EXISTENCE OF SOLUTIONS TO PARABOLIC PROBLEMS WITH NONSTANDARD GROWTH AND IRREGULAR OBSTACLES

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Abstract. In this paper, we establish the existence theory to nonlinear parabolic problems with nonstandard $p(x,t)$-growth conditions and irregular obstacles related to

$$\partial_t u - \operatorname{div} a(x,t,Du) \;=\; f - \operatorname{div}\left(|F|^{p(x,t)-2}F\right) \text{ in } \Omega_T.$$

1. Introduction

The aim of this paper is to establish the existence of solutions to parabolic obstacle problems related to

$$\partial_t u - \operatorname{div} a(x,t,Du) \;=\; f - \operatorname{div}\left(|F|^{p(x,t)-2}F\right) \text{ in } \Omega_T. \tag{1.1}$$

The motivation of this paper and the study of problems with nonstandard growth and irregular obstacles is on the one hand based on mathematical aspects, on the other hand the consideration of problems in the sense of (1.1) are motivated by issues of life sciences. We refer to [13, 34] for an overview of the classical theory and applications. Moreover, obstacle problems have been exploited in nonlinear potential theory for approximating supersolutions by solutions to obstacle problems, see [31, 33, 35]. Up to now, the theory for elliptic problems is well understood, as well the theory for elliptic obstacle problems, see e.g. [6, 14, 22, 37]. Therefore, parabolic problems arouse interest more and more in mathematics during the last years. Moreover, parabolic problems are motivated by physical aspects. In particular, evolutionary equations and systems can be used to model physical processes, e.g. heat conduction or diffusion processes. There are many open problems, e.g. with regard to the Navier-Stokes equation, the basic equation of fluid mechanics. Some properties of solutions of the system of a modified Navier-Stokes equation, describing electro-rheological fluids are studied in [4]. Such fluids, which are recently of high technological interest, because of their ability to change the mechanical properties under the influence of exterior electro-magnetic field, see [30, 41]. For example, many electro-rheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field, see [42]. Most of the known results concern the stationary models with $p(x)$-growth, see e.g. [1, 2, 3]. In the context of parabolic problems with $p(x,t)$-growth conditions, applications are e.g. the models for flows in porous media [8, 32]. Moreover, parabolic equations and systems with $p(x,t)$-growth were studied intensively in the last years, cf. e.g. [9, 11, 12, 16, 25, 26, 29, 45, 46].

First existence results for parabolic problems with time-independent obstacles have been achieved in the linear case by Lions and Stampacchia [38] and for more general parabolic problems by Brezis [14]. Obstacle functions that depend in some sense continuously on time are treated in [15]. The article [6] by Alt and Luckhaus contains existence results for elliptic and parabolic problems in great generality, but the results on obstacle problems are limited to time-independent or bounded obstacle functions. In the parabolic setting however, a comprehensive theory is available

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only under certain restrictions on the obstacle. An important break-through in the parabolic case succeeded to Bögelein, Duzaar and Mingione in [17]. Here, we want to highlight that in [17] the authors established the first existence result to parabolic problems with irregular obstacles, which are not necessarily non-increasing in time. They consider general obstacles with the only additional assumption that the time derivative of the obstacle lies in $L^{p'}$. This is required since their method relies on a time mollification argument, combined with a maximum construction in order to recover the obstacle condition, where the pointwise maximum construction is not compatible with distributional time derivatives. Moreover, they established the Calderón-Zygmund theory for a large class of parabolic obstacle problems, i.e. they proved that the (spatial) gradient of solutions is as integrable as that of the assigned obstacles. Then, Scheven considered a more general class of obstacles in [43, 44]. He introduced a new concept of solution to parabolic obstacle problems of p-Laplacian type with highly irregular obstacles, the so-called *localizable solutions*, see Definition 1.6. The main feature of localizable solutions is that they solve the obstacle problem locally, which is a priori not clear by the formulation of the problem, cf. the remarks preceding Definition 1.6. This new concept allows to consider more general settings, i.e. it is no more necessary to assume that the time derivative of the obstacle function lies in $L^{p'}$. It suffices to consider obstacles with distributional time derivatives. Moreover, we want to emphasize that the concept of localizable solutions allows to prove more general regularity results. Scheven also proved Calderón-Zygmund estimates for parabolic obstacle problems. The main difference between the result of Scheven and the result of Bögelein, Duzaar and Mingione is that in [17] they need an additional assumption on the boundary data, which seems to be unnatural for proving regularity in the interior. The reason for the additional assumption on the boundary data arises from the fact that the formulation of the obstacle problem is not of local nature. Bögelein, Duzaar and Mingione used a complex approximation argument to approximate the solutions by more regular ones and since the given solution was not known to be localizable, this approximation procedure had to be implemented on the whole domain. This problem could be avoid by the concept of localizable solutions. The concept of localizable solutions allows also to establish further regularity results, e.g. the higher integrability of solutions and the Hölder continuity of the spatial gradient of the solution u [18, 28]. Here, we will also use this concept to prove the existence of solutions to parabolic obstacle problems related to (1.1). Moreover, we highlight that the concept of localizable solutions permits to derive some regularity results for general parabolic obstacle problems with nonstandard growth. More precisely, the higher integrability of solutions and the Calderón-Zygmund theory, cf. [24, 25, 26, 27, 29]. Finally, we want to mention that beside the results we referred, the regularity of parabolic problems with irregular obstacle has been studied very intensive in the last years, cf. [10, 19, 20, 21, 36].

1.1. **General assumptions.** We consider a bounded domain $\Omega \subset \mathbb{R}^n$ of dimension $n \geq 2$ and we write $\Omega_T := \Omega \times (0,T)$ for the space-time cylinder over Ω of the height $T > 0$. In this paper, u_t respectively $\partial_t u$ denotes the partial derivate with respect to the time variable t and Du denotes the one with respect to the space variable x.

The setting. First of all, we should mention that we denote by $\partial_{\mathcal{P}}\Omega_T = (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T))$ the **parabolic boundary** of Ω_T. Furthermore, we write $z = (x,t)$ for points in $\mathbb{R}^{n+1}$. We shall consider vector-fields $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ which are assumed to be Carathéodory functions - i.e. $a(z,w)$ is measurable in the first argument for every $w \in \mathbb{R}^n$ and continuous in the second one for a.e. $z \in \Omega_T$ - and satisfy the following nonstandard **growth** and **monotonicity properties**, for some growth exponent $p : \Omega_T \to (\frac{2n}{n+2}, \infty)$ and structure constants $0 < \nu \leq 1 \leq L$ and $\mu \in [0,1]$:

$$|a(z,w)| \leq \; L(1+|w|)^{p(z)-1}, \tag{1.2}$$

$$(a(z,w)-a(z,w_0)) \cdot (w-w_0) \geq \nu(\mu^2+|w|^2+|w_0|^2)^{\frac{p(z)-2}{2}}|w-w_0|^2 \tag{1.3}$$

for all $z \in \Omega_T$ and $w,\ w_0 \in \mathbb{R}^n$. Furthermore, the growth exponent $p : \Omega_T \to (\frac{2n}{n+2}, \infty)$ satisfies the following conditions: There exist constants $\gamma_1,\ \gamma_2 < \infty$, such that

$$\frac{2n}{n+2} < \gamma_1 \le p(z) \le \gamma_2 \text{ and } |p(z_1) - p(z_2)| \le \omega(d_{\mathcal{P}}(z_1, z_2)) \tag{1.4}$$

holds for any choice of $z_1,\ z_2 \in \Omega_T$, where $\omega : [0, \infty) \to [0, 1]$ denotes a modulus of continuity. More precisely, we shall assume that $\omega(\cdot)$ is a concave, non-decreasing function with $\lim_{\rho \downarrow 0} \omega(\rho) = 0 = \omega(0)$. Moreover, the **parabolic distance** is given by $d_{\mathcal{P}}(z_1, z_2) := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$ for $z_1 = (x_1, t_1),\ z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$. In addition, for the modulus of continuity $\omega(\cdot)$ we assume the following **weak logarithmic continuity condition** to hold:

$$\limsup_{\rho \downarrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) < +\infty. \tag{1.5}$$

By virtue of (1.5) we may assume for a constant $L_1 > 0$ depending on $\omega(\cdot)$ that

$$\omega(\rho) \log\left(\frac{1}{\rho}\right) \le L_1, \text{ for all } \rho \in (0, 1]. \tag{1.6}$$

At this stage it is worth to mention that assuming the existence of such $\gamma_1,\ \gamma_2$ is not restrictive, since the results we are going to prove are of local nature. We mention that the previous lower bound on γ_1 is a typical assumption in the regularity theory of nonlinear parabolic equations and systems. Moreover, we denote by p_1 and p_2 the infimum resp. supremum of $p(\cdot)$ with respect to the domain we are going to deal with, e.g. $p_1 := \inf_{\Omega_T} p(\cdot),\ p_2 := \sup_{\Omega_T} p(\cdot)$. Finally, we point out that (1.3) implies, by using (1.2) and Young's inequality, the **coercivity property**

$$a(z, w) \cdot w \ge \frac{\nu}{c(\gamma_1, \gamma_2)} |w|^{p(z)} - c(\gamma_1, \gamma_2, \nu, L)\ \forall\ z \in \Omega_T \text{ and } w \in \mathbb{R}^n. \tag{1.7}$$

1.2. **The function spaces.** The spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ stand for the usual Lebesgue and Sobolev spaces.

Parabolic Lebesgue-Orlicz spaces. We start by the definition of the nonstandard $p(z)$-Lebesgue space. The space $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ is defined as the set of those measurable functions $v : \Omega_T \to \mathbb{R}^k$ for $k \in \mathbb{N}$, which satisfy $|v|^{p(\cdot)} \in L^1(\Omega_T, \mathbb{R}^k)$, i.e.

$$L^{p(z)}(\Omega_T, \mathbb{R}^k) := \left\{ v : \Omega_T \to \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |v|^{p(\cdot)}\, \mathrm{d}z < +\infty \right\}.$$

The set $L^{p(\cdot)}(\Omega_T, \mathbb{R}^k)$ equipped with the **Luxemburg norm**

$$\|v\|_{L^{p(\cdot)}(\Omega_T)} := \inf\left\{ \lambda > 0 : \int_{\Omega_T} \left|\frac{v}{\lambda}\right|^{p(\cdot)} \mathrm{d}z \le 1 \right\}$$

becomes a Banach space. This space is reflexive, see [5]. For the elements of $L^{p(\cdot)}(\Omega_T, \mathbb{R}^k)$ the **generalized Hölder's inequality** holds in the following form: If $f \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^k)$, $g \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^k)$, where $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$, we have

$$\left| \int_{\Omega_T} fg\, \mathrm{d}z \right| \le \left(\frac{1}{\gamma_1} + \frac{\gamma_2 - 1}{\gamma_2} \right) \|f\|_{L^{p(\cdot)}(\Omega_T)} \|g\|_{L^{p'(\cdot)}(\Omega_T)}, \tag{1.8}$$

see also [5]. Notice that the norm $\|\cdot\|_{L^{p(\cdot)}(\Omega_T)}$ can be estimated as follows:

$$-1 + \|v\|_{L^{p(\cdot)}(\Omega_T)}^{\gamma_1} \le \int_{\Omega_T} |v|^{p(\cdot)}\, \mathrm{d}z \le \|v\|_{L^{p(\cdot)}(\Omega_T)}^{\gamma_2} + 1. \tag{1.9}$$

Finally, for the right-hand side of (1.1) we assume

$$F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n) \text{ and } f \in L^{\gamma_1'}(0, T; W^{-1,\gamma_1'}(\Omega)). \tag{1.10}$$

Parabolic Sobolev-Orlicz spaces. First, we introduce nonstandard parabolic Sobolev spaces for fixed $t \in (0, T)$. From (1.4), we know that $p(\cdot, t)$ satisfy $|p(x_1, t) - p(x_2, t)| \le \omega(|x_1 - x_2|)$ for any choice of $x_1,\ x_2 \in \Omega$ and for every $t \in (0, T)$.

For every fixed $t \in (0,T)$, we define the Banach space $W^{1,p(\cdot,t)}(\Omega)$ as follows: $W^{1,p(\cdot,t)}(\Omega) := \{u \in L^{p(\cdot,t)}(\Omega,\mathbb{R}) \mid Du \in L^{p(\cdot,t)}(\Omega,\mathbb{R}^n)\}$ equipped with the norm

$$\|u\|_{W^{1,p(\cdot,t)}(\Omega)} := \|u\|_{L^{p(\cdot,t)}(\Omega)} + \|Du\|_{L^{p(\cdot,t)}(\Omega)}.$$

In addition, $W_0^{1,p(\cdot,t)}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot,t)}(\Omega)$ and denote by $W^{1,p(\cdot,t)}(\Omega)'$ its dual. For every $t \in (0,T)$ the inclusion $W_0^{1,p(\cdot,t)}(\Omega) \subset W_0^{1,\gamma_1}(\Omega)$ holds. Now, we consider more general **nonstandard parabolic Sobolev** spaces without fixed t. By $W_g^{p(\cdot)}(\Omega_T)$ we denote the Banach space

$$W_g^{p(\cdot)}(\Omega_T) := \left\{u \in [g + L^1(0,T;W_0^{1,1}(\Omega))] \cap L^{p(\cdot)}(\Omega_T) \mid Du \in L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)\right\}$$

equipped by the norm

$$\|u\|_{W^{p(\cdot)}(\Omega_T)} := \|u\|_{L^{p(\cdot)}(\Omega_T)} + \|Du\|_{L^{p(\cdot)}(\Omega_T)}.$$

If $g = 0$ we write $W_0^{p(\cdot)}(\Omega_T)$ instead of $W_g^{p(\cdot)}(\Omega_T)$. Here, it is worth to mention that the notion $(u-g) \in W_0^{p(\cdot)}(\Omega_T)$ respectively $u \in g + W_0^{p(\cdot)}(\Omega_T)$ to indicate that u agrees with g on the lateral boundary of the cylinder Ω_T, i.e. $u \in W_g^{p(\cdot)}(\Omega_T)$. We are now ready to give the definition of a weak solution to the nonstandard parabolic equation (1.1):

Definition 1.1. We identify a function $u \in L^1(\Omega_T)$ as a **weak solution** of the parabolic equation (1.1), if and only if $u \in C^0([0,T];L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ and

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z,Du)\cdot D\varphi]\ \mathrm{d}z = -\int_{\Omega_T} \left[f\cdot\varphi + |F|^{p(\cdot)-2}F\cdot D\varphi\right]\ \mathrm{d}z \tag{1.11}$$

holds, whenever $\varphi \in C_0^\infty(\Omega_T)$.

Our next aim is to introduce the dual space of $W_0^{p(\cdot)}(\Omega_T)$. Therefore, we denote by $W^{p(\cdot)}(\Omega_T)'$ the dual of the space $W_0^{p(\cdot)}(\Omega_T)$. Assume that $v \in W^{p(\cdot)}(\Omega_T)'$. Then, there exist functions $v_i \in L^{p'(\cdot)}(\Omega_T)$, $i = 0,1,...,n$, such that

$$\langle\!\langle v,w\rangle\!\rangle_{\Omega_T} = \int_{\Omega_T}\left(v_0 w + \sum_{i=1}^n v_i D_i w\right)\ \mathrm{d}z\ \forall\ w \in W_0^{p(\cdot)}(\Omega_T), \tag{1.12}$$

see [7]. Here and in the following, we will write $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\Omega_T}$ for the pairing between $W^{p(\cdot)}(\Omega_T)'$ and $W_0^{p(\cdot)}(\Omega_T)$. Furthermore, if $v \in W^{p(\cdot)}(\Omega_T)'$, we define the norm

$$\|v\|_{W^{p(\cdot)}(\Omega_T)'} = \sup\{\langle\!\langle v,w\rangle\!\rangle_{\Omega_T}\, | w \in W_0^{p(\cdot)}(\Omega_T),\ \|w\|_{W_0^{p(\cdot)}(\Omega_T)} \le 1\}.$$

Notice, whenever (1.12) holds, we can write $v = v_0 - \sum_{i=1}^n D_i v_i$, where $D_i v_i$ has to be interpreted as a distributional derivate. By

$$w \in W(\Omega_T) := \left\{w \in W^{p(\cdot)}(\Omega_T) | w_t \in W^{p(\cdot)}(\Omega_T)'\right\}$$

we mean that there exists $w_t \in W^{p(\cdot)}(\Omega_T)'$, such that

$$\langle\!\langle w_t,\varphi\rangle\!\rangle_{\Omega_T} = -\int_{\Omega_T} w\cdot\varphi_t\ \mathrm{d}z \text{ for all } \varphi \in C_0^\infty(\Omega_T).$$

The previous equality makes sense due to the inclusions

$$W^{p(\cdot)}(\Omega_T) \hookrightarrow L^2(\Omega_T) \cong (L^2(\Omega_T))' \hookrightarrow W^{p(\cdot)}(\Omega_T)'$$

which allow us to identify w as an element of $W^{p(\cdot)}(\Omega_T)'$. Next, we refer the properties of the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\Omega_T}$.

Proposition 1.2 (Proposition 2.2, [26]). *Let $u,w \in W_0^{p(\cdot)}(\Omega_T)$, $v,\tilde{v} \in W^{p(\cdot)}(\Omega_T)'$, $\zeta \in C_{cpt}^\infty(\Omega)$ and $a \in \mathbb{R}$, then the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\Omega_T}$ between $W^{p(\cdot)}(\Omega_T)'$ and $W_0^{p(\cdot)}(\Omega_T)$ has the following properties:*

(i) $\left\langle\!\left\langle v,a^2u\right\rangle\!\right\rangle_{\Omega_T} = \langle\!\langle av,au\rangle\!\rangle_{\Omega_T} = \left\langle\!\left\langle a^2v,u\right\rangle\!\right\rangle_{\Omega_T} = a^2\,\langle\!\langle v,u\rangle\!\rangle_{\Omega_T}$,

(ii) $\langle\!\langle v,w+u\rangle\!\rangle_{\Omega_T} = \langle\!\langle v,w\rangle\!\rangle_{\Omega_T} + \langle\!\langle v,u\rangle\!\rangle_{\Omega_T}$,

(iii) $\langle\!\langle v+\tilde{v},w\rangle\!\rangle_{\Omega_T} = \langle\!\langle v,u\rangle\!\rangle_{\Omega_T} + \langle\!\langle \tilde{v},u\rangle\!\rangle_{\Omega_T}$.

If $\partial_t w, \partial_t u \in W^{p(\cdot)}(\Omega_T)'$, we have also

(iv) $\langle\!\langle \partial_t(\zeta(w-u)), \zeta(w-u)\rangle\!\rangle_{\Omega_T} = \langle\!\langle \partial_t(w-u), \zeta^2(w-u)\rangle\!\rangle_{\Omega_T}$

in the distributional sense. □

Finally, from the definition of the norm $\|\cdot\|_{W^{p(\cdot)}(\Omega_T)'}$, we can conclude that for the elements of $W^{p(\cdot)}(\Omega_T)$ the following estimate holds: If $f \in W_0^{p(\cdot)}(\Omega_T)$, $g \in W^{p(\cdot)}(\Omega_T)'$ we have

$$\langle\!\langle f, g\rangle\!\rangle_{\Omega_T} \le c(\gamma_1, \gamma_2)\|f\|_{W^{p(\cdot)}(\Omega_T)}\|g\|_{W^{p(\cdot)}(\Omega_T)'}, \tag{1.13}$$

see [26]. Notice also that in the case $p(\cdot) = \text{const.}$, we deal with the standard Lebesgue and Sobolev spaces. This means for example, if $p(\cdot) = \gamma_1$, then we have $W^{\gamma_1}(\Omega_T) = L^{\gamma_1}(0,T; W^{1,\gamma_1}(\Omega))$. Consequently, the dual space of $W^{\gamma_1}(\Omega_T)$ is given by $W^{\gamma_1}(\Omega_T)' = L^{\gamma_1'}(0,T; W^{-1,\gamma_1'}(\Omega))$.

