A GENERAL CLASS OF FREE BOUNDARY PROBLEMS FOR FULLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this paper we consider the fully nonlinear parabolic free boundary problem

$$\begin{cases} F(D^2u) - \partial_t u = 1 & \text{a.e. in } Q_1 \cap \Omega \\ |D^2u| + |\partial_t u| \le K & \text{a.e. in } Q_1 \setminus \Omega, \end{cases}$$

where $K > 0$ is a positive constant, and Ω is an (unknown) open set.

Our main result is the optimal regularity for solutions to this problem: namely, we prove that $W^{2,n}_x \cap W^{1,n}_t$ solutions are locally $C^{1,1}_x \cap C^{0,1}_t$ inside Q_1. A key starting point for this result is a new BMO-type estimate which extends to the parabolic setting the main result in [4].

Once optimal regularity for u is obtained, we also show regularity for the free boundary $\partial\Omega \cap Q_1$ under the extra condition that $\Omega \supset \{u \neq 0\}$, and a uniform thickness assumption on the coincidence set $\{u = 0\}$,

1. Introduction and main result

1.1. **Background.** This paper is the parabolic counterpart of our earlier work [10] on fully nonlinear elliptic free boundary problems of obstacle type. The problem at hand concerns very generalized version of free boundary problems that have been in focus in the last two decades.

The particular application, in the linear theory, is related to "inverse Cauchy-Kowalevskya theory". This amounts to showing that if a domain $\Omega \subset \mathbb{R}^{n+1}$ admits a solution to the overdetermined problem

$$\Delta u - \partial_t u = 1 \quad \text{in } \Omega, \qquad u = \nabla u = 0 \quad \text{on } \partial\Omega,$$

then both the solution and the boundary must be reasonably smooth. Notice that, by Cauchy-Kowalevskaya theory, it is well-known that for smooth enough boundaries there is a solution to the above problem in a neighborhood of $\partial\Omega$, hence the question asked here is the converse.

In this paper we shall consider a much more general version of this question, allowing fully-nonlinear parabolic equations of the type $\mathcal{H}(u) := F(D^2u) - \partial_t u$, as well as a more general equation, see (1.1) below.

1.2. **Setting of the problem.** We will use $Q_r(X) := B_r(x) \times (t - r, t) \subset \mathbb{R}^n \times \mathbb{R}$ to denote the parabolic ball of radius r centered at a point $X = (x, t) \in \mathbb{R}^{n+1}$, and we will use the notation $Q_r = Q_r(0)$.

Our starting point will be a $W^{2,n}_x(Q_1) \cap W^{1,n}_t(Q_1)$ function $u : Q_1 \to \mathbb{R}$ satisfying

$$\begin{cases} \mathcal{H}(u) = 1 & \text{a.e. in } Q_1 \cap \Omega, \\ |\tilde{D}^2u| \le K & \text{a.e. in } Q_1 \setminus \Omega, \end{cases} \tag{1.1}$$

where $\tilde{D}^2u = (D^2_x u, D_t u) \in \mathbb{R}^{n^2+1}$, $\mathcal{H}(u) := F(D^2u) - \partial_t u$, $K > 0$, and $\Omega \subset \mathbb{R}^{n+1}$ is some unknown open set. Since, by assumption, $\tilde{D}^2u$ is bounded in the complement of Ω, we see that

$\mathcal{H}(u)$ is bounded inside the whole Q_1 and u is a so-called "strong L^n solution" to a fully nonlinear parabolic equation with bounded right hand side [7]. We refer to [12, 7] as basic references to parabolic fully nonlinear equations and viscosity methods.

The above free boundary problem has a very general form and encompasses several other free boundaries of obstacle type. In the elliptic case, it has been recently studied by the authors in [10]. We also refer to several other articles concerning similar type of problems: for elliptic case see [5], [1], and for parabolic case see [6], [2]. One may find applications and relevant discussions about these kinds of problems in these articles.

Since most of the results follow the same line of arguments (sometimes with obvious modifications) as that of its elliptic counterpart done in [10], here we have decided not to enter into the details of the proof as they can be worked out in a similar way as in the elliptic case. Instead, we shall give the outline of the proofs and point out all the necessary changes. For the reader unfamiliar with these techniques, we suggest first to read [10].

Going back to our problem, we observe that, if $u \in W^{2,n}_x \cap W^{1,n}_t$, then $\tilde{D}^2 u = 0$ a.e. inside $\{u = 0\}$, and $D^2 u = 0$ a.e. inside $\{\nabla u = 0\}$ (here and in the sequel, ∇u denotes only the spatial gradient of u). In particular we easily deduce that (1.1) includes, as special cases, both $\mathcal{H}(u) = \chi_{\{u\neq 0\}}$ and $\mathcal{H}(u) = \chi_{\{\nabla u\neq 0\}}$.

We assume that:

(H0) $F(0) = 0$.

