ON THE GEOMETRY OF GRADIENT EINSTEIN-TYPE MANIFOLDS

GIOVANNI CATINO[1], PAOLO MASTROLIA[2], DARIO D. MONTICELLI[3] AND MARCO RIGOLI[4],

ABSTRACT. In this paper we introduce the notion of Einstein-type structure on a Riemannian manifold (M, g), unifying various particular cases recently studied in the literature, such as gradient Ricci solitons, Yamabe solitons and quasi-Einstein manifolds. We show that these general structures can be locally classified when the Bach tensor is null.

1. INTRODUCTION AND MAIN RESULTS

In the last years there has been an increasing interest in the study of Riemannian manifolds endowed with metrics satisfying some structural equations, possibly involving curvature and some globally defined vector fields. These objects naturally arise in several different frameworks; the most important and well studied examples are *Ricci solitons* (see e.g. [19], [28], [27], [26], [6], [4] and references therein). Other examples are, for instance, *Ricci almost solitons* ([29]), *Yamabe solitons* ([16], [8]), *Yamabe quasi-solitons* ([21], [32]), *conformal gradient solitons* ([30], [12]), *quasi-Einstein manifolds* ([23], [10], [13], [20]), *ρ-Einstein solitons* ([14], [15]).

In this paper we study Riemannian manifolds satisfying a general structural condition that includes all the aforementioned examples as particular cases, in order to hopefully provide a useful compendium that also gives a summary and unification of classification problems thoroughly studied over the past years.

Towards this aim we consider a smooth, connected Riemannian manifold (M, g) of dimension $m \geq 3$, and we denote with Ric and S the corresponding *Ricci tensor* and *scalar curvature*, respectively (see the next section for the details). We denote with $\operatorname{Hess}(f)$ the Hessian of a function $f \in C^\infty(M)$ and with $\mathcal{L}_X g$ the Lie derivative of the metric g in the direction of the vector field X. We introduce the following

Definition 1.1. *We say that (M, g) is an* Einstein-type manifold *(or, equivalently, that (M, g) supports an* Einstein-type structure*) if there exist $X \in \mathfrak{X}(M)$ and $\lambda \in C^\infty(M)$ such that*

$$\alpha \operatorname{Ric} + \frac{\beta}{2}\mathcal{L}_X g + \mu X^\flat \otimes X^\flat = (\rho S + \lambda) g, \tag{1.1}$$

for some constants $\alpha, \beta, \mu, \rho \in \mathbb{R}$, with $(\alpha, \beta, \mu) \neq (0, 0, 0)$. If $X = \nabla f$ for some $f \in C^\infty(M)$, we say that (M, g) is a gradient Einstein-type manifold*. Accordingly equation (1.1) becomes*

$$\alpha \operatorname{Ric} + \beta \operatorname{Hess}(f) + \mu df \otimes df = (\rho S + \lambda) g, \tag{1.2}$$

for some $\alpha, \beta, \mu, \rho \in \mathbb{R}$.

Here $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M and $X^\flat$ the 1-form metrically dual to X.

We note that, from the definition, the term ρS could clearly be absorbed into the function λ. However, we keep them separate in order to explicitly include and highlight the case of ρ-Einstein solitons.

In the present paper we focus our analysis on the gradient case.

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[1]Politecnico di Milano, Italy. Email: giovanni.catino@polimi.it. Partially supported by GNAMPA, section "Calcolo delle Variazioni, Teoria del Controllo e Ottimizzazione", and GNAMPA, project "Equazioni differenziali con invarianze in Analisi Globale".
[2]Università degli Studi di Milano, Italy. Email: paolo.mastrolia@gmail.com. Supported by GNAMPA project "Analisi Globale ed Operatori Degeneri".
[3]Politecnico di Milano, Italy. Email: dario.monticelli@polimi.it. Partially supported by GNAMPA, section "Equazioni Differenziali e Sistemi Dinamici" and GNAMPA project "Analisi Globale ed Operatori Degeneri".
[4]Università degli Studi di Milano, Italy. Email: marco.rigoli@unimi.it.

Leaving aside the case $\beta = 0$ that will be addressed separately, see Proposition 5.7 below, we say that the gradient Einstein-type manifold (M, g) is *nondegenerate* if $\beta \neq 0$ and $\beta^2 \neq (m-2)\alpha\mu$; otherwise, that is if $\beta \neq 0$ and $\beta^2 = (m-2)\alpha\mu$ we have a *degenerate* gradient Einstein-type manifold. Note that, in this last case, necessarily α and μ are not null. The above terminology is justified by the next observation:

$$\begin{gathered}(M, g) \text{ is conformally Einstein if and only if} \\ \text{for some } \alpha, \beta, \mu \neq 0,\ (M, g) \text{ is a degenerate, gradient Einstein-type manifold.}\end{gathered} \tag{1.3}$$

For the proof and for the notion of conformally Einstein manifold see Section 2 below.

In case f is constant we say that the Einstein-type structure is *trivial*. Note that, since $m \geq 3$, in this case (M, g) is Einstein. However, the converse is generally false; indeed, if (M, g) is Einstein, then for some constant $\Lambda \in \mathbb{R}$ we have $\mathrm{Ric} = \Lambda g$ and inserting into (1.2) we obtain

$$\beta\, \mathrm{Hess}(f) + \mu df \otimes df = (\rho S + \lambda - \Lambda\alpha) g.$$

Thus, if $\rho \neq 0$, (M, g) is a Yamabe quasi-soliton and f is not necessarily constant (see [21], [32]). We will also deal with the case $\alpha = 0$ separately, see Theorem 1.4 below. We explicitly remark that, from the definition, α and β cannot both be equal to zero.

As we have already noted, the class of manifolds satisfying Definition 1.1 gives rise to the previously quoted examples by specifying, in general not in a unique way, the values of the parameters and possibly the function λ. In particular we have:

(1) Einstein manifolds: $(\alpha, \beta, \mu, \rho) = \left(1, 0, 0, \frac{1}{m}\right), \lambda = 0$ (or, equivalently for $m \geq 3$, $\rho = 0$ and $\lambda = \frac{S}{m}$);
(2) Ricci solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, 0), \lambda \in \mathbb{R}$;
(3) Ricci almost solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, 0), \lambda \in C^\infty(M)$;
(4) Yamabe solitons: $(\alpha, \beta, \mu, \rho) = (0, 1, 0, 1), \lambda \in \mathbb{R}$;
(5) Yamabe quasi-solitons: $(\alpha, \beta, \mu, \rho) = \left(0, 1, -\frac{1}{k}, 1\right), k \in \mathbb{R} \setminus \{0\}, \lambda \in \mathbb{R}$;
(6) conformal gradient solitons: $(\alpha, \beta, \mu, \rho) = (0, 1, 0, 0), \lambda \in C^\infty(M)$;
(7) quasi-Einstein manifolds: $(\alpha, \beta, \mu, \rho) = \left(1, 1, -\frac{1}{k}, 0\right)$, $\lambda \in \mathbb{R}$, $k \neq 0$;
(8) ρ-Einstein solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, \rho)$, $\rho \neq 0, \lambda \in \mathbb{R}$.

Of course one may wonder about the existence of Einstein-type structures. We know from the literature positive answers to the various examples that we mentioned earlier. For the general case we can consider three different necessary conditions; the first two are the general integrability conditions (4.5) and (4.6) contained in Theorem 4.4 below. The third comes from the simple observation that, in case $\mu \neq 0$, tracing equation (1.2) and defining $u = e^{\frac{\mu}{\beta} f}$, the existence of a gradient Einstein-type structure on (M, g) yields the existence of a positive solution of

$$Lu = \Delta u - \frac{\mu}{\beta^2}[m\lambda + (m\rho - \alpha)S]u = 0,$$

so that, by a well-known spectral result (see for instance Fischer-Colbrie–Schoen [17], or Moss-Piepenbrink [25]), the operator L is stable, or, in other words, the spectral radius of L, $\lambda_1^L(M)$, is nonnegative. Here we will not further pursue this direction.

As it appears from Definition 1.1, the fact that (M, g) is an Einstein-type manifold can be interpreted as a prescribed condition on the Ricci tensor of g (see for instance the nice survey [3]), that is, on the "trace part" of the Riemann tensor. Thus, it is reasonable to expect classification and rigidity results for these structures only assuming further conditions on the traceless part of the Riemann tensor, i.e. on the Weyl tensor. Indeed, most of the aforementioned papers pursue this direction, for instance, assuming that (M, g) is locally conformally flat or has harmonic Weyl tensor. In the spirit of the recent work of H.-D. Cao and Q. Chen [7], we study the class of gradient Einstein-type manifolds with vanishing Bach tensor along the integral curves of f. We note that this condition is weaker than local conformal flatness (see Section 2).

It turns out that, as in the case of gradient Ricci solitons (see [6], [7] and [5]), the leading actor is a three tensor, D, that plays a fundamental role in relating the Einstein-type structure to the geometry of the underlying manifold. D naturally appears when writing the first two integrability conditions for the structure defining the differential system (1.2). Quite unexpectedly, the constant ρ and the function λ have no influence on this relation.