Obstacle function, boundary, initial values and energy bound. At this stage, we state the assumptions for the obstacle function, boundary data, initial values and the obstacle constraint. These assumptions we need to define the function spaces in which we will formulate the obstacle problems. Therefore, we consider on the lateral boundary $\partial\Omega \times (0,T)$ **Dirichlet boundary data** given by

$$g \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \text{ and } \partial_t g \in L^{\gamma_1'}(0,T; W^{-1,\gamma_1'}(\Omega)) \tag{1.14}$$

and **initial values** $g(\cdot, 0) \in L^2(\Omega)$. The **obstacle constraint** will be given by a function $\psi : \Omega_T \to \mathbb{R}$ with

$$\psi \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \text{ and } \partial_t \psi \in L^{\gamma_1'}(0,T; W^{-1,\gamma_1'}(\Omega)). \tag{1.15}$$

For the boundary and initial values, we assume the **compatibility conditions**

$$g \ge \psi \text{ on } \partial\Omega \times (0,T) \text{ and } g(\cdot,0) \ge \psi(\cdot,0) \text{ a.e. on } \Omega, \tag{1.16}$$

where the first one is to be understood in the L^1-$W_0^{1,1}$-sense, i.e. $(\psi - g)_+ \in W_0^{p(\cdot)}(\Omega_T)$. Now, we are in a situation to introduce the function spaces in which we will formulate the obstacle problem. These spaces are defined as follows:

$$\mathcal{K}_{\psi,g}(\Omega_T) := \Big\{u \in C^0([0,T]; L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T),\ u \ge \psi \text{ a.e. on } \Omega_T\Big\},$$

and the function space

$$\mathcal{K}'_{\psi,g}(\Omega_T) := \Big\{u \in \mathcal{K}_{\psi,g}(\Omega_T) \mid \partial_t u \in W^{p(\cdot)}(\Omega_T)'\Big\},$$

whose members play the role of admissible comparison functions.

1.3. **Parabolic obstacle problems with nonstandard p(z)-growth.**

The main problem we are going to deal with, are the obstacle problems. More precisely, problems with irregular time dependent obstacles $\psi : \Omega_T \to \mathbb{R}$. The **variational inequality** that we have in mind can be written in two different ways.

Definition 1.3. We identify a function $u \in \mathcal{K}'_{\psi,g}(\Omega_T)$ as a solution of the **strong formulation of the variational inequality** if $u(\cdot,0) = g(\cdot,0)$ and

$$\begin{aligned}\langle\!\langle \partial_t u, v-u\rangle\!\rangle_{\Omega_T} &+ \int_{\Omega_T} a(z,Du)\cdot D(v-u)\,\mathrm{d}z \\ &\ge \int_{\Omega_T} |F|^{p(\cdot)-2}F\cdot D(v-u) + f(v-u)\,\mathrm{d}z,\end{aligned} \tag{1.17}$$

holds for all comparison functions $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$.

It turns out that in our situation, the solution to the obstacle problem does not necessarily possess a time derivative in the distributional space $W^{p(\cdot)}(\Omega_T)'$, but only satisfies $u \in \mathcal{K}_{\psi,g}(\Omega_T)$. In this case, only the following formulation makes sense:

Definition 1.4. We identify a function $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ as a solution of the **weak formulation of the variational inequality** if

$$\begin{aligned}\langle\!\langle \partial_t v, v-u\rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z,Du)\cdot D(v-u)\,\mathrm{d}z + \|v(\cdot,0)-g(\cdot,0)\|^2_{L^2(\Omega)} \\ \geq \int_{\Omega_T} f(v-u) + |F|^{p(\cdot)-2}F\cdot D(v-u)\,\mathrm{d}z\end{aligned} \tag{1.18}$$

holds for all test functions $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$.

Remark 1.5. Although not always explicitly stated, when referring to an initial condition of the type

$$u(\cdot,0) = g(\cdot,0) \text{ a.e. on } \Omega$$

we shall always mean

$$\frac{1}{h}\int_0^h \int_\Omega |u-g(\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } h\downarrow 0. \tag{1.19}$$

In particular, when $u \in C^0([0,T];L^2(\Omega))$, then (1.19) is obviously equivalent with saying $u(\cdot,0)=g(\cdot,0)$.

1.4. **The concept of localizable solutions.** The concept of localizable solutions goes back to Ch. Scheven, see [43, 44], and the idea of this concept is the following: In the general situation that we are considering, the solutions do not necessarily satisfy $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$, so that the variational inequality can only be written in the weak formulation (1.18). However, this formulation does not seem to be the most suitable notion of solution, since it is not of local nature. More precisely, for a given parabolic cylinder $\mathcal{O}_I = \mathcal{O}\times(t_1,t_2)\subset\Omega_T$, it is a priori not clear that the restriction $u_{|_{\mathcal{O}_I}}$ of a solution u to the weak formulation of the variational inequality (1.18) again satisfies a variational inequality on $\mathcal{O}_I$. Even more, it is unclear if the space $\mathcal{K}'_{\psi,u}(\mathcal{O}_I)$ of admissible comparison maps is not empty. In fact, it is not evident from the formulation (1.18) that there exists any map that agrees with u on the lateral boundary of $\mathcal{O}_I$ and at the same time possesses a time derivative in the distributional space $W^{p(\cdot)}(\Omega_T)'$, which would be necessary for the construction of suitable comparison maps. These considerations motivate the following concept of a localizable solution to a parabolic obstacle problem.

Definition 1.6. We say that $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ is a **localizable solution** of the weak formulation (1.18) of the obstacle problem if for every parabolic cylinder $\mathcal{O}_I := \mathcal{O}\times(t_1,t_2)\subset\Omega_T$, where $\mathcal{O}=\tilde{\mathcal{O}}\cap\Omega$ with a Lipschitz regular domain $\tilde{\mathcal{O}}\subset\mathbb{R}^n$ and a time interval $I=(t_1,t_2)\subset(0,T)\subset\mathbb{R}$, the following two conditions hold.

(i) The map u satisfies the **extension property**, i.e. there holds $\mathcal{K}'_{\psi,u}(\mathcal{O}_I)\neq\emptyset$.
(ii) For all comparison maps $v\in\mathcal{K}'_{\psi,u}(\mathcal{O}_I)$, there holds

$$\begin{aligned}\langle\!\langle \partial_t v, v-u\rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} a(z,Du)\cdot D(v-u)\,\mathrm{d}z + \|(v-u)(\cdot,t_1)\|^2_{L^2(\mathcal{O})} \\ \geq \int_{\mathcal{O}_I} f(v-u) + |F|^{p(\cdot)-2}F\cdot D(v-u)\,\mathrm{d}z,\end{aligned} \tag{1.20}$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{O}_I}$ denotes the dual pairing between $W^{p(\cdot)}(\mathcal{O}_I)'$ and $W_0^{p(\cdot)}(\mathcal{O}_I)$.

1.5. **Statement of the result.** Our first existence result holds on any bounded domains $\Omega\subset\mathbb{R}^n$ if the obstacle function satisfies a certain approximation assumption. This is in particular the case for a general obstacle if $\partial\Omega$ fulfills some weak regularity property. For the most general form of our first existence theorem, we assume that the obstacle function

$$\psi \in C^0([0,T];L^2(\Omega))\cap W^{p(\cdot)}(\Omega_T) \text{ with } \partial_t\psi \in L^{\gamma_1'}(\Omega_T), \tag{1.21}$$

can be approximated by more regular obstacle functions

$$\psi_i \in C^0([0,T];L^2(\Omega))\cap W^{p(\cdot)}(\Omega_T) \text{ with } \partial_t\psi_i \in L^{\gamma_1'}(\Omega_T) \tag{1.22}$$

with an additional regularity property

$$|\partial_t\psi_i| + (1+|D\psi_i|)^{p(\cdot)}|D^2\psi_i| \in L^{\gamma_1'}(\Omega_T) \tag{1.23}$$

for $i \in \mathbb{N}$, which approximate ψ in the sense

$$\begin{cases} \psi_i \to \psi & \text{strongly in } W^{p(\cdot)}(\Omega_T) \text{ and } L^\infty(0,T;L^2(\Omega)), \\ \partial_t\psi_i \to \partial_t\psi & \text{strongly in } L^{\gamma_1'}(\Omega_T), \end{cases} \tag{1.24}$$

as $i \to \infty$. This approximation assumption can be omitted under mild assumptions on the boundary of the domain Ω. Our result on existence and uniqueness reads as follows.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4)-(1.5). Furthermore, assume that $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies the growth and monotonicity condition (1.2) and (1.3). Moreover, suppose that the inhomogeneities*

$$F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n) \text{ and } f \in L^{\gamma_1'}(\Omega_T), \tag{1.25}$$

the boundary data g satisfying

$$g \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \text{ with } \partial_t g \in L^{\gamma_1'}(\Omega_T), \tag{1.26}$$

and the obstacle function ψ satisfying (1.21) are given. Further, suppose that the compatibility conditions (1.16) with $g(\cdot,0) \in L^2(\Omega)$ is valid. In addition, assume that the approximation assumption stated in (1.22)-(1.24) hold. Finally, suppose that

$$\begin{aligned} |\operatorname{div}_x a(x,t,w)| &\le L\log(1+|w|)(1+|w|)^{p(x,t)-1}, \\ |D_w a(x,t,w)| &\le L(1+|w|)^{p(x,t)-2} \end{aligned} \tag{1.27}$$

are valid for all $(x,t) \in \Omega_T$ and $w \in \mathbb{R}^n$. Then, there exists a localizable solution $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ - in the sense of Definition 1.6 - to the obstacle problem (1.18) with $u(\cdot,0) = g(\cdot,0)$. Moreover, this solution satisfies the energy estimate

$$\sup_{t\in(0,T)} \int_\Omega |u(\cdot,t)|^2 \,\mathrm{d}x + \int_{\Omega_T} |Du|^{p(\cdot)} \,\mathrm{d}z \le cM, \tag{1.28}$$

with a constant c, which only depends on $(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$, where $M \ge 1$ is defined as follows

$$M := \int_{\Omega_T} \Psi^{p(\cdot)} + |\partial_t\psi|^{\gamma_1'} + |f|^{\gamma_1'} \,\mathrm{d}z + \|g(\cdot,0)\|^2_{L^2(\Omega)} + \|g\|^2_{L^\infty(0,T;L^2(\Omega))} + 1 \tag{1.29}$$

with $\Psi := 1 + |D\psi| + |F| + |Dg| + |g|$. The localizable solution u constructed above is unique and even more strongly, every solution to the weak formulation (1.18) of the obstacle problem coincides with u.

Moreover, we have a result that holds on any, maybe highly irregular domains $\Omega \subset \mathbb{R}^n$ and for general obstacle functions $\psi \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ with $\partial_t\psi \in L^{\gamma_1'}(\Omega_T)$. Since in this general situation, we can approximate ψ only locally by functions with better regularity and integrability properties, we can show strong convergence to a solution only on every compactly contained subdomain $\Omega' \Subset \Omega$. Consequently, the limit map solves the variational inequality only on such subsets and the question of uniqueness remains open.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4)-(1.5). Then, assume that $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies the growth and monotonicity condition (1.2) and (1.3) and additional (1.27). Moreover, suppose for the inhomogeneity F and f that (1.25) is valid. Furthermore, assume that the obstacle function ψ and boundary data g satisfy (1.21) and (1.26). Finally, the initial values $g(\cdot,0) \in L^2(\Omega)$ satisfy the compatibility conditions (1.16). Then, there exists a map $u \in L^\infty(0,T,L^2(\Omega)) \cap W^{p(\cdot)}_g(\Omega_T)$ with $u \ge \psi$ a.e. on Ω_T, that solves the obstacle problem (1.18) in the following sense. For every Lipschitz regular domain $\Omega' \Subset \Omega$, there holds $u \in C^0([0,T]; L^2(\Omega'))$ and*

$u|_{\Omega'_T}$ is a localizable solution - in the sense of Definition 1.6 - to the obstacle problem (1.18) with the initial values $u(\cdot,0)=g(\cdot,0)$. Moreover, it satisfies the energy estimate (1.28).

Plan of the paper. Finally, we briefly describe the strategy of the proof to our main result and the technical novelties of the paper. We start with some useful preliminary results before we are able to show the existence results of Theorem 1.7 and Theorem 1.8. After we have shown the needed technical tools, we will refer in Section 3 the existence of solutions to the parabolic equation (1.1) under certain boundary and initial data conditions from [26], see also [25]. First of all, we refer the existence of a weak solution to the Dirichlet problem of (1.1). In [26] we dealed similar to Antontsev and Shmarev, with the Galerkin approximation. Then, we will give the existence of a weak solution to the Cauchy-Dirichlet problem of (1.1) with general boundary data. In Section 4, we will establish existence results to the strong formulation of the variational inequality (1.17) with regular obstacles via penalization. Moreover, we will expand this result to irregular obstacles by the theory of localizable solutions, see Section 5. Here, we will gain the existence and uniqueness result of Theorem 1.7. Finally in Section 6, we will proof the existence of localizable solutions to the parabolic obstacle problem (1.18) on arbitrary domains with general obstacle functions of Theorem 1.8.

Remark 1.9. Here, we want to mention that additional assumption (1.27) is not necessary if the obstacle ψ satisfies

$$\partial_t\psi - \operatorname{div} a(z, D\psi) \in L^{\gamma_1'}(\Omega_T).$$

2. Preliminaries and notations

Moreover, since weak solutions u of parabolic equations possess only weak regularity properties with respect to the time variable t, i.e. they are not assumed to be weakly differentiable, in principle it is not possible to use the solution u itself (also disregarding boundary values) as a test-function in the weak formulation of the parabolic equation. In order to be nevertheless able to test the equation properly, we smooth the solution u with respect to the time direction t using the so-called **Steklov averages**. Hence, we introduce the following: The Steklov averages of a function $f\in L^1(\Omega_T)$ are defined as

$$[f]_h(x,t) := \begin{cases} \frac{1}{h}\displaystyle\int_t^{t+h} f(x,s)\,ds & \text{for } t\in(0,T-h),\\ 0 & \text{for } t\in[T-h,T), \end{cases} \tag{2.1}$$

for $x\in\Omega$, for all $t\in(0,T)$ and $0<h<T$. Assuming that $u\in C^0([0,T];L^2(\Omega))\cap W^{p(\cdot)}(\Omega_T)$ is a weak solution to the parabolic equation (1.1) the Steklov average $[u]_h$ satisfies the corresponding equation

$$\int_{\Omega\times\{t\}} \partial_t([u]_h)\cdot\varphi + [a(\cdot,Du)]_h\cdot D\varphi\,\mathrm{d}x = \int_{\Omega\times\{t\}} [f]_h\cdot\varphi + [F^{p(\cdot)-2}F]_h\cdot D\varphi\,\mathrm{d}x \tag{2.2}$$

for a.e. $t\in(0,T)$ and all $\varphi\in C_0^\infty(\Omega)$.

2.1. Compact embedding - Compactness Theorem.

Since $L^2(\Omega)$ is a Hilbert space which is identified with its dual

$$L^2(\Omega)\cong(L^2(\Omega))'$$

and in which $L^{p(\cdot,t)}(\Omega)$ is dense and continuously embedded $\forall t\in[0,T]$, where $p(\cdot,t)>2n/(n+2)$, see [23, Lemma 5.5], we have

$$L^{p(\cdot,t)}(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow L^{p'(\cdot,t)}(\Omega)$$

for all $t\in[0,T]$. The fact that $L^2(\Omega)\cong(L^2(\Omega))'$ can be demonstrated by the Riesz representation theorem. We denote the dual of $W_0^{1,p(\cdot,t)}(\Omega)$ by $W^{1,p(\cdot,t)}(\Omega)'$ and the natural pairing between $W^{1,p(\cdot,t)}(\Omega)'$ and $W_0^{1,p(\cdot,t)}(\Omega)$ by $\langle\cdot,\cdot\rangle$. Moreover, we have the embeddings $W_0^{1,p(\cdot,t)}(\Omega) \subset L^2(\Omega)$ and $(L^2(\Omega))' \subset W^{1,p(\cdot,t)}(\Omega)'$. Therefore, we can conclude

$$W_0^{1,p(\cdot,t)}(\Omega) \hookrightarrow L^2(\Omega) \cong (L^2(\Omega))' \hookrightarrow W^{1,p(\cdot,t)}(\Omega)',$$

where the injections are compact. This also allows us to identify the duality product $\langle \cdot, \cdot \rangle$ with the inner product between $L^2(\Omega)$ and $W_0^{1,p(\cdot,t)}(\Omega)$, i.e.

$$f(v) = \langle f, v \rangle = \langle f, v \rangle_{L^2(\Omega)} = \int_\Omega f \cdot v \, \mathrm{d}x \tag{2.3}$$

whenever $f \in L^2(\Omega) \subset W^{1,p(\cdot,t)}(\Omega)'$ and $v \in W_0^{1,p(\cdot,t)}(\Omega)$ and $t \in [0,T]$. Next, we consider the Banach space

$$W_0(\Omega_T) := \left\{ w \in W_0^{p(\cdot)}(\Omega_T) | w_t \in W^{p(\cdot)}(\Omega_T)' \right\}.$$

Now, from [25, 26] we could refer the following result.

Lemma 2.1. *Let $n \geq 2$. Assume that the exponent function $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4)-(1.5). Then $W(\Omega_T)$ is contained in $C^0([0,T]; L^2(\Omega))$. Moreover, if $u \in W_0(\Omega_T)$ then $t \mapsto \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is absolutely continuous on $[0,T]$,*

$$\frac{\mathrm{d}}{\mathrm{d}z} \int_\Omega |u(\cdot,t)|^2 \, \mathrm{d}x = 2 \langle \partial_t u(\cdot,t), u(\cdot,t) \rangle, \text{ for a.e. } t \in [0,T],$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{1,p(\cdot,t)}(\Omega)'$ and $W_0^{1,p(\cdot,t)}(\Omega)$. Moreover, there is a constant c such that

$$\|u\|_{C^0([0,T];L^2(\Omega))} \leq c \|u\|_{W(\Omega_T)}$$

for every $u \in W_0(\Omega_T)$.

Now, we are in the situation to refer the compactness theorem in the sense of Aubin and Lions, see [25, 26].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ an open, bounded Lipschitz domain with $n \geq 2$ and $p(\cdot) > \frac{2n}{n+2}$ satisfying (1.4)-(1.5). Furthermore, define $\hat{p}(\cdot) := \max\{2, p(\cdot)\}$. Then, the inclusion $W(\Omega_T) \hookrightarrow L^{\hat{p}(\cdot)}(\Omega_T)$ is compact.*

2.2. **Technical tools.** First of all, we recall that under the additional regularity assumption $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$, both formulations (1.17) and (1.18) are equivalent. This result reads as follows.

Corollary 2.3 ([26], Corollary 3.8)**.** *A function $u \in \mathcal{K}'_{\psi,g}(\Omega_T)$ satisfies the strong formulation (1.17) of the obstacle problem if and only if it satisfies the weak formulation (1.18).*

Our next problem is, that we need a Poincaré inequality, but in the parabolic case, there does not exist such a global estimate. It is only possible to use the elliptic Poincaré inequality slicewise for a.e. times t. For parabolic problems with nonstandard growth, it is not allowed to apply such an estimate, not even slicewise. There exists just a "Luxemburg-version", see [7], i.e. $\|u\|_{L^{p(x)}(\Omega)} \leq c\|Du\|_{L^{p(x)}(\Omega)}$ for all $u \in W_0^{1,p(x)}(\Omega)$, where $c > 0$. But we need a "classical" Poincaré type inequality. The desired result is given by the following lemma, which is stated in [26].

Lemma 2.4 ([26], Lemma 3.9)**.** *Let $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain and $\gamma_2 := \sup_{\Omega_T} p(\cdot)$. Assume that $u \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ and the exponent $p(\cdot)$ satisfies the conditions (1.4)-(1.5). Then, there exists a constant $c = c(n, \gamma_1, \gamma_2, \mathrm{diam}(\Omega), \omega(\cdot))$, such that the following two versions of the* ***Poincaré type estimate*** *are valid:*

$$\int_{\Omega_T} |u|^{p(\cdot)} \, \mathrm{d}z \leq \; c \left(\|u\|^{\frac{4\gamma_2}{n+2}}_{L^\infty(0,T;L^2(\Omega))} + 1 \right) \left(\int_{\Omega_T} |Du|^{p(\cdot)} + \; 1 \, \mathrm{d}z \right) \tag{2.4}$$

and

$$\|u\|^{\gamma_1}_{L^{p(z)}(\Omega_T)} \leq \; c \left(\|u\|^{\frac{4\gamma_2}{n+2}}_{L^\infty(0,T;L^2(\Omega))} + 1 \right) \left(\int_{\Omega_T} |Du|^{p(\cdot)} + \; 1 \, \mathrm{d}z \right). \tag{2.5}$$

One main aim of this paper is to show an existence result to degenerate parabolic obstacle problems on irregular domains via localizable solutions, see Section 5. For the proof of the extension property of a map u, which is mentioned in Definition 1.6, we will need a more general existence result with more general boundary data. Hence, we need an other Poincaré type estimate. But this Poincaré type estimate can be only stated in a local version on a cylinder $Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, \rho^2) \subset \mathbb{R}^{n+1}$ with a small radius $\rho \in (0,1)$, see the following Lemma.

