hold:

(a) $\nabla(u^\xi_y)(t) = e(u)\,(y + t\xi)\,\xi \cdot \xi$ *for $\mathcal{L}^1$-a.e. $t \in \Omega^\xi_y$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$;*

(b) $J_{u^\xi_y} = \left(J^\xi_u\right)^\xi_y$ *for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, where*

$$J^\xi_u := \{x \in J_u : \, [u](x) \cdot \xi \neq 0\};$$

(c) *for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$*

$$\mathcal{H}^{n-1}(J_u \setminus J^\xi_u) = 0. \tag{1.7}$$

1.5 GBD-functions

We now summarize the definition and the main properties of GBD-functions, referring to [25] for more details. The space $GBD(\Omega)$ is defined as follows (see [25, Definition 4.1] for related comments).

Definition 1.2. An $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}^n$ belongs to $GBD(\Omega)$ if there exists $\lambda_u \in \mathcal{M}_b^+(\Omega)$ such that the following equivalent conditions hold for every $\xi \in \mathbb{S}^{n-1}$:

(a) for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$, the partial derivative $D_\xi(\tau(u\cdot\xi))$ belongs to $\mathcal{M}_b(\Omega)$ and its total variation satisfies

$$|D_\xi(\tau(u\cdot\xi))|(B) \le \lambda_u(B), \tag{1.8}$$

for every Borel set $B \subset \Omega$;

(b) for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the function u_y^ξ belongs to $BV_{loc}(\Omega_y^\xi)$ and for every Borel set $B \subset \Omega$ it satisfies

$$\int_{\Omega^\xi} \Big(|Du_y^\xi|(B_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{u_y^\xi})\Big) d\mathcal{H}^{n-1} \le \lambda_u(B), \tag{1.9}$$

where we have set

$$J^1_{u_y^\xi} := \{t \in J_{u_y^\xi} : |[u_y^\xi](t)| \ge 1\}.$$

The space $GSBD(\Omega)$ is the set of all functions $u \in GBD(\Omega)$ such that for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the function u_y^ξ belongs to $SBV_{loc}(\Omega_y^\xi)$.

For every $u \in GBD(\Omega)$ one can define the approximate one-sided limits $u^\pm$ on regular submanifolds [25, Theorem 5.2].

Theorem 1.3. *Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be a C^1-submanifold of dimension $n-1$ with unit normal ν. Then for $\mathcal{H}^{n-1}$-a.e. $x \in M$ there exist $u_M^+(x)$, $u_M^-(x) \in \mathbb{R}^n$ such that*

$$\operatorname*{ap\,lim}_{\substack{\pm(y-x)\cdot\nu(x)>0\\ y\to x}} u(y) = u_M^\pm(x)\,. \tag{1.10}$$

Moreover for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$u_M^\pm(y+t\xi)\cdot\xi = \operatorname*{ap\,lim}_{\substack{\sigma_y^\xi(t)(s-t)>0\\ s\to t}} u_y^\xi(s) \quad \textit{for every } t \in M_y^\xi\,, \tag{1.11}$$

where ap lim *denotes the approximate limit and* $\sigma\colon M \to \{-1,1\}$ *is defined by* $\sigma(x) := \operatorname{sign}(\xi\cdot\nu(x))$.

One can also introduce the jump function $[u] := u^+ - u^-$ and the approximate jump set J_u [25, Definition 2.4], which turns out to be $(\mathcal{H}^{n-1}, n-1)$-rectifiable [25, Section 6].

Let $\xi \in \mathbb{S}^{n-1}$ and let

$$J_u^\xi := \{x \in J_u : u^+(x)\cdot\xi - u^-(x)\cdot\xi \neq 0\}\,. \tag{1.12}$$

Then for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ we have

$$(J_u^\xi)_y^\xi = J_{u_y^\xi}\,, \tag{1.13}$$

$$u^\pm(y+t\xi)\cdot\xi = (u_y^\xi)^\pm(t) \quad \text{for every } t \in (J_u)_y^\xi\,, \tag{1.14}$$

where the normals to J_u and $J_{u_y^\xi}$ are oriented so that $\xi\cdot\nu_u \geq 0$ and $\nu_{u_y^\xi} = 1$ (see [25, Theorem 8.1]).

For $u \in GBD(\Omega)$ the approximate symmetric gradient $e(u)$ in the sense of [13, Definition 8.1] exists and belongs to $L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ (see [25, Theorem 9.1]). Moreover for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ one has

$$(e(u))_y^\xi = \nabla u_y^\xi \quad \mathcal{L}^1\text{-a.e. on } \Omega_y^\xi. \tag{1.15}$$

Let us define $GSBD^p(\Omega)$ for $1 < p < +\infty$ by

$$GSBD^p(\Omega) := \big\{u \in GSBD(\Omega) : e(u) \in L^p(\Omega, \mathbb{M}^{n\times n}_{sym}) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\big\}.$$

Using the Fubini Theorem one can show that

$$\mathcal{H}^{n-1}(J_u \setminus J_u^\xi) = 0, \tag{1.16}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$.

1.5.1 Continuity of the trace

The rest of the section is devoted to the proof of some fine properties of GBD functions. Such results are included in the paper [38].

With the following lemma we deduce the existence of an orthonormal basis $(e_i)_{i=1}^n$ for which (1.16) holds for every $\xi \in D := \{e_i \text{ for } i = 1,\dots,n,\ e_i \pm e_j \text{ for } 1 \leq i < j \leq n\}$. We denote by μ the invariant Radon measure on the rotation group $SO(n)$ with $\mu(SO(n)) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$.

Lemma 1.4. *Let $\xi_1, \dots, \xi_k \in \mathbb{S}^{n-1}$. Then each $\xi \in \{R\xi_1, \dots, R\xi_k\}$ satisfies equation (1.16), for μ-a.e. $R \in SO(n)$.*

Proof. Let $N \subset \mathbb{S}^{n-1}$ be the set where (1.16) fails and let

$$M_j := \{R \in SO(n) : R\xi_j \in N\}.$$

For $j = 1, \dots, k$ we have

$$\mu(M_j) = \mathcal{H}^{n-1}(N) = 0.$$

Therefore for every $R \notin \bigcup_{j=1}^k M_j$ we find that $R\xi_1, \dots, R\xi_k \notin N$ and this concludes the proof. □

The following remark is about the extension by zero of GBD-functions.

Remark 1.5. Assume that Ω has Lipschitz boundary and consider a bounded open set $\hat{\Omega}$ with $\Omega \subset \hat{\Omega}$. Let $u \in GBD(\Omega) \cap L^1(\Omega, \mathbb{R}^n)$ and let us define $\hat{u} \colon \hat{\Omega} \to \mathbb{R}^n$ by $\hat{u} := u$ in Ω and by $\hat{u} := 0$ outside of Ω. Then the extension $\hat{u}$ belongs to $GBD(\hat{\Omega})$. Indeed, for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the slice u_y^ξ belongs to $BV(\Omega_y^\xi)$. Since Ω has Lipschitz boundary, for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the set Ω_y^ξ has finitely many connected components, so that $\hat{u}_y^\xi \in BV(\mathbb{R})$. Moreover an easy computation and the coarea formula show that

$$\int_{\hat{\Omega}^\xi} \Big(|D\hat{u}_y^\xi|(B_y^\xi \setminus J^1_{\hat{u}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{u}_y^\xi})\Big) d\mathcal{H}^{n-1} \leq \lambda_u(B \cap \Omega) + \mathcal{H}^{n-1} \lfloor \partial\Omega(B),$$

for every Borel set $B \subset \hat{\Omega}$ and for λ_u satisfying (1.9).

The next result provides an estimate for the trace $\mathrm{tr}(u)$ at the boundary $\partial\Omega$ of a function u belonging to $GSBD(\Omega) \cap L^1(\Omega, \mathbb{R}^n)$.

Lemma 1.6. *Assume that Ω has Lipschitz boundary and define $\tau(s) := \frac{1}{\pi} arctg\,(s)$ for $s \in \mathbb{R}$. Then there exists a constant $c(\Omega) < +\infty$, depending on Ω, such that*

$$\int_{\partial\Omega} \tau(|tr(u)|) d\mathcal{H}^{n-1} \leq c(\Omega) \Big(||u||_{L^1(\Omega, \mathbb{R}^n)} + \lambda_u(\Omega) \Big) \tag{1.17}$$

holds for every $u \in GSBD(\Omega) \cap L^1(\Omega, \mathbb{R}^n)$ and for $\lambda_u \in \mathcal{M}_b^+(\Omega)$ satisfying (1.9).

Proof. It is not restrictive to assume that Ω has the form

$$\{y + t\eta \in \mathbb{R}^n : y \in B^\eta,\ 0 < t < a(y)\} \tag{1.18}$$

and that u has compact support in $\Omega \cup \mathrm{graph}(a)$, where $\eta \in \mathbb{S}^{n-1}$, $B^\eta \subset \Pi^\eta$ is a relatively open ball, and $a \colon \overline{B^\eta} \to \mathbb{R}$ is a Lipschitz function. Indeed, let $(A_i)_{i=1}^k$ be an open covering of $\partial\Omega$ in a way that $A_i \cap \Omega$ has the form (1.18). Let $A_0 \subset\subset \Omega$ be such that $(A_i)_{i=0}^k$ covers $\overline{\Omega}$. Let us consider also a partition of unity $(\varphi_i)_{i=0}^k$, such that $\varphi_i \in C_c^\infty(A_i)$, $0 \leq \varphi_i \leq 1$, and $\sum_{i=0}^k \varphi_i = 1$ on $\overline{\Omega}$. Then each $\varphi_i u$ belongs to $GSBD(A_i \cap \Omega) \cap L^1(A_i \cap \Omega, \mathbb{R}^n)$ and has compact support in $A_i \cap \overline{\Omega}$. Moreover $\varphi_i u$ satisfies (1.9) with $\lambda_u(B)$ replaced by

$$||\nabla\varphi_i||_{L^\infty(A_i)} \int_B |u|dx + \lambda_u(B), \tag{1.19}$$

for every Borel set $B \subset A_i \cap \Omega$. Note that the measure defined in (1.19) belongs to $\mathcal{M}_b^+(A_i \cap \Omega)$.

Using the triangle inequality for τ and inequality (1.17) for $\varphi_i u$ with the measure (1.19), we obtain

$$\begin{aligned}\int_{\partial\Omega} \tau(|\mathrm{tr}(u)|)d\mathcal{H}^{n-1} &\leq \sum_{i=1}^k \int_{A_i\cap\partial\Omega} \tau(|\mathrm{tr}(\varphi_i u)|)d\mathcal{H}^{n-1} \\ &\leq c\Big(||u||_{L^1(\Omega,\mathbb{R}^n)} + \lambda_u(\Omega)\Big),\end{aligned}$$

where $c < +\infty$ depends on Ω and $(\varphi_i)_{i=1}^k$.

Let us prove now (1.17) under the assumption that Ω has the form (1.18) and that u has compact support on $\Omega \cup \mathrm{graph}(a)$. We may also assume that there exists a basis $(\eta_i)_{i=1}^n$ such that $M := \mathrm{graph}(a)$ is still a Lipschitz graph in the direction determined by each η_i and that $\nu(x) \cdot \eta_i > \delta > 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in M$, where δ is constant and ν is normal to M.

Therefore we obtain

$$\int_{\partial\Omega} \tau(|\mathrm{tr}(u)|)d\mathcal{H}^{n-1} = \int_M \tau(|\mathrm{tr}(u)|)d\mathcal{H}^{n-1} \leq \int_M \tau(c\sum_{i=1}^n |\mathrm{tr}(u)\cdot\eta_i|)d\mathcal{H}^{n-1}, \tag{1.20}$$

where $c < +\infty$ depends only on $(\eta_i)_{i=1}^n$. The very definition of τ implies that

$$\int_M \tau(c\sum_{i=1}^n |\mathrm{tr}(u)\cdot\eta_i|)d\mathcal{H}^{n-1} \leq \int_M c\sum_{i=1}^n |\tau(\mathrm{tr}(u)\cdot\eta_i)|d\mathcal{H}^{n-1}, \tag{1.21}$$

where the constant $c < +\infty$ can possibly change from the first to the second term.

Since Theorem 1.3 and the choice of $(\eta_i)_{i=1}^n$ ensure that $\tau(\mathrm{tr}(u)\cdot\eta_i) = \mathrm{tr}(\tau(u\cdot\eta_i))$ holds for $\mathcal{H}^{n-1}$-a.e. $x \in M$, we deduce by (1.20) and (1.21) that

$$\int_{\partial\Omega} \tau(|\mathrm{tr}(u)|)d\mathcal{H}^{n-1} \leq c\sum_{i=1}^n \int_M |\mathrm{tr}(\tau(u\cdot\eta_i))|d\mathcal{H}^{n-1}. \tag{1.22}$$

We observe now that $\tau(u \cdot \eta_i)$ belongs to $L^1(\Omega)$ and its derivative $D_{\eta_i}\tau(u \cdot \eta_i)$ belongs to $\mathcal{M}_b^+(\Omega)$, so that [45, Lemma 1.1] yields

$$\begin{aligned} c\sum_{i=1}^n \int_M |\mathrm{tr}(\tau(u \cdot \eta_i))| d\mathcal{H}^{n-1} &= c\sum_{i=1}^n \int_{\partial\Omega} |\mathrm{tr}(\tau(u \cdot \eta_i))| d\mathcal{H}^{n-1} \\ &\leq c\sum_{i=1}^n c(\Omega, \eta_i)\Big(||\tau(u \cdot \eta_i)||_{L^1(\Omega)} + |D_{\eta_i}\tau(u \cdot \eta_i)|(\Omega)\Big) \\ &\leq c\Big(||u||_{L^1(\Omega,\mathbb{R}^n)} + \lambda_u(\Omega)\Big), \end{aligned} \tag{1.23}$$

where $c < +\infty$ depends on Ω and $\lambda_u \in \mathcal{M}_b^+(\Omega)$ satisfies (1.8). Inequality (1.17) follows from (1.22) and (1.23). □

Remark 1.7. Let $u \in GSBD(\Omega) \cap L^1(\Omega, \mathbb{R}^n)$ with $\mathcal{H}^{n-1}(J_u) < +\infty$ and let us define

$$\tilde{\lambda}_u(B) := \int_B |e(u)| dx + \mathcal{H}^{n-1}(J_u \cap B), \tag{1.24}$$

for every $B \subset \Omega$ Borel set. Then (1.13), (1.15), and the coarea formula imply that $\tilde{\lambda}_u$ satisfies (1.9).

The following theorem concerns the continuity of the trace operator. For the proof we follow the lines of [44, Section 3.2]. We recall that a sequence $\mu_k \in \mathcal{M}_b^+(\Omega)$ weakly* converges in $(C_b^0)'$ to $\mu \in \mathcal{M}_b^+(\Omega)$ if

$$\int_\Omega \varphi d\mu_k \to \int_\Omega \varphi d\mu,$$

for every bounded continuous function φ defined on Ω.

Theorem 1.8 (Continuity of the trace)**.** *Let us assume that Ω has Lipschitz boundary. Let u_k, u belong to $GSBD(\Omega) \cap L^1(\Omega, \mathbb{R}^n)$ with $\mathcal{H}^{n-1}(J_{u_k}), \mathcal{H}^{n-1}(J_u) < +\infty$, and let*

$$u_k \to u \quad \text{in } L^1(\Omega, \mathbb{R}^n) \qquad \text{and} \qquad \tilde{\lambda}_{u_k} \rightharpoonup \tilde{\lambda}_u \quad \text{weakly* in } (C_b^0)', \tag{1.25}$$

where $\tilde{\lambda}$ has been introduced in (1.24). Then

$$\int_{\partial\Omega} |tr(u_k) - tr(u)| \wedge 1 d\mathcal{H}^{n-1} \to 0. \tag{1.26}$$

Proof. Let $\eta > 0$ and let $\Omega_0 \subset\subset \Omega$ be such that

$$\tilde{\lambda}_u(\Omega \setminus \overline{\Omega}_0) \leq \eta \qquad \text{and} \qquad \tilde{\lambda}_u(\partial\Omega_0) = 0. \tag{1.27}$$

Let $\varphi_0 \in C_c^\infty(\Omega)$ be such that $\varphi_0 = 1$ on Ω_0 and $0 \le \varphi \le 1$, and let $\psi_0 := 1 - \varphi_0$.

By (1.25) and (1.27) we obtain for k large

$$\int_\Omega |u_k - u| dx \le \frac{\eta}{1 + ||\nabla \psi_0||_{L^\infty(\Omega)}} \tag{1.28}$$

$$\tilde{\lambda}_{u_k}(\Omega \setminus \overline{\Omega}_0) \le \tilde{\lambda}_u(\Omega \setminus \overline{\Omega}_0) + \eta \le 2\eta. \tag{1.29}$$

Applying inequality (1.17) to the function $(u_k - u)\psi_0$ we find

$$\begin{aligned}
&\int_{\partial\Omega} \tau(|\mathrm{tr}(u_k) - \mathrm{tr}(u)|) d\mathcal{H}^{n-1} \le \\
\le\ & c\Big(||(u_k - u)\psi_0||_{L^1(\Omega,\mathbb{R}^n)} + \int_\Omega |e((u_k - u)\psi_0)| dx + \mathcal{H}^{n-1}(J_{(u_k-u)\psi_0}) \Big) \\
\le\ & c\Big(||u_k - u||_{L^1(\Omega,\mathbb{R}^n)} + \int_{\Omega\setminus\overline{\Omega}_0} |e(u_k)| dx + \int_{\Omega\setminus\overline{\Omega}_0} |e(u)| dx \\
& + ||u_k - u||_{L^1(\Omega,\mathbb{R}^n)} ||\nabla\psi_0||_{L^\infty(\Omega)} + \mathcal{H}^{n-1}(J_{u_k} \cap (\Omega \setminus \overline{\Omega}_0)) \\
& + \mathcal{H}^{n-1}(J_u \cap (\Omega \setminus \overline{\Omega}_0)) \Big) \\
\le\ & c\Big(||u_k - u||_{L^1(\Omega,\mathbb{R}^n)} (1 + ||\nabla\psi_0||_{L^\infty(\Omega)}) + \tilde{\lambda}_{u_k}(\Omega \setminus \overline{\Omega}_0) + \tilde{\lambda}_u(\Omega \setminus \overline{\Omega}_0) \Big) \le 4c\eta,
\end{aligned}$$

where in last inequalities we have used (1.27)–(1.29). Since $\eta > 0$ is arbitrary we deduce that $\tau(|\mathrm{tr}(u_k) - \mathrm{tr}(u)|) \to 0$ in $L^1_{\mathcal{H}^{n-1}}(\partial\Omega)$. Finally using the dominated convergence theorem we obtain (1.26). □

1.6 Compactness results

This section is devoted to recall some compactness results. We start with the following theorem which guarantees compactness of sequences as consequence of compactness of one-dimensional slices (see [1]).

For every set $\mathcal{F} \subset L^1(\Omega)$ we define $\mathcal{F}_y^\xi := \{u_y^\xi : u \in \mathcal{F}\}$, for $\xi \in S^{n-1}$ and $y \in \Pi^\xi$.

Theorem 1.9. *Let $\mathcal{F}$ be an equibounded subset of $L^\infty(\Omega)$. Assume that there exist n linearly independent unit vectors ξ which satisfy the following property: for every $\delta > 0$ there exists an equibounded subset $\mathcal{F}_\delta$ of $L^\infty(\Omega)$ such that $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}_\delta$ with respect to the $L^1(\Omega)$ distance and $(\mathcal{F}_\delta)_y^\xi$ is pre-compact in $L^1(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$. Then $\mathcal{F}$ is pre-compact in $L^1(\Omega)$.*

A slight generalization of the previous theorem is the following proposition, whose assumptions avoid the requirement of L^∞ bounds and concern only the components $u \cdot \xi$ of u and the corresponding slices in the same direction ξ (see [25, Lemma

10.7]). We recall that a modulus of continuity is an increasing continuous function $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(0) = 0$.

Proposition 1.10. *Let U be a set of $\mathcal{L}^n$-measurable functions from Ω into $\mathbb{R}^n$ and let $\psi_0\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing continuous function satisfying*

$$\lim_{s\to+\infty} \psi_0(s) = +\infty.$$

Assume that there exist $M \in \mathbb{R}^+$ such that

$$\int_\Omega \psi_0(|u|)dx \le M$$

holds for every $u \in U$ and a modulus of continuity $\hat{\omega}$ such that

$$|hs| \le \hat{\omega}(h)\psi_0(s)$$

holds for every $0 < h < 1$ and for every $s \in \mathbb{R}^+$. Assume also that for every $\delta > 0$ we can find a modulus of continuity ω_δ such that for every $\xi \in \mathbb{S}^{n-1}$ there exists a set V_δ^ξ of $\mathcal{L}^n$-measurable functions from Ω into $\mathbb{R}$ with the following properties:

(a) *for every $u \in U$ there exists $v \in V_\delta^\xi$ with*

$$\int_{\mathbb{R}^n} |u(x)\cdot\xi - v(x)|dx \le \delta;$$

(b) *for every $v \in V_\delta^\xi$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have*

$$\int_{\mathbb{R}} |v_y^\xi(t+h) - v_y^\xi(t)|dt \le \omega_\delta(h)$$

for every $0 < h < 1$.

Then every sequence in U has a subsequence that converges strongly in $L^1(\Omega, \mathbb{R}^n)$ to an $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}^n$.

The following lemma estimates the modulus of continuity in L^1 of the translations of BV functions when $n = 1$ (see [25, Lemma 10.8]).

Lemma 1.11. *Let $z \in BV(\mathbb{R})$. Assume that there exist two constants $a > 0$ and $b > 0$ such that*

$$|Dz|(\mathbb{R} \setminus J_z^1) + \mathcal{H}^0(J_z^1) \le a \qquad \text{and} \qquad ||z||_{L^\infty(\mathbb{R})} \le b.$$

Then

$$\int_{\mathbb{R}} |z(t+h) - z(t)|dt \le (a + 2ab)h$$

for every $h > 0$.

Finally we recall a compactness result for $GSBD$ (see [25, Theorem 11.3]).

Theorem 1.12. *Let u_k be a sequence in $GSBD(\Omega)$. Suppose that there exist a constant $M \in \mathbb{R}^+$ and two increasing continuous functions $\psi_0 \colon \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi_1 \colon \mathbb{R}^+ \to \mathbb{R}^+$, with*

$$\lim_{s \to +\infty} \psi_0(s) = +\infty \qquad \textit{and} \qquad \lim_{s \to +\infty} \frac{\psi_1(s)}{s} = +\infty,$$

such that

$$\int_\Omega \psi_0(|u_k|)dx + \int_{Omega} \psi_(|e(u_k)|)dx + \mathcal{H}^{n-1}(J_{u_k}) \leq M$$

for every k. Then there exist a subsequence, still denoted by u_k, and a function $u \in GSBD(\Omega)$, such that

$$\begin{gathered} u_k \to u \quad \mathcal{L}^n\text{-a.e. on } \Omega, \\ e(u_k) \rightharpoonup e(u) \quad \text{weakly in } L^1(\Omega, \mathbb{M}^{n\times n}_{sym}), \\ \mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(J_u). \end{gathered}$$

If, in addition,

$$\lim_{s \to +\infty} \frac{\psi_0(s)}{s} = +\infty$$

holds, then $u_k \in L^1(\Omega, \mathbb{R}^n)$ for every k, $u \in L^1(\Omega, \mathbb{R}^n)$, and the subsequence converges strongly in $L^1(\Omega, \mathbb{R}^n)$.

1.7 A density result for SBV

We recall next a density result in SBV [23, Theorem 3.1], for which we need to introduce further terminology. We say that $u \in SBV(\Omega, \mathbb{R}^n)$ is a piecewise smooth SBV-function if $u \in W^{m,\infty}(\Omega \setminus \overline{J_u}, \mathbb{R}^n)$ for every m, $\mathcal{H}^{n-1}((\overline{J_u} \cap \Omega) \setminus J_u) = 0$, and the set $\overline{J_u} \cap \Omega$ is a finite union of closed pairwise disjoint $(n-1)$-simplexes intersected with Ω.

Theorem 1.13. *Assume that Ω has Lipschitz boundary. Let u belong to the space $SBV^2 \cap L^\infty(\Omega, \mathbb{R}^n)$. Then there exists a sequence (u_k) of piecewise smooth SBV-functions such that*

(1) $||u_k - u||_{L^2(\Omega, \mathbb{R}^n)} \to 0$,

(2) $||\nabla u_k - \nabla u||_{L^2(\Omega, \mathbb{M}^{n\times n})} \to 0$,

(3) $\limsup_k \int_{\overline{A}\cap J_{u_k}} \varphi(x, u_k^+, u_k^-, \nu_{u_k}) d\mathcal{H}^{n-1} \leq \int_{\overline{A}\cap J_u} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1}$,

for every open set $A \subset \Omega$ and for every function $\varphi\colon \Omega\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^{n-1} \to [0,+\infty)$ upper semicontinuous and such that

$$\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu) \quad \text{for } x \in \Omega,$$
$$\limsup_{\substack{(y,a',b',\mu)\to(x,a,b,\nu)\\ y\in\Omega}} \varphi(y, a', b', \mu) < +\infty \quad \text{for } x \in \partial\Omega,$$

for every $a, b \in \mathbb{R}^n$, and $\nu \in \mathbb{S}^{n-1}$.

Remark 1.14. Note that if $\Omega \subset \mathbb{R}^n$ is an open cube, then the intersection $\overline{J_{u_k}} \cap \Omega$ is a polyhedron. Therefore, adapting the arguments in [23, Remark 3.5] and [22, Corollary 3.11] we can construct a new approximating sequence $(\tilde{u}_k)$ satisfying all requirements of Theorem 1.13 and such that $J_{\tilde{u}_k} \subset\subset \Omega$.

1.8 Γ-convergence

In this last section we briefly recall the definition and the main properties of Γ-convergence, for whose exhaustive treatment we refer to the book [24].

Definition 1.15. Given a metric space (X, d) and a sequence of functionals F_k, F defined on X with values in $\overline{\mathbb{R}}$ we say that F_k Γ-converges to F if for every $u \in X$ the following properties hold:

(a) for every sequence u_k with $u_k \to u$ we have $F(u) \leq \liminf_{k\to+\infty} F_k(u_k)$;

(b) there exists a sequence u_k with $u_k \to u$ such that $\limsup_{k\to+\infty} F_k(u_k) \leq F(u)$.

The most valuable property of the Γ-convergence concerns the convergence of minima and minimizers.

Theorem 1.16. *Let (X, d) be a metric space and let $F_k\colon X \to \overline{\mathbb{R}}$ be a sequence of equi-mildly cohercive functions, that is there exists a nonempty compact set $K \subset X$ such that $\inf_X F_k = \inf_K F_k$. Let $F = \Gamma\text{-}\lim_{k\to+\infty} F_k$, then*

$$\exists \min_X F = \lim_{k\to+\infty} \inf_X F_k.$$

Moreover, if (u_k) is a precompact sequence such that

$$\lim_{k\to+\infty} F_k(u_k) = \lim_{k\to+\infty} \inf_X F_k$$

then every limit of a subsequence of (u_k) is a minimum point for F.

Chapter 2

Asymptotic behaviour of certain damage models: the case of antiplane shear

2.1 Overview of the chapter

Damage models are used to describe the progressive degradation and failure in engineering materials such as metal, concrete, or rocks. The standard presentation of damage problems describes the state of the elastic body by means of two functions: the displacement u and the internal variable v.

In this chapter we consider a variational damage model for homogeneous isotropic materials in the case of antiplane shear. Our model depends on three small parameters δ_k, ε_k, and η_k, which are related respectively to the cost of the damage, to the width of the damaged regions, and to the minimum elasticity constant attained in the damaged regions. Denoting by $\alpha := \lim_{k\to+\infty} \eta_k/\delta_k$ and $\beta := \lim_{k\to+\infty} \delta_k/\varepsilon_k$ the asymptotic ratios as these parameters tend to zero, we analyse the limit behaviour of the damage model as $\alpha, \beta \in [0, +\infty]$ vary. We rigorously obtain, by Γ-convergence techniques, limit models for brittle fracture, for fracture with a cohesive zone, or for perfect plasticity, according to the relative magnitude of the three parameters.

The chapter is organized as follows. In Section 2.2 we describe the setting of the problem. In Section 2.3 we discuss the one-dimensional case. Section 2.4.1 is devoted to the proof of the liminf inequality in the n-dimensional case, while Section 2.4 to the construction of the corresponding recovery sequence. Finally in Section 2.5 we deal with the compactness result and the convergence of minima and minimizers.

We introduce in Subsection 2.2.2 the results published in [26], obtained in collaboration with Gianni Dal Maso and based on [37]. Here the pointwise constraint $||\nabla v||_{L^\infty(\Omega)} \leq 1/\varepsilon_k$ is supposed to penalize the spatial variation of the internal damage variable, under the assumption $\beta = 1$. Subsequently, with [39] the previous study is generalized to the case $\beta \neq 1$ and a technical hypothesis (see (2.44)) is removed; it is also considered the case when the penalization constraint is replaced by the penalization term of integral type in the total energy $\int_\Omega \varepsilon_k^{p-1}|\nabla v|^p dx$, $1 < p < +\infty$. These results are described in Subsection 2.2.1. From a technical point of view in the two subsections distinguished approaches are proposed for the regime which leads to the cohesive model.

2.2 The Γ-convergence result

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\delta_k > 0$, $\varepsilon_k \geq 0$, $\eta_k \geq 0$ be infinitesimal sequences. We assume that the limits

$$\alpha := \lim_{k\to+\infty} \frac{\eta_k}{\delta_k} \qquad \text{and} \qquad \beta := \lim_{k\to+\infty} \frac{\delta_k}{\varepsilon_k} \tag{2.1}$$

exist. We also introduce a parameter $1 < p \leq +\infty$ which will be involved in the penalization condition on the spatial variation of the internal variable v.

2.2.1 The case $p < +\infty$

Fixed $1 < p < +\infty$, our purpose is to study the Γ-limit in $L^1(\Omega)\times L^1(\Omega)$ of the sequence of functionals $F_k\colon L^1(\Omega)\times L^1(\Omega) \to [0,+\infty]$ defined by

$$F_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(v|\nabla u|^2 + \frac{\psi(v)}{\delta_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v|^p \Big) dx & \text{if } (u,v) \in H^1(\Omega)\times V_{\eta_k}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

where $c > 0$,

$$\psi \in C([0,1]) \text{ is strictly decreasing with } \psi(1) = 0, \tag{2.3}$$

$$V_{\eta_k} := \big\{ v \in W^{1,p}(\Omega) : \eta_k \leq v \leq 1 \ \mathcal{L}^n\text{-a.e. in } \Omega \big\}. \tag{2.4}$$

When $0 < \alpha < +\infty$ and $0 < \beta < +\infty$ we define $\Phi_{\alpha,\beta}\colon L^1(\Omega) \mapsto [0,+\infty]$ by

$$\Phi_{\alpha,\beta}(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

where

$$a_\beta := 2\Big(\frac{q}{\beta}\Big)^{\frac{1}{q}}(\gamma p)^{\frac{1}{p}}\int_0^1 \psi^{\frac{1}{q}}ds, \quad b_\alpha := 2(\alpha\psi(0))^{\frac{1}{2}}, \quad \text{and} \quad \frac{1}{p}+\frac{1}{q}=1. \tag{2.6}$$

In the limiting case when $\alpha = 0$ and $0 < \beta < +\infty$ we define

$$\Phi_{0,\beta}(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) & \text{if } u \in GSBV^2(\Omega)\cap L^1(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.7}$$

If $\alpha = +\infty$ or $\beta = 0$ we define

$$\Phi_{\alpha,\beta}(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx & \text{if } u \in H^1(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.8}$$

It remains to define the functional $\Phi_{\alpha,\beta}$ when $0 \le \alpha < +\infty$ and $\beta = +\infty$. When $\alpha = 0$ and $\beta = +\infty$ we set

$$\Phi_{0,\infty}(u) := \begin{cases} 0 & \text{if } u \in L^1(\Omega),\\ +\infty & \text{otherwise,}\end{cases} \tag{2.9}$$

whereas for $0 < \alpha < +\infty$ and $\beta = +\infty$ we set

$$\Phi_{\alpha,\infty}(u) := \begin{cases} \displaystyle\int_\Omega f_\alpha(|\nabla u|)dx + b_\alpha |D^s u|(\Omega) & \text{if } u \in BV(\Omega),\\ +\infty & \text{otherwise,}\end{cases} \tag{2.10}$$

where $f_\alpha(t) = t^2$ for $0 \le t < b_\alpha/2$ and $f_\alpha(t) = b_\alpha(t - b_\alpha/4)$ for $t \ge b_\alpha/2$.

The following Γ-convergence result holds.

Theorem 2.1. *Assume* (2.1)–(2.4) *and assume that* Ω *has Lipschitz boundary. The* Γ*-limit of* (F_k) *in* $L^1(\Omega)\times L^1(\Omega)$ *exists and is given by*

$$F_{\alpha,\beta}(u,v) := \begin{cases} \Phi_{\alpha,\beta}(u) & \textit{if } v = 1\ \mathcal{L}^n\textit{-a.e. in } \Omega,\\ +\infty & \textit{otherwise.}\end{cases} \tag{2.11}$$

Theorem 2.1 directly follows from the estimates for the functionals

$$F'_{\alpha,\beta} := \Gamma\text{-}\liminf_{k\to+\infty} F_k \qquad \text{and} \qquad F''_{\alpha,\beta} := \Gamma\text{-}\limsup_{k\to+\infty} F_k \tag{2.12}$$

stated in the following theorems.

Theorem 2.2. *Assume* (2.1)–(2.4). *Let* $(u,v) \in L^1(\Omega)\times L^1(\Omega)$ *be such that the functional* $F'_{\alpha,\beta}(u,v)$ *is finite. Then* $v=1$ $\mathcal{L}^n$*-a.e. in* Ω *and*

$$\Phi_{\alpha,\beta}(u) \le F'_{\alpha,\beta}(u,1). \tag{2.13}$$

Theorem 2.3. *Assume* (2.1)–(2.4) *and assume that* Ω *has Lipschitz boundary. For every* $u \in L^1(\Omega)$ *the following estimate holds*

$$F''_{\alpha,\beta}(u,1) \le \Phi_{\alpha,\beta}(u). \tag{2.14}$$

Theorem 2.2 is an immediate consequence of the following proposition.

Proposition 2.4. *Assume* (2.1)–(2.4). *Let* (u_k,v_k) *be a sequence in the space* $L^1(\Omega)\times L^1(\Omega)$ *such that*

$$(u_k,v_k) \to (u,v) \text{ in } L^1(\Omega)\times L^1(\Omega), \tag{2.15}$$

$$(F_k(u_k,v_k)) \text{ is bounded.} \tag{2.16}$$

Then $v=1$ $\mathcal{L}^n$*-a.e. in* Ω *and*

$$\Phi_{\alpha,\beta}(u) \le \liminf_{k\to+\infty} \int_\Omega \Big(v_k|\nabla u_k|^2 + \frac{\psi(v_k)}{\delta_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx. \tag{2.17}$$

Moreover, when $0 \le \alpha < +\infty$ *and* $0 < \beta < +\infty$ *the following estimates hold*

$$\int_\Omega |\nabla u|^2 dx \le \liminf_{k\to+\infty} \int_\Omega v_k|\nabla u_k|^2 dx, \tag{2.18}$$

$$a_\beta \mathcal{H}^{n-1}(J_u) \le \liminf_{k\to+\infty} \int_\Omega \Big(\frac{\psi(v_k)}{\delta_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx; \tag{2.19}$$

estimate (2.18) also holds if $\alpha = +\infty$ *or* $\beta = 0$.

We shall prove the one-dimensional case of Proposition 2.4 and Theorem 2.3 in Section 2.3, whereas the n-dimensional case will be studied in Section 2.4.