(H1) F is uniformly elliptic with ellipticity constants $0 < \lambda_0 \le \lambda_1 < \infty$, that is,

$$\mathscr{P}^-(P_1 - P_2) \le F(P_1) - F(P_2) \le \mathscr{P}^+(P_1 - P_2)$$

for any P_1, P_2 symmetric, where $\mathscr{P}^-$ and $\mathscr{P}^+$ are the extremal Pucci operators: given a symmetric matrix M one defines

$$\mathscr{P}^-(M) := \inf_{\lambda_0\,\mathrm{Id}\le N\le \lambda_1\,\mathrm{Id}} \mathrm{trace}(NM), \qquad \mathscr{P}^+(M) := \sup_{\lambda_0\,\mathrm{Id}\le N\le \lambda_1\,\mathrm{Id}} \mathrm{trace}(NM),$$

where N in the formula above is symmetric as well.

(H2) F is either convex or concave.

Under assumptions (H0)-(H2) above, strong L^n solutions are also viscosity solutions [5], and hence regularity results for parabolic fully nonlinear equations [12, 13] show that $u \in W^{2,p}_x(Q_\rho) \cap W^{1,p}_t(Q_\rho)$ for all $\rho \in (0,1)$ and $p < \infty$.

Our first result concern the optimal $C^{1,1}_x \cap C^{0,1}_t$-regularity for u. Once this will be done, we will be able to study the regularity of the free boundary.

1.3. **Main results.** Our first result concerns the optimal regularity of solutions to (1.1):

Theorem 1.1. *(Interior $C^{1,1}_x \cap C^{0,1}_t$ regularity) Let $u : Q_1 \to \mathbb{R}$ be a $W^{2,n}_x \cap W^{1,n}_t$ solution of* (1.1)*, and assume that F satisfies (H0)-(H2). Then there exists a constant $\bar{C} = \bar{C}(n, \lambda_0, \lambda_1, \|u\|_\infty) > 0$ such that*

$$|\tilde{D}^2 u| \le \bar{C}, \qquad \text{in } Q_{1/2}.$$

To state our result on the regularity of the free boundary, we need to introduce the concept of minimal diameter: for any set $E \subset \mathbb{R}^n$ let $\mathrm{MD}(E)$ denote the smallest possible distance between two parallel hyperplanes containing E. Then, given a point $X^0 = (x^0, t^0) \in \mathbb{R}^{n+1}$, we define

$$\delta_r(u, X^0) := \inf_{t\in[t_0-r^2,t_0+r^2]} \frac{\mathrm{MD}\big(\Lambda \cap \big(B_r(x^0) \times \{t\}\big)\big)}{r}, \qquad \Lambda := Q_1 \setminus \Omega. \tag{1.2}$$

In other words, $\delta_r(u, X^0)$ measures the thickness of the complement of Ω at all time levels $t \in (t^0 - r^2, t^0 + r^2)$, around the point x^0. Notice that δ_r depends on u since Ω does. In particular, we observe that if u solves (1.1) for some set Ω, then $u_r(y, \tau) := u(x + ry, t + r^2\tau)/r^2$ solves (1.1) with

$$\Omega_r := \{(y, \tau) \,:\, (x + ry, t + r^2\tau) \in \Omega\}$$

in place of Ω, and δ_r enjoys the scaling property $\delta_1(u_r, 0) = \delta_r(u, X)$, $X = (x, t)$.

Our result provides regularity for the free boundary under a uniform thickness condition. As a corollary of our result, we deduce that Lipschitz free boundaries are C^1, and hence smooth [8].

Theorem 1.2. *(Free boundary regularity) Let $u : Q_1 \to \mathbb{R}$ be a $W^{2,n}_x \cap W^{1,n}_t$ solution of* (1.1). *Assume that F is convex and satisfies (H0)-(H1), and that $\Omega \supset \{u \neq 0\}$. Suppose further that there exists $\varepsilon > 0$ such that*

$$\delta_r(u, z) > \varepsilon \qquad \forall\, r < 1/4,\, \forall\, z \in \partial\Omega \cap Q_r(0).$$

Then $\partial\Omega \cap Q_{r_0}(0)$ is a C^1-graph in space-time, where r_0 depends only on ε and the data.

The paper is organized as follows:

In Section 2 we prove Theorem 1.1. Then in Section 3 we investigate the non-degeneracy of solutions, and classify global solutions under a suitable thickness assumption. In Section 4 we show directional monotonicity for local solutions which gives Lipschitz (and then C^1) regularity for the free boundary, as shown in Section 5.

Acknowledgements: We thank Guido De Philippis for useful discussions concerning the proof of Lemma 6.2. A. Figalli was partially supported by NSF grant DMS-1262411.

2. Proof of Theorem 1.1

The proof of this theorem follows the same line of ideas as its elliptic counterpart [10]. First one starts from a BMO-type estimate on D^2u, and then one shows a dichotomy that either u has quadratic growth away from a free boundary point X^0, or the density of the set Λ at X^0 vanishes fast enough to assure the quadratic bound.

In [10] the following result was a consequence of the BMO-type estimate proved in [4, Theorem A] (see [10, Lemma 2.3]). Since we could not find a reference for this estimate in the parabolic case, we prove this result in the appendix. We notice that our proof is much simpler than the one in [4] and actually gives a new proof of the results there (see Remark 6.3).