Lemma 2.5 ([26], Lemma 3.11). *Assume that* $\frac{2n}{n+2} < \gamma_1 \le \gamma_2 < \infty$ *and* $\omega : [0,\infty) \to [0,1]$ *satisfies (1.5). Then, there exists* $\theta_0 = \theta_0(n,\gamma_1) \in (0,1)$*, such that for any* $\theta \in (0,\theta_0]$ *the following holds: There exists* $\rho_0 = \rho_0(\theta,\omega(\cdot)) \in (0,1]$*, such that for any cylinders* $Q_\rho(z_0) \subset \mathbb{R}^{n+1}$ *with radius* $\rho \in (0,\rho_0]$, $p : Q_\rho(z_0) \to [\gamma_1,\gamma_2]$ *satisfies (1.4) and* $v \in C^0([t_0-\rho^2,t_0]; L^2(B_\rho(x_0))) \cap W_0^{p(\cdot)}(Q_\rho(z_0))$ *the following* ***Poincaré type estimate*** *holds:*

$$\int_{Q_\rho(z_0)} |v|^{p(\cdot)} \,\mathrm{d}z \le c \left(\sup_{t_0-\rho^2 \le t \le t_0} \|v(\cdot,t)\|^\theta_{L^2(B_\rho(x_0))} + 1 \right) \left(\int_{Q_\rho(z_0)} |Dv|^{p(\cdot)} + 1 \,\mathrm{d}z \right) \tag{2.6}$$

with a constant $c = c(n,\gamma_1,\gamma_2,L_1)$.

Remark 2.6. Under the assumption of Lemma 2.5, we infer from (2.6) by using (1.9) the Poincaré type estimate

$$\|v\|^{\gamma_1}_{L^{p(z)}(Q_\rho(z_0))} \le c \left(\sup_{t_0-\rho^2 \le t \le t_0} \|v(\cdot,t)\|^\theta_{L^2(B_\rho(x_0))} + 1 \right) \left(\int_{Q_\rho(z_0)} |Dv|^{p(\cdot)} \,\mathrm{d}z + 1 \right), \tag{2.7}$$

holds for every radius $\rho \le \rho_0$ and any $0 < \theta \le \theta_0$ with a constant $c = c(n,\gamma_1,\gamma_2,L_1)$.

Comparison principle and comparison estimate. In this section we refer a comparison principle, which will be a key tool for constructing comparison maps that almost everywhere satisfy the obstacle constraint $v \ge \psi$.

Lemma 2.7 ([26], Lemma 3.15). *Let* $\Omega \subset \mathbb{R}^n$ *with* $n \ge 2$ *and* $p : \Omega_T \to [\gamma_1,\gamma_2]$ *satisfies (1.4)-(1.5). Moreover, suppose that* $\psi, v \in W(\Omega_T)$ *satisfy in the weak sense*

$$\begin{cases} \partial_t \psi - \operatorname{div}\ a(z,D\psi) & \le \partial_t v - \operatorname{div}\ a(z,Dv) \text{ in } \Omega_T, \\ \psi & \le v \text{ on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{2.8}$$

where (1.3) are in force. Then, there holds $\psi \le v$ *a.e. on* Ω_T.

Moreover, we refer the following comparison estimate, which will be used to transfer estimates from homogeneous equation to weak solutions to obstacle problem. The following lemma will provide the comparison between an obstacle problem and a suitable parabolic equation stated and is established in [26].

Lemma 2.8 ([26], Lemma 3.16). *Let* $\rho \in (0,1]$. *Assume that the assumptions (1.2)-(1.3) with exponents (1.4)-(1.5) are in force. Moreover, suppose that the obstacle function* ψ *and the inhomogeneities* F, f *satisfy (1.21) and (1.25). Further, suppose that* $v \in W(Q_\rho(z_0))$ *solves the parabolic equation*

$$\partial_t v - \operatorname{div} a(z,Dv) = \partial_t \psi - \operatorname{div} a(z,D\psi) \text{ in } Q_\rho(z_0) \tag{2.9}$$

and that $u \in \mathcal{K}_{\psi,v}(Q_\rho(z_0))$ *is a solution to the variational inequality*

$$\begin{aligned} \langle\!\langle \partial_t w, w-u \rangle\!\rangle_{Q_\rho(z_0)} &+ \int_{Q_\rho(z_0)} a(z,Du)\cdot D(w-u)\,\mathrm{d}z \\ &+ \frac{1}{2}\|(w-u)(\cdot,t_1)\|^2_{L^2(B_\rho(x_0))} \\ &\ge \int_{Q_\rho(z_0)} |F|^{p(\cdot)-2}F\cdot D(w-u) + f\cdot(w-u)\,\mathrm{d}z \end{aligned} \tag{2.10}$$

with $t_1 = t_0 - \rho^2$ *for all comparison functions* $w \in \mathcal{K}'_{\psi,v}(Q_\rho(z_0))$. *Then, for any* $\tilde{\kappa} \in (0,1)$, *there exists a constant* $c_{\tilde{\kappa}} = c(\tilde{\kappa}, n, \gamma_1, \gamma_2, \nu, L) \geq 1$, *such that the* ***comparison estimate***

$$\begin{aligned} \int_{Q_\rho(z_0)} |D(u-v)|^{p(\cdot)}\, dz &\leq \tilde{\kappa} \int_{Q_\rho(z_0)} (\mu + |Du|)^{p(\cdot)}\, dz \\ &\quad + c_{\tilde{\kappa}} \int_{Q_\rho(z_0)} |D\psi|^{p(\cdot)} + |F|^{p(\cdot)} + |\partial_t \psi|^{\gamma_1'} + |f|^{\gamma_1'} + 1\, dz \end{aligned} \tag{2.11}$$

holds. Moreover, for every $p(\cdot) > \frac{2n}{n+2}$, *we have the energy estimate*

$$\begin{aligned} \int_{Q_\rho(z_0)} |Dv|^{p(\cdot)}\, dz &\leq c \int_{Q_\rho(z_0)} (\mu + |Du|)^{p(\cdot)}\, dz \\ &\quad + c \int_{Q_\rho(z_0)} |D\psi|^{p(\cdot)} + |F|^{p(\cdot)} + |\partial_t \psi|^{\gamma_1'} + |f|^{\gamma_1'} + 1\, dz \end{aligned} \tag{2.12}$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L) \geq 1$.

Minty type lemma. The next lemma is a slightly modified version of Browder-Minty's Lemma, which employs a certain monotonicity condition to justify the passage to weak limits. An elliptic version of this Lemma can be found in [34] and some Minty type Lemma for parabolic problems with p-growth is established in [17]. Our Minty type Lemma reads as follows.

Lemma 2.9 ([26], Lemma 2.13)**.** *Suppose that* $p(\cdot) > \frac{2n}{n+2}$ *satisfies (1.4)-(1.5) and*

$$\mathcal{C} \subset \left\{ v \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) : v_t \in W^{p(\cdot)}(\Omega_T)' \right\}$$

is closed and convex. Moreover, let $A : \mathcal{C} \to W^{p(\cdot)}(\Omega_T)'$ *be a monotone operator which is continuous on a finite dimensional subspaces of* $\mathcal{C}$. *Here, monotonicity has to be understood in the sense that*

$$\langle\langle Av - A\tilde{v}, v - \tilde{v} \rangle\rangle_{\Omega_T} \geq 0 \ \forall\ v,\ \tilde{v} \in \mathcal{C}.$$

Finally, let $B : W^{p(\cdot)}(\Omega_T) \to \mathbb{R}$ *be a continuous linear operator. Then, for* $u \in \mathcal{C}$

$$\langle\langle \partial_t u + Au, v - u \rangle\rangle_{\Omega_T} \geq B(v-u) \ \forall\ v \in \mathcal{C} \tag{2.13}$$

holds if and only if

$$\langle\langle \partial_t v + Av, v - u \rangle\rangle_{\Omega_T} + \frac{1}{2} \|(v-u)(\cdot,0)\|^2_{L^2(\Omega)} \geq B(v-u) \ \forall\ v \in \mathcal{C}. \tag{2.14}$$

3. Existence results to degenerate parabolic equations with nonstandard growth

In this section, we will refer from [26] (see also [25]) some existence results to degenerate parabolic equations. These results we will use to obtain our existence theorems. For the proofs of Lemma 3.1 and Lemma 3.2 and of the local versions of Corollary 3.3 and Corollary 3.4 we refer to [26]. The starting point is to consider the initial data problem of (1.1)

$$\begin{cases} \partial_t u - \operatorname{div} a(z, Du) = & f - \operatorname{div}(|F|^{p(\cdot)-2} F) \text{ in } \Omega_T, \\ u = & 0 \text{ on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = & g(\cdot, 0) \text{ on } \Omega \times \{0\}. \end{cases} \tag{3.1}$$

The approach to prove the existence to the Dirichlet problem is to construct a solution, which solve the problem (3.1). In [26] we start by constructing a sequence of the Galerkin's approximations, where the limit of this sequence is equal to the solution in (3.1). Then, we had shown that this approximate solution converges to a general solution, where we used some energy bounds, which derive by the proof and finally, the compact embedding of Theorem 2.2 yield the desired convergence of the approximate solutions to general solutions. The results reads as follows.

Lemma 3.1 ([26], Lemma 4.1)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4)-(1.5). Then, suppose that the vector-field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies, for a given function $v \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t v \in L^{\gamma_1'}(0,T; W^{-1,\gamma_1'}(\Omega))$, the growth condition*

$$|a(z,w)| \le c(\gamma_2, L)\left((1+|w|)^{p(\cdot)-1} + |v|^{p(\cdot)-1}\right) \tag{3.2}$$

and the monotonicity property

$$\nu(\mu^2 + |w+v|^2 + |w_0+v|^2)^{\frac{p(\cdot)-2}{2}}|w-w_0|^2 \le (a(z,w) - a(z,w_0)) \cdot (w - w_0) \tag{3.3}$$

for all $z \in \Omega_T$ and $w, w_0 \in \mathbb{R}^n$. Moreover, let (1.10) and $g(\cdot,0) \in L^2(\Omega)$ hold. Then, there exists a weak solution $u \in W_0(\Omega_T)$ of the parabolic boundary problem (3.1) and this solution satisfies the following energy estimate

$$\sup_{0\le t\le T}\int_\Omega |u(\cdot,t)|^2 \,\mathrm{d}x + \int_{\Omega_T} |Du|^{p(\cdot)} \,\mathrm{d}z \le c\left(\int_{\Omega_T} 1 + |F|^{p(\cdot)} + |v|^{p(\cdot)} \,\mathrm{d}z \right.$$
$$\left. + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(0,T;W^{-1,\gamma_1'}(\Omega))} + \|g(\cdot,0)\|^2_{L^2(\Omega)} + 1\right)$$

with $u(\cdot,0) = g(\cdot,0)$ and a constant $c = c(n, \gamma_1, \gamma_2, \operatorname{diam}(\Omega))$.

Moreover, the existence of solutions to initial value problem (3.1) can be extend to general boundary problems. Therefore, we consider the **Cauchy-Dirichlet problem** of the parabolic problem (1.1):

$$\begin{cases} \partial_t u - \operatorname{div}\ a(z, Du) &= f - \operatorname{div}\,(|F|^{p(\cdot)-2}F) \text{ in } \Omega_T \\ u &= g \text{ on } \partial\Omega \times (0,T) \\ u(\cdot,0) &= g(\cdot,0) \text{ on } \Omega \times \{0\}\,. \end{cases} \tag{3.4}$$

We used the result of Lemma 3.1 to the Cauchy-Dirichlet problem (3.4) to get existence of solutions to (1.1) with general boundary data. Therefore, we have to change the problem (3.4) into a problem comparing to (3.1). Then, we can conclude the existence of solution to the modified problem. Hence, we get the existence result to the primal Cauchy-Dirichlet problem (3.4). This result is stated in the following lemma.

Lemma 3.2 ([26], Lemma 4.3)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4)-(1.5). Then, suppose that the vector-field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies the growth condition (1.2) and the monotonicity condition (1.3). Moreover, let (1.10) fulfilled. Furthermore, the boundary data g satisfy (1.14). Then, there exists a weak solution $u \in C^0([0,T]; L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T)$ with $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$ of the parabolic Cauchy-Dirichlet problem (3.4) and this solution satisfies the following energy estimate*

$$\sup_{0\le t\le T}\int_\Omega |u(\cdot,t)|^2 \,\mathrm{d}x + \int_{\Omega_T} |Du|^{p(\cdot)} \,\mathrm{d}z \le c\left(\|g(\cdot,0)\|^2_{L^2(\Omega)} + \|g\|^2_{L^\infty-L^2} + \mathcal{M}_g\right),$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$ and $\mathcal{M}_g$ is defined as follows

$$\mathcal{M}_g := \int_{\Omega_T} 1 + |F|^{p(\cdot)} + |Dg|^{p(\cdot)} \,\mathrm{d}z + \|f\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'} + \|\partial_t g\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'} + 1.$$

Finally, we need local versions of Lemma 3.2, since we need more general data to prove the existence of localizable solutions. But these existence results we get only local on a cylinder $Q_\rho(z_0) = (t_0 - \rho^2, t_0) \times B_\rho(x_0) \subset \Omega_T$, with radius $\rho \le \rho_0(\theta, \omega(\cdot)) \in (0,1]$, where the maximal radius ρ_0 and $\theta \le \theta_0(n, \gamma_1) \in (0,1)$ are introduced in Lemma 2.5. Before, we are able to prove a local versions of Lemma 3.2, we have to prove a local versions of Lemma 3.1. Therefore, we begin by considering the Dirichlet problem for the parabolic equation (1.1):

$$\begin{cases} \partial_t u - \operatorname{div} a(z,Du) = & f - \operatorname{div}(|F|^{p(\cdot)-2}F) \text{ in } Q_\rho(z_0), \\ u = & 0 \text{ on } \partial B_\rho(x_0)\times(t_0-\rho^2,t_0), \\ u(\cdot,t_0-\rho^2) = & g(\cdot,0) \text{ on } B_\rho(x_0)\times\left\{t_0-\rho^2\right\}. \end{cases} \tag{3.5}$$

Corollary 3.3 ([26], Corollary 4.4)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open, bounded Lipschitz domain and $p:\Omega_T\to[\gamma_1,\gamma_2]$ satisfies (1.4)-(1.5). Then, suppose that the vector-field $a:\Omega_T\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathéodory function and satisfies, for a given function $v\in W(\Omega_T)$, the growth condition (3.2) and the (3.3) monotonicity property. Moreover, let $F\in L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)$, $f\in W^{p(\cdot)}(\Omega_T)'$ and $g(\cdot,0)\in L^2(\Omega)$ hold. Then, there exist $\theta_0=\theta_0(n,\gamma_1)\in(0,1)$ and a radius $\rho_0=\rho_0(\theta,\omega(\cdot))\in(0,1]$ with $\theta\le\theta_0$, such that the following holds: Whenever $0<\rho\le\rho_0$, there exists a weak solution $u\in W_0(Q_\rho(z_0))$ of the parabolic boundary problem (3.5) and this solution satisfies the following energy estimate*

$$\sup_{t\in(t_0-\rho^2,t_0)}\int_{B_\rho(x_0)}|u(\cdot,t)|^2\,\mathrm{d}x+\int_{Q_\rho(z_0)}|Du|^{p(\cdot)}\,\mathrm{d}z\le c\Bigg(\int_{Q_\rho(z_0)}1+|F|^{p(\cdot)}+|v|^{p(\cdot)}\,\mathrm{d}z \\ +\|f\|_{W^{p(\cdot)}(Q_\rho(z_0))'}^{\frac{\gamma_1+1}{\gamma_1-1}}+\|g(\cdot,0)\|_{L^2(B_\rho(x_0))}^2+1\Bigg)$$

with a constant $c=c(n,\gamma_1,\gamma_2,\nu,L,L_1)$.

Now, we give a local existence result with more general data similar to the Cauchy-Dirichlet problem 3.4. Therefore, we consider the local Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div}\ a(z,Du) &= f - \operatorname{div}\ (|F|^{p(\cdot)-2}F) \text{ in } Q_\rho(z_0), \\ u &= g \text{ on } \partial B_\rho(x_0)\times(t_0-\rho^2,t_0), \\ u(\cdot,t_0-\rho^2) &= g(\cdot,0) \text{ on } B_\rho(x_0)\times\left\{t_0-\rho^2\right\}. \end{cases} \tag{3.6}$$

Corollary 3.4 ([26], Corollary 4.5)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open, bounded Lipschitz domain and $Q_\rho(z_0)\subset\Omega_T$. Assume that $p:\Omega_T\to[\gamma_1,\gamma_2]$ satisfies (1.4)-(1.5). Then, suppose that the vector-field $a:\Omega_T\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathéodory function and satisfies (1.2)-(1.3). Moreover, assume that $F\in L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)$, $f\in W^{p(\cdot)}(\Omega_T)'$, $g\in W(\Omega_T)$ and $g(\cdot,0)\in L^2(\Omega)$ are in force. Then, there exist $\theta_0=\theta_0(n,\gamma_1)\in(0,1)$ and a radius $\rho_0=\rho_0(\theta,\omega(\cdot))\in(0,1]$ with $\theta\le\theta_0$, such that the following holds: Whenever $0<\rho\le\rho_0$, there exists a weak solution $u\in W(Q_\rho(z_0))$ of the local parabolic boundary problem (3.6). Moreover, this solution satisfies the following energy estimate*

$$\sup_{t\in(t_0-\rho^2,t_0)}\int_{B_\rho(x_0)}|u(\cdot,t)|^2\,\mathrm{d}x+\int_{Q_\rho(z_0)}|Du|^{p(\cdot)}\,\mathrm{d}z\le\ c\cdot\mathcal{M}_{local} \tag{3.7}$$

with a constant $c=c(n,\gamma_1,\gamma_2,\nu,L,L_1)$, where $\mathcal{M}_{local}$ is defined as follows

$$\mathcal{M}_{local}:=1+\int_{Q_\rho(z_0)}|F|^{p(\cdot)}+|Dg|^{p(\cdot)}\,\mathrm{d}z+\|f\|_{W^{p(\cdot)}(Q_\rho(z_0))'}^{\frac{\gamma_1+1}{\gamma_1-1}} \\ +\|g(\cdot,0)\|_{L^2(\Omega)}^2+\|g\|_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}^2+\|\partial_t g\|_{W^{p(\cdot)}(Q_\rho(z_0))'}^{\frac{\gamma_1+1}{\gamma_1-1}}.$$

4. Existence of strong solutions to degenerate problem with regular obstacles

The first step to the existence in degenerate problem with irregular obstacles and nonstandard growth, is to consider more regular data. In this situation, we can deduce from the existence results of the previous section the following lemma. This lemma will play an important role for the proof of the existence of localizable solutions to nonlinear problems with irregular obstacles.

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^n$ an open, bounded Lipschitz domain and $p:\Omega_T\to[\gamma_1,\gamma_2]$ satisfies (1.4)-(1.5). Suppose that the vector-field $a:\Omega_T\times\mathbb{R}^n\to\mathbb{R}^n$ is a*

Carathéodory function and satisfies (1.2) and (1.3). Moreover, assume that the obstacle function ψ, the inhomogeneities F, f and the boundary data g satisfy (1.21), (1.25), (1.26) and $g(\cdot,0) \in L^2(\Omega)$. Furthermore, suppose that the additional regularity assumptions

$$\partial_t \psi - \operatorname{div} a(z, D\psi) \in L^{\gamma_1'}(\Omega_T), \ \operatorname{div}(|F|^{p(\cdot)-2}F) \in L^{\gamma_1'}(\Omega_T) \tag{4.1}$$

are in force and that the compatibility condition (1.16) is valid. Then, there exists a solution $u \in \mathcal{K}'_{\psi,g}(\Omega_T)$ of the strong formulation of variational inequality (1.17). Moreover, there exists a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$, such that the energy estimate

$$\sup_{t\in(0,T)} \int_\Omega |u(\cdot,t)|^2 \, dx + \int_{\Omega_T} |Du|^{p(\cdot)} \, dz \le c\mathcal{E} \tag{4.2}$$

holds with

$$\begin{aligned}\mathcal{E} \equiv & \|g(\cdot,0)\|^2_{L^2(\Omega)} + \|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t\psi - \operatorname{div} a(z,D\psi)\|^{\gamma_1'}_{L^{\gamma_1'}(\Omega_T)} \\ & + \|f - \operatorname{div}(|F|^{p(\cdot)-2}F)\|^{\gamma_1'}_{L^{\gamma_1'}(\Omega_T)} + \int_{\Omega_T} |Dg|^{p(\cdot)} \, dz + \|\partial_t g\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'}.\end{aligned}$$

Proof. The proof is divided into several steps. We begin with

Step 1: Regularization. We will revert the existence of a solution to the obstacle problem to the existence of solutions to certain penalized parabolic equations by certain approximation scheme. To construct such a penalization, we define $\zeta_\delta \in W^{1,\infty}(\mathbb{R})$, such that $\zeta_\delta(t) := 0$ if $t \in (-\infty, -\delta]$, $\zeta_\delta(t) := 1 + \frac{t}{\delta}$ if $t \in (-\delta, 0)$ and $\zeta_\delta(t) := 1$ if $t \in [0,\infty)$, for $\delta \in (0,1]$. Then, by $u_\delta \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}_g(\Omega_T)$ and $\partial_t u_\delta \in W^{p(\cdot)}(\Omega_T)'$ we denote the solution to the following Cauchy-Dirichlet problem:

$$\begin{cases} \partial_t u_\delta - \operatorname{div} a(z, Du_\delta) & = \zeta_\delta(\psi - u_\delta)\tilde{\Psi}_+ - \operatorname{div}(|F|^{p(\cdot)-2}F) + f \text{ in } \Omega_T \\ u_\delta & = g \text{ on } \partial\Omega \times (0,T) \\ u_\delta(\cdot,0) & = g(\cdot,0) \text{ on } \Omega \times \{0\}, \end{cases} \tag{4.3}$$

where $\tilde{\Psi} := \partial_t\psi - \operatorname{div} a(z,D\psi) + \operatorname{div}(|F|^{p(\cdot)-2}F) - f$. The existence of u_δ follows from Lemma 3.2. Notice that, we write $k_+ := \max\{k, 0\}$ and $k_- := \max\{-k, 0\}$.