2.2.2 The case $p = +\infty$

In [26] the limiting case $p=+\infty$ when $\beta = 1$ is faced. In order to give a complete frame we state now the Γ-convergence results when $p=+\infty$ for different values of α and β.

We define $F_k\colon L^1(\Omega)\times L^1(\Omega)\to[0,+\infty]$ by

$$F_k(u,v):=\begin{cases}\displaystyle\int_\Omega\Big(v|\nabla u|^2+\frac{\psi(v)}{\delta_k}\Big)dx & \text{if } (u,v)\in H^1(\Omega)\times V_k,\\ +\infty & \text{otherwise,}\end{cases}$$

where,

$$\psi\in C([0,1]) \text{ is strictly decreasing with } \psi(1)=0,$$
$$V_k:=\Big\{v\in W^{1,\infty}(\Omega):\eta_k\le v\le 1,\ |\nabla v|\le\frac{1}{\varepsilon_k}\ \mathcal{L}^n\text{-a.e. in }\Omega\Big\}.$$

Let α,β be defined as in (2.1) and $\Phi_{\alpha,\beta}$ be defined as in (2.5)–(2.10) with the only modification that a_β and b_α are now set equal to

$$a_\beta:=\frac{2}{\beta}\int_0^1\psi\,ds,\qquad b_\alpha:=2(\alpha\psi(0))^{\frac{1}{2}}. \tag{2.20}$$

Under these hypotheses Theorems 2.1-2.4 holds. Proofs are similar to the ones given below for $p<+\infty$ and will be in part omitted. We will provide in details the estimate from below in dimension one when $0<\alpha<+\infty$ and $0<\beta<+\infty$ contained in [26], representing an alternative approach to that proposed in [39]. We also give the proof of the estimate from above, which turns out to be slightly simpler in the case $p=+\infty$.

2.3 Proof in the one-dimensional case

2.3.1 The case $p<+\infty$

Let us fix $1<p<+\infty$ and start proving the liminf inequality in the case $n=1$.

Proof of Proposition 2.4. Let (u_k,v_k) be a sequence satisfying (2.15) and (2.16) with bounding constant C. First we note that (2.16) and (2.3) imply $v=1$ $\mathcal{L}^1$-a.e. in Ω. This in particular concludes the proof in the case with $\alpha=0$ and $\beta=+\infty$.

Let now $\alpha=+\infty$. Up to subsequences we can suppose that the lower limit in the right-hand side of (2.18) is a limit and that $\eta_k>0$. We are going to prove that the sequence $(|\nabla u_k|)$ is equi-integrable. Let $A\subset\Omega$ be a measurable set, then the Hölder inequality and (2.16) imply

$$\begin{aligned}\int_A|\nabla u_k|dx &\le \Big(\int_\Omega v_k|\nabla u_k|^2dx\Big)^{\frac{1}{2}}\Big(\int_A 1/v_k dx\Big)^{\frac{1}{2}}\\ &\le C^{\frac{1}{2}}\Big(\int_{A\cap\{v_k\ge1/2\}}1/v_kdx+\int_{A\cap\{v_k<1/2\}}1/v_kdx\Big)^{\frac{1}{2}}\end{aligned}$$

$$\leq C^{\frac{1}{2}}\Big(2\mathcal{L}^1(A) + \frac{1}{\psi(1/2)}\frac{\delta_k}{\eta_k}\int_\Omega \frac{\psi(v_k)}{\delta_k}dx\Big)^{\frac{1}{2}}$$
$$\leq C^{\frac{1}{2}}\Big(2\mathcal{L}^1(A) + \frac{C}{\psi(1/2)}\frac{\delta_k}{\eta_k}\Big)^{\frac{1}{2}}. \tag{2.21}$$

Given $\sigma > 0$, the inequality $\frac{C}{\psi(1/2)}\frac{\delta_k}{\eta_k} \leq \frac{\sigma^2}{2C}$ is true for k large since $\alpha = +\infty$. Therefore $\mathcal{L}^1(A) < \frac{\sigma^2}{4C}$ implies the last term in (2.21) is less than σ for k large. Using for the first terms of the sequence the absolute continuity of the Lebesgue integral, we conclude that $(|\nabla u_k|)$ is equi-integrable. Now the Dunford-Pettis Theorem implies $u \in W^{1,1}(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^1(\Omega)$. By a classical lower semicontinuity result (see, for instance, [19, Theorem 2.3.1]) finally we obtain (2.18) and then $u \in H^1(\Omega)$.

Let $0 \leq \alpha < +\infty$ and $0 \leq \beta < +\infty$. In what follows we shall use the notation $I(x,\mu)$ for the interval $(x-\mu, x+\mu)$, whereas we shall write $F_k(u,v,I)$ to indicate the functional in (2.2) when the integrals are defined on the set I.

Proof of (2.18). Let $x_0 \in \Omega$ and $\mu > 0$ be such that u is absolutely continuous in $I(x_0,\mu) \subset \Omega$. Now the same argument used by Ambrosio and Tortorelli in [9, Lemma 4.2] and [10, Lemma 2.1] works here with obvious adaptations. We conclude that $u \in H^1(I(x_0,\mu))$ and (2.18) holds in $I(x_0,\mu)$.

Proof of (2.19). Let now x_0 be a point such that u is not absolutely continuous in any interval of the form $I(x_0,\mu)$. We sketch the argument proposed by Ambrosio and Tortorelli in [10, Lemma 2.1] in order to prove that there are only finitely many points of such a type.

Let $\mu > 0$ small enough; since u is not absolutely continuous in $I(x_0,\mu/2)$, the infimum $\inf\limits_{I(x_0,\mu/2)} v_k$ tends to 0 and this guarantees the existence for every k of a point $x_0-\mu/2 < x_k < x_0+\mu/2$ such that $v_k(x_k) \to 0$. Moreover, up to subsequences, $v_k \to 1$ $\mathcal{L}^1$-a.e. in Ω, so that we can find two points $x_0-\mu < y_1 < x_k < y_2 < x_0+\mu$ with $v_k(y_1) \to 1$ and $v_k(y_2) \to 1$. The Young inequality now gives

$$F_k(u_k,v_k,I(x_0,\mu)) \geq$$
$$\geq (\gamma p)^{\frac{1}{p}}\Big(\frac{q\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_{y_1}^{x_k}\psi(v_k)^{\frac{1}{q}}|\nabla v_k|dx + (\gamma p)^{\frac{1}{p}}\Big(\frac{q\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_{x_k}^{y_2}\psi(v_k)^{\frac{1}{q}}|\nabla v_k|dx$$
$$\geq (\gamma p)^{\frac{1}{p}}\Big(\frac{q\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_{v_k(x_k)}^{v_k(y_1)}\psi(s)^{\frac{1}{q}}ds + (\gamma p)^{\frac{1}{p}}\Big(\frac{q\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_{v_k(x_k)}^{v_k(y_2)}\psi(s)^{\frac{1}{q}}ds. \tag{2.22}$$

Passing to the lower limit in the previous inequality we obtain

$$\liminf_{k\to+\infty} F_k(u_k,v_k,I(x_0,\mu)) \geq a_\beta > 0 \tag{2.23}$$

in the case $0 < \beta < +\infty$. Since the left-hand side in (2.23) is bounded by (2.16), the number of disjoint intervals such as $I(x_0, \mu)$ is bounded by a constant independent by μ. This implies $u \in SBV(\Omega)$ and (2.19) follows. From (2.18) we also deduce $u \in SBV^2(\Omega)$.

In the case $\beta = 0$ we achieve a contradiction since the left-hand side of (2.22) is bounded by (2.16), whereas the right-hand side tends to infinity. Therefore, each point of Ω satisfies the previous step, so that $u \in H^1(\Omega)$ and (2.18) holds.

Proof of (2.17) *in the case* $0 < \beta < +\infty$. First we note that (2.18) and (2.19) lead to (2.17) in the case $\alpha = 0$, $0 < \beta < +\infty$.

It remains to consider the case $0 < \alpha < +\infty$, $0 < \beta < +\infty$. We shall define suitably six points in place of y_1, x_k, y_2; in this way we determine some intervals we shall study separately. In the external intervals, we shall be able to repeat the previous argument by Ambrosio and Tortorelli, the two in-between intervals will be neglected, and the central one will give rise to the cohesive term.

Let $x_0 \in J_u$ and assume $u^-(x_0) < u^+(x_0)$. Let $0 < \sigma < |[u(x_0)]|/2$ and let $\mu > 0$ be such that $|u(x) - u^{\pm}(x_0)| < \sigma/2$ for $0 < |x - x_0| \le \mu/2$; since $u_k \to u$ $\mathcal{L}^1$-a.e. in Ω up to subsequences, it is not restrictive to assume $u_k(x_0 \pm \mu/2) \to u(x_0 \pm \mu/2)$. We prove that there exist six points $y_1 < x_k^1 \le \tilde{x}_k^1 < \tilde{x}_k^2 \le x_k^2 < y_2$ in the interval $I(x_0, \mu)$, such that

$$\lim_{k\to+\infty} v_k(y_1) = \lim_{k\to+\infty} v_k(y_2) = 1,$$
$$\lim_{k\to+\infty} v_k(x_k^1) = \lim_{k\to+\infty} v_k(x_k^2) = 0,$$
$$u_k(\tilde{x}_k^1) = u^-(x_0) + \sigma, \quad u_k(\tilde{x}_k^2) = u^+(x_0) - \sigma.$$

Let us define

$$\tilde{x}_k^1 := \max\{x \in [x_0 - \mu/2, x_0 + \mu/2] : u_k(x) \le u^-(x_0) + \sigma\}.$$

Since $|u_k(x_0 \pm \mu/2) - u^{\pm}(x_0)| < \sigma$ for k large, the continuity of u_k implies that $\tilde{x}_k^1$ is well-defined, that $\tilde{x}_k^1 < x_0 + \mu/2$, and that $u_k(\tilde{x}_k^1) = u^-(x_0) + \sigma$.

We now verify that $x_0 \le \liminf_{k\to+\infty} \tilde{x}_k^1$. If not up to subsequences we have $\tilde{x}_k^1 < c_0 < x_0$, where c_0 is a constant. Using the definition of $\tilde{x}_k^1$ we obtain, as $k \to +\infty$, that $u(x) \ge u^-(x_0) + \sigma$ in (c_0, x_0). As $x \to x_0^-$ we get a contradiction.

We claim now that

$$\limsup_{k\to+\infty} \inf_{[x_0-\mu/2, \tilde{x}_k^1]} v_k \le 0. \tag{2.24}$$

By contradiction we assume that the opposite inequality holds. By this and

(2.16) we have, up to subsequences, that

$$\int_{x_0-\mu/2}^{\tilde{x}_k^1} |\nabla u_k|^2 dx \quad \text{is bounded.} \tag{2.25}$$

Let us verify now that $\limsup_{k\to+\infty} \tilde{x}_k^1 \leq x_0$. We argue again by contradiction and suppose $\tilde{x}_k^1 > c_1 > x_0$, where c_1 is a constant. Up to subsequences the integral $\int_{x_0-\mu/2}^{c_1} |\nabla u_k|^2 dx$ is bounded by (2.25), so that u is continuous in x_0 and this contradicts the assumption $x_0 \in J_u$. Therefore we conclude $\tilde{x}_k^1 \to x_0$.

Now, by the absolute continuity of u_k and the Hölder inequality we obtain for every $y \in (x_0 - \mu/2, \tilde{x}_k^1)$

$$|u_k(\tilde{x}_k^1) - u_k(y)| \leq |\tilde{x}_k^1 - y|^{\frac{1}{2}} \Big(\int_y^{\tilde{x}_k^1} |\nabla u_k|^2 dx \Big)^{\frac{1}{2}} \leq c_2 |\tilde{x}_k^1 - y|^{\frac{1}{2}}, \tag{2.26}$$

where in the last inequality $c_2 < +\infty$ is a constant and we have used (2.25). Let us fix $y \in (x_0 - \mu, x_0)$ such that $u_k(y) \to u(y)$; then $y \in (x_0 - \mu, \tilde{x}_k^1)$ for k large, so that inequality

$$|u^-(x_0) + \sigma - u_k(y)| \leq c_2 |\tilde{x}_k^1 - y|^{\frac{1}{2}}$$

follows from $u_k(\tilde{x}_k^1) = u^-(x_0) + \sigma$ and (2.26). Passing to the limit first as $k \to +\infty$ and then as $y \to x_0^-$ we achieve a contradiction and the claim (2.24) is proved.

By (2.24) we are able to find a sequence $x_0 - \mu/2 \leq x_k^1 \leq \tilde{x}_k^1$ such that $v_k(x_k^1) \to 0$. Since $v_k \to 1$ $\mathcal{L}^1$-a.e. in Ω, we also find a point $y_1 \in (x_0 - \mu, x_0 - \mu/2)$ such that $v_k(y_1) \to 1$.

Let us define now

$$\tilde{x}_k^2 := \min\{x \in [\tilde{x}_k^1, x_0 + \mu/2] : u_k(x) \geq u^+(x_0) - \sigma\}.$$

We can easily prove that it is well-defined, that $u_k(\tilde{x}_k^2) = u^+(x_0) - \sigma$, and that $\tilde{x}_k^2 \to x_0$. Note that the convergence $\tilde{x}_k^2 \to x_0$ implies the convergence $\tilde{x}_k^1 \to x_0$.

As before we can also prove that

$$\limsup_{k\to+\infty} \inf_{[\tilde{x}_k^2, x_0+\mu/2]} v_k = 0$$

and the existence of x_k^2 and y_2 follows.

Now let us proceed with the computation. In the intervals (y_1, x_k^1) and (x_k^2, y_2) we can repeat the argument by Ambrosio and Tortorelli in (2.23), so that

$$\liminf_{k\to+\infty} F_k(u_k, v_k, (y_1, x_k^1) \cup (x_k^2, y_2)) \geq a_\beta. \tag{2.27}$$

It remains to estimate the functional in the interval $I_k := (\tilde{x}_k^1, \tilde{x}_k^2)$. Let us define

$$W_k := \{w \in H^1(I_k),\ w(\tilde{x}_k^1) = u^-(x_0) + \sigma,\ w(\tilde{x}_k^2) = u^+(x_0) - \sigma\},$$

$$Z_k := \{z \in W^{1,p}(I_k),\ \eta_k \le z \le 1\ \mathcal{L}^1\text{-a.e. on } I_k\},$$

$$H_k(w,z) := \int_{I_k} \Big(z|\nabla w|^2 + \frac{\psi(z)}{\delta_k}\Big)dx, \quad \text{for } (w,z) \in W_k \times Z_k,$$

$$h_k(z) := \min_{w \in W_k} H_k(w,z), \quad \text{for } z \in Z_k.$$

By elementary computation we find that this minimum is achieved and that

$$h_k(z) = \frac{([u](x_0) - 2\sigma)^2}{\displaystyle\int_{I_k} \frac{1}{z} dx} + \int_{I_k} \frac{\psi(z)}{\delta_k} dx. \tag{2.28}$$

Let now $0 < \lambda < 1$. We observe that

$$\int_{\{x \in I_k : v_k \ge \lambda\}} \frac{1}{v_k} dx \le \frac{\mathcal{L}^1(I_k)}{\lambda},$$

$$\int_{\{x \in I_k : v_k < \lambda\}} \frac{1}{v_k} dx \le \frac{1}{\psi(\lambda)} \frac{\delta_k}{\eta_k} \Big(\int_{I_k} \frac{\psi(v_k)}{\delta_k} dx \Big).$$

We use the previous inequalities to estimate the functional $F_k(u_k, v_k, I_k)$:

$$\begin{aligned} F_k(u_k, v_k, I_k) &\ge H_k(u_k, v_k) \\ &\ge h_k(v_k) \\ &\ge \frac{([u(x_0)] - 2\sigma)^2}{\dfrac{\mathcal{L}^1(I_k)}{\lambda} + \dfrac{1}{\psi(\lambda)} \dfrac{\delta_k}{\eta_k} \Big(\displaystyle\int_{I_k} \frac{\psi(v_k)}{\delta_k} dx \Big)} + \int_{I_k} \frac{\psi(v_k)}{\delta_k} dx \\ &\ge 2\Big(\frac{\eta_k \psi(\lambda)}{\delta_k}\Big)^{\frac{1}{2}} ([u(x_0)] - 2\sigma) - \frac{\eta_k \psi(\lambda) \mathcal{L}^1(I_k)}{\lambda \delta_k}, \end{aligned}$$

where to get the last inequality we have minimized in $[0, \infty[$ the function

$$t \mapsto \frac{([u(x_0)] - 2\sigma)^2}{\dfrac{\mathcal{L}^1(I_k)}{\lambda} + \dfrac{1}{\psi(\lambda)} \dfrac{\delta_k}{\eta_k} t} + t.$$

Passing to the limit first as $k \to +\infty$, then as $\lambda \to 0$, and finally as $\sigma \to 0$ we obtain

$$\liminf_{k \to +\infty} F_k(u_k, v_k, I_k) \ge b_\alpha |[u(x_0)]|. \tag{2.29}$$

Inequalities (2.18) for the set $I(x_0, \mu)$, (2.27), and (2.29) lead to (2.17).

It remains to study the case $0 < \alpha < +\infty$, $\beta = +\infty$. By [15, Theorem 2.1] the functional $\Phi_{\alpha,\infty}$ is weakly* lower semicontinuous in $BV(\Omega)$ and strongly lower semicontinuous in $L^1(\Omega)$, so that it is sufficient to prove that

$$\liminf_{k\to+\infty} \Phi_{\alpha,\infty}(u_k) \le \liminf_{k\to+\infty} F_k(u_k, v_k). \tag{2.30}$$

In order to simplify the notation we set $A_k := \{|\nabla u_k| < b_\alpha/2\}$; we compute the integrals of $f_\alpha(|\nabla u_k|)$ on A_k and on A_k^c.

Let us fix $0 < \lambda < \mu < 1$. First we note that the convergence in measure $v_k \to 1$ implies

$$\int_{A_k\cap\{v_k<\mu\}} f_\alpha(|\nabla u_k|)dx + \int_{A_k\cap\{v_k\ge\mu\}} f_\alpha(|\nabla u_k|)dx \le \int_{A_k\cap\{v_k\ge\mu\}} f_\alpha(|\nabla u_k|)dx + o(1). \tag{2.31}$$

On A_k^c we have

$$\begin{aligned}\int_{A_k^c\cap\{v_k\ge\mu\}} f_\alpha(|\nabla u_k|)dx &+ \int_{A_k^c\cap\{\lambda<v_k<\mu\}} f_\alpha(|\nabla u_k|)dx + \int_{A_k^c\cap\{v_k\le\lambda\}} f_\alpha(|\nabla u_k|)dx \le \\ &\le \int_{A_k^c\cap\{v_k\ge\mu\}} |\nabla u_k|^2 dx + \int_{A_k^c\cap\{\lambda<v_k<\mu\}} b_\alpha\Big(|\nabla u_k| - \frac{b_\alpha}{4}\Big)dx \\ &\quad + \int_{A_k^c\cap\{v_k\le\lambda\}} b_\alpha\Big(|\nabla u_k| - \frac{b_\alpha}{4}\Big)dx,\end{aligned} \tag{2.32}$$

where we have used the definition of f_α and the fact that $b_\alpha(t - b_\alpha/4) \le t^2$ for $t \ge b_\alpha/2$. The last term in (2.32) by the Hölder inequality is less than or equal to

$$\begin{aligned}&\int_{A_k^c\cap\{v_k\ge\mu\}} |\nabla u_k|^2 dx + b_\alpha\Big(\int_\Omega v_k|\nabla u_k|^2dx\Big)^{\frac12}\Big(\int_{\{\lambda<v_k<\mu\}} \frac{1}{v_k}dx\Big)^{\frac12} \\ &+b_\alpha\Big(\int_{\{v_k\le\lambda\}} v_k|\nabla u_k|^2dx\Big)^{\frac12}\Big(\int_{\{v_k\le\lambda\}} \frac{1}{v_k}dx\Big)^{\frac12} \\ \le\ &\int_{A_k^c\cap\{v_k\ge\mu\}} |\nabla u_k|^2 dx + b_\alpha\Big(\frac{C}{\lambda}\Big)^{\frac12}\mathcal{L}^1(\{v_k<\mu\})^{\frac12} \\ &+b_\alpha\Big(\frac{1}{\psi(\lambda)}\frac{\delta_k}{\eta_k}\Big)^{\frac12}\Big(\int_{\{v_k\le\lambda\}} v_k|\nabla u_k|^2dx\Big)^{\frac12}\Big(\int_{\{v_k\le\lambda\}} \frac{\psi(v_k)}{\delta_k}dx\Big)^{\frac12},\end{aligned} \tag{2.33}$$

where the last inequality follows from property (2.16) and an easy computation. Finally from the Cauchy inequality and the convergence in measure $v_k \to 1$ we find that the last term in (2.33) is less than or equal to

$$\int_{A_k^c\cap\{v_k\ge\mu\}} |\nabla u_k|^2 dx + \frac{b_\alpha}{2}\Big(\frac{1}{\psi(\lambda)}\frac{\delta_k}{\eta_k}\Big)^{\frac12}\int_{\{v_k\le\lambda\}}\Big(v_k|\nabla u_k|^2 + \frac{\psi(v_k)}{\delta_k}\Big)dx + o(1). \tag{2.34}$$

From (2.31) and (2.34) we deduce

$$
\begin{aligned}
&\Phi_{\alpha,\infty}(u_k) \leq \\
\leq\ & \int_{\{v_k\geq\mu\}} |\nabla u_k|^2 dx + \frac{b_\alpha}{2}\Big(\frac{1}{\psi(\lambda)}\frac{\delta_k}{\eta_k}\Big)^{\frac{1}{2}} \int_{\{v_k\leq\lambda\}} \Big(v_k|\nabla u_k|^2 + \frac{\psi(v_k)}{\delta_k}\Big) dx + o(1) \\
\leq\ & \frac{1}{\mu}\int_{\{v_k\geq\mu\}} \Big(v_k|\nabla u_k|^2 + \frac{\psi(v_k)}{\delta_k}\Big) dx \\
& + \frac{b_\alpha}{2}\Big(\frac{1}{\psi(\lambda)}\frac{\delta_k}{\eta_k}\Big)^{\frac{1}{2}} \int_{\{v_k\leq\lambda\}} \Big(v_k|\nabla u_k|^2 + \frac{\psi(v_k)}{\delta_k}\Big) dx + o(1) \\
\leq\ & \max\Big(\frac{1}{\mu}, \frac{b_\alpha}{2}\Big(\frac{1}{\psi(\lambda)}\frac{\delta_k}{\eta_k}\Big)^{\frac{1}{2}}\Big) F_k(u_k, v_k) + o(1).
\end{aligned}
$$

Passing to the limit first as $k \to +\infty$ and then as $\lambda \to 0$, $\mu \to 1$ we obtain (2.30). □

Let us complete the one-dimensional case of the Γ-convergence result by proving the upper estimate.

Proof of Theorem 2.3. The cases $\alpha = +\infty$ or $\beta = 0$ are trivial since $F'_{\alpha,\beta}(u,v) < +\infty$ implies $u \in H^1(\Omega)$ and $v = 1$ $\mathcal{L}^1$-a.e. in Ω.

Let now $0 \leq \alpha < +\infty$ and $0 < \beta < +\infty$ and let u be such that $\Phi_{\alpha,\beta}(u,1) < +\infty$. A truncation argument shows that in dimension $n = 1$ a function u such that $\Phi_{0,\beta}(u,1) < +\infty$ actually belongs to $SBV^2(\Omega)$. Therefore, both for $\alpha = 0$ and for $0 < \alpha < +\infty$, we start with a function $u \in SBV^2(\Omega)$; for simplicity we also suppose $J_u = \{\overline{x}\}$. Let (σ_k^α) and (μ_k) be positive infinitesimal sequences which we shall specify later and let

$$
A_k := (\overline{x} - \sigma_k^\alpha, \overline{x} + \sigma_k^\alpha) \quad \text{and} \quad B_k := (\overline{x} - \sigma_k^\alpha - \mu_k, \overline{x} - \sigma_k^\alpha) \cup (\overline{x} + \sigma_k^\alpha, \overline{x} + \sigma_k^\alpha + \mu_k).
$$

Let us define u_k by u out of A_k and linking linearly in A_k.

Let $f(\rho) := \psi(1-\rho)$, $g(\rho) := \dfrac{1}{\int_0^{1-\rho} \psi^{-\frac{1}{p}} ds}$, and $h := (fg)^{\frac{1}{2}}$ for $0 < \rho < 1$; we note that h is strictly increasing and that h and f/g are infinitesimal in 0. Then the sequence $\rho_k := h^{-1}(\delta_k)$ is infinitesimal and

$$
\frac{f(\rho_k)}{\delta_k} \to 0, \qquad \frac{\delta_k}{g(\rho_k)} \to 0. \tag{2.35}
$$

We now set v_k equal to η_k in A_k and equal to $1 - \rho_k$ out of $A_k \cup B_k$.

In order to define v_k everywhere, we first consider the following Cauchy problem

$$
\begin{cases}
w_k' = \Big(\dfrac{q}{\gamma p \delta_k}\Big)^{\frac{1}{p}} \varepsilon_k^{-\frac{1}{q}} \psi(w_k)^{\frac{1}{p}} \\
w_k(0) = \eta_k.
\end{cases} \tag{2.36}
$$

Since $\eta_k < 1$ and ψ is continuous and strictly positive in $[0,1)$, the previous problem has only one solution w_k in the interval $[0, T_k)$, where $T_k \in (0,+\infty]$ is defined by

$$T_k := \Big(\frac{\gamma p \delta_k}{q}\Big)^{\frac{1}{p}} \varepsilon_k^{\frac{1}{q}} \int_{\eta_k}^{1} \psi^{-\frac{1}{p}} ds.$$

Precisely, the solution w_k is obtained by taking the inverse of the function

$$z \in [\eta_k, 1) \mapsto \Big(\frac{\gamma p \delta_k}{q}\Big)^{\frac{1}{p}} \varepsilon_k^{\frac{1}{q}} \int_{\eta_k}^{z} \psi^{-\frac{1}{p}} ds \in [0, T_k).$$

By this we can define

$$v_k(x) := w_k(|x - \overline{x}| - \sigma_k^\alpha) \text{ on } B_k, \tag{2.37}$$

$$\mu_k := \Big(\frac{\gamma p \delta_k}{q}\Big)^{\frac{1}{p}} \varepsilon_k^{\frac{1}{q}} \int_{\eta_k}^{1-\rho_k} \psi^{-\frac{1}{p}} ds, \tag{2.38}$$

where μ_k is infinitesimal by (2.35).

Then $(u_k, v_k) \in H^1(\Omega) \times V_{\eta_k}$ and $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. An easy computation shows that

$$\int_{\Omega \setminus A_k} v_k |\nabla u_k|^2 dx \leq \int_\Omega |\nabla u|^2 dx, \tag{2.39}$$

$$\int_{\Omega \setminus (A_k \cup B_k)} \frac{\psi(v_k)}{\delta_k} dx \leq \frac{\psi(1-\rho_k)}{\delta_k} \mathcal{L}^1(\Omega), \tag{2.40}$$

$$\int_{A_k} \Big(v_k |\nabla u_k|^2 + \frac{\psi(v_k)}{\delta_k}\Big) dx = \frac{\eta_k}{2\sigma_k^\alpha} (u(\overline{x} + \sigma_k^\alpha) - u(\overline{x} - \sigma_k^\alpha))^2 + 2\psi(\eta_k) \frac{\sigma_k^\alpha}{\delta_k}. \tag{2.41}$$

We note that the integral in (2.40) tends to 0 by (2.35). If $\alpha = 0$ we take σ_k^0 such that $\eta_k / \sigma_k^0 \to 0$ and $\sigma_k^0 / \delta_k \to 0$; by this choice the integral in (2.41) converges to 0. Whereas if $0 < \alpha < +\infty$ we define $\sigma_k^\alpha := \frac{1}{2}(\frac{\alpha}{\psi(0)})^{\frac{1}{2}} |[u(\overline{x})]| \delta_k$ and the integral in (2.41) tends to $b_\alpha |[u(\overline{x})]|$.

Let us compute now the integral on B_k. Thanks to the choice of w_k the Young inequality holds with equality, so that

$$\begin{aligned} \int_{B_k} \Big(\frac{\psi(v_k)}{\delta_k} + \gamma \varepsilon_k^{p-1} |\nabla v_k|^p\Big) dx &= 2(\gamma p)^{\frac{1}{p}} \Big(\frac{q \varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}} \int_0^{\mu_k} \psi(w_k)^{\frac{1}{q}} w_k' \, dx \\ &= 2(\gamma p)^{\frac{1}{p}} \Big(\frac{q \varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}} \int_{\eta_k}^{1-\rho_k} \psi(s)^{\frac{1}{q}} ds. \end{aligned} \tag{2.42}$$

As $k \to +\infty$ this term tends to a_β and the proof is complete.

Let us consider now the case $\alpha = 0$, $\beta = +\infty$. First we suppose that u is piecewise constant with $J_u = \{\overline{x}\}$. If this is the case we define all parameters as before, so that by repeating the computations in (2.39)–(2.42) we obtain that $F''_{0,\infty}(u,1)$ is null. In the general case when $u \in L^1(\Omega)$ we argue by approximation with piecewise constant functions; since $F''_{0,\infty}$ is lower semicontinuous we achieve the same conclusion as before.

The last case to study is $0 < \alpha < +\infty$, $\beta = +\infty$. By [15, Theorem 3.1] if we prove that for every $u \in SBV^2(\Omega)$ we have

$$F''_{\alpha,\infty}(u,1) \leq \int_\Omega |\nabla u|^2 dx + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^0 \tag{2.43}$$

we are done, since the left-hand side is lower semicontinuous in $L^1(\Omega)$ and the lower semicontinuous envelope of the right-hand side is $\Phi_{\alpha,\infty}$. Inequality (2.43) is easily proved by defining all parameters as before and repeating the computation in (2.39)–(2.42). □

2.3.2 The case $p = +\infty$

The proofs proposed in the previous subsection can be easily adapted to the case $p = +\infty$. In this subsection we provide the proofs in the case $0 < \beta < +\infty$, showing an alternative argument for the estimate of the cohesive term in the case $0 < \alpha < +\infty$. For simplicity we assume $\delta_k = \varepsilon_k$, which corresponds to $\beta = 1$, so that we can omit β from the notation.

We also assume that ψ satisfies a very mild technical condition, which is fulfilled in the standard examples $\psi(z) = 1 - z^r$, with $r > 0$: for every $c \geq 0$

$$\text{the equation} \quad s^2\psi'(s) = -c \quad \text{has a finite number of solutions.} \tag{2.44}$$

This condition will be used under the regime $0 < \alpha < +\infty$ in order to obtain a lower estimate involving $\sum_{x \in J_u} |[u](x)|$.

Proof of Proposition 2.4. It is sufficient to prove the statement when Ω is an interval, since the left-hand sides of (2.18), (2.19) and (2.17) are σ-additive with respect to Ω, whereas the right-hand sides are σ-superadditive. Therefore we can assume $\Omega =]0,1[$.

Let (u_k, v_k) be a sequence satisfying (2.15) and (2.16) with bounding constant c. Note that $\psi(v_k) \to 0$ in $L^1(\Omega)$ by (2.2) and (2.16); as $\psi(v_k) \to \psi(v)$ in $L^1(\Omega)$ we deduce $v = 1$ $\mathcal{L}^1$-a.e. on Ω.

Proof of (2.18). It is not restrictive to assume that the lower limit in the right-hand side of (2.18) is actually a limit. Let us divide the proof into two steps.

(a) Since v_k is a Lipschitz function, the set

$$B_k = \{x \in \overline{\Omega} : v_k(x) > 1/2\}$$

is relatively open in $\overline{\Omega}$. By Chebyshev inequality we get

$$\psi(1/2)\mathcal{L}^1(B_k^c) \le \int_0^1 \psi(v_k)dx,$$

so that (2.2) and (2.16) imply

$$\mathcal{L}^1(B_k^c) \to 0. \tag{2.45}$$

We write

$$B_k = \bigcup_{1\le j\le N_k} I_j^k \cup \bigcup_{j>N_k} J_j^k, \tag{2.46}$$

where $I_1^k, \dots, I_{N_k}^k$ are the connected components of B_k such that $\mathcal{L}^1(I_j^k) \ge \varepsilon_k/4$, whereas J_j^k are the connected components satisfying the opposite inequality. Let a_j^k and b_j^k be the end points of the interval I_j^k. By changing the numeration, we may assume that $0 \le a_1^k \le b_1^k < a_2^k < b_2^k < \dots < a_{N_k}^k \le b_{N_k}^k \le 1$. Moreover we set $b_0^k := 0$ and $a_{N_k+1}^k := 1$.

By definition $v_k \le 1/2$ on B_k^c; moreover $v_k \le 3/4$ on each J_j^k, since at least one end point belongs to B_k^c, the length of J_j^k is less than $\varepsilon_k/4$, and $|\nabla v_k| \le 1/\varepsilon_k$ $\mathcal{L}^1$-a.e. in Ω by (2.2), (2.4), and (2.16). Then $v_k \le 3/4$ in $[b_j^k, a_{j+1}^k]$ for $j = 0, \dots, N_k$. From this estimate and from (2.16) it follows that

$$\sum_{j>N_k} \mathcal{L}^1(J_j^k) \le \frac{\varepsilon_k c}{C_1}, \tag{2.47}$$

where $C_1 := \psi(3/4)$.

Let us show that (N_k) is bounded. To this aim we choose a point r_j in each interval $[b_{j-1}^k, a_j^k]$. We have $v_k \le 7/8$ in $]r_j - \frac{\varepsilon_k}{8}, r_j + \frac{\varepsilon_k}{8}[$, since $v_k(r_j) \le 3/4$ and $|\nabla v_k| \le 1/\varepsilon_k$ $\mathcal{L}^1$-a.e. in Ω. Then

$$\frac{1}{\varepsilon_k}\int_{r_j-\frac{\varepsilon_k}{8}}^{r_j+\frac{\varepsilon_k}{8}} \psi(v_k)dx \ge C_2,$$

where $C_2 := 1/4\psi(7/8)$. We note that the intervals $]r_j - \frac{\varepsilon_k}{8}, r_j + \frac{\varepsilon_k}{8}[$ are pairwise disjoint, since $\mathcal{L}^1(I_j^k) \ge \varepsilon_k/4$. By summing on the index j we find

$$C_2(N_k + 1) \le c.$$

This shows that (N_k) is a bounded sequence of integers. Up to subsequences, we can assume $N_k = N$ for a certain N; by compactness we can also assume the existence of the limits

$$\lim_{k\to+\infty} b_j^k =: b_j \quad \text{and} \quad \lim_{k\to+\infty} a_j^k =: a_j, \tag{2.48}$$

with $0 = b_0 \le a_1 \le b_1 \le \cdots \le a_N \le b_N \le a_{N+1} = 1$. Now, by (2.45) and (2.47) we have

$$\sum_{j=0}^{N}(a_{j+1}^k - b_j^k) = \mathcal{L}^1(B_k^c) + \sum_{j>N} \mathcal{L}^1(J_j^k) \to 0; \tag{2.49}$$

it follows that $b_j = a_{j+1}$, for $j = 0, \ldots, N$. Let $0 = x_0 < x_1 < \cdots < x_m = 1$ be an increasing enumeration of the set $\{b_0, \ldots, b_N\}$.