In all this section, u is as in the statement of Theorem 1.1. With no loss of generality, we will carry out the proof at the origin, by letting $X^0 = (0, 0)$.

We say that P is a "parabolic (second order) polynomial" if it is of the form

$$P(x, t) = a_0 + \langle b_0, x\rangle + \langle M_0x, x\rangle + c_0t, \qquad a_0, c_0 \in \mathbb{R},\, b_0 \in \mathbb{R}^n,\, M_0 \in \mathbb{R}^{n\times n}.$$

Lemma 2.1. *There exist a constant $C = C(n, \lambda_0, \lambda_1, \|u\|_\infty)$, and a family of parabolic polynomials $\{P_r\}_{r\in(0,1)}$ solving $\mathcal{H}(P_r) = 0$, such that*

$$\sup_{Q_r(0)} |u - P_r| \le Cr^2, \qquad \forall\, r \in (0, 1). \tag{2.1}$$

Consequently

$$\sup_{Q_r(0)} |u| \le (Cr^2 + |\tilde{D}^2 P_r|), \qquad \forall\, r \in (0, 1). \tag{2.2}$$

The first statement in the Lemma is proven in Appendix (see (6.1) and Lemma 6.2 there), while the second estimate is a straightforward consequence of the first one. It should be remarked that these polynomials P_r need not to be unique.

Define

$$A_r := \left\{(x, t) : \ (rx, r^2 t) \in Q_r \setminus \Omega\right\} \subset Q_1 \qquad \forall\, r < 1/4. \tag{2.3}$$

We shall prove that if $|P_r|$ is sufficiently large then the Lebesgue measure of A_r has to decay geometrically.

Proposition 2.2. *Let P_r be as in Lemma 2.1 and set $\tilde{P}_r := \tilde{D}^2 P_r$. There exists $M > 0$ universal such that, for any $r \in (0, 1/8)$, if $|\tilde{P}_r| \ge M$ then*

$$|A_{r/2}| \le \frac{|A_r|}{2^{n+1}}.$$

The proof of the proposition follows the same lines of ideas as that of [10, Proposition 2.4]. However, since the changes are not completely straightforward, for the reader's convenience we present the proof here.

Proof. Set $u_r(y, t) := u(ry, r^2)/r^2$ and let

$$u_r(y, t) = P_r(y, t) + v_r(y, t) + w_r(y, t), \tag{2.4}$$

where v_r is defined as the solution of

$$\begin{cases} \mathcal{H}(P_r + v_r) - 1 = 0 & \text{in } Q_1, \\ v_r(y, t) = u_r(y, t) - P_r(y, t) & \text{on } \partial_p Q_1, \end{cases} \tag{2.5}$$

where $\partial_p Q_1$ denotes the parabolic boundary of Q_1, and by definition $w_r := u_r - P_r - v_r$.

Set $f_r := \mathcal{H}(u_r) \in L^\infty(B_1)$ (recall that $|\tilde{D}^2 u_r| \le K$ a.e. inside A_r). Notice that, since $f_r = 1$ outside A_r,

$$\mathcal{H}(u_r) - \mathcal{H}(P_r + v_r) = (f_r - 1)\chi_{A_r},$$

so it follows by (H1) that w_r solves

$$\begin{cases} \mathscr{P}^-(D^2 w_r) - \partial_t w_r \le (f_r - 1)\chi_{A_r} \le \mathscr{P}^+(D^2 w_r) - \partial_t w_r & \text{in } Q_1, \\ w_r = 0 & \text{on } \partial_p Q_1. \end{cases} \tag{2.6}$$

Hence, we can apply the ABP estimate [12, Theorem 3.14] to deduce that

$$\sup_{Q_1} |w_r| \le C\|\chi_{A_r}\|_{L^{n+1}(Q_1)} = C|A_r|^{1/(n+1)}. \tag{2.7}$$

Also, since $\mathcal{H}(P_r) = 0$ and v_r is universally bounded on $\partial_p Q_1$ (see (2.1) and (2.5)), by the parabolic Evans-Krylov's theorem [9] applied to (2.5) we have

$$\|\tilde{D}^2 v_r\|_{C^{0,\alpha}(Q_{3/4})} \le C. \tag{2.8}$$

This implies that w_r solves the fully nonlinear equation with Hölder coefficients

$$G(D^2w_r,X)-\partial_t w_r-\partial_t(v_r+P_r)=(f_r-1)\chi_{A_r}\quad \text{in } Q_{3/4},\qquad G(M,X):=F(D^2P_r+D^2v_r(x)+M)-1.$$

Since $G(0,X)=0$, we can apply [12, Theorem 5.6] with $p=n+2$ and (2.7) to obtain

$$\int_{Q_{1/2}} |\tilde{D}^2 w_r|^{n+2} \le C\left(\|w_r\|_{L^\infty(Q_{3/4})}+\|\chi_{A_r}\|_{L^{2n}(Q_{3/4})}\right)^{n+2} \le C\,|A_r| \tag{2.9}$$

(recall that $|A_r|\le |Q_1|$).