Step 2: Obstacle constraint. Our first aim is to show that $u_\delta \ge \psi$ on Ω_T for any $\delta \in (0,1]$. We start by rewriting the weak formulation of the Cauchy-Dirichlet problem (4.3) in its Steklov-form. Then, for a.e. $\tau \in (0,T)$ we have

$$\begin{aligned}\int_\Omega \left(\partial_t[u_\delta]_h \cdot \varphi + [a(z,Du_\delta)]_h \cdot D\varphi\right)(\cdot,\tau) \, dx = & \int_\Omega \varphi \left(\left[\zeta_\delta(\psi - u_\delta)\tilde{\Psi}_+\right]_h \right. \\ & \left. - \left[\operatorname{div}\left(|F|^{p(\cdot)-2}F\right)\right]_h + [f]_h \right)(\cdot,\tau) \, dx\end{aligned} \tag{4.4}$$

for all test functions $\varphi \in W^{1,p(\cdot,\tau)}_0(\Omega)$. First, we add on both sides the term

$$-\int_\Omega \left(\partial_t[\psi]_h \varphi + [a(z,D\psi)]_h \cdot D\varphi\right)(\cdot,\tau) \, dx$$

and then, multiply the resulting equation by -1. Next, we integrate by parts and utilize the fact that $\operatorname{div}([a(z,D\psi)]_h) = [\operatorname{div} a(z,D\psi)]_h$. Hence, we have

$$\begin{aligned}\int_\Omega & \left(\partial_t[\psi - u_\delta]_h \cdot \varphi + ([a(z,D\psi)]_h - [a(z,Du_\delta)]_h) \cdot D\varphi\right)(\cdot,\tau) \, dx \\ & = \int_\Omega \varphi \left(\left[\tilde{\Psi}\right]_h - \left[\zeta_\delta(\psi - u_\delta)\tilde{\Psi}_+\right]_h \right)(\cdot,\tau) \, dx.\end{aligned} \tag{4.5}$$

Now, we choose as admissible test function $\varphi = ([\psi - u_\delta]_h)_+$. We note that this particular choice is allowed, since $u_\delta = g \ge \psi$ on $\partial\Omega \times (0,T)$ in the sense of traces. For $t \in (0,T)$ we integrate both sides with respect to τ over $(0,t)$. Here, denotes Ω_t the cylinder $\Omega \times (0,t)$. Therefore, it yields for the first term on the left-hand side of (4.5) that

$$\int_{\Omega_t} \partial_t[\psi - u_\delta]_h \cdot \varphi \, dz = \frac{1}{2}\int_\Omega |([\psi - u_\delta]_h)_+|^2(\cdot,t) - |([\psi - u_\delta]_h)_+|^2(\cdot,0) \, dx.$$

Combining this with (4.5) and then, using the Steklov average property as $h \downarrow 0$. Together with $u_\delta(\cdot,0) = g(\cdot,0) \geq \psi(\cdot,0)$, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |(\psi-u_\delta)_+|^2(\cdot,t)\ \mathrm{d}x + \int_{\Omega_t} (a(z,D\psi)-a(z,Du_\delta))\cdot D(\psi-u_\delta)_+\ \mathrm{d}z \\ = \int_{\Omega_t} \left(\tilde\Psi - \zeta_\delta(\psi-u_\delta)\tilde\Psi_+\right)(\psi-u_\delta)_+\ \mathrm{d}z \\ = \int_{\Omega_t}\left(\tilde\Psi-\tilde\Psi_+\right)(\psi-u_\delta)_+\ \mathrm{d}z.\end{aligned}$$

Here, we have used that $(\psi-u_\delta)_+(z) > 0$ for some $z\in\Omega_t$ implies $\zeta_\delta(\psi-u_\delta)=1$. Moreover, $\tilde\Psi-\tilde\Psi_+ = -\tilde\Psi_-$ is valid, since $\tilde\Psi-\tilde\Psi_+ = \tilde\Psi - \max\left\{\tilde\Psi,0\right\} = \min\left\{\tilde\Psi,0\right\} = -\max\left\{-\tilde\Psi,0\right\} = -\tilde\Psi_-$. Thus, it gains

$$\begin{aligned}\frac{1}{2}\int_\Omega |(\psi-u_\delta)_+|^2(\cdot,t)\ \mathrm{d}x + \int_{\Omega_t} (a(z,D\psi)-a(z,Du_\delta))\cdot D(\psi-u_\delta)_+\ \mathrm{d}z \\ = -\int_{\Omega_t}\tilde\Psi_-(\psi-u_\delta)_+\ \mathrm{d}z \leq 0.\end{aligned}$$

Since, the second integral on the left side is non-negative, that is obvious by (1.3), we can conclude

$$\int_\Omega |(\psi-u_\delta)_+|^2(\cdot,t)\ \mathrm{d}x = 0 \text{ for a.e. } t\in(0,T),$$

which implies that

$$u_\delta \geq \psi \text{ on } \Omega_T. \tag{4.6}$$

Step 3: Energy bounds, weak and strong convergence. The next step is devoted to the derivation of uniform bounds with respect to δ for u_δ in $W^{p(\cdot)}(\Omega_T)$ and $L^\infty(0,T;L^2(\Omega))$ and for $\partial_t u_\delta$ in $W^{p(\cdot)}(\Omega_T)'$. As before, we start with the Steklov-formulation (4.4). But now, we add on both sides

$$-\int_\Omega \partial_t[g]_h(\cdot,\tau)\varphi\ \mathrm{d}x$$

to infer that

$$\begin{aligned}\int_\Omega \left(\partial_t[u_\delta-g]_h\cdot\varphi + [a(z,Du_\delta)]_h\cdot D\varphi\right)(\cdot,\tau)\ \mathrm{d}x \\ = \int_\Omega \varphi\left(\left[\zeta_\delta(\psi-u_\delta)\tilde\Psi_+\right]_h - \partial_t[g]_h - \left[\operatorname{div}\left(|F|^{p(\cdot)-2}F\right)\right]_h + [f]_h\right)(\cdot,\tau)\ \mathrm{d}x\end{aligned}$$

holds for all $\varphi\in W_0^{1,p(\cdot,\tau)}(\Omega)$ and for a.e. $\tau\in(0,T)$. In this equation we choose the admissible test function $\varphi = [u_\delta - g]_h$ and recall again that $u_\delta = g$ on $\partial\Omega\times(0,T)$. Next, we integrate with respect to τ over $(0,t)$, where $t\in(0,T)$. Hence, it follows for the first term on the left-hand side that

$$\begin{aligned}\int_{\Omega_t}\partial_t[u_\delta-g]_h\cdot\varphi\ \mathrm{d}z =& \frac{1}{2}\int_{\Omega_t}\partial_t[u_\delta-g]_h^2\ \mathrm{d}z \\ =& \frac{1}{2}\int_\Omega [u_\delta-g]_h^2(\cdot,t)\ \mathrm{d}x - \frac{1}{2}\int_\Omega [u_\delta-g]_h^2(\cdot,0)\ \mathrm{d}x.\end{aligned}$$

At this stage, we want to pass the limit $h\downarrow 0$. The only term which causes some problems is the one involving $\partial_t[g]_h$. Here, we have to apply (2.3) for $f\in L^2(\Omega)\subset W^{-1,\gamma_1'}(\Omega)$, $v\in W_0^{1,\gamma_1}(\Omega)$. This implies, in the case $p(\cdot)\equiv$const., that

$$\int_{\Omega_T} f\cdot v\ \mathrm{d}z = \int_0^T \langle f(s),v\rangle_{L^2(\Omega)}\,ds = \langle\langle f,v\rangle\rangle_{\Omega_T}, \tag{4.7}$$

for $f\in L^2(\Omega_T)$, $v\in L^{\gamma_1}(0,T;W_0^{1,\gamma_1}(\Omega))$. Therefore, we can obtain

$$\lim_{h\downarrow 0}\int_{\Omega_t}\partial_t[g]_h\cdot\varphi\ \mathrm{d}z = \langle\langle g_t,\varphi\rangle\rangle_{\Omega_t},$$

since $g_t \in L^{\gamma_1'}(0,T;W^{-1,\gamma_1'}(\Omega))$. Keeping in mind that the initial value of u_δ satisfies $u_\delta(\cdot,0)=g(\cdot,0)$ on Ω, this leads us to

$$\begin{aligned}\frac{1}{2}\int_\Omega |u_\delta-g|^2(\cdot,t)\,\mathrm{d}x+&\int_{\Omega_t} a(z,Du_\delta)\cdot Du_\delta\,\mathrm{d}z\\ &=\int_{\Omega_t}\left(\zeta_\delta(\psi-u_\delta)\tilde{\Psi}_+-\operatorname{div}\left(|F|^{p(\cdot)-2}F\right)+f\right)(u_\delta-g)\,\mathrm{d}z\\ &+\int_{\Omega_t} a(z,Du_\delta)\cdot Dg\,\mathrm{d}z-\langle\!\langle g_t,u_\delta-g\rangle\!\rangle_{\Omega_t}.\end{aligned}$$

Using Young's inequality with an arbitrary $\varepsilon>0$ and (1.13). Then, utilizing the standard Poincaré inequality slicewise, the growth assumption (1.2) on $a(\cdot)$, Young's inequality once more and (1.9), we get for any $\varepsilon>0$ that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_\delta-g|^2(\cdot,t)\,\mathrm{d}x+\int_{\Omega_t} a(z,Du_\delta)\cdot Du_\delta\,\mathrm{d}z\\ &\le c_\varepsilon\|\partial_t g\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'}+\varepsilon c\left(\int_{\Omega_t}|Du_\delta|^{p(\cdot)}+|D(u_\delta-g)|^{\gamma_1}+1\,\mathrm{d}z+1\right)\\ &+c_\varepsilon\left(\int_{\Omega_t}|\tilde{\Psi}|^{\gamma_1'}+|f-\operatorname{div}(|F|^{p(\cdot)-2}F)|^{\gamma_1'}+|Dg|^{p(\cdot)}\,\mathrm{d}z\right)\end{aligned}$$

with constants $c=c(n,\gamma_1,\gamma_2,L,\operatorname{diam}(\Omega))$ and $c_\varepsilon=c_\varepsilon(\frac{1}{\varepsilon},\gamma_1,\gamma_2)$. Then, we use the coercivity property (1.7) to bound the second integral on the left-hand side from below. Proceeding this way and recalling the definition of Ψ we find for $\varepsilon>0$ and a.e. $t\in(0,T)$ that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_\delta(\cdot,t)|^2\,\mathrm{d}x+\frac{\nu}{c(\gamma_1,\gamma_2)}\int_{\Omega_t}|Du_\delta|^{p(\cdot)}\;-c\,\mathrm{d}z\\ &\le\|g\|^2_{L^\infty(0,t;L^2(\Omega))}+\varepsilon c\left(\int_{\Omega_t}|Du_\delta|^{p(\cdot)}+|Dg|^{p(\cdot)}+1\,\mathrm{d}z+1\right)\\ &+c_\varepsilon\left(\int_{\Omega_t}|\partial_t\psi-\operatorname{div}a(z,D\psi)|^{\gamma_1'}+|f-\operatorname{div}(|F|^{p(\cdot)-2}F)|^{\gamma_1'}+|Dg|^{p(\cdot)}\mathrm{d}z+\|\partial_t g\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'}\right)\end{aligned}$$

with constants $c=c(n,\gamma_1,\gamma_2,L,\operatorname{diam}(\Omega))$ and $c_\varepsilon=c_\varepsilon(\frac{1}{\varepsilon},\gamma_1,\gamma_2)$. Choosing ε small enough we can reabsorb $\int_{\Omega_t}|Du_\delta|^{p(\cdot)}\,\mathrm{d}z$ on the left-hand side, e.g. $\varepsilon c\le\frac{\nu}{2c(\gamma_1,\gamma_2)}$. Then, taking the supremum over $t\in(0,T)$ in the first integral and $t=T$ in the second one, we finally arrive at

$$\sup_{t\in(0,T)}\int_\Omega|u_\delta(\cdot,t)|^2\,\mathrm{d}x+\int_{\Omega_T}|Du_\delta|^{p(\cdot)}\,\mathrm{d}z\le c\mathcal{E},\tag{4.8}$$

where $c=c(n,\gamma_1,\gamma_2,\nu,L,\operatorname{diam}(\Omega))$. By the Poincaré type inequality (2.5), (1.9) and (4.8), we also get the following uniform $L^{p(\cdot)}$-bound for u_δ

$$\|u_\delta\|_{L^{p(\cdot)}(\Omega_T)}\le\; c\mathcal{E}^{(\frac{4\gamma_2}{n+2}+1)\frac{1}{\gamma_1}},\tag{4.9}$$

with a constant $c=c(n,\gamma_1,\gamma_2,\nu,L,\operatorname{diam}(\Omega),\omega(\cdot))$.

Finally, we want to derive an uniform bound for $\partial_t u_\delta$ in $W^{p(\cdot)}(\Omega_T)'$. For this aim we consider $\varphi\in W_0^{p(\cdot)}(\Omega_T)$. From the weak formulation of (4.3) we get, by using the generalized Hölder's inequality (1.8) and (1.2), the following estimate:

$$\begin{aligned}|\,\langle\!\langle\partial_t u_\delta,\varphi\rangle\!\rangle_{\Omega_T}|&\le\int_{\Omega_T}|a(z,Du_\delta)|\cdot|D\varphi|+\left[|\tilde{\Psi}|+|f-\operatorname{div}(|F|^{p(\cdot)-2}F)|\right]\cdot|\varphi|\,\mathrm{d}z\\ &\le L\int_{\Omega_T}\left(|\tilde{\Psi}|+|f-\operatorname{div}(|F|^{p(\cdot)-2}F)|+(1+|Du_\delta|)^{p(\cdot)-1}\right)(|D\varphi|+|\varphi|)\,\mathrm{d}z.\end{aligned}$$

Here, we should mention that $\tilde{\Psi}\in L^{p'(\cdot)}(\Omega_T)$, since $L^{\gamma_1'}(\Omega_T)\subset L^{p'(\cdot)}(\Omega_T)$ is dense. Therefore, we are allowed to use the generalized Hölder's inequality (1.8). This yields

$$|\,\langle\!\langle\partial_t u_\delta,\varphi\rangle\!\rangle_{\Omega_T}|\le c\left(\|Du_\delta+1\|_{L^{p(\cdot)}}+\|\tilde{\Psi}\|_{L^{p'(\cdot)}}+\|f-\operatorname{div}(|F|^{p(\cdot)-2}F)\|_{L^{p'(\cdot)}}\right)\|\varphi\|_{W^{p(\cdot)}},$$

where $c = c(\gamma_1, \gamma_2, L)$. Now, we estimate the term $\|Du_\delta + 1\|_{L^{p(\cdot)}(\Omega_T)}$ from above

$$\|Du_\delta + 1\|_{L^{p(\cdot)}(\Omega_T)} \le \; c(\gamma_1, \gamma_2) \left(\int_{\Omega_T} |Du_\delta|^{p(\cdot)} \, \mathrm{d}z + 1 \right)^{\frac{1}{\gamma_1}} \le c(\gamma_1, \gamma_2, \mathcal{E}),$$

where we used (1.9) and (4.8). Therefore, we can conclude that

$$|\langle\!\langle \partial_t u, \varphi \rangle\!\rangle_{\Omega_T}| \le c(\gamma_1, \gamma_2, L, \mathcal{E}, \|\tilde{\Psi}\|_{L^{p'(\cdot)}}, \|f - \operatorname{div}(|F|^{p(\cdot)-2}F)\|_{L^{p'(\cdot)}}) \|\varphi\|_{W^{p(\cdot)}(\Omega_T)}.$$

This shows $\partial_t u_\delta \in W^{p(\cdot)}(\Omega_T)'$ with the estimate

$$\|\partial_t u_\delta\|_{W^{p(\cdot)}(\Omega_T)'} \le c(\gamma_1, \gamma_2, L, \mathcal{E}, \|\tilde{\Psi}\|_{L^{p'(\cdot)}}, \|f - \operatorname{div}(|F|^{p(\cdot)-2}F)\|_{L^{p'(\cdot)}}). \tag{4.10}$$

Due to the uniform bounds in (4.8), (4.9) and (4.10), there exists a subsequence, also labeled with δ, and a function $u \in W^{p(\cdot)}(\Omega_T) \cap L^\infty(0, T; L^2(\Omega))$ with $u = g$ on $\partial\Omega \times (0, T)$ and $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$, such that

$$\begin{cases} u_\delta \to u & \text{strongly in } L^{\hat{p}(\cdot)}(\Omega_T), \\ u_\delta \rightharpoonup^* u & \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\ Du_\delta \rightharpoonup Du & \text{weakly in } L^{p(\cdot)}(\Omega_T, \mathbb{R}^n), \\ \partial_t u_\delta \rightharpoonup \partial_t u & \text{weakly in } W^{p(\cdot)}(\Omega_T)'. \end{cases}$$

Note that the convergence $u_\delta \to u$ in $L^{\hat{p}(\cdot)}(\Omega_T)$ is strong due to the compactness of the embedding $W(\Omega_T) \hookrightarrow L^{\hat{p}(\cdot)}(\Omega_T)$ which results from the Aubin-Lions Theorem 2.2.