Let $\sigma > 0$ be such that $x_{i-1} + \sigma < x_i - \sigma$ for $i = 1, \ldots, m$. For large values of k we have a_j^k, $b_j^k \notin [x_{i-1} + \sigma, x_i - \sigma]$. Using (2.49) and (2.48), we can deduce that for every k and every i there exists j such that

$$[x_{i-1} + \sigma, x_i - \sigma] \subset]a_j^k, b_j^k[;$$

therefore $v_k > 1/2$ in $[x_{i-1} + \sigma, x_i - \sigma]$, for $i = 1, \ldots, m$. By (2.2) and (2.16) we find

$$\int_{x_{i-1}+\sigma}^{x_i-\sigma} |\nabla u_k|^2 dx \le 2c, \tag{2.50}$$

i.e., (∇u_k) is bounded in $L^2(x_{i-1} + \sigma, x_i - \sigma)$, for $i = 1, \ldots, m$.

(b) Using the Poincaré-Wirtinger inequality, we deduce from (2.15) and (2.50) that (u_k) is bounded in $H^1(]x_{i-1} + \sigma, x_i - \sigma[)$. This ensures that $u \in H^1(x_{i-1} + \sigma, x_i - \sigma)$ and that $u_k \rightharpoonup u$ weakly in $H^1(x_{i-1} + \sigma, x_i - \sigma)$.

By the Severini-Egorov Theorem for every $\mu > 0$ there exists a measurable set $A_\mu \subset [x_{i-1} + \sigma, x_i - \sigma]$, with $\mathcal{L}^1(A_\mu) < \mu$, such that, up to a subsequence, $v_k \to 1$ uniformly in $[x_{i-1} + \sigma, x_i - \sigma] \setminus A_\mu$. Then, fixed $\delta > 0$, we have $v_k > 1 - \delta$ in $[x_{i-1} + \sigma, x_i - \sigma] \setminus A_\mu$ for large k. By the weak lower semicontinuity of the L^2-norm, we have

$$(1-\delta) \int_{[x_{i-1}+\sigma, x_i-\sigma]\setminus A_\mu} |\nabla u|^2 dx \le \liminf_{k\to+\infty} \int_{x_{i-1}+\sigma}^{x_i-\sigma} v_k |\nabla u_k|^2 dx.$$

We pass to the limit first as $\delta \to 0$ and then as $\mu \to 0$; adding on the index i we find

$$\sum_{i=1}^{m} \int_{x_{i-1}+\sigma}^{x_i-\sigma} |\nabla u|^2 dx \le \liminf_{k\to+\infty} \sum_{i=1}^{m} \int_{x_{i-1}+\sigma}^{x_i-\sigma} v_k |\nabla u_k|^2 dx. \tag{2.51}$$

As $\sigma \to 0$, from (2.16) we obtain $u \in H^1(x_{i-1}, x_i)$ for $i = 1, \ldots, m$. Inequality (2.18) follows.

Proof of (2.19). If u is continuous in a certain x_i, then $u \in H^1(x_{i-1}, x_{i+1})$ and we can remove x_i from the list. Therefore it is not restrictive to assume that every x_i is a jump point for u, for $i = 1, \dots, m-1$, so that $\mathcal{H}^0(J_u) = m-1$. Fix $\sigma > 0$ such that $2\sigma < x_i - x_{i-1}$ for every i and let

$$\delta_k^i = \min\{v_k(x) : x \in [x_i - \tfrac{\sigma}{2}, x_i + \tfrac{\sigma}{2}]\}.$$

Let us prove that $\delta_k^i \to 0$ as $k \to +\infty$; by contradiction, we suppose that there exists a subsequence of (δ_k^i), not relabeled, and a constant $K > 0$ such that $\delta_k^i > K$ for every k, i.e., $v_k > K > 0$ in $[x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]$. By repeating the argument used in steps (a) and (b) we find that $u \in H^1(x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2})$ and this contradicts the assumption that x_i is a jump point.

Now let t_k^i be a minimum point for v_k in $[x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]$. For large value of k we have $[t_k^i - \varepsilon_k(1-\delta_k^i), t_k^i + \varepsilon_k(1-\delta_k^i)] \subset \,]x_i - \sigma, x_i + \sigma[$. Since $v_k(t_k^i) = \delta_k^i$ and $|\nabla v_k| \le 1/\varepsilon_k$ $\mathcal{L}^1$-a.e. in Ω, it follows that $v_k \le \frac{1}{\varepsilon_k}|x - t_k^i| + \delta_k^i$. Since ψ is decreasing we deduce

$$2\int_{\delta_k^i}^1 \psi(s)ds = \frac{1}{\varepsilon_k}\int_{t_k^i-\varepsilon_k(1-\delta_k^i)}^{t_k^i+\varepsilon_k(1-\delta_k^i)} \psi\Big(\frac{|x-t_k^i|}{\varepsilon_k} + \delta_k^i\Big)dx \le \frac{1}{\varepsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(v_k)dx;$$

adding with respect to i and passing to the lower limit we obtain (2.19).

Proof of (2.17). In the case $\alpha = 0$ inequality (2.17) is obtained by adding (2.18) and (2.19).

Let $\alpha > 0$. Up to subsequences, we have $u_k \to u$ $\mathcal{L}^1$-a.e. on Ω; we write $J_u = \{x_1 \dots x_{m-1}\}$, where $0 = x_0 < x_1 < \dots < x_{m-1} < x_m = 1$, and we choose $\sigma > 0$, with $2\sigma < x_i - x_{i-1}$, such that

$$u_k(x_i - \sigma) \to u(x_i - \sigma) \quad \text{and} \quad u_k(x_{i-1} + \sigma) \to u(x_{i-1} + \sigma) \quad \text{for } i = 1, \dots, m. \tag{2.52}$$

We want to estimate from below the integrals

$$I_k^i := \int_{x_i-\sigma}^{x_i+\sigma} v_k(\nabla u_k)^2 dx + \frac{1}{\varepsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(v_k)dx. \tag{2.53}$$

To this aim fix $1 \le i \le m-1$ and for k large we define

$$W_k := \{w \in H^1(x_i - \sigma, x_i + \sigma),\ w(x_i - \sigma) = u_k(x_i - \sigma),\ w(x_i + \sigma) = u_k(x_i + \sigma)\},$$

$$Z_k := \{z \in W^{1,\infty}(x_i - \sigma, x_i + \sigma),\ \eta_k \le z \le 1,\ |\nabla z| \le 1/\varepsilon_k\ \mathcal{L}^1\text{-a.e. on }]x_i - \sigma, x_i + \sigma[\},$$

$$H_k(w, z) := \int_{x_i-\sigma}^{x_i+\sigma} z|\nabla w|^2 dx + \frac{1}{\varepsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(z)dx, \quad \text{for } (w, z) \in W_k \times Z_k,$$

$$h_k(z) := \min_{w \in W_k} H_k(w, z).$$

By elementary computation we find that this minimum is achieved and that

$$h_k(z) = \frac{{\beta_k}^2}{\displaystyle\int_{x_i-\sigma}^{x_i+\sigma} \frac{1}{z} dx} + \frac{1}{\varepsilon_k} \int_{x_i-\sigma}^{x_i+\sigma} \psi(z) dx, \tag{2.54}$$

where

$$\beta_k := |u_k(x_i + \sigma) - u_k(x_i - \sigma)|. \tag{2.55}$$

Let z_k be a minimum point for h_k in Z_k. It follows from the definition of h_k and from (2.53) that

$$h_k(z_k) \leq I_k^i. \tag{2.56}$$

We note that h_k is invariant with respect to symmetric rearrangements of z, therefore we can assume that z_k is symmetric with respect to x_i and nondecreasing on $[x_i, x_i + \sigma[$. Now we want to prove that z_k is piecewise affine.

First of all, by monotonicity and continuity, the sets

$$A_k := \{z_k = \eta_k\} \cap [x_i, x_i + \sigma[\quad \text{and} \quad B_k := \{z_k = 1\} \cap [x_i, x_i + \sigma[$$

are closed intervals of $[x_i, x_i + \sigma[$. Let us define

$$C_k := \{\eta_k < z_k < 1, \ |\nabla z_k| < 1/\varepsilon_k\} \cap [x_i, x_i + \sigma[,$$
$$U_{j,k} := \{\eta_k + \tfrac{1}{j} < z_k < 1 - \tfrac{1}{j}\} \cap [x_i, x_i + \sigma[, \quad E_{j,k} = \{|\nabla z_k| < \tfrac{1}{\varepsilon_k} - \tfrac{1}{j}\} \cap U_{j,k},$$

so that C_k is the union of the sets $E_{j,k}$ for $j \in \mathbb{N}$. For every j, $U_{j,k}$ is open in $[x_i, x_i + \sigma[$ and $E_{j,k}$ is measurable. Suppose $\mathcal{L}^1(C_k) > 0$ and fix j such that $\mathcal{L}^1(E_{j,k}) > 0$; let φ be a Lipschitz function such that

$$\{\varphi \neq 0\} \subset U_{j,k} \quad \text{and} \quad |\nabla \varphi| \leq 1_{E_{j,k}} \ \mathcal{L}^1\text{-a.e. on } \mathbb{R}; \tag{2.57}$$

then $z_k + t\varphi \in Z_k$ for t small enough. So 0 is a a minimizer for the function $t \mapsto h_k(z_k + t\varphi)$ and, imposing that 0 is a critical point, we find

$$\int_{U_{j,k}} \left[\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} \right] \varphi dx = 0, \tag{2.58}$$

where $\lambda_k := \beta_k^2 \big(2 \int_{x_i}^{x_i+\sigma} \frac{1}{z_k} dx\big)^{-2}$. Let us prove that

$$\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} = 0 \quad \mathcal{L}^1\text{-a.e. on } E_{j,k}, \tag{2.59}$$

arguing by contradiction. Let

$$E^+_{j,k} := E_{j,k} \cap \left\{ \frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} > 0 \right\}$$

and suppose $\mathcal{L}^1(E^+_{j,k}) > 0$. By the continuity of z_k and ψ' and by the Lebesgue Differentiation Theorem there exist $x_0 \in E^+_{j,k}$ and $\delta > 0$ such that

$$[x_0 - \delta, x_0 + \delta] \subset U_{j,k} \cap \left\{ \frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} > 0 \right\} \quad \text{and} \quad \mathcal{L}^1\big(E_{j,k} \cap [x_0 - \delta, x_0 + \delta]\big) > 0.$$

Now let y be such that

$$\mathcal{L}^1\big(E_{j,k} \cap [x_0 - \delta, y]\big) = \mathcal{L}^1\big(E_{j,k} \cap [y, x_0 + \delta]\big),$$

and let

$$\theta(x) := \mathcal{L}^1\big(E_{j,k} \cap [x_0 - \delta, y] \cap [x_0 - \delta, x]\big) - \mathcal{L}^1\big(E_{j,k} \cap [x_0 - \delta, x] \cap [y, x_0 + \delta]\big),$$

for $x \in [x_i, x_i + \sigma[$. In particular θ is a Lipschitz function satisfying (2.57), so that (2.58) implies

$$\int_{x_0-\delta}^{x_0+\delta} \left[\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} \right] \theta dx = 0; \tag{2.60}$$

since $\theta \geq 0$, $\theta(y) > 0$, and $\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\varepsilon_k} > 0$ in $[x_0 - \delta, x_0 + \delta]$ the integral in (2.60) is positive and we get a contradiction. This concludes the proof of (2.59).

From (2.59) it follows that z_k maps C_k into the set of solutions of the equation $s^2\psi'(s) = -\lambda_k\varepsilon_k$, where $\lambda_k\varepsilon_k$ is infinitesimal since (λ_k) is bounded. Then, assumption (2.44) implies that z_k takes only a finite number of different values on C_k and, by monotonicity and continuity, C_k is a finite union of intervals. It follows that $[x_i, x_i + \sigma[$ can be written as union of a finite number of intervals, where either z_k is constant or $\nabla z_k = 1/\varepsilon_k$.

We now estimate from below $h_k(z_k)$. In order to simplify the computation, we suppose that z_k assumes a unique value ξ_k in C_k, $\eta_k < \xi_k < 1$, so that C_k is an interval. Let $\alpha_k := \mathcal{L}^1(A_k)$ and $\gamma_k := \mathcal{L}^1(C_k)$; since $\nabla z_k = 1/\varepsilon_k$ in $[x_i, x_i + \sigma[\setminus(A_k \cup B_k \cup C_k)$, the measure of $[x_i, x_i + \sigma[\setminus(A_k \cup B_k \cup C_k)$ is $-\varepsilon_k\eta_k + \varepsilon_k$ so that $\mathcal{L}^1(B_k) = \sigma - \gamma_k - \alpha_k + \varepsilon_k\eta_k - \varepsilon_k$.

By (2.54) we get

$$h_k(z_k) \quad = \quad \frac{{\beta_k}^2}{2\alpha_k\frac{1-\eta_k}{\eta_k} + 2\gamma_k\frac{1-\xi_k}{\xi_k} + \zeta_k} + 2\alpha_k\frac{\psi(\eta_k)}{\varepsilon_k} + 2\gamma_k\frac{\psi(\xi_k)}{\varepsilon_k} + \kappa_k$$

$$\geq \quad \frac{{\beta_k}^2}{2\frac{1-\eta_k}{\eta_k}(\alpha_k+\gamma_k)+\zeta_k} + 2(\alpha_k+\gamma_k)\frac{\psi(\xi_k)}{\varepsilon_k} + \kappa_k,$$

where $\zeta_k = 2\sigma + 2\varepsilon_k\eta_k - 2\varepsilon_k - 2\varepsilon_k \log\eta_k$ and $\kappa_k = 2\int_{\eta_k}^1 \psi(s)ds$.

The map

$$t \mapsto \frac{{\beta_k}^2}{t+\zeta_k} + \frac{\eta_k}{\varepsilon_k}\frac{\psi(\xi_k)}{1-\eta_k}t + \kappa_k$$

can be estimated differently in the cases $\alpha = +\infty$ and $0 < \alpha < +\infty$.

If $\alpha = +\infty$, by (2.16), (2.53), and (2.56) we find

$$\frac{\beta_k^2}{\zeta_k} \leq h_k(z_k) \leq I_k^i \leq c.$$

By (2.52), this implies, as $k \to +\infty$,

$$\frac{(u(x_i+\sigma)-u(x_i-\sigma))^2}{2\sigma} \leq c.$$

As $\sigma \to 0$, we obtain $|[u(x_i)]| = 0$; this contradicts our assumption that x_i is a jump point and proves that $\mathcal{H}^0(J_u) = 0$, so that $u \in H^1(\Omega)$ and (2.17) follows from (2.18).

If $0 < \alpha < +\infty$ we have

$$2\beta_k\left(\frac{\psi(\xi_k)}{1-\eta_k}\frac{\eta_k}{\varepsilon_k}\right)^{\frac{1}{2}} - \frac{\psi(\xi_k)}{1-\eta_k}\frac{\eta_k}{\varepsilon_k}\zeta_k + \kappa_k \leq h_k(z_k) \leq I_k^i,$$

then taking $k \to +\infty$ and summing on the index i we get

$$\sum_{i=1}^{m-1} 2\left[(\alpha\psi(0))^{\frac{1}{2}}|u(x_i+\sigma)-u(x_i-\sigma)| - \alpha\psi(0)\sigma + \int_0^1 \psi(s)ds\right]$$
$$\leq \sum_{i=1}^{m-1} \liminf_{k\to+\infty} \int_{x_i-\sigma}^{x_i+\sigma} \left[v_k(\nabla u_k)^2 dx + \frac{\psi(v_k)}{\varepsilon_k}\right]dx. \tag{2.61}$$

By adding (2.51) and (2.61), as $\sigma \to 0$, we obtain (2.17). □

Let us give below the proof of the Γ-limsup inequality.

Proof of Theorem 2.3. Let us consider $u \in SBV^2(\Omega)$. We are going to construct a recovery sequence converging to $(u,1)$ in $L^1(\Omega)\times L^1(\Omega)$.

The case $\alpha = +\infty$ is trivial since the right-hand side of (2.14) is finite if and only if $u \in H^1(\Omega)$ and in this case it is sufficient to choose the recovery sequence identically equal to $(u,1)$.

Now we suppose $\alpha < +\infty$. In order to simplify the discussion we assume u has only one jump point $\overline{x}$. Let (σ_k^α) be an infinitesimal sequence and let

$$A_k := [\overline{x} - \sigma_k^\alpha, \overline{x} + \sigma_k^\alpha] \quad \text{and} \quad B_k := [\overline{x} - \sigma_k^\alpha - \varepsilon_k(1-\eta_k), \overline{x} + \sigma_k^\alpha + \varepsilon_k(1-\eta_k)];$$

moreover let us define v_k by η_k in A_k, by 1 out of B_k, and connecting linearly in $B_k \setminus A_k$; finally let us define u_k by u out of A_k and linking linearly in A_k.

Then $(u_k, v_k) \in H^1(\Omega) \times V_k$ and $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. We have

$$\lim_k \int_{\Omega \setminus A_k} \Big(v_k |\nabla u_k|^2 + \frac{1}{\varepsilon_k}\psi(v_k)\Big) dx = \int_\Omega |\nabla u|^2 dx + 2\int_0^1 \psi(s) ds,$$

$$\int_{A_k} \Big(v_k |\nabla u_k|^2 + \frac{1}{\varepsilon_k}\psi(v_k)\Big) dx = \frac{\eta_k}{2\sigma_k^\alpha}(u(\overline{x} + \sigma_k^\alpha) - u(\overline{x} - \sigma_k^\alpha))^2 + 2\psi(\eta_k)\frac{\sigma_k^\alpha}{\varepsilon_k}. \quad (2.62)$$

If $\alpha = 0$ we take σ_k^0 such that $\eta_k/\sigma_k^0 \to 0$ and $\sigma_k^0/\varepsilon_k \to 0$; by this choice the integral in (2.62) converges to 0. Whereas if $0 < \alpha < +\infty$ we define $\sigma_k^\alpha := \frac{1}{2}(\frac{\alpha}{\psi(0)})^{\frac{1}{2}} |[u(\overline{x})]| \varepsilon_k$ and the integral in (2.62) tends to $b_\alpha |[u(\overline{x})]|$. □

The following remark exhibits an example, in the case $n > 1$, of a function $u \in GSBV(\Omega) \setminus BV(\Omega)$ for which $F_0(u, 1) < +\infty$.

Remark 2.5. Let us note that, if $n > 1$, then the inequality $F_0(u, 1) < +\infty$ does not imply $u \in BV(\Omega)$ nor $u \in L^2(\Omega)$. Indeed, let Ω be a bounded open set in $\mathbb{R}^n$ and consider a sequence of pairwise disjoint balls $B_{r_i}(x_i)$, contained in Ω, with centres x_i and radii $r_i := 2^{-i}$. Moreover assume that also the balls $B_{3r_i}(x_i)$ are contained in Ω and pairwise disjoint. Let $u \in L^1(\Omega)$ be defined by

$$u(x) := \begin{cases} a_i & \text{if } x \in B_{r_i}(x_i), \\ 0 & \text{otherwise}, \end{cases} \quad (2.63)$$

where $a_i := 2^{(n-1)i}$. Clearly $u \in L^1(\Omega) \setminus L^2(\Omega)$. Moreover u belongs to $GSBV(\Omega)$ but does not to $BV(\Omega)$ since

$$|D^j u|(\Omega) = \sum_{i=1}^{+\infty} a_i {r_i}^{n-1} = +\infty.$$

Let $\sigma \geq 2$, $\varepsilon_k := 2^{-nk}$, and $\eta_k := \varepsilon_k^\sigma$; this implies $\alpha = 0$. Let us show that $F_0'(u, 1) < +\infty$. To this aim let us consider $\delta_k := 2^{nk(1-\sigma)}$ and let us define u_k as a_i in $B_{r_i - \delta_k}(x_i)$, 0 out of $B_{r_i + \delta_k}(x_i)$, and with constant slope in $B_{r_i+\delta_k}(x_i) \setminus B_{r_i - \delta_k}(x_i)$, for $i \leq k$; we set $u_k := 0$ otherwise. Let v_k be defined as η_k in $B_{r_i+\delta_k}(x_i) \setminus B_{r_i-\delta_k}(x_i)$, with constant slope in $(B_{r_i+\delta_k+\varepsilon_k(1-\eta_k)}(x_i) \setminus B_{r_i+\delta_k}(x_i)) \cup (B_{r_i-\delta_k}(x_i) \setminus$

$B_{r_i-\delta_k-\varepsilon_k(1-\eta_k)}(x_i))$, for $i \le k$, and as 1 otherwise. Note that $(u_k, v_k) \in H^1(\Omega)\times V_k$ and $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega)\times L^1(\Omega)$. A direct computation shows that

$$\liminf_{k\to+\infty} F_k(u_k, v_k) < +\infty,$$

so that $F_0(u, 1) < +\infty$.

2.4 Proof in the n-dimensional case

We are now concerned with the general case $n > 1$. Let us prove first the liminf inequality.

2.4.1 The estimate from below

In this subsection we use the slicing argument (see Section 1.4) to prove the estimate from below (2.13) when $n > 1$. We also make use of the fine properties of GBV-functions collected in [7, Theorem 4.34].

In order to obtain the Γ-$\liminf$ inequality it is sufficient to prove Proposition 2.4.

Proof of Proposition 2.4. The case $\alpha = +\infty$ and the case $0 \le \alpha < +\infty$, $\beta = +\infty$ can be faced as for $n = 1$.

We shall prove the theorem in the case $1 < p < \infty$ for $0 \le \alpha < +\infty$ under the assumption $\delta_k = \varepsilon_k$ (then $\beta = 1$ will be omitted as usual from the notation). Indeed first this case models each one with $0 \le \alpha < +\infty$ and $0 < \beta < +\infty$. With obvious modification one can extend the proof to the regime $\beta = 0$ and to the case $p = +\infty$.

Let (u_k, v_k) be a sequence satisfying (2.15) and (2.16) with bounding constant c; as in the one-dimensional case we can deduce that $v = 1$ $\mathcal{L}^n$-a.e. in Ω. In the first part of the proof we assume that (u_k) is bounded in $L^\infty(\Omega)$ and we want to prove that $u \in SBV^2(\Omega)$.

Proof of (2.18) in the bounded case. Given $\xi \in S^{n-1}$, we extract a subsequence (u_r, v_r) of (u_k, v_k) such that

$$((u_r)^\xi_y, (v_r)^\xi_y) \to (u^\xi_y, 1) \text{ in } L^1(\Omega^\xi_y)\times L^1(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \tag{2.64}$$

and

$$\lim_{r\to+\infty} \int_\Omega v_r |\nabla u_r \cdot \xi|^2 dx = \liminf_{k\to+\infty} \int_\Omega v_k |\nabla u_k \cdot \xi|^2 dx. \tag{2.65}$$

Let $0<\kappa<1$; by the Fubini Theorem and (1.5) we can write

$$\int_{\Pi^\xi} d\mathcal{H}^{n-1}(y)\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\kappa\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)\Big)dt\leq$$
$$\leq\int_\Omega\Big(v_r|\nabla u_r|^2+\kappa\Big(\frac{\psi(v_r)}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla v_r|^p\Big)\Big)dx\leq c,$$

where the last inequality follows from (2.16). From the Fatou Lemma it follows that

$$\int_{\Pi^\xi}\liminf_{r\to+\infty}\Big[\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\kappa\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)\Big)dt\Big]d\mathcal{H}^{n-1}(y)$$

is bounded, so that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$

$$\liminf_{r\to+\infty}\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\kappa\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)\Big)dt<+\infty. \tag{2.66}$$

Let $F_{y,r}$ be the one-dimensional functional on the set Ω_y^ξ, defined by

$$F_{y,r}(w,z):=\begin{cases}\displaystyle\int_{\Omega_y^\xi}\Big(z|\nabla w|^2+\frac{\varphi(z)}{\varepsilon_r}+\tilde\gamma\,\varepsilon_r^{p-1}|\nabla z|^p\Big)dt & \text{if } (w,z)\in H^1(\Omega_y^\xi)\times V_{y,r},\\ +\infty & \text{otherwise,}\end{cases} \tag{2.67}$$

where $\varphi:=\kappa\psi$, $\tilde\gamma:=\kappa\gamma$, and

$$V_{y,r}:=\Big\{z\in W^{1,p}(\Omega_y^\xi):\eta_r\leq z\leq 1\ \mathcal{H}^1\text{-a.e. in }\Omega_y^\xi\Big\}. \tag{2.68}$$

The corresponding Γ-$\liminf$ will be denoted by $F'_{y,\alpha}$.

For $0<\alpha<+\infty$ let $\Phi_{y,\alpha}\colon L^1(\Omega_y^\xi)\mapsto[0,+\infty]$ be defined by

$$\Phi_{y,\alpha}(w):=\begin{cases}\displaystyle\int_{\Omega_y^\xi\setminus J_w}|\nabla w|^2dx+a\mathcal{H}^0(J_w)+b_\alpha\int_{J_w}|[w]|d\mathcal{H}^0 & \text{if } w\in SBV^2(\Omega_y^\xi)\\ +\infty & \text{otherwise,}\end{cases}$$

where a and b_α are defined as in (2.6) with φ and $\tilde\gamma$ which replaces ψ and γ.

In the limiting case $\alpha=0$ we define

$$\Phi_{y,0}(w):=\begin{cases}\displaystyle\int_{\Omega_y^\xi\setminus J_w}|\nabla w|^2dx+a\mathcal{H}^0(J_w) & \text{if } w\in SBV^2(\Omega_y^\xi)\cap L^1(\Omega_y^\xi)\\ +\infty & \text{otherwise,}\end{cases}$$

For $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we can find a subsequence (u_m, v_m) of (u_r, v_r) such that

$$\lim_{m\to+\infty} \int_{\Omega_y^\xi} \Big((v_m)_y^\xi |\nabla((u_m)_y^\xi)|^2 + \kappa\Big(\frac{\psi(v_m)_y^\xi}{\varepsilon_m} + \gamma\, \varepsilon_m^{p-1} |\nabla((v_m)_y^\xi)|^p\Big)\Big) dt =$$
$$= \liminf_{r\to+\infty} \int_{\Omega_y^\xi} \Big((v_r)_y^\xi |\nabla((u_r)_y^\xi)|^2 + \kappa\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} |\nabla((v_r)_y^\xi)|^p\Big)\Big) dt, \quad (2.69)$$

so that (2.64) and (2.69) in particular imply

$$F'_{y,\alpha}(u_y^\xi, 1) \le \lim_{m\to+\infty} F_{y,m}((u_m)_y^\xi, (v_m)_y^\xi) < +\infty,$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Applying Theorem 2.2 in the case $n = 1$ we obtain that $u_y^\xi \in SBV^2(\Omega_y^\xi)$,

$$\Phi_{y,\alpha}(u_y^\xi) \le F'_{y,\alpha}(u_y^\xi, 1), \quad (2.70)$$

and that (2.18) is true for $((u_m)_y^\xi, (v_m)_y^\xi)$.

Now let us prove that $u \in SBV(\Omega)$. Let $M < +\infty$ be such that $||u_m||_{L^\infty(\Omega)} \le M$ for every m. Then decomposing the derivative of u_y^ξ (see [7, Section 3.9]) we get

$$|D(u_y^\xi)|(\Omega_y^\xi) = \int_{\Omega_y^\xi \setminus J_{u_y^\xi}} |\nabla(u_y^\xi)| dt + \sum_{J_{u_y^\xi}} |[u_y^\xi]|$$
$$\le \mathcal{L}^1(\Omega_y^\xi) + \int_{\Omega_y^\xi \setminus J_{u_y^\xi}} |\nabla(u_y^\xi)|^2 dt + 2M\mathcal{H}^0(J_{u_y^\xi}) \le A[1 + F'_{y,\alpha}(u_y^\xi, 1)],$$

where in the last inequality $A := \mathrm{diam}(\Omega) + 1 + \frac{2M}{a}$ and we have used (2.70). Since (u_r) does not depend on y, we can integrate on the projection $\pi^\xi(\Omega)$ of Ω on Π^ξ and we obtain

$$\begin{aligned} & \int_{\pi^\xi(\Omega)} |D(u_y^\xi)|(\Omega_y^\xi) d\mathcal{H}^{n-1}(y) \\ \le\ & A\, \mathcal{H}^{n-1}(\Pi^\xi(\Omega)) + A \int_{\Pi^\xi} \liminf_{r\to+\infty} F_{y,r}((u_r)_y^\xi, (v_r)_y^\xi) d\mathcal{H}^{n-1}(y) \\ \le\ & A\mathcal{H}^{n-1}(\Pi^\xi(\Omega)) + Ac < +\infty. \end{aligned}$$

By taking $\xi = e_1, \dots, e_n$, the elements of the canonical basis of $\mathbb{R}^n$, we get $u \in BV(\Omega)$ by [7, Remark 3.104]; since $u_y^\xi \in SBV^2(\Omega_y^\xi)$, we obtain also $u \in SBV(\Omega)$ by [7, Theorem 3.108].

From (2.18) applied to $((u_m)_y^\xi, (v_m)_y^\xi)$ and from (2.69) it follows that

$$\int_{\Omega_y^\xi \setminus J_{u_y^\xi}} |\nabla(u_y^\xi)|^2 dt \le$$

$$\leq \liminf_{r\to+\infty}\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\kappa\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)\Big)dt,$$

for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$. Integrating on Π^ξ and applying the Fatou Lemma we get

$$\int_{\Pi^\xi}d\mathcal{H}^{n-1}(y)\int_{\Omega_y^\xi\setminus J_{u_y^\xi}}|\nabla(u_y^\xi)|^2dt\leq$$

$$\leq \liminf_{r\to+\infty}\int_{\Pi^\xi}d\mathcal{H}^{n-1}(y)\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\kappa\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)\Big)dt$$
$$\leq \liminf_{k\to+\infty}\int_{\Pi^\xi}d\mathcal{H}^{n-1}(y)\int_{\Omega_y^\xi}(v_k)_y^\xi|\nabla((u_k)_y^\xi)|^2dt+\kappa\,c,$$

where the last inequality follows from (2.16) and (2.65). We observe that (u_k,v_k) does not depend on κ; as $\kappa\to0$ in the previous inequality we find

$$\int_\Omega|\nabla u\cdot\xi|^2dx\leq\liminf_{k\to+\infty}\int_\Omega v_k|\nabla u_k\cdot\xi|^2dx, \tag{2.71}$$

using (1.5) and the Fubini Theorem. By taking $\xi=e_1,\dots,e_n$ and summing the results we obtain (2.18).

Proof of (2.19) *in the bounded case.* Given $\xi\in S^{n-1}$, the first subsequence (u_r,v_r) of (u_k,v_k) is now chosen so that (2.64) holds and (2.65) is replaced by

$$\lim_{r\to+\infty}\int_{\Pi^\xi}\Big[\int_{\Omega_y^\xi}\Big(\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt\Big]d\mathcal{H}^{n-1}(y)=$$
$$=\liminf_{k\to+\infty}\int_{\Pi^\xi}\Big[\int_{\Omega_y^\xi}\Big(\frac{\psi(v_k)_y^\xi}{\varepsilon_k}+\gamma\,\varepsilon_k^{p-1}|\nabla((v_k)_y^\xi)|^p\Big)dt\Big]d\mathcal{H}^{n-1}(y). \tag{2.72}$$

Let $0<\kappa<1$; by the Fubini Theorem and the Fatou Lemma we find

$$\int_{\Pi^\xi}\liminf_{r\to+\infty}\Big[\int_{\Omega_y^\xi}\Big(\kappa\,(v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt\Big]d\mathcal{H}^{n-1}(y)$$

and this implies, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$,

$$\liminf_{r\to+\infty}\int_{\Omega_y^\xi}\Big(\kappa\,(v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt<+\infty.$$

It follows that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ there exists a subsequence (u_m,v_m) of (u_r,v_r) such that

$$\lim_{m\to+\infty}\int_{\Omega_y^\xi}\Big(\kappa\,(v_m)_y^\xi|\nabla((u_m)_y^\xi)|^2+\frac{\psi(v_m)_y^\xi}{\varepsilon_m}+\gamma\,\varepsilon_m^{p-1}|\nabla((v_m)_y^\xi)|^p\Big)dt$$

$$= \liminf_{r\to+\infty} \int_{\Omega_y^\xi} \Big(\kappa\, (v_r)_y^\xi |\nabla((u_r)_y^\xi)|^2 + \frac{\psi(v_r)_y^\xi}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} |\nabla((v_r)_y^\xi)|^p \Big) dt. \tag{2.73}$$

Let us consider the one-dimensional functional $F_{y,r}$ defined in (2.67) with $\varphi := \psi$ and $\tilde{\gamma} := \gamma$.

By (2.64) and (2.73) the sequence $F_{y,m}((\kappa^{1/2} u_m)_y^\xi, (v_m)_y^\xi)$ is bounded, so that Theorem 2.2 in the case $n = 1$ implies that inequality (2.19) holds for the sequence $((\kappa^{1/2} u_m)_y^\xi, (v_m)_y^\xi)$; using formula (2.73) we get

$$a\mathcal{H}^0(J_{u_y^\xi}) \le \liminf_{r\to+\infty} \int_{\Omega_y^\xi} \Big(\kappa\, (v_r)_y^\xi |\nabla((u_r)_y^\xi)|^2 + \frac{\psi(v_r)_y^\xi}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} |\nabla((v_r)_y^\xi)|^p \Big) dt.$$

Let us observe that (u_r) does not depend on y. Then we can integrate on Π^ξ both sides of the previous inequality and apply the Fatou Lemma

$$\begin{aligned} & a \int_{\Pi^\xi} \mathcal{H}^0(J_{u_y^\xi}) d\mathcal{H}^{n-1}(y) \le \\ \le\ & \liminf_{r\to+\infty} \int_{\Pi^\xi} d\mathcal{H}^{n-1}(y) \int_{\Omega_y^\xi} \Big(\kappa\, (v_r)_y^\xi |\nabla((u_r)_y^\xi)|^2 + \frac{\psi(v_r)_y^\xi}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} |\nabla((v_r)_y^\xi)|^p \Big) dt \\ \le\ & \liminf_{k\to+\infty} \int_{\Pi^\xi} d\mathcal{H}^{n-1}(y) \int_{\Omega_y^\xi} \Big(\frac{\psi(v_k)_y^\xi}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla((v_k)_y^\xi)|^p \Big) dt + \kappa\, c, \end{aligned}$$

by (2.16) and (2.72). As $\kappa \to 0$, using (1.3) and (1.6) we find

$$a \int_{J_u} |\nu_u \cdot \xi| d\mathcal{H}^{n-1} \le \liminf_{k\to+\infty} \int_\Omega \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p \Big) dx \le c. \tag{2.74}$$

Applying (2.74) with $\xi = e_1, \ldots, e_n$ we get $\mathcal{H}^{n-1}(J_u) < +\infty$. Since we have already proved that $u \in SBV(\Omega)$, we deduce from (2.16) and (2.18) that $u \in SBV^2(\Omega)$.

In order to obtain (2.19) we use a particular case of the localization method developed in [19, Theorem 2.3.1]. First we note that (2.74) holds also for an open set $A \subset \Omega$, hence

$$a \int_{J_u \cap A} |\nu_u \cdot \xi| d\mathcal{H}^{n-1} \le \liminf_{k\to+\infty} \int_\Omega \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p \Big) dx. \tag{2.75}$$

Since ν_u is a Borel function with values in S^{n-1}, there exists a sequence (ω_j) of simple functions with values in S^{n-1} converging to ν_u pointwise $\mathcal{H}^{n-1}$-a.e. in J_u. We can write $\omega_j = \xi_j^1 1_{B_j^1} + \cdots + \xi_j^{m_j} 1_{B_j^{m_j}}$, where ξ_j^i are unit vectors and $B_j^1, \ldots, B_j^{m_j}$

is a Borel partition of J_u. By the dominated convergence theorem we have

$$\lim_{j\to+\infty}\sum_{i=1}^{m_j}\int_{B_j^i}|\nu_u\cdot\xi_j^i|d\mathcal{H}^{n-1}=\mathcal{H}^{n-1}(J_u). \tag{2.76}$$

For every j we can find $A_j^1,\dots,A_j^{m_j}$ a family of pairwise disjoint open subsets of Ω such that $\mathcal{H}^{n-1}((A_j^i\cap J_u)\triangle B_j^i)\le 1/(jm_j)$. Then (2.76) holds with B_j^i replaced by $J_u\cap A_j^i$. Since by (2.75)

$$a\sum_{i=1}^{m_j}\int_{J_u\cap A_j^i}|\nu_u\cdot\xi_j^i|d\mathcal{H}^{n-1}\le\liminf_{k\to+\infty}\int_\Omega\Big(\frac{\psi(v_k)}{\varepsilon_k}+\gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx,$$

we obtain (2.19) as $j\to+\infty$.