Our main purpose is to give local characterizations of complete, noncompact, nondegenerate gradient Einstein-type manifolds. Denoting with B the Bach tensor of (M, g) (see Section 2), our first result is

Theorem 1.2. *Let (M, g) be a complete, noncompact, nondegenerate gradient Einstein-type manifold of dimension $m \geq 3$. If $B(\nabla f, \cdot) = 0$ and f is a proper function, then, in a neighbourhood of every regular level set of f, the manifold (M, g) is locally a warped product with $(m-1)$-dimensional Einstein fibers.*

In dimension four we improve this result, obtaining

Corollary 1.3. *Let (M^4, g) be a complete, noncompact nondegenerate gradient Einstein-type manifold of dimension four. If $B(\nabla f, \cdot) = 0$ and f is a proper function, then, in a neighbourhood of every regular level set of f, the manifold (M, g) is locally a warped product with three-dimensional fibers of constant curvature. In particular, (M^4, g) is locally conformally flat.*

As we will show in Section 7, the properness assumption is satisfied by some important subclasses of Einstein-type manifolds, under quite natural geometric assumptions. As a consequence, in the case of gradient Ricci solitons, we recover a local version of the results in [7] and [5], while, in the cases of ρ-Einstein solitons and Ricci almost solitons, we prove two new classification theorems (see Theorem 7.1 and 7.2).

In the special case $\alpha = 0$ (which includes Yamabe solitons, Yamabe quasi-solitons and conformal gradient solitons) we give a version of Theorem 1.2 in the following local result that provides a very precise description of the metric in this situation. Note that Theorem 1.4 and Corollary 1.5 also apply to the compact case.

Theorem 1.4. *Let (M, g) be a complete gradient Einstein-type manifold of dimension $m \geq 3$ with $\alpha = 0$. Then, in a neighbourhood of every regular level set of f, the manifold (M, g) is locally a warped product with $(m-1)$-dimensional fibers. More precisely, every regular level set Σ of f admits a maximal open neighborhood $U \subset M^m$ on which f only depends on the signed distance r to the hypersurface Σ. In addition, the potential function f can be chosen in such a way that the metric g takes the form*

$$g = dr \otimes dr + \left(\frac{f'(r)}{f'(0)} e^{\mu f(r)} \right)^2 g^{\Sigma} \quad \text{on } U, \tag{1.4}$$

where g^{Σ} is the metric induced by g on Σ. As a consequence, f has at most two critical points on M^m and we have the following cases:

(1) *If f has no critical points, then (M, g) is globally conformally equivalent to a direct product $I \times N^{m-1}$ of some interval $I = (t_*, t^*) \subseteq \mathbb{R}$ with a $(m-1)$-dimensional complete Riemannian manifold (N^{m-1}, g^N). More precisely, the metric takes the form*
$$g = u^2(t) \left(dt^2 + g^N \right),$$
where $u : (t_*, t^*) \to \mathbb{R}$ is some positive smooth function.

(2) *If f has only one critical point $O \in M^m$, then (M, g) is globally conformally equivalent to the interior of a Euclidean ball of radius $t^* \in (0, +\infty]$. More precisely, on $M^m \setminus \{O\}$, the metric takes the form*
$$g = v^2(t) \left(dt^2 + t^2 g^{\mathbb{S}^{m-1}} \right),$$
where $v : (0, t^*) \to \mathbb{R}$ is some positive smooth function and $\mathbb{S}^{m-1}$ denotes the standard unit sphere of dimension $m-1$. In particular (M, g) is complete, noncompact and rotationally symmetric.

(3) *If the function f has two critical points $N, S \in M^m$, then (M, g) is globally conformally equivalent to $\mathbb{S}^m$. More precisely, on $M^m \setminus \{N, S\}$, the metric takes the form*

$$g = w^2(t)\left(dt^2 + \sin^2(t)\, g^{\mathbb{S}^{m-1}}\right),$$

where $w : (0, \pi) \to \mathbb{R}$ is some smooth positive function. In particular (M, g) is compact and rotationally symmetric.

In this case, we can obtain a stronger global result, just assuming nonnegativity of the Ricci curvature; namely we have the following

Corollary 1.5. *Any nontrivial, complete, gradient Einstein type manifold of dimension $m \geq 3$ with $\alpha = 0$ and nonnegative Ricci curvature is either rotationally symmetric or it is isometric to a Riemannian product $\mathbb{R} \times N^{m-1}$, where N^{m-1} is an $(m-1)$-dimensional Riemannian manifold with nonnegative Ricci curvature.*

The proof of Theorem 1.4 follows immediately from [12] by substituting $u = e^{\mu f}$ in the equation. This result covers the cases of Yamabe solitons [8] and conformal gradient solitons [12]. Concerning Yamabe quasi-solitons, Corollary 1.5 improves the results in [21]. In particular, this shows that most of the assumptions in [21, Theorem 1.1] are not necessary.

The paper is organized as follows. In Section 2 we recall some useful definitions and properties of various geometric tensors and fix our conventions and notation. In Section 3 we collect some useful commutation relations for covariant derivatives of functions and tensors. In Section 4 we prove the two aforementioned integrability conditions that follow directly from the Einstein-type structures. In Section 5 we compute the squared norm of the tensor D in terms of D itself, the Bach tensor B and the potential function f. In Section 6 we relate the tensor D to the geometry of the regular level sets of the potential function f. Finally, in Section 7 we prove Theorem 1.2 and Corollary 1.3, and we give some geometric applications in the special cases of gradient Ricci solitons, ρ-Einstein solitons and Ricci almost solitons.

2. Definitions and notation

In this section we recall some useful definitions and properties of various geometric tensors and fix our conventions and notation.

To perform computations, we freely use the method of the moving frame referring to a local orthonormal coframe of the m-dimensional Riemannian manifold (M, g). We fix the index range $1 \leq i, j, \ldots \leq m$ and recall that the Einstein summation convention will be in force throughout.

We denote with R the *Riemann curvature tensor* (of type $(1, 3)$) associated to the metric g, and with Ric and S the corresponding *Ricci tensor* and *scalar curvature*, respectively. The components of the $(0, 4)$-versions of the Riemann tensor and of the *Weyl tensor* W are related by the formula:

$$R_{ijkt} = W_{ijkt} + \frac{1}{m-2}(R_{ik}\delta_{jt} - R_{it}\delta_{jk} + R_{jt}\delta_{ik} - R_{jk}\delta_{it}) - \frac{S}{(m-1)(m-2)}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}) \tag{2.1}$$

and they satisfy the symmetry relations

$$R_{ijkt} = -R_{jikt} = -R_{ijtk} = R_{ktij}, \tag{2.2}$$

$$W_{ijkt} = -W_{jikt} = -W_{ijtk} = W_{ktij}. \tag{2.3}$$

A computation shows that the Weyl tensor is also totally trace-free. The *Schouten tensor* A is defined by

$$\mathrm{A} = \mathrm{Ric} - \frac{S}{2(m-1)}g. \tag{2.4}$$

Tracing we have $\mathrm{tr}(\mathrm{A}) = A_{tt} = \frac{(m-2)}{2(m-1)}S$.

***Remark* 2.1.** Some authors adopt a different convention and define the Schouten tensor as $\frac{1}{m-2}A$.

According to this convention the (components of the) Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R_{itjt} = R_{titj}$ and $S = R_{tt}$. We note that, in terms of the Schouten tensor and of the Weyl tensor, the Riemann curvature tensor can be expressed in the form

$$\mathrm{R} = \mathrm{W} + \frac{1}{m-2}\mathrm{A} \owedge g, \tag{2.5}$$

where $\owedge$ is the Kulkarni-Nomizu product; in components,

$$R_{ijkt} = W_{ijkt} + \frac{1}{m-2}(A_{ik}\delta_{jt} - A_{it}\delta_{jk} + A_{jt}\delta_{ik} - A_{jk}\delta_{it}). \tag{2.6}$$

Next we introduce the *Cotton tensor* C as the obstruction to the commutativity of the covariant derivative of the Schouten tensor, that is

$$C_{ijk} = A_{ij,k} - A_{ik,j} = R_{ij,k} - R_{ik,j} - \frac{1}{2(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}). \tag{2.7}$$

We also recall that the Cotton tensor, for $m \geq 4$, can be defined as one of the possible divergences of the Weyl tensor; precisely

$$C_{ijk} = \left(\frac{m-2}{m-3}\right)W_{tikj,t} = -\left(\frac{m-2}{m-3}\right)W_{tijk,t}. \tag{2.8}$$

A computation shows that the two definitions (for $m \geq 4$) coincide (see again [24]).

***Remark* 2.2.** It is worth to recall that the Cotton tensor is skew-symmetric in the second and third indices (i.e. $C_{ijk} = -C_{ikj}$) and totally trace-free (i.e. $C_{iik} = C_{iki} = C_{kii} = 0$).