Step 4: Continuity in time and initial values. First of all, we note also that $u \in C^0([0, T]; L^2(\Omega))$ by Lemma 2.1 and the fact that we assume $p(\cdot) > \frac{2n}{n+2}$. Furthermore, the strong convergence $u_\delta \to u$ in $L^{\hat{p}(\cdot)}(\Omega_T)$ together with (4.6), i.e. $u_\delta \ge \psi$ a.e. on Ω_T, ensure that also $u \ge \psi$ a.e. on Ω_T, since

$$0 \le \lim_{\delta \downarrow 0} \int_{\Omega_T} (u_\delta - \psi) \mathbb{1}_{\{u < \psi\}} \, \mathrm{d}z \; = \int_{\Omega_T} (u - \psi) \mathbb{1}_{\{u < \psi\}} \, \mathrm{d}z,$$

where $\mathbb{1}_{\{u<\psi\}}$ is the characteristic function of the set $\{z \in \Omega_T : u(z) < \psi(z)\}$. Hence, we have $u \in \mathcal{K}'_{\psi,g}(\Omega_T)$. Next, we want to show that $u(\cdot, 0) = g(\cdot, 0)$ in the usual L^2-sense. For $\delta > 0$, $h > 0$ and $0 < t \le h$ we have by the standard Hölder's inequality, (4.10) and the fact that $W^{1,p(\cdot,t)}(\Omega)' \subseteq W^{-1,p_2'}(\Omega)$ for every $t \in (0, T)$ and $W^{p(\cdot)}(\Omega_T)' \subseteq L^{p_2'}(0, T; W^{-1,p_2'}(\Omega))$

$$\begin{aligned} \|u_\delta(\cdot, t) - g(\cdot, 0)\|_{W^{-1,p_2'}(\Omega)} &\le \int_0^t \|\partial_\tau u_\delta(\cdot, \tau)\|_{W^{-1,p_2'}(\Omega)} \, \mathrm{d}\tau \\ &\le h^{\frac{1}{p_2}} \left(\int_0^T \|\partial_\tau u_\delta(\cdot, \tau)\|^{p_2'}_{W^{-1,p_2'}(\Omega)} \, \mathrm{d}\tau \right)^{\frac{1}{p_2'}} \\ &\le c h^{\frac{1}{p_2}} \|\partial_\tau u_\delta(\cdot, \tau)\|_{W^{p(\cdot)}(\Omega_T)'} \le c h^{\frac{1}{p_2}}, \end{aligned}$$

and therefore, we can conclude that

$$\left\| \frac{1}{h} \int_0^h u_\delta(\cdot, t) \, \mathrm{d}t - g(\cdot, 0) \right\|^{p_2'}_{W^{-1,p_2'}(\Omega)} \le \frac{1}{h} \int_0^h \|u_\delta(\cdot, t) - g(\cdot, 0)\|^{p_2'}_{W^{-1,p_2'}(\Omega)} \mathrm{d}t \le c h^{\frac{1}{p_2 - 1}},$$

where $c(\gamma_1, \gamma_2, L, M)$. This implies in particular that $[u_\delta]_h(\cdot, 0) \to g(\cdot, 0)$ as $h \downarrow 0$ uniformly with respect to δ in $W^{-1,p_2'}(\Omega)$. Therefore,

$$\lim_{h \downarrow 0} \left\| \frac{1}{h} \int_0^h u(\cdot, t) \, \mathrm{d}t - g(\cdot, 0) \right\|^{p_2'}_{W^{-1,p_2'}} = \lim_{h \downarrow 0} \lim_{\delta \downarrow 0} \left\| \frac{1}{h} \int_0^h u_\delta(\cdot, t) \, \mathrm{d}t - g(\cdot, 0) \right\|^{p_2'}_{W^{-1,p_2'}} = 0,$$

proving that u admits the initial trace $g(\cdot, 0)$ in the sense of $W^{-1,p_2'}(\Omega)$. Moreover, since $u \in C^0([0, T]; L^2(\Omega))$, it has also a strong trace in the sense of (1.19). Due to the uniqueness of traces the previous estimate implies that the initial datum $g(\cdot, 0)$ is indeed assumed in the sense of (1.19), i.e.

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h \|u(\cdot, t) - g(\cdot, 0)\|^2_{L^2(\Omega)} \, \mathrm{d}t \; = 0. \tag{4.11}$$

Step 5: Variational inequality for the limit map. Finally, the last step is to address the variational inequality for the limit map. Therefore, we let $\delta \downarrow 0$ in the Cauchy-Dirichlet problem (4.3) to show that u is in fact the desired solution to the obstacle problem. To this aim we take $\varphi = v - u_\delta$ with $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$ as test function in (4.3) and apply Lemma 2.9, i.e. the Minty type Lemma, to the monotone operator $A : w \mapsto -\operatorname{div} a(z, Dw) - \zeta_\delta(\psi - w)\tilde{\Psi}_+$ with the obvious choice for the linear continuous operator B determined by $-\operatorname{div}(|F|^{p(\cdot)-2}F) + f$ and the closed convex set $\mathcal{C} = \mathcal{K}'_{\psi,g}(\Omega_T)$ to infer

$$\begin{aligned}&\langle\!\langle \partial_t v, v - u_\delta \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Dv) \cdot D(v - u_\delta)\, \mathrm{d}z + \frac{1}{2}\|v(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)} \\ &\geq \int_{\Omega_T} |F|^{p(\cdot)-2}F \cdot D(v - u_\delta) + f(v - u_\delta)\, \mathrm{d}z + \int_{\Omega_T} \zeta_\delta(\psi - v)\tilde{\Psi}_+(v - u_\delta)\, \mathrm{d}z\end{aligned} \tag{4.12}$$

for any choice of $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$. In the preceding inequality we now would like to get rid of the last term on the right-hand side as $\delta \downarrow 0$. To this aim we fix a cut-off function in space $\eta_\delta \in C_0^\infty(\Omega)$, such that $0 \leq \eta_\delta \leq 1$, $\eta_\delta \equiv 1$ on $\Omega \backslash \Omega^\delta$ and $\delta|D\eta_\delta| \leq c$, where $\Omega^\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$. Recall that Ω is an open, bounded Lipschitz regular domain. For $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$ we now define $v_\delta := v + \delta\eta_\delta$ and choose $v \equiv v_\delta$ in (4.12) to get (note that $\partial_t v_\delta = \partial_t v$)

$$\begin{aligned}\langle\!\langle \partial_t v, v - u_\delta \rangle\!\rangle_{\Omega_T} + &\int_{\Omega_T} a(z, Dv) \cdot D(v - u_\delta)\, \mathrm{d}z + \frac{1}{2}\|v(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)} \\ &\geq \int_{\Omega_T} |F|^{p(\cdot)-2}F \cdot D(v - u_\delta) + f(v - u_\delta)\, \mathrm{d}z \\ - \langle\!\langle \partial_t v, \delta\eta_\delta \rangle\!\rangle_{\Omega_T} &+ \int_{\Omega_T} a(z, Dv) \cdot D(v - u_\delta) - a(z, Dv_\delta) \cdot D(v_\delta - u_\delta)\, \mathrm{d}z \\ &+ \frac{1}{2}\left(\|v(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)} - \|v_\delta(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)}\right) \\ + \int_{\Omega_T} &|F|^{p(\cdot)-2}F \cdot \delta D\eta_\delta + f\delta\eta_\delta\, \mathrm{d}z + \int_{\Omega_T} \zeta_\delta(\psi - v_\delta)\tilde{\Psi}_+(v_\delta - u_\delta)\, \mathrm{d}z.\end{aligned} \tag{4.13}$$

Now, we want to ensure that the last five terms - those in the last three lines of (4.13) - disappear in the limit $\delta \downarrow 0$. For the first one we have by (1.13), that $\langle\!\langle \partial_t v, \delta\eta_\delta \rangle\!\rangle_{\Omega_T} \leq c(\gamma_1, \gamma_2)\|v_t\|_{W^{p(\cdot)}(\Omega_T)'}\|\delta\eta_\delta\|_{W^{p(\cdot)}(\Omega_T)} \to 0$ as $\delta \downarrow 0$. Note that the convergence follows from the facts that $\delta\eta_\delta \to 0$ in $L^{p(\cdot)}(\Omega_T)$ and moreover, $|\delta D\eta_\delta| \leq c$ and $\delta D\eta_\delta \equiv 0$ on $\Omega \backslash \Omega^\delta$ and hence also $\delta D\eta_\delta \to 0$ in $L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)$. For the second term in (4.13), we first note that $Dv_\delta = Dv$ on $\Omega_T \backslash \Omega_T^\delta$, where $\Omega_T^\delta := \Omega^\delta \times (0, T)$. Then, using the growth condition (1.2) of $a(z, \cdot)$, the fact that $|Dv_\delta| \leq |Dv| + c$ and generalized Hölder's inequality (1.8) we get

$$\begin{aligned}&\left|\int_{\Omega_T} a(z, Dv) \cdot D(v - u_\delta) - a(z, Dv_\delta) \cdot D(v_\delta - u_\delta)\, \mathrm{d}z\right| \\ &= \left|\int_{\Omega_T^\delta} a(z, Dv) \cdot D(v - u_\delta) - a(z, Dv_\delta) \cdot D(v_\delta - u_\delta)\, \mathrm{d}z\right| \\ &\leq \int_{\Omega_T^\delta} |a(z, Dv)|\,(|Dv| + Du_\delta) + |a(z, Dv_\delta)|\,(|Dv_\delta| + |Du_\delta|)\, \mathrm{d}z \\ &\leq c\||a(z, Dv)| + |a(z, Dv_\delta)|\|_{L^{p'(\cdot)}(\Omega_T^\delta)}\|(|Dv| + |Du_\delta| + c)\|_{L^{p(\cdot)}(\Omega_T^\delta)} \to 0,\end{aligned}$$

as $\delta \downarrow 0$, where the convergence follows from $|\Omega_T^\delta| \to 0$. Here, we also used for the convergence that Du_δ is bounded in $L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)$. The terms involving $g(\cdot, 0)$, F, f are treated similarly. Notice also that

$$\left(\|v(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)} - \|v_\delta(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)}\right) \to 0,$$

since v_δ tends to v as $\delta \downarrow 0$. Finally, for the last term on the right-hand side we get in the limit $\delta \downarrow 0$ that

$$\begin{aligned}\int_{\Omega_T} \zeta_\delta(\psi - v_\delta)\tilde{\Psi}_+(v_\delta - u_\delta)\,\mathrm{d}z &= \int_{\Omega_T^\delta} \zeta_\delta(\psi - v_\delta)\tilde{\Psi}_+(v_\delta - u_\delta)\,\mathrm{d}z \\ &\geq \int_{\Omega_T^\delta} \zeta_\delta(\psi - v_\delta)\tilde{\Psi}_+(\psi - u_\delta)\,\mathrm{d}z \\ &\geq -\int_{\Omega_T^\delta} \tilde{\Psi}_+|\psi - u_\delta|\,\mathrm{d}z \geq -\|\tilde{\Psi}_+\|_{L^{p'(\cdot)}(\Omega_T^\delta)}\|(|\psi| + |u_\delta|)\|_{L^{p(\cdot)}(\Omega_T^\delta)} \to 0\end{aligned}$$

since, u_δ is bounded in $L^{p(\cdot)}(\Omega_T)$. Here, we used that $\zeta_\delta(\psi - v_\delta) = \zeta_\delta(\psi - v - \delta\eta_\delta) = \zeta_\delta(\psi - v - \delta) = 0$ on $\Omega_T \backslash \Omega_T^\delta$, since $v_\delta = v + \delta\eta_\delta$, $\eta_\delta \equiv 1$ on $\Omega \backslash \Omega^\delta$, $\psi - v \leq 0$ a.e. on Ω_T and $\zeta_\delta(t) = 0$ if $t \in (-\infty, -\delta]$. Therefore, using the preceding observations together with the strong convergence of u_δ in $L^{p(\cdot)}(\Omega_T)$ and the weak convergence of Du_δ in $L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)$ we can pass to the limit $\delta \downarrow 0$ in (4.13) to obtain

$$\begin{aligned}\langle\!\langle \partial_t v, v - u\rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Dv)\cdot D(v - u)\,\mathrm{d}z + \frac{1}{2}\|v(\cdot, 0) - g(\cdot, 0)\|^2_{L^2(\Omega)} \\ \geq \int_{\Omega_T} |F|^{p(\cdot)-2}F\cdot D(v - u) + f(v - u)\,\mathrm{d}z,\end{aligned}$$

for all $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$. A second application of Minty's lemma, now to the monotone operator $A : w \mapsto -\operatorname{div} a(z, Dw)$, the same linear continuous operator B and the same closed convex set $\mathcal{C}$ as above, yields that

$$\begin{aligned}\langle\!\langle \partial_t u, v - u\rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du)\cdot D(v - u)\,\mathrm{d}z \\ \geq \int_{\Omega_T} |F|^{p(\cdot)-2}F\cdot D(v - u) + f(v - u)\,\mathrm{d}z,\end{aligned}$$

for all $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$. Hence, we conclude that u is the (unique) solution to the considered obstacle problem satisfying the asserted estimate. The latter assertion follows from (4.8), (4.9) and (4.10) and the lower semi continuity of the involved norms with respect to weak convergence. This finishes the proof of the lemma. □

The next part of this section is concerned with the following refinement of the above result in Lemma 4.1. It will be crucial in the proof of uniqueness of localizable solutions, since it will enable us to test certain regularized variational inequalities with less regular comparison functions whose distributional time derivative might not be contained in $W^{p(\cdot)}(\Omega_T)'$.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ an open, bounded Lipschitz domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4)-(1.5). Suppose that the vector-field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies (1.2) and (1.3). Moreover, let (1.16) with $g(\cdot, 0) \in L^2(\Omega)$, (1.21), (1.25), (1.26) and (4.1) hold. Then, there exists a solution $u \in \mathcal{K}'_{\psi,g}(\Omega_T)$, which satisfies the strong formulation of the variational inequality (1.17)* ***more generally*** *for all comparison function $v \in W_g^{p(\cdot)}(\Omega_T)$ with $v \geq \psi$ a.e. on Ω_T. Moreover, the energy estimate (4.2) holds true.*

Proof. We begin by defining u_δ as the solution to the Cauchy-Dirichlet problem (4.3), where u_δ is constructed as in the proof of Lemma 4.1. Then, our assumptions implies that $\tilde{\Psi} \in L^{\gamma_1'}(\Omega_T)$ and the preceding Lemma assured that, there exists a solution $u_\delta \in C^0([0, T]; L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T)$ with $\partial_t u_\delta \in W^{p(\cdot)}(\Omega_T)'$, where the solutions satisfies the obstacle constraint $u_\delta \geq \psi$ a.e. on Ω_T for every $\delta > 0$ and they satisfy the uniform energy bound

$$\sup_{\delta > 0}\left\{\sup_{t \in (0,T)} \int_\Omega |u_\delta(\cdot, t)|^2\,\mathrm{d}x \; + \int_{\Omega_T} |Du_\delta|^{p(\cdot)}\,\mathrm{d}z \; + \|\partial_t u_\delta\|_{W^{p(\cdot)}(\Omega_T)'}\right\} < \infty \tag{4.14}$$

cf. (4.8)-(4.10). These bounds and the Aubin-Lions compactness argument from Theorem 2.2 imply the convergences

$$\begin{cases} u_{\delta_i} \to u & \text{strongly in } L^{\hat{p}(\cdot)}(\Omega_T) \\ u_{\delta_i} \rightharpoonup^* u & \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)) \\ Du_{\delta_i} \rightharpoonup Du & \text{weakly in } W^{p(\cdot)}(\Omega_T) \\ \partial_t u_{\delta_i} \rightharpoonup \partial_t u & \text{weakly in } W^{p(\cdot)}(\Omega_T)' \end{cases} \tag{4.15}$$

as $i \to \infty$ for some sequence $\delta_i \downarrow 0$ and a limit map u with $u \in W^{p(\cdot)}_g(\Omega_T)$ and $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$. Further, the Lemma 2.1, applied to $u - g$, yields $u \in C^0([0,T];L^2(\Omega))$, and the limit map attains the prescribed initial values $u(\cdot,0) = g(\cdot,0)$.

Step 1: Strong convergence of the gradient. Now, our aim is to derive the even stronger convergence $Du_{\delta_i} \to Du$ with respect to the $L^{p(\cdot)}$-norm and then, the claimed variational inequality. For the proof of the strong convergence, we test the equation (4.3) for u_{δ_i} with the testing function $(u_{\delta_i} - u) \in W^{p(\cdot)}_0(\Omega_T)$, which yields for all $i \in \mathbb{N}$

$$\begin{aligned} \int_{\Omega_T} a(z,Du_{\delta_i})D(u_{\delta_i}-u)\,\mathrm{d}z &= \langle\!\langle \partial_t u_{\delta_i}, u - u_{\delta_i}\rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} \zeta_{\delta_i}(\psi - u_{\delta_i})\tilde{\Psi}_+(u_{\delta_i}-u)\,\mathrm{d}z \\ &+ \int_{\Omega_T} |F|^{p(\cdot)-2}F\cdot D(u_{\delta_i}-u) \;+ f(u_{\delta_i}-u)\,\mathrm{d}z =: I_i + II_i + III_i. \end{aligned} \tag{4.16}$$

Here, we apply Lemma 2.1 to I_i. Hence, we can conclude that

$$\begin{aligned} \limsup_{i\to\infty} I_i &= \limsup_{i\to\infty}\left(\langle\!\langle \partial_t u, u - u_{\delta_i}\rangle\!\rangle_{\Omega_T} - \frac{1}{2}\int_0^T \partial_t \|(u_{\delta_i}-u)(\cdot,t)\|^2_{L^2(\Omega)}\,\mathrm{d}z \right) \\ &= -\frac{1}{2}\limsup_{i\to\infty} \|(u_{\delta_i}-u)(\cdot,T)\|^2_{L^2(\Omega)} \le 0, \end{aligned} \tag{4.17}$$

where we used that $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$, $u_{\delta_i} \rightharpoonup u$ weakly in $W^{p(\cdot)}(\Omega_T)$ according to (4.15) and $u_{\delta_i}(\cdot,0) = g(\cdot,0)$. Next, we use the fact $0 \le \zeta_{\delta_i} \le 1$, generalized Hölder's inequality (1.8) and the strong convergence $u_{\delta_i} \to u$ in $L^{p(\cdot)}(\Omega_T)$ by (4.15), which gives

$$|II_i| \le \|\tilde{\Psi}_+\|_{L^{p'(\cdot)}(\Omega_T)}\|u_{\delta_i}-u\|_{L^{p(\cdot)}(\Omega_T)} \to 0, \text{ as } i\to\infty. \tag{4.18}$$

Finally, the weak convergence $u_{\delta_i} \rightharpoonup u$ in $W^{p(\cdot)}(\Omega_T)$, that holds by (4.15), implies

$$III_i \to 0 \text{ as } i \to \infty. \tag{4.19}$$

Plugging (4.17), (4.18) and (4.19) into (4.16), we arrive at

$$\limsup_{i\to\infty}\int_{\Omega_T} a(z,Du_{\delta_i})\cdot D(u_{\delta_i}-u)\,\mathrm{d}z \le 0.$$

Moreover, we note that the weak convergence $Du_{\delta_i} \rightharpoonup Du$ in $L^{p(\cdot)}(\Omega_T)$ yields

$$\limsup_{i\to\infty}\int_{\Omega_T} a(z,Du)\cdot D(u_{\delta_i}-u)\,\mathrm{d}z = 0.$$

Joining the two preceding formulae affords

$$\limsup_{i\to\infty}\int_{\Omega_T} (a(z,Du_{\delta_i}) - a(z,Du))\cdot D(u_{\delta_i}-u)\,\mathrm{d}z \le 0.$$

In view of the monotonicity condition (1.3), this implies

$$\limsup_{i\to\infty}\int_{\Omega_T} (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{p(\cdot)-2}{2}}|Du_{\delta_i} - Du|^2\,\mathrm{d}z = 0,$$

from which we can infer the desired convergence

$$u_{\delta_i} \to u \text{ strongly in } W^{p(\cdot)}(\Omega_T), \text{ as } i\to\infty, \tag{4.20}$$

since in the case $p(\cdot) \ge 2$ this implication is immediate, because of

$$\int_{\Omega_T} |Du_{\delta_i} - Du|^{p(\cdot)}\,\mathrm{d}z \le \int_{\Omega_T} (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{p(\cdot)-2}{2}}|Du_{\delta_i} - Du|^2\,\mathrm{d}z,$$

while for $\frac{2n}{n+2} < p(\cdot) < 2$, we have to use the following calculus:

$$\begin{aligned} |Du_{\delta_i} - Du|^{p(\cdot)} &= (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{p(\cdot)(p(\cdot)-2)}{4}} |Du_{\delta_i} - Du|^{p(\cdot)} \\ &\quad \times (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{p(\cdot)(2-p(\cdot))}{4}} \\ &\le c\,(\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{(p(\cdot)-2)}{2}} |Du_{\delta_i} - Du|^2 \\ &\quad + c\,(\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{p(\cdot)}{2}} \end{aligned}$$

with a constant $c = c(\gamma_1, \gamma_2)$, where we applied the Young's inequality with exponents $p(\cdot)/2 + (2-p(\cdot))/2 = 1$. Next, we estimate the second term on the right-hand side as follows:

$$\begin{aligned} (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{p(\cdot)}{2}} &= (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{(p(\cdot)-2)}{2}} (\mu^2 + |Du_{\delta_i}|^2 + |Du|^2) \\ &\le 2\,(\mu^2 + |Du_{\delta_i}|^2 + |Du|^2)^{\frac{(p(\cdot)-2)}{2}} |Du_{\delta_i} - Du|^2 \\ &\quad + 3\,(\mu + |Du|)^{p(\cdot)}, \end{aligned}$$

where we utilized the fact that $p(\cdot) - 2 < 0$. Combining the last two estimate, integrating over Ω_T and using the energy bound (4.14) yields the claim.