We are now ready to conclude the proof: since $|\tilde{D}^2u_r|\le K$ a.e. inside A_r (by (1.1)), recalling (2.4) we have

$$\int_{A_r\cap Q_{1/2}} |\tilde{D}^2v_r+\tilde{D}^2w_r+\tilde{P}_r|^{n+2} = \int_{A_r\cap Q_{1/2}} |\tilde{D}^2u_r|^{n+2} \le K^{n+2}|A_r|.$$

Therefore, by (2.8) and (2.9),

$$\begin{aligned}
|A_r\cap Q_{1/2}|\,|\tilde{P}_r|^{n+2} &= \int_{A_r\cap Q_{1/2}} |\tilde{P}_r|^{n+2}\\
&\le 3^{2n}\left(\int_{A_r\cap Q_{1/2}} |\tilde{D}^2v_r|^{n+2} + \int_{A_r\cap Q_{1/2}} |\tilde{D}^2w_r|^{n+2} + K^{n+2}|A_r|\right)\\
&\le 3^{n+2}\left(|A_r\cap Q_{1/2}|\,\|\tilde{D}^2v_r\|_{L^\infty(Q_{1/2})} + \int_{Q_{1/2}} |\tilde{D}^2w_r|^{n+2} + K^{n+2}|A_r|\right)\\
&\le C\,|A_r\cap Q_{1/2}| + C\,|A_r|,
\end{aligned}$$

which gives

$$|A_r\cap Q_{1/2}(0)|\,|\tilde{P}_r|^{n+2} \le C|A_r|.$$

Hence, if $|\tilde{P}_r|$ is sufficiently large so that $C\le \frac{1}{4^{n+1}}|\tilde{P}_r|^{n+2}$ we get

$$|A_r\cap Q_{1/2}(0)|\,|\tilde{P}_r|^{n+2} \le \frac{1}{4^{n+1}}|\tilde{P}_r|^{n+2}|A_r|.$$

Since $|A_{r/2}|=2^{n+1}|A_r\cap Q_{1/2}(0)|$, this gives the desired result. □

2.1. **Proof of Theorem 1.1.** Taking $M>0$ as in Proposition 2.2, we have that one of the following hold:

(i) $\liminf_{k\to\infty}|P_{2^{-k}}|\le 3M$,
(ii) $\liminf_{k\to\infty}|P_{2^{-k}}|\ge 3M$.

Then, one consider the two case separately and, arguing exactly as in the proof of Theorem 1.2 in [10] one obtains the desired result. (We notice that the reference [3, Theorem 3] in that proof is to be replaced by [13, Theorem 1.1].)

3. Non-degeneracy and global solutions

3.1. **Local non-degeneracy.** The $C_x^{1,1} \cap C_t^{0,1}$-regularity proved in the previous section implies that u cannot grow more than quadratically in space and linearly in time away from the free boundary. Non-degeneracy means that a solution always grows exactly at such a rate, and this property is extremely useful for proving smoothness of the free boundary, both in the elliptic and the parabolic case.

As shown in [10, Section 3] non-degeneracy fails in general for the elliptic case, hence for our problem as well. Nevertheless, the non-degeneracy does hold for the case $\Omega \supset \{\nabla u \neq 0\}$, see [10, Lemma 3.1]. We now show that this non-degeneracy result still holds in the parabolic case:

Lemma 3.1. *Let $u : Q_1 \to \mathbb{R}$ be a $W_x^{2,n} \cap W_t^{1,n}$ solution of (1.1), assume that F satisfies (H0)-(H2), and that $\Omega \supset \{\nabla u \neq 0\}$. Then, for any $X^0 = (x^0, t^0) \in \overline{\Omega} \cap Q_{1/2}$,*

$$\max_{\partial_p Q_r(X^0)} u \geq u(X^0) + \frac{r^2}{2n\lambda_1 + 1} \qquad \forall\, r \in (0, 1/4).$$

Proof. For

$$v(x) := u(x) - \frac{|x - x^0|^2 - (t - t^0)}{2n\lambda_1 + 1}, \qquad X^0 \in \Omega \cap Q_{1/2},$$

one readily verifies that $\mathcal{H}(v) \geq 0$ in $Q_r(X^0)$. Then, by the very same argument as in the proof of [10, Lemma 3.1] we deduce that[1]

$$\max_{\partial_p Q_r(X^0)} v = \sup_{Q_r(X^0)} v,$$

and the result follows easily. By continuity the lemma holds for $X^0 \in \overline{\Omega} \cap Q_{1/2}$ □

3.2. **Classification of global solutions.** As already discussed in the previous section, to have non-degeneracy of solutions we need to assume that $\Omega \supset \{\nabla u \neq 0\}$. In the elliptic case this assumption is also sufficient to classify global solutions with a "thick free boundary" (see [10, Proposition 3.2]). However, in the parabolic case the situation is much more complicated: indeed, while global solutions of the elliptic problem with "thick free boundary" are convex and one-dimensional, in the parabolic case we have non-convex solutions. For instance the function

$$u(x,t) = \begin{cases} -2t - x_1^2/2 & \text{if } x_1 > 0, \\ -2t & \text{if } x_1 \leq 0, \end{cases}$$

is a solution of (1.1) on the whole $\mathbb{R}^{n+1}$ with $F(D^2u) = \Delta u$ and $\Omega := \{x_1 > 0\} = \{\nabla u \neq 0\}$. In order to avoid this kind of examples, here we shall only consider the case $\Omega \supset \{u \neq 0\}$.

