LOWER SEMICONTINUITY FOR NON-COERCIVE POLYCONVEX INTEGRALS IN THE LIMIT CASE

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In memory of Riccardo Ricci

Abstract. Lower semicontinuity results for polyconvex functionals of the Calculus of Variations along sequences of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ in $W^{1,m}$, $2 \leq m \leq n$, weakly converging in $W^{1,m-1}$ are established.

In addition, for $m = n + 1$, we also consider the autonomous case for weakly converging maps in $W^{1,n-1}$.

1. Introduction

Let m, n be positive integers, let Ω be a bounded open set of $\mathbb{R}^n$ and let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a *map* in the *Sobolev space* $W^{1,p}(\Omega, \mathbb{R}^m)$ for some $p \geq 1$. The *functional* associated to the map u is an integral of the type

$$F(u) = \int_\Omega f\big(x, u(x), \mathcal{M}^\ell(\nabla u(x))\big)\, dx\,, \tag{1.1}$$

where throughout the paper $\ell := \min\{m, n\}$ and $\mathcal{M}^\ell(\mathbb{A})$ denotes the vector whose components are all the *minors* of order up to ℓ of the matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, i.e.,

$$\mathcal{M}^\ell(\mathbb{A}) = (\mathbb{A}, \mathrm{adj}_2\mathbb{A}, \ldots, \mathrm{adj}_i\mathbb{A}, \ldots, \mathrm{adj}_\ell\mathbb{A})\,.$$

The celebrated result by Morrey (see [30] and [31]) establishes that the *quasi-convexity* of the energy density

$$g(\mathbb{A}) = f(x_0, u_0, \mathcal{M}^\ell(\mathbb{A}))$$

for $\mathcal{L}^n$ a.e. (x_0, u_0) is a necessary condition for the functional F to be (sequentially) lower semicontinuous in the weak* $W^{1,\infty}$ topology.

Since the seminal works of Morrey [30] and of Acerbi & Fusco [2] several authors investigated the sufficiency of quasi-convexity for the lower semicontinuity of F under various conditions (cp. [7, 17, 12, 18, 19, 26, 27, 28, 29]). However, due to the high generality of the quasi-convexity assumption, in all those contributions (and in all known results) some polynomial growth of the energy density (depending on the topology considered) with respect to the gradient variable is required.

A relevant subclass of quasi-convex functions arising in applications to continuum mechanics and geometric measure theory, [7, 23], is given by *polyconvex* integrands introduced by Ball [5], i.e., those energy densities f such that $f(x_0, u_0, \cdot)$ is convex for every point (x_0, u_0). In this case weak lower semicontinuity holds under weaker assumptions concerning both the growth of the integrands and the underlying topology following the results of Dacorogna & Marcellini [8] (cp. [1, 3, 4, 6, 9, 11, 13, 14, 15, 21, 22, 24, 25]).

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In this paper we follow this line of research. More precisely, we investigate the lower semicontinuity properties of energies as in (1.1), with densities f satisfying

(Hp) $f = f(x,u,\xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0,\infty)$ is $C^0(\Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma)$ and $f(x,u,\cdot)$ is convex for all $(x,u) \in \Omega \times \mathbb{R}^m$ (see (2.1) for the definition of σ),

along sequences

(Seq) $(u_j)_j \subset W^{1,\ell}(\Omega,\mathbb{R}^m)$ satisfying

$$u_j \rightharpoonup u \quad \text{in } W^{1,\ell-1}. \tag{1.2}$$

As pointed out by Malý in [26], lower semicontinuity fails if the requirements of (Seq) above are relaxed to weak topology in $W^{1,p}$ for $p < \ell - 1$ even for integrands depending only on the minors. We remark that if $\ell \geq 3$ condition (Seq) is equivalent to

$$u_j \to u \text{ in } L^1 \text{ and } \sup_j \|u_j\|_{W^{1,\ell-1}} < \infty, \tag{1.3}$$

while if $\ell = 2$ it is stronger and indeed it is equivalent to

$$u_j \to u \text{ in } L^1,\ \sup_j \|u_j\|_{W^{1,1}} < \infty, \text{ and } (\nabla u_j)_j \text{ is equi-integrable.}$$

Let us first describe our contributions in the case when the co-domain dimension m is less than or equal to the domain dimension n, i.e., $2 \leq m \leq n$ (cp. with Theorem 1.4 below for a sharper result in case $m = n$). In particular, in Theorem 1.1 we prove the following Serrin type result building upon the chain-rule argument introduced in [4], that extends to this setting the ideas of [32]. However, contrary to the above mentioned results, we also need to assume Lipschitz continuity in the variable u.

Theorem 1.1. *Let $2 \leq m \leq n$, let f satisfy* (Hp)*, and suppose in addition that*

$$f(\cdot,\cdot,\xi) \quad \text{is locally Lipschitz continuous for all } \xi \in \mathbb{R}^\sigma. \tag{1.4}$$

Then, for every sequence $(u_j)_j \subset W^{1,m}(\Omega,\mathbb{R}^m)$ satisfying (Seq) *we have*

$$F(u) \leq \liminf_j F(u_j).$$

In the autonomous case $f = f(\xi)$ the conclusions of Theorem 1.1 has been established in [11, Theorem 3.1] under the only assumption (1.3) (which is weaker than (Seq) for $m = 2$ as noted above).

The conclusions of Theorem 1.1 can be extended to the class of densities that are approximated from below by those satisfying (1.4). It was established in [20, Theorem 7] that demi-coercive integrands, i.e. coercive up to addition of null Lagrangeans, belong to the latter class. More precisely, supposing that

(Dem) there exist functions $\alpha : \Omega \times \mathbb{R}^m \to \mathbb{R}^\sigma$, $\beta, \gamma : \Omega \times \mathbb{R}^m \to \mathbb{R}$, with $\beta > 0$ such that

$$f(x,u,\xi) + \langle \alpha(x,u), \xi \rangle \geq \beta(x,u)|\xi| + \gamma(x,u), \tag{1.5}$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma$,

we can establish the next result.

Theorem 1.2. *Let $2 \leq m \leq n$, and let f enjoy* (Hp) *and* (Dem)*.*

Then, for every sequence $(u_j)_j \subset W^{1,m}(\Omega,\mathbb{R}^m)$ satisfying (Seq) *we have*

$$F(u) \leq \liminf_j F(u_j).$$

Furthermore, due to (1.2), we can slightly weaken (Dem) and deduce the following corollary in which demicoercivity is required only for higher order minors.

Corollary 1.3. *Let $2 \le m \le n$, let f enjoy* (Hp) *and*

$$f(x,u,\xi) + \langle \alpha(x,u), \xi \rangle \ge \beta(x,u)|\eta| + \gamma(x,u), \tag{1.6}$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma$ and for some functions $\alpha : \Omega \times \mathbb{R}^m \to \mathbb{R}^\sigma$, $\beta, \gamma : \Omega \times \mathbb{R}^m \to \mathbb{R}$, with $\beta > 0$, where $\xi = (\bar{\xi}, \eta) \in \mathbb{R}^{\sigma-\tau} \times \mathbb{R}^\tau$ with $\tau = \binom{n}{m}$.

Then, for every sequence $(u_j)_j \subset W^{1,m}(\Omega, \mathbb{R}^m)$ satisfying (Seq) *we have*

$$F(u) \le \liminf_j F(u_j).$$

Note that the assumption that f is bounded from below cannot be dropped. An example is given by taking the demicoercive integrand $f(\xi) = -(\det \xi)_+$, $\xi \in \mathbb{R}^{2\times 2}$, and the sequence $u_j(x,y) = y^j(\sin(jx), \cos(jx))$, $(x,y) \in (0,1)^2$.

In case the dimensions n and m are equal we can actually remove the Lipschitz continuity assumption on f as in Theorem 1.1 obtaining the following sharp result.

Theorem 1.4. *Let $2 \le m = n$, and let f enjoy* (Hp).

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ satisfying (Seq) *we have*

$$F(u) \le \liminf_j F(u_j). \tag{1.7}$$

The result above can be extended to energy densities that can be approximated from below by those satisfying (Hp). In particular, our result can be used to deal with some integrands such that $f(\cdot, \cdot, \mathcal{M}^n(\mathbb{A})) \to +\infty$ if $\det \mathbb{A} \to 0^+$. For $n = m = 3$, these integrands appear in problems of non-linear elasticity.

Theorem 1.4 builds upon the recent contribution [11, Theorem 1.1] where an additional technical hypothesis on the integrand was assumed (see case (a) in the proof of Theorem 1.4 below). Actually, [11, Theorem 1.1] was proven under the convergence conditions in (1.3).

Finally, let us discuss the case $m > n$. To our knowledge the best known result is [18, Corollary 4.2] where lower semicontinuity in the weak $W^{1,p}$ topology, $p > n-1$, is established for autonomous densities (see also [28], [17], [18, Remark 4.3] for related results). Moreover, the counterexample [18, Example 4.4] shows that even a smooth dependence of the integrand with respect to the variable u is forbidden.