Proof of (2.17) *in the bounded case.* If $\alpha=0$ inequality (2.17) can be obtained by adding (2.18) and (2.19).

Let now $0<\alpha<+\infty$. Given $\xi\in S^{n-1}$, we choose a subsequence (u_r,v_r) of (u_k,v_k) such that (2.64) holds and

$$\begin{aligned}&\lim_{r\to+\infty}\int_{\Pi^\xi}d\mathcal{H}^{n-1}(y)\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla(u_r)_y^\xi|^2+\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt\\&=\liminf_{k\to+\infty}\int_{\Pi^\xi}d\mathcal{H}^{n-1}(y)\int_{\Omega_y^\xi}\Big((v_k)_y^\xi|\nabla(u_k)_y^\xi|^2+\frac{\psi(v_k)_y^\xi}{\varepsilon_k}+\gamma\,\varepsilon_k^{p-1}|\nabla((v_k)_y^\xi)|^p\Big)dt.\end{aligned}$$

By (1.5), using the Fubini Theorem and the Fatou Lemma we get

$$\int_{\Pi^\xi}\liminf_{r\to+\infty}\Big[\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt\Big]d\mathcal{H}^{n-1}(y)\le c$$

and then for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ we have

$$\liminf_{r\to+\infty}\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt<+\infty.$$

Let $F_{y,r}$ be the one-dimensional functional defined in (2.67), where $\varphi:=\psi$ and $\tilde\gamma:=\gamma$. For $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ we can find a subsequence (u_m,v_m) of (u_r,v_r) such that

$$\begin{aligned}&\lim_{m\to+\infty}\int_{\Omega_y^\xi}\Big((v_m)_y^\xi|\nabla((u_m)_y^\xi)|^2+\frac{\psi(v_m)_y^\xi}{\varepsilon_m}+\gamma\,\varepsilon_m^{p-1}|\nabla((v_m)_y^\xi)|^p\Big)dt=\\&=\liminf_{r\to+\infty}\int_{\Omega_y^\xi}\Big((v_r)_y^\xi|\nabla((u_r)_y^\xi)|^2+\frac{\psi(v_r)_y^\xi}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla((v_r)_y^\xi)|^p\Big)dt;\end{aligned} \tag{2.77}$$

then Theorem 2.2 in the case $n = 1$ implies

$$\Phi_{y,\alpha}(u_y^\xi) \le \liminf_{r\to+\infty} F_{y,r}((u_r)_y^\xi, (v_r)_y^\xi).$$

Let us observe that (u_r) does not depend on y; integrating on Π^ξ both sides of the previous inequality and applying the Fatou Lemma we get

$$\int_{\Pi^\xi} \Phi_{y,\alpha}(u_y^\xi) d\mathcal{H}^{n-1}(y) \le \tag{2.78}$$
$$\le \liminf_{k\to+\infty} \int_{\Pi^\xi} d\mathcal{H}^{n-1}(y) \int_{\Omega_y^\xi} \left((v_k)_y^\xi |\nabla((u_k)_y^\xi)|^2 + \frac{\psi(v_k)_y^\xi}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla((v_k)_y^\xi)|^p \right) dt.$$

We now apply the localization method to the measure $\mu = \mathcal{L}^n \lfloor \Omega + \mathcal{H}^{n-1} \lfloor J_u$ instead of $\mathcal{H}^{n-1}\lfloor J_u$. Since (2.78) holds with an open set $A \subset \Omega$ in place of Ω, by (1.3)–(1.6) and by the Fubini Theorem we get

$$\int_A \Big[|\nabla u \cdot \xi|^2 1_{\Omega\setminus J_u} + |\nu_u \cdot \xi| (a + b_\alpha |[u]|) 1_{J_u} \Big] d\mu$$
$$\le \liminf_{k\to+\infty} \int_A \Big[v_k |\nabla u_k \cdot \xi|^2 + \frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p \Big] dx. \tag{2.79}$$

Let us define $\omega := \nu_u$ on J_u, $\omega := \nabla u/|\nabla u|$ on $\{\nabla u \ne 0\} \setminus J_u$, and $\omega := e_1$ elsewhere. Since ω is a μ-measurable function with values in S^{n-1}, there exists a sequence (ω_j) of simple functions with values in S^{n-1}, converging to ω μ-a.e. in Ω. We can write $\omega_j = \xi_j^1 1_{B_j^1} + \dots + \xi_j^{m_j} 1_{B_j^{m_j}}$, where ξ_j^i are unit vectors and $B_j^1, \dots, B_j^{m_j}$ is a Borel partition of Ω. By the dominated convergence theorem we have

$$\lim_{j\to+\infty} \sum_{i=1}^{m_j} \int_{B_j^i} \Big[|\nabla u \cdot \xi_j^i|^2 1_{\Omega\setminus J_u} + |\nu_u \cdot \xi_j^i| (a + b_\alpha |[u]|) 1_{J_u} \Big] d\mu = \Phi_\alpha(u). \tag{2.80}$$

For every j we can find a family $A_j^1, \dots, A_j^{m_j}$ of pairwise disjoint open subsets of Ω such that $\mu(A_j^i \triangle B_j^i) \le 1/(j m_j)$. Then (2.80) holds with B_j^i replaced by A_j^i. By (2.79) we find

$$\sum_{i=1}^{m_j} \int_{A_j^i} \Big[|\nabla u \cdot \xi_j^i|^2 1_{\Omega\setminus J_u} + |\nu_u \cdot \xi_j^i| (a + b_\alpha |[u]|) 1_{J_u} \Big] d\mu$$
$$\le \liminf_{k\to+\infty} \int_\Omega \Big[v_k |\nabla u_k|^2 + \frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p \Big] dx$$

and we obtain (2.17) as $j \to +\infty$.

The general case. We now remove the assumption that (u_k) is bounded in

$L^\infty(\Omega)$. Let us fix $M > 0$ and let us consider the sequence of truncated functions $u_k^M = (-M \vee u) \wedge M$. We have that $u_k^M \to u^M$ in $L^1(\Omega)$, $v_k \to 1$ in $L^1(\Omega)$, and by (2.16)

$$F_k(u_k^M, v_k) \le F_k(u_k, v_k) \le c.$$

From the proof in the bounded case it follows that $u^M \in SBV^2(\Omega)$ and that

$$\int_\Omega |\nabla u^M|^2 dx \le \liminf_{k\to+\infty} \int_\Omega v_k |\nabla u_k|^2 dx, \tag{2.81}$$

$$a\mathcal{H}^{n-1}(J_{u^M}) \le \liminf_{k\to+\infty} \int_\Omega \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p\Big) dx. \tag{2.82}$$

This implies $u \in GSBV(\Omega)$. As $|\nabla u^M| = |\nabla u| 1_{\{|u|\le M\}}$ by Theorem [7, Theorem 4.34], using the monotone convergence theorem we obtain

$$\int_\Omega |\nabla u|^2 dx = \lim_{M\to+\infty} \int_\Omega |\nabla u^M|^2 dx,$$

which together with (2.81) proves (2.18). Moreover, taking $M \to +\infty$ in (2.82) we find (2.19). Therefore $u \in GSBV^2(\Omega)$.

Let us prove now (2.17). When $\alpha = 0$, this inequality can be obtained by adding (2.18) and (2.19).

Let $0 < \alpha < +\infty$. The proof in the bounded case, applied to (u_k^M, v_k), gives

$$\Phi_\alpha(u^M) \le \liminf_{k\to+\infty} \int_\Omega \Big[v_k |\nabla u_k|^2 + \frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p\Big] dx \le c. \tag{2.83}$$

Since $u^M \in SBV^2(\Omega)$, inequality (2.83) gives $|Du^M|(\Omega) \le \mathcal{L}^n(\Omega) + c \max(1, 1/b_\alpha)$ for every $M > 0$. From $u^M \to u$ in $L^1(\Omega)$, we conclude that $u \in BV(\Omega)$ and $u^M \rightharpoonup u$ weakly* in $BV(\Omega)$. Using the Closure Theorem for SBV [7, Theorem 4.7], we deduce from (2.83) that $u \in SBV^2(\Omega)$. Estimate (2.83), as $M \to +\infty$, leads to (2.17). □

2.4.2 The estimate from above

Now our purpose is to prove the Γ-lim sup inequality. In order to work with more regular functions and jump sets, we first introduce an approximation result. The following theorem is a small modification of a theorem due to Cortesani and Toader (see [23, Theorem 3.1] and Section 1.7).

Theorem 2.6. *Let $Q \subset \mathbb{R}^n$ be an open cube, let $1 < p \le 2$, and let u belong to $SBV^p(Q, \mathbb{R}^n) \cap L^\infty(Q, \mathbb{R}^n)$. Then for every $\varepsilon > 0$ there exist a function $v \in SBV^p(Q, \mathbb{R}^n)$ and a set $S = \cup_{i=1}^m S_i$, with S_i closed and pairwise disjoint $(n-1)$-*

simplexes contained in Q*, such that*

(a) $\mathcal{H}^{n-1}(S \setminus J_v) = 0$*;*

(b) $v \in W^{k,\infty}(Q \setminus S, \mathbb{R}^n)$ *for every* k*;*

(c) $||v - u||_{L^1(Q,\mathbb{R}^n)} < \varepsilon$*;*

(d) $||\nabla v - \nabla u||_{L^p(Q,\mathbb{M}^{n\times n})} < \varepsilon$;

(e) $\mathcal{H}^{n-1}(J_v) < \mathcal{H}^{n-1}(J_u) + \varepsilon$;

(f) $\displaystyle\int_{J_v} |[v] \odot \nu_v| d\mathcal{H}^{n-1} < \int_{J_u} |[u] \odot \nu_u| d\mathcal{H}^{n-1} + \varepsilon$.

Proof. Using [23, Theorem 3.1] and [23, Remark 3.5] we can find a function $w \in SBV^p(Q, \mathbb{R}^n)$ and a set $T = \cup_{i=1}^m T_i$, not necessarily contained in Q, with T_i closed and pairwise disjoint $(n-1)$-simplexes, such that conditions (a)–(f) hold for w in place of v and $T \cap Q$ in place of S. Since $T \cap Q$ is a polyhedron, we can adapt the arguments in [23, Remark 3.5] to obtain a function v and a set $S \subset Q$ satisfying conditions (a)–(f). □

Let us focus now on the Γ-limsup inequality.

Proof of Theorem 2.3. Given $u \in L^1(\Omega)$ such that $\Phi_{\alpha,\beta}(u) < +\infty$, we have to construct a recovery sequence (u_k, v_k) converging to $(u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$.

Let us assume first $1 < p < +\infty$. The cases $\alpha = +\infty$ or $\beta = 0$ are trivial since in these cases $\Phi_{\alpha,\beta}(u) < +\infty$ is finite if and only if $u \in H^1(\Omega)$, and in this case it is sufficient to define $(u_k, v_k) := (u, 1)$.

Let now $0 \leq \alpha < +\infty$ and $0 < \beta < +\infty$. Let $u \in GSBV^2(\Omega) \cap L^1(\Omega)$ and we consider first the case $u \in L^\infty(\Omega)$, so that u belongs in effect to $SBV^2(\Omega) \cap L^\infty(\Omega)$.

It is enough to prove (2.14) for a cube Q and for a function u satisfying properties (a) and (b) of Theorem 2.6. Indeed, if Ω is an arbitrary bounded open set Ω with Lipschitz boundary and $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, then a local reflection argument provides an extension of u to a function $\tilde{u} \in SBV^2(Q) \cap L^\infty(Q)$ such that $\mathcal{H}^{n-1}(J_{\tilde{u}} \cap \partial\Omega) = 0$. Through this paragraph we shall write explicitly the domain of the integrals in the functionals (2.2), (2.5), (2.7), (2.11), and (2.12). By Theorem 2.6 for every k we can find a function $w_k \in SBV^2(Q)$ satisfying properties (a)–(f). Assuming that (2.14) holds for w_k, we have $F''_{\alpha,\beta,Q}(w_k, 1) \leq \Phi_{\alpha,\beta,Q}(w_k)$. Then by the lower semicontinuity of $F''_{\alpha,\beta,Q}$ we obtain

$$F''_{\alpha,\beta,Q}(\tilde{u}, 1) \leq \limsup_{k\to\infty} \Phi_{\alpha,\beta,Q}(w_k)$$

$$\begin{aligned} &\le \limsup_{k\to\infty}\Big[\Phi_{\alpha,\beta,Q}(\tilde u)+\frac{1}{k^2}+\frac{2}{k}||\nabla\tilde u||_{L^2(Q,\mathbb{R}^n)}+\frac{a_\beta+b_\alpha}{k}\Big] \\ &= \Phi_{\alpha,\beta,Q}(\tilde u). \end{aligned} \tag{2.84}$$

Let us check that this implies $F''_{\alpha,\beta,\Omega}(u,1)\le\Phi_{\alpha,\beta,\Omega}(u)$. By Theorem 2.2 and inequality (2.84) we have

$$\begin{gathered} F_{\alpha,\beta,Q}(\tilde u,1)=\Phi_{\alpha,\beta,\Omega}(u)+\Phi_{\alpha,\beta,Q\setminus\overline{\Omega}}(\tilde u), \\ \Phi_{\alpha,\beta,\Omega}(u)\le F'_{\alpha,\beta,\Omega}(u,1), \quad \Phi_{\alpha,\beta,Q\setminus\overline{\Omega}}(\tilde u)\le F'_{\alpha,\beta,Q\setminus\overline{\Omega}}(\tilde u), \end{gathered} \tag{2.85}$$

so that

$$F_{\alpha,\beta,Q}(\tilde u,1)\le F'_{\alpha,\beta,\Omega}(u,1)+F'_{\alpha,\beta,Q\setminus\overline{\Omega}}(\tilde u). \tag{2.86}$$

Moreover [24, Proposition 6.17] implies

$$F''_{\alpha,\beta,\Omega}(u,1)+F'_{\alpha,\beta,Q\setminus\overline{\Omega}}(\tilde u)\le F_{\alpha,\beta,Q}(\tilde u,1);$$

this estimate together with (2.85) and (2.86) gives $F''_{\alpha,\beta,\Omega}(u,1)=\Phi_{\alpha,\beta,\Omega}(u)$.

Therefore, in the rest of the proof we assume that $\Omega=Q$, $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$, and that properties (a) and (b) of Theorem 2.6 hold for u. Finally, in order to simplify the computation, we suppose that S is a unique $(n-1)$-simplex and that $S\subset\{x_n=0\}$. We write a point $x\in\mathbb{R}^n$ as $x=(\overline{x},x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ and we orient J_u so that $\nu_u=(0,1)$. Let

$$\Omega^\pm:=\big\{x\in\Omega:\pm x_n>0\big\}$$

and let L be the maximum between the Lipschitz constants of u in Ω^+ and Ω^-.

Let us define $\sigma_k^\alpha(\overline{x}):=\frac{1}{2}\delta_k(\frac{\alpha}{\psi(0)})^{1/2}|[u(\overline{x},0)]|$ for $x=(\overline{x},x_n)\in\Omega$ in the case $0<\alpha<+\infty$; whereas for $\alpha=0$ we define σ_k^0 as any sequence of constant functions such that $\eta_k/\sigma_k^0\to 0$ and $\sigma_k^0/\delta_k\to 0$. We observe that σ_k^α is Lipschitz since u^+ and u^- are; moreover in the case $0<\alpha<+\infty$ we have $\sigma_k^\alpha(\overline{x})=0$ for $\overline{x}\in\partial S$, where ∂S is the boundary of S in the relative topology of $\mathbb{R}^{n-1}\times\{0\}$.

Let

$$\begin{gathered} A_k:=\Big\{x\in\mathbb{R}^n:(\overline{x},0)\in S,\ |x_n|<\sigma_k^\alpha(\overline{x})\Big\}, \\ A'_k:=\Big\{x\in\mathbb{R}^n:(\overline{x},0)\notin S,\ d(x,\partial S)<\sigma_k^\alpha(\overline{x})\Big\}, \end{gathered}$$

where $d(x,S)$ is the distance from the point x to the set S. The closure of $A_k\cup A'_k$ is contained in Ω for k large.

Let

$$u_k(\overline{x}, x_n) := \begin{cases} \dfrac{x_n + \sigma_k^\alpha}{2\sigma_k^\alpha}(u(\overline{x}, \sigma_k^\alpha) - u(\overline{x}, -\sigma_k^\alpha)) + u(\overline{x}, -\sigma_k^\alpha) & \text{if } x \in A_k, \\ u(x) & \text{if } x \in \Omega \setminus (A_k \cup A_k'). \end{cases}$$

Here and henceforth σ_k^α denotes $\sigma_k^\alpha(\overline{x})$. Let us verify that $u_k \in W^{1,\infty}(\Omega \setminus A_k')$. If $x = (\overline{x}, x_n) \in A_k$, we have

$$\begin{aligned} & |D_n u_k(\overline{x}, x_n)| \\ = & \left| \frac{u(\overline{x}, \sigma_k^\alpha) - u(\overline{x}, -\sigma_k^\alpha)}{2\sigma_k^\alpha} \right| \\ = & \left| \frac{u(\overline{x}, \sigma_k^\alpha) - u^+(\overline{x}, 0)}{2\sigma_k^\alpha} + \frac{u^+(\overline{x}, 0) - u^-(\overline{x}, 0)}{2\sigma_k^\alpha} + \frac{u^-(\overline{x}, 0) - u(\overline{x}, -\sigma_k^\alpha)}{2\sigma_k^\alpha} \right| \\ \leq & \ L + \frac{|[u(\overline{x}, 0)]|}{2\sigma_k^\alpha}, \end{aligned} \tag{2.87}$$

where the last inequality follows from the Lipschitz continuity of u on $\Omega^\pm$. Using the previous estimate we also obtain

$$\begin{aligned} & |D_j u_k(\overline{x}, x_n)| \\ \leq & \left| \frac{x_n}{\sigma_k^\alpha} D_j \sigma_k^\alpha \frac{u(\overline{x}, \sigma_k^\alpha) - u(\overline{x}, -\sigma_k^\alpha)}{2\sigma_k^\alpha} \right| + \left| D_j u(\overline{x}, -\sigma_k^\alpha) - D_n u(\overline{x}, -\sigma_k^\alpha) D_j \sigma_k^\alpha \right| \\ & + \left| D_j u(\overline{x}, \sigma_k^\alpha) + D_n u(\overline{x}, \sigma_k^\alpha) D_j \sigma_k^\alpha - D_j u(\overline{x}, -\sigma_k^\alpha) + D_n u(\overline{x}, -\sigma_k^\alpha) D_j \sigma_k^\alpha \right| \\ \leq & \ D_j \sigma_k^\alpha \Big(\frac{|[u(\overline{x}, 0)]|}{2\sigma_k^\alpha} + 4L \Big) + 3L, \end{aligned} \tag{2.88}$$

for $j = 1, \dots, n-1$ and for every $(\overline{x}, x_n) \in A_k$.

By the definition of σ_k^α and the boundedness of u, the quotient $|[u(\overline{x}, 0)]|/\sigma_k^\alpha$ is bounded uniformly with respect to $\overline{x}$; since $D_j \sigma_k^\alpha \leq (\frac{\alpha}{\psi(0)})^{1/2} L \delta_k$, we deduce from (2.87) and (2.88) that $u_k \in W^{1,\infty}(\Omega \setminus A_k')$, so that in the case $0 < \alpha < \infty$ we obtain $u_k \in W^{1,\infty}(\Omega)$. In the case $\alpha = 0$ inequalities (2.87) and (2.88) imply that u_k is Lipschitz continuous in $\{x \in \Omega : (\overline{x}, 0) \in S\}$, with Lipschitz constant $(M/\sigma_k^0) + 3nL$, where $M := ||u||_{L^\infty(\Omega)}$.

To prove that u_k is Lipschitz continuous in $\Omega \setminus A_k'$ we will show that

$$|u_k(x) - u_k(y)| \leq \Big(\frac{4M}{\sigma_k^0} + 12nL \Big) (|\overline{x} - \overline{y}| + |x_n - y_n|) \quad \text{for } x, y \in \Omega \setminus A_k'. \tag{2.89}$$

Let $x, y \in A_k \cup B_k \cup B_k'$. It is enough to prove (2.89) when x_n and y_n have the same sign. Indeed, if $(\overline{x},0)\in S$ we can write

$$|u_k(x)-u_k(y)|\le |u_k(\overline{x},x_n)-u_k(\overline{x},y_n)|+|u_k(\overline{x},y_n)-u_k(\overline{y},y_n)| \tag{2.90}$$

and the estimate for the first term in the right-hand side comes from the Lipschitz continuity of u_k in $\{x\in\Omega:(\overline{x},0)\in S\}$. If $(\overline{x},0)\notin S$ and $(\overline{y},0)\notin S$, then

$$|u_k(x)-u_k(y)|=|u(x)-u(y)|\le |u(\overline{x},x_n)-u(\overline{x},y_n)|+|u(\overline{x},y_n)-u(\overline{y},y_n)|.$$

Since the segment with end points $(\overline{x},x_n)$ and $(\overline{x},y_n)$ is contained in $\Omega\setminus S$, the first term in the right-hand side is estimated by $L|x_n-y_n|$, whereas the second term is estimated by $L|\overline{x}-\overline{y}|$ due to the Lipschitz continuity of u in $\Omega^\pm$.

Therefore, it is enough to prove (2.89) when $x_n>0$ and $y_n>0$. If $y_n>\sigma_k^0$, then we can write (2.90) and the right-hand side reduces to $|u_k(\overline{x},x_n)-u_k(\overline{x},y_n)|+|u(\overline{x},y_n)-u(\overline{y},y_n)|$. The second term is estimated by L as before. If $(\overline{x},0)\in S$ the first term is estimated using the Lipschitz continuity of u_k in $\{x\in\Omega:(\overline{x},0)\in S\}$. If $(\overline{x},0)\notin S$, the first term can be written as $|u(\overline{x},x_n)-u(\overline{x},y_n)|$, which is estimated by $L|x_n-y_n|$, since $x,y\in\Omega^+$.

It remains to consider the case $0<x_n<\sigma_k^0$ and $0<y_n<\sigma_k^0$. If $(\overline{x},0),(\overline{y},0)\in S$ then $x,y\in A_k$ and the estimate has already been proved. If $(\overline{x},0),(\overline{y},0)\notin S$ then $|u_k(x)-u_k(y)|=|u(x)-u(y)|$, which can be estimated by the Lipschitz continuity of u in Ω^+. Assume now $(\overline{x},0)\notin S$ and $(\overline{y},0)\in S$. Let $(\overline{z},0)$ be an element of ∂S in the segment of end points $\overline{x}$ and $\overline{y}$, and let $z:=(\overline{z},\sigma_k^0)$. Then

$$|u_k(x)-u_k(y)|\le |u(x)-u(z)|+|u_k(z)-u_k(y)|\le\Big(\frac{M}{\sigma_k^0}+3nL\Big)(|x-z|+|z-y|). \tag{2.91}$$

We have

$$\begin{aligned}|x-z|+|z-y| &\le |\overline{x}-\overline{z}|+|x_n-\sigma_k^0|+|\overline{z}-\overline{y}|+|y_n-\sigma_k^0|\\ &= |\overline{x}-\overline{z}|+|\overline{z}-\overline{y}|+2|x_n-\sigma_k^0|+|x_n-y_n|;\end{aligned} \tag{2.92}$$

since $x\notin A_k'$ we obtain

$$(\sigma_k^0)^2\le |(\overline{x},x_n)-(\overline{z},0)|^2\le |\overline{x}-\overline{z}|^2+x_n^2,$$

so that we can estimate $(\sigma_k^0-x_n)^2$ as follows

$$(\sigma_k^0-x_n)^2\le(\sigma_k^0)^2-x_n^2\le|\overline{x}-\overline{z}|^2. \tag{2.93}$$

Inequality (2.89) follows from (2.91), (2.92), (2.93), and from $|\overline{x}-\overline{z}|+|\overline{z}-\overline{y}|=|\overline{x}-\overline{y}|$.

This concludes the proof of the Lipschitz continuity of u_k in $\Omega \setminus A'_k$. We are now in a position to apply the McShane Theorem, so that there exists a function, still denoted u_k, that extends u_k to A'_k and has the same Lipschitz constant as u_k, i.e.,

$$|u_k(x) - u_k(y)| \leq \Big(\frac{4M}{\sigma_k^0} + 12nL\Big)(|\overline{x} - \overline{y}| + |x_n - y_n|) \quad \text{for } x, y \in \Omega. \tag{2.94}$$

From the definition of u_k we immediately deduce that $u_k \to u$ in $L^1(\Omega)$.

Let now ρ_k, w_k and μ_k be defined as in the one-dimensional case by (2.35), (2.36), and (2.38); we are able to define now

$$B_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ 0 \leq |x_n| - \sigma_k^\alpha(\overline{x}) \leq \mu_k\Big\},$$
$$B'_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \notin S,\ 0 \leq d(x, \partial S) - \sigma_k^\alpha(\overline{x}) \leq \mu_k\Big\},$$

and

$$v_k(x) := \begin{cases} \eta_k & \text{if } x \in A_k \cup A'_k, \\ w_k(|x_n| - \sigma_k^\alpha(\overline{x})) & \text{if } x \in B_k, \\ w_k(d(x, \partial S) - \sigma_k^\alpha(\overline{x})) & \text{if } x \in B'_k, \\ 1 - \rho_k & \text{otherwise.} \end{cases}$$

By this choice $\eta_k \leq v_k \leq 1$ $\mathcal{L}^n$-a.e. in Ω, $v_k \in W^{1,p}(\Omega)$, and $v_k \to 1$ in $L^1(\Omega)$.

Let us proceed with the computation. The sequence $F_k(u_k, v_k)$ can be written now as

$$\begin{aligned} F_k(u_k, v_k) &= \int_\Omega v_k |\nabla u_k|^2 dx + \int_{\Omega \setminus (B_k \cup B'_k)} \frac{\psi(v_k)}{\delta_k} dx + \int_{B_k} \Big(\frac{\psi(v_k)}{\delta_k} \\ &\quad + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p\Big) dx + \int_{B'_k} \Big(\frac{\psi(v_k)}{\delta_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p\Big) dx. \end{aligned} \tag{2.95}$$

As for the first term of the previous expression we note that

$$\begin{aligned} \int_{A_k} \eta_k |\nabla u_k|^2 dx &= \int_{A_k} \eta_k (D_n u_k)^2 dx + \sum_{j=1}^{n-1} \int_{A_k} \eta_k (D_j u_k)^2 dx \\ &\leq \int_{J_u} \eta_k \frac{(u(\overline{x}, \sigma_k^\alpha) - u(\overline{x}, -\delta_k^\alpha))^2}{2\sigma_k^\alpha} d\mathcal{H}^{n-1} + c\eta_k, \end{aligned} \tag{2.96}$$

for a suitable constant $c + \infty$; if $\alpha = 0$ the right-hand side of the previous inequality tends to 0, since $u \in L^\infty(\Omega)$ and $\eta_k / \sigma_k^0 \to 0$; if $0 < \alpha < \infty$, by the dominated convergence theorem it tends to

$$\frac{b_\alpha}{2}\int_{J_u}|[u]|d\mathcal{H}^{n-1}.$$

In the case $\alpha = 0$, when $A'_k \neq \emptyset$, we get by (2.94)

$$\int_{A'_k}\eta_k|\nabla u_k|^2dx \leq c\frac{\eta_k}{(\sigma_k^0)^2}\mathcal{L}^n(A'_k) + c\eta_k, \tag{2.97}$$

where $c < +\infty$ is constant. First we note that $A'_k \subset (\partial S)_{\sigma_k^0}$, where $(\partial S)_{\sigma_k^0} := \{x \in \mathbb{R}^n : d(x, \partial S) < \sigma_k^0\}$. From a well-known result about the Minkowski content, (see, for instance, [7, Theorem 2.106]), we can write

$$\mathcal{L}^n(A'_k) \leq O((\sigma_k^0)^2),$$

so that the integral in (2.97) tends to 0. Finally let us note that

$$\int_{\Omega\setminus(A_k\cup A'_k)} v_k|\nabla u|^2dx \leq \int_\Omega |\nabla u|^2dx.$$

Taking into account the computation in (2.40), we deduce for $0 < \alpha < +\infty$

$$\int_\Omega v_k|\nabla u_k|^2dx + \int_{\Omega\setminus(B_k\cup B'_k)}\frac{\psi(v_k)}{\delta_k}dx \leq \int_\Omega|\nabla u|^2dx + b_\alpha\int_{J_u}|[u]|d\mathcal{H}^{n-1} + o(1); \tag{2.98}$$

whereas if $\alpha = 0$ we find

$$\int_\Omega v_k|\nabla u_k|^2dx + \int_{\Omega\setminus(B_k\cup B'_k)}\frac{\psi(v_k)}{\delta_k}dx \leq \int_\Omega|\nabla u|^2dx + o(1). \tag{2.99}$$

Let us consider now the integral on B_k in (2.95). By the choice of B_k and v_k we obtain

$$\begin{aligned}
&\int_{B_k}\Big(\frac{\psi(v_k)}{\delta_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \leq \\
\leq\;& 2(\gamma p)^{\frac{1}{p}}\Big(\frac{q\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_{J_u}\Big[\int_0^{\mu_k}\psi(w_k)^{\frac{1}{q}}w'_k\,dx_n\Big]d\mathcal{H}^{n-1}(\overline{x}) \\
=\;& 2(\gamma p)^{\frac{1}{p}}\Big(\frac{q\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\Big(\int_{\eta_k}^{1-\rho_k}\psi^{\frac{1}{q}}ds\Big)\mathcal{H}^{n-1}(J_u).
\end{aligned} \tag{2.100}$$

Moreover coarea formula implies

$$\int_{B'_k}\Big(\frac{\psi(v_k)}{\delta_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \leq$$

$$
\begin{aligned}
&\le c_1\Big(\frac{\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_{\sigma_k^\alpha}^{\sigma_k^\alpha+\mu_k}\psi(w_k(t-\sigma_k^\alpha))^{\frac{1}{q}}w_k'(t-\sigma_k^\alpha)\mathcal{H}^{n-1}(\{d(x,\partial J_u)=t\})dt\\
&\le c_2(\sigma_k^\alpha+\mu_k)\Big(\frac{\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}\int_0^{\mu_k}\psi(w_k)^{\frac{1}{q}}w_k'\,dt\le c_3(\sigma_k^\alpha+\mu_k)\Big(\frac{\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}}, \qquad (2.101)
\end{aligned}
$$

where $c_1, c_2, c_3 < +\infty$ are constant and we have used the fact that

$$\mathcal{H}^{n-1}(\{d(x,\partial J_u)=t\}) = O(t).$$

The last term in (2.100) tends to 0 by the choice of β, σ_k^α, and μ_k. By (2.98), (2.99), (2.100), and (2.101) we obtain (2.14).

In the general case when $u \notin L^\infty(\Omega)$, we obtain (2.14) through a truncation argument.

Let now $0<\alpha<+\infty$, $\beta=+\infty$; as in the case $n=1$ it is sufficient to prove by [15, Theorem 3.1] that for every $u \in SBV^2(\Omega)$ we have

$$F''_{\alpha,\infty}(u,1) \le \int_\Omega |\nabla u|^2 dx + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1}.$$

We define all parameters as in the previous case; the computations in (2.98), (2.99), and (2.101) give the same results as before, whereas the last term in (2.100) tends to 0 since $\beta=+\infty$. Estimate (2.14) follows.

We conclude the proof of the estimate from above by studying the case $\alpha=0$, $\beta=+\infty$. We shall prove that $F''_{0,\infty}(u,1)=0$ for every $u\in L^1(\Omega)$.

Since $F''_{0,\infty}$ is lower semicontinuous, it is sufficient to prove the estimate on a set which is dense in $L^1(\Omega)$. To this aim we consider the set of functions which are constant on finitely many disjoint balls and null otherwise. For simplicity we consider only the case of a function u which is constant on a ball B well-contained in Ω and null out of A. Let σ_k^0, ρ_k, w_k, and μ_k be defined as before; let φ_k be a cut-off function such that $\varphi_k=1$ on $(\partial B)_{\sigma_k^0/2}$, $\varphi_k=0$ out of $(\partial B)_{\sigma_k^0}$, and $|\nabla\varphi_k|\le 4/\sigma_k^0$, where $(\partial B)_r := \{d(x,\partial B)<r\}$. We define $u_k := (1-\varphi_k)u$ and v_k as η_k on $(\partial B)_{\sigma_k^0}$, as $1-\rho_k$ out of $(\partial B)_{\sigma_k^0+\mu_k}$, and as $w_k(d(x,\partial B)-\sigma_k^0)$ in $(\partial B)_{\sigma_k^0+\mu_k}\setminus(\partial B)_{\sigma_k^0}$. By this choice $u_k\in H^1(\Omega)$, $v_k\in V_{\eta_k}$ and $(u_k,v_k)\to(u,1)$ in $L^1(\Omega)\times L^1(\Omega)$. Let us proceed with the computation.

We have that

$$
\begin{aligned}
\int_\Omega v_k|\nabla u_k|^2dx + \int_{(\partial B)_{\sigma_k^0}}\frac{\psi(v_k)}{\delta_k}dx &\le \Big(16u^2\frac{\eta_k}{(\sigma_k^0)^2}+\frac{\psi(\eta_k)}{\delta_k}\Big)\mathcal{L}^n((\partial B)_{\sigma_k^0})\\
&\le c_1\frac{\eta_k}{\sigma_k^0}+c_2\frac{\sigma_k^0}{\delta_k},
\end{aligned}
$$

where c_1, c_2 are constant; the last term in the previous expression tends to 0 by the choice of σ_k^0.

Since ρ_k satisfies (2.35) we also obtain

$$\int_{\Omega\setminus(\partial B)_{\sigma_k^0+\mu_k}} \frac{\psi(1-\rho_k)}{\delta_k} dx \leq o(1).$$

Finally we note that

$$\int_{(\partial B)_{\sigma_k^0+\mu_k}\setminus(\partial B)_{\sigma_k^0}} \Big(\frac{\psi(v_k)}{\delta_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v_k|^p\Big) dx =$$
$$= c_3\Big(\frac{\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}} \int_0^{\mu_k} \psi(w_k)^{\frac{1}{q}} w_k'(t+\sigma_k^0)^{n-1}\, dt \leq c_4\Big(\frac{\varepsilon_k}{\delta_k}\Big)^{\frac{1}{q}},$$

where c_3, c_4 are constant; since $\beta = +\infty$ also the last term in the previous expression tends to 0. Equality $F''_{0,\infty}(u,1) = 0$ follows.