Since we will use minimal diameter to measure sets (recall (1.2)), we need some classical facts about their stability properties. First of all we recall that, for polynomial global solutions $P_2 = \sum_j a_j x_j^2 + bt$ (with $A = \mathrm{diag}(a_j)$, and b such that $F(A) - b = 1$), one has

$$\delta_r(P_2, 0) = 0. \tag{3.1}$$

[1]The proof of this fact is a consequence of the strong maximum principle: if there exists an interior maximum point $Y \in Q_r(X^0)$, since $\nabla v(Y) = 0$ and $\Omega \supset \{\nabla u \neq 0\}$, one deduces that $Y \in \Omega \cap Q_r(X^0)$. Hence, because v is a subsolution and $\nabla u = 0$ outside Ω, v must be constant inside $Q_r(X^0)$ and the result follows (see the proof of [10, Lemma 3.1] for more details).

Also, the scaling and stability estimate

$$\delta_r(u, X) = \delta_1(u_r, 0), \qquad \limsup_{r\to 0} \delta_r(u, X^0) \leq \delta_1(u_0, 0) \tag{3.2}$$

holds whenever $u_r(y, \tau) = u(x + ry, t + r^2\tau)/r^2$ converges in C^1 to some function u_0.

In the next proposition we classify global solution with a "thick free boundary". We notice that assumption (3.3) below allows us to exclude the family of global solutions $u_\sigma(t, x) = -(t - \sigma)_+$, $\sigma \in \mathbb{R}$.

Proposition 3.2. *Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be a $W^{2,n}$ solution of* (1.1) *on the whole $\mathbb{R}^{n+1}$, assume that F is convex and satisfies (H0)-(H1), and that $\Omega \supset \{u \neq 0\}$. Furthermore, assume that there exists $\epsilon_0 > 0$ such that*

$$\delta_r(u, X^0) \geq \epsilon_0 \qquad \forall\, r > 0, \forall\, X^0 \in \partial\Omega. \tag{3.3}$$

Then u is time-independent. In particular, by the elliptic case [10, Proposition 3.2]*, u is a half-space solution, i.e., up to a rotation, $u(x) = \gamma[(x_1)_+]^2/2$, where $\gamma \in (1/\lambda_1, 1/\lambda_0)$ is such that $F(\gamma e_1 \otimes e_1) = 1$.*

Proof. Let $m := \sup_{\mathbb{R}^{n+1}} \partial_t u$ (notice that m is finite by Theorem 1.1) and consider a sequence $m_j = \partial_t u(X^j)$ such that $m_j \to m$.

We now perform the scaling

$$u_j(x, t) := \frac{u(d_j x + x^j, d_j^2 t + t^j)}{d_j^2},$$

where $X^j = (x^j, t^j)$ and $d_j := \mathrm{dist}(X^j, \partial\Omega)$.

The functions u_j still satisfy (1.1). Also, since $u = 0$ on $\partial\Omega$ it follows by the $C^{1,1}_x \cap C^{0,1}_t$ regularity of u that u_j are uniformly bounded, hence, up to subsequences, they converge to another global solution u_∞ which satisfies $\partial_t u_\infty(0) = m$. By (3.2) and the assumption (3.3) we obtain

$$\delta_r(u_\infty, X^0) \geq \epsilon_0 \qquad \forall\, r > 0, \forall\, X^0 \in \partial\Omega_\infty, \tag{3.4}$$

where Ω_∞ is the limit, as $j \to \infty$, of the family of open sets

$$\Omega_j := \big\{(x, t) \,:\, (d_j x + x^j, d_j^2 t + t^j) \in \Omega\big\}.$$

Let us observe that, by the condition $\Omega \supset \{u \neq 0\}$ we get $u_\infty(t, x) = 0$ on $\partial\Omega_\infty$.

In addition $\partial_t u_\infty$ is a solution of the uniformly parabolic linear operator $F_{ij}(D^2 u_\infty)\partial_{ij} - \partial_t$ inside Ω_∞. Hence, since $\partial_t u_\infty \leq m$ and $\partial_t u_\infty(0) = m$, by the strong maximum principle we deduce that $\partial_t u_\infty$ is constant inside the connected component of Ω_∞ containing 0 (call it Ω_0).