For $m = n+1$ a geometric argument allows us to reduce the problem to the case of equal dimensions, and to prove the following result in the autonomous setting under the only assumption (Seq). This is ,to the best of our knowledge, the first result establishing lower semicontinuity in the critical case when the dimension of the co-domain is strictly greater than the one of the domain.

Theorem 1.5. *Let $2 \le m = n+1$, let $f : \mathbb{R}^\sigma \to [0,\infty)$ be convex, and*

$$F(u) = \int_\Omega f\big(\mathcal{M}^n(\nabla u(x))\big)\, dx\,.$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^{n+1})$ satisfying (Seq) *we have*

$$F(u) \le \liminf_j F(u_j).$$

Eventually, we resume the structure of the paper. In Section 2 we introduce the notation, recall several auxiliary results and prove some technical lemmas. Section 3 is devoted to prove Theorem 1.1 and Theorem 1.2. In Section 4 we give the proof of Theorem 1.4; and finally in Section 5 we prove Theorem 1.5.

2. Definitions and preliminary results

The aim of this section is to introduce some notations and to recall some basic definition and results which will be used in the sequel.

We begin with some algebraic notation. Let $n, m \geq 2$ and $\mathbb{M}^{m\times n}$ be the linear space of all $m \times n$ real matrices. For $\mathbb{A} \in \mathbb{M}^{m\times n}$, we denote $\mathbb{A} = (\mathbb{A}^i_j)$, $1 \leq i \leq m$, $1 \leq j \leq n$, where upper and lower indices correspond to rows and columns respectively.

The euclidean norm of $\mathbb{A}$ will be denoted by $|\mathbb{A}|$. The number of all minors up to the order $\ell = \min\{m, n\}$ of any matrix in $\mathbb{M}^{m\times n}$ is given by

$$\sigma := \sum_{i=1}^{\ell} \binom{m}{i}\binom{n}{i}. \tag{2.1}$$

We shall also adopt the following notations. We set

$$I_{l,k} = \{\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l : 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_l \leq k\},$$

where $1 \leq l \leq k$. If $\alpha \in I_{l,m}$ and $\beta \in I_{l,n}$, then $M_{\alpha,\beta}(\mathbb{A}) = \det(\mathbb{A}^{\alpha_i}_{\beta_j})$.

By $\mathcal{M}_l(\mathbb{A})$ we denote the vector whose components are all the minors of order l, and by $\mathcal{M}^l(\mathbb{A})$ the vector of all minors of order up to l, for every $l \in \{1, \ldots, \ell\}$.

As usual, $Q_r(x)$, $B_r(x)$ denote the open euclidean cube, ball in $\mathbb{R}^n$, $n \geq 2$, with side r, radius r and center the point x, respectively. The center shall not be indicated explicitly if it coincides with the origin.

We shall often deal in what follows with convergences of measures. As usual, we shall name local weak* convergence of Radon measures the one defined by duality with $C_c(\Omega)$, and weak* convergence the one defined by duality with $C_0(\Omega)$ on the subset of finite Radon measures.

2.1. **Approximation of convex functions.** We survey now on an approximation theorem for convex functions, due to De Giorgi, that plays an important role in the framework of lower semicontinuity problems (see [10]). Given a convex function $f : \mathbb{R}^k \to \mathbb{R}$, $k \geq 1$, consider the affine functions $\xi \to a_j + \langle b_j, \xi\rangle$, with $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R}^k$, given by

$$a_j := \int_{\mathbb{R}^k} f(\eta)\big((k+1)\alpha_j(\eta) + \langle \nabla\alpha_j(\eta), \eta\rangle\big)d\eta \tag{2.2}$$

$$b_j := -\int_{\mathbb{R}^k} f(\eta)\nabla\alpha_j(\eta)d\eta, \tag{2.3}$$

where, for all $j \in \mathbb{N}$, $\alpha_j(\xi) := j^n\alpha(j(q_j - \xi))$, $(q_j)_j \subset \mathbb{Q}^k$ and $\alpha \in C^1_0(\mathbb{R}^k)$ is a non negative function such that $\int_{\mathbb{R}^k} \alpha(\eta)d\eta = 1$.

Lemma 2.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function and a_j, b_j be defined as in* (2.2)-(2.3)*. Then,*

$$f(\xi) = \sup_{j\in\mathbb{N}} \left(a_j + \langle b_j, \xi\rangle\right), \quad \textit{for all } \xi \in \mathbb{R}^k.$$

The main feature of the approximation above is the explicit dependence of the coefficients a_j and b_j on f. In particular, if f depends on the lower order variables (x, u) regularity properties of the coefficients a_j and b_j with respect to (x, u) can be easily deduced from related hypotheses satisfied by f thanks to formulas (2.2) and (2.3) and Lemma 2.1. We thus have the following approximation result:

Theorem 2.2. *Let $f = f(x, u, \xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0, \infty)$, be a continuous function, convex in the last variable ξ. Then, there exist two sequences of compactly supported continuous functions*

$a_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $b_i : \Omega \times \mathbb{R}^m \to \mathbb{R}^\sigma$ *such that, setting for every* $i \in \mathbb{N}$,

$$f_i(x,u,\xi) := \big(a_i(x,u) + \langle b_i(x,u), \xi\rangle\big)_+, \tag{2.4}$$

then

$$f(x,u,\xi) = \sup_i f_i(x,u,\xi).$$

Moreover, for every $i \in \mathbb{N}$ *there exists a positive constant* C_i *such that*

(a) f_i *is continuous, convex in* ξ *and*

$$0 \le f_i(x,u,\xi) \le C_i(1+|\xi|) \quad \textit{for all } (x,u,\xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma; \tag{2.5}$$

(b) *if* ω_i *denotes a modulus of continuity of* $a_i + |b_i|$ *we have*

$$|f_i(x,u,\xi_1) - f_i(y,v,\xi_2)| \le C_i|\xi_1 - \xi_2| + \omega_i(|x-y| + |u-v|)(1 + \min\{|\xi_1|, |\xi_2|\}) \tag{2.6}$$

for all (x,u,ξ_1) *and* $(y,v,\xi_2) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma$.

The compactness of the supports of a_i and b_i is obtained by first approximating f with a monotone sequence $f_j(x,u,\xi) := m_j(x,u)f(x,u,\xi)$, where $m_j \in C_c(\Omega \times \mathbb{R}^m)$, $m_j = 1$ on $\Omega_j \times \{|u| < j\}$ and $m_j = 0$ on $\Omega \times \mathbb{R}^m \setminus (\Omega_{j+1} \times \{|u| < j+1\})$, $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ a family of open sets invading Ω; and then applying to each f_j De Giorgi's approximation result in Lemma 2.1.

2.2. **Convergence of minors.** Let us start recalling the following lemma (see [21, Lemma 2.2], and also [6, Corollary 3.3] for $m = n$).

Lemma 2.3. *Let* $2 \le l \le \ell$, u_j, $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ *be maps such that*

(a) $(u_j)_j$ *converges to* u *in* $L^\infty(\Omega, \mathbb{R}^m)$;

(b) $\sup_j \|\mathcal{M}^l(\nabla u_j)\|_{L^1} < \infty$.

Then, $\mathcal{M}^l(\nabla u_j) \rightharpoonup \mathcal{M}^l(\nabla u)$ *weakly** *in the sense of measures on* Ω.

For sequences satisfying weaker assumptions than those in Lemma 2.3 we can still determine the absolutely continuous part, with respect to the Lebesgue measure, of the limit measures of the sequence of minors as proven by Celada and Dal Maso in [6, Lemma 1.2] (see also [16]). We state their result in the form needed for our purposes.

Lemma 2.4. *Let* $n \ge 2$ *and* $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ *be bounded in* L^∞, *weakly converging in* $W^{1,n-1}$ *to* u, *and such that* $\det \nabla u_j \overset{*}{\rightharpoonup} \mu$ *locally as Radon measures.*

Then, the absolutely continuous part μ^a *of* μ *satisfies* $\mu^a = \det \nabla u \, d\mathcal{L}^n$.

We recall also Zhang's biting lemma for minors in the form needed in the subsequent sections (cp. with [33, Theorem 2.1] for the full statement).

Theorem 2.5. *Let* $2 \le m \le n$, $U \subset \mathbb{R}^n$ *be a bounded open smooth set, and* $(u_j)_j \subset W^{1,m-1}(U, \mathbb{R}^m)$ *be weakly converging to* u *in* $W^{1,m-1}$.

Then, there exists a subsequence (not relabelled for convenience) and a decreasing family $(U_h)_h$ *of Borel subsets of* U *such that* $\mathcal{L}^n(U_h) \downarrow 0$ *and*

$$\mathcal{M}^{m-1}(\nabla u_j) \rightharpoonup \mathcal{M}^{m-1}(\nabla u) \quad L^1(U \setminus U_h), \ \textit{for all } h \in \mathbb{N}.$$

We will also need the following result proved in [6, Lemma 3.2].