We are now ready to define the *Bach tensor* B, originally introduced by Bach in [1] in the study of conformal relativity. Its components are

$$B_{ij} = \frac{1}{m-2}(C_{jik,k} + R_{kt}W_{ikjt}), \tag{2.9}$$

that, in case $m \geq 4$, by (2.8) can be alternatively written as

$$B_{ij} = \frac{1}{m-3}W_{ikjt,tk} + \frac{1}{m-2}R_{kt}W_{ikjt}. \tag{2.10}$$

Note that if (M, g) is either locally conformally flat (i.e. $C = 0$ if $m = 3$ or $W = 0$ if $m \geq 4$) or Einstein, then $B = 0$. A computation shows that the Bach tensor is symmetric (i.e. $B_{ij} = B_{ji}$) and evidently trace-free (i.e. $B_{ii} = 0$). As a consequence we observe that we can write

$$B_{ij} = \frac{1}{m-2}(C_{ijk,k} + R_{kl}W_{ikjl}).$$

We recall that

Definition 2.3. *The manifold (M, g) is* conformally Einstein *if its metric g can be pointwise conformally deformed to an Einstein metric $\widetilde{g}$.*

We observe that, if $\widetilde{g} = e^{2a\varphi}g$, for some $\varphi \in C^\infty(M)$ and some constant $a \in \mathbb{R}$, then its Ricci tensor $\widetilde{\mathrm{Ric}}$ is related to that of g by the well-known formula (see [2])

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (m-2)a\,\mathrm{Hess}(\varphi) + (m-2)a^2 d\varphi \otimes \varphi - \Big[(m-2)a^2|\nabla\varphi|^2 + a\Delta\varphi\Big]g. \tag{2.11}$$

Here the various operators (and for their precise definitions see Section 3) are defined with respect to the metric g.

Now we can easily prove statement (1.3); indeed, suppose that $\beta \neq 0$ and $\beta^2 = (m-2)\alpha\mu$, that is, the Einstein-type structure is degenerate. Tracing (1.2) we obtain

$$\frac{1}{\alpha}(\rho S + \lambda) = \frac{1}{m}\left(S + \frac{\beta}{\alpha}\Delta f + \frac{\mu}{\alpha}|\nabla f|^2\right). \tag{2.12}$$

Choose $\varphi = f$ and $a = -\frac{\beta}{(m-2)\alpha}$ in (2.11) to obtain

$$\widetilde{\mathrm{Ric}} = \frac{1}{\alpha}\left[\frac{\beta^2}{(m-2)\alpha} - \mu\right] df \otimes df + \frac{1}{\alpha}(\rho S + \lambda) g + \frac{\beta}{(m-2)\alpha}\left(\Delta f - \frac{\beta}{\alpha}|\nabla f|^2\right) g. \tag{2.13}$$

Inserting (2.12) into (2.13) and using the fact that the Einstein-type structure is degenerate yields

$$\widetilde{\mathrm{Ric}} = \frac{1}{\alpha}\left[\frac{\beta^2}{(m-2)\alpha} - \mu\right] df \otimes df + \frac{1}{m}\left[S + 2\frac{\beta}{\alpha}\frac{m-1}{m-2}\Delta f - \frac{\mu}{\alpha}(m-1)|\nabla f|^2\right] g.$$

Hence, since $\beta^2 = (m-2)\alpha\mu$,

$$\widetilde{\mathrm{Ric}} = \frac{1}{m}\left[S + 2\frac{\beta}{\alpha}\frac{m-1}{m-2}\Delta f - \frac{\mu}{\alpha}(m-1)|\nabla f|^2\right] g, \tag{2.14}$$

that is, $\widetilde{g} = e^{-\frac{2\beta}{(m-2)\alpha}f} g$ is an Einstein metric (this was also obtained in Theorem 1.159 of [2]).

Viceversa, suppose that $\widetilde{g} = e^{2af} g$, $a \neq 0$, is an Einstein metric, so that, for some $\Lambda \in \mathbb{R}$, $\widetilde{\mathrm{Ric}} = \Lambda \widetilde{g}$. From (2.11)

$$\mathrm{Ric} - (m-2)a\,\mathrm{Hess}(f) + (m-2)a^2 df \otimes df = \left[\Lambda e^{2af} + (m-2)a^2|\nabla f|^2 + a\Delta f\right] g. \tag{2.15}$$

Tracing we get

$$\frac{S}{m-1} = \left[(m-2)a^2|\nabla f|^2 + a\Delta f\right] + a\Delta f + \frac{m}{m-1}\Lambda e^{2af}.$$

Thus, inserting into (2.15),

$$\mathrm{Ric} - (m-2)a\,\mathrm{Hess}(f) + (m-2)a^2 df \otimes df = \left(\frac{S}{m-1} - a\Delta f - \frac{\Lambda}{m-1}e^{2af}\right) g.$$

We choose $\alpha = 1$, $\beta = -(m-2)a$, $\mu = (m-2)a^2$, $\rho = \frac{1}{m-1}$ and $\lambda(x) = -a\Delta f - \frac{\Lambda}{m-1}e^{2af}$. We note that $\beta \neq 0$ and

$$\beta^2 = (m-2)^2 a^2 = (m-2)\alpha\mu,$$

so that the above choice of α, β, μ, ρ and λ yields a degenerate Einstein-type structure.

To conclude we note that the equivalence of degenerate gradient Ricci solitons and conformally Einstein metrics is well-known in conformal geometry (see [11, 22]).

3. Some basics on moving frames and commutation rules

In this section we collect some useful commutation relations for covariant derivatives of functions and tensors that will be used in the rest of the paper. All of these formulas are well-known to experts.

Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. For the sake of completeness (see [24] for details) we recall that, having fixed a (local) orthonormal coframe $\{\theta^i\}$, with dual frame $\{e_i\}$, then the corresponding *Levi-Civita connection forms* $\{\theta^i_j\}$ are the 1-forms uniquely defined by the requirements

$$d\theta^i = -\theta^i_j \wedge \theta^j \quad \text{(first structure equations)}, \tag{3.1}$$

$$\theta^i_j + \theta^j_i = 0. \tag{3.2}$$

The *curvature forms* $\{\Theta^i_j\}$ associated to the connection are the 2-forms defined via the *second structure equations*

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j. \tag{3.3}$$

They are skew-symmetric (i.e. $\Theta^i_j + \Theta^j_i = 0$) and they can be written as

$$\Theta^i_j = \frac{1}{2}R^i_{jkt}\theta^k \wedge \theta^t = \sum_{k<t} R^i_{jkt}\theta^k \wedge \theta^t, \tag{3.4}$$

where R^i_{jkt} are precisely the coefficients of the ((1,3)-version of the) Riemann curvature tensor.

The *covariant derivative of a vector field* $X \in \mathfrak{X}(M)$ is defined by

$$\nabla X = (dX^i + X^j\theta^i_j) \otimes e_i = X^i_k \theta^k \otimes e_i,$$

while the *covariant derivative of a 1-form* ω is defined by

$$\nabla\omega = (d\omega_i - w_j\theta^j_i) \otimes \theta^i = \omega_{ik}\theta^k \otimes \theta^i.$$

The *divergence* of the vector field $X \in \mathfrak{X}(M)$ is the trace of the endomorphism $(\nabla X)^\sharp : TM \to TM$, that is,

$$\operatorname{div} X = \operatorname{tr}(\nabla X)^\sharp = g(\nabla_{e_i}X, e_i) = X^i_i. \tag{3.5}$$

For a smooth function f we can write

$$df = f_i\theta^i, \tag{3.6}$$

for some smooth coefficients $f_i \in C^\infty(M)$. The *Hessian* of f, $\operatorname{Hess}(f)$, is the $(0,2)$-tensor defined as

$$\operatorname{Hess}(f) = \nabla df = f_{ij}\theta^j \otimes \theta^i, \tag{3.7}$$

with

$$f_{ij}\theta^j = df_i - f_t\theta^t_i. \tag{3.8}$$

Note that (see Lemma 3.1 below)

$$f_{ij} = f_{ji}.$$

The *Laplacian* of f, Δf, is the trace of the Hessian, in other words

$$\Delta f = \operatorname{tr}(\operatorname{Hess}(f)) = f_{ii}.$$

The moving frame formalism reveals extremely useful in determining the commutation rules of geometric tensors. Some of them will be essential in our computations.

Lemma 3.1. *If $f \in C^3(M)$ then:*

$$f_{ij} = f_{ji}; \tag{3.9}$$

$$f_{ijk} = f_{jik}; \tag{3.10}$$

$$f_{ijk} = f_{ikj} + f_tR_{tijk}; \tag{3.11}$$

$$\begin{aligned} f_{ijk} = {} & f_{ikj} + f_tW_{tijk} + \frac{1}{m-2}(f_tR_{tj}\delta_{ik} - f_tR_{tk}\delta_{ij} + f_jR_{ik} - f_kR_{ij}) \\ & - \frac{S}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}); \end{aligned} \tag{3.12}$$

$$f_{ijk} = f_{ikj} + f_tW_{tijk} + \frac{1}{m-2}(f_tA_{tj}\delta_{ik} - f_tA_{tk}\delta_{ij} + f_jA_{ik} - f_kA_{ij}); \tag{3.13}$$

In particular, tracing (3.11) we deduce

$$f_{itt} = f_{tti} + f_tR_{ti}. \tag{3.14}$$

For the Riemann curvature tensor we recall the classical Bianchi identities, that in our formalism become

$$R_{ijkt} + R_{itjk} + R_{iktj} = 0 \quad \text{(First Bianchi Identities)}; \tag{3.15}$$

$$R_{ijkt,l} + R_{ijlk,t} + R_{ijtl,k} = 0 \quad \text{(Second Bianchi Identities)}. \tag{3.16}$$

For the derivatives of the curvature we have the well known formulas

Lemma 3.2.