In the case $p = +\infty$ one can reproduce the same arguments and computations as before. For convenience of the reader, we just provide below the slight modifications to make to the definitions of B_k, B_k', and v_k in the regimes $0 \leq \alpha < +\infty$ when $\delta_k = \varepsilon_k$ (and then β=1). The sets B_k and B_k' can be redefined as follows

$$B_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ \sigma_k^\alpha(\overline{x}) \leq |x_n| \leq \sigma_k^\alpha(\overline{x}) + \frac{\varepsilon_k(1-\eta_k)}{c_{k,\alpha}}\Big\},$$
$$B_k' := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \notin S,\ \sigma_k^\alpha(\overline{x}) \leq d(x,\partial S) \leq \sigma_k^\alpha(\overline{x}) + \frac{\varepsilon_k(1-\eta_k)}{c_{k,\alpha}}\Big\},$$

where $c_{k,\alpha} := 1$ for $\alpha = 0$, whereas $c_{k,\alpha} := 1 - \varepsilon_k(\frac{\alpha}{\psi(0)})^{1/2}L$ for $0 < \alpha < \infty$; finally v_k can be set equal to

$$v_k(x) := \begin{cases} \eta_k & \text{if } x \in A_k \cup A_k', \\ \eta_k + \dfrac{c_{k,\alpha}}{\varepsilon_k}(|x_n| - \sigma_k^\alpha) & \text{if } x \in B_k, \\ \eta_k + \dfrac{c_{k,\alpha}}{\varepsilon_k}(d(x,\partial S) - \sigma_k^\alpha) & \text{if } x \in B_k', \\ 1 & \text{otherwise.} \end{cases}$$

□

2.5 Convergence of minimizers

Throughout this section we assume $1 < p \leq +\infty$ and we use the notation F_k to indicate both the functionals introduced in Sections 2.2.1 and 2.2.2 respectively for $1 < p < +\infty$ and for $p = +\infty$. The most important result of the chapter is the following theorem on the convergence of minimizers of some variational problems involving the functionals F_k and $F_{\alpha,\beta}$.

Theorem 2.7. *Let $r > 1$; let (δ_k), (ε_k), and (η_k) be infinitesimal sequences of positive numbers, and let $g \in L^r(\Omega)$. For every k, let (u_k, v_k) be a minimizer of the functional*

$$F_k(u,v) + \int_\Omega |u-g|^r dx \tag{2.102}$$

with the constraint $\eta_k \le v \le 1$. Then $v_k \to 1$ strongly in $L^1(\Omega)$ and a subsequence of (u_k) converges strongly in $L^r(\Omega)$ to a minimizer u of the following limit problem:

$$\min_{u\in SBV^2(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} + \int_\Omega |u-g|^r dx \Big),$$

$$\text{if } 0 < \alpha, \beta < +\infty,$$

$$\min_{u\in GSBV^2(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + a_\beta \mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^r dx \Big),$$

$$\text{if } \alpha = 0,\ 0 < \beta < +\infty,$$

$$\min_{u\in H^1(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u-g|^r dx \Big), \text{ if } \alpha = +\infty \text{ or } \beta = 0,$$

$$\min_{u\in L^1(\Omega)} \Big(\int_\Omega |u-g|^r dx \Big), \text{ if } \alpha = 0,\ \beta = +\infty,$$

$$\min_{u\in BV(\Omega)} \Big(\int_\Omega f_\alpha(|\nabla u|) dx + b_\alpha |D^s u|(\Omega) + \int_\Omega |u-g|^r dx \Big),$$

$$\text{if } 0 < \alpha < +\infty,\ \beta = +\infty.$$

Moreover for every α and β the minimum values of (2.102) tend to the minimum value of the limit problem.

In order to prove Theorem 2.7 we need a compactness result, whose proof makes use of Theorem 1.9 about the compactness of sequences and slices. Our compactness result is then given by the following theorem.

Lemma 2.8. *Let $\alpha > 0$ or $\beta < +\infty$. Let (u_k, v_k) be a sequence in $L^1(\Omega) \times L^1(\Omega)$ such that (u_k) is bounded in $L^1(\Omega)$ and*

$$\liminf_{k\to+\infty} F_k(u_k, v_k) < +\infty.$$

Then there exists a subsequence (u_j, v_j) of (u_k, v_k) and a function $u \in GSBV(\Omega) \cap L^1(\Omega)$ such that $u_j \to u$ $\mathcal{L}^n$-a.e. on Ω and $v_j \to 1$ in $L^1(\Omega)$.

If $0 < \alpha < +\infty$ *and* $\beta = +\infty$*, or* $\alpha = +\infty$*, the convergence* $u_j \to u$ *is also in* $L^1(\Omega)$*.*

The previous lemma does not apply when $\alpha = 0$ and $\beta = +\infty$, but we shall be able to prove Theorem 2.7 also in this case.

Proof. We can suppose, up to subsequences, that $F_k(u_k, v_k)$ is bounded by a constant $M < +\infty$; in particular then $v_k \to 1$ in $L^1(\Omega)$. Now let $0 \le \alpha < +\infty$ and $0 \le \beta < +\infty$.

We divide the proof into three steps.

The bounded case for $n = 1$. Let $n = 1$ and let (u_k) be bounded in $L^\infty(\Omega)$. It is not restrictive to assume $\Omega = (0, 1)$; if this is not the case we prove the statement for each connected component and then we use a diagonal argument.

Repeating the first part of the proof of Theorem 2.2 in the case $n = 1$, we can find $m + 1$ points $0 = x_0 < \cdots < x_m = 1$ such that ∇u_k is bounded in $L^2(x_i + \mu, x_{i+1} - \mu)$ uniformly with respect to k, $\mu > 0$, and $i = 0, \ldots, m - 1$. This implies by assumption that u_k is bounded in $H^1(x_i + \mu, x_{i+1} - \mu)$ uniformly with respect to k, μ, and i. For every $\mu > 0$, we can find a subsequence of (u_k), not relabeled, that converges in $L^2(x_i + \mu, x_{i+1} - \mu)$, for $i = 0, \ldots, m - 1$. Then by a diagonal argument we extract a further subsequence (u_j) of (u_k) that converges in $L^1(\Omega)$ to some $u \in L^\infty(\Omega)$. From this convergence and from Proposition 2.4 we also deduce $u \in SBV^2(\Omega)$.

The bounded case for $n > 1$. Let $n > 1$ and let (u_k) be bounded in $L^\infty(\Omega)$.

Let $\xi \in \mathbb{R}^n$ be a unit vector and let $F_{y,k}, V_{y,k}$ be defined as in (2.67) and (2.68) in the case $1 < p < +\infty$ (obvious modification can be provided to prove the case $p = +\infty$).

Moreover we set

$$A_k := \{y \in \Pi^\xi : F_{y,k}((u_k)^\xi_y, (v_k)^\xi_y) \le L\},$$

where L is a fixed constant, so that by the Chebyshev inequality we obtain

$$\mathcal{H}^{n-1}((A_k)^c) \le \frac{M}{L}.$$

Let $\delta > 0$; we can choose L so that $\mathrm{diam}(\Omega) cM/L < \delta$, with $c := \sup_k ||u_k||_{L^\infty}$. Let us define

$$(w_k)^\xi_y(t) := \begin{cases} (u_k)^\xi_y & \text{if } y \in A_k, \\ 0 & \text{otherwise} \end{cases}$$

and let $w_k(y+t\xi) := (w_k)^\xi_y(t)$, for $y \in \Pi^\xi$ and $t \in \Omega^\xi_y$. Then

$$||w_k - u_k||_{L^1(\Omega)} \le c \operatorname{diam}(\Omega)\mathcal{H}^{n-1}((A_k)^c) < \delta.$$

Let $\mathcal{F} := (u_k)$ and $\mathcal{F}_\delta := (w_k)$, then $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}_\delta$ with respect to the $L^1(\Omega)$ distance; moreover $\mathcal{F}_\delta$ is pre-compact by the first part of the proof. From Theorem 1.9, we deduce the existence of a function $u \in L^\infty(\Omega)$ and of a subsequence (u_j, v_j) of (u_k, v_k) such that $(u_j, v_j) \to (u, 1)$ in $L^1(\Omega)\times L^1(\Omega)$ and $||u||_{L^\infty(\Omega)} \le c$. Since

$$F'_{\alpha,\beta}(u,1) \le \lim_{j\to\infty} F_j(u_j, v_j) \le M,$$

by Theorem 2.1 we conclude $u \in GSBV^2(\Omega)\cap L^\infty(\Omega)$, i.e., $u \in SBV^2(\Omega)\cap L^\infty(\Omega)$.

The general case. For every $\mu \in \mathbb{N}$ we can consider $u^\mu_k := (-\mu \vee u_k)\wedge \mu$, then

$$F_k(u^\mu_k, v_k) \le F_k(u_k, v_k)$$

and by the first part of the proof there exists a subsequence (u^μ_j) of (u^μ_k) and a function $u_\mu \in SBV^2(\Omega)\cap L^\infty(\Omega)$, with $||u_\mu||_{L^\infty(\Omega)} \le \mu$, such that $u^\mu_j \to u_\mu$ in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω. This implies that the complement of the set

$$A := \big\{x \in \Omega : (u^\mu_j(x)) \text{ converges for every } \mu \in \mathbb{N}\big\}$$

is negligible. Let us observe that

$$\big(u_\mu(x)\big)^\lambda = \lim_{j\to\infty} \big(u^\mu_j(x)\big)^\lambda = \lim_{j\to\infty} u^\lambda_j(x) = u_\lambda(x) \quad \text{for every } \mu > \lambda. \tag{2.103}$$

We claim that the subset of A

$$E := \big\{x \in A : |u_\lambda(x)| = \lambda \text{ for every } \lambda \in \mathbb{N}\big\}$$

has measure zero. Indeed, for every $\lambda \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$\mathcal{L}^n(E) \le \mathcal{L}^n\big(\big\{|u^\lambda_j| > \lambda - \varepsilon\big\}\big) \le \frac{1}{\lambda-\varepsilon}\int_\Omega |u_j|dx \le \frac{c}{\lambda-\varepsilon}$$

for j large enough, where c is the bounding constant of (u_j) in $L^1(\Omega)$; as $\varepsilon \to 0$ and $\lambda \to \infty$ we obtain $\mathcal{L}^n(E) = 0$. Let now $x \in A \setminus E$, so that there exists $\lambda \in \mathbb{N}$ with $|u_\lambda(x)| < \lambda$; this condition, together with equalities (2.103) gives $u_\mu(x) = u_\lambda(x)$ for every $\mu > \lambda$.

Let us define for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$u(x) := \lim_{\lambda \to \infty} u_\lambda(x),$$

then by (2.103) u_λ coincides with the truncated function u^λ $\mathcal{L}^n$-a.e. in Ω. This implies that $u_j \to u$ $\mathcal{L}^n$-a.e. in Ω; since (u_λ) is contained in $SBV(\Omega)$ we deduce that $u \in GSBV(\Omega)$. Finally, since u_j^λ is uniformly bounded in $L^1(\Omega)$ with respect to λ and j, we also conclude that $u \in L^1(\Omega)$.

Let $\alpha = +\infty$. Repeating the computation in (2.21) we deduce by assumptions that (u_k) is bounded in $BV(\Omega)$. This implies the existence of a function u to which u_k converges in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω, up to subsequences. The same argument works in the case $0 < \alpha < +\infty$, $\beta = +\infty$. □

To prove Theorem 2.7 we shall consider the functionals $F_{r,k}\colon L^r(\Omega)\times L^1(\Omega) \to [0, +\infty]$ defined by

$$F_{r,k}(u, v) := F_k|_{L^r(\Omega)\times L^1(\Omega)},$$

where the functionals (F_k) are defined in Section 2.2.

The second step in the proof of Theorem 2.7 is the following lemma.

Lemma 2.9. *Under the hypotheses of Theorem 2.7, the functionals $F_{r,k}$ Γ-converge in $L^r(\Omega)\times L^1(\Omega)$ to the functional $F_{r,\alpha,\beta} := F_{\alpha,\beta}|_{L^r(\Omega)\times L^1(\Omega)}$, where $F_{\alpha,\beta}$ is defined in Section 2.2.*

Proof. Let $F'_{r,\alpha,\beta}$ and $F''_{r,\alpha,\beta}$ be the Γ-lim inf and the Γ-lim sup of $F_{r,k}$ in the space $L^r(\Omega)\times L^1(\Omega)$ and let $(u, v) \in L^r(\Omega)\times L^1(\Omega)$.

Proof of the estimate from below. The Γ-lim inf inequality follows from $F'_{r,\alpha,\beta} \geq F'_{\alpha,\beta}$ (see, for instance, [24, Proposition 6.3]) and from Theorem 2.1.

Proof of the estimate from above. Let $u \in GSBV^2(\Omega) \cap L^r(\Omega)$ with $F_{\alpha,\beta}(u, 1) < +\infty$. First we suppose $u \in L^\infty(\Omega)$. Theorem 2.3 ensures the existence of a sequence $(u_k, v_k) \in H^1(\Omega)\times V_k(\Omega)$ such that $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega)\times L^1(\Omega)$ and

$$\lim_{k\to\infty} F_k(u_k, v_k) = F_{\alpha,\beta}(u, 1).$$

The Γ-lim sup inequality follows from this equality, from the convergence of the truncated functions $u_k^M \to u$ in $L^r(\Omega)$ with $M := ||u||_{L^\infty(\Omega)}$, and from the fact that $F_{r,k}(u_k^M, v_k) \leq F_k(u_k, v_k)$.

In the general case when $u \notin L^\infty(\Omega)$ the Γ-lim sup inequality follows from the previous step applied to the truncated function u^M, from the lower semicontinuity of $F''_{r,\alpha,\beta}$ and from the fact that $F_{\alpha,\beta}(u^M, 1) \leq F_{\alpha,\beta}(u, 1)$. □

Let us define the sequence of functionals

$$G_k(u,v) := F_k(u,v) + \int_\Omega |u-g|^r dx, \tag{2.104}$$

$$G_{\alpha,\beta}(u,v) := F_{\alpha,\beta}(u,v) + \int_\Omega |u-g|^r dx, \tag{2.105}$$

where $u, v \in L^1(\Omega)$.

Lemma 2.10. *Let $1 \le r < +\infty$ and let $g \in L^r(\Omega)$. Then the functionals G_k in (2.104) Γ-converge in $L^1(\Omega)\times L^1(\Omega)$ to the functional $G_{\alpha,\beta}$ in (2.105).*

Proof. Let $G'_{\alpha,\beta}$ and $G''_{\alpha,\beta}$ be the Γ-$\liminf$ and the Γ-$\limsup$ of G_k in the space $L^1(\Omega)\times L^1(\Omega)$. First we observe that the functional $H : L^1(\Omega)\times L^1(\Omega) \to [0,+\infty]$ defined by

$$H(u,v) := \int_\Omega |u-g|^r dx$$

is lower semicontinuous.

In the case $r = 1$ the functional H is continuous; since (F_k) Γ-converges to $F_{\alpha,\beta}$ by Theorem 2.1, we can apply [24, Proposition 6.21] about the sum of Γ-limits to conclude that G_k Γ-converges to $F_{\alpha,\beta} + H$.

Let $r > 1$. Since H is not continuous, we need a different argument. To this aim we introduce $G''_{r,\alpha,\beta}$, the Γ-$\limsup$ of G_k in $L^r(\Omega)\times L^1(\Omega)$.

If $(u,v) \in (L^1(\Omega) \setminus L^r(\Omega))\times L^1(\Omega)$ we obtain by [24, Proposition 6.17]

$$+\infty = F_{\alpha,\beta}(u,v) + H(u,v) \le G'_{\alpha,\beta}(u,v);$$

let now $(u,v) \in L^r(\Omega)\times L^1(\Omega)$. By [24, Proposition 6.3, 6.17, and 6.21], by Theorem 2.1, and by Lemma 2.9 we can deduce that

$$\begin{aligned} F_{\alpha,\beta}(u,v) + H(u,v) &\le G'_{\alpha,\beta}(u,v) \le G''_{\alpha,\beta}(u,v) \le G''_{r,\alpha,\beta}(u,v) \\ &= F_{r,\alpha,\beta}(u,v) + H(u,v) = F_{\alpha,\beta}(u,v) + H(u,v), \end{aligned}$$

so that the functionals G_k Γ-converge to the functional $G_{\alpha,\beta}$. □

Remark 2.11. In Theorem 2.7 we assume $\eta_k > 0$ only to guarantee the existence of a minimum point for G_k. In the case $\eta_k \ge 0$, the thesis of Theorem 2.7 continues to hold if (u_k, v_k) is a sequence which satisfies

$$\lim_{k\to\infty} G_k(u_k, v_k) - \inf_{L^r(\Omega)\times L^1(\Omega)} G_k = 0.$$

The proof is essentially the same.

We are now in a position to prove Theorem 2.7.

Proof. We fix k and prove that each functional G_k, defined in (2.104), attains its minimum. Let (u_j, v_j) be a sequence such that

$$\lim_{j\to\infty} G_k(u_j, v_j) = \inf_{L^r(\Omega)\times L^1(\Omega)} G_k.$$

Since $(G_k(u_j, v_j))$ is bounded, from the definition of G_k we deduce $(u_j, v_j) \in H^1(\Omega)\times V_k$. In particular (u_j) is bounded in $L^r(\Omega)$ and (∇u_j) is bounded in $L^2(\Omega, \mathbb{R}^n)$; this implies that (u_j) is bounded in $H^1(\Omega)$.

Then we can find a function $u \in H^1(\Omega) \cap L^r(\Omega)$ and a subsequence of (u_j), not relabeled, such that $u_j \rightharpoonup u$ weakly in $H^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω. From the boundedness of (v_j) in $W^{1,p}(\Omega)$ we can deduce the existence of a function $v \in W^{1,p}(\Omega)$, with $\eta_k \le v \le 1$ and of a subsequence of (v_j), not relabeled, such that $v_j \to v$ in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω. By [19, Theorem 2.3.1] and by the Fatou lemma, this implies that the estimates

$$\int_\Omega |\nabla u|^2 v dx \le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 v_j dx, \quad \int_\Omega |u-g|^r dx \le \liminf_{j\to\infty} \int_\Omega |u_j - g|^r dx \tag{2.106}$$

hold, so that we obtain

$$G_k(u, v) \le \lim_{j\to\infty} G_k(u_j, v_j) = \inf_{L^r(\Omega)\times L^1(\Omega)} G_k.$$

This shows that the infimum of G_k is achieved.

Let $\alpha > 0$ or $\beta < +\infty$ and let (u_k, v_k) be a minimizer of G_k, which obviously belongs to $H^1(\Omega)\times V_k$. Since the sequence $(F_k(u_k, v_k))$ is bounded, by the compactness theorem 2.8 there exists a function $u \in GBV(\Omega) \cap L^r(\Omega)$ and a subsequence of (u_k, v_k), not relabeled, such that $u_k \to u$ $\mathcal{L}^n$-a.e. in Ω and $v_k \to 1$ in $L^1(\Omega)$. Let us prove that $u_k \to u$ in $L^1(\Omega)$. By the the dominated convergence theorem we get $\int_\Omega |u_k - u| 1_{B_k^c} dx \to 0$, where $B_k := \{|u_k - u| > 1\}$; moreover using the Hölder inequality we obtain

$$\int_{B_k} |u_k - u| dx \le \Big(||u_k - g||_{L^r(\Omega)} + ||u - g||_{L^r(\Omega)}\Big)\mathcal{L}^n(B_k)^{1-\frac{1}{r}} \le 2||g||_{L^r(\Omega)}\mathcal{L}^n(B_k)^{1-\frac{1}{r}},$$

where the last inequality follows from the estimate $G_k(u_k, v_k) \le G_k(0, 1) = ||g||^r_{L^r(\Omega)}$ and from (2.106). Since $u_k \to u$ in measure we conclude that $\mathcal{L}^n(B_k) \to 0$ and the convergence $u_k \to u$ in $L^1(\Omega)$ follows.

By the Γ-convergence of G_k to $G_{\alpha,\beta}$ (Lemma 2.10) and by a general property of Γ-convergence (see Section 1.8), we find that $(u, 1)$ is a minimizer for $G_{\alpha,\beta}$. Moreover we have the convergence of minimum values and the convergence of minimizers in $L^1(\Omega)\times L^1(\Omega)$.

Let us prove now that $u_k \to u$ in $L^r(\Omega)$, up to subsequences. Since

$$F_{\alpha,\beta}(u,1) + \int_\Omega |u-g|^r dx = \lim_{k\to\infty} \Big(F_k(u_k,v_k) + \int_\Omega |u_k-g|^r dx \Big),$$

$$F_{\alpha,\beta}(u,1) \le \liminf_{k\to\infty} F_k(u_k,v_k), \quad \text{and} \quad \int_\Omega |u-g|^r dx \le \liminf_{k\to\infty} \int_\Omega |u_k-g|^r dx,$$

we obtain

$$\int_\Omega |u-g|^r dx = \lim_{k\to\infty} \int_\Omega |u_k-g|^r dx. \tag{2.107}$$

This fact, together with the $\mathcal{L}^n$-a.e. convergence in Ω of $u_k - g$ to $u - g$, implies that $u_k \to u$ in $L^r(\Omega)$ by the generalized dominated convergence theorem.

We suppose now that $\alpha = 0$ and $\beta = +\infty$. We fix k and we consider a minimizer $(u_k,v_k) \in H^1(\Omega)\times V_{\eta_k}$ of G_k. Since $G_k(u_k,v_k)$ is bounded, we can find a subsequence of u_k, not relabelled, and a function $u \in L^r(\Omega)$ to which u_k converges weakly in $L^r(\Omega)$. Therefore we have

$$\int_\Omega |u-g|^r dx \le \liminf_{k\to+\infty} \int_\Omega |u_k-g|^r dx \le \liminf_{k\to+\infty} G_k(\tilde{u}_k,\tilde{v}_k) = \int_\Omega |u-g|^r dx,$$

where we have chosen $(\tilde{u}_k,\tilde{v}_k) \to (u,1)$ in $L^1(\Omega)\times L^1(\Omega)$ with $\lim_{k\to+\infty} G_k(\tilde{u}_k,\tilde{v}_k) = G_{\alpha,\beta}(u,1)$.

Since now $u_k - g \rightharpoonup u - g$ weakly in $L^r(\Omega)$ and $||u_k-g||_{L^r(\Omega)} \to ||u-g||_{L^r(\Omega)}$ we also conclude that $u_k \to u$ strongly in $L^r(\Omega)$.

Again by the Γ-convergence of G_k to $G_{\alpha,\beta}$ and [24, Corollary 7.20], we find that $(u,1)$ is a minimizer for $G_{\alpha,\beta}$, so that $u = g$ $\mathcal{L}^n$-a.e. in Ω. □

Chapter 3

A density result for the space of Generalised Special Functions of Bounded Deformation

3.1 Overview of the chapter

The space of Generalised Special Functions with Bounded Deformation has been recently introduced in [25] as the natural functional framework for weak formulations of variational problems arising in fracture mechanics in the setting of linearized elasticity. Roughly speaking, it provides the natural completion of SBD when no uniform bounds in L^∞ can be assumed for the problem at hand, analogously to SBV and its counterpart $GSBV$. For preliminary results and notation about $GSBD$-functions we refer to [25] and to Sections 1.5 and 1.6.

In this chapter we present an approximation result for functions $u\colon \Omega \to \mathbb{R}^n$ belonging to the space $GSBD(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$ with $e(u)$ square integrable and $\mathcal{H}^{n-1}(J_u)$ finite. The approximating functions u_k are piecewise continuous functions such that $u_k \to u$ in $L^2(\Omega, \mathbb{R}^n)$, $e(u_k) \to e(u)$ in $L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$, $\mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \to 0$, and $\int_{J_{u_k}\cup J_u} |u_k^\pm - u^\pm| \wedge 1 \, d\mathcal{H}^{n-1} \to 0$. Two applications of this result to the Ambrosio-Tortorelli convergence will be presented in the next chapter.

The chapter is composed of four sections. In Section 3.2 we state the density theorem, which is the main result of the chapter. Following the approach used by Chambolle in [20, 21] for the SBD context, we divide the proof into three steps. The first step is faced in Section 3.3, where a first unified approximation of the energies with bad constants is provided. The second step and the third step are described in Section 3.4. The former consists in proving a further unified approximation for the energies with the right constants, the latter in the application of the Compactness Theorem 1.12 for $GSBD$.

The results presented in this chapter will appear in [38].

3.2 The density theorem

Let us assume $n \geq 2$. In this section we present the main result of the chapter: the approximation theorem for $GSBD$ functions.

Theorem 3.1 (Density)**.** *Assume that Ω has Lipschitz boundary. Let u belong to $GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$. Then there exists a sequence $(u_k) \subset SBV^2(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ such that each J_{u_k} is contained in the union S_k of a finite number of closed connected pieces of C^1-hypersurfaces, each u_k belongs to $W^{1,\infty}(\Omega \setminus S_k, \mathbb{R}^n)$, and the following properties hold:*

(1) $||u_k - u||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,

(2) $||e(u_k) - e(u)||_{L^2(\Omega,\mathbb{M}^{n\times n}_{sym})} \to 0$,

(3) $\mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \to 0$,

(4) $\displaystyle\int_{J_{u_k}\cup J_u} |u_k^\pm - u^\pm| \wedge 1 \, d\mathcal{H}^{n-1} \to 0.$

We remark that Theorem 3.1 can be combined with the SBV density theorem by Cortesani and Toader [23, Theorem 3.1] (see also [22] and Section 1.7) to obtain better approximating functions.

A useful tool for the proof of Theorem 3.1 is the following lemma, which allows us to substitute a $GSBD^2$-function with another function of the same type, defined in a larger set, in a way that the norm of the function and of its approximate symmetric gradient, the measure of the jump set, and the trace on $\partial\Omega$ do not increase too much.

Lemma 3.2. *Assume that Ω has Lipschitz boundary. Let $Q \colon \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ be a positive definite quadratic form and let $u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$. Then for every $\varepsilon > 0$ we can find a Lipschitz open set $\hat{\Omega}$ with $\Omega \subset\subset \hat{\Omega}$, and a function $\hat{u} \in GSBD^2(\hat{\Omega}) \cap L^2(\hat{\Omega}, \mathbb{R}^n)$, such that*

(1) $||\hat{u} - u||_{L^2(\Omega,\mathbb{R}^n)} < \varepsilon$,

(2) $\displaystyle\int_{\hat{\Omega}} Q(e(\hat{u}))\, dx \leq \int_\Omega Q(e(u))\, dx + \varepsilon$,

(3) $\mathcal{H}^{n-1}(J_{\hat{u}}) \leq \mathcal{H}^{n-1}(J_u) + \varepsilon$,

(4) $\mathcal{H}^{n-1}(J_{\hat{u}} \cap \partial\Omega) = 0$,

(5) $\int_{\partial\Omega} |\hat{u} - tr(u)| \wedge 1 \, d\mathcal{H}^{n-1} < \varepsilon$.

Proof. For the first three properties of the lemma we follow the proof of [20, Lemma 3.2] and we only summarize the essential lines. Property (4) will be an easy consequence of a well-known result in Measure Theory. Eventually, property (5) will be obtained through Theorem 1.8.

Since Ω has Lipschitz boundary, we can cover $\partial\Omega$ with open sets $(A_i)_{i=1}^k$, in a way that each $A_i \cap \Omega$ is the subgraph of a Lipschitz function $f_i \colon \Pi^{\xi_i} \to \mathbb{R}$, for a suitable $\xi_i \in \mathbb{S}^{n-1}$. Then we consider an open set $A_0 \subset\subset \Omega$, such that $\overline{\Omega} \subset \bigcup_{i=0}^k A_i$.

We define

$$u_t^0 := u \quad \text{in } A_0$$
$$u_t^i(x) := u(x - t\,\xi_i) \quad \text{for } x \in A_i \cap (\Omega + [0,t)\xi_i),$$

for t small enough; we extend u_t^i by 0 in the rest of A_i.

Clearly we are going to glue the functions u_t^i together through a partition of unity, but the choice of the partition has to be done properly in view of property (3).

We choose a partition of unity $(\varphi_i)_{i=0}^k$ subordinate to $(A_i)_{i=1}^k$ in a way that $\sum_{i=0}^k \varphi_i = 1$ on $\overline{\Omega}$ and

$$\mathcal{H}^{n-1}\Big(J_u \cap \bigcup_{i=0}^k \overline{\{0 < \varphi_i < 1\}}\Big) \leq \frac{\varepsilon}{2(k+1)}; \tag{3.1}$$

this is possible through [20, Lemma 3.3] applied to the positive Borel measure $\mathcal{H}^{n-1} \lfloor J_u$, which is finite on $\mathbb{R}^n$. We set

$$u_{\bar{t}} := \sum_{i=0}^k u_{t_i}^i \varphi_i \qquad \text{and} \qquad \Omega_{\bar{t}} := A_0 \cup \Big(\bigcup_{i=1}^k (A_i \cap (\Omega + [0,t_i)\xi_i))\Big),$$

where we have set $\bar{t} = (t_1, \dots, t_k)$ and each t_i is small. Arguing as in [20, Lemma 3.2] we prove that the pair $(u_{\bar{t}}, \Omega_{\bar{t}})$ satisfies properties (1)–(3) for $\bar{t}$ small enough.

Proof of (4). Let us fix $i = 1, \dots, k$, then for every $t \in \mathbb{R}$ we have

$$\mathcal{H}^{n-1}(J_{u_t^i} \cap \partial\Omega) = \mathcal{H}^{n-1}(J_{u_t^i} \cap A_i \cap \partial\Omega) = \mathcal{H}^{n-1}(J_u \cap ((A_i \cap \partial\Omega) - t\xi_i)). \tag{3.2}$$

Since the measure $\mathcal{H}^{n-1} \lfloor J_u$ is finite, a classical result of measure theory implies that the pairwise disjoint Borel sets $((A_i \cap \partial\Omega) - t\xi_i)_t$ are $\mathcal{H}^{n-1} \lfloor J_u$-negligible, except for a countable set of indices $t \in \mathbb{R}$. This proves that $u_{\bar{t}}$ also satisfies property (4) for $\mathcal{L}^k$-a.e. $\bar{t} \in \mathbb{R}^k$.

Proof of (5). First we note that

$$\int_{\partial\Omega} \tau(|\mathrm{tr}(u_{\bar{t}}) - \mathrm{tr}(u)|) d\mathcal{H}^{n-1} \leq \sum_{i=1}^{k} \int_{\partial\Omega \cap \{\varphi_i \neq 0\}} \tau(|\mathrm{tr}(u^i_{t_i}) - \mathrm{tr}(u)|) d\mathcal{H}^{n-1},$$

where $\tau(s) := \frac{1}{\pi} \mathrm{arctg}\,(s)$ for $s \in \mathbb{R}$. Let us fix $i = 1, \ldots, k$ and let us define $M := \partial\Omega \cap \{\varphi_i \neq 0\}$. Let $\Omega_1 \subset\subset A_i$ be such that $\partial\Omega_1$ is smooth, $M \subset\subset (\Omega_1 \cap \partial\Omega)$, and $\mathcal{H}^{n-1}(\partial\Omega_1 \cap J_u) = 0$.

We aim to apply Theorem 1.8 to the functions $u^i_{t_i}, u$ on the set $\Omega_1 \cap \Omega$. Clearly we have $u^i_{t_i} \to u$ in $L^1(\Omega_1 \cap \Omega, \mathbb{R}^n)$ and $e(u^i_{t_i}) \to e(u)$ in $L^1(\Omega_1 \cap \Omega, \mathbb{R}^n)$ by the L^1-continuity of the translations. It remains to check that

$$\int_{J_{u^i_{t_i}} \cap \Omega_1 \cap \Omega} \psi d\mathcal{H}^{n-1} \to \int_{J_u \cap \Omega_1 \cap \Omega} \psi d\mathcal{H}^{n-1}, \tag{3.3}$$

for every $\psi \in C^0_b(\Omega_1 \cap \Omega)$. Fixed $\psi \in C^0_b(\Omega_1 \cap \Omega)$, one easily shows that

$$\psi(x + t_i\xi_i)\chi_{\Omega_1 \cap \Omega}(x + t_i\xi_i) \to \psi(x)\chi_{\Omega_1 \cap \Omega}(x)$$

when $x \in J_u \setminus \partial\Omega_1$. By our assumptions on Ω_1 we find that $\mathcal{H}^{n-1}$-a.e. $x \in J_u$ is out of $\partial\Omega_1$. By the dominated convergence theorem we eventually obtain (3.3) and finally Theorem 1.8 gives the continuity of the trace. We conclude that there exists $\bar{t}$ small enough such that properties (1)–(5) hold for the pair $(u_{\bar{t}}, \Omega_{\bar{t}})$. □

3.3 A first unified approximation of the energies with bad constants

The proof of Theorem 3.1 is quite technical, so we break it into three steps. The first step is the following theorem, which will give a rough and unified approximation of the energies.

Theorem 3.3. *Assume that Ω has Lipschitz boundary and let $u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$. Then there exists a sequence $(u_k) \subset SBV^2(\Omega, \mathbb{R}^n) \cap L^2(\Omega, \mathbb{R}^n)$ such that J_{u_k} is contained in the union Σ_k of a finite number of $(n-1)$-dimensional closed cubes, $u_k \in W^{1,\infty}(\Omega \setminus \Sigma_k, \mathbb{R}^n)$, and the following properties hold:*

(1) $||u_k - u||_{L^2(\Omega, \mathbb{R}^n)} \to 0$,

(2) $\limsup_{k\to+\infty}\Big(\int_\Omega Q_n(e(u_k))\,dx + \mathcal{H}^{n-1}(\Sigma_k)\Big) \le \int_\Omega Q_n(e(u))\,dx + c_1\mathcal{H}^{n-1}(J_u)$.
Here c_1 is a positive constant depending only on the dimension n and Q_n is the positive definite quadratic form on $\mathbb{M}^{n\times n}_{sym}$ defined by

$$Q_n(A) := \frac{3(n-2)}{2}\sum_{i=1}^n a_{i,i}^2 + Tr(AA^t) + \frac{1}{2}(Tr(A))^2, \quad \text{for } A \in \mathbb{M}^{n\times n}_{sym}, \quad (3.4)$$

where $Tr(A)$ denotes the trace of the matrix A;

(3) $\int_{\partial\Omega} |tr(u_k) - tr(u)| \wedge 1\, d\mathcal{H}^{n-1} \to 0$,

(4) *if $(\Gamma_i)_{i=1}^{+\infty}$ is a fixed sequence of C^1-manifolds contained in Ω, then (u_k) can be chosen such that also $\mathcal{H}^{n-1}(\Sigma_k \cap \Gamma_i) = 0$, for $i = 1, \dots, +\infty$.*

Proof. We follow the lines of [20, Proof of Theorem 1]. We first substitute the function u with a similar function $\hat{u}$ defined on a larger set $\hat{\Omega}$. Then we discretize $\hat{u}$ on a suitable lattice and interpolate it with a continuous function. Finally the approximating function will be obtained redefining the interpolating function on some cubes of the lattice which intersect $J_{\hat{u}}$.

Let $u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$, let $\varepsilon > 0$, and let $\hat{u}$ and $\hat{\Omega}$ as in Lemma 3.2. By Lemma 1.4 we can find a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ such that, for every vector e in the set

$$D := \{e_i,\ i = 1, \dots, n,\ e_i \pm e_j,\ 1 \le i < j \le n\},$$

one has

$$\mathcal{H}^{n-1}(\{x \in J_{\hat{u}} : [\hat{u}](x)\cdot e = 0\}) = 0.$$

For each small discretization step $h > 0$ and for each $y \in [0,1)^n$, we define the discretized function of $\hat{u}$

$$\hat{u}_h^y(\xi) := \hat{u}(hy + \xi), \quad \text{for } \xi \in h\mathbb{Z}^n \cap (\hat{\Omega} - hy).$$

We also define the continuous interpolation of $\hat{u}_h^y$

$$w_h^y(x) := \sum_{\xi \in h\mathbb{Z}^n \cap \hat{\Omega}} \hat{u}_h^y(\xi)\Delta\Big(\frac{x - (\xi + hy)}{h}\Big) \quad \text{for } x \in \Omega,$$

where

$$\Delta(x) := \prod_{i=1}^n (1 - |x_i|)^+.$$

We note that $w_h^y \in W^{1,\infty}(\Omega,\mathbb{R}^n)$. In view of the definition of the discrete energies we introduce

$$J^\tau := \bigcup_{x\in J_{\hat u}} [x, x-\tau] \quad \text{for } \tau \in \mathbb{R}^n,$$
$$l^y_{e,h}(\xi) := \chi_{J^{he}}(hy+\xi) \quad \text{for } \xi \in h\mathbb{Z}^n \text{ and } e \in D.$$

In what follows ξ is intended to belong to $h\mathbb{Z}^n$.