Lemma 2.6. *Let* $(\mu_k)_k$ *be a sequence of signed Radon measures on* Ω. *Assume that*

(a) *there exists* $T \in \mathcal{D}'(\Omega)$ *such that* $\mu_k \to T$ *in the sense of distributions on* Ω;

(b) *there exists a positive Radon measure* ν *such that* $\mu_k^+ \rightharpoonup \nu$ *(locally) weakly** *in the sense of measures on* Ω.

Then, there exists a Radon measure μ such that $T = \mu$ on Ω and $\mu_k \rightharpoonup T$ locally weakly* in the sense of measures on Ω.

2.3. **Two truncation results for minors.** In this section we give two truncation results. The first statement below, that is instrumental to prove Theorem 1.1, follows from Lemma 2.3 and by refining De Giorgi's truncation method on the codomain as employed by Malý [26, Theorem 3.1].

Proposition 2.7. *Let $2 \le m \le n$ and let $(u_j)_j \subset W^{1,m}(\Omega, \mathbb{R}^m)$ and u in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ be such that $u_j \rightharpoonup u$ in $W^{1,m-1}(\Omega, \mathbb{R}^m)$.*

Set $\bar{u}_j := (u_j^2, \dots, u_j^m)$, $D_\alpha := (\partial_{\alpha_1}, \dots, \partial_{\alpha_m})$ for all $\alpha \in I_{m,n}$, and let $(w_j)_j \subset W^{1,m-1}(\Omega, \mathbb{R}^\tau)$, $\tau = \# I_{m,n}$, such that

$$\text{either } \sup_j \|w_j\|_{W^{1,m-1}} < \infty \text{ if } m \ge 3 \text{ or } (\nabla w_j)_j \text{ is equi-integrable if } m = 2,$$

and satisfying

$$\sup_j \int_\Omega \Big| \sum_{\alpha \in I_{m,n}} \det \Big[D_\alpha w_j^\alpha, D_\alpha u_j^2, \dots, D_\alpha u_j^m \Big] \Big| \, dx < \infty. \tag{2.7}$$

Then, there exist sequences $(\bar{v}_j)_j \in W^{1,m}(\Omega, \mathbb{R}^{m-1})$ and $s_j \downarrow 0$ such that

$$\bar{v}_j \to \bar{u} := (u^2, \dots, u^m) \quad \text{in } L^\infty(\Omega, \mathbb{R}^{m-1}), \tag{2.8}$$

and

$$\{x \in \Omega : \bar{u}_j(x) \ne \bar{v}_j(x)\} \subseteq A_j := \{x \in \Omega : |\bar{u}_j(x) - \bar{u}(x)| > s_j\}. \tag{2.9}$$

Moreover, setting $v_j = (u_j^1, \bar{v}_j)$ we have

$$v_j \rightharpoonup u \qquad \text{in } W^{1,m-1}(\Omega, \mathbb{R}^m), \tag{2.10}$$

$$\mathcal{M}^{m-1}(\nabla v_j) \rightharpoonup \mathcal{M}^{m-1}(\nabla u) \quad \text{weakly}^* \text{ in the sense of measures,} \tag{2.11}$$

and

$$\lim_j \int_{A_j} \left(1 + \big|\mathcal{M}^{m-1}(\nabla v_j)\big| + \Big(\sum_{\alpha \in I_{m,n}} \det \Big[D_\alpha w_j^\alpha, D_\alpha \bar{v}_j^2, \dots, D_\alpha \bar{v}_j^m \Big] \Big)_+ \right) dx = 0. \tag{2.12}$$

Proof. With the same notations as above, we first note that up to a subsequence not relabeled for convenience we can assume that

$$\lim_j \mathcal{L}^n \big(\{y \in \Omega : |\bar{u}_j - \bar{u}| > 2^{-2j+1}\} \big) = 0. \tag{2.13}$$

Moreover thanks to the boundedness of the sequence and (2.7), for every $j \in \mathbb{N}$ we can choose a $k_j \in \{j+1, \dots, 2j\}$ such that

$$\int_{\{2^{-k_j} < |\bar{u}_j - \bar{u}| \le 2^{-k_j+1}\}} \Big(\Big| \sum_{\alpha \in I_{m,n}} \det \Big[D_\alpha w_j^\alpha, D_\alpha u_j^2, \dots, D_\alpha u_j^m \Big] \Big| + |\nabla u_j|^{m-1} \Big) dx \le \frac{C}{j}. \tag{2.14}$$

Let now $\psi : \mathbb{R}^+ \to \mathbb{R}$ be defined as

$$\psi(s) := \begin{cases} 1 & 0 \le s \le 1 \\ 2 - s & 1 \le s \le 2 \\ 0 & s \ge 2, \end{cases}$$

and set

$$\bar{v}_j := \bar{u} + \psi\big(2^{k_j} |\bar{u}_j - \bar{u}|\big)(\bar{u}_j - \bar{u}).$$

Clearly,

$$\bar v_j = \bar u_j \quad \text{on the set} \quad \{|\bar u_j - \bar u| \le 2^{-k_j}\} \tag{2.15}$$

and

$$|\bar v_j - \bar u| \le 2^{-k_j+1},$$

so that $(\bar v_j)_j$ converges to $\bar u$ in L^∞, i.e. (2.8) and (2.9) are satisfied with $s_j := 2^{-k_j}$. Moreover,

$$\nabla \bar v_j = \nabla \bar u + R_j[\bar u_j - \bar u](\nabla \bar u_j - \nabla \bar u), \tag{2.16}$$

where

$$R_j[y] := \psi(2^{k_j}|y|)\,\mathrm{Id}_{m-1} + 2^{k_j}\,\psi'(2^{k_j}|y|)\,\frac{y\otimes y}{|y|} \in \mathbb{R}^{(m-1)\times(m-1)}.$$

In particular, setting

$$R_j := R_j[\bar u_j - \bar u],$$

then for some dimensional constant C

$$\big|\mathcal{M}^{m-1}(R_j)\big| \le C. \tag{2.17}$$

The above equation and a standard computation show that

$$|\nabla \bar v_j| \le C(1+|\nabla \bar u_j|) \qquad \text{and} \qquad \big|\mathcal{M}^{m-1}(\nabla v_j)\big| \le C\big(1+\big|\mathcal{M}^{m-1}(\nabla u_j)\big|\big) \tag{2.18}$$

where $v_j = (u_j^1, \bar v_j)$ and C depends also on $\|\nabla \bar u\|_{L^\infty}$. Equation (2.18) implies (2.10) while (2.11) follows by (2.10) and (2.8) by a standard induction and integration by part argument (this follows for instance by an inspection of the proof of [11, Corollary 2.6] noting that it is enough that all but one components of u_j converge in L^∞).

We are thus left to show (2.12). To this end, notice that, thanks to (2.16),

$$\begin{aligned}\Big[D_\alpha w_j^\alpha, D_\alpha \bar v_j^2, \dots, D_\alpha \bar v_j^m\Big]^T &= \Big[D_\alpha w_j^\alpha, R_j D_\alpha u_j^2, \dots, R_j D_\alpha u_j^m\Big]^T \\ &\quad + \Big[D_\alpha w_j^\alpha, \big(\mathrm{Id}_{m-1} - R_j\big) D_\alpha u^2, \dots, \big(\mathrm{Id}_{m-1} - R_j\big) D_\alpha u^m\Big]^T,\end{aligned}$$

where $\mathbb{B}^T$ denotes the transpose of $\mathbb{B} \in \mathbb{R}^{m\times m}$.