$$R_{ijkt,lr} - R_{ijkt,rl} = R_{sjkt}R_{silr} + R_{iskt}R_{sjlr} + R_{ijst}R_{sklr} + R_{ijks}R_{stlr}. \tag{3.17}$$

$$R_{ij,k} - R_{ik,j} = -R_{tijk,t} = R_{tikj,t}; \tag{3.18}$$

$$R_{ij,kt} - R_{ij,tk} = R_{likt}R_{lj} + R_{ljkt}R_{li}. \tag{3.19}$$

The First Bianchi Identities imply that

$$C_{ijk}+C_{jki}+C_{kij}=0. \tag{3.20}$$

From the definition of the Cotton tensor we also deduce that

$$C_{ijk,t}=A_{ij,kt}-A_{ik,jt}=R_{ij,kt}-R_{ik,jt}-\frac{1}{2(m-1)}(S_{kt}\delta_{ij}-S_{jt}\delta_{ik}). \tag{3.21}$$

On the other hand, by Lemma 3.2 and Schur's identity $S_i=\frac{1}{2}R_{ik,k}$,

$$R_{ik,jk}=R_{ik,kj}+R_{tijk}R_{tk}+R_{tkjk}R_{ti}=\frac{1}{2}S_{ij}-R_{tk}R_{itjk}+R_{it}R_{tj}. \tag{3.22}$$

This enables us to obtain the following expression for the divergence of the Cotton tensor:

$$C_{ijk,k}=R_{ij,kk}-\frac{m-2}{2(m-1)}S_{ij}+R_{tk}R_{itjk}-R_{it}R_{tj}-\frac{1}{2(m-1)}\Delta S\delta_{ij}. \tag{3.23}$$

The previous relation also shows that

$$C_{ijk,k}=C_{jik,k}, \tag{3.24}$$

thus confirming the symmetry of the Bach tensor, see (2.9).

Taking the covariant derivative of (3.20) and using (3.24) we also deduce

$$C_{kij,k}=0. \tag{3.25}$$

4. The tensor D and the integrability conditions

The main result of this section concerns two natural integrability conditions that follow directly from the Einstein-type structure; as in the case of Ricci solitons and Yamabe (quasi)-solitons, there is a natural tensor that turns out to play a fundamental role in relating the Einstein-type structure to the geometry of the underlying manifold. Quite surprisingly, as it is shown in Theorem 4.4, the presence of the constant ρ and of the function λ seems to be completely irrelevant.

Let (M,g) be gradient Einstein-type manifold of dimension $m\geq 3$. Equation (1.2) in components reads as

$$\alpha R_{ij}+\beta f_{ij}+\mu f_i f_j=(\rho S+\lambda)\delta_{ij}. \tag{4.1}$$

Tracing the previous relation we immediately deduce that

$$(\alpha-m\rho)S+\beta\Delta f+\mu|\nabla f|^2=m\lambda. \tag{4.2}$$

Definition 4.1. *We define the tensor D by its components*

$$D_{ijk}=\frac{1}{m-2}(f_kR_{ij}-f_jR_{ik})+\frac{1}{(m-1)(m-2)}f_t(R_{tk}\delta_{ij}-R_{tj}\delta_{ik})-\frac{S}{(m-1)(m-2)}(f_k\delta_{ij}-f_j\delta_{ik}). \tag{4.3}$$

Note that D is skew-symmetric in the second and third indices (i.e. $D_{ijk}=-D_{ikj}$) and totally trace-free (i.e. $D_{iik}=D_{iki}=D_{kii}=0$).

Remark 4.2. We explicitly note that our conventions for the Cotton tensor and for the tensor D differ from those in [7].

Lemma 4.3. *Let (M,g) be a gradient Einstein-type manifold of dimension $m\geq 3$. The tensor D can be written in the next three equivalent ways:*

$$D_{ijk}=\frac{\beta}{\alpha}\left[\frac{1}{m-2}(f_jf_{ik}-f_kf_{ij})+\frac{1}{(m-1)(m-2)}f_t(f_{tj}\delta_{ik}-f_{tk}\delta_{ij})-\frac{\Delta f}{(m-1)(m-2)}(f_j\delta_{ik}-f_k\delta_{ij})\right], \tag{4.4}$$

where E_{ij} *are the components of the* Einstein tensor *(see* [2]*) defined as*

$$E_{ij} = R_{ij} - \frac{S}{2}\delta_{ij}.$$

The proof is just a simple computation, using the definitions of the tensors involved, equation (4.1) and equation (4.2).

The following theorem should be compared with Lemma 3.1 and equation (4.1) in [7], with Lemma 2.4 and equation (2.12) in [5] and with Proposition 2.2 in [21]. This result highlights the geometric relevance of D in this general situation and shows that, even in this more general framework, similar structural equations hold.

Theorem 4.4. *Let* (M, g) *be a gradient Einstein-type manifold with* $\beta \neq 0$ *of dimension* $m \geq 3$. *Then the following integrability conditions hold:*

$$\alpha C_{ijk} + \beta f_t W_{tijk} = \left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right] D_{ijk}, \tag{4.5}$$

$$\alpha B_{ij} = \frac{1}{m-2}\left\{\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right] D_{ijk,k} + \beta\left(\frac{m-3}{m-2}\right) f_t C_{jit} - \mu f_t f_k W_{itjk}\right\}. \tag{4.6}$$

Proof. We begin with the covariant derivative of equation (4.1) to get

$$\alpha R_{ij,k} + \beta f_{ij,k} + \mu(f_{ik}f_j + f_i f_{jk}) = (\rho S_k + \lambda_k)\delta_{ij}. \tag{4.7}$$

Skew-symmetrizing with respect to j and k and using (3.11) we obtain

$$\alpha(R_{ij,k} - R_{ik,j}) + \beta f_t R_{tijk} + \mu(f_{ik}f_j - f_{ij}f_k) = \rho(S_k\delta_{ij} - S_j\delta_{ik}) + (\lambda_k\delta_{ij} - \lambda_j\delta_{ik}). \tag{4.8}$$

To get rid of the two terms on the right-hand side of equation (4.8) we proceed as follows: first we trace the equation with respect to i and j and we use Schur's identity $S_k = 2R_{tk,t}$ to deduce

$$[\alpha - 2\rho(m-1)]S_k = 2\beta f_t R_{tk} + 2(m-1)\lambda_k - 2\mu(f_t f_{tk} - \Delta f f_k); \tag{4.9}$$

secondly, from equations (4.1) and (4.2) we respectively have

$$f_{tk} = \frac{1}{\beta}[(\rho S + \lambda)\delta_{tk} - \alpha R_{tk} - \mu f_t f_k] \tag{4.10}$$

and

$$\Delta f = \frac{1}{\beta}\left[(m\rho - \alpha)S + m\lambda - \mu|\nabla f|^2\right]. \tag{4.11}$$

Inserting the two previous relations in (4.9) and simplifying we deduce the following important equation

$$[\alpha - 2\rho(m-1)]S_k = 2\left(\beta + \frac{\alpha\mu}{\beta}\right) f_t R_{tk} + 2(m-1)\lambda_k - \frac{2\mu}{\beta}[\alpha - \rho(m-1)]S f_k + \frac{2\mu}{\beta}(m-1)\lambda f_k. \tag{4.12}$$

From (2.1) and (4.4) we deduce that

$$f_t R_{tijk} = f_t W_{tijk} - D_{ijk} - \frac{1}{m-1}(f_t R_{tk}\delta_{ij} - f_t R_{tj}\delta_{ik}). \tag{4.13}$$

Inserting now (4.13), (2.7) and (4.12) into (4.8) and simplifying we get (4.5).

Taking the divergence of equation (4.5) we obtain

$$\alpha C_{ijk,k} - \beta f_{tk} W_{itjk} - \beta\left(\frac{m-3}{m-2}\right) f_t C_{jit} = \left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right] D_{ijk,k}; \tag{4.14}$$

using the definition of the Bach tensor (2.9), equation (4.10) and the symmetries of W we immediately deduce (4.6). □

Remark 4.5. Equation (4.12) is the analogue of the fundamental $S_k = 2f_t R_{tk}$, valid for every gradient Ricci soliton.

***Remark* 4.6.** In case $\beta = 0$ (and thus $\alpha \neq 0$), by direct calculations, using (2.7), (4.3) and (4.1), one can show that $D = 0$ and equations (4.5) and (4.6) take the form

$$\alpha C_{ijk} = -\mu(f_j f_{ik} - f_k f_{ij}) - \frac{\mu}{m-1} f_t(f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) + \frac{\mu\,\Delta f}{m-1}(f_j\delta_{ik} - f_k\delta_{ij}),$$
$$\alpha B_{ij} = \frac{1}{m-2}\{\alpha C_{ijk,k} - \mu f_t f_k W_{itjk}\}.$$

5. Vanishing of the tensor D

In this section we compute the squared norm of the tensor D in terms of D itself, the Bach tensor B and the potential function f. Moreover, under the assumption of Theorem 1.2, we prove the vanishing of D. We begin with

Lemma 5.1. *Let (M, g) be a nondegenerate gradient Einstein-type manifold of dimension $m \geq 3$. If $\alpha \neq 0$,*

$$\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 = -\beta(m-2)f_i f_j B_{ij} + \frac{\beta}{\alpha}\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right](f_i f_j D_{ijk})_k, \tag{5.1}$$

while if $\alpha = 0$

$$\left(\frac{m-2}{2}\right)|D|^2 = -(m-2)f_i f_j B_{ij} + (f_i f_j C_{ijk})_k. \tag{5.2}$$