We are now in a position to define the discrete energies

$$E_1^{y,h}(\hat\Omega) := h^n \sum_{e\in D} \sum_{\substack{\xi\in\hat\Omega-hy \\ \xi\in\hat\Omega-hy-he}} \alpha(e) \frac{((\hat u_h^y(\xi+he)-\hat u_h^y(\xi))\cdot e)^2}{h^2}\Big(1-l^y_{e,h}(\xi)\Big), \tag{3.5}$$

$$E_2^{y,h}(\hat\Omega) := \tilde c_1 h^n \sum_{e\in D} \sum_{\substack{\xi\in\hat\Omega-hy \\ \xi\in\hat\Omega-hy-he}} \frac{l^y_{e,h}(\xi)}{|e|h}, \tag{3.6}$$

where $(\alpha(e))_{e\in D}$ are positive parameters, chosen in a way that we shall be able to keep the constant 1 for the bulk term in estimate (2). Precisely, we define $\alpha(e) := n-1$ if $e=e_i$, for $i=1,\dots,n$ and $\alpha(e) := 1/4$ for $1\le i<j\le n$. Moreover $\tilde c_1$ is a constant depending only on the dimension n which will be chosen later. We also set $\hat e := e/|e|$.

The first part of the proof is devoted to the choice of a suitable $y\in[0,1)^n$, and a suitable subsequence of h, not relabelled, such that the following properties hold:

(1') $||w_h^y - \hat u||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,

(2') $\lim_{h\to+\infty} \Big[E_1^{y,h}(\hat\Omega)+E_2^{y,h}(\hat\Omega)\Big] \le \int_{\hat\Omega} Q_n(e(\hat u))dx + c_1\mathcal{H}^{n-1}(J_{\hat u})$, where $c_1<+\infty$ depends on $\tilde c_1$,

(3'a) $\int_{\partial\Omega} |w_h^y-\hat u|\wedge 1\, d\mathcal{H}^{n-1} \to 0$,

(3'b) $E_2^{y,h}((\partial\Omega)_{nh}) \to 0$. Here $(\partial\Omega)_{nh} := \{x\in\mathbb{R}^n : \mathrm{d}(x,\partial\Omega) < nh\}$ and the expression $E_2^{y,h}((\partial\Omega)_{nh})$ means that $(\partial\Omega)_{nh}$ replaces $\hat\Omega$ in the definition (3.6);

(4') if $(\Gamma_i)_{i=1}^{+}\infty$ is a fixed sequence of C^1-manifold contained in Ω, then y and the subsequence of h can be chosen such that also $\mathcal{H}^{n-1}((hy+h\mathbb{Z}^n+[0,h)e_j)\cap\Gamma_i)=0$, for $i=1,\dots,+\infty$ and $j=1,\dots,n$.

The first part of the proof (properties (1') and (2')) is analogous to that in [20, Theorem 1]. We summarize it for completeness and for future convenience.

Proof of (1'). By the very definition of w_h^y, the Fubini Theorem, and a change of variable we find

$$
\begin{aligned}
&\int_{[0,1)^n} dy \int_\Omega |w_h^y(x) - \hat{u}(x)|^2 \, dx \leq \\
\leq \; & \int_{[0,1)^n} dy \int_\Omega \sum_{\xi \in h\mathbb{Z}^n \cap \hat{\Omega}} \Delta\Big(\frac{x - (\xi + hy)}{h}\Big) |\hat{u}(\xi + hy) - \hat{u}(x)|^2 \, dx \\
\leq \; & \sum_{\xi \in h\mathbb{Z}^n \cap \hat{\Omega}} \int_\Omega dx \int_{\frac{x-\xi}{h} - [0,1)^n} \Delta(z) |\hat{u}(x - hz) - \hat{u}(x)|^2 \, dz \\
\leq \; & \int_{(-1,1)^n} \Delta(z) dz \int_\Omega |\hat{u}(x - hz) - \hat{u}(x)|^2 \, dx
\end{aligned}
$$

where to infer the last inequality we notice that the sets $\frac{x-\xi}{h} - [0,1)^n$ are pairwise disjoint as ξ varies in $h\mathbb{Z}^n \cap \hat{\Omega}$. The last term in the previous inequality converges to 0 by the dominated convergence theorem. Then property (1') is satisfied for a subsequence of h, not relabelled, and for y varying in a subset of $[0,1)^n$ with full measure.

Proof of (2'). Let us estimate

$$
\int_{[0,1)^n} E_j^{y,h}(\hat{\Omega}) dy, \tag{3.7}
$$

for $j = 1, 2$. For convenience we introduce $I_z^e := \{s \in \mathbb{R} : z + s\hat{e} \in \hat{\Omega}\}$ and $I_{z,h}^e := \{s \in \mathbb{R} : z + s\hat{e} \in \hat{\Omega}, z + (s + h|e|)\hat{e} \in \hat{\Omega}\}$. First a change of variable gives

$$
\begin{aligned}
&\int_{[0,1)^n} E_1^{y,h}(\hat{\Omega}) \, dy = \\
= \; & \sum_{e \in D} \alpha(e) \sum_{\xi \in h\mathbb{Z}^n} \int_{\xi + h[0,1)^n} \chi_{\hat{\Omega} \cap (\hat{\Omega} - he)}(x) \frac{|(\hat{u}(x + he) - \hat{u}(x)) \cdot e|^2}{h^2} (1 - \chi_{J^{he}}(x)) \, dx \\
= \; & \sum_{e \in D} \alpha(e) \int_{\Pi^e} dz \int_{I_{z,h}^e} \frac{|\hat{u}_z^e(s + h|e|) - \hat{u}_z^e(s)|^2}{h^2} (1 - \chi_{J^{he}}(z + s\hat{e})) \, ds.
\end{aligned} \tag{3.8}
$$

As in the SBD-case [20], when $\hat{u} \in GSBD^2(\hat{\Omega}) \cap L^2(\hat{\Omega}, \mathbb{R}^n)$ the slice $\hat{u}_z^e(s) := \hat{u}(z + s\hat{e}) \cdot \hat{e}$ belongs to $SBV^2(I_z^e)$, for $e \in D$ and for $\mathcal{H}^{n-1}$-a.e. $z \in \Pi^e$. Noticing that $\chi_{J^{he}}(z + s\hat{e}) = 0$ is equivalent to $J_{\hat{u}_z^e} \cap [s, s + h|e|] = 0$, we deduce that (3.8) is less than or equal to

$$
\sum_{e \in D} \alpha(e) \int_{\Pi^e} dz \int_{I_z^e} \Big|\frac{\partial \hat{u}_z^e}{\partial s}(t)\Big|^2 \, dt \leq \int_{\hat{\Omega}} \sum_{e \in D} \alpha(e) |e(\hat{u}) e \cdot e|^2 dx, \tag{3.9}
$$

where we have used (1.15). Eventually the very definitions of $\alpha(e)$ and Q_n give

$$\sum_{e\in D}\alpha(e)|e(\hat{u})e\cdot e|^2 = Q_n(e(\hat{u})),$$

so that

$$\int_{[0,1)^n} E_1^{y,h}(\hat{\Omega})\,dy \le \int_{\hat{\Omega}} Q_n(e(\hat{u}))\,dx. \tag{3.10}$$

The same argument applied to $E_2^{y,h}$ gives

$$\int_{[0,1)^n} E_2^{y,h}(\hat{\Omega})\,dy = \sum_{e\in D}\tilde{c}_1\int_{\Pi^e}dz\int_{I^e_{z,h}}\frac{\chi_{J^{he}(z+s\hat{e})}}{|e|h}ds \le \sum_{e\in D}\tilde{c}_1\mathcal{H}^0(J_{\hat{u}^e_z}) \le c_1\mathcal{H}^{n-1}(J_{\hat{u}}) \tag{3.11}$$

where $c_1 := \tilde{c}_1\max_{|\nu|=1}(\sum_{e\in D}|\nu\cdot e|/|e|)$ and we have used (1.13).

For technical reasons, which will be clear at the end of the proof, it is convenient to prove properties (3'a)–(4') before completing the proof of (2').

Proof of (3'a). Using the very definition of w_h^y and defining $z := (x-\xi)/h - y$ we obtain

$$\begin{aligned}
&\int_{[0,1)^n}dy\int_{\partial\Omega}|w_h^y(x)-\hat{u}(x)|\wedge 1\,d\mathcal{H}^{n-1}(x) \le \\
\le\;& \sum_{\xi\in h\mathbb{Z}^n\cap\hat{\Omega}}\int_{[0,1)^n}dy\int_{\partial\Omega\cap(\xi+hy+h(-1,1)^n)}|\hat{u}(\xi+hy)-\hat{u}(x)|\wedge 1\,d\mathcal{H}^{n-1}(x)\\
\le\;& \sum_{\xi\in h\mathbb{Z}^n\cap\hat{\Omega}}\int_{\partial\Omega\cap(\xi+h(-1,2)^n)}d\mathcal{H}^{n-1}(x)\int_{\frac{x-\xi}{h}-[0,1)^n}|\hat{u}(x-hz)-\hat{u}(x)|\wedge 1\,dz\\
\le\;& \sum_{\xi\in h\mathbb{Z}^n\cap\hat{\Omega}}\int_{\partial\Omega\cap(\xi+h(-1,2)^n)}d\mathcal{H}^{n-1}(x)\int_{(-2,2)^n}|\hat{u}(x-hz)-\hat{u}(x)|\wedge 1\,dz\\
\le\;& c\int_{\partial\Omega}d\mathcal{H}^{n-1}(x)\fint_{B(x,ch)}|\hat{u}(x')-\hat{u}(x)|\wedge 1\,dx',
\end{aligned}$$

where $c<+\infty$ depends only on the dimension n.

Now, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$ we obtain

$$\fint_{B(x,ch)}|\hat{u}(x')-\hat{u}(x)|\wedge 1\,dx' \to 0,$$

by Theorem 1.3 and property (4) of Lemma 3.2 applied to $\hat{u}$. Eventually the dominated convergence theorem implies $\int_{\partial\Omega}|w_h^y(x)-\hat{u}(x)|\wedge 1\,d\mathcal{H}^{n-1}(x)\to 0$ in $L^1([0,1)^n)$.

Hence property (3'a) holds for a subsequence of h, not relabelled, and y in a subset of $[0,1)^n$ with full measure.

Proof of (3'b). This step requires a computation analogous to that in (3.11), which leads to

$$\int_{[0,1)^n} E_2^{y,h}((\partial\Omega)_{nh})\,dy \leq c_1\mathcal{H}^{n-1}(J_{\hat{u}}\cap(\partial\Omega)_{nh}). \tag{3.12}$$

Since $\hat{u}$ satisfies property (4) of Lemma 3.2, we find that $E_2^{y,h}((\partial\Omega)_{nh})$ converges to 0 in $L^1([0,1)^n)$ and then a subsequence of h and a set of full measure of $[0,1)^n$ satisfy (3'b).

Proof of (4'). Let us fix $i=1,\dots,+\infty$, $j=1,\dots,n$, and let us consider the set

$$\Gamma_i \cap \bigcup_{\substack{y_j\in[0,1)\\ \xi_j\in h\mathbb{Z}}} \{x\in\mathbb{R}^n : x_j = hy_j+\xi_j\}.$$

Since $\bigcup_{\xi_j\in h\mathbb{Z}}\{x\in\mathbb{R}^n : x_j = hy_j+\xi_j\}$ are disjoint sets as y_j varies in $[0,1)$ and since the measure $\mathcal{H}^{n-1}\lfloor\Gamma_i$ is finite, we infer for $\mathcal{H}^{n-1}$-a.e. $y_j\in[0,1)$ the following holds

$$\mathcal{H}^{n-1}(\bigcup_{\xi_j\in h\mathbb{Z}}(\Gamma_i\cap\{x\in\mathbb{R}^n : x_j = hy_j+\xi_j\}))=0.$$

Taking the union as $i=1,\dots,+\infty$ and $j=1,\dots,n$ we obtain (4').

Continuation of the proof of (2'). Let us consider the subsequence of h given by the proofs of (1'), (3'a), (3'b), and (4') and write inequalities (3.10) and (3.11) for this subsequence. Now we are in the position to apply the Fatou Lemma, so that

$$\int_{[0,1)^n} \liminf_{h\to 0}\Big[E_1^{y,h}(\hat{\Omega})+E_2^{y,h}(\hat{\Omega})\Big]dy \leq \int_{\hat{\Omega}} Q_n(e(\hat{u}))\,dx + c_1\mathcal{H}^{n-1}(J_{\hat{u}}).$$

Eventually we can find $y\in[0,1)^n$ and a further subsequence of h, not relabelled, such that properties (1')–(4') hold. In what follows we shall omit y, writing, *e.g.*, w_h in place of w_h^y.

In this second part of the proof we redefine the function w_h within some cubes. Precisely, we say that a hypercube

$$C = \xi + hy + [0,h)^n$$

is "bad" if either $J_{\hat{u}}$ crosses an edge of C

$$\xi + hy + h\eta + [0, he_i],\ \text{where } i=1,\dots,n \text{ and } \eta\in\{0,1\}^n \text{ with } \eta_i = 0 \tag{3.13}$$

(namely if $l_{e_i,h}(\xi+h\eta) = \chi_{J^{he_i}}(\xi+hy+h\eta)=1$), or $J_{\hat{u}}$ crosses a diagonal of a

2-dimensional face

$$\xi + hy + h\eta + [0, h(e_i + e_j)], \text{ where } i < j \text{ and } \eta \in \{0,1\}^n \text{ with } \eta_i = \eta_j = 0 \quad (3.14)$$

(namely if $l_{e_i+e_j,h}(\xi + h\eta) = \chi_{J^{h(e_i+e_j)}}(\xi + hy + h\eta) = 1$), or

$$\xi + hy + h\eta + [he_j, he_j + h(e_i - e_j)], \text{ where } i < j \text{ and } \eta \in \{0,1\}^n \text{ with } \eta_i = \eta_j = 0 \quad (3.15)$$

(namely if $l_{e_i-e_j,h}(\xi + h\eta + he_j) = \chi_{J^{h(e_i-e_j)}}(\xi + hy + h\eta + he_j) = 1$). We define $v_h := 0$ in every bad hypercube and $v_h := w_h$ otherwise.

Thanks to the previous definition the following properties hold:

(1'') $||w_h - v_h||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,

(2'') the constant $\tilde{c}_1(n)$ in (3.6) can be chosen in a way that $\int_\Omega Q_n(e(v_h))dx + \mathcal{H}^{n-1}(\overline{J_{v_h}}) \leq E_1^{y,h}(\hat{\Omega}) + E_2^{y,h}(\hat{\Omega})$,

(3'') $\int_{\partial\Omega} |w_h - \mathrm{tr}(v_h)| \wedge 1 \, d\mathcal{H}^{n-1} \to 0$, where $\mathrm{tr}(v_h)$ is the trace from the interior of Ω.

The proof of (1'') and of (2'') work as in [20, 21] since the definition of v_h and of the discrete energies are the same. Let us prove now (3'').

Proof of (3''). First we note that

$$\int_{\partial\Omega} |w_h - \mathrm{tr}(v_h)| \wedge 1 \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\{\partial\Omega \cap \bigcup_{C \text{ bad cube}} C\})$$

and that for each cube we have

$$\mathcal{H}^{n-1}(\{\partial\Omega \cap C\}) \leq ch^{n-1}, \quad (3.16)$$

where c depends on Ω. Now the contribution of a bad cube C to $E_2^h((\partial\Omega)_{nh})$ is given by

$$\frac{h^{n-1}}{2^{n-1}} \sum_{i=1}^{n} \sum_{\substack{\eta \in \{0,1\}^n \\ \eta_i = 0}} l_{e_i,h}(\xi + h\eta) + $$
$$+ \frac{h^{n-1}}{2^{n-2}} \sum_{1 \leq i < j \leq n} \sum_{\substack{\eta \in \{0,1\}^n \\ \eta_i = \eta_j = 0}} \frac{l_{e_i+e_j,h}(\xi + h\eta) + l_{e_i-e_j,h}(\xi + h\eta + he_j)}{\sqrt{2}}, \quad (3.17)$$

where the coefficients take into account the fact that each edge is common to 2^{n-1} hypercubes and a diagonal of a 2-face is common to 2^{n-2} hypercubes. Since at least one of the $l_{e,h}$ in the sum is equal to 1, we find that the term in (3.17) is greater than or equal to $\frac{h^{n-1}}{2^{n-1}}$. Hence by this and (3.16) we find

$$\sum_{C \text{ bad cube}} \mathcal{H}^{n-1}(\{\partial\Omega \cap C\}) \leq cE_2^h((\partial\Omega)_{nh}),$$

for a suitable constant $c < +\infty$ depending on Ω. Thanks to property (3'b) we eventually obtain (3'').

Finally properties (1')–(4'), (1'')–(3''), and (1)–(5) of Lemma 3.2 yield (1)–(4). □

3.4 A unified approximation of the energies with the right constants

With the next theorem we provide a further approximation of the given function in a way that the unified estimate for the bulk and the surface energies has now the right coefficients. The proof follows the line of [20, Theorem 2].

Theorem 3.4. *Assume that Ω has Lipschitz boundary. Let $u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n)$. Then there exists a sequence $(u_k) \subset SBV^2(\Omega, \mathbb{R}^n) \cap L^2(\Omega, \mathbb{R}^n)$ such that J_{u_k} is contained in the union S_k of a finite number of closed connected pieces of C^1-hypersurfaces, $u_k \in W^{1,\infty}(\Omega \setminus S_k, \mathbb{R}^n)$, and the following properties hold:*

(1) $||u_k - u||_{L^2(\Omega,\mathbb{R}^n)} \to 0$,

(2) $\limsup_{k\to+\infty} \Big(\int_\Omega Q_n(e(u_k))\,dx + \mathcal{H}^{n-1}(S_k) \Big) \leq \int_\Omega Q_n(e(u))\,dx + \mathcal{H}^{n-1}(J_u)$,

(3) $\int_{J_u} |u_k^\pm - u^\pm| \wedge 1 \, d\mathcal{H}^{n-1} \to 0$,

(4) $\mathcal{H}^{n-1}(J_u \setminus J_{u_k}) \to 0$, *where Q_n is defined in (3.4).*

Proof. Since J_u is $(\mathcal{H}^{n-1}, n-1)$-rectifiable, we can find a sequence (Γ_i) of C^1-hypersurfaces such that $\mathcal{H}^{n-1}(J_u \setminus \bigcup_{i=1}^{+} \infty\Gamma_i) = 0$. We fix now $\varepsilon > 0$ and use a Besicovitch recovering argument, as in [20, Theorem 2], to find a sequence of pairwise disjoint closed balls $\overline{B_j} \subset \Omega$ and an index j_0 such that

(a) for every j there exists i_j for which Γ_{i_j} divides B_j into two connected components,

(b) $\mathcal{H}^{n-1}(J_u \cap \partial B_j) = 0$,

(c) $\mathcal{H}^{n-1}(J_u \setminus \bigcup_{j\geq 1} \overline{B_j}) = 0$,

(d) $\sum_{j>j_0} \mathcal{H}^{n-1}(J_u \cap B_j) < \varepsilon$,

(e) $\mathcal{H}^{n-1}((J_u \triangle \Gamma_{i_j}) \cap \overline{B_j}) \leq \frac{\varepsilon}{1-\varepsilon}\mathcal{H}^{n-1}(J_u \cap \overline{B_j})$, for $j = 1, \dots, j_0$.

Applying Theorem 3.3 in both of connected components of $B_j \setminus \Gamma_{i_j}$, we find a sequence of functions u_k^j defined $\mathcal{L}^n$-a.e. on B_j for which property (1) of Theorem 3.3 holds in B_j, property (3) holds in ∂B_j and in Γ_{i_j}, property (4) holds for the sequence (Γ_i) introduced above, and

$$\begin{aligned}\limsup_{k\to+\infty} \int_{B_j} Q_n(e(u_k^j))dx + \mathcal{H}^{n-1}(\overline{J_{u_k^j} \cap B_j}) \ &\leq \ \int_{B_j} Q_n(e(u))dx + \mathcal{H}^{n-1}(J_u \cap B_j) \\ &\quad + c\frac{\varepsilon}{1-\varepsilon}\mathcal{H}^{n-1}(J_u \cap B_j), \qquad (3.18)\end{aligned}$$

for a suitable universal constant $c < +\infty$. Defined

$$A_t := \Big\{x \in \mathbb{R}^n : \mathrm{dist}\Big(x, \Omega \setminus \bigcup_{j=1}^{j_0} \overline{B_j}\Big) < t\Big\},$$

we observe that

$$\mathcal{H}^{n-1}\Big(J_u \cap \bigcap_{t>0} A_t\Big) = \mathcal{H}^{n-1}\Big(J_u \setminus \bigcup_{j=1}^{j_0} \overline{B_j}\Big) < \varepsilon \quad \text{and} \quad \lim_{t\to 0} \int_{A_t \cap \bigcup_{j=1}^{j_0} B_j} Q_n(e(u))dx = 0,$$

therefore we can choose $t > 0$ such that

$$\int_{A_t \cap \bigcup_{j=1}^{j_0} B_j} Q_n(e(u))dx < \varepsilon \quad \text{and} \quad \mathcal{H}^{n-1}(J_u \cap A_t) < \varepsilon. \tag{3.19}$$

Let (u_k^0) be the sequence obtained applying Theorem 3.3 in $A_t \cap \Omega$. Then using (3.19) we find

$$\limsup_{k\to+\infty} \int_{A_t\cap\Omega} Q_n(e(u_k^0))dx + \mathcal{H}^{n-1}(\overline{J_{u_k^0}}) \leq \int_{A_t\cap\Omega} Q_n(e(u))dx + c\varepsilon. \tag{3.20}$$

Now we construct a suitable partition of unity to glue together the functions u_k^j. For $j = 0, \dots, j_0$ we find a compact set K_j, with $\overline{A_t}^c \cap B_j \subset\subset K_j \subset\subset B_j$, such that

$$\mathcal{H}^{n-1}((B_j \setminus K_j) \cap \Gamma_{i_j}) < \frac{\varepsilon}{j_0}. \tag{3.21}$$

Let $\varphi_j \in C_c^\infty(B_j)$ for $j = 1, \dots, j_0$ such that $\varphi_j = 1$ in K_j and $0 \leq \varphi \leq 1$. Let also $\varphi_0 \in C_c^\infty(A_t)$ be such that $\varphi_0 := 1 - \varphi_j$ in B_j and $\varphi_0 := 1$ in $\Omega \setminus \bigcup_{j=1}^{j_0} B_j$.

We finally define

$$u_k := \sum_{j=0}^{j_0} \varphi_j u_k^j.$$

Then property (1) is satisfied by construction. As for property (2), inequalities (3.18), (3.19), and (3.20) yield

$$\limsup_{k\to+\infty} \int_\Omega Q_n(e(u_k))dx + \mathcal{H}^{n-1}(\overline{J_{u_k}}) \le \int_\Omega Q_n(e(u))dx + \mathcal{H}^{n-1}(\overline{J_u}) + c\varepsilon,$$

where $c < +\infty$ is a universal constant.

Let us prove property (3). Using (c), (d), and (e) we find

$$\begin{aligned}\int_{J_u} |u_k^\pm - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} &\le \int_{J_u \cap \bigcup_{j=1}^{j_0}(B_j \cap \Gamma_{i_j})} |u_k^\pm - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} + c\varepsilon \\ &\le \sum_{j=1}^{j_0} \int_{B_j\cap\Gamma_{i_j}} |u_k^\pm - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} + c\varepsilon. \end{aligned} \tag{3.22}$$

The very definition of u_k implies now that (3.22) is less than or equal to

$$\begin{aligned} &\sum_{j=1}^{j_0}\sum_{l=0}^{j_0} \int_{B_j\cap\Gamma_{i_j}} \varphi_l |u_k^{l\,\pm} - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} + c\varepsilon \\ = &\sum_{j=1}^{j_0} \Big(\int_{B_j\cap\Gamma_{i_j}} \varphi_0 |u_k^{0\,\pm} - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} \\ &+ \int_{B_j\cap\Gamma_{i_j}} \varphi_j |u_k^{j\,\pm} - u^\pm| \wedge 1\, d\mathcal{H}^{n-1}\Big) + c\varepsilon \\ \le &\sum_{j=1}^{j_0} \int_{B_j\cap\Gamma_{i_j}} |u_k^{j\,\pm} - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} + c\varepsilon, \end{aligned}$$

where $c < +\infty$ and the last two inequalities follow from the assumptions on φ_j and from (3.21). By the definition of u_k^j, passing to the limit as $k \to +\infty$ we find

$$\limsup_{k\to+\infty} \int_{J_u} |u_k^\pm - u^\pm| \wedge 1\, d\mathcal{H}^{n-1} \le c\varepsilon.$$

Eventually a diagonalization argument conclude the proof of properties (2) and (3).

Now property (4) easily follows from property (3). Indeed, the measure $\mathcal{H}^{n-1}\lfloor J_u$

is absolutely continuous with respect to the measure defined by

$$\nu(B) := \int_{B \cap J_u} |[u]| \wedge 1 d\mathcal{H}^{n-1},$$

for every Borel set $B \subset \Omega$. Moreover

$$\int_{J_u \setminus J_{u_k}} |[u]| \wedge 1 d\mathcal{H}^{n-1} \to 0 \tag{3.23}$$

holds true by property (3); this yields property (4) and concludes the proof. □

We are now in a position to prove the Density Theorem 3.1. The proof follows the lines of [20, Theorem 3].

Proof of the Density Theorem 3.1. Let us consider the sequence (u_k) given by Theorem 3.4. Using the compactness result for $GSBD$ [25, Theorem 11.3] we infer that a subsequence of (u_k), not relabelled, satisfies

$$e(u_k) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega, \mathbb{M}^{n\times n}_{sym}), \tag{3.24}$$

$$\int_\Omega Q_n(e(u))dx \leq \liminf_{k\to+\infty} \int_\Omega Q_n(e(u_k))dx, \tag{3.25}$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(J_{u_k}). \tag{3.26}$$

From property (2) of Theorem 3.4 and from (3.25) and (3.26) we deduce

$$\int_\Omega Q_n(e(u))dx = \lim_{k\to+\infty} \int_\Omega Q_n(e(u_k))dx, \tag{3.27}$$

$$\mathcal{H}^{n-1}(J_u) = \lim_{k\to+\infty} \mathcal{H}^{n-1}(J_{u_k}). \tag{3.28}$$

Now (3.24) and (3.27) yield property (2) of the thesis. Property (3) follows from property (4) of Theorem 3.4 and from (3.28). To obtain property (4) it is sufficient to use property (3) of Theorem 3.4 and the already proved property (3) of the thesis. □

Chapter 4

Asymptotic behaviour of certain damage model: the general case

4.1 Overview of the chapter

In this Chapter we deal with two applications of the density result for $GSBD$ presented in Chapter 3. Precisely, we extend some results presented in the asymptotic study of Chapter 2 to the vector-valued case, in the framework of Linearized Elasticity.

We consider damage energies of Ambrosio-Tortorelli type (4.1), depending on two small parameters η_k and ε_k (we assume $\delta_k = \varepsilon_k$ using the notation of Chapter 2). We first analyze the asymptotic behaviour of the models under the regime $\eta_k/\varepsilon_k \to 0$, as $\eta_k, \varepsilon_k \to 0$ (Section 4.2). The limit energy (see 4.2), rigorously obtained via Γ-convergence, involves a functional used in some brittle fracture models. This functional is finite when valued on functions u running in the space $GSBD^2(\Omega)$, i.e., on special generalised fields with bounded deformation such that the symmetric gradient $e(u)$ is square integrable and the jump set J_u has finite $(n-1)$-Hausdorff measure in $\mathbb{R}^n$. This represents the vector counterpart of the Γ-convergence result in $GSBV(\Omega)$ proved by Ambrosio and Tortorelli in [9, 10].

The second regime we consider corresponds to $\eta_k = \varepsilon_k$, with $\varepsilon_k \to 0$ (Section 4.3). The limit energy (4.40) now includes a further surface term depending linearly on the amplitude of the jump of u. The field u is therefore required to be slightly more regular: it belongs to the subspace $SBD^2(\Omega)$ of special fields with bounded deformation with $e(u)$ square integrable and J_u having finite $(n-1)$-Hausdorff measure in $\mathbb{R}^n$.

The Chapter is organised as follows: in Section 4.2 we focus on the extension to the vector-valued case of the classic Ambrosio-Tortorelli result (see Theorem 2.1

of Chapter 2, regime corresponding to $\eta_k/\varepsilon_k \to 0$, $\delta_k = \varepsilon_k$). The Γ-convergence result (Theorem 4.1) is proved as usual through a lower estimate, based on a slicing argument (Theorem 4.3), and an upper estimate, for which the contribution of the Density Theorem 3.1 turns out to be crucial (Theorem 4.4). The proof of the compactness (Proposition 4.5) and the convergence of minimizers (Corollary 4.2) complete the result and the section.

Section 4.3 studies the vector-valued counterpart, under the regime $\eta_k = \delta_k = \varepsilon_k$, of Theorem 2.1 described in Chapter 2. The main result of the section is the convergence Theorem 4.7. The liminf inequality (Theorem 4.8) is now performed through more global arguments with respect to Theorem 4.1 of the previous section. The more delicate limsup inequality is finally discussed in Remark 4.10 and is proved under suitable hypotheses in Theorem 4.9.

The results stated in Section 4.2 will be appear in [38]. Those of Section 4.3 are contained in [31] and are obtained in collaboration with Matteo Focardi.

4.2 Application 1: approximation of brittle fracture energies

Throughout the chapter we shall assume $n \geq 2$. In this section we compute the Γ-limit in $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ of the sequence of functionals

$$G_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(\mathscr{Q}(v,e(u)) + \frac{\psi(v)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v|^p + |u-g|^2\Big)\,dx & \text{if } (u,v)\in H^1(\Omega,\mathbb{R}^n)\times V_{\eta_k}, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1}$$

where

(a) $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\varepsilon_k > 0$, $\eta_k \geq 0$ are infinitesimal sequences with $\eta_k/\varepsilon_k \to 0$,

(b) $\mathscr{Q}\colon \mathbb{R}\times\mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ is lower semicontinuous,

(c) for every $s\in\mathbb{R}$, the function $\mathscr{Q}(s,\cdot)$ is a positive definite quadratic form on $\mathbb{M}^{n\times n}_{sym}$,

(d) there exist two constants $0 < c_1, c_2 < +\infty$, such that $c_1 s|A|^2 \leq \mathscr{Q}(s,A) \leq c_2 s|A|^2$, for every $s\in\mathbb{R}$ and $A\in\mathbb{M}^{n\times n}_{sym}$,

(e) $\psi \in C([0,1])$ is strictly decreasing with $\psi(1)=0$ and $g\in L^2(\Omega,\mathbb{R}^n)$,

(f) $a, p \in \mathbb{R}$ with $a > 0$ and $p > 1$ (the extension to the case $p = +\infty$ immediately follows),

(g) $V_{\eta_k} := \big\{ v \in W^{1,p}(\Omega) : \eta_k \le v \le 1 \ \mathcal{L}^n\text{-a.e. in } \Omega \big\}$.

We also define the functional $\Psi \colon L^1(\Omega, \mathbb{R}^n) \to [0, +\infty]$ by

$$\Psi(u) := \begin{cases} \displaystyle\int_\Omega \mathscr{Q}(e(u))dx + a\mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^2 dx \\ \qquad\qquad\qquad\qquad \text{if } u \in GSBD^2(\Omega) \cap L^2(\Omega, \mathbb{R}^n), \\ +\infty \qquad\qquad\qquad\quad \text{otherwise,} \end{cases} \tag{4.2}$$

where $\mathscr{Q}(e(u)) := \mathscr{Q}(1, e(u))$ and

$$a := 2q^{\frac{1}{q}}(\gamma p)^{\frac{1}{p}} \int_0^1 \psi^{\frac{1}{q}} ds, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.3}$$

Then the following result holds.

Theorem 4.1. *Assume (a)–(g) and assume that Ω has Lipschitz boundary. Then the Γ-limit of (G_k) in $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$ is given by*

$$G(u, v) := \begin{cases} \Psi(u) & \textit{if } v = 1 \ \mathcal{L}^n\textit{-a.e. in } \Omega, \\ +\infty & \textit{otherwise.} \end{cases}$$

The previous theorem, together with a compactness result for the functionals G_k (Proposition 4.5), will give in turn the convergence of minima and minimizers in the space $L^2(\Omega, \mathbb{R}^n) \times L^1(\Omega)$.

Corollary 4.2. *Assume (a)–(g) and assume that Ω has Lipschitz boundary. For every k, let (u_k, v_k) be a minimizer of the problem*

$$\min_{(u,v) \in H^1(\Omega, \mathbb{R}^n) \times V_{\eta_k}} \int_\Omega \Big(\mathscr{Q}(v, e(u)) + \frac{\psi(v)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v|^p + |u-g|^2 \Big) dx. \tag{4.4}$$

Then $v_k \to 1$ in $L^1(\Omega)$ and a subsequence of (u_k) converges in $L^2(\Omega, \mathbb{R}^n)$ to a minimizer u of the following problem

$$\min_{u \in GSBD(\Omega)} \Big(\int_\Omega \mathscr{Q}(e(u))dx + \alpha\mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^2 dx \Big). \tag{4.5}$$

Moreover the minimum values in (4.4) tend to the minimum value in (4.5).

As usual, we shall prove Theorem 4.1 giving a lower estimate for the Γ-lower limit of G_k and an upper estimate for the Γ-upper limit of G_k. To simplify the notation we introduce the functionals $F_k\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)\to[0,+\infty]$ and $\Phi\colon L^1(\Omega,\mathbb{R}^n)\to[0,+\infty]$ defined by

$$F_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(\mathscr{Q}(v,e(u)) + \frac{\psi(v)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v|^p\Big)dx & \text{if } (u,v)\in H^1(\Omega,\mathbb{R}^n)\times V_{\eta_k},\\ +\infty & \text{otherwise,}\end{cases}$$

$$\Phi(u) := \begin{cases} \displaystyle\int_\Omega \mathscr{Q}(e(u))dx + a\mathcal{H}^{n-1}(J_u) & \text{if } u\in GSBD^2(\Omega)\cap L^1(\Omega,\mathbb{R}^n),\\ +\infty & \text{otherwise.}\end{cases}$$

For technical reasons which will be clear in the last part of the proof, we first study the Γ-lower limit of F_k in the space $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ (Theorem 4.3) and the Γ-upper limit of (the restriction of) F_k in the space $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ (Theorem 4.4).

Theorem 4.3. *Assume* (a)–(g). *Let* $(u,v)\in L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *and let* (u_k,v_k) *be a sequence such that*

$$(u_k,v_k)\to(u,v) \text{ in } L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega), \tag{4.6}$$

$$(F_k(u_k,v_k)) \text{ is bounded.} \tag{4.7}$$

Then $u\in GSBD^2(\Omega)\cap L^1(\Omega,\mathbb{R}^n)$, $v=1$ $\mathcal{L}^n$*-a.e. in* Ω, *and*

$$\int_\Omega \mathscr{Q}(e(u))dx \le \liminf_{k\to+\infty}\int_\Omega \mathscr{Q}(v_k,e(u_k))dx, \tag{4.8}$$

$$a\mathcal{H}^{n-1}(J_u) \le \liminf_{k\to+\infty}\int_\Omega\Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx. \tag{4.9}$$

Proof. The convergence $v_k\to1$ in $L^1(\Omega)$ is an immediate consequence of (4.6) and (4.7). In the first part of the proof we argue by slicing following the lines of Proposition 2.4.