Now an elementary computation based on the formula for the determinant of the sum of two matrices (cp. [7, Proposition 5.67]) shows that

$$\begin{aligned}&\int_{\{2^{-k_j}<|\bar u_j - \bar u| \le 2^{-k_j+1}\}} \Big(\sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_j^\alpha, D_\alpha \bar v_j^2, \dots, D_\alpha \bar v_j^m\Big]\Big)_+ dx \\ &\quad\le \int_{\{2^{-k_j}<|\bar u_j - \bar u| \le 2^{-k_j+1}\}} \big|\det R_j\big| \cdot \Big|\sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_j^\alpha, D_\alpha u_j^2, \dots, D_\alpha u_j^m\Big]\Big| dx \\ &\qquad + C\Bigg(\int_{\{2^{-k_j}<|\bar u_j - \bar u| \le 2^{-k_j+1}\}} (1+|\nabla u_j|^{m-1})\,dx\Bigg)^{\frac{m-2}{m-1}}\end{aligned} \tag{2.19}$$

where C depends on $\|\nabla u\|_{L^\infty}$ and $\|w_j\|_{W^{1,m-1}}$. By (2.17), $|\det R_j| \le C$, hence, (2.14), (2.18) and (2.19) imply that

$$\lim_j \int_{\{2^{-k_j}<|\bar u_j - \bar u| \le 2^{-k_j+1}\}} \big|\mathcal{M}^{m-1}(\nabla v_j)\big| + \Big(\sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_j^\alpha, D_\alpha u_j^2, \dots, D_\alpha u_j^n\Big]\Big)_+ dx = 0, \tag{2.20}$$

where we have also used that $\mathcal{L}^n\big(\{2^{-k_j} < |\bar{u}_j - \bar{u}| \leq 2^{-k_j+1}\}\big) \to 0$. In addition, since equality $\bar{v}_j = \bar{u}$ holds true on the set $\{2^{-k_j+1} < |\bar{u}_j - \bar{u}|\}$, we infer that

$$\begin{aligned}
&\int\limits_{\{2^{-k_j+1}<|\bar{u}_j-\bar{u}|\}} \big|\mathcal{M}^{m-1}(\nabla v_j)\big| + \Big(\sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_j^\alpha, D_\alpha \bar{v}_j^2, \dots, D_\alpha \bar{v}_j^m\Big]\Big)_+ dx \\
&\quad = \int\limits_{\{2^{-k_j+1}<|\bar{u}_j-\bar{u}|\}} \big|\mathcal{M}^{m-1}\big(\nabla(u_j^1, \bar{u})\big)\big| + \Big(\sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_j^\alpha, D_\alpha \bar{u}^2, \dots, D_\alpha \bar{u}^m\Big]\Big)_+ dx \\
&\quad \leq C \int\limits_{\{2^{-k_j+1}<|\bar{u}_j-\bar{u}|\}} \big(1 + |\nabla u_j^1| + |\nabla w_j|\big)\, dx, \qquad (2.21)
\end{aligned}$$

for some constant C depending on $\|\nabla \bar{u}\|_{L^\infty}$.

Finally, (2.12) immediately follows from (2.13), (2.20) and (2.21) thanks to the assumptions on $(w_j)_j$, the fact that u_j is weakly convergent in $W^{1,m-1}$ (so that $|\nabla u_j|$ is equi-integrable) and since $k_j \leq 2j$. □

The ensuing proposition, that can be proven analogously to Proposition 2.7, is a slight improvement of a well-known result by Fusco and Hutchinson (see [21, Proposition 2.5], and also [11, Proposition 2.8] for a variant under weaker assumptions).

Proposition 2.8. *Let $2 \leq m$, n and let $(u_j)_j \subset W^{1,\ell}(\Omega, \mathbb{R}^m)$ and u in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ be such that $u_j \to u$ in $L^1(\Omega, \mathbb{R}^m)$ and*

$$\sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty.$$

Then there exist sequences $(v_j)_j \in W^{1,\ell}(\Omega, \mathbb{R}^m)$ and $s_j \downarrow 0$ such that

$$v_j \to u \quad L^\infty(\Omega, \mathbb{R}^m), \tag{2.22}$$

$$\mathcal{M}^\ell(\nabla v_j) \rightharpoonup \mathcal{M}^\ell(\nabla u) \quad \text{weakly}^* \text{ in the sense of measures}, \tag{2.23}$$

$$\{x \in \Omega : u_j(x) \neq v_j(x)\} \subseteq A_j := \{x \in \Omega : |u_j(x) - u(x)| > s_j\}, \tag{2.24}$$

and

$$\lim_j \int_{A_j} \Big(1 + |\mathcal{M}^\ell(\nabla v_j)|\Big)\, dx = 0. \tag{2.25}$$

Moreover, if $u_j \rightharpoonup u$ in $W^{1,\ell-1}(\Omega, \mathbb{R}^m)$, then also $v_j \rightharpoonup u$ in $W^{1,\ell-1}(\Omega, \mathbb{R}^m)$.

2.4. **A blow-up type lemma.** The contents of the next lemma show that to infer the lower semicontinuity inequality

$$F(u) \leq \liminf_j F(u_j), \tag{2.26}$$

for functionals F defined as in (1.1), with integrands f satisfying (Hp) and along sequences $(u_j)_j \subset W^{1,\ell}(\Omega, \mathbb{R}^m)$ satisfying (Seq) we can always reduce ourselves to affine target maps. This was first observed in [19] and we refer to [11, Lemma 2.11] for a proof in case $3 \leq \ell$ (the other case follows similarly).

Lemma 2.9. *Suppose that for $\mathcal{L}^n$ a.e. $x_0 \in \Omega$, and for all sequences $\varepsilon_k \downarrow 0$ and $(u_k)_k \subset W^{1,\ell}(Q_1, \mathbb{R}^m)$ such that*

$$u_k \rightharpoonup u_0 := \nabla u(x_0) \cdot y \quad \text{in } W^{1,\ell-1}(Q_1, \mathbb{R}^m),$$

we have

$$\liminf_k \int_{Q_1} f\big(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, u_k, \mathcal{M}^\ell(\nabla u_k)\big)\, dy \geq f\big(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))\big), \tag{2.27}$$

then the lower semicontinuity inequality (2.26) *holds.*

3. The case $m \leq n$

In this section we prove Theorem 1.1. The argument is based on the chain rule formula in the spirit of [4] and on Proposition 2.7.

Proof of Theorem 1.1. We divide the proof in some steps. First note that, thanks to Theorem 2.2, it will be enough to prove the Theorem for functionals of the form

$$F(u) = \int_\Omega f(x, u(x), \mathcal{M}^m(\nabla u(x)))\, dx$$

with

$$f(x, u, \xi) = \big(a(x, u) + \langle b(x, u), \xi\rangle\big)_+$$

satisfying (2.5) and (2.6), with a and b continuous and compactly supported. Moreover, thanks to Lemma 2.9 it is enough to show that if $\varepsilon_k \downarrow 0$ and $(u_k)_k \subset W^{1,m}(Q_1, \mathbb{R}^m)$ is a sequence such that

$$u_k \rightharpoonup u_0 := \nabla u(x_0) \cdot y \text{ in } W^{1,m-1}(Q_1, \mathbb{R}^m),$$

then

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, u_k, \mathcal{M}^m(\nabla u_k))\, dy \geq f(x_0, u(x_0), \mathcal{M}^m(\nabla u(x_0))). \tag{3.1}$$

We assume without loss of generality that $x_0 = 0$ and $u(x_0) = 0$. Moreover we can also safely assume that

$$\sup_k \int_{Q_1} f(\varepsilon_k\, y, \varepsilon_k\, u_k, \mathcal{M}^m(\nabla v_k))\, dy \leq C \tag{3.2}$$

and that the liminf is actually a limit. We also recall the notation $\xi = (\bar{\xi}, \eta) \in \mathbb{R}^{\sigma-\tau} \times \mathbb{R}^\tau$ so that we can write

$$f(x, u, \xi) = \big(a(x, u) + \langle \bar{b}(x, u), \bar{\xi}\rangle + \langle c(x, u), \eta\rangle\big)_+. \tag{3.3}$$

Step 1. *Truncation.* We show that we can replace the sequence $(u_k)_k$ with a new sequence $(\widetilde{u}_k)_k \subset W^{1,m}$ which is uniformly bounded in L^∞. Since $u_k \rightharpoonup u_0$ in $W^{1,m-1}$ we clearly have

$$\sup_k \int_{Q_1} |\nabla u_k|^{m-1} \leq C. \tag{3.4}$$

Let us now take $M \geq 1 + \|u_0\|_{L^\infty}$ and $j \in \mathbb{N}$, $j \geq 1$. Then we can find $j_k \in \{1, \cdots, j\}$ such that

$$\int_{Q_1 \cap \{M^{j_k} \leq |u_k| \leq M^{j_k+1}\}} |\nabla u_k|^{m-1} \leq \frac{C}{j}.$$

Let us now set $\widetilde{u}_k := \pi_{M^{j_k}}(u_k)$ where

$$\pi_M(u) := \begin{cases} u & \text{if } |u| \leq M \\ M\dfrac{u}{|u|} & \text{if } |u| \geq M. \end{cases} \tag{3.5}$$

Clearly $|\widetilde{u}_k| \le M^j$, $\widetilde{u}_k = u_k$ on $\{|u_k| \le M^{j_k}\}$ and

$$\begin{aligned}\int_{Q_1\cap\{\widetilde{u}_k\neq u_k\}} |\nabla \widetilde{u}_k|^{m-1} &\le \int_{Q_1\cap\{M^{j_k}\le|u_k|\le M^{j_k+1}\}} |\nabla u_k|^{m-1} + \frac{1}{M^{m-1}}\int_{Q_1\cap\{M^{j_k+1}\le|u_k|\}\}} |\nabla u_k|^{m-1}\\ &\le \frac{C}{j} + \frac{C}{M^{m-1}} := \eta_{M,j}\end{aligned} \tag{3.6}$$

where we have used that $\mathrm{Lip}\,(\pi_{M^{j_k}}) \le 1$, $\mathrm{Lip}\Big(\pi_{M^{j_k}}\big|_{\{|u|\ge M^{j_k+1}\}}\Big) \le 1/M$ and (3.4). Hence since $\mathcal{M}_m(\nabla\widetilde{u}_k) = 0$ on $\{\widetilde{u}_k \neq u_k\}$ we get thanks to (3.6) and by taking into account (3.3) and the boundedness of $\bar{b}$ that

$$\int_{Q_1} f\big(\varepsilon_k y, \varepsilon_k \widetilde{u}_k, \mathcal{M}^m(\nabla\widetilde{u}_k)\big)dx \le \int_{Q_1} f\big(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^m(\nabla u_k)\big)dx + \widetilde{\eta}_{M,j}.$$

with $\widetilde{\eta}_{M,j} \downarrow 0$ as $M, j \uparrow +\infty$. It will be thus enough to prove (3.1) with $\widetilde{u}_k$ instead of u_k. For notational simplicity we will re-name $\widetilde{u}_k$ as u_k.