Proof. We observe that, since D is totally trace-free and $D_{ijk} = -D_{ikj}$,

$$|D|^2 = D_{ijk}D_{ijk} = \frac{1}{m-2}D_{ijk}(f_k R_{ij} - f_j R_{ik}) = \frac{1}{m-2}(f_k R_{ij} D_{ijk} + f_j R_{ik} D_{ikj}),$$

so that

$$|D|^2 = \frac{2}{m-2} f_k R_{ij} D_{ijk}. \tag{5.3}$$

The nondegeneracy condition $\beta - \frac{(m-2)\alpha\mu}{\beta} \neq 0$ implies that, using (4.5) and the definition of the Bach tensor, we can write

$$\begin{aligned}
\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 &= f_k R_{ij}(\alpha C_{ijk} + \beta f_t W_{tijk}) \\
&= \alpha f_k R_{ij} C_{ijk} - \beta f_i f_j R_{tk} W_{itjk} \\
&= \alpha f_k R_{ij} C_{ijk} - \beta(m-2) f_i f_j B_{ij} + \beta f_i f_j C_{ijk,k}.
\end{aligned}$$

By the symmetries of the Cotton tensor we also have

$$\begin{aligned}
f_i f_j C_{ijk,k} &= f_i (f_j C_{ijk})_k - f_i f_{jk} C_{ijk} \\
&= (f_i f_j C_{ijk})_k - f_{ik} f_j C_{ijk} \\
&= (f_i f_j C_{ijk})_k + f_{ij} f_k C_{ijk},
\end{aligned}$$

therefore we obtain

$$\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 = \alpha f_k R_{ij} C_{ijk} - \beta(m-2) f_i f_j B_{ij} + \beta (f_i f_j C_{ijk})_k + \beta f_{ij} f_k C_{ijk}. \tag{5.4}$$

If $\alpha = 0$, using equation (4.1) in (5.4) we immediately get

$$\left(\frac{m-2}{2}\right)|D|^2 = -(m-2) f_i f_j B_{ij} + (f_i f_j C_{ijk})_k,$$

that is (5.2).

If $\alpha \neq 0$, using equations (4.1) and (4.5) in (5.4) and simplifying we deduce

$$\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 = -\beta(m-2)f_if_jB_{ij} + \frac{\beta}{\alpha}\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right](f_if_jD_{ijk})_k, \tag{5.5}$$

that is, equation (5.1). □

Remark 5.2. In case $\alpha \neq 0$ equation (5.1) can be obtained in a direct way: one takes the second integrability condition (4.6), multiplies both members by f_if_j and simplifies, using the symmetries of the tensors involved and equation (4.5).

Theorem 5.3. *Let (M, g) be a complete nondegenerate gradient Einstein-type manifold of dimension $m \geq 3$. If $B(\nabla f, \cdot) = 0$ and f is proper, then $D = 0$.*

Proof. We define the vector field $Y = Y(\alpha)$ of components

$$Y_k = \begin{cases} \frac{\beta}{\alpha} f_if_jD_{ijk} & \text{if } \alpha \neq 0; \\ f_if_jC_{ijk} & \text{if } \alpha = 0. \end{cases} \tag{5.6}$$

By the symmetries of D and C we immediately have

$$g(Y, \nabla f) = 0. \tag{5.7}$$

If $B(\nabla f, \cdot) = 0$ and $\alpha \neq 0$, from equation (5.1) we obtain

$$\left(\frac{m-2}{2}\right)|D|^2 = \frac{\beta}{\alpha}(f_if_jD_{ijk})_k, \tag{5.8}$$

while if $\alpha = 0$ from equation (5.2) we deduce

$$\left(\frac{m-2}{2}\right)|D|^2 = (f_if_jC_{ijk})_k. \tag{5.9}$$

In both cases

$$\left(\frac{m-2}{2}\right)|D|^2 = \operatorname{div} Y. \tag{5.10}$$

Let now c be a regular value of f and Ω_c and Σ_c be, respectively, the corresponding sublevel set and level hypersurface, i.e. $\Omega_c = \{x \in M : f(x) \leq c\}$, $\Sigma_c = \{x \in M : f(x) = c\}$. Integrating equation (5.10) on Ω_c and using the divergence theorem we get

$$\int_{\Omega_c} \left(\frac{m-2}{2}\right)|D|^2 = \int_{\Omega_c} \operatorname{div} Y = \int_{\Sigma_c} g(Y, \nu),$$

where ν is the unit normal to Σ_c. Since ν is in the direction of ∇f, using (5.7) and letting $c \to +\infty$ we immediately deduce

$$\int_M \left(\frac{m-2}{2}\right)|D|^2 = 0, \tag{5.11}$$

which implies $D = 0$ on M. □

Remark 5.4. The validity of Theorem 5.3 is based on that of the divergence theorem in this situation. Thus, instead of using properness of f, we can use Theorem A of [18] to obtain the above conclusion, that is $D \equiv 0$, under the following assumptions: for some $p > 1$, M is p-parabolic and the vector field $Y \in L^q(M)$, where q is the conjugate exponent of p. We note that a sufficient condition for p-parabolicity is

$$\frac{1}{\operatorname{vol}(\partial B_r)^{\frac{1}{p-1}}} \notin L^1(+\infty)$$

(see e.g. [31]), and, according to (5.6), $Y \in L^q(M)$ in case for some pair of conjugate exponents P, P' we have

$$|\nabla f| \in L^{2Pq}(M) \qquad \text{and} \quad |D| \in L^{P'q}(M) \ \text{ if } \alpha \neq 0$$

or

$$|\nabla f| \in L^{2Pq}(M) \qquad \text{and} \quad |C| \in L^{P'q}(M) \ \text{ if } \alpha = 0.$$

Remark 5.5. A simple computation using the definition of the tensor D gives

$$f_i D_{ijk} = \frac{1}{m-1}(f_t f_k R_{tj} - f_t f_j R_{tk}), \tag{5.12}$$

and then

$$f_i f_j D_{ijk} = \frac{1}{m-1}\Big(\mathrm{Ric}\,(\nabla f, \nabla f) f_k - |\nabla f|^2 f_t R_{tk}\Big). \tag{5.13}$$

This shows that, in the case $\alpha \neq 0$, the vector field Y defined in (5.6) can be expressed in the remarkable form

$$Y = \frac{\beta}{\alpha(m-1)}\Big[\mathrm{Ric}\,(\nabla f, \nabla f)\nabla f - |\nabla f|^2 \Big(\mathrm{Ric}\,(\nabla f, \cdot)^\sharp\Big)\Big], \tag{5.14}$$

where $\sharp$ denotes the usual musical isomorphism.

Moreover, in the special case of a gradient Ricci soliton (M, g, f, λ), using the fundamental relation $S_k = 2 f_t R_{tk}$, the vector field Y can also be written in the equivalent form

$$Y = \frac{1}{2(m-1)}\Big[g(\nabla S, \nabla f)\nabla f - |\nabla f|^2 \nabla S\Big].$$

We also observe that

$$g(Y, \nabla f) = 0, \ \ g(Y, \nabla S) = \frac{1}{2(m-1)}\Big[g(\nabla S, \nabla f)^2 - |\nabla S|^2 |\nabla f|^2\Big] \leq 0$$

and that

$$|Y|^2 = \frac{1}{4(m-1)^2}|\nabla f|^2\Big[|\nabla S|^2|\nabla f|^2 - g(\nabla S, \nabla f)^2\Big] = -\frac{1}{2(m-1)}|\nabla f|^2 g(Y, \nabla S).$$

Remark 5.6. In case $\beta = 0$ and $\mu \neq 0$, using Remark 4.6 and arguing as in Lemma 5.1, one can obtain the following identity

$$\frac{\alpha}{2\mu}|C|^2 = (m-2) f_i f_j B_{ij} - (f_i f_j C_{ijk})_k.$$

Then, following the proof of Theorem 5.3, we obtain

Proposition 5.7. *Let (M, g) be a complete nondegenerate gradient Einstein-type manifold of dimension $m \geq 3$ and with $\beta = 0$. If $B(\nabla f, \cdot) = 0$ and f is proper, then $C = 0$.*

6. D and the geometry of the level sets of f

In this section we relate the tensor D to the geometry of the regular level sets of the potential function f. Our first result highlights, in the case $\alpha \neq 0$, the link between the squared norm of the tensor D and the second fundamental form of the level sets of f. This should be compared with [7, Proposition 3.1] and [6, Lemma 4.1]. For the case $\alpha = 0$ we refer to [21, Proposition 2.3].

From now on, we extend our index convention assuming $1 \leq i, j, k, \ldots \leq m$ and $1 \leq a, b, c, \ldots \leq m-1$.

Proposition 6.1. *Let (M, g) be a complete m-dimensional ($m \geq 3$) gradient Einstein-type manifold with $\alpha, \beta \neq 0$. Let c be a regular value of f and let $\Sigma_c = \{x \in M | f(x) = c\}$ be the corresponding level hypersurface. For $p \in \Sigma_c$ choose an orthonormal frame such that $\{e_1, \ldots, e_{m-1}\}$ are tangent to Σ_c and $e_m = \frac{\nabla f}{|\nabla f|}$ (i.e., $\{e_1, \ldots, e_{m-1}, e_m\}$ is a local first order frame along f). Then, in p, the squared norm of the tensor D can be written as*

$$|D|^2 = \left(\frac{\beta}{\alpha}\right)^2 \frac{2|\nabla f|^4}{(m-2)^2}|h_{ab} - h\delta_{ab}|^2 + \frac{2|\nabla f|^2}{(m-1)(m-2)} R_{am}R_{am}, \tag{6.1}$$

where h_{ab} are the coefficients of the second fundamental tensor and h is the mean curvature of Σ_c.