Therefore, integrating u_∞ in the direction t gives

$$u_\infty(t, x) = mt + U(x) \quad \text{inside } \Omega_0, \qquad u_\infty = 0 \quad \text{on } \partial\Omega_0. \tag{3.5}$$

We claim that $m = 0$. Indeed, suppose by contradiction that $m \neq 0$. Then, for any point $(\bar{x}, \bar{t}) \in \Omega_0$ it follows by (3.5) that: (a) either there exists $t' \in \mathbb{R}$ such that $(\bar{x}, t') \in \partial\Omega_0$; (b) or $\{\bar{x}\} \times \mathbb{R} \subset \Omega_0$. Thanks to the thickness assumption (3.4) we see that $\nabla u_\infty = 0$ on $\partial\Omega_0$, so in case (a) we obtain that $\nabla U(\bar{x}) = \nabla u_\infty(\bar{x}, t') = 0$. Hence, by the arbitrariness of $\bar{x}$, we can write

$$\Omega_0 = \Omega_1 \cup \Omega_2,$$

where $\nabla u_\infty \equiv 0$ in Ω_1, and Ω_2 is a cylinder of the form $V \times \mathbb{R}$ with $V \subset \mathbb{R}^n$. So, it follows from (3.5) that $u_\infty = 0$ on $\partial\Omega_2$, which is incompatible with the fact that $u_\infty(t, x) = mt + U(x)$

inside Ω_2 (and so, by continuity, also on $\partial\Omega_2$) unless $m = 0$. This proves the claim, showing that $\sup_{\mathbb{R}^{n+1}} \partial_t u = 0$.

By a completely symmetric argument we obtain $\inf_{\mathbb{R}^{n+1}} \partial_t u = 0$. Thus $\partial_t u = 0$, which implies that u is time-independent and therefore, by [10, Proposition 3.2], up to a rotation u is of the form $u(x) = \gamma[(x_1)_+]^2/2 + c$ $\gamma \in (1/\lambda_1, 1/\lambda_0)$ is such that $F(\gamma e_1 \otimes e_1) = 1$ and $c \in \mathbb{R}$. Since $\Omega \supset \{u \neq 0\}$ we see that $c = 0$, which proves the result. □

4. Local solutions and directional monotonicity

In this section we shall prove a directional monotonicity for solutions to our equations. In the next section we will use Lemma 4.2 below to show that, if u is close enough to a half-space solution $\gamma[(x_1)_+]^2$ in a ball B_r, then for any $e = (e_x, e_t) \in \mathbb{S}^n$ with $e \cdot (e_1, 0) \geq s > 0$ we have $C_0\partial_e u - u \geq 0$ inside $B_{r/2}$.

Lemma 4.1. *Let $u : Q_1 \to \mathbb{R}$ be a $W^{2,n}_x \cap W^{1,n}_t$ solution of (1.1) with $\Omega \supset \{u \neq 0\}$. Then, under the conditions of Theorem 1.2 we have*

$$\lim_{\Omega \ni X \to \partial\Omega} \partial_t u(X) = 0.$$

Proof. The proof of this lemma follows easily by a contradiction argument, along with scaling and blow-up. Indeed, given a sequence $X^j \to \partial\Omega$ such that $|\partial_t u(X^j)| \geq c > 0$, then one may scale at X^j with $d_j = \mathrm{dist}(X^j, \partial\Omega)$ and define $u_j(X) := \left[u(d_j x + x^j, d_j^2 t + t^j) - u(X^j)\right]/d_j^2$ to end up with a global solution u_∞ with the property $\partial_t u_\infty(0) \neq 0$, contradicting Proposition 3.2. □

The proof of the following result is a minor modification of the one of [10, Lemma 4.1], so we just give a sketch of the proof.

Lemma 4.2. *Let $u : Q_1 \to \mathbb{R}$ be a $W^{2,n}_x \cap W^{1,n}_t$ solution of (1.1) with $\Omega \supset \{u \neq 0\}$. Assume that for some space-time direction $e = (e_x, e_t)$ with $|e| = 1$ we have $C_0\partial_e u - u \geq -\varepsilon_0$ in Q_1 for some $C_0, \varepsilon_0 \geq 0$, and that F is convex and satisfies (H0)-(H1). Then $C_0\partial_e u - u \geq 0$ in $Q_{1/2}$ provided $\varepsilon_0 \leq \frac{1}{4(2n\lambda_1+1)}$.*

Proof. Since F is convex, for any matrix M we can choose an element P^M inside $\partial F(M)$ (the subdifferential of F at M) in such a way that the map $M \mapsto P^M$ is measurable, and we define the measurable uniformly elliptic coefficients

$$a_{ij}(x,t) := (P^{D^2u(x,t)})_{ij} \in \partial F(D^2u(x,t)).$$

As in the proof of [10, Lemma 4.1], by the convexity of F if follows that, in the viscosity sense,

$$a_{ij}\partial_{ij}(\partial_e u) - \partial_t(\partial_e u) \leq 0 \qquad \text{in } \Omega \tag{4.1}$$

and

$$a_{ij}\partial_{ij} u - \partial_t u \geq 1 \qquad \text{in } \Omega. \tag{4.2}$$

Now, let us assume by contradiction that there exists $X^0 = (x^0, t^0) \in Q_{1/2}$ such that $C_0\partial_e u(X^0) - u(X^0) < 0$, and consider the function

$$w(X) := C_0\partial_e u(X) - u(X) + \frac{|x - x^0|^2 - (t - t_0)}{2n\lambda_1 + 1}.$$