Step 2. *Re-writing the functional through the chain-rule.* Using that $(u_k)_k \subset W^{1,m}$, that c is Lipschitz continuous and the multi-linearity and antisimmetry of the determinant, following [4] we can write

$$\begin{aligned}&\Big\langle c(\varepsilon_k y, \varepsilon_k u_k(y)), \mathcal{M}_m(\nabla u_k(y))\Big\rangle\\ &= \sum_{\alpha\in I_{m,n}} c^\alpha\big(\varepsilon_k y, \varepsilon_k u_k(y)\big)\det\left[\frac{\partial(u_k^1,\dots,u_k^m)}{\partial(y_{\alpha_1},\dots,y_{\alpha_m})}\right]\\ &= \sum_{\alpha\in I_{m,n}}\Big(\det\Big[D_\alpha w_k^\alpha, D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big] - \det\Big[(D_\alpha C^\alpha)(\varepsilon_k y,\varepsilon_k u_k), D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big]\Big),\end{aligned} \tag{3.7}$$

where we have set

$$C^\alpha(x,u) = \int_0^{u^1} c^\alpha(x,t,u^2,\dots,u^m)dt \tag{3.8}$$

and

$$w_k^\alpha(y) = \frac{1}{\varepsilon_k}C^\alpha(\varepsilon_k y, \varepsilon_k u_k(y)). \tag{3.9}$$

Notice that

$$|Dw_k^\alpha(y)| \le \varepsilon_k\|u_k^1\|_{L^\infty}\,\mathrm{Lip}(c^\alpha)\Big(1+\sum_{i=2}^m |Du_k^i(y)|\Big) + \|c^\alpha\|_{L^\infty}|Du_k^1(y)| \tag{3.10}$$

and therefore

$$w_k^\alpha \rightharpoonup c^\alpha(0,0)u_0^1 \qquad \text{in } W^{1,m-1}(\Omega,\mathbb{R}^m). \tag{3.11}$$

Moreover,

$$\big\|(D_\alpha C^\alpha)(\varepsilon_k\,\cdot, \varepsilon_k u_k(\cdot)\|_{L^\infty} \le \varepsilon_k\|u_k^1\|_{L^\infty}\mathrm{Lip}(c^\alpha)$$

so that recalling (3.3), equation (3.7) and the bounds of the sequence in $W^{1,m-1}$ imply,

$$\begin{aligned}
&\liminf_{k\to\infty}\int_{Q_1} f\big(\varepsilon_k y,\varepsilon_k u_k,\mathcal{M}^m(\nabla u_k)\big)dx\\
&\geq \liminf_{k\to\infty}\int_{Q_1}\Big(a(\varepsilon_k y,\varepsilon_k u_k)+\big\langle \bar b(\varepsilon_k y,\varepsilon_k u_k),\mathcal{M}^{m-1}(\nabla u_k)\big\rangle\\
&\quad+\sum_{\alpha\in I_{m,n}}\det\Big[D_\alpha w_k^\alpha,D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big]\Big)_+dx-\limsup_{k\to\infty}\big\|(D_\alpha C^\alpha)(\varepsilon_k\,\cdot,\varepsilon_k u_k(\cdot))\big\|_{L^\infty}\big\|u_k\big\|_{W^{1,m-1}}\\
&=\liminf_{k\to\infty}\int_{Q_1}\Big(a(\varepsilon_k y,\varepsilon_k u_k)+\big\langle \bar b(\varepsilon_k y,\varepsilon_k u_k),\mathcal{M}^{m-1}(\nabla u_k)\big\rangle+\sum_{\alpha\in I_{m,n}}\det\Big[D_\alpha w_k^\alpha,D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big]\Big)_+dx.
\end{aligned}\tag{3.12}$$

Note that since we are assuming that the limit is finite, the boundedness in $W^{1,m-1}\cap L^\infty$ of the sequence $(u_k)_k$ and (3.12) imply

$$\sup_k\int_{Q_1}\Big(\sum_{\alpha\in I_{m,n}}\det\Big[D_\alpha w_k^\alpha,D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big]\Big)_+\leq C,\tag{3.13}$$

for some constant C depending on $\|a\|_{L^\infty}$, $\|\bar b\|_{L^\infty}$, $\sup_k\|u_k\|_{W^{1,m-1}}$ (and not on k).

Step 3. L^1 *boundedness.* We are going to improve (3.13) to

$$\sup_k\int_{Q_\rho}\left|\sum_{\alpha\in I_{m,n}}\det\Big[D_\alpha w_k^\alpha,D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big]\right|\leq C_\rho\qquad\text{for all } 0<\rho<1,\tag{3.14}$$

for some constant C_ρ independent of k. For, notice that the order one distributions

$$\begin{aligned}
C_0^1(Q_1)\ni\varphi\longmapsto T_k(\varphi)&:=\sum_{\alpha\in I_{m.n}}\int_{Q_1}\varphi\det\Big[D_\alpha w_k^\alpha,D_\alpha u_k^2,\dots,D_\alpha u_k^m\Big]\\
&=\sum_{\alpha\in I_{m.n}}\int_{Q_1}u_k^2\det\Big[D_\alpha w_k^\alpha,D_\alpha\varphi,\dots,D_\alpha u_k^m\Big],
\end{aligned}\tag{3.15}$$

are such that $|T_k(\varphi)|\leq C\|\nabla\varphi\|_{L^\infty}$, where C depends on the L^∞ and $W^{1,m-1}$ bounds on the sequence u_k. In particular they converge (up to subsequences) as order one distributions, since by (3.13) their positive parts converge as measures, (3.14) is a consequence of Lemma 2.6 and the Uniform Boundedness Principle.

Step 4. *Conclusion.* We now apply Proposition 2.7 in the open set Q_ρ, $\rho<1$ to construct a sequence $(\bar v_k)_k$ which satisfies conclusions (2.8), (2.9) (2.11) and (2.12). In particular, setting $v_k=(u_k^1,\bar v_k)$ we have

$$v_k\rightharpoonup u_0\quad\text{in } W^{1,m-1}(Q_\rho,\mathbb{R}^m),\qquad \bar u_k\to\bar u_0:=(u_0^2,\cdots,u_0^m)\quad\text{in } L^\infty(Q_\rho,\mathbb{R}^{m-1})\ .$$

By (2.12) and the boundedness of a and b,

$$\begin{aligned}
&\limsup_k\int_{A_k}\Big(a(\varepsilon_k y,\varepsilon_k v_k)+\big\langle\bar b(\varepsilon_k y,\varepsilon_k v_k),\mathcal{M}^{m-1}(\nabla v_k)\big\rangle+\sum_{\alpha\in I_{m,n}}\det\Big[D_\alpha w_k^\alpha,D_\alpha v_k^2,\dots,D_\alpha v_k^m\Big]\Big)_+\\
&\qquad\leq C\,\limsup_k\int_{A_k}\left(1+|\mathcal{M}^{m-1}(\nabla v_k)|+\Big(\sum_{\alpha\in I_{m,n}}\det\Big[D_\alpha w_k^\alpha,D_\alpha v_k^2,\dots,D_\alpha v_k^m\Big]\Big)_+\right)=0
\end{aligned}$$

where $A_k = \{\bar v_k \neq \bar u_k\}$. Hence, thanks to the positivity of the integrand, (3.12) and the above equation we have