Proof of (4.8). We fix $\xi\in\mathbb{R}^n$, $\xi\neq0$. We are going to prove that $u\in GSBD(\Omega)$ and that satisfies

$$\int_\Omega (e(u)\xi\cdot\xi)^2dx \le \liminf_{k\to+\infty}\int_\Omega v_k(e(u_k)\xi\cdot\xi)^2dx. \tag{4.10}$$

To this aim we first extract a subsequence (u_r,v_r) of (u_k,v_k) such that

$$((u_r)^\xi_y,(v_r)^\xi_y)\to(u^\xi_y,1) \text{ in } L^1(\Omega^\xi_y)\times L^1(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y\in\Omega^\xi \tag{4.11}$$

and

$$\lim_{r\to+\infty}\int_\Omega v_r(e(u_r)\xi\cdot\xi)^2dx=\liminf_{k\to+\infty}\int_\Omega v_k(e(u_k)\xi\cdot\xi)^2dx. \tag{4.12}$$

Fixed $0<\kappa<1$, the Fubini Theorem, [5, Structure Theorem 4.5], and (4.7) imply

$$\int_{\Omega^\xi}\Big[\int_{\Omega^\xi_y}\Big((v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\kappa\Big(\frac{\psi(v_r)^\xi_y}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla(v_r)^\xi_y|^p\Big)\Big)dt\Big]d\mathcal{H}^{n-1}(y)\leq$$
$$\leq\int_\Omega\Big(v_r(e(u_r)\xi\cdot\xi)^2+\kappa\Big(\frac{\psi(v_r)}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla v_r|^p\Big)\Big)dx\leq c, \tag{4.13}$$

where $c<+\infty$ is constant. Using the previous inequality and the Fatou Lemma, for $\mathcal{H}^{n-1}$-a.e. $y\in\Omega^\xi$ we can find a subsequence (u_m,v_m) of (u_r,v_r) such that

$$\lim_{m\to+\infty}\int_{\Omega^\xi_y}\Big((v_m)^\xi_y\left|\nabla((u_m)^\xi_y)\right|^2+\kappa\Big(\frac{\psi(v_m)^\xi_y}{\varepsilon_m}+\gamma\,\varepsilon_m^{p-1}|\nabla(v_m)^\xi_y|^p\Big)\Big)dt=$$
$$=\liminf_{r\to+\infty}\int_{\Omega^\xi_y}\Big((v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\kappa\Big(\frac{\psi(v_r)^\xi_y}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla(v_r)^\xi_y|^p\Big)\Big)dt \tag{4.14}$$

and the last term is finite. Since (4.11) and (4.14) hold, we can apply the scalar result Proposition 2.4 to $((u_m)^\xi_y,(v_m)^\xi_y)$, so that $u^\xi_y\in SBV^2(\Omega^\xi_y)$ and

$$\int_{\Omega^\xi_y}|\nabla(u^\xi_y)|^2dt\leq\liminf_{m\to+\infty}\int_{\Omega^\xi_y}(v_m)^\xi_y|\nabla((u_m)^\xi_y)|^2dt, \tag{4.15}$$

$$a\mathcal{H}^{n-1}(J_{u^\xi_y})\leq\liminf_{m\to+\infty}\int_{\Omega^\xi_y}\Big(\frac{\psi(v_m)^\xi_y}{\varepsilon_m}+\gamma\,\varepsilon_m^{p-1}|\nabla((v_m)^\xi_y)|^p\Big)dt. \tag{4.16}$$

To check that $u\in GSBD(\Omega)$, we observe the following inequalities hold

$$\int_{\Omega^\xi}\Big(|D(u^\xi_y)|(\Omega^\xi_y\setminus J_{u^\xi_y})+\mathcal{H}^0(J_{u^\xi_y})\Big)d\mathcal{H}^{n-1}(y)\leq$$
$$\leq\int_{\Omega^\xi}\Big(\mathcal{L}^1(\Omega^\xi_y)+\int_{\Omega^\xi_y\setminus J_{u^\xi_y}}|\nabla(u^\xi_y)|^2dt+\mathcal{H}^0(J_{u^\xi_y})\Big)d\mathcal{H}^{n-1}(y)\leq$$
$$\leq\int_{\Omega^\xi}c\Big[1+\liminf_{r\to+\infty}\int_{\Omega^\xi_y}\Big((v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\kappa\Big(\frac{\psi(v_r)}{\varepsilon_r}+\gamma\,\varepsilon_r^{p-1}|\nabla(v_r)|^p\Big)\Big)dt\Big],$$

where $c:=\operatorname{diam}(\Omega)+1+a$ and we have used (4.13)–(4.16). The last term in the previous estimate is bounded by (4.13) and this gives $u\in GSBD(\Omega)$.

Now we integrate on Ω^ξ both sides of (4.15); by (4.12)–(4.14), (1.15), and the Fubini Theorem we find (4.10) as $\kappa\to0$.

Now we observe that

$$\int_\Omega(e(u)\xi\cdot\xi-w)^2dx\leq\liminf_{k\to+\infty}\int_\Omega v_k(e(u_k)\xi\cdot\xi-w)^2dx \tag{4.17}$$

follows from (4.10) for every $w \in L^2(\Omega)$. Indeed, (4.17) trivially holds if w is piecewise constant on a Lipschitz partition of Ω; then a density argument proves (4.17) for an arbitrary $w \in L^2(\Omega)$.

The next step is to deduce by (4.17) that

$$e(u_k)v_k^{\frac{1}{2}} \rightharpoonup e(u) \;\text{ weakly in } L^2(\Omega, \mathbb{M}^{n\times n}_{sym}). \tag{4.18}$$

To this aim, we first extract a subsequence (u_l, v_l) of (u_k, v_k) such that $v_l \to 1$ $\mathcal{L}^n$-a.e. in Ω and $e(u_l)v_l^{\frac{1}{2}} \rightharpoonup A$ weakly in $L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$, for a suitable function A in $L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$. Now we apply (4.17) to $w = A\xi\cdot\xi - tz$, for $t \in \mathbb{R}$ and $z \in L^2(\Omega)$. After an easy computation we find

$$\int_\Omega ((e(u)-A)\xi\cdot\xi)^2 dx + 2t\int_\Omega z(e(u)-A)\xi\cdot\xi dx \le \liminf_{l\to+\infty}\int_\Omega v_l((e(u_l)-A)\xi\cdot\xi)^2 dx.$$

As $t \to \pm\infty$, the previous inequality leads to a contradiction unless $\int_\Omega z(e(u) - A)\xi\cdot\xi dx = 0$ for every $z \in L^2(\Omega)$ and every $\xi \in \mathbb{R}^n$, namely unless $e(u) = A$ $\mathcal{L}^n$-a.e. in Ω. Therefore (4.18) holds true.

We use now the Egorov Theorem to find, in correspondence of $\mu > 0$, a Borel set $B_\mu \subset \Omega$ such that $\mathcal{L}^n(\Omega\setminus B_\mu) < \mu$ and $v_k > 1-\mu$ on B_μ for k large. An easy computation then shows that

$$e(u_k)\chi_{B_\mu} \rightharpoonup e(u)\chi_{B_\mu} \;\text{ weakly in } L^2(\Omega, \mathbb{M}^{n\times n}_{sym}). \tag{4.19}$$

We are now in a position to apply [19, Theorem 2.3.1], so that

$$\int_{B_\mu} \mathscr{Q}(e(u)) \le \liminf_{k\to+\infty}\int_\Omega \mathscr{Q}(v_k, e(u_k)\chi_{B_\mu})dx \le \int_\Omega \mathscr{Q}(v_k, e(u_k))dx.$$

By the absolute continuity of the Lebesgue integral the left-hand side of the previous inequality tends to $\int_\Omega \mathscr{Q}(e(u))dx$ as $\mu \to 0$, and this concludes the proof of (4.8).

Proof of (4.9). For this part we refer to Theorem 2.4. We only point out that arguing again by slicing, using (1.13) and the coarea formula, we find

$$\alpha\int_{J_u^\xi} |\nu_u\cdot\xi| d\mathcal{H}^{n-1} \le \liminf_{k\to+\infty}\int_\Omega \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx, \tag{4.20}$$

namely the set J_u^ξ replaces the set J_u appearing in (2.74). Nevertheless, inequality (4.20) still holds true with J_u in place of J_u^ξ by (1.16), being the set

$$\{\xi \in \mathbb{S}^{n-1} : \mathcal{H}^{n-1}(J_u\setminus J_u^\xi) = 0\}$$

dense in $\mathbb{S}^{n-1}$. Eventually, inequality (4.9) follows from this and from a classical localization argument. □

Let us prove now the upper estimate. We denote by F_2'' the Γ-lim sup of F_k in $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$.

Theorem 4.4. *Assume (a)–(g) and assume that Ω has Lipschitz boundary. Then*

$$F_2''(u,1) \leq \Phi(u), \tag{4.21}$$

for every $u \in GSBD^2(\Omega)\cap L^2(\Omega,\mathbb{R}^n)$.

Proof. The crucial point of this proof is the approximation of a function u in $GSBD^2(\Omega)\cap L^2(\Omega,\mathbb{R}^n)$ with more regular functions, through the Density Theorem 3.1. Precisely, it provides a sequence $u_k \in SBV^2(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n)$ such that

$$u_k \to u \text{ in } L^2(\Omega,\mathbb{R}^n) \qquad \text{and} \qquad \Phi(u_k)\to\Phi(u), \tag{4.22}$$

so that if we prove that u_k satisfies (4.21), then also u satisfies (4.21), being F_2'' lower semicontinuous in $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$.

The proof of (4.22) for functions in $SBV^2(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n)$ is now standard (see, for instance, [20, 21]). Let us give a brief description of the construction of the recovery sequence, following the approach of Theorem 2.3.

Using a local reflection argument we reduce to prove the statement for Ω open cube in $\mathbb{R}^n$. Now Theorem 1.13 and Remark 1.14 allow us to assume in addition that $\overline{J_u}$ is contained in Ω and that u satisfies properties (1)–(3) of Theorem 1.13. Moreover, it is not restrictive to consider only the case when $\overline{J_u}$ is a $(n-1)$-simplex, which we denote by S.

Let us fix a sequence of constants σ_k such that $\eta_k/\sigma_k\to 0$ and $\sigma_k/\varepsilon_k\to 0$. We introduce now the sets A_k, A_k', B_k, and B_k', defined precisely in Theorem 2.3. Here we just recall that $A_k\cup A_k'$ is a neighborhood of S such that

$$\mathcal{L}^n(A_k)\leq c\sigma_k \quad \text{and} \quad \mathcal{L}^n(A_k')\leq c\sigma_k^2 \tag{4.23}$$

and the set $B_k\cup B_k'$ is a layer which envelops $A_k\cup A_k'$ and satisfies

$$\mathcal{L}^n(B_k)\leq c\varepsilon_k \quad \text{and} \quad \mathcal{L}^n(B_k')\leq c\varepsilon_k^2, \tag{4.24}$$

for a suitable constant $c<+\infty$.

Also the definition of the recovery sequence (u_k, v_k) is given in analogy with Theorem 2.3. In particular u_k is set equal to u out of $A_k\cup A_k'$ and it is a linear link in A_k in the direction of e_n. With this definition u_k is a Lipschitz function in $\Omega \setminus A'_k$ with constant c/σ_k, where $c < +\infty$. To check this it is sufficient to apply the arguments given in (2.87)–(2.94) to each components u^i_k of u_k. Thanks to the Mc Shane Theorem we are now able to define u_k also in A'_k in a way that

$$|Du_k| \le c/\sigma_k \ \mathcal{L}^n\text{-a.e. in } \Omega. \tag{4.25}$$

In addition, we define v_k by η_k in $A_k \cup A'_k$, by 1 out of $A_k \cup A'_k \cup B_k \cup B'_k$, and in a way that, in terms of energy, the transition in $B_k \cup B'_k$ is optimal.

As for the computation of $F_k(u_k, v_k)$, we only observe that

$$\int_{A_k \cup A'_k} \mathscr{Q}(\eta_k, e(u_k))dx \to 0, \tag{4.26}$$

by (4.23), (4.25), and by the convergence $\eta_k/\sigma_k \to 0$. This concludes the proof, since the computation for the other terms work as in Theorem 2.3. □

Let us prove the Γ-convergence Theorem 4.1 for (G_k).

Proof of Theorem 4.1. Let us introduce $H\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega) \to [0,+\infty]$, defined by

$$H(u,v) := \begin{cases} \displaystyle\int_\Omega |u-g|^2 dx & \text{if } u \in L^2(\Omega,\mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.27}$$

On the one hand we notice that

$$F' + H \le G', \tag{4.28}$$

where F', G' represent the Γ-lower limits of F_k and G_k in $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ and we have used the fact that H is lower semicontinuous in $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$. Then if $(u,v) \in L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ satisfies $G'(u,v) < +\infty$, one deduces by Theorem 4.3 that u belongs to $GSBD^2(\Omega) \cap L^2(\Omega,\mathbb{R}^n)$, $v = 1$ $\mathcal{L}^n$-a.e., and

$$\Psi(u) = \Phi(u) + H(u,1) \le G'(u,1).$$

On the other hand if $u \in GSBD^2(\Omega) \cap L^2(\Omega,\mathbb{R}^n)$, then the continuity of H in $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ and Theorem 4.4 yield

$$G''(u,1) \le G''_2(u,1) = F''_2(u,1) + H(u,1) \le \Phi(u) + H(u,1) = \Psi(u), \tag{4.29}$$

where G'', G''_2 represent the Γ-upper limits of G_k in $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ and in $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$. The thesis follows from (4.28) and (4.29). □

A key point for the proof of Corollary 4.2 is the compactness of a minimizing sequence. This is obtained in the following proposition, through a characterization which relates compactness of sequences to compactness of slices (see [1, Theorem 6.6], [25, Theorem 10.7], and Section 1.6).

Proposition 4.5. *Let* $(u_k, v_k) \in L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$ *be such that* $(G_k(u_k, v_k))$ *is bounded. Then* $v_j \to 1$ *in* $L^1(\Omega)$ *and a subsequence* (u_j) *of* (u_k) *converges in* $L^1(\Omega, \mathbb{R}^n)$ *to a function* $u \in L^2(\Omega, \mathbb{R}^n)$*.*

Proof. The proof follows the lines of [25, Theorem 11.1]. It is sufficient to prove the statement for any open set which is relatively compact in Ω. Furthermore we assume that Ω is a finite union of open rectangles and we extend each function by zero out of Ω. Let $M < +\infty$ be such that $G_k(u_k, v_k) \le M$.

Since $(F_k(u_k, v_k))$ is bounded, the sequence v_k converges to 1 in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω, up to subsequences. We fix now $k \in \mathbb{N}$ and $\xi \in \mathbb{S}^{n-1}$. For $y \in \Omega^\xi$ we consider the one-dimensional functional $F_{y,k} \colon L^1(\Omega^\xi_y) \times L^1(\Omega^\xi_y) \to \mathbb{R}$ defined by

$$F_{y,k}(w, z) := \begin{cases} \displaystyle\int_{\Omega^\xi_y} \Big(z\,|\nabla w|^2 + \frac{\psi(z)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1} |\nabla(z)|^p \Big) dt & \\ & \text{if } (w,z) \in H^1(\Omega^\xi_y) \times V_{y,\eta_k}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $V_{y,\eta_k} := \Big\{ z \in W^{1,p}(\Omega^\xi_y) : \eta_k \le z \le 1 \ \mathcal{H}^1\text{-a.e. in } \Omega^\xi_y \Big\}$. We also define for every $\lambda > 0$

$$\hat{A}^{\xi,\lambda}_k := \Big\{ y \in \Omega^\xi : (u_k)^\xi_y \in H^1(\Omega^\xi_y), F_{y,k}((u_k)^\xi_y, (v_k)^\xi_y) \le \lambda \Big\}, \quad \hat{B}^{\xi,\lambda}_k := \Omega^\xi \setminus \hat{A}^{\xi,\lambda}_k,$$
$$A^{\xi,\lambda}_k := \Big\{ x \in \Omega : \Pi^\xi(x) \in \hat{A}^{\xi,\lambda}_k \Big\}, \quad B^{\xi,\lambda}_k := \Big\{ x \in \Omega : \Pi^\xi(x) \in \hat{B}^{\xi,\lambda}_k \Big\},$$

being $\Pi^\xi(x)$ the projection of x on the plane Π^ξ. Since $(F_k(u_k, v_k))$ is bounded, the Chebychev Inequality and the Fubini Theorem yield

$$\mathcal{L}^n(B^{\xi,\lambda}_k) \le \mathrm{diam}(\Omega) \frac{c}{\lambda}. \tag{4.30}$$

Here and henceforth c represents a finite constant; in particular $c(\delta)$ will denote its possible dependence on δ. For $\mu > 0$ and $t \in \mathbb{R}$, we introduce the truncation function $\tau_\mu(t) := -\mu \vee t \wedge \mu$ and we set

$$w^{\xi,\lambda}_{k,\mu} := \begin{cases} \tau_\mu(u_k \cdot \xi) & \text{in } A^{\xi,\lambda}_k, \\ 0 & \text{in } B^{\xi,\lambda}_k. \end{cases}$$

Let

$$\phi(t) := \int_0^t \psi^{\frac{1}{q}} ds \quad \text{for } t \in [0,1]$$

and let $\tilde{c}$ be a constant which uniformly bounds $\phi(v_k)$. For $\delta > 0$ we are able to find λ_δ and μ_δ large enough to guarantee

$$\tilde{c}||u_k \cdot \xi - w_{k,\mu_\delta}^{\xi,\lambda_\delta}||_{L^1(\mathbb{R}^n)} < \delta \tag{4.31}$$

uniformly with respect to k. Indeed, let $\mu_\delta > 0$ be such that $s \leq \frac{\delta}{4M} s^2$ for $s \geq \mu_\delta$ and let λ_δ be such that $\mu_\delta \mathcal{L}^n(B_k^{\xi,\lambda_\delta}) \leq \delta/2$, (this is possible by (4.30)). Therefore we find

$$\begin{aligned}
\int_\Omega |u \cdot \xi - w_{k,\mu_\delta}^{\xi,\lambda_\delta}|dx &= \int_{\{|u\cdot\xi|>\mu_\delta\}} |u \cdot \xi - w_{k,\mu_\delta}^{\xi,\lambda_\delta}|dx + \int_{B_k^{\xi,\lambda_\delta} \cap \{|u\cdot\xi|\leq\mu_\delta\}} |u \cdot \xi|dx \\
&\leq 2\int_{\{|u\cdot\xi|>\mu_\delta\}} |u|dx + \mu_\delta \mathcal{L}^n(B_k^{\xi,\lambda_\delta}) \\
&\leq \frac{\delta}{2M} \int_\Omega |u|^2 dx + \frac{\delta}{2} = \delta.
\end{aligned}$$

For simplicity in what follows we write w_k in place of $w_{k,\mu_\delta}^{\xi,\lambda_\delta}$.

In order to apply Proposition 1.10, we set

$$U := (\phi(v_k)u_k), \qquad V_\delta^\xi := (\phi(v_k)w_k),$$

and we show that for every k and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ we have

$$\int_\mathbb{R} |(\phi(v_k)w_k)_y^\xi(t+h) - (\phi(v_k)w_k)_y^\xi(t)|dt \leq \omega_\delta(h) \quad \text{for } h \in (0,1), \tag{4.32}$$

for a suitable modulus of continuity ω_δ independent on k, y, and ξ. To this aim we check that for every k and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the function $(\phi(v_k)w_k)_y^\xi$ satisfies all requirements of Lemma 1.11, uniformly with respect to k and y.

First note that for every k and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ the function $(\phi(v_k)w_k)_y^\xi$ belongs to $SBV^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, that $\mathcal{H}^0((J_{w_k})_y^\xi) \leq c$, and that $||(\phi(v_k)w_k)_y^\xi||_{L^\infty(\mathbb{R})} \leq c(\delta)$. Moreover the Young Inequality, the estimate $\phi(t) \leq ct$, and the Hölder Inequality yield

$$\begin{aligned}
\int_{\Omega_y^\xi} |\nabla((\phi(v_k)w_k)_y^\xi)|dt &\leq c(\delta) \int_{\Omega_y^\xi} \Big(\frac{\psi((v_k)_y^\xi)}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla((v_k)_y^\xi)|^p\Big) dt \\
&\quad + c(\operatorname{diam}(\Omega))^{\frac{1}{2}} \Big(\int_{\Omega_y^\xi} (v_k)_y^\xi \left|\nabla(w_k)_y^\xi\right|^2 dt\Big)^{\frac{1}{2}} \leq c(\delta).
\end{aligned}$$

We are now in a position to apply Lemma 1.11, so that (4.32) holds with $\omega_\delta(h) := c(\delta)h$. Through Proposition 1.10, inequalities (4.31) and (4.32) imply the existence of a subsequence $(\phi(v_j)u_j)$ of $(\phi(v_k)u_k)$ and of a function $\tilde{u} \in L^1(\Omega, \mathbb{R}^n)$ such that $\phi(v_j)u_j \to \tilde{u}$ in $L^1(\Omega, \mathbb{R}^n)$. The Fatou Lemma also gives $\tilde{u} \in L^2(\Omega, \mathbb{R}^n)$. Eventually the thesis follows for $u := \tilde{u}/\phi(1)$. □

We conclude proving Corollary 4.2.

Proof of Corollary 4.2. Let us fix k and check that the functional G_k achieves its infimum. If (u_j, v_j) is a minimizing sequence for G_k, the sequence (u_j) belongs to $H^1(\Omega, \mathbb{R}^n)$, is bounded in $L^2(\Omega, \mathbb{R}^n)$, and the sequence of symmetric gradients $e(u_j)$ is bounded in $L^2(\Omega, \mathbb{M}^{n\times n}_{sym})$. By Korn's inequality this implies that (u_j) is bounded in $H^1(\Omega, \mathbb{R}^n)$, so that there exist a subsequence of (u_j), not relabelled, and a function $u \in H^1(\Omega, \mathbb{R}^n)$ such that $u_j \rightharpoonup u$ weakly in $H^1(\Omega, \mathbb{R}^n)$.

Being (v_j) bounded in $W^{1,p}(\Omega)$ we also infer that there exists a further subsequence of (v_j), not relabelled, and a function $v \in V_{\eta_k}$ such that

$$v_j \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega) \text{ and } \mathcal{L}^n\text{-a.e. in } \Omega.$$

By the Ioffe-Olech semicontinuity theorem (see, for instance, [19, Theorem 2.3.1.]) and the Fatou lemma we deduce that

$$\begin{aligned} \int_\Omega \mathscr{Q}(v, e(u))dx &\leq \liminf_{j\to+\infty} \int_\Omega \mathscr{Q}(v_j, e(u_j))dx \\ \int_\Omega |u-g|^2 dx &\leq \liminf_{j\to+\infty} \int_\Omega |u_j - g|^2 dx \end{aligned} \tag{4.33}$$

hold, therefore (u, v) minimizes G_k.

Now a sequence (u_k, v_k) of minimizers of G_k is compact in $L^1(\Omega, \mathbb{R}^n)\times L^1(\Omega)$ by Proposition 4.5. Let $(u, 1)$ be the limit point of a subsequence, not relabelled, of (u_k, v_k). By Theorem 4.1 and by a general result of Γ-convergence (see Section 1.8), we infer that $(u, 1)$ is a minimizer for G and that the convergence of minimum values holds.

To conclude the proof it remains to show that $u_k \to u$ in $L^2(\Omega, \mathbb{R}^n)$. To this aim it is sufficient to prove that

$$\int_\Omega |u_k - g|^2 dx \to \int_\Omega |u-g|^2 dx. \tag{4.34}$$

By the convergence of the minimum values $G_k(u_k, v_k) \to G(u, v)$, the following inequalities

$$\Phi(u) \leq \liminf_{k\to+\infty} F_k(u_k, v_k) \quad \text{and} \quad \int_\Omega |u-g|^2 dx \leq \liminf_{k\to+\infty} \int_\Omega |u_k - g|^2 dx$$

(holding true by Theorem 4.3 and the lower semicontinuity of H) are actually equalities. This gives (4.34) and concludes the proof. □

4.3 Application 2: approximation of cohesive fracture energies

We conclude the chapter showing the second application of the density result proved in Chapter 3, which generalizes Theorem 2.1 to the vector-valued case for the regime given by $0 < \alpha < +\infty$ and $0 < \beta < +\infty$.

4.3.1 The main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 < p < +\infty$, and let $\varepsilon_k > 0$ be an infinitesimal sequence.

Consider the sequence of functionals $F_k\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega) \to [0,+\infty]$ defined by

$$F_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(\mathscr{Q}(v, e(u)) + \frac{\psi(v)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v|^p\Big) dx & \text{if } (u,v) \in H^1(\Omega,\mathbb{R}^n)\times V_{\varepsilon_k}, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.35}$$

where $0 < \gamma < +\infty$ and

$$\psi \in C^0([0,1]) \text{ is strictly decreasing with } \psi(1) = 0, \tag{4.36}$$

$$V_{\varepsilon_k} := \big\{v \in W^{1,p}(\Omega) : \varepsilon_k \leq v \leq 1 \ \mathcal{L}^n\text{-a.e. in } \Omega\big\}. \tag{4.37}$$

Moreover, the function $\mathscr{Q} : (0,1]\times\mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ satisfies

(H1) $\mathscr{Q}$ is lower semicontinuous and for every $\mathbb{A} \in \mathbb{M}^{n\times n}_{sym}$ the function $\mathscr{Q}(\cdot,\mathbb{A})$ is continuous as $s \uparrow 1$;

(H2) for every $s \in (0,1]$, the function $\mathscr{Q}(s,\cdot)$ is a positive definite quadratic form;

(H3) for every $s \in (0,1]$ and $\mathbb{A} \in \mathbb{M}^{n\times n}_{sym}$, the following inequalities hold

$$c_1 s|\mathbb{A}|^2 \leq \mathscr{Q}(s,\mathbb{A}) \leq c_2 s|\mathbb{A}|^2, \tag{4.38}$$

for suitable positive constants c_1 and c_2;

(H4) the quadratic forms $s^{-1}\mathscr{Q}(s,\cdot)$ converge uniformly on compact sets of $\mathbb{M}^{n\times n}_{sym}$ to some function $\mathscr{Q}_0$ as $s\downarrow 0^+$.

Note that by items (H3) and (H4) above $\mathscr{Q}_0$ is a quadratic form satisfying

$$c_1|\mathbb{A}|^2 \le \mathscr{Q}_0(\mathbb{A}) \le c_2|\mathbb{A}|^2 \quad \text{for every } \mathbb{A}\in\mathbb{M}^{n\times n}_{sym}.$$

In particular, $\mathscr{Q}_0^{1/2}$ is a norm on $\mathbb{M}^{n\times n}_{sym}$, and

$$c_3^{-1} s\,\mathscr{Q}_0(\mathbb{A}) \le \mathscr{Q}(s,\mathbb{A}) \le c_3\, s\,\mathscr{Q}_0(\mathbb{A}) \quad \text{for all } (s,\mathbb{A})\in(0,1]\times\mathbb{M}^{n\times n}_{sym}, \tag{4.39}$$

with $c_3 := c_2\, c_1^{-1} \ge 1$.

Remark 4.6. Let us stress that thanks to (H2) and (H3), assumption (H4) is rather natural as it is satisfied by families $\varepsilon_k^{-1}\mathscr{Q}(\varepsilon_k,\cdot)$, $\varepsilon_k\downarrow 0^+$, up to the extraction of subsequences.

For instance, given $\mathscr{Q}_0$ and $\mathscr{Q}_1$ two coercive quadratic forms on $\mathbb{M}^{n\times n}_{sym}$, the family $\mathscr{Q}(s,\mathbb{A}) := s(s\mathscr{Q}_1(\mathbb{A}) + (1-s)\mathscr{Q}_0(\mathbb{A}))$ satisfies all the assumptions (H1)–(H4) above.

The asymptotic behaviour of the family (F_k) is described in terms of the functional $\Phi\colon L^1(\Omega,\mathbb{R}^n)\to[0,+\infty]$ given by

$$\Phi(u) := \begin{cases} \displaystyle\int_\Omega \mathscr{Q}_1(e(u))dx + a\mathcal{H}^{n-1}(J_u) + b\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot\nu_u)d\mathcal{H}^{n-1} & \\ & \text{if } u\in SBD^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.40}$$

where we have set $\mathscr{Q}_1(\mathbb{A}) := \mathscr{Q}(1,\mathbb{A})$ for all $\mathbb{A}\in\mathbb{M}^{n\times n}_{sym}$, and

$$a := 2q^{1/q}(\gamma p)^{1/p}\int_0^1 \psi^{1/q}(s)\,ds, \quad b := 2\psi^{1/2}(0), \quad \text{with } \frac{1}{p}+\frac{1}{q} = 1. \tag{4.41}$$

The Γ-limit of F_k is identified in suitable subspaces of $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ (cp. with Theorem 4.7 and Remark 4.10 below).

Theorem 4.7. *Assume the conditions in* (4.35)-(4.41) *to be satisfied, and let* Ω *be a bounded open set with Lipschitz boundary. The* Γ*-limit of* (F_k) *in the strong* $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *topology is given on the subspace* $L^\infty(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *by*

$$F(u,v) := \begin{cases} \Phi(u) & \text{if } v = 1\ \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.42}$$

As usual, we shall prove the previous result by showing separately a lower bound inequality and an upper bound inequality. To this aim we define

$$F' := \Gamma\text{-}\liminf_{k\to+\infty} F_k \qquad \text{and} \qquad F'' := \Gamma\text{-}\limsup_{k\to+\infty} F_k. \tag{4.43}$$

Then, Theorem 4.7 follows from the ensuing two statements, in which on one hand we establish the lower bound inequality in full generality, and on the other hand we prove the upper bound inequality on L^∞ (and SBV) due to a difficulty probably of technical nature (see Remark 4.10).

Theorem 4.8. *Assume* (4.35)-(4.41). *Let* $(u,v) \in L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ *be such that* $F'(u,v)$ *is finite. Then,* $v = 1$ $\mathcal{L}^n$*-a.e. in* Ω *and*

$$\Phi(u) \leq F'(u,1). \tag{4.44}$$

Theorem 4.9. *Assume* (4.35)-(4.41) *and assume that* Ω *is a bounded open set with Lipschitz boundary. Then, for every* $u \in L^\infty(\Omega,\mathbb{R}^n)$ *we have*

$$F''(u,1) \leq \Phi(u). \tag{4.45}$$

4.3.2 Proof of the main results

We start off by establishing the lower bound estimate. We need to introduce further notation: we consider the strictly increasing map $\phi\colon [0,1] \to [0,+\infty)$ defined by

$$\phi(t) := \int_0^t \psi^{1/q}(s)\, ds \qquad \text{for every } t \in [0,1]. \tag{4.46}$$

Proof of Theorem 4.8. By the definition of Γ-$\liminf$ it is enough to prove that if (u,v) belongs to $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ and if $(u_k,v_k) \in L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ is a sequence such that

$$(u_k,v_k) \to (u,v) \text{ in } L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega), \tag{4.47}$$

$$\sup_k F_k(u_k,v_k) \leq L < +\infty, \tag{4.48}$$

then $u \in SBD^2(\Omega)$, $v = 1$ $\mathcal{L}^n$-a.e. in Ω, and the ensuing estimates hold true with $\lambda \in (0,1)$

$$\liminf_{k\to+\infty} \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k))dx \geq \int_\Omega \mathscr{Q}_1(e(u))dx, \tag{4.49}$$

$$\liminf_{k\to+\infty} \int_{\Omega\setminus\Omega_k^\lambda} \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\,\varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \geq 2q^{1/q}(\gamma\, p)^{1/p}(\phi(1)-\phi(\lambda))\mathcal{H}^{n-1}(J_u), \tag{4.50}$$

and with fixed $\delta > 0$ there is $\lambda_\delta > 0$ such that for all $\lambda \in (0, \lambda_\delta)$

$$\liminf_{k\to+\infty} \int_{\Omega_k^\lambda} \Big(\mathscr{Q}(v_k, e(u_k)) + \frac{\psi(v_k)}{\varepsilon_k}\Big) dx \geq 2\psi^{1/2}(\lambda) \int_{J_u} \mathscr{Q}_0^{1/2}([u] \odot \nu_u) d\mathcal{H}^{n-1} + O(\delta), \tag{4.51}$$

where we have set $\Omega_k^\lambda := \{v_k \leq \lambda\}$. Given (4.49)-(4.51) for granted, we conclude (4.44) by letting first $\lambda \downarrow 0$ and then $\delta \downarrow 0$.

In order to simplify the notation, we set

$$\begin{aligned} I_k^1 &:= \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k))\, dx, \\ I_k^2 &:= \int_{\Omega\setminus\Omega_k^\lambda} \left(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1} |\nabla v_k|^p\right) dx, \\ I_k^3 &:= \int_{\Omega_k^\lambda} \Big(\mathscr{Q}(v_k, e(u_k)) + \frac{\psi(v_k)}{\varepsilon_k}\Big)\, dx. \end{aligned}$$

Clearly, if (u_k, v_k) satisfies (4.47) and (4.48), then $v_k \to v = 1$ in $L^1(\Omega)$. The fact that u belongs to $SBD^2(\Omega)$ and inequalities (4.49) and (4.50) can be obtained as a by-product of a slicing argument, following the lines of Theorem 4.3. Here, we pursue a global approach, arguing as in [30, Lemma 3.2.1] (see also [29]).

We first notice that (u_k) is pre-compact in the weak* topology of $BD(\Omega)$. To verify this it is sufficient to prove that

$$\sup_k \int_\Omega |e(u_k)| dx < +\infty. \tag{4.52}$$

Now, on one hand by (4.38) and the Jensen inequality we have

$$I_k^1 = \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k)) dx \geq c_1\, \lambda \int_{\Omega\setminus\Omega_k^\lambda} |e(u_k)|^2 dx \geq \frac{c_1\, \lambda}{\mathcal{L}^n(\Omega)} \Big(\int_{\Omega\setminus\Omega_k^\lambda} |e(u_k)| dx\Big)^2, \tag{4.53}$$

and on the other hand by the Cauchy-Schwartz inequality we find

$$\begin{aligned} I_k^3 = \int_{\Omega_k^\lambda} \Big(\mathscr{Q}(v_k, e(u_k)) + \frac{\psi(v_k)}{\varepsilon_k}\Big) dx &\geq c_1\, \varepsilon_k \int_{\Omega_k^\lambda} |e(u_k)|^2 dx + \frac{\psi(\lambda)}{\varepsilon_k} \mathcal{L}^n(\Omega_k^\lambda) \\ &\geq 2(c_1\, \psi(\lambda))^{1/2} \int_{\Omega_k^\lambda} |e(u_k)| dx. \end{aligned} \tag{4.54}$$

Estimates (4.53), (4.54) together with (4.48) eventually imply

$$\int_\Omega |e(u_k)| dx \leq c\, ((I_k^1)^{1/2} + I_k^3) \leq c,$$

for some positive constant $c = c(\Omega, \lambda, \psi, L, c_1)$. In conclusion, (4.52) follows.

From (4.52), as u_k converges to u in $L^1(\Omega, \mathbb{R}^n)$, we deduce that $u \in BD(\Omega)$ and that actually $u_k \rightharpoonup u$ weakly*-$BD(\Omega)$.

Proof of estimate (4.49) *and that* $u \in SBD^2(\Omega)$. We construct a function $\tilde{u}_k$ in a way that it is null near the jump set J_u of u and coincides with u_k elsewhere.