Remark 6.2. Note that $|h_{ab}-h\delta_{ab}|^2$ is the squared norm of the traceless second fundamental tensor Φ of components $\Phi_{ab}=h_{ab}-h\delta_{ab}$.

Proof. First of all, we observe that, in the chosen frame, we have

$$df=f_a\theta^a+f_m\theta^m=|\nabla f|\theta^m, \tag{6.2}$$

since $f_a=0$, $a=1,\dots,m-1$.

The second fundamental tensor II of the immersion $\Sigma_c\hookrightarrow M$ is

$$II=h_{ab}\theta^b\otimes\theta^a\otimes\nu,$$

where the coefficients $h_{ab}=h_{ba}$ are defined as

$$\nabla e_m=\nabla\nu=\theta^a_m\otimes e_a=-\theta^m_a\otimes e_a=-h_{ab}\theta^b\otimes e_a \tag{6.3}$$

(see also [24]), so that

$$h_{ab}=g(II(e_a,e_b),\nu)=-g(\nabla_{e_a}\nu,e_b)=-(\nabla\nu)^\flat(e_a,e_b). \tag{6.4}$$

In the present setting we have

$$\nabla\nu=\frac{1}{|\nabla f|}\nabla(\nabla f)+\nabla\left(\frac{1}{|\nabla f|}\right)\otimes\nabla f$$

and

$$(\nabla\nu)^\flat=\frac{1}{|\nabla f|}\operatorname{Hess}(f)+d\left(\frac{1}{|\nabla f|}\right)\otimes df,$$

thus, using equation (4.1), we deduce

$$h_{ab}=-\frac{1}{|\nabla f|}f_{ab}=\frac{1}{\beta|\nabla f|}[\alpha R_{ab}-(\rho S+\lambda)\delta_{ab}], \tag{6.5}$$

The mean curvature h is defined as $h=\frac{1}{m-1}h_{aa}$; tracing equation (6.5) we get

$$h=\frac{1}{\beta|\nabla f|}\left[\left(\frac{\alpha}{m-1}-\rho\right)S-\frac{\alpha}{m-1}R_{mm}-\lambda\right]. \tag{6.6}$$

Now we compute the squared norm of the traceless second fundamental tensor Φ:

(6.7)

$$\begin{aligned}|h_{ab}-h\delta_{ab}|^2&=|h_{ab}|^2-2hh_{aa}+(m-1)h^2=|h_{ab}|^2-(m-1)h^2\\&=\frac{1}{\beta^2|\nabla f|^2}\left\{[\alpha R_{ab}-(\rho S+\lambda)\delta_{ab}]^2-(m-1)\left[\left(\frac{\alpha}{m-1}-\rho\right)S-\frac{\alpha}{m-1}R_{mm}-\lambda\right]^2\right\}\\&=\frac{\alpha^2}{\beta^2|\nabla f|^2}\left\{|\mathrm{Ric}|^2-2R_{am}R_{am}-(R_{mm})^2-\frac{1}{m-1}\left[S^2-2SR_{mm}+(R_{mm})^2\right]\right\}\\&=\frac{\alpha^2}{\beta^2|\nabla f|^2}\left[|\mathrm{Ric}|^2-2R_{am}R_{am}-\frac{m}{m-1}(R_{mm})^2-\frac{1}{m-1}S^2+\frac{2}{m-1}SR_{mm}\right].\end{aligned}$$

On the other hand, from the definition of D we have

$$
\begin{aligned}
(6.8)\quad |D|^2 &= \frac{(f_kR_{ij}-f_jR_{ik})^2}{(m-2)^2} + \frac{(f_tR_{tk}\delta_{ij}-f_tR_{tj}\delta_{ik})^2}{(m-1)^2(m-2)^2} + \frac{S^2}{(m-1)^2(m-2)^2}(f_k\delta_{ij}-f_j\delta_{ik})^2 \\
&+ \frac{2}{(m-1)(m-2)^2}(f_kR_{ij}-f_jR_{ik})(f_tR_{tk}\delta_{ij}-f_tR_{tj}) \\
&- \frac{2S}{(m-1)(m-2)^2}(f_kR_{ij}-f_jR_{ik})(f_k\delta_{ij}-f_j\delta_{ik}) \\
&- \frac{2S}{(m-1)^2(m-2)^2}(f_tR_{tk}\delta_{ij}-f_tR_{tj}\delta_{ik})(f_k\delta_{ij}-f_j\delta_{ik}) \\
&= \frac{2|\nabla f|^2}{(m-2)^2}\Big(|\mathrm{Ric}|^2 - R_{am}R_{am} - R_{mm}R_{mm}\Big) + \frac{2|\nabla f|^2}{(m-1)(m-2)^2}(R_{am}R_{am}+R_{mm}R_{mm}) \\
&+ \frac{2S^2}{(m-1)(m-2)^2}|\nabla f|^2 + \frac{4|\nabla f|^2}{(m-1)(m-2)^2}\Big(SR_{mm} - (R_{mm})^2 - R_{am}R_{am}\Big) \\
&- \frac{4S|\nabla f|^2}{(m-1)(m-2)^2}(S-R_{mm}) - \frac{4S|\nabla f|^2}{(m-1)(m-2)^2}R_{mm}.
\end{aligned}
$$

Symplifying, rearranging and comparing (6.7) and (6.8) we arrive at

$$
(6.9)\qquad \frac{(m-2)^2}{2|\nabla f|^2}|D|^2 = \left(\frac{\beta}{\alpha}\right)^2 |\nabla f|^2 |h_{ab}-h\delta_{ab}|^2 + \left(\frac{m-2}{m-1}\right)R_{am}R_{am},
$$

which easily implies equation (6.1).

□

Proposition 6.1 is one of the key ingredients in the proof of the following theorem, which generalizes [7, Proposition 3.2] (compare also with in [21, Proposition 2.4]). Our proof is similar to those in [7] and [21], but the presence of μ and the nonconstancy of λ require extra care, in particular in showing that S is constant on Σ_c.

Theorem 6.3. *Let (M,g) be a complete m-dimensional, $m \geq 3$, gradient Einstein-type manifold with $\alpha, \beta \neq 0$ and tensor $D \equiv 0$. Let c be a regular value of f and let $\Sigma_c = \{x \in M | f(x) = c\}$ be the corresponding level hypersurface. Choose any local orthonormal frame such that $\{e_1, \dots, e_{m-1}\}$ are tangent to Σ_c and $e_m = \frac{\nabla f}{|\nabla f|}$ (i.e., $\{e_1, \dots, e_{m-1}, e_m\}$ is a first order frame along f). Then*

1. *$|\nabla f|^2$ is constant on Σ_c;*
2. *$R_{am} = R_{ma} = 0$ for every $a = 1, \dots, m-1$ and e_m is an eigenvector of* Ric*;*
3. *Σ_c is totally umbilical;*
4. *the mean curvature h is constant on Σ_c;*
5. *the scalar curvature S and λ are constant on Σ_c;*
6. *Σ_c is Einstein with respect to the induced metric;*
7. *on Σ_c the (components of the) Ricci tensor of M can be written as $R_{ab} = \frac{S-\Lambda_1}{m-1}\delta_{ab}$, where $\Lambda_1 \in \mathbb{R}$ is an eigenvalue of multiplicity 1 or m (and in this latter case $S = m\Lambda_1$); in either case e_m is an eigenvector associated to Λ_1.*

Proof. If $D = 0$, from Proposition 6.1 we immediately deduce that

$$
(6.10)\qquad h_{ab} - h\delta_{ab} = 0,
$$

that is, property (3), and

$$
(6.11)\qquad R_{am} = 0, \qquad a = 1, \dots, m-1.
$$

From (6.10) a simple computation using (6.5) and (6.6) shows that

$$
(6.12)\qquad R_{ab} = \frac{S - R_{mm}}{m-1}\delta_{ab},
$$

which also implies

$$\operatorname{Ric}(\nu,\nu)=\frac{R_{ij}f_if_j}{|\nabla f|^2}=R_{mm}=R_{mm}|\nu|^2; \tag{6.13}$$

this complete the proof of (2). To prove (1) we take the covariant derivative of $\beta|\nabla f|^2$ and use (4.1):

$$\begin{aligned}\beta\left(|\nabla f|^2\right)_k&=2\beta f_if_{ik}\\&=2\left[\left(\rho S+\lambda-\mu|\nabla f|^2\right)f_k-\alpha f_tR_{tk}\right]\\&=2\left[\left(\rho S+\lambda-\mu|\nabla f|^2\right)f_k-\alpha f_cR_{ck}-\alpha|\nabla f|R_{mk}\right];\end{aligned}$$

evaluating the previous relation at $k=a$ and using property (2) we immediately get

$$\left(|\nabla f|^2\right)_a=0,$$

that is (1). To prove (4) we start from Codazzi equations, that in our setting read

$$-R_{mabc}=h_{ab,c}-h_{ac,b}; \tag{6.14}$$

tracing with respect to a and c we get

$$-R_{maba}=-R_{mkbk}+R_{mmbm}=h_{ab,a}-h_{aa,b},$$

that is, using (2),

$$0=-R_{mb}=h_{ab,a}-h_{aa,b}. \tag{6.15}$$

On the other hand, from (3) we have

$$h_{ab,a}=h_b$$

and

$$h_{aa,b}=(m-1)h_b,$$

so that (6.15) immediately implies

$$0=(m-2)h_b,\quad b=1,\dots,m-1, \tag{6.16}$$

that is (4). To show the validity of (5) we first observe that, evaluating (4.12) at $k=a$ and using (2), we deduce

$$[\alpha-2\rho(m-1)]S_a-2(m-1)\lambda_a=0,$$

which implies

$$[\alpha-2\rho(m-1)]S-2(m-1)\lambda=\text{const.}\quad\text{on }\Sigma_c. \tag{6.17}$$