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_{Q_1} f\big(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}_m(\nabla v_k)\big) \\
&= \liminf_{k\to\infty} \int_{Q_1} \Big(a(\varepsilon_k y, \varepsilon_k u_k) + \big\langle \bar b(\varepsilon_k y, \varepsilon_k u_k), \mathcal{M}^{m-1}(\nabla u_k)\big\rangle + \sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_k^\alpha, D_\alpha u_k^2, \dots, D_\alpha u_k^m\Big]\Big)_+ \\
&\geq \liminf_{k\to\infty} \int_{Q_\rho\setminus A_k} \Big(a(\varepsilon_k y, \varepsilon_k u_k) + \big\langle \bar b(\varepsilon_k y, \varepsilon_k u_k), \mathcal{M}^{m-1}(\nabla u_k)\big\rangle + \sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_k^\alpha, D_\alpha u_k^2, \dots, D_\alpha u_k^m\Big]\Big)_+ \\
&= \liminf_{k\to\infty} \int_{Q_\rho} \Big(a(\varepsilon_k y, \varepsilon_k v_k) + \big\langle \bar b(\varepsilon_k y, \varepsilon_k v_k), \mathcal{M}^{m-1}(\nabla v_k)\big\rangle + \sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_k^\alpha, D_\alpha v_k^2, \dots, D_\alpha v_k^m\Big]\Big)_+ \\
&= \liminf_{k\to\infty} \int_{Q_\rho} \Big(a(0,0) + \big\langle \bar b(0,0), \mathcal{M}^{m-1}(\nabla v_k)\big\rangle + \sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_k^\alpha, D_\alpha v_k^2, \dots, D_\alpha v_k^m\Big]\Big)_+,
\end{aligned}
$$

where in the last step we have used that the sequence v_k is bounded in $W^{1,m-1}\cap L^\infty(Q_\rho, \mathbb{R}^m)$ and that the coefficients a, b are uniformly continuous. Let now $\varphi \in C_c^1(Q_\rho)$, $0 \leq \varphi \leq 1$, then

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_{Q_\rho} \Big(a(0,0) + \big\langle \bar b(0,0), \mathcal{M}^{m-1}(\nabla v_k)\big\rangle + \sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_k^\alpha, D_\alpha v_k^2, \dots, D_\alpha v_k^m\Big]\Big)_+ \\
&\geq \liminf_{k\to\infty} \int_{Q_\rho} \Big(a(0,0) + \big\langle \bar b(0,0), \mathcal{M}^{m-1}(\nabla v_k)\big\rangle + \sum_{\alpha\in I_{m,n}} \det\Big[D_\alpha w_k^\alpha, D_\alpha v_k^2, \dots, D_\alpha v_k^m\Big]\Big)\varphi \\
&= \int_{Q_\rho} \Big(a(0,0) + \big\langle \bar b(0,0), \mathcal{M}^{m-1}(\nabla v_0)\big\rangle\Big)\varphi + \lim_k \int_{Q_\rho} \sum_{\alpha\in I_{m,n}} v_k^2 \det\Big[D_\alpha w_k^\alpha, D_\alpha \varphi, \dots, D_\alpha v_k^m\Big]\Big) \\
&= \int_{Q_\rho} \Big(a(0,0) + \big\langle \bar b(0,0), \mathcal{M}^{m-1}(\nabla u_0)\big\rangle\Big)\varphi + \int_{Q_\rho} \sum_{\alpha\in I_{m,n}} u_0^2 \det\Big[c^\alpha(0,0) D_\alpha u_0^1, D_\alpha \varphi, \dots, D_\alpha v_k^m\Big]\Big) \\
&= \int_{Q_\rho} \Big(a(0,0) + \big\langle \bar b(0,0), \mathcal{M}^{m-1}(\nabla u_0)\big\rangle + \big\langle c(0,0), \mathcal{M}_m(\nabla u_0)\big\rangle\Big)\varphi,
\end{aligned}
$$

where we have used (2.11), (3.11). Taking the supremum on $\varphi \in C_c(Q_\rho)$, $0 \leq \varphi \leq 1$, and letting $\rho \uparrow 1$ we then get

$$
\begin{aligned}
\liminf_{k\to\infty} \int_{Q_1} f\big(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}_m(\nabla v_k)\big) &\geq \Big(a(0,0) + \big\langle b(0,0), \mathcal{M}^{m-1}(\nabla u_0)\big\rangle + \big\langle c(0,0), \mathcal{M}_m(\nabla u_0)\big\rangle\Big)_+ \\
&= f\big(0, 0, \mathcal{M}^m(\nabla u(0))\big),
\end{aligned}
$$

which concludes the proof. □

3.1. **The demi-coercive case.** In this subsection we address the demi-coercive case by proving both Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. We use [20, Theorem 7] to find a sequence of positive convex functions $f_j \in C^\infty(\Omega\times\mathbb{R}^m\times\mathbb{R}^\sigma)$ such that $f = \sup_j f_j$. The conclusion then follows at once from Theorem 1.1 applied to each f_j, and then by taking the supremum on j. □

Before proving Corollary 1.3 we recall the notation $\xi = (\bar\xi, \eta) \in \mathbb{R}^{\sigma-\tau}\times\mathbb{R}^\tau$.

Proof of Corollary 1.3. Given any $\delta > 0$ let $f_\delta(x,u,\xi) := f(x,u,\xi) + \delta|\eta|$, then each f_δ satisfies assumption (Dem). In particular, by Theorem 1.2 we get for all $\delta > 0$

$$\liminf_j \int_\Omega f(x,u_j,\mathcal{M}^m(\nabla u_j))\,dx + \delta \sup_j \|\mathcal{M}^{m-1}(\nabla u_j)\|_{L^1}$$
$$\geq \liminf_j \int_\Omega f_\delta(x,u_j,\mathcal{M}^m(\nabla u_j))\,dx \geq \int_\Omega f(x,u,\mathcal{M}^m(\nabla u))\,dx,$$

and the conclusion follows at once from the last inequality thanks to (1.2) by letting $\delta \downarrow 0$. □

4. The case of equal dimensions $m = n$

In this section we prove Theorem 1.4, to this end let us introduce the notation $\xi = (\bar\xi, \eta) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$ if $\xi \in \mathbb{R}^\sigma$ and $m = n$.

Proof of Theorem 1.4. As in the proof of Theorem 1.1 we can assume that

$$f(x,u,\xi) = \big(a(x,u) + \langle \bar b(x,u), \bar\xi\rangle + c(x,u)\,\eta\big)_+$$

where $a, c : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $\bar b : \Omega \times \mathbb{R}^n \to \mathbb{R}^{\sigma-1}$ are continuous and compactly supported. By Lemma 2.9 to infer (1.7) we are left with proving

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, u_k, \mathcal{M}^n(\nabla u_k))\,dy \geq f(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))). \tag{4.1}$$

along sequences satisfying

$$u_k \to u_0 := \nabla u(x_0)\cdot y \quad L^1, \text{ and } \sup_k \|u_k\|_{W^{1,n-1}} < \infty,$$

for all points x_0 of approximate differentiability of u. We will set without loss of generality $x_0 = 0$ and $u(x_0) = 0$. As usual we can assume that the left hand side in (4.1) is a limit. We distinguish two cases

(a) $c(0,0) = 0$,

(b) $c(0,0) \neq 0$.

Our main new contribution is a strategy to handle case (a), that was trivialized in [11, Theorem 1.1] by means of a mild technical assumption. We repeat the proof of case (b) given in [11, Theorem 1.1] as well for the readers convenience.

In what follows we shall give the proof in case $m \geq 3$ for which Theorem 2.5 is instrumental. The remaining case $m = 2$ can be handled more easily with similar arguments by taking advantage of the equi-integrability assumption on $(\nabla u_k)_k$.

Proof in case (a): In this case the function $f(0,0,\bar\xi,\eta)$ does not depend on η, therefore, to simplify the notation in the rest of the proof, we introduce the (convex) function

$$g(\bar\xi) := f(0,0,\bar\xi,\eta).$$

Next, we employ Zhang's biting lemma for minors Theorem 2.5 to select a sequence $(U_h)_h$ of Borel subsets of Q_1 such that $\mathcal{L}^n(U_h) \downarrow 0$ and $(\mathcal{M}^{n-1}(\nabla u_k))_k$ converges to $\mathcal{M}^{n-1}(\nabla u_0)$ weakly in $L^1(Q \setminus U_h)$ for every h. Fix now $M > \|u_0\|_{L^\infty} + 1$ and set

$$u_{k,M} = \pi_M(u_k), \tag{4.2}$$

where π_M is defined as in (3.5). Then, as $f \geq 0$, we have for all k

$$\begin{aligned}\int_{Q_1\setminus U_h} & f(\varepsilon_k y, \varepsilon_k u_{k,M}, \mathcal{M}^n(\nabla u_{k,M}))\, dy \\ & \leq \int_{Q_1} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\, dy \\ & + \int_{\{y\in Q_1\setminus U_h:\, |u_k|\geq M\}} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^{n-1}(\nabla u_k), 0)\, dy \\ & \leq \int_{Q_1} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\, dy + C\,\eta_{h,M},\end{aligned} \tag{4.3}$$

where we have set $C = \|a_i\|_{L^\infty} + \|b_i\|_{L^\infty}$ and

$$\eta_{h,M} := \sup_k \int_{\{y\in Q_1\setminus U_h:\, |u_k|\geq M\}} (1 + |\mathcal{M}^{n-1}(\nabla u_k)|)\, dy.$$

Note that by the equi-integrability of $(\mathcal{M}^{n-1}(\nabla u_k))_k$ on $Q_1 \setminus U_h$ for each h, we have that $\eta_{h,M} \downarrow 0$ as $M \uparrow \infty$ for all h. We divide the rest of the proof in two steps.