Recalling the very definition of ϕ in (4.46) we have that $\phi(v_k) \in W^{1,p}(\Omega)$, and moreover, Young inequality and the BV Coarea Formula yield

$$\begin{aligned} I_k^2 &\geq q^{1/q}(\gamma p)^{1/p} \int_{\Omega\setminus\Omega_k^\lambda} \psi^{1/q}(v_k)|\nabla v_k| dx \\ &= q^{1/q}(\gamma p)^{1/p} \int_{\Omega\setminus\Omega_k^\lambda} |\nabla(\phi(v_k))| dx = q^{1/q}(\gamma p)^{1/p} \int_{\phi(\lambda)}^{\phi(1)} \mathrm{Per}\,(\{\phi(v_k) > t\}, \Omega) dt. \end{aligned} \tag{4.55}$$

Fix $\lambda' \in (\lambda, 1)$, the Mean Value theorem ensures for every $k \in \mathbb{N}$ the existence of $t_k \in (\phi(\lambda), \phi(\lambda'))$ such that

$$\int_{\phi(\lambda)}^{\phi(1)} \mathrm{Per}\,(\{\phi(v_k) > t\}, \Omega) dt \geq (\phi(\lambda') - \phi(\lambda)) \mathrm{Per}\,(\{\phi(v_k) > t_k\}, \Omega). \tag{4.56}$$

Set $\lambda_k := \phi^{-1}(t_k)$, then note that $\Omega \setminus \Omega_k^{\lambda_k} = \{\phi(v_k) > t_k\}$ is a set of finite perimeter satisfying by the latter inequality and (4.48)

$$\mathrm{Per}\,(\Omega \setminus \Omega_k^{\lambda_k}, \Omega) \leq c \tag{4.57}$$

for some $c = c(\lambda, \lambda', \phi, L)$. Let now $\tilde{u}_k := \chi_{\Omega\setminus\Omega_k^{\lambda_k}} u_k$, then the Chain Rule Formula in BV [7, Theorem 3.96] yields that $\tilde{u}_k \in SBV(\Omega, \mathbb{R}^n)$ with

$$D\tilde{u}_k = \chi_{\Omega\setminus\Omega_k^{\lambda_k}} \nabla u_k \mathcal{L}^n \llcorner \Omega + u_k \otimes \nu_{\partial^*\Omega_k^{\lambda_k}} \mathcal{H}^{n-1} \llcorner \partial^*\Omega_k^{\lambda_k}.$$

In particular, $\mathcal{H}^{n-1}(J_{\tilde{u}_k} \setminus \partial^*\Omega_k^{\lambda_k}) = 0$, then by (4.53), (4.55) and (4.57) the functions $\tilde{u}_k$ satisfy

$$\int_\Omega |e(\tilde{u}_k)|^2 dx + \mathcal{H}^{n-1}(J_{\tilde{u}_k}) \leq c \tag{4.58}$$

for some $c = c(\lambda, \lambda', \phi, L, c_1) < +\infty$, and in addition

$$\|\tilde{u}_k - u\|_{L^1(\Omega,\mathbb{R}^n)} \leq \|u_k - u\|_{L^1(\Omega,\mathbb{R}^n)} + \int_{\Omega_k^\lambda} |u| dx. \tag{4.59}$$

As $v_k \to 1$ in $L^1(\Omega)$ we find $\mathcal{L}^n(\Omega_k^\lambda) \downarrow 0$, thus (4.59) implies that $\tilde{u}_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$. Since we have established that $u \in BD(\Omega)$, it is easy to deduce from the

SBD Compactness Theorem [14, Theorem 1.1] (see also [20, Lemma 5.1]) and from inequality (4.58) that actually $u \in SBD^2(\Omega)$, with

$$e(\tilde{u}_k) \rightharpoonup e(u) \quad \text{weakly in } L^2(\Omega, \mathbb{M}^{n\times n}_{sym}), \tag{4.60}$$

and

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{\tilde{u}_k}). \tag{4.61}$$

Eventually, by taking into account that

$$\liminf_{k\to+\infty} \int_{\Omega\setminus\Omega_k^\lambda} \mathscr{Q}(v_k, e(u_k))dx = \liminf_{k\to+\infty} \int_{\Omega} \mathscr{Q}(v_k, e(\tilde{u}_k))dx,$$

(4.49) follows from (4.60), from the convergence $v_k \to 1$ in $L^1(\Omega)$, and from [19, Theorem 2.3.1].

Proof of estimate (4.50). Regrettably, inequality (4.50) is not a straightforward consequence of the previous arguments. Indeed, (4.55), (4.56), (4.61) and $\mathcal{H}^{n-1}(J_{\tilde{u}_k} \setminus \partial^*\Omega_k^{\lambda_k}) = 0$ lead to an estimate differing from (4.50) by a multiplicative factor 2 on the left-hand side. Therefore, we need a more accurate argument. To this aim, we note that by (4.55) and the Fatou Lemma we have

$$\liminf_{k\to\infty} I_k^2 \geq q^{1/q}(\gamma\, p)^{1/p} \int_{\phi(\lambda)}^{\phi(1)} \liminf_{k\to\infty} \mathrm{Per}\left(\{\phi(v_k) > t\}, \Omega\right) dt,$$

then in order to conclude (4.50) it suffices to prove that

$$\liminf_{k} \mathrm{Per}\left(\{\phi(v_k) > t\}, \Omega\right) \geq 2\mathcal{H}^{n-1}(J_u) \quad \text{ for all } t \in (\phi(\lambda), \phi(1)). \tag{4.62}$$

This follows via a slicing argument as established in [30, Lemma 3.2.1]. We report in what follows the proof of estimate (4.62) for the sake of completeness.

Fixed $t \in (\phi(\lambda), \phi(1))$ for which the right-hand side of (4.62) is finite, we define $\tau := \phi^{-1}(t)$ and $U_k^\tau := \Omega \setminus \Omega_k^\tau$. For every open subset $A \subset \Omega$ and vector $\xi \in \mathbb{S}^{n-1}$, we claim that

$$\liminf_{k} \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) \geq 2 \int_{\pi_\xi(A)} \mathcal{H}^0(J_{u_y^\xi} \cap A) d\mathcal{H}^{n-1}, \tag{4.63}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi(A)$ (recall the notations and the results in Theorem 1.1). Given (4.63) for granted, the Coarea Formula for rectifiable sets and the Fatou lemma yield the following lower semicontinuity estimate

$$\liminf_{k} \mathrm{Per}\left(\{\phi(v_k) > \phi(\tau)\}, A\right) =$$

$$= \liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) \geq 2 \int_{\pi_\xi(A)} \mathcal{H}^0(J_{u_y^\xi} \cap A) d\mathcal{H}^{n-1} = 2 \int_{J_u^\xi \cap A} |\nu_u \cdot \xi| d\mathcal{H}^{n-1}. \tag{4.64}$$

Since $\mathcal{H}^{n-1}(J_u \setminus J_u^\xi) = 0$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ (see (1.7)), we infer from (4.64)

$$\liminf_k \operatorname{Per}\left(\{\phi(v_k) > \phi(\tau)\}, A\right) \geq 2 \int_{J_u \cap A} |\nu_u \cdot \xi| d\mathcal{H}^{n-1}. \tag{4.65}$$

In conclusion, inequality (4.62) follows from (4.65) by passing to the supremum on a sequence (ξ_r) dense in $\mathbb{S}^{n-1}$ and applying [7, Lemma 2.35], since the function

$$A \to \liminf_k \operatorname{Per}\left(\{\phi(v_k) > \phi(\tau)\}, A\right)$$

is superadditive on disjoint open subsets of Ω.

Let us finally prove (4.63). Note that there exists a subsequence (u_r, v_r) of (u_k, v_k) such that

$$\liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) = \lim_r \mathcal{H}^{n-1}(J_{\chi_{U_r^\tau}} \cap A), \tag{4.66}$$

$$\left((u_r)_y^\xi, (v_r)_y^\xi\right) \to \left(u_y^\xi, 1\right) \text{ in } L^1(\Omega_y^\xi) \times L^1(\Omega_y^\xi), \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_\xi(\Omega), \tag{4.67}$$

and with fixed $\eta > 0$, for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi(\Omega)$ we find

$$\liminf_r \left(\eta \int_{A_y^\xi} \left((v_r)_y^\xi \left|\nabla((u_r)_y^\xi)\right|^2 + \frac{\psi\left((v_r)_y^\xi\right)}{\varepsilon_r} + \right.\right.$$
$$\left.\left.\gamma\, \varepsilon_r^{p-1} \left|\nabla((v_r)_y^\xi)\right|^p\right) dt + \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap A)\right) < +\infty, \tag{4.68}$$

by (4.38), (4.48), our choice of τ, and the Fatou lemma.

Fix $y \in \pi_\xi(\Omega)$ be satisfying (4.67), (4.68), and assume also that $\mathcal{H}^0\left(J_{u_y^\xi} \cap A\right)$ is strictly positive. Moreover, up to extracting a further subsequence (depending on y and not relabeled for convenience), we may suppose that the lower limit in (4.68) is actually a limit.

Let $\{t_1, ..., t_l\}$ be an arbitrary subset of $J_{u_y^\xi} \cap A$, and let $(I_i)_{1 \leq i \leq l}$ be a family of pairwise disjoint open intervals such that $t_i \in I_i$, $I_i \subset\subset A_y^\xi$. Then, for every $1 \leq i \leq l$, we claim that

$$s_i := \limsup_r \inf_{I_i} (v_r)_y^\xi = 0.$$

Indeed, if s_h was strictly positive for some $h \in \{1, ..., l\}$, then

$$\inf_{I_h} (v_j)_y^\xi \geq \frac{s_h}{2}$$

for a suitable subsequence (v_j) of (v_r), and thus (4.68) would give

$$\int_{I_h} \left| \nabla((u_j)_y^\xi) \right|^2 dt \leq c,$$

for some constant c. Hence, Rellich-Kondrakov's theorem and (4.67) would imply the slice u_y^ξ to be in $W^{1,1}(I_h, \mathbb{R}^n)$, which is a contradiction since by assumption $\mathcal{H}^0 \left(J_{u_y^\xi} \cap I_h \right) > 0$. So let $t_r^i \in I_i$ be such that

$$\lim_r (v_r)_y^\xi (t_r^i) = 0,$$

and α_i, $\beta_i \in I_i$, with $\alpha_i < t_r^i < \beta_i$, be such that

$$\lim_r (v_r)_y^\xi (\alpha_i) = \lim_r (v_r)_y^\xi (\beta_i) = 1.$$

Then, there follows

$$\liminf_r \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap I_i) \geq 2.$$

Hence, the subadditivity of the inferior limit and the arbitrariness of l yield

$$\liminf_r \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap A) \geq 2\mathcal{H}^0(J_{u_y^\xi} \cap A).$$

Therefore, we obtain

$$\liminf_r \Bigg(\eta \int_{A_y^\xi} \left((v_r)_y^\xi \left| \nabla((u_r)_y^\xi) \right|^2 + \frac{\psi\left((v_r)_y^\xi \right)}{\varepsilon_r} + \gamma\, \varepsilon_r^{p-1} \left| \nabla((v_r)_y^\xi) \right|^p \right) dt + \\ + \mathcal{H}^0(J_{\chi_{(U_r^\tau)_y^\xi}} \cap A) \Bigg) \geq 2\mathcal{H}^0(J_{u_y^\xi} \cap A),$$

which integrated on $\pi_\xi(A)$ gives

$$\liminf_k \mathcal{H}^{n-1}(J_{\chi_{U_k^\tau}} \cap A) \geq 2 \int_{\pi_\xi(A)} \mathcal{H}^0(J_{u_y^\xi} \cap A) d\mathcal{H}^{n-1} - \eta c$$

for some positive constant $c = c(L)$. As $\eta \downarrow 0$ we find (4.63).

Proof of estimate (4.51). We employ the blow-up technique introduced by Fonseca and Müller in [32]. First, we observe that by the Cauchy-Schwartz inequality we have

$$I_k^3 \geq \varepsilon_k \int_{\Omega_k^\lambda} \frac{\mathscr{Q}(v_k, e(u_k))}{v_k} dx + \frac{\psi(\lambda)}{\varepsilon_k} \mathcal{L}^n(\Omega_k^\lambda) \geq 2\psi^{1/2}(\lambda) \int_{\Omega_k^\lambda} \left(\frac{\mathscr{Q}(v_k, e(u_k))}{v_k} \right)^{1/2} dx, \tag{4.69}$$

thus in order to get (4.51) it suffices to show that for all $\delta > 0$ there is $\lambda_\delta > 0$ such that for $\lambda \in (0, \lambda_\delta)$ we have

$$\liminf_k \int_{\Omega_k^\lambda} \left(\frac{\mathscr{Q}(v_k, e(u_k))}{v_k} \right)^{1/2} dx \geq \int_{J_u} \mathscr{Q}_0^{1/2}([u] \odot \nu) d\mathcal{H}^{n-1} + O(\delta). \tag{4.70}$$

Actually the uniform convergence on compact sets of $\mathbb{M}^{n\times n}_{sym}$ assumed in (H4) above implies that, with fixed $\delta > 0$, for some $\lambda_\delta > 0$ and all $\lambda \in (0, \lambda_\delta)$ we have

$$\begin{aligned}
\int_{\Omega_k^\lambda} \left(\frac{\mathscr{Q}(v_k, e(u_k))}{v_k} \right)^{1/2} dx &= \int_{\Omega_k^\lambda} \mathscr{Q}^{1/2}_{v_k(x)} \Big(\frac{e(u_k)}{|e(u_k)|} \Big) |e(u_k)| dx \\
&\geq \int_{\Omega_k^\lambda} \left(\mathscr{Q}_0^{1/2} \Big(\frac{e(u_k)}{|e(u_k)|} \Big) - \delta \right) |e(u_k)| dx \geq \int_{\Omega_k^\lambda} \mathscr{Q}_0^{1/2}(e(u_k)) dx - \delta \, |Eu_k|(\Omega),
\end{aligned}$$

where we have set $\mathscr{Q}_s(\mathbb{A}) := s^{-1} \mathscr{Q}(s, \mathbb{A})$. Thus, inequality (4.70) is reduced to prove

$$\liminf_k \int_{\Omega_k^\lambda} \mathscr{Q}_0^{1/2}(e(u_k)) dx \geq \int_{J_u} \mathscr{Q}_0^{1/2}([u] \odot \nu) d\mathcal{H}^{n-1}, \tag{4.71}$$

being $\delta > 0$ arbitrary and $(|Eu_k|(\Omega))$ being bounded as shown in (4.52).

Let (u_r) be a subsequence of (u_k) such that

$$\liminf_k \int_{\Omega_k^\lambda} \mathscr{Q}_0^{1/2}(e(u_k)) dx = \lim_r \int_{\Omega_r^\lambda} \mathscr{Q}_0^{1/2}(e(u_r)) dx.$$

In order to prove (4.71), for every Borel set $A \subseteq \Omega$ we introduce

$$\mu_r(A) := \int_{\Omega_r^\lambda \cap A} \mathscr{Q}_0^{1/2}(e(u_r)) dx,$$

$$\theta_r(A) := \int_A \mathscr{Q}_0^{1/2}(e(u_r)) dx,$$

and

$$\zeta_r(A) := F_r(u_r, v_r, A),$$

where $F_r(\cdot, \cdot, A)$ denotes the functional defined in (4.35) with the set of integration Ω replaced by A.

It is evident that the former set functions are finite Borel measures, with (μ_r),

(θ_r) and (ζ_r) actually equi-bounded in mass thanks to inequalities (4.48) and (4.52). Hence, up to subsequences not relabelled for convenience, we may suppose that

$$\mu_r \rightharpoonup \mu, \quad \theta_r \rightharpoonup \theta, \quad \text{and} \quad \zeta_r \rightharpoonup \zeta \quad \text{weakly}^* \text{ in } \mathcal{M}_b^+(\Omega), \tag{4.72}$$

for some μ, θ and $\zeta \in \mathcal{M}_b^+(\Omega)$, respectively.

Being

$$\lim_r \mu_r(\Omega) \geq \mu(\Omega),$$

to infer (4.71) we need only to show that

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u} \geq \mathscr{Q}_0^{1/2}([u] \odot \nu_u) \quad \mathcal{H}^{n-1}\text{-a.e. in } J_u, \tag{4.73}$$

where $\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}$ is the Radon-Nikodým derivative of μ with respect to $\mathcal{H}^{n-1} \llcorner J_u$.

We shall prove the latter inequality for the subset of points x_0 in J_u for which the Radon-Nikodým derivatives

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0), \ \frac{d\theta}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0), \ \frac{d\zeta}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0), \tag{4.74}$$

exist finite,

$$\frac{d\mathscr{Q}_0^{1/2}(\frac{dEu}{d|Eu|})|Eu|}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \mathscr{Q}_0^{1/2}([u] \odot \nu_u)(x_0) \tag{4.75}$$

and

$$\lim_{\rho \to 0} \frac{\mathcal{H}^{n-1}(J_u \cap Q_\nu(x_0, \rho))}{\rho^{n-1}} = 1, \tag{4.76}$$

where $\nu := \nu_u(x_0)$, Q_ν is any unitary cube centred in the origin with one face orthogonal to ν, and $Q_\nu(x_0, \rho) := x_0 + \rho\, Q_\nu$. Formula (4.76) is a consequence of the $(\mathcal{H}^{n-1}, n-1)$ rectifiability of J_u (see [7, Theorem 2.83]). Note that all the conditions above define a set of full measure in J_u.

By selecting one of such points $x_0 \in J_u$, we get

$$\begin{aligned} \frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_u}(x_0) &= \lim_{\rho \to 0} \frac{\mu(Q_\nu(x_0, \rho))}{\rho^{n-1}} = \lim_{\substack{\rho \in I \\ \rho \to 0}} \lim_{r \to +\infty} \frac{\mu_r(Q_\nu(x_0, \rho))}{\rho^{n-1}} \\ &= \lim_{\substack{\rho \in I \\ \rho \to 0}} \lim_{r \to +\infty} \frac{1}{\rho^{n-1}} \left(\theta_r(Q_\nu(x_0, \rho)) - \theta_r(Q_\nu(x_0, \rho) \setminus \Omega_r^\lambda) \right), \end{aligned} \tag{4.77}$$

where

$$I := \Big\{ \rho \in (0, \frac{2}{\sqrt{n}} \text{dist}(x_0, \partial\Omega)) : \ \mu(\partial Q_\nu(x_0, \rho)) = \theta(\partial Q_\nu(x_0, \rho)) =$$

$$= \zeta(\partial Q_\nu(x_0,\rho)) = 0\Big\}.$$

Note that I is a subset of radii of full measure in $(0, \frac{2}{\sqrt{n}}\mathrm{dist}(x_0,\partial\Omega))$, and that the second equality in (4.77) easily follows from the convergence $\mu_r \rightharpoonup \mu$ weakly* in $\mathcal{M}_b^+(\Omega)$.

Further, we claim that

$$\lim_{\substack{\rho\in I\\ \rho\to 0}} \lim_{r\to+\infty} \frac{\theta_r(Q_\nu(x_0,\rho)\setminus\Omega_r^\lambda)}{\rho^{n-1}} = 0. \tag{4.78}$$

Indeed, the Hölder inequality, the very definition of F_k in (4.35), and (4.39) imply

$$\begin{aligned}
\frac{\theta_r(Q_\nu(x_0,\rho)\setminus\Omega_r^\lambda)}{\rho^{n-1}} &= \frac{1}{\rho^{n-1}}\int_{Q_\nu(x_0,\rho)\setminus\Omega_r^\lambda} \mathscr{Q}_0^{1/2}(e(u_r))dx\\
&\le \frac{c_3^{1/2}}{\rho^{n-1}}\int_{Q_\nu(x_0,\rho)\setminus\Omega_k^\lambda} \mathscr{Q}_{v_r(x)}^{1/2}(e(u_r))dx\\
&\le \Big(c_3\frac{\mathcal{L}^n(Q_\nu(x_0,\rho)\setminus\Omega_r^\lambda)}{\rho^{n-1}}\Big)^{1/2}\Big(\frac{1}{\rho^{n-1}}\int_{Q_\nu(x_0,\rho)\setminus\Omega_r^\lambda} \mathscr{Q}_{v_r(x)}(e(u_r))dx\Big)^{1/2}\\
&\le (c_3\rho)^{1/2}\lambda^{-1/2}\Big(\frac{F_r(u_r,v_r,Q_\nu(x_0,\rho))}{\rho^{n-1}}\Big)^{1/2} = (c_3\rho)^{1/2}\lambda^{-1/2}\Big(\frac{\zeta_r(Q_\nu(x_0,\rho))}{\rho^{n-1}}\Big)^{1/2}.
\end{aligned}$$

Finally, equality (4.78) is a consequence of the latter estimate and condition (4.74).

By taking (4.78) into account, (4.77) rewrites as

$$\frac{d\mu}{d\mathcal{H}^{n-1}\llcorner J_u}(x_0) = \frac{d\theta}{d\mathcal{H}^{n-1}\llcorner J_u}(x_0). \tag{4.79}$$

The convergence of the symmetrized distributional derivatives, i.e.

$$Eu_r \rightharpoonup Eu \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega,\mathbb{M}^{n\times n}_{sym})$$

is a result of (4.47) and (4.52), in turn implying that

$$\theta(Q_\nu(x_0,\rho)) \ge \int_{Q_\nu(x_0,\rho)} \mathscr{Q}_0^{1/2}\Big(\frac{dEu}{d|Eu|}\Big)d|Eu| \tag{4.80}$$

by the convexity of $\mathscr{Q}_0^{1/2}$ and the stated convergence. Thus, by (4.75) and (4.80) we get

$$\frac{d\theta}{d\mathcal{H}^{n-1}\llcorner J_u}(x_0) \ge \liminf_{\rho\to 0}\frac{1}{\rho^{n-1}}\int_{Q_\nu(x_0,\rho)} \mathscr{Q}_0^{1/2}\Big(\frac{dEu}{d|Eu|}\Big)d|Eu| = \mathscr{Q}_0^{1/2}([u]\odot\nu_u)(x_0). \tag{4.81}$$

Eventually, (4.79) and (4.81) conclude the proof of (4.73), and then of (4.71). □

The proof of the Γ-lim sup inequality in Theorem 4.9 takes advantage of the Density Theorems 3.1 for $GSBD(\Omega)$ and for $SBV(\Omega,\mathbb{R}^n)$ [23, Theorem 3.1] (see Theorem 1.13).

Remark 4.10. The Γ-lim sup inequality in Theorem 4.9 is stated only for fields in the subspace $L^\infty(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ of $L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ since Theorem 3.1 does not guarantee the convergence

$$\int_{J_{u_k}\cup J_u} |[u_k]-[u]|\, d\mathcal{H}^{n-1} \to 0 \tag{4.82}$$

for every u in $SBD^2(\Omega)\cap L^2(\Omega,\mathbb{R}^n)$. If (4.82) was true, then Theorem 3.1 combined with Theorem 1.13 would allow us to prove the Γ-lim sup inequality for those fields u that are piecewise smooth. In such a case, the construction of recovery sequences follows quite classical lines, and by density the Γ-lim sup inequality in $L^2(\Omega,\mathbb{R}^n)\times L^1(\Omega)$ would be completely proved.

Nevertheless, this argument applies to fields in $L^\infty(\Omega,\mathbb{R}^n)$ since the approximating sequence (u_k) in Theorem 3.1 is constructed in a way that $\|u_k\|_{L^\infty(\Omega,\mathbb{R}^n)} \le \|u\|_{L^\infty(\Omega,\mathbb{R}^n)}$.

The same conclusion of Theorem 4.9 can be drawn for all fields in $SBV^2(\Omega,\mathbb{R}^n)$. Indeed, the functional in (4.40) is continuous on sequences of truncations, therefore the conclusion follows by Theorem 1.13 and a diagonal argument. In this respect, take also into account the equality $GSBV^2(\Omega,\mathbb{R}^n)\cap BD(\Omega) = SBV^2(\Omega,\mathbb{R}^n)$.

Finally let us prove the upper bound estimate.

Proof of Theorem 4.9. Let $u\in SBD^2(\Omega)\cap L^\infty(\Omega,\mathbb{R}^n)$, then by the lower semicontinuity of F'' and Theorem 3.1 it is not restrictive to assume that u belongs to $SBV^2\cap L^\infty(\Omega,\mathbb{R}^n)$. By a local reflection argument we can also assume that $\Omega\subset\mathbb{R}^n$ is a open cube and again by the lower semicontinuity of F'', by Theorem 1.13, and by Remark 1.14 we can reduce ourselves to prove (4.45) for a piecewise smooth SBV-function u with $\overline{J_u}\subset\Omega$. Finally, up to a truncation argument, condition $u\in L^\infty(\Omega,\mathbb{R}^n)$ is preserved.

For the construction of the recovery sequence we shall follow the lines of Theorem 2.3. For convenience of the reader we recall the main steps.

Since $\overline{J_u}$ is a finite union of closed pairwise disjoint $(n-1)$-simplexes well-contained in Ω, we reduce to study the case when $S := \overline{J_u}$ is a $(n-1)$-simplex. In order to simplify the computation we also assume $S\subset\{x_n=0\}$, we denote the generic point $x\in\mathbb{R}^n$ by $x=(\overline{x},x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$, and we orient J_u so that $\nu_u=(0,1)$.

Let

$$\Omega^\pm := \{x \in \Omega : \pm x_n > 0\}$$

and let L be the maximum between the Lipschitz constants of u in Ω^+ and Ω^-. Let also

$$\sigma_k(\overline{x}) := \frac{\varepsilon_k}{2\psi(0)^{1/2}} \mathscr{Q}_0^{1/2}([u(\overline{x},0)] \odot e_n), \tag{4.83}$$

for every $(\overline{x}, x_n) \in \Omega$. Being u^+ and u^- Lipschitz functions, we deduce that σ_k is in turn a Lipschitz function and that

$$|\nabla \sigma_k(\overline{x})| \leq c\,\varepsilon_k, \tag{4.84}$$

for $\mathcal{L}^n$-a.e. $(\overline{x}, x_n) \in \Omega$ and for a suitable constant $c = c(\psi, L, \mathscr{Q}_0) > 0$. Moreover, $\sigma_k = 0$ on ∂S, where ∂S is the boundary of S in the relative topology of $\mathbb{R}^{n-1} \times \{0\}$.

We set for $\rho \in (0,1)$

$$f(\rho) := \psi(1-\rho), \quad g(\rho) := \Big(\int_0^{1-\rho} \psi^{-1/p}(s)\, ds \Big)^{-1}, \quad \text{and} \quad h(\rho) := (f \cdot g)^{1/2}(\rho),$$

and we introduce the infinitesimal sequence $\rho_k := h^{-1}(\varepsilon_k)$ having the property that

$$\frac{f(\rho_k)}{\varepsilon_k} = \frac{\varepsilon_k}{g(\rho_k)} \to 0 \quad \text{as } k \uparrow \infty. \tag{4.85}$$

Denote by w_k the only solution of the following Cauchy problem in the interval $[0, T_k)$,

$$\begin{cases} w_k' = \Big(\dfrac{q}{\gamma p}\Big)^{1/p} \varepsilon_k^{-1} \psi^{1/p}(w_k) \\ w_k(0) = \varepsilon_k, \end{cases} \tag{4.86}$$

where $T_k \in (0, +\infty]$ is given by

$$T_k := \Big(\frac{\gamma p}{q}\Big)^{1/p} \varepsilon_k \int_{\varepsilon_k}^1 \psi^{-1/p}(s)\, ds.$$

Furthermore, define $\mu_k \in (0, T_k)$

$$\mu_k := \Big(\frac{\gamma p}{q}\Big)^{1/p} \varepsilon_k \int_{\varepsilon_k}^{1-\rho_k} \psi^{-1/p}(s)\, ds, \tag{4.87}$$

thus μ_k is infinitesimal by (4.85).

We are now in a position to introduce the sets

$$A_k := \Big\{ x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ |x_n| < \sigma_k(\overline{x}) \Big\},$$

$$B_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ 0 \le |x_n| - \sigma_k(\overline{x}) \le \mu_k\Big\},$$
$$C_k := \Big\{x \in \mathbb{R}^n : (\overline{x}, 0) \notin S,\ d(x, \partial S) \le \mu_k\Big\},$$

where $d(x, \partial S)$ is the distance of the point x from the set ∂S.

Consider the sequence (u_k, v_k) defined by

$$u_k(\overline{x}, x_n) := \begin{cases} \dfrac{x_n + \sigma_k(\overline{x})}{2\sigma_k(\overline{x})}(u(\overline{x}, \sigma_k(\overline{x})) - u(\overline{x}, -\sigma_k(\overline{x}))) + u(\overline{x}, -\sigma_k(\overline{x})) & \\ & \text{if } x \in A_k, \\ u(x) & \text{if } x \in \Omega \setminus A_k, \end{cases}$$

and

$$v_k(x) := \begin{cases} \varepsilon_k & \text{if } x \in A_k, \\ w_k(|x_n| - \sigma_k(\overline{x})) & \text{if } x \in B_k, \\ w_k(d(x, \partial S) - \sigma_k(\overline{x})) & \text{if } x \in C_k, \\ 1 - \rho_k & \text{otherwise.} \end{cases}$$

Then, $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega)$, moreover we shall show that it provides a recovery sequence following the arguments used in (2.87)–(2.94). First note that, for every component u_k^i of u_k for $\mathcal{L}^n$-a.e. $(\overline{x}, x_n) \in A_k$ we have that

$$\begin{aligned} |D_j u_k^i(\overline{x}, x_n)| \le\ & \left| \frac{x_n}{\sigma_k(\overline{x})} D_j \sigma_k(\overline{x}) \frac{u^i(\overline{x}, \sigma_k(\overline{x})) - u^i(\overline{x}, -\sigma_k(\overline{x}))}{2\sigma_k(\overline{x})} \right| \\ & + \Big| D_j u^i(\overline{x}, -\sigma_k(\overline{x})) - D_n u^i(\overline{x}, -\sigma_k(\overline{x})) D_j \sigma_k(\overline{x}) \Big| \\ & + \Big| D_j u^i(\overline{x}, \sigma_k(\overline{x})) + D_n u^i(\overline{x}, \sigma_k(\overline{x})) D_j \sigma_k(\overline{x}) \\ & - D_j u^i(\overline{x}, -\sigma_k(\overline{x})) + D_n u^i(\overline{x}, -\sigma_k(\overline{x})) D_j \sigma_k(\overline{x}) \Big| \\ & \le |D_j \sigma_k(\overline{x})| \Big(\frac{|[u^i(\overline{x}, 0)]|}{2\sigma_k(\overline{x})} + 4L \Big) + 3L \le c, \end{aligned} \tag{4.88}$$

where $j = 1, \dots, n-1$, and

$$\begin{aligned} |D_n u_k^i(\overline{x}, x_n)| &= \left| \frac{u^i(\overline{x}, \sigma_k(\overline{x})) - u^i(\overline{x}, -\sigma_k(\overline{x}))}{2\sigma_k(\overline{x})} \right| \\ &= \left| \frac{u^i(\overline{x}, \sigma_k(\overline{x})) - u^{i+}(\overline{x}, 0)}{2\sigma_k(\overline{x})} + \frac{u^{i+}(\overline{x}, 0) - u^{i-}(\overline{x}, 0)}{2\sigma_k(\overline{x})} + \frac{u^{i-}(\overline{x}, 0) - u^i(\overline{x}, -\sigma_k(\overline{x}))}{2\sigma_k(\overline{x})} \right| \\ &\le L + \frac{|[u^i(\overline{x}, 0)]|}{2\sigma_k(\overline{x})} \le \frac{c}{\varepsilon_k}; \end{aligned} \tag{4.89}$$

in the previous estimates $c = c(L)$ and we have used (4.84). In particular, we deduce that u_k is a Lipschitz function.

For what the computation of the energy $F_k(u_k, v_k)$ is concerned we shall mainly focus on the term

$$\int_{A_k} \mathscr{Q}(v_k, e(u_k))dx.$$

The others are estimated in an elementary way following Theorem 2.3. More precisely, we have

$$\begin{aligned}\limsup_k \int_{\Omega\setminus A_k} \mathscr{Q}(v_k, e(u_k))dx &= \limsup_k \int_{\Omega\setminus A_k} \mathscr{Q}(v_k, e(u))dx \\ &\leq \int_{\Omega} \mathscr{Q}_1(e(u))dx \end{aligned} \tag{4.90}$$

by dominated convergence thanks to assumptions (H1) and (H3); then as a result of a straightforward calculation we infer

$$\begin{aligned}&\limsup_k \int_{A_k} \frac{\psi(v_k)}{\varepsilon_k}dx \leq \\ &\leq \lim_k \frac{\psi(\varepsilon_k)}{\psi(0)^{1/2}} \int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot e_n)\, d\mathcal{H}^{n-1} = \frac{b}{2}\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot e_n)\, d\mathcal{H}^{n-1}; \end{aligned} \tag{4.91}$$

furthermore from the very definition of w_k and (4.87) we find

$$\begin{aligned}&\int_{B_k} \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \leq \\ &\leq (1+O(\varepsilon_k))(\gamma p)^{1/p} q^{1/q} \Big(\int_{\varepsilon_k}^{1-\rho_k} \psi^{1/q}(s)\, ds\Big)\mathcal{H}^{n-1}(J_u); \end{aligned} \tag{4.92}$$

finally by the Coarea formula and again by the definition of w_k it follows that

$$\int_{C_k} \Big(\frac{\psi(v_k)}{\varepsilon_k} + \gamma\, \varepsilon_k^{p-1}|\nabla v_k|^p\Big)dx \leq c\,\mu_k \int_{\varepsilon_k}^{1-\rho_k} \psi^{1/q}(s)\ ds \leq c\,\mu_k, \tag{4.93}$$

where $c < +\infty$. Therefore, by collecting (4.90)-(4.93), to conclude we need only to verify that

$$\lim_k \int_{A_k} \mathscr{Q}(v_k, e(u_k))dx = \frac{b}{2}\int_{J_u} \mathscr{Q}_0^{1/2}([u]\odot e_n)d\mathcal{H}^{n-1}.$$

To this aim, observe first that assumption (H3), the very definition of u_k, v_k and estimates (4.88), (4.89) imply, as $k \uparrow +\infty$,

$$\int_{A_k} \mathscr{Q}(v_k, e(u_k))dx = \int_{A_k} \mathscr{Q}\Big(\varepsilon_k, \frac{1}{2}\Lambda(D_n u_k^1, \ldots, D_n u_k^{n-1}, 2D_n u_k^n)\Big)dx + o(1),$$

where $\Lambda\colon \mathbb{R}^n \to \mathbb{M}^{n\times n}_{sym}$ is defined by

$$(\Lambda(x_1,\dots,x_n))_{i\,j} := 0 \quad \text{if } i,j<n, \quad (\Lambda(x_1,\dots,x_n))_{i\,n} := x_i \quad \text{if } i\leq n. \tag{4.94}$$

In addition, the definition of σ_k in (4.83) and an easy computation yields

$$\begin{aligned}&\int_{A_k} \mathscr{Q}\Big(\varepsilon_k, \frac{1}{2}\Lambda(D_n u_k^1,\dots,D_n u_k^{n-1}, 2D_n u_k^n)\Big)dx = \\ &= \frac{b}{2}\int_{J_u} \mathscr{Q}_{\varepsilon_k}(\zeta_k(\overline{x}))\cdot \mathscr{Q}_0^{-1/2}([u](\overline{x},0)\odot e_n)d\mathcal{H}^{n-1},\end{aligned}$$

where

$$\begin{aligned}\zeta_k(\overline{x}) := \frac{1}{2}\Lambda\Big(&u^1(\overline{x},\sigma_k(\overline{x})) - u^1(\overline{x},-\sigma_k(\overline{x})),\dots,u^{n-1}(\overline{x},\sigma_k(\overline{x})) - u^{n-1}(\overline{x},-\sigma_k(\overline{x})),\\ &2(u^n(\overline{x},\sigma_k(\overline{x})) - u^n(\overline{x},-\sigma_k(\overline{x})))\Big).\end{aligned}$$

Eventually, the conclusion follows by (4.94), by (H4), and by the dominated convergence theorem as (ζ_k) converges uniformly to $[u](\cdot,0)\odot e_n$ on S as $k\uparrow\infty$. □

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