Thanks to (4.1), (4.2), and assumption (H1) (which implies that $\lambda_0 \,\mathrm{Id} \le a_{ij} \le \lambda_1 \,\mathrm{Id}$) we deduce that w is a supersolution of the linear operator $\mathscr{L} := a_{ij}\partial_{ij} - \partial_t$, hence, by the minimum principle,

$$\min_{\partial_p(\Omega\cap Q_1(Y^0))} w = \min_{\Omega\cap Q_1(Y^0)} w \le w(Y^0) < 0.$$

By Lemma 4.1 and the assumption $\Omega \supset \{u \neq 0\}$ we have $\partial_t u = u = |\nabla u| = 0$ on $\partial\Omega$, therefore $w \ge 0$ on $\partial\Omega$. Thus, since $|x - x^0|^2 - (t - t^0) \ge 1/4$ on $\partial_p Q_1^-$ it follows that

$$0 > \min_{\partial_p Q_{1/2}^-(X^0)} w \ge -\varepsilon_0 + \frac{1}{4(2n\lambda_1 + 1)},$$

a contradiction if $\varepsilon_0 \le \frac{1}{4(2n\lambda_1+1)}$. □

5. Proof of Theorem 1.2

The proof of this theorem is very similar to the proof of [10, Theorem 1.3]. Indeed, take $X^0 = (x^0, t^0) \in \partial\Omega \cap Q_{1/8}$, and rescale the solution around X^0, that is $u_r(x,t) := [u(rx + x^0, r^2 t + t^0) - u(x^0, t^0) - r\nabla u(x^0, t^0)\cdot x]/r^2$.

Because of the uniform $C_x^{1,1} \cap C_t^{0,1}$ estimate provided by Theorem 1.1 and the thickness assumption on the free boundary of u, we can find a sequence $r_j \to 0$ such that u_{r_j} converges locally uniformly to a global solution u_∞ of the form $u_\infty(x) = \gamma[(x\cdot e_{X^0})_+]^2/2$ with $\gamma \in [1/\lambda_1, 1/\lambda_0]$ and $e_{X^0} \in \mathbb{S}^{n-1}$ (see Proposition 3.2).

Notice now that, for any $s \in (0,1)$, we can find a large constant C_s such that

$$C_s\partial_e u_\infty - u_\infty \ge 0 \qquad \text{inside } B_1$$

for all directions $e = (e_x, e_t) \in \mathbb{S}^n$ such that $e \cdot (e_{X^0}, 0) \ge s$, hence by the C_x^1 convergence of u_{r_j} to u_∞ and Lemmas 4.1 and 4.2 we deduce that

$$C_s\partial_e u_{r_j} - u_{r_j} \ge 0 \qquad \text{in } Q_{1/2}, \tag{5.1}$$

and since $u_{r_j}(0) = 0$ a simple ODE argument shows that $u_{r_j} \ge 0$ in $Q_{1/4}$.

Using (5.1) again, this implies that $\partial_e u_{r_j}$ inside $Q_{1/4}$, and so in terms of u we deduce that there exists $r = r(s) > 0$ such that

$$\partial_e u \ge 0 \qquad \text{inside } Q_r(X^0)$$

for all $e \in \mathbb{S}^n$ such that $e\cdot(e_{X^0}, 0) \ge s$.

A simple compactness argument shows that r is independent of the point x, which implies that the free boundary is s-Lipschitz. Since s can be taken arbitrarily small (provided one reduces the size of r), this actually proves that the free boundary is C^1. Higher regularity is then classical.

6. Appendix: Parabolic BMO estimates

Let $u : Q_1 \to \mathbb{R}$ satisfy $|u| \le 1$ and $|\mathcal{H}(u)| \le M$. Up to replacing u by $u(x/R, t/R^2)$ and $\mathcal{H}$ by $(F(R^2\,\cdot)/R^2 - \partial_t)$ with R a large fixed constant, we can assume that $|\mathcal{H}(u)| \le \delta$ with δ a small constant to be fixed later. Observe that, with this scaling, the ellipticity remains the same.

Let us first state a standard stability result.

Lemma 6.1. *(Compactness) Let $\varepsilon > 0$, and u be such that $u : Q_1 \to \mathbb{R}$ satisfy $|u| \le 1$. Let further $v : Q_{1/2} \to \mathbb{R}$ solve*

$$\begin{cases} \mathcal{H}(v) = 0 & \text{in } Q_{1/2}, \\ v = u & \text{on } \partial_p Q_{1/2}. \end{cases}$$

Then there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|u - v| \le \varepsilon \qquad \text{in } Q_{1/2},$$

provided $|\mathcal{H}(u)| \le \delta$.

The proof of the lemma is based on a standard compactness argument, using that both u and v are uniformly Hölder continuous (in (x,t)-variables) inside $Q_{1/2}$; see [12], Lemma 5.1.