Step 1. *Freezing of the coefficients.* In view of (2.6) in Theorem 2.2 and the boundedness of $(u_k)_k$ in $W^{1,n-1}$, we have for each given $L > 0$

$$\begin{aligned}\int_{Q_1\setminus U_h} & f(\varepsilon_k y, \varepsilon_k u_{k,M}, \mathcal{M}^n(\nabla u_{k,M}))\, dy \\ & \geq \int_{\{y\in Q_1\setminus U_h:\, |\det\nabla u_{k,M}|\leq L\}} f\big(\varepsilon_k y, \varepsilon_k u_{k,M}, \mathcal{M}^n(\nabla u_{k,M})\big)\, dy \\ & \geq \int_{\{y\in Q_1\setminus U_h:\, |\det\nabla u_{k,M}|\leq L\}} g\big(\mathcal{M}^{n-1}(\nabla u_{k,M})\big)\, dy - (1+L)\,\omega(\varepsilon_k(\sqrt{n}+M)).\end{aligned} \tag{4.4}$$

Step 2. *Conclusion in case (a).*

We first recall Hadamard's inequality

$$|\det \mathbb{A}| \leq c(n)|\mathbb{A}|^n \quad \text{for all } \mathbb{A} \in \mathbb{R}^{n\times n}.$$

In particular, from this and Chebychev's inequality we infer that

$$\begin{aligned}\mathcal{L}^n(\{y \in Q_1 :\, |\det \nabla u_{k,M}| \geq L\}) & \leq \mathcal{L}^n\big(\, \{y\in Q_1 :\, |\nabla u_{k,M}|^n \geq c(n)^{-1}L\}\,\big) \\ & \leq \left(\frac{c(n)}{L}\right)^{1-\frac{1}{n}} \int_{Q_1} |\nabla u_k|^{n-1}\, dy \leq C\, L^{\frac{1}{n}-1}.\end{aligned}$$

In turn, the previous estimate and the equi-integrability of $(\mathcal{M}^{n-1}(\nabla u_k))_k$ on $L^1(Q_1\setminus U_h)$ give that

$$\delta_{h,M,L} := \sup_k \int_{\{y\in Q_1\setminus U_h:\, |\det\nabla u_{k,M}|\geq L\}} (1 + |\mathcal{M}^{n-1}(\nabla u_k)|)\, dy$$

goes to 0 as $L \uparrow \infty$ for all h and M.

Thus, by collecting inequalities (4.3), (4.4) we find

$$\begin{aligned}\int_{Q_1} & f(\varepsilon_k\, y, \varepsilon_k\, u_k, \mathcal{M}^n(\nabla u_k))\, dy \\ & \geq \int_{Q_1\setminus U_h} g(\mathcal{M}^{n-1}(\nabla u_{k,M}))\, dy - (1+L)\,\omega_i(\varepsilon_k(\sqrt{m}+M)) - C\,(\eta_{h,M} + \delta_{h,L,M})\end{aligned}$$

$$\geq \int_{Q_1\setminus U_h} g(\mathcal{M}^{n-1}(\nabla u_k))\,dy - (1+L)\,\omega(\varepsilon_k(\sqrt{m}+M)) - 2C\,(\eta_{h,M}+\delta_{h,L,M}), \tag{4.5}$$

from which we infer

$$\begin{aligned}\liminf_k \int_{Q_1} & f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\,dy \\ &\geq \liminf_k \int_{Q_1\setminus U_h} g(\mathcal{M}^{n-1}(\nabla u_k))\,dy - 2C\,(\eta_{h,M}+\delta_{h,L,M}) \\ &\geq \mathcal{L}^n(Q_1\setminus U_h)\, g(\mathcal{M}^{n-1}(\nabla u(0))) - 2C\,(\eta_{h,M}+\delta_{h,L,M}) \\ &= \mathcal{L}^n(Q_1\setminus U_h)\, f(0,0,\mathcal{M}^n(\nabla u(0))) - 2C\,(\eta_{h,M}+\delta_{h,L,M}). \end{aligned} \tag{4.6}$$

The last equality follows from the very definition of g, the last but one inequality instead is a consequence of the weak convergence of $(\mathcal{M}^{n-1}(\nabla u_k))_k$ to $\mathcal{M}^{n-1}(\nabla u(0))$ in $L^1(Q_1\setminus U_h)$. The conclusion in case (a) then follows from (4.6) by letting first $L\uparrow\infty$, then $M\uparrow\infty$ and finally $h\uparrow\infty$.

Proof in case (b): Without loss of generality we may assume $c(0,0))>0$. Otherwise, we replace the functions $u_k=(u_k^1,\ldots,u_k^n)$ with $(-u_k^1,u_k^2,\ldots,v_k^n)$, the coefficient $c(x,u)$ with $-c(x,-u^1,\ldots,u^n)$ and the remaining coefficients a and b accordingly.

Fix now $M>\|u_0\|_{L^\infty}+1$ and consider the functions $u_{k,M}$ defined in (4.2). Then, as f is non-negative, for all k we have

$$\int_{\{y\in Q_1\colon |u_k|\leq M\}} f(\varepsilon_k y, \varepsilon_k u_{k,M}, \mathcal{M}^n(\nabla u_{k,M}))\,dy \leq \int_{Q_1} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\,dy. \tag{4.7}$$

Therefore, since the sequence $(u_{k,M})_k$ is bounded in $W^{1,n-1}(Q_1,\mathbb{R}^n)$ we deduce that

$$\sup_k \int_{\{y\in Q_1\colon |u_k|\leq M\}} \left(c(\varepsilon_k y,\varepsilon_k u_k)\det\nabla u_{k,M}\right)_+\,dy<\infty.$$

Recalling the choice $c(0,0)>0$, the continuity of c yields for tk sufficiently large

$$\sup_k \int_{\{y\in Q_1\colon |u_k|\leq M\}} \left(\det\nabla u_{k,M}\right)_+\,dy<\infty,$$

in turn implying

$$\sup_k \int_{Q_1} \left(\det\nabla u_{k,M}\right)_+\,dy<\infty.$$

Arguing as in Step 3 in the proof of Theorem 1.1, an application of Lemma 2.6 gives that, up to a subsequence not relabeled for convenience, the sequence $(\det\nabla u_{k,M})_k$ converges locally weakly* in the sense of measures in Q_1. In particular, $(\det\nabla u_{k,M})_k$ is bounded in $L^1_{\mathrm{loc}}(Q_1)$. Hence, with fixed $\rho\in(0,1)$, Proposition 2.8 provides sequences $s_k\downarrow 0$ and $(v_k)_k$ in $W^{1,n}(Q_\rho,\mathbb{R}^n)$ satisfying conclusions (2.23), (2.24) and (2.25) there. Note that, for k sufficiently large, recalling the choice of M, we have

$$\{y\in Q_\rho:\ |u_k(y)|>M\}\subseteq A_k=\{y\in Q_\rho:\ |u_k(y)-u_0(y)|>s_k\}\,.$$

Therefore, estimate (2.5) and equation (4.7) imply

$$\begin{aligned}
&\int_{Q_\rho} f(\varepsilon_k y, \varepsilon_k v_k, \mathcal{M}^n(\nabla v_k))\, dy \\
&\leq \int_{Q_\rho \setminus A_k} f(\varepsilon_k y, \varepsilon_k u_{k,M}, \mathcal{M}^n(\nabla u_{k,M}))\, dy + C \int_{A_k} (1 + |\mathcal{M}^n(\nabla v_k)|)\; dy \\
&\leq \int_{\{y \in Q_\rho :\, |u_k| \leq M\}} f(\varepsilon_k y, \varepsilon_k u_{k,M}, \mathcal{M}^n(\nabla u_{k,M}))\, dy + C \int_{A_k} (1 + |\mathcal{M}^n(\nabla v_k)|)\; dy \\
&\leq \int_{Q_1} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\, dy + C \int_{A_k} (1 + |\mathcal{M}^n(\nabla v_k)|)\; dy.
\end{aligned}$$

The convergence of $(v_k)_k$ to u_0 in L^∞, the latter inequality, (2.25) and (2.6) imply

$$\liminf_k \int_{Q_1} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\, dy \geq \liminf_k \int_{Q_\rho} f(0, 0, \mathcal{M}^n(\nabla v_k))\, dy.$$

In turn, from this and by taking into account that $(\mathcal{M}^n(\nabla v_k))_k$ converges to $\mathcal{M}^n(\nabla u(0))$ weakly* in the sense of measures on Q_ρ, by the convexity of $f(0,0,\cdot)$ we get

$$\liminf_k \int_{Q_1} f(\varepsilon_k y, \varepsilon_k u_k, \mathcal{M}^n(\nabla u_k))\, dy \geq \rho^n\, f(0, 0, \mathcal{M}^n(\nabla u(0))),$$

from which the conclusion follows straightforwardly as $\rho \uparrow 1$. □

5. The autonomous case $m = n + 1$

In this last section we prove Theorem 1.5.