From equation (6.6), the constancy of h and of $|\nabla f|$ on Σ_c also give that

$$[\alpha-\rho(m-1)]S-\alpha R_{mm}-(m-1)\lambda=\text{const.}\quad\text{on }\Sigma_c. \tag{6.18}$$

Combining (6.17) and (6.18) we arrive at

$$S-2R_{mm}=\text{const.}\quad\text{on }\Sigma_c. \tag{6.19}$$

Now we evaluate (4.12) at $k=m$, we use (2) and rearrange to deduce

(6.20)

$$\begin{aligned}{[\alpha-2\rho(m-1)]S_m}&=2\left(\beta+\frac{\alpha\mu}{\beta}\right)|\nabla f|R_{mm}+2(m-1)\lambda_m-\frac{2\mu|\nabla f|}{\beta}\{[\alpha-\rho(m-1)]S-(m-1)\lambda\}\\&=2\beta|\nabla f|R_{mm}+2(m-1)\lambda_m-\frac{2\mu|\nabla f|}{\beta}\{[\alpha-\rho(m-1)]S-\alpha R_{mm}-(m-1)\lambda\}.\end{aligned}$$

Since by (1) and (6.18) the quantity $\frac{2\mu|\nabla f|}{\beta}\{[\alpha-\rho(m-1)]S-\alpha R_{mm}-(m-1)\lambda\}$ is constant on Σ_c we infer

$$[\alpha-2\rho(m-1)]S_m-2\beta|\nabla f|R_{mm}-2(m-1)\lambda_m=\text{const.}\quad\text{on }\Sigma_c. \tag{6.21}$$

Now we take the covariant derivative of (6.21) and evaluate at $k=a$ to obtain

$$[\alpha-2\rho(m-1)]S_{ma}-2\beta|\nabla f|R_{mm,a}-2(m-1)\lambda_{ma}=0\quad\text{on }\Sigma_c; \tag{6.22}$$

but $S_{ma}=S_{am}$ and $\lambda_{ma}=\lambda_{am}$, thus (6.22) can be written as

$$\{[\alpha-2\rho(m-1)]S-2(m-1)\lambda\}_{am}=2\beta|\nabla f|R_{mm,a}\quad\text{on }\Sigma_c, \tag{6.23}$$

which implies, by (6.17), that

$$R_{mm}=\text{const.}\quad\text{on }\Sigma_c. \tag{6.24}$$

The previous relation, (6.19) and (6.17) show that S and λ are constant on Σ_c, that is (5). To prove (6) we start from the Gauss equations

$$^{\Sigma_c}R_{abcd}=R_{abcd}+h_{ac}h_{bd}-h_{ad}h_{bc},$$

which by property (3) can be rewritten as

$$^{\Sigma_c}R_{abcd}=R_{abcd}+h^2(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}). \tag{6.25}$$

Tracing equation (6.25) with respect to b and d gives

$$^{\Sigma_c}R_{ac}=R_{ac}-R_{amcm}+(m-2)h^2\delta_{ac}; \tag{6.26}$$

tracing again we deduce

$$^{\Sigma_c}S=S-2R_{mm}+(m-1)(m-2)h^2=\text{const.}\quad\text{on }\Sigma_c. \tag{6.27}$$

Now a simple computation using decomposition (2.1) of the Riemann tensor, equation (6.12) and the fact that $W_{amcm}=0$ (see Proposition 6.4) shows that

$$R_{amcm}=\frac{1}{m-1}R_{mm}\delta_{ac}. \tag{6.28}$$

Next, inserting (6.12) and (6.28) into (6.26), we get

$$^{\Sigma_c}R_{ac}=\left[\frac{S-2R_{mm}}{m-1}+(m-2)h^2\right]\delta_{ac}, \tag{6.29}$$

which shows the validity of (6). Now (7) is an easy consequence of the other properties. □

The next two results are the analogue of [7, Lemma 4.2] and [7, Lemma 4.3], respectively.

Proposition 6.4. *Let (M,g) be a complete noncompact m-dimensional ($m\geq 3$) nondegenerate Einstein-type manifold with $\alpha\neq 0$. If $D=0$ then $C=0$, unless f is locally constant.*

Proof. First of all, by analyticity, it is sufficient to prove the result where $\{\nabla f\neq 0\}$. We choose a local first order frame along f (so that $f_a=0$, $a=1,\dots,m-1$ and $f_m=|\nabla f|$). The vanishing of D implies, by the first integrability condition (4.5), that

$$\alpha C_{ijk}+\beta f_t W_{tijk}=0,$$

which implies, since $\alpha\neq 0$,

$$C_{ijk}=-\frac{\beta}{\alpha}f_t W_{tijk} \tag{6.30}$$

and consequently

$$f_i C_{ijk}=f_m C_{mjk}=|\nabla f|C_{mjk}=0,\ \ j,k=1,\dots,m; \tag{6.31}$$

thus

$$C_{mjk} = 0 \tag{6.32}$$

at all points where $|\nabla f| \neq 0$. Using (3) and (4) of Theorem 6.3 we have

$$h_{ab,c} = 0, \tag{6.33}$$

and from the Codazzi equations we get

$$-R_{mabc} = h_{ab,c} - h_{ac,b} = 0; \tag{6.34}$$

since also $R_{am} = 0$ by (2) of Theorem 6.3, from the decomposition (2.1) we easily deduce

$$W_{ambc} = 0, \tag{6.35}$$

which implies by (6.30) that

$$C_{abc} = 0. \tag{6.36}$$

By the symmetries of C, to conclude it only remains to show that $C_{abm} = 0 = C_{amb}$. First we observe that $R_{am} = 0$ implies, by the definition of covariant derivative,

$$\begin{aligned} 0 &= dR_{am} \\ &= R_{km}\theta^k_a + R_{ak}\theta^k_m + R_{am,k}\theta^k \\ &= R_{bm}\theta^b_a + R_{mm}\theta^m_a + R_{ab}\theta^b_m + R_{am}\theta^m_m + R_{am,k}\theta^k \\ &= R_{mm}\theta^m_a + R_{ab}\theta^b_m + R_{am,k}\theta^k, \end{aligned}$$

so that, using (6.12),

$$\begin{aligned} R_{am,k}\theta^k = R_{am,b}\theta^b + R_{am,m}\theta^m &= R_{ab}\theta^m_b - R_{mm}\theta^m_a \\ &= \left(\frac{S - R_{mm}}{m-1}\delta_{ab}\right)\theta^m_b - R_{mm}\theta^m_a \\ &= \left(\frac{S - mR_{mm}}{m-1}\right)\theta^m_a. \end{aligned} \tag{6.37}$$

Now we want to show that $R_{am,m} = 0$. To see that we first evaluate equation (4.1) for $i = a$ and $j = m$, obtaining $f_{am} = 0$; then we take the covariant derivative of the same equation:

$$\alpha R_{ij,k} + \beta f_{ijk} + \mu(f_{ik}f_j + f_i f_{jk}) = (\rho S_k + \lambda_k)\delta_{ij}, \tag{6.38}$$

which for $i = k = m$, $j = a$ gives (using $f_{am} = 0$)

$$\alpha R_{am,m} = -\beta f_{mam}; \tag{6.39}$$

but

$$f_{mam} = f_{mma} + f_i R_{imam} = f_{mma},$$

while (4.2) and Theorem 6.3 tell us that the (globally defined) quantity Δf is constant on Σ_c, so that

$$(\Delta f)_a = 0. \tag{6.40}$$

On the other hand, from (4.1) and (6.12) we deduce

$$\beta f_{ab} = -\frac{1}{m-1}\{[\alpha - \rho(m-1)]S - \alpha R_{mm} - (m-1)\lambda\}\delta_{ab} \tag{6.41}$$

which implies, by tracing, that

$$\beta(\Delta f - f_{mm}) = \text{const.} \quad \text{on } \Sigma_c; \tag{6.42}$$

in particular

$$f_{mam} = f_{mma} = (\Delta f)_a = 0, \tag{6.43}$$

and thus

$$R_{am,m} = 0. \tag{6.44}$$

Getting back to equation (6.37) we now have

$$R_{am,b}\theta^b = \left(\frac{S - mR_{mm}}{m-1}\right)\theta_a^m, \tag{6.45}$$

and thus

$$\begin{aligned} R_{am,b} &= \left(\frac{S - mR_{mm}}{m-1}\right)\theta_a^m(e_b) \\ &= \frac{1}{|\nabla f|}\left(\frac{mR_{mm} - S}{m-1}\right)f_{ab}. \end{aligned} \tag{6.46}$$