Step 2: Variational inequality for the limit map. Now, we are in a position to prove the variational inequality (1.17) for the limit map, for every comparison function $v \in W_g^{p(\cdot)}(\Omega_T)$ with $v \ge \psi$ a.e. on Ω_T. The basic idea is to test the weak formulation of the parabolic Cauchy-Dirichlet problem (4.3) for the approximating functions u_{δ_i} with the test functions $v - u$ and then to pass to the limit. However, some effort has to be made in order to deal with the correction term $\zeta_\delta(\psi - u_\delta)\tilde{\Psi}_+$ appearing in the equation. For this aim, we fix a cut-off function in space $\eta_\delta \in C_0^\infty(\Omega)$, such that $0 \le \eta_\delta \le 1$, $\eta_\delta \equiv 1$ on $\Omega\backslash\Omega^\delta$ and $|D\eta_\delta| \le \frac{c}{\delta}$, where $\Omega^\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$. Now, we fix an arbitrary $v \in W_g^{p(\cdot)}(\Omega_T)$ with $v \ge \psi$ a.e. on Ω_T and define $v_\delta := v + \delta\eta_\delta \in W_g^{p(\cdot)}(\Omega_T)$. Notice that the choice of η_δ implies

$$\int_{\Omega_T} |Dv_\delta - Dv|^{p(\cdot)}\, \mathrm{d}z = T \int_{\Omega\backslash\Omega^\delta} |\delta D\eta_\delta|^{p(\cdot)}\, \mathrm{d}x \le c^{\gamma_2} T |\Omega\backslash\Omega^\delta| \to 0,$$

as $\delta \downarrow 0$, and consequently

$$v_\delta \to v \text{ strongly in } W^{p(\cdot)}(\Omega_T), \text{ as } \delta \downarrow 0. \tag{4.21}$$

Next, we test the equation (4.3) for u_{δ_i} with $v_{\delta_i} - u_{\delta_i} \in W_0^{p(\cdot)}(\Omega_T)$, this yields

$$\begin{aligned} \langle\!\langle \partial_t u_{\delta_i}, v_{\delta_i} - u_{\delta_i} \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_{\delta_i}) \cdot D(v_{\delta_i} - u_{\delta_i})\, \mathrm{d}z \\ = \int_{\Omega_T} |F|^{p(\cdot)-2} F \cdot D(v_{\delta_i} - u_{\delta_i}) + f \cdot (v_{\delta_i} - u_{\delta_i})\, \mathrm{d}z \\ + \int_{\Omega_T} \zeta_{\delta_i}(\psi - u_{\delta_i}) \tilde{\Psi}_+ (v_{\delta_i} - u_{\delta_i})\, \mathrm{d}z. \end{aligned} \tag{4.22}$$

Our next aim is to show that the last integral appearing on the right-hand side is non-negative in the limit. Here, we have to consider the two cases $v_{\delta_i} \le u_{\delta_i}$ and $v_{\delta_i} > u_{\delta_i}$. Keeping in mind that the function ζ_{δ_i} is monotonously non-decreasing, we know that

$$\int_{\Omega_T} \zeta_{\delta_i}(\psi - u_{\delta_i}) \tilde{\Psi}_+ (v_{\delta_i} - u_{\delta_i})\, \mathrm{d}z \ge \int_{\Omega_T} \zeta_{\delta_i}(\psi - v_{\delta_i}) \tilde{\Psi}_+ (v_{\delta_i} - u_{\delta_i})\, \mathrm{d}z.$$

In the case $v_{\delta_i} > u_{\delta_i}$, we know that $-v_{\delta_i} < -u_{\delta_i}$ and therefore, $\zeta_{\delta_i}(\psi - u_{\delta_i}) \ge \zeta_{\delta_i}(\psi - v_{\delta_i})$. While in the case $v_{\delta_i} \le u_{\delta_i}$, we know that $-v_{\delta_i} \ge -u_{\delta_i}$ and therefore, $\zeta_{\delta_i}(\psi - u_{\delta_i}) \le \zeta_{\delta_i}(\psi - v_{\delta_i})$. But we have also that $(v_{\delta_i} - u_{\delta_i}) \le 0$ and finally, $\zeta_{\delta_i}(\psi - u_{\delta_i})(v_{\delta_i} - u_{\delta_i}) \ge \zeta_{\delta_i}(\psi - v_{\delta_i})(v_{\delta_i} - u_{\delta_i})$. By the choice of v and the definition of v_{δ_i}, there holds $\psi - v_{\delta_i} \le -\delta_i \eta_{\delta_i} = -\delta_i$ on $\Omega\backslash\Omega^{\delta_i} \times (0, T)$, and therefore, the integrand of the last integral vanishes on the set $\Omega\backslash\Omega^{\delta_i}\times(0,T)$. This implies

$$\begin{aligned}\int_{\Omega_T}\zeta_{\delta_i}(\psi-u_{\delta_i})\tilde{\Psi}_+(v_{\delta_i}-u_{\delta_i})\,\mathrm{d}z&=\int_{\Omega_T^{\delta_i}}\zeta_\delta(\psi-u_{\delta_i})\tilde{\Psi}_+(v_{\delta_i}-u_{\delta_i})\,\mathrm{d}z\\&\geq-\left|\int_{\Omega_T^{\delta_i}}\tilde{\Psi}_+(v_{\delta_i}-u_{\delta_i})\,\mathrm{d}z\right|\\&\geq-\|\tilde{\Psi}_+\|_{L^{p'(\cdot)}(\Omega_T^{\delta_i})}\|(|v_{\delta_i}|+|u_{\delta_i}|)\|_{L^{p(\cdot)}(\Omega_T^{\delta_i})}\to0\end{aligned}$$

as $i\to\infty$, where the convergence follows from $|\Omega_T^{\delta_i}|\to0$ and the uniform bounds (4.14). Joining this with (4.22) of the last integral and using (4.20), (4.21) and (4.15), we have

$$\begin{aligned}\langle\!\langle\partial_t u,v-u\rangle\!\rangle_{\Omega_T}+\int_{\Omega_T}a(z,Du)\cdot D(v-u)\,\mathrm{d}z&=\lim_{i\to\infty}\langle\!\langle\partial_t u_{\delta_i},v_{\delta_i}-u_{\delta_i}\rangle\!\rangle_{\Omega_T}\\&\quad+\lim_{i\to\infty}\int_{\Omega_T}a(z,Du_{\delta_i})\cdot D(v_{\delta_i}-u_{\delta_i})\,\mathrm{d}z\\&\geq\lim_{i\to\infty}\int_{\Omega_T}|F|^{p(\cdot)-2}F\cdot D(v_{\delta_i}-u_{\delta_i})+f(v_{\delta_i}-u_{\delta_i})\,\mathrm{d}z\\&=\int_{\Omega_T}|F|^{p(\cdot)-2}F\cdot D(v-u)+f(v-u)\,\mathrm{d}z.\end{aligned}$$

This finishes the proof of the lemma. □

5. Proof of Theorem 1.7: Existence and uniqueness of localizable solutions

In this section, we want to consider a more general domain with irregular boundary. To this aim, we need to assume an additional approximation assumption on the obstacle function ψ, which are introduced in (1.22)-(1.24). More precisely, we give the proof of the existence and uniqueness Theorem 1.7.

Proof of Theorem 1.7. The proof is divided into several steps. We begin with

Step 1: Regularization. We start by assuming that the obstacle function ψ can be approximated by more general functions as in (1.22) with an additional regularity property (1.23), which approximate ψ in the sense of (1.24). Next, we define boundary data $\tilde{g}_i:=\tilde{g}-\psi+\psi_i$ with $\tilde{g}:=\max\{g,\psi\}$, which adapted to ψ_i and respects the obstacle constraint $\tilde{g}_i\geq\psi_i$ a.e. on Ω_T. In addition, $\tilde{g}_i$ satisfies $\tilde{g}_i\in C^0([0,T];L^2(\Omega))\cap W^{p(\cdot)}(\Omega_T)$ with $\partial_t\tilde{g}_i\in L^{\gamma_1'}(\Omega_T)$ and the convergence

$$\begin{cases}\tilde{g}_i\to\tilde{g}&\text{strongly in }W^{p(\cdot)}(\Omega_T)\text{ and }L^\infty(0,T;L^2(\Omega)),\\\partial_t\tilde{g}_i\to\partial_t\tilde{g}&\text{strongly in }L^{\gamma_1'}(\Omega_T),\end{cases}\tag{5.1}$$

as $i\to\infty$, which follows directly from (1.24). Here, we should also mentioned that by the compatibility condition (1.16), we have $\tilde{g}_i=g-\psi+\psi_i$ on $\partial\Omega\times(0,T)$ and $\tilde{g}_i(\cdot,0)=g(\cdot,0)-\psi(\cdot,0)+\psi_i(\cdot,0)$ on Ω. Therefore by (5.1) resp. (1.24), we have $\tilde{g}_i\to g$ on $\partial\Omega\times(0,T)$ and $\tilde{g}_i(\cdot,0)\to g(\cdot,0)$ on Ω as $i\to\infty$. Next, we need a mollifier to regularize the inhomogeneity. Therefore, we choose a standard, radially symmetric mollifier, e.g. a Friedrich's mollifier, $\phi\in C_0^\infty(B_1)$, $\phi\geq0$ with $\int_{B_1}\phi\,\mathrm{d}x=1$ and let $\phi_{\delta_i}(x):=\delta_i^{-n}\phi(x/\delta_i)$ for some sequence $\delta_i\in(0,1)$, which tends to zero as $i\to\infty$. Now, we extend F by zero outside of Ω_T over the whole $\mathbb{R}^{n+1}$, then we let $F_i(\cdot,t):=F(\cdot,t)*\phi_{\delta_i}$ for almost every $t\in(0,T)$ and $f_i(\cdot,t):=f(\cdot,t)-\operatorname{div}\left((\delta_i^2+|F_i|^2)^{\frac{p(\cdot)-2}{2}}F_i\right)$. These functions satisfy $f_i\in L^{\gamma_1'}(\Omega_T)$ and converge in the sense

$$f_i\to f-\operatorname{div}(|F|^{p(\cdot)-2}F)\text{ strongly in }W^{p(\cdot)}(\Omega_T)',\tag{5.2}$$

as $i\to\infty$. Moreover, by standard results on mollifications, we have the convergence

$$F_i(\cdot,t)\to F\text{ strongly in }L^{p(\cdot)}(\Omega_T,\mathbb{R}^n),\tag{5.3}$$

since $\delta_i \downarrow 0$, as $i \to \infty$ and therefore, $(\delta_i^2 + |F_i|^2)^{\frac{p(\cdot)-2}{2}} F_i \to |F|^{p(\cdot)-2}F$ strongly in $L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$. Moreover, we can infer

$$\begin{aligned}|\operatorname{div} a(z, D\psi_i)| &\le |\operatorname{div}_x a(z, D\psi_i)| + |D^2\psi_i| \cdot |D_w a(z, D\psi_i)| \\ &\le L(1 + |D\psi_i|)^{p(\cdot)}(1 + |D^2\psi_i|) \in L^{\gamma_1'}(\Omega_T).\end{aligned} \tag{5.4}$$

This holds true by the property (1.23) of ψ_i and (1.27). Finally, we define $u_i \in \mathcal{K}'_{\psi_i,\tilde{g}_i}(\Omega_T)$ as the solution to the regularized problem

$$\langle\!\langle \partial_t u_i, v - u_i \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_i) \cdot D(v - u_i) \, \mathrm{d}z \ge \int_{\Omega_T} f_i \cdot (v - u_i) \, \mathrm{d}z \tag{5.5}$$

for every $v \in W^{p(\cdot)}_{\tilde{g}_i}(\Omega_T)$ with $v \ge \psi_i$ a.e. on Ω_T, where we impose the initial and boundary values

$$u_i(\cdot, 0) = \tilde{g}_i(\cdot, 0) \text{ on } \Omega \times \{0\} \text{ and } u_i = \tilde{g}_i \text{ on } \partial\Omega \times (0, T). \tag{5.6}$$

By Lemma 4.2 we know, that this solution exists, since f_i and ψ_i satisfy (4.1). This is fulfilled by (1.23) and (5.4).

Step 2: Energy bounds and weak convergence. Next, we will deduce an energy estimate to get the energy bounds and finally, to conclude the weak convergence. For this aim, we define abbreviate $\Omega_t := \Omega \times (0, t)$ with arbitrary fixed time $t \in (0, T)$ and $v := u_i + (\tilde{g}_i - u_i)\mathbb{1}_{(0,t)}(\tau) \in W^{p(\cdot)}_{u_i}(\Omega_T)$ as comparison function in the variational inequality (5.5). The map v is admissible as comparison function, since $v \ge \psi_i$ a.e. on Ω_t and $\mathcal{K}'_{\psi_i,\tilde{g}_i}(\Omega_T)$ is dense in $\mathcal{K}_{\psi_i,\tilde{g}_i}(\Omega_T)$. Hence, we derive from (5.5) that

$$\langle\!\langle \partial_t u_i, \tilde{g}_i - u_i \rangle\!\rangle_{\Omega_t} + \int_{\Omega_t} a(z, Du_i) \cdot D(\tilde{g}_i - u_i) \, \mathrm{d}z \ge \int_{\Omega_t} f_i \cdot (\tilde{g}_i - u_i) \, \mathrm{d}z.$$

Moreover, we add $-\langle\!\langle \partial_t \tilde{g}_i, \tilde{g}_i - u_i \rangle\!\rangle_{\Omega_t}$ on both sides and multiply the inequality by -1, thus it follows

$$\begin{aligned}\langle\!\langle \partial_t u_i - \partial_t \tilde{g}_i, u_i - \tilde{g}_i \rangle\!\rangle_{\Omega_t} &+ \int_{\Omega_t} a(z, Du_i) \cdot D(u_i - \tilde{g}_i) \, \mathrm{d}z \\ &\le \int_{\Omega_t} f_i \cdot (u_i - \tilde{g}_i) \, \mathrm{d}z + \langle\!\langle \partial_t \tilde{g}_i, \tilde{g}_i - u_i \rangle\!\rangle_{\Omega_t} \\ &= \int_{\Omega_t} f_i \cdot (u_i - \tilde{g}_i) \, \mathrm{d}z + \int_{\Omega_t} \partial_t \tilde{g}_i \cdot (\tilde{g}_i - u_i) \, \mathrm{d}z.\end{aligned} \tag{5.7}$$

Here, we should mention that we are allowed to identify the duality product by the inner product between $L^2(\Omega_T)$ and $L^{\gamma_1}(0, T; W_0^{1,\gamma_1}(\Omega))$, since $\tilde{g}_i - u_i \in W_0^{p(\cdot)}(\Omega_T)$, $W_0^{p(\cdot)}(\Omega_T) \subseteq L^{\gamma_1}(0, T; W_0^{1,\gamma_1}(\Omega)) \subset L^{\gamma_1'}(\Omega_T)$ and $\partial_t \tilde{g}_i \in L^{\gamma_1'}(\Omega_T) \subset W^{\gamma_1}(\Omega_T)')$, cf. (4.7). Furthermore, we can conclude by means of Lemma 2.1

$$\langle\!\langle \partial_t u_i - \partial_t \tilde{g}_i, u_i - \tilde{g}_i \rangle\!\rangle_{\Omega_t} = \frac{1}{2} \int_{\Omega \times \{t\}} |u_i - \tilde{g}_i|^2 \, \mathrm{d}x,$$

where we used the fact $u_i(\cdot, 0) = \tilde{g}_i(\cdot, 0)$ that holds by (5.6). Next, we will estimate the second term on the left-hand side of (5.7) from below. Furthermore, we use first the coercivity property (1.7), then the growth condition (1.2) and finally Young's inequality. Hence, we can conclude

$$\int_{\Omega_t} a(z, Du_i) D(u_i - \tilde{g}_i) \, \mathrm{d}z \ge \frac{\nu}{2c(n, \gamma_1, \gamma_2)} \int_{\Omega_t} |Du_i|^{p(\cdot)} \, \mathrm{d}z - c \int_{\Omega_t} (1 + |D\tilde{g}_i|^{p(\cdot)}) \, \mathrm{d}z.$$

Now, we combine the last three equations and use Young's inequality several times, then it follows for any $\varepsilon > 0$ that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\times\{t\}} |u_i-\tilde g_i|^2 \,\mathrm{d}x \;+\frac{\nu}{2c}\int_{\Omega_t} |Du_i|^{p(\cdot)} \,\mathrm{d}z \\
&\le \int_{\Omega_t} f_i\cdot(u_i-\tilde g_i)\,\mathrm{d}z + \int_{\Omega_t} \partial_t\tilde g_i\cdot(\tilde g_i-u_i)\,\mathrm{d}z \;+\; c\int_{\Omega_t}(1+|D\tilde g_i|^{p(\cdot)})\,\mathrm{d}z \\
&\le c_\varepsilon\left(\|f\|^{\gamma_1'}_{L^{\gamma_1'}(\Omega_t)} + \|\partial_t\tilde g_i\|^{\gamma_1'}_{L^{\gamma_1'}(\Omega_t)} + \int_{\Omega_t}|(\delta_i^2+|F_i|^2)^{\frac{p(\cdot)-2}{2}}F_i|^{p'(\cdot)}\,\mathrm{d}z\right) \\
&+\varepsilon c\left(\|(u_i-\tilde g_i)\|^{\gamma_1}_{L^{\gamma_1}(\Omega_t)} + \int_{\Omega_t}|D(u_i-\tilde g_i)|^{p(\cdot)}\,\mathrm{d}z\right) + \; c\int_{\Omega_t}(1+|D\tilde g_i|^{p(\cdot)})\,\mathrm{d}z,
\end{aligned}$$

with a constant $c_\varepsilon = c(\varepsilon, n, \gamma_1, \gamma_2, \nu, L)$. Moreover, we apply the standard Poincaré inequality slicewise to the right-hand side and choose ε, such that $(\frac{\nu}{2c} - 2\varepsilon c) \le \frac{1}{2}$, where $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$. Thus we have

$$\begin{aligned}
\int_{\Omega\times\{t\}} |u_i-\tilde g_i|^2\,\mathrm{d}x + \int_{\Omega_t}|Du_i|^{p(\cdot)}\,\mathrm{d}z \le\; & c\|f\|^{\gamma_1'}_{L^{\gamma_1'}(\Omega_t)} \;+\; c\|\partial_t\tilde g_i\|^{\gamma_1'}_{L^{\gamma_1'}(\Omega_T)} \\
&+c\int_{\Omega_t}|D\tilde g_i|^{p(\cdot)} + |(\delta_i^2+|F_i|^2)^{\frac{p(\cdot)-2}{2}}F_i|^{p'(\cdot)} + 1\,\mathrm{d}z
\end{aligned}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$. Combining this with the definition of $\tilde g_i$ and the convergences in (5.1) and (5.3), we can conclude the following energy estimate

$$\sup_{t\in(0,T)}\int_\Omega |u_i(\cdot,t)|^2\,\mathrm{d}x \;+\int_{\Omega_T}|Du_i|^{p(\cdot)}\,\mathrm{d}z \;\le\; cM \tag{5.8}$$

for all sufficiently large $i \in \mathbb{N}$ with a constant $c = c(n, \gamma_1, \gamma_2, \nu,\ L, \operatorname{diam}(\Omega))$. Furthermore, we use the Poincaré type inequality (2.5) to infer an uniform $L^{p(\cdot)}$-bound. This yields

$$\|u_i\|^{\gamma_1}_{L^{p(\cdot)}(\Omega_T)} \le\; cM^{\frac{2\gamma_2}{n+2}}\int_{\Omega_T}|Du_i|^{p(\cdot)} + |D\tilde g_i|^{p(\cdot)} + |\tilde g_i|^{p(\cdot)} + 1\,\mathrm{d}z \;\le\; cM^{\frac{2\gamma_2}{n+2}+1} \tag{5.9}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega), \omega(\cdot))$. From the last two estimates, we can conclude that the functions u_i are uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and in $W^{p(\cdot)}(\Omega_T)$. Therefore and by the convergence property of the function $\tilde g_i$ in (5.1), we may find a function $u \in L^\infty(0,T;L^2(\Omega))\cap W^{p(\cdot)}_{\tilde g}(\Omega_T)$, such that - possibly after passing to a subsequence - there holds

$$\begin{cases} u_i \rightharpoonup u & \text{weakly in } L^{p(\cdot)}(\Omega_T), \\ Du_i \rightharpoonup Du & \text{weakly in } L^{p(\cdot)}(\Omega_T,\mathbb{R}^n), \\ u_i \rightharpoonup^* u & \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{5.10}$$

as $i \to \infty$. Finally, we can infer - with respect to the weak convergence - from (5.8) and (5.9) that the energy estimate (1.28) holds true. Furthermore, the growth assumption (1.2) of $a(\cdot)$ and (5.8) imply that the sequence $\{a(z, Du_i)\}_{i\in\mathbb{N}}$ is bounded in $L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$. Consequently, after passing to a subsequence once more, we can find a limit map $A_0 \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$ with

$$a(z, Du_i) \rightharpoonup A_0 \text{ weakly in } L^{p'(\cdot)}(\Omega_T,\mathbb{R}^n) \text{ as } i\to\infty. \tag{5.11}$$