Proof of Theorem 1.5. By Lemma 2.1 it is sufficient to establish the lower semicontinuity property for integrands of the form

$$f(\xi) = \Big(a + \langle b, \mathcal{M}^{n-1}(\xi)\rangle + \langle c, \mathcal{M}_n(\xi)\rangle\Big)_+.$$

Moreover we can assume that $c \neq 0$, the other case being elementary.

The main idea is to reduce to the case $c = e_1 = (1, 0, \dots, 0)$ via a change of variable in the codomain. The geometric intuition behind this reduction is that, up to signs, $\mathcal{M}_n(\nabla u)$ is parallel to the normal vector to the image of u and hence $\langle c, \mathcal{M}_n(\nabla u)\rangle$ is its component along the direction of c. By a suitable change of coordinates we can make then c parallel to e_1. More precisely, let us show that there exists an invertible matrix $\mathbb{A} \in \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$\langle c, \mathcal{M}_n(\nabla w)\rangle = \langle e_1, \mathcal{M}_n(\nabla(\mathbb{A}w))\rangle = \det(\nabla(\overline{\mathbb{A}w})),$$

for all $w \in W^{1,n}(\Omega, \mathbb{R}^{n+1})$, where $\overline{v} := (v^1, \dots, v^n)$, $v \in \mathbb{R}^{n+1}$. Indeed, by the Cauchy-Binet formula (see [7, Proposition 5.66])

$$\mathcal{M}_n(\nabla(\mathbb{A}u)) = \mathrm{adj}\mathbb{A} \cdot \mathcal{M}_n(\nabla u)$$

where we recall that $\mathrm{adj}\mathbb{A} = (\det \mathbb{A})\, \mathbb{A}^{-1}$. Hence, for $c \neq 0$ it is straightforward to find such a matrix $\mathbb{A}$.

Therefore, we can assume that the functional F satisfies

$$F(u) = \int_\Omega f\big(\mathcal{M}^n(\nabla u)\big) = \int_\Omega \big(a + \langle b, \mathcal{M}^{n-1}(\nabla u)\rangle + \det(\nabla \overline{u})\big)_+. \tag{5.1}$$

In addition, by Lemma 2.9 we may suppose that $\Omega = Q_1$ and $u_k \rightharpoonup u_0 := \nabla u(0) \cdot y$ in $W^{1,n-1}$.

We now divide the proof in two steps.

Step 1. *Truncation.* As in Step 1 of the proof Theorem 1.1 we show that we can replace the sequence u_k with a sequence v_k such that $\overline{v}_k$ is uniformly bounded. Indeed, since $u_k \rightharpoonup u_0$ in $W^{1,n-1}$ we have

$$\sup_k \int_{Q_1} |\nabla u_k|^{n-1} \leq C. \tag{5.2}$$

With fixed $M \geq 1 + \|u_0\|_{L^\infty}$ and $j \in \mathbb{N}$, we can find $j_k \in \{1, \cdots, j\}$ such that

$$\int_{Q_1 \cap \{M^{j_k} \leq |\overline{u}_k| \leq M^{j_k+1}\}} |\nabla u_k|^{n-1} \leq \frac{C}{j}. \tag{5.3}$$

Let us now define v_k as follows: $\overline{v}_k = \pi_{M^{j_k}}(\overline{u}_k)$, where we recall the definition of π_M in (3.5), and $v_k^{n+1} = u_k^{n+1}$. Clearly, $|\overline{v}_k| \leq M^j$, $v_k = u_k$ on $\{|\overline{u}_k| \leq M^{j_k}\}$. For $n \geq 3$ we estimate the minors of order $n-1$ of ∇v_k on the region where v_k differs from u_k as follows

$$\begin{aligned}
\int_{Q_1 \cap \{v_k \neq u_k\}} |\mathcal{M}_{n-1}(\nabla v_k)| &\leq C \int_{Q_1 \cap \{M^{j_k} \leq |\overline{u}_k| \leq M^{j_k+1}\}} |\nabla v_k|^{n-1} + \int_{Q_1 \cap \{M^{j_k+1} \leq |\overline{u}_k|\}\}} |\mathcal{M}_{n-1}(\nabla v_k)| \\
&\leq C \int_{Q_1 \cap \{M^{j_k} \leq |\overline{u}_k| \leq M^{j_k+1}\}} |\nabla u_k|^{n-1} + \frac{C}{M^{n-2}} \int_{Q_1 \cap \{M^{j_k+1} \leq |\overline{u}_k|\}\}} |\nabla u_k|^{n-1} \\
&\leq \frac{C}{j} + \frac{C}{M^{n-2}} := \eta_{M,j},
\end{aligned} \tag{5.4}$$

where we have used inequalities (5.2), (5.3), the fact that $\mathrm{Lip}\,(\pi_{M^{j_k}}) \leq 1$, and the following pointwise estimate in the region $\{|\overline{u}_k| \geq M^{j_k+1}\}$:

$$|\mathcal{M}_{n-1}(\nabla v_k)| \leq C \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I|=n-1}} \prod_{i \in I} |\nabla v_k^i| \leq \frac{C}{M^{n-2}} \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I|=n-1}} \prod_{i \in I} |\nabla u_k^i| \leq \frac{C}{M^{n-2}} |\nabla u_k|^{n-1}.$$

Note that in the second inequality above we have used that $\mathrm{Lip}\Big(\pi_{M^{j_k}}\big|_{\{|u| \geq M^{j_k+1}\}}\Big) \leq M^{-1}$, and that in each product there are at least $n-2$ indices i that are less than or equal to n.

By taking into account (5.1), (5.4), the equi-integrability of $\mathcal{M}_{n-2}$, and that $\det(\nabla \overline{v}_k) = 0$ on $\{v_k \neq u_k\}$ we get

$$\int_{Q_1} f\big(\mathcal{M}^n(\nabla v_k)\big) \leq \int_{Q_1} f\big(\mathcal{M}^n(\nabla u_k)\big) + \tilde{\eta}_{M,j},$$

with $\tilde{\eta}_{M,j} \downarrow 0$ as $M, j \uparrow +\infty$. The equi-integrability of ∇u_k provides the same conclusion for $n = 2$. It will be thus enough to prove lower semicontinuity along sequences whose first n-components are bounded in L^∞.

Step 2. *Conclusion.* Let $(u_k)_k$ be a sequence such that $\|u_k\|_{W^{1,n-1}}$ and $\|\overline{u}_k\|_{L^\infty}$ are uniformly bounded with

$$\liminf_k F(u_k) = \lim_k F(u_k) < \infty.$$

In particular, we have that

$$\sup_k \int_{Q_1} \big(\det(\nabla \overline{v}_k)\big)_+ < \infty.$$

As in Step 3 of Theorem 1.1 an integration by parts implies that the order 1 distributions

$$T_k(\varphi) := \int_{Q_1} \varphi \det(\nabla \overline{v}_k)$$

are bounded. Hence, up to a subsequence, by Lemma 2.6 we obtain that $\det(\nabla\overline{v}_k))_k$ is locally weakly* converging as measures to some measure μ. Furthermore, $\mu^a = \det(\nabla\overline{u}_0)\,d\mathcal{L}^n$ by Lemma 2.4, and

$$\mathcal{M}^{n-1}(\nabla u_k) \overset{*}{\rightharpoonup} \mathcal{M}^{n-1}(\nabla u)$$

by the usual integration by parts argument. In conclusion, for every continuous function $0 \le \varphi \le 1$ with compact support in Q_1 we have that

$$\begin{aligned}\liminf_k F(u_k) &\geq \liminf_k \int_{Q_1} \big(a + \langle b, \mathcal{M}^{n-1}(\nabla u_k)\rangle + \det(\nabla\overline{u}_k)\big)\varphi\,dx \\ &= \int_{Q_1} \big(a + \langle b, \mathcal{M}^{n-1}(\nabla u_0)\rangle + \det(\nabla\overline{u}_0)\big)\varphi\,dx + \int_{Q_1} \varphi\,d\mu^s,\end{aligned} \tag{5.5}$$

where $\mu = \det(\nabla\overline{u}_0)\,d\mathcal{L}^n + \mu^s$. By taking the supremum on all such φ's we get

$$\liminf_k F(u_k) \geq \int_{Q_1} f(\nabla u_0)dx + (\mu^s)^+(Q_1) \geq F(u_0).$$

This conclude the proof. □

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