Schur's identity implies

$$S_m = 2R_{im,i} = 2R_{am,a} + 2R_{mm,m}; \tag{6.47}$$

from the definition of C we have, using (6.12) and (6.46),

$$\begin{aligned} C_{abm} &= R_{ab,m} - R_{am,b} - \frac{1}{2(m-1)}S_m\delta_{ab} \\ &= \frac{S_m - R_{mm,m}}{m-1}\delta_{ab} + \frac{1}{|\nabla f|}\left(\frac{s - mR_{mm}}{m-1}\right)f_{ab} - \frac{1}{2(m-1)}S_m\delta_{ab} \\ &= \frac{1}{2(m-1)}S_m\delta_{ab} - \frac{1}{m-1}R_{mm,m}\delta_{ab} + \frac{1}{|\nabla f|}\left(\frac{S - mR_{mm}}{m-1}\right)f_{ab}. \end{aligned} \tag{6.48}$$

Using (6.47), (6.46) and (6.41) into (6.48) we arrive at

$$\begin{aligned} C_{abm} &= \frac{1}{m-1}R_{cm,c}\delta_{ab} + \frac{1}{|\nabla f|}\left(\frac{S - mR_{mm}}{m-1}\right)f_{ab} \\ &= -\frac{1}{m-1}\frac{1}{|\nabla f|}(S - mR_{mm,m})f_{ab} + \frac{1}{|\nabla f|}\left(\frac{S - mR_{mm}}{m-1}\right)f_{ab} \\ &= 0, \end{aligned} \tag{6.49}$$

concluding the proof. □

In dimension four, we can prove the following

Corollary 6.5. *Let (M^4, g) be a complete noncompact nondegenerate Einstein-type manifold of dimension four with $\alpha \neq 0$. If $D = 0$ then $W = 0$, unless f is locally constant.*

Proof. From Proposition 6.4, we know that $C_{ijk} = 0$. Hence, from (4.5), we deduce $f_t W_{tijk} = 0$ for any $i, j, k = 1, \ldots, 4$. For any $p \in M^4$ such that $\nabla f(p) \neq 0$, we choose an orthonormal frame $\{e_1, \ldots, e_4\}$ such that $e_4 = \frac{\nabla f}{|\nabla f|}$, thus we have

$$W_{4ijk}(p) = 0, \qquad \text{for} \quad i, j, k = 1, \ldots 4\,.$$

It remains to show that $W_{abcd}(p) = 0$ for any $a, b, c, d = 1, 2, 3$. This follows from the symmetries and the traceless property of the Weyl tensor (for instance, see [7, Lemma 4.3]). □

7. Proof of the main theorems and some geometric applications

In this last section we first prove Theorem 1.2 and Corollary 1.3. Then, we give some geometric applications in the special cases of gradient Ricci solitons, ρ-Einstein solitons and Ricci almost solitons. We begin with

Proof of Theorem 1.2. From Theorem 5.3 we know that the tensor D has to vanish on M. Let Σ be a regular level set of the function $f : M^m \to \mathbb{R}$, i.e. $|\nabla f| \neq 0$ on Σ, which exists by Sard's Theorem and the fact that f is nontrivial. By Theorem 6.3 (1) we have that $|\nabla f|$ has to be constant on Σ. Thus, in a neighborhood U of Σ which does not contain any critical point of f, the potential function f only depends on the signed distance r to the hypersurface Σ. Hence, by a suitable change of variable, we can express the metric g_{ij} as

$$ds^2 = dr^2 + g_{ab}(r,\theta)d\theta^a \otimes d\theta^b \ , \quad r_* < r < r^* \ ,$$

for some maximal $r_* \in [-\infty, 0)$ and $r^* \in (0, \infty]$, where $(\theta^2, \cdots, \theta^m)$ is any local coordinates system on the level surface Σ. Moreover, by Theorem 6.3 (3)-(4), we have

$$\frac{\partial}{\partial r} g_{ab} = -2h_{ab} = \phi(r) g_{ab} \ ,$$

where $\phi(r) = -2h(r)$. Thus, it follows easily that

$$g_{ab}(r,\theta) = e^{\Phi(r)} g_{ab}(0,\theta),$$

where

$$\Phi(r) = \int_0^r \phi(r)\, dr.$$

This proves that on U the metric g takes the form of a warped product metric:

$$ds^2 = dr^2 + w(r)^2 g^E \ , \quad r \in (r_*, r^*) \ ,$$

where w is some positive smooth function on U, and $g^E = g^\Sigma$ is the metric defined on the level surface Σ, which is Einstein, by Theorem 6.3 (6). This concludes the proof of Theorem 1.2.

Proof of Corollary 1.3. The proof of Corollary 1.3 follows from all the previous considerations combined with Corollary 6.5.

□

Next we show that the properness assumption on the potential function f in Theorem 1.2 is automatically satisfied by some classes of Einstein-type manifolds.

First of all, let (M, g) be a complete, noncompact, *gradient Ricci soliton* with potential function f. Then, it is well known that f is always proper, provided that the soliton is either shrinking [9, Theorem 1.1], or steady with positive Ricci curvature and scalar curvature attaining its maximum at some point [6, Proposition 2.3] or expanding with nonnegative Ricci curvature [5, Lemma 5.5]. Hence, in these cases, Theorem 1.2 provides a local version of the classification results obtained in [7] and [5].

Secondly, if (M, g) is a complete, noncompact, *gradient shrinking ρ-Einstein soliton* with $\rho > 0$ and bounded scalar curvature, then it follows by [15, Lemma 3.2] that the potential function f is proper. Hence, Theorem 1.2 implies the following

Theorem 7.1. *Let (M, g) be a complete, noncompact gradient shrinking ρ-Einstein soliton of dimension $m \geq 3$ with bounded scalar curvature and $\rho > 0$. If $B(\nabla f, \cdot) = 0$, then around any regular point of f the manifold (M, g) is locally a warped product with $(m-1)$-dimensional Einstein fibers.*

Finally, we want to show the following result concerning *gradient Ricci almost solitons* which are "strongly" shrinking.

Theorem 7.2. *Let (M, g) be a complete, noncompact gradient Ricci almost soliton of dimension $m \geq 3$ with bounded Ricci curvature and with $\lambda \geq \underline{\lambda} > 0$, for some $\underline{\lambda}$. If $B(\nabla f, \cdot) = 0$, then around any regular point of f the manifold (M, g) is locally a warped product with $(m-1)$-dimensional Einstein fibers.*

Proof. By Theorem 1.2 it is sufficient to show that under these assumptions the potential function is proper. To do this we will apply a second variation argument as in [9, Theorem 1.1]. Let $r(x) = \mathrm{dist}(x, o)$, for some fixed origin $o \in M$. We will show that, for $r(x) \gg 1$,

$$f(x) \geq \frac{1}{2} \underline{\lambda}\big(r(x) - c\big)^2 ,$$

for some positive constant $c > 0$ depending only on m and on the geometry of g on the unit ball $B_o(1)$. Let $\gamma(s)$, $0 \leq s \leq s_0$ for some $s_0 > 0$, be any minimizing unit speed geodesic starting from $o = \gamma(0)$ and let $\dot\gamma(s)$ be the unit tangent vector of γ. Then by the second variation of the arc length, we have

$$\int_0^{s_0} \phi^2(s)\,\mathrm{Ric}(\dot\gamma, \dot\gamma)\, ds \leq (m-1) \int_0^{s_0} |\dot\phi(s)|^2\, ds ,$$

for every nonnegative function $\phi : [0, s_0] \to \mathbb{R}$. We choose $\phi(s) = s$ on $[0,1]$, $\phi(s) = 1$ on $[1, s_0 - 1]$ and $\phi(s) = s_0 - s$ on $[s_0 - 1, s_0]$. Then, since the solitons has bounded Ricci curvature, one has

$$\int_0^{s_0} \mathrm{Ric}(\dot\gamma, \dot\gamma)\, ds \;\leq\; 2(m-1) + \max_{B_1(o)} |\,\mathrm{Ric}\,| + \max_{B_1(\gamma(s_0))} |\,\mathrm{Ric}\,| \;\leq\; C ,$$

for some positive constant C independent of s_0. On the other hand, from the soliton equation, we have

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} f \;=\; \lambda - \mathrm{Ric}(\dot\gamma, \dot\gamma) .$$

Integrating along γ, we get

$$\dot f\big(\gamma(s_0)\big) - \dot f\big(\gamma(0)\big) \;=\; \int_0^{s_0} \lambda\, ds - \int_0^{s_0} \mathrm{Ric}(\dot\gamma, \dot\gamma)\, ds \geq \underline{\lambda}\, s_0 - C .$$

Integrating again, we obtain the desired estimate

$$f\big(\gamma(s_0)\big) \;\geq\; \frac{1}{2}\underline{\lambda}\big(s_0 - c\big)^2 ,$$

for some constant c. This concludes the proof of the theorem. □

Remark 7.3. As it is clear from the above proof, in case $\underline{\lambda} = \underline{\lambda}(r)$ is such that $\frac{1}{\underline{\lambda}(r)} = o\big(\frac{1}{r^2}\big)$ as $r \to +\infty$ we have $f(r) \to +\infty$ as $r \to +\infty$. This suffices to prove 7.2.

To conclude, we note that Ricci almost solitons which are warped product were constructed in [29, Remark 2.6].

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