Step 3: Strong convergence. The next aim is to show the strong convergence. Therefore, we choose the comparison function $v := u_k - \psi_k + \psi_i \in W^{p(\cdot)}(\Omega_T)$, for arbitrary $i,\ k \in \mathbb{N}$, in the variational inequality (5.5). This function is admissible as comparison function for u_i, since $u_k = \tilde g_k = \tilde g - \psi + \psi_k$ holds on $\partial_{\mathcal{P}}\Omega_T$, which implies $v = \tilde g - \psi + \psi_i = \tilde g_i$ on $\partial_{\mathcal{P}}\Omega_T$, and further, the obstacle constraint $v \ge \psi_i$

a.e. on Ω_T is valid. Therefore, we can conclude from (5.5) that

$$\begin{aligned}\langle\!\langle \partial_t u_i, u_i - \psi_i - u_k + \psi_k \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_i) \cdot D(u_i - \psi_i - u_k + \psi_k)\, \mathrm{d}z \\ \le \int_{\Omega_T} f_i \cdot (u_i - \psi_i - u_k + \psi_k)\, \mathrm{d}z.\end{aligned} \tag{5.12}$$

Next, we define functions $\hat{u}_i := u_i - \psi_i$ for all $i \in \mathbb{N}$ and similarly, $\hat{u} := u - \psi$. Notice that $\hat{u}_i = \hat{u}$ holds on $\partial_{\mathcal{P}}\Omega_T$ for all $i \in \mathbb{N}$, since $\hat{u}_i = \tilde{g}_i - \psi_i = g - \psi$ on $\partial_{\mathcal{P}}\Omega_T$. By the convergence of ψ_i and u_i in (1.24) and (5.10), we have

$$\hat{u}_i \rightharpoonup \hat{u} \text{ weakly in } W^{p(\cdot)}(\Omega_T) \tag{5.13}$$

as $i \to \infty$. Now, we can conclude from the preceding inequality the following estimate

$$\begin{aligned}\langle\!\langle \partial_t \hat{u}_i, \hat{u}_i - \hat{u}_k \rangle\!\rangle_{\Omega_T} &+ \int_{\Omega_T} a(z, Du_i) \cdot D(u_i - u)\, \mathrm{d}z \\ &\le \int_{\Omega_T} a(z, Du_i) \cdot D(\hat{u}_k - u + \psi_i)\, \mathrm{d}z \\ &\quad + \int_{\Omega_T} f_i \cdot (\hat{u}_i - \hat{u}_k)\, \mathrm{d}z \ + \langle\!\langle \partial_t \psi_i, \hat{u}_i - \hat{u}_k \rangle\!\rangle_{\Omega_T}.\end{aligned} \tag{5.14}$$

Since the indices $i,\ k \in \mathbb{N}$ are arbitrary, we can exchange i and k in (5.12) and add the term $\langle\!\langle \partial_t \psi_k, \hat{u}_k - \hat{u}_i \rangle\!\rangle_{\Omega_T}$ on both sides of (5.12). Then, we add the resulting estimate with (5.14) and get

$$\begin{aligned}\frac{1}{2}\int_{\Omega \times \{T\}} |\hat{u}_i - \hat{u}_k|^2\, \mathrm{d}x &+ \int_{\Omega_T} a(z, Du_i) \cdot D(u_i - u) + a(z, Du_k) \cdot D(u_k - u)\, \mathrm{d}z \\ &\le \int_{\Omega_T} (f_i - f_k) \cdot (\hat{u}_i - \hat{u}_k)\, \mathrm{d}z + \langle\!\langle \partial_t \psi_i - \partial_t \psi_k, \hat{u}_i - \hat{u}_k \rangle\!\rangle_{\Omega_T} \\ &\quad + \int_{\Omega_T} (a(z, Du_i) - a(z, Du_k)) \cdot D(\psi_i - \psi_k)\, \mathrm{d}z \\ &\quad + \int_{\Omega_T} a(z, Du_i) \cdot D(u_k - u) + a(z, Du_k) \cdot D(u_i - u)\, \mathrm{d}z,\end{aligned}$$

where we used Lemma 2.1. Next, we utilize the strong convergence of $\psi_k \to \psi$ and $f_k \to f - \operatorname{div}(|F|^{p(\cdot)-2}F)$ as $k \to \infty$ according to (1.24) and (5.2). In addition, we apply the weak convergence of Du_i stated in (5.10), (5.11) and (5.13). Thus, we can deduce the following estimate

$$\begin{aligned}\int_{\Omega_T} a(z, Du_i) \cdot D(u_i - u)\, \mathrm{d}z \ &+ \limsup_{k\to\infty} \int_{\Omega_T} a(z, Du_k) \cdot D(u_k - u)\, \mathrm{d}z \\ &\le \int_{\Omega_T} (f_i - f + \operatorname{div}(|F|^{p(\cdot)-2}F)) \cdot (\hat{u}_i - \hat{u})\, \mathrm{d}z + \langle\!\langle \partial_t \psi_i - \partial_t \psi, \hat{u}_i - \hat{u} \rangle\!\rangle_{\Omega_T} \\ &\quad + \int_{\Omega_T} (a(z, Du_i) - A_0) \cdot D(\psi_i - \psi) \ + \ A_0 \cdot D(u_i - u)\, \mathrm{d}z.\end{aligned}$$

Finally, we let $i \to \infty$. Therefore, we arrive at

$$\limsup_{i\to\infty} \int_{\Omega_T} a(z, Du_i) \cdot D(u_i - u)\, \mathrm{d}z \le 0, \tag{5.15}$$

where we exploited (1.24), (5.2), (5.10) and (5.13). Then, by the weak convergence of $Du_i \rightharpoonup Du$ in $L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)$ as $i \to \infty$, there holds

$$\limsup_{i\to\infty} \int_{\Omega_T} a(z, Du) \cdot D(u_i - u)\, \mathrm{d}z = 0. \tag{5.16}$$

Combining (5.15) and (5.16) we conclude that

$$\limsup_{i\to\infty} \int_{\Omega_T} (a(z, Du_i) - a(z, Du)) \cdot D(u_i - u)\, \mathrm{d}z \le 0.$$

Finally, we know by the monotonicity condition (1.3) that the left-hand side of the preceding inequality is non-negativ, such that

$$\limsup_{i\to\infty}\int_{\Omega_T}(\tilde{\mu}^2+|Du_i|^2+|Du|^2)^{\frac{p(\cdot)-2}{2}}|Du_i-Du|^2\,\mathrm{d}z=0,$$

which yields the desired strong convergence

$$u_i\to u \text{ strongly in } W^{p(\cdot)}(\Omega_T) \tag{5.17}$$

as $i\to\infty$. This is obvious because we can conclude

$$\begin{aligned}&\limsup_{i\to\infty}\int_{\Omega_T}(\tilde{\mu}^2+|Du_i|^2+|Du|^2)^{\frac{p(\cdot)-2}{2}}|Du_i-Du|^2\,\mathrm{d}z\\&\geq\limsup_{i\to\infty}\int_{\Omega_T}|Du_i-Du|^{p(\cdot)}\,\mathrm{d}z=0.\end{aligned}$$

This last implication is straightforward in the case $p(\cdot)\geq 2$, while for exponents $p(\cdot)<2$, it follows from the same calculus as in the proof of Lemma 4.2, see page 20. Possibly after extracting another subsequence, we also have $Du_i\to Du$ a.e. on Ω_T and we can conclude that

$$a(\cdot,Du_i)\to a(\cdot,Du) \text{ for almost every } z\in\Omega_T,\text{ as } i\to\infty. \tag{5.18}$$

Step 4: Continuity in time and initial values. Now, we have to show the continuity in time and initial values. For this aim, we choose a comparison function $v:=u_i+(\hat{u}_k-\hat{u}_i)\mathbb{1}_{(0,\tau)}(t)\in W^{p(\cdot)}_{u_i}(\Omega_T)$ for an arbitrary time $\tau\in(0,T)$ and where $\hat{u}_i:=u_i-\psi_i$ for all $i\in\mathbb{N}$. This function we use to test the variational inequality (5.5) for u_i. Moreover, that we use $\partial_t\psi_i,f_i\in L^{p'(\cdot)}(\Omega_T)$, since $\partial_t\psi_i,f_i\in L^{\gamma_1'}(\Omega_T)$. Thus, it follows by the generalized Hölder's inequality (1.8) that

$$\begin{aligned}&\langle\!\langle\partial_t\hat{u}_i,\hat{u}_i-\hat{u}_k\rangle\!\rangle_{\Omega_\tau}\leq\int_{\Omega_\tau}a(z,Du_i)\cdot D(\hat{u}_k-\hat{u}_i)\,\mathrm{d}z-\int_{\Omega_\tau}(f_i+\partial_t\psi_i)\cdot(\hat{u}_k-\hat{u}_i)\,\mathrm{d}z\\&\leq 2\|f_i\|_{L^{p'(\cdot)}(\Omega_\tau)}\|\hat{u}_k-\hat{u}_i\|_{L^{p(\cdot)}(\Omega_\tau)}+c(\gamma_1,\gamma_2)\|\partial_t\psi_i\|_{L^{p'(\cdot)}(\Omega_\tau)}\|\hat{u}_k-\hat{u}_i\|_{L^{p(\cdot)}(\Omega_\tau)}\\&+c(\gamma_2,L)\|(1+|Du_i|)^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega_\tau)}\|D(\hat{u}_k-\hat{u}_i)\|_{L^{p(\cdot)}(\Omega_\tau)},\end{aligned}$$

where we also used the growth assumption (1.2). Now, we can estimate the right-hand side of the preceding inequality from above, by exchanging Ω_τ by the Ω_T. Then, notice that the right-hand side is independent from $\tau\in(0,T)$. Moreover, we note that the right-hand side vanishes as $i,k\to\infty$, since the strong convergence (5.17) of u_i in $W^{p(\cdot)}(\Omega_T)$ implies, in combination with the convergence of ψ_i in (1.24), that $\{D\hat{u}_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)$. Therefore, it yields $\limsup_{i,k\to\infty}\langle\!\langle\partial_t\hat{u}_i,\hat{u}_i-\hat{u}_k\rangle\!\rangle_{\Omega_\tau}\leq 0$. Applying furthermore Lemma 2.1 and keeping in mind, that $\hat{u}_i=\hat{u}_k$ on $\partial_{\mathcal{P}}\Omega_T$, we conclude

$$\limsup_{i,k\to\infty}\sup_{\tau\in(0,\tau)}\int_{\Omega}|(\hat{u}_i-\hat{u}_k)(\cdot,\tau)|^2\,\mathrm{d}x=2\limsup_{i,k\to\infty}\sup_{\tau\in(0,\tau)}\langle\!\langle\partial_t\hat{u}_i-\partial_t\hat{u}_k,\hat{u}_i-\hat{u}_k\rangle\!\rangle=0.$$

Thus, we have established that $\{\hat{u}_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T];L^2(\Omega))$, which implies in view of the strong convergence $\psi_i\to\psi$ in $C^0([0,T];L^2(\Omega))$ according to (1.24), that

$$u_i\to u \text{ strongly in } C^0([0,T];L^2(\Omega)), \tag{5.19}$$

as $i\to\infty$ and in particular $u\in C^0([0,T];L^2(\Omega))$. Because of $u_i(\cdot,0)=\tilde{g}_i(\cdot,0)\to\tilde{g}(\cdot,0)=g(\cdot,0)$ as $i\to\infty$, the convergence (5.19) implies, that u attains the prescribed initial values $u(\cdot,0)=g(\cdot,0)$.

Step 5: Proof of the extension property. For our next aim, we fix a domain $\mathcal{O}:=\tilde{\mathcal{O}}\cap\Omega$, where $\tilde{\mathcal{O}}\subset\mathbb{R}^n$ is any Lipschitz regular domain which is contained in a ball with radius $\rho_0=\rho_0(\theta,\omega(\cdot))\in(0,1]$ with $\theta\leq\theta_0=\theta_0(n,\gamma_1)\in(0,1)$ is given by Lemma 2.5. Moreover, we choose a time interval $I:=(t_1,t_2)\subset(0,T)$ and define $\mathcal{O}_I:=\mathcal{O}\times I$. Then, we consider the parabolic boundary value problems

$$\begin{cases}\partial_t w_i-\operatorname{div}a(z,Dw_i) &=\partial_t\psi_i-\operatorname{div}a(z,D\psi_i) \text{ in } \mathcal{O}_I,\\ \qquad\qquad w_i &=u_i \qquad\qquad\qquad\quad \text{on } \partial_{\mathcal{P}}\mathcal{O}_I,\end{cases} \tag{5.20}$$

where we defined w_i as the solutions to (5.20). By Corollary 3.4 we know that, there exists a solution $w_i \in W(\mathcal{O}_I)$. In addition, the comparison principle in Lemma 2.7 yields the obstacle constraint $w_i \geq \psi_i$ a.e. on $\mathcal{O}_I$. Furthermore, we have the energy bound (2.12) from the comparison Lemma 2.8. This yields

$$\int_{\mathcal{O}_I} |Dw_i|^{p(\cdot)} dz \leq c \int_{\mathcal{O}_I} (\mu + |Du_i|)^{p(\cdot)} dz + c \Bigg[\int_{\mathcal{O}_I} |\partial_t \psi_i|^{\gamma_1'} + |f|^{\gamma_1'} \, \mathrm{d}z + \int_{\mathcal{O}_I} |D\psi_i|^{p(\cdot)} + |(\delta_i^2 + |F_i|^2)^{\frac{p(\cdot)-2}{2}} F_i|^{p'(\cdot)} + 1 \, \mathrm{d}z \Bigg],$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L)$, for every $i \in \mathbb{N}$. Here, we have to mention that we replaced $|F|^{p(\cdot)}$ by $|(\delta_i^2 + |F_i|^2)^{\frac{p(\cdot)-2}{2}} F_i|^{\frac{p(\cdot)}{p(\cdot)-1}}$. This is possible, since we consider $(\delta_i^2 + |F_i|^2)^{\frac{p(\cdot)-2}{2}} F_i$ instead of $|F|^{p(\cdot)-2}F$. Later, by passing to the limit, we will see that we gain accurately (2.12). Combining this with the bound (5.8), the convergences of ψ_i in (1.24) and of F_i in (5.3), we infer

$$\limsup_{i\to\infty} \int_{\mathcal{O}_I} |Dw_i|^{p(\cdot)} \, \mathrm{d}z \leq cM \tag{5.21}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L)$. Next, we utilize the Poincaré type inequality (2.7), as follows

$$\|w_i\|_{L^{p(\cdot)}(\mathcal{O}_I)} \leq c \int_{\mathcal{O}_I} |Dw_i|^{p(\cdot)} + |Du_i|^{p(\cdot)} + |u_i|^{p(\cdot)} + \; 1 \; dz \; + \; c$$

for every $i \in \mathbb{N}$, where $c = c(n, \gamma_1, \gamma_2, L, L_1, M)$. This $L^{p(\cdot)}$-bound for w_i together with (1.9), (5.8), (5.9) and (5.10) imply that

$$\begin{aligned} \limsup_{i\to\infty} \|w_i\|_{L^{p(\cdot)}(\mathcal{O}_I)} &\leq c \limsup_{i\to\infty} \int_{\mathcal{O}_I} |Dw_i|^{p(\cdot)} + |Du_i|^{p(\cdot)} + |u_i|^{p(\cdot)} + 1 \, \mathrm{d}z \\ &\leq cM^{(\frac{2\gamma_2}{n+2}+1)\frac{\gamma_2}{\gamma_1}}, \end{aligned} \tag{5.22}$$

where the constant c depends on $n, \gamma_1, \gamma_2, \nu, L, L_1, \theta, M$. Finally, the equation (5.20) gives [cf. e.g. proof of (4.10)] the following

$$\begin{aligned} \limsup_{i\to\infty} \|\partial_t w_i\|_{W^{p(\cdot)}(\mathcal{O}_I)'} \leq c \limsup_{i\to\infty} \int_{\mathcal{O}_I} 1 + |Dw_i|^{p(\cdot)} + |D\psi_i|^{p(\cdot)} \, \mathrm{d}z \\ + c \limsup_{i\to\infty} \|\partial_t \psi_i\|_{L^{p'(\cdot)}(\mathcal{O}_I)} \leq \; c \end{aligned} \tag{5.23}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1, M, \|\partial_t \psi\|_{L^{p'(\cdot)}})$, where we used the fact $\partial_t \psi_i \in L^{\gamma_1'}(\mathcal{O}_I)$ implies $\partial_t \psi_i \in L^{p'(\cdot)}(\mathcal{O}_I)$. Notice also, that we gain from the energy bound (3.7) that w_i is bounded in $L^\infty(I; L^2(\mathcal{O}))$, i.e.

$$\sup_{t\in I} \int_{\mathcal{O}} |w_i(\cdot,t)|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}_I} |Dw_i|^{p(\cdot)} \, \mathrm{d}z \leq cM \tag{5.24}$$

for every $i \in \mathbb{N}$. Due to the bounds (5.21), (5.22), (5.23) and (5.24) and the compactness argument from Theorem 2.2, we can find a limit map $w \in W_u^{p(\cdot)}(\mathcal{O}_I)$ with $\partial_t w \in W^{p(\cdot)}(\mathcal{O}_I)'$, such that

$$\begin{cases} w_i \to w & \text{strongly in } L^{\hat{p}(\cdot)}(\mathcal{O}_I, \mathbb{R}), \\ Dw_i \to Dw & \text{weakly in } L^{p(\cdot)}(\mathcal{O}_I, \mathbb{R}^n), \\ \partial_t w_i \to \partial_t w & \text{weakly in } W^{p(\cdot)}(\mathcal{O}_I)', \\ w_i \rightharpoonup^* w & \text{weakly}^* \text{ in } L^\infty(I; L^2(\mathcal{O})), \end{cases} \tag{5.25}$$

as $i \to \infty$, possibly after extraction of a suitable subsequence. In addition, we may assume $w_i \to w$ and $\psi_i \to \psi$ a.e. on $\mathcal{O}_I$ as $i \to \infty$. Therefore, we can conclude that w satisfies the obstacle constraint $w \geq \psi$ a.e. on $\mathcal{O}_I$.

Now, we want to remove the smallness condition on $\mathcal{O}$, i.e. we want to consider also domains $\mathcal{O}$, which are not contained in a ball of radius $\rho_0 = \rho_0(\theta, \omega(\cdot)) \in (0,1]$. Therefore, we have to cover the domain $\mathcal{O}$ by smaller subdomains which allow the application of Corollary 3.4. Thus, we choose a family of disjoint open cubes $\{C_\rho(x_j)\}_{j=1}^\infty$ with $\rho \leq \rho_0$ and $x_j \in \mathbb{R}^n$, such that $\bigcup_{j=1}^\infty C_\rho(x_j) = \mathbb{R}^n \setminus N$, where

$\mathcal{L}^n(N) = 0$. Then, we consider $\mathcal{O}_j = \mathcal{O} \cap C_\rho(x_j) \subset \Omega_T$, $= 1, ..., N$, such that $\mathcal{O} = \bigcup_{j=1}^N \mathcal{O}_j \backslash N$. Then, we consider the parabolic boundary value problems

$$\begin{cases} \partial_t w_i^j - \operatorname{div} a(z, Dw_i^j) & = \partial_t \psi_i - \operatorname{div} a(z, D\psi_i) \text{ in } \mathcal{O}_j \times I, \\ \qquad\qquad w_i^j & = u_i \qquad\qquad\qquad\quad \text{on } \partial_{\mathcal{P}}\mathcal{O}_j \times I, \end{cases} \tag{5.26}$$

where we defined w_i^j as the solutions to (5.26). By Corollary 3.4 we know that, there exists a solution $w_i^j \in W(\mathcal{O}_j \times I)$. In addition, the comparison principle in Lemma 2.7 yields the obstacle constraint $w_i^j \geq \psi_i$ a.e. on $\mathcal{O}_j \times I$. At this stage, we have to mention that the conclusions from above are available. The next step is to compose the functions w_i^j. This is possible, since the boundary values of w_i^j are equal to the boundary data of its "'neighbors"'. Thus, we have $\breve{w}_i = \sum_j^N \mathbb{1}_{\mathcal{O}_j} w_i^j$, where $\mathbb{1}_{\mathcal{O}_j}$ is the characteristic function of the set $\mathcal{O}_j$. This allows us to infer from (5.21) that

$$\limsup_{i\to\infty} \int_{\mathcal{O}_I} |D\breve{w}_i|^{p(\cdot)} \,\mathrm{d}z \leq \limsup_{i\to\infty} \sum_j^N \int_{\mathcal{O}_j \times I} |Dw_i^j|^{p(\cdot)} \,\mathrm{d}z \leq c \sum_j^N \int_{\mathcal{O}_j \times I} (\mu + |Du|)^{p(\cdot)} \,\mathrm{d}z$$
$$+ c \left(\sum_j^N \int_{\mathcal{O}_j \times I} |\partial_t \psi|^{\gamma_1'} + |f|^{\gamma_1'} \,\mathrm{d}z + \sum_j^N \int_{\mathcal{O}_j \times I} |D\psi|^{p(\cdot)} + |F|^{p(\cdot)} + 1 \,\mathrm{d}z \right) \leq cM$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L)$. Similar to (5.22), (5.23) and (5.24), we get the same convergences as in (5.25), i.e. we can find a limit map $\breve{w} \in W_u^{p(\cdot)}(\mathcal{O}_I)$ with $\partial_t \breve{w} \in W^{p(\cdot)}(\mathcal{O}_I)'$, such that

$$\begin{cases} \breve{w}_i \to \breve{w} & \text{strongly in } L^{\hat{p}(\cdot)}(\mathcal{O}_I, \mathbb{R}), \\ D\breve{w}_i \to D\breve{w} & \text{weakly in } L^{p(\cdot)}(\mathcal{O}_I, \mathbb{R}^n), \\ \partial_t \breve{w}_i \to \partial_t \breve{w} & \text{weakly in } W^{p(\cdot)}(\mathcal{O}_I)', \\ \breve{w}_i \rightharpoonup^* \breve{w} & \text{weakly}^* \text{ in } L^\infty(I; L^2(\mathcal{O})), \end{cases}$$

as $i \to \infty$, possibly after extraction of a suitable subsequence. In addition, we may assume $\breve{w}_i \to \breve{w}$ and $\psi_i \to \psi$ a.e. on $\mathcal{O}_I$ as $i \to \infty$. Therefore, we can conclude that $\breve{w}$ satisfies the obstacle constraint $\breve{w} \geq \psi$ a.e. on $\mathcal{O}_I$.

From now on, we will use the preceding results [(5.21)-(5.25)] for any fix domain $\mathcal{O} := \tilde{\mathcal{O}} \cap \Omega$, where $\tilde{\mathcal{O}}$ is any Lipschitz regular domain. Here, we should accentuate that the function $\breve{w}$ is not the solution of (5.20) with the smallness condition on the domain $\mathcal{O}$. The function $\breve{w}$ is only on $\mathcal{O}_I$ (without smallness condition), which satisfies the obstacle constraint, the boundary data and the needed regularity properties. This is important for the construction of the extension map. Next, we relabel $\breve{w}$ by w and use the conclusions (5.21)-(5.25) for the function $\breve{w}_i$ from above on a domain $\mathcal{O}$ without the smallness assumption, where we also relabel $\breve{w}_i$ by w_i. But we keep in mind that the function w don't solve (5.20) on $\mathcal{O}_I$. Our next desired aim is to show that $w \in \mathcal{K}'_{\psi,u}(\mathcal{O}_I)$. Therefore, we have only to show that $w \in C^0([t_1, t_2]; L^2(\mathcal{O}))$. Unfortunately, this property we can not conclude from Lemma 2.1, since w does not vanish on $\partial\mathcal{O} \times (t_1, t_2)$. Thus, we apply Lemma 2.1 to the function $(1 - \zeta(x))\, w(x, t)$, where $\zeta \in C^\infty(\tilde{\mathcal{O}})$, $0 \leq \zeta \leq 1$ is a suitable cut-off function with $\zeta \equiv 1$ on $\partial\tilde{\mathcal{O}}$. Hence, $(1 - \zeta)\, w$ vanish on $\partial\tilde{\mathcal{O}} \times (t_1, t_2)$ and we can infer from Lemma 2.1 that $(1 - \zeta)w \in C^0([t_1, t_2]; L^2(\mathcal{O}))$. Here we should also mention, that in the case $\partial\Omega \cap \partial\mathcal{O} \neq \emptyset$, we apply Lemma 2.1 to $(1 - \zeta(x))(w(x, t) - g(x, t))$, which vanishes on $\partial\Omega \cap \partial\mathcal{O}$, and use the regularity assumptions on g. Next, we choose $\zeta_\varepsilon \in C^\infty(\tilde{\mathcal{O}})$, $0 \leq \zeta_\varepsilon \leq 1$ with

$$\zeta_\varepsilon \equiv \begin{cases} 0 & \text{on } \mathcal{O} \backslash U^\varepsilon, \\ 1 & \text{on } \partial\mathcal{O}, \end{cases} \quad \text{and } |D\zeta_\varepsilon| \leq \frac{c}{\varepsilon}, \tag{5.27}$$

where we defined $U^\varepsilon := \left\{ x \in \mathcal{O} : \operatorname{dist}(x, \partial\tilde{\mathcal{O}}) < \varepsilon \right\}$ for any $\varepsilon > 0$. Furthermore, we write $U_I^\varepsilon := U^\varepsilon \times (t_1, t_2) \subset \mathcal{O}_I$. Now, we will apply a version of the Poincaré type inequality (2.5) to $w_i - u_i \in C^0([t_1,t_2];L^2(\mathcal{O})) \cap W_0^{p(\cdot)}(\mathcal{O}_I)$. This is possible since we assumed that $\mathcal{O} = \tilde{\mathcal{O}} \cap \Omega$ with a Lipschitz regular domain $\tilde{\mathcal{O}} \subset \mathbb{R}^n$ and yields

$$\|w_i - u_i\|_{L^{p(\cdot)}(U_I^\varepsilon)} \le c \cdot \varepsilon \left[M^{\frac{2\gamma_2}{n+2}} \left(\int_{U_I^\varepsilon} |Dw_i - Du_i|^{p(\cdot)} + 1 \,\mathrm{d}z \right) \right]^{\frac{1}{\gamma_1}}, \tag{5.28}$$

where $c = c(n,\gamma_1,\gamma_2,\nu,L,L_1)$. Note that it is crucial that $\tilde{\mathcal{O}}$ is a Lipschitz regular domain in order to gain a factor ε in (5.28). Consequently, we can conclude from (5.27), (5.28) and (1.13) that, there holds

$$\begin{aligned} \|D(\zeta_\varepsilon^2(w_i - u_i))\|_{L^{p(\cdot)}(U_I^\varepsilon)} \le & 2\|\zeta_\varepsilon D\zeta_\varepsilon \cdot (w_i - u_i)\|_{L^{p(\cdot)}(U_I^\varepsilon)} \\ & + \|\zeta_\varepsilon^2 \cdot D(w_i - u_i)\|_{L^{p(\cdot)}(U_I^\varepsilon)} \le \; cM^{(\frac{2\gamma_2}{n+2}+1)\frac{1}{\gamma_1}} \end{aligned} \tag{5.29}$$

for sufficiently large $i \in \mathbb{N}$, where we used (5.8), (5.21) and Young's inequality for the last step. Next, we choose a comparison function $v := u_i + (w_i - u_i)\zeta_\varepsilon^2(x)\mathbb{1}_{(t_1,\tau)}(t) \in W_{u_i}^{p(\cdot)}(\Omega_T)$ for an arbitrary time $\tau \in (t_1,t_2)$. Note that, v holds the obstacle constraint $v \ge \psi_i$ a.e. on $\mathcal{O}_I$, since $w_i \ge \psi_i$ and $u_i \ge \psi_i$ a.e. on $\mathcal{O}_I$. Now, we test (5.5) with v and infer

$$\begin{aligned} &-\langle\!\langle \partial_t u_i, \zeta_\varepsilon^2(w_i - u_i)\rangle\!\rangle_{\mathcal{O}_I} - \int_{\mathcal{O}_I} a(z,Du_i) \cdot D(\zeta_\varepsilon^2(w_i - u_i)) \,\mathrm{d}z \\ &\le cM^{(\frac{2\gamma_2}{n+2}+1)\frac{1}{\gamma_1}} \left(\|f\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|(\delta_i^2 + |F_i|^2)^{\frac{p(\cdot)-2}{2}} F_i\|_{L^{p'(\cdot)}(U_I^\varepsilon)} \right) \end{aligned}$$

with a constant $c = c(n,\gamma_1,\gamma_2,\nu,L,L_1)$. Here, we used the generalized Hölder's inequality (1.8) and finally, (5.8), (5.21), $\zeta_\varepsilon^2 \le 1$, (5.28) and (5.29). In addition, we test the weak formulation of (5.20) with the test-function $(w_i - u_i)\zeta_\varepsilon^2(x)\mathbb{1}_{(t_1,\tau)}(t) \in W_0^{p(\cdot)}(U_I^\varepsilon)$, which yields

$$\begin{aligned} \langle\!\langle \partial_t w_i, \zeta_\varepsilon^2(w_i - u_i)\rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} a(z,Dw_i) D(\zeta_\varepsilon^2(w_i - u_i)) \; dz &= \langle\!\langle \partial_t \psi_i, (\zeta_\varepsilon^2(w_i - u_i))\rangle\!\rangle_{\mathcal{O}_I} \\ &\quad + \int_{\mathcal{O}_I} a(z,D\psi_i) \cdot D(\zeta_\varepsilon^2(w_i - u_i)) \; dz \\ &\le \; cM^{(\frac{2\gamma_2}{n+2}+1)\frac{1}{\gamma_1}} \left(\|a(z,D\psi_i)\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|\partial_t \psi_i\|_{L^{p'(\cdot)}(U_I^\varepsilon)} \right) \end{aligned}$$

with a constant $c = c(n,\gamma_1,\gamma_2,\nu,L,L_1)$. Here, we employed the generalized Hölder's inequality (1.8), the Poincaré type inequality (2.5), (5.8), (5.21) and (5.29). Adding the last two inequalities, using Proposition 1.2(iv), we arrive

$$\langle\!\langle \partial_t(\zeta_\varepsilon(w_i - u_i)), \zeta_\varepsilon(w_i - u_i)\rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} (a(z,Dw_i) - a(z,Du_i)) D(\zeta_\varepsilon^2(w_i - u_i)) \,\mathrm{d}z \le \Xi$$

for every $i \in \mathbb{N}$, every $\tau \in (t_1,t_2)$ and $\varepsilon > 0$, where

$$\begin{aligned} \Xi := cM^{(\frac{2\gamma_2}{n+2}+1)\frac{1}{\gamma_1}} \Big(& \|a(z,D\psi_i)\|_{L^{p'(\cdot)}} + \|\partial_t \psi_i\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|f\|_{L^{p'(\cdot)}(U_I^\varepsilon)} \\ & + \|(\delta_i^2 + |F_i|^2)^{\frac{p(\cdot)-2}{2}} F_i\|_{L^{p'(\cdot)}(U_I^\varepsilon)} \Big). \end{aligned} \tag{5.30}$$

From this estimate, we can conclude the following inequality

$$\begin{aligned} &\langle\!\langle \partial_t(\zeta_\varepsilon(w_i - u_i)), \zeta_\varepsilon(w_i - u_i)\rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} (a(z,Dw_i) - a(z,Du_i)) \cdot \zeta_\varepsilon^2 D(w_i - u_i) \; dz \\ &\le \Xi - \int_{\mathcal{O}_I} (a(z,Dw_i) - a(z,Du_i)) \cdot 2\zeta_\varepsilon D\zeta_\varepsilon (w_i - u_i) \; dz \\ &\le \Xi + \frac{c}{\varepsilon} \left(\|a(z,Dw_i)\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|a(z,Du_i)\|_{L^{p'(\cdot)}(U_I^\varepsilon)} \right) \|w_i - u_i\|_{L^{p(\cdot)}(U_I^\varepsilon)} \end{aligned}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L)$. Finally, we utilize Lemma 2.1 and the monotonicity condition (1.3) to the left-hand side and the Young's inequality with exponents γ_1 and γ_1' and (1.9) to the right-hand side, then we have

$$\sup_{\tau\in(t_1,t_2)} \int_{\mathcal{O}\times\{\tau\}} |\zeta_\varepsilon(w_i - u_i)|^2 \, dx \le \frac{c}{\varepsilon^{\gamma_1}} \|w_i - u_i\|^{\gamma_1}_{L^{p(\cdot)}(U_I^\varepsilon)} + \Xi$$
$$+ c\left(\|a(z, Dw_i)\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|a(z, Du_i)\|_{L^{p'(\cdot)}(U_I^\varepsilon)}\right)^{\gamma_1'}$$

for $i \in \mathbb{N}$, where Ξ is defined in (5.30). Now, we use the growth condition (1.2) and pass to the limit $i \to \infty$, apply Fatou's Lemma to the left-hand side and employ (5.28) and the convergences (1.24), (5.2), (5.17) and (5.25) on the right-hand side. Then, we can conclude

$$\begin{aligned}
&\sup_{\tau\in(t_1,t_2)} \int_{\mathcal{O}\times\{\tau\}} |\zeta_\varepsilon(w - u)|^2 \, dx \le cM^{\frac{2\gamma_2}{n+2}} \int_{U_I^\varepsilon} |Dw - Du|^{p(\cdot)} + 1 \, dz \\
&+ cM\left(\|(1 + |D\psi|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|\partial_t\psi\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|f\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|F\|_{L^{p(\cdot)}(U_I^\varepsilon)}\right) \\
&+ c\left(\|(1 + |Dw|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(U_I^\varepsilon)} + \|(1 + |Du|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(U_I^\varepsilon)}\right)^{\gamma_1'} =: I_\varepsilon + II_\varepsilon + III_\varepsilon
\end{aligned} \tag{5.31}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L)$. Finally, we want to shows that the right-hand side can be made arbitrarily small by choosing $\varepsilon > 0$ small enough. This allows us to conclude that by the absolute continuity of the integral the expressions I_ε, II_ε and III_ε tend to zero as $\varepsilon \downarrow 0$. Therefore, we get from (5.31) the following

$$\sup_{\tau\in(t_1,t_2)} \int_{\mathcal{O}\times\{\tau\}} |\zeta_\varepsilon(w - u)|^2 \, dx \to 0 \text{ as } \varepsilon \downarrow 0$$

and since, we already know that $u \in C^0([0, T], L^2(\Omega))$, we are able to deduce that

$$\sup_{\tau\in(t_1,t_2)} \int_{\mathcal{O}\times\{\tau\}} |\zeta_\varepsilon w|^2 \, dx \to 0 \text{ as } \varepsilon \downarrow 0.$$

Moreover, as mentioned above, there holds $(1 - \zeta_\varepsilon)w \in C^0([0, T]; L^2(\Omega))$ as a consequence of the Interpolation Lemma 2.1. In view of the above convergence we deduce $w \in C^0([t_1, t_2]; L^2(\mathcal{O}))$ and thus conclude the proof of the claimed property $w \in \mathcal{K}'_{\psi,u}(\mathcal{O}_I)$ of the extension map.

Step 6: Variational inequality for the limit map. First, we fix a Lipschitz regular domain $\tilde{\mathcal{O}} \subset \mathbb{R}^n$ contained in a ball of radius $\rho_0 = \rho_0(\theta, \omega(\cdot))$ with $\theta \le \theta_0(n, \gamma_1) \in (0, 1)$ from Lemma 2.5 and a time interval $I = (t_1, t_2) \subset (0, T)$ and abbreviate $\mathcal{O} := \tilde{\mathcal{O}} \cap \Omega$. As always, $\mathcal{O}_I$ denotes the space-time cylinder $\mathcal{O} \times I$. Next, we choose an arbitrary comparison map $v \in \mathcal{K}'_{\psi,u}(\mathcal{O}_I)$, which exists by Step 5. Furthermore, we choose a cut-off function $\zeta \in C_0^\infty(\tilde{\mathcal{O}})$, $0 \le \zeta \le 1$, which will be specified later and define an admissible comparison function in the variational inequality (5.5) for u_i. In addition, we let $v_i := \zeta^2(v - \psi + \psi_i) + (1 - \zeta^2)u_i \in W^{p(\cdot)}_{u_i}(\mathcal{O}_I)$ for $i \in \mathbb{N}$, and we extend the function by u_i on $\Omega_T \setminus \mathcal{O}_I$. This function satisfies the obstacle condition $v_i \ge \psi_i$, since it is a convex combination of the functions $v - \psi + \psi_i$ and u_i, where both satisfy the same obstacle constraint. Thus, v_i is the desired function. For this reason, we have the following variational inequality

$$\langle\!\langle \partial_t u_i, v_i - u_i \rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} a(z, Du_i) \cdot D(v_i - u_i) \, \mathrm{d}z \ge \int_{\mathcal{O}_I} f_i \cdot (v_i - u_i) \, \mathrm{d}z \tag{5.32}$$

for $i \in \mathbb{N}$. Utilizing the definition of v_i, the first integral can be rewritten by Proposition 1.2 as follows:

$$\begin{aligned}
\langle\!\langle \partial_t u_i, v_i - u_i \rangle\!\rangle_{\mathcal{O}_I} &= \langle\!\langle \partial_t(v - \psi + \psi_i), v_i - u_i \rangle\!\rangle_{\mathcal{O}_I} \\
&\quad - \langle\!\langle \partial_t(\zeta(v - \psi + \psi_i - u_i)), \zeta(v - \psi + \psi_i - u_i) \rangle\!\rangle_{\mathcal{O}_I} \\
&=: I_i - II_i.
\end{aligned} \tag{5.33}$$

In order to calculate the limit of the first term, we observe

$$v_i - u_i = \zeta^2(v - \psi + \psi_i - u_i) \overset{i\to\infty}{\longrightarrow} \zeta^2(v-u) \text{ in } W_0^{p(\cdot)}(\mathcal{O}_I) \tag{5.34}$$

because of (1.24) and (5.17). Combining this with the strong convergence $\partial_t\psi_i \to \partial_t\psi$ in $L^{\gamma_1'}(\mathcal{O}_I)$ according to (1.24) and (5.19), we can conclude

$$\lim_{i\to\infty} I_i = \langle\!\langle \partial_t v, \zeta^2(v-u)\rangle\!\rangle_{\mathcal{O}_I} \tag{5.35}$$

Next, we use Lemma 2.1 and then the convergence $\psi_i \to \psi$ and $u_i \to u$ in $L^\infty(0,T;L^2(\Omega))$ according to (1.24) and (5.19), with the result

$$\begin{aligned}-II_i =&\frac{1}{2}\|\zeta(v-\psi+\psi_i-u_i)(\cdot,t_1)\|^2_{L^2(\mathcal{O})} - \frac{1}{2}\|\zeta(v-\psi+\psi_i-u_i)(\cdot,t_2)\|^2_{L^2(\mathcal{O})}\\ \le&\frac{1}{2}\|\zeta(v-\psi+\psi_i-u_i)(\cdot,t_1)\|^2_{L^2(\mathcal{O})} \longrightarrow \frac{1}{2}\|\zeta(v-u)(\cdot,t_1)\|^2_{L^2(\mathcal{O})}\end{aligned} \tag{5.36}$$

as $i\to\infty$. Now, we use (5.35) and (5.36) while passing to the limits in (5.33), so we have

$$\limsup_{i\to\infty} \langle\!\langle \partial_t u_i, v_i - u_i\rangle\!\rangle_{\mathcal{O}_I} \le \langle\!\langle \partial_t v, \zeta^2(v-u)\rangle\!\rangle_{\mathcal{O}_I} + \frac{1}{2}\|\zeta(v-u)(\cdot,t_1)\|^2_{L^2(\mathcal{O})}.$$

Combining this with the variational inequality (5.32) and applying the convergences $a(\cdot, Du_i) \to a(\cdot, Du)$ for a.e. $z\in\Omega_T$ by (5.18), $f_i \to f - \operatorname{div}(|F|^{p(\cdot)-2}F)$ by (5.2) and the convergence (5.34) of $v_i - u_i$. Hence, we can conclude that

$$\begin{aligned}\langle\!\langle \partial_t v, \zeta^2(v-u)\rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} a(z,Du)\cdot D(\zeta^2(v-u))\,\mathrm{d}z \;+\; \frac{1}{2}\|\zeta(v-u)(\cdot,t_1)\|^2_{L^2(\mathcal{O})}\\ \ge \int_{\mathcal{O}_I} f\zeta^2(v-u) + |F|^{p(\cdot)-2}F\cdot D(\zeta^2(v-u))\,\mathrm{d}z.\end{aligned} \tag{5.37}$$

Finally, we choose cut-off functions $\zeta_\varepsilon \in C_0^\infty(\tilde{\mathcal{O}})$, $0\le\zeta_\varepsilon\le 1$ with $\zeta_\varepsilon\equiv 1$ on the set $\tilde{\mathcal{O