Recall that P is a parabolic polynomial if it is of the form

$$P(x,t) = a_0 + \langle b_0, x\rangle + \langle M_0 x, x\rangle + c_0 t, \qquad a_0, c_0 \in \mathbb{R},\ b_0 \in \mathbb{R}^n,\ M_0 \in \mathbb{R}^{n\times n}.$$

We now prove by induction the following result:

Lemma 6.2. *Let $u : Q_1 \to \mathbb{R}$ be a solution to our problem* (1.1)*, with $|u| \le 1$. Then there exists $\rho > 0$ universal such that*

$$|u(X) - P_k(X)| \le \rho^{2k} \qquad \text{inside } Q_{\rho^k} \quad \forall\, k \in \mathbb{N},$$

where P_k is a parabolic polynomial such that $\mathcal{H}(P_k) = 0$.

A straight forward implication of this result is that there is a universal constant $C = 1/\rho^2$ such that

$$|u(X) - P_r(X)| \le C r^2 \qquad \text{inside } Q_r \quad \forall\, 0 < r < 1, \tag{6.1}$$

where P_r is a parabolic second order polynomial such that $\mathcal{H}(P_r) = 0$. This in turn implies an L^p-BMO type result, see the corollary below.

Proof. (of Lemma 6.2) Since the result is obviously true for $k = 0$ (just take $P_0 = 0$), we prove the inductive step. So, let us assume that the result is true for k and we prove it for $k+1$.

Define $u_k(X) := \frac{u(\rho^k x, \rho^{2k} t) - P_k(\rho^k x, \rho^{2k} t)}{\rho^{2k}}$. Then, by the inductive hypothesis $|u_k| \le 1$ inside Q_1. In addition

$$|\mathcal{H}_k(u_k)| \le \delta, \qquad \mathcal{H}_k(v) := F(D^2 v + D^2 P_k) - \partial_t P_k - \partial_t v.$$

Observe that $\mathcal{H}_k$ keeps the same ellipticity as $\mathcal{H}$. Hence we can apply the lemma above to deduce that

$$|u_k - v_k| \le \varepsilon \qquad \text{in } Q_{1/2},$$

where v_k solves

$$\begin{cases} \mathcal{H}_k(v_k) = 0 & \text{in } Q_{1/2}, \\ v_k = u_k & \text{on } \partial_p Q_{1/2}. \end{cases}$$

Since $\|v_k\|_{L^\infty(Q_{1/2})} \le \|u_k\|_{L^\infty(Q_1)} \le 1$, by interior $C^{2,1}_\alpha$ estimates we get

$$\|v_k\|_{C^{2,1}_\alpha(Q_{1/4})} \le C_0.$$

Let $\hat{P}_k$ be the "parabolic" second order Taylor expansion of v_k at $(0,0)$, and notice that $\mathcal{H}_k(\hat{P}_k) = \mathcal{H}_k(v_k(0,0)) = 0$. Then

$$|v_k - \hat{P}_k| \le C_0 \rho^{2+\alpha} \qquad \text{inside } Q_\rho,$$

which gives

$$|u_k - \hat{P}_k| \le C_0\rho^{2+\alpha} + \varepsilon \qquad \text{inside } Q_\rho.$$

In particular, if we choose ρ sufficiently small so that $C_0\rho^\alpha \le 1/2$ and then $\varepsilon \le \rho^2/2$ we arrive at

$$|u_k(X) - \hat{P}_k(X)| \le \rho^2 \qquad \text{inside } Q_\rho,$$

or equivalently (recalling the definition of u_k)

$$|u(X) - P_{k+1}(X)| \le \rho^{2(k+1)}, \qquad P_{k+1}(X) := P_k(X) + \rho^{2k}\hat{P}_k(x/\rho^k, t/\rho^{2k}).$$

Also, since $\mathcal{H}_k(\hat{P}_k) = 0$ we will have

$$\mathcal{H}(P_{k+1}) = F(D^2P_{k+1}) - \partial_t P_{k+1} = F(D^2P_k + D^2\hat{P}_k) - \partial_t P_k - \partial_t\hat{P}_k = \mathcal{H}_k(\hat{P}_k) = 0$$

which concludes the proof of the inductive step.

□

Remark 6.3. *As a corollary of our result we deduce L^p-BMO estimates on $\tilde{D}^2u$ ($p \in (1,\infty)$) for solutions to general elliptic/parabolic operators of type $F = F(\tilde{D}^2u, \nabla u, u, X)$ provided F is Hölder continuous and $u \in C_x^{1,\alpha}$.*

Indeed, if $F = F(D^2u, X)$, since $u_k(X) := \frac{u(\rho^k x, \rho^2 t) - P_k(\rho^k x, \rho^2 t)}{\rho^{2k}}$ satisfies $|u_k| \le 1$ and $|\mathcal{H}_k(u_k)| \le \delta$ inside Q_1, by interior $W_p^{2,1}$ estimates we get

$$\|\tilde{D}^2u_k\|_{L^p(Q_{1/2})} \le C,$$

that is

$$\frac{1}{|Q_{\rho^k/2}|}\int_{Q_{\rho^k/2}} |\tilde{D}^2u - \tilde{D}^2P_k|^p \le C \qquad \forall\, k \in \mathbb{N}.$$

For general operators F it suffices to apply the above argument to $G(M,X) := F(M, Du(X), u(X), X)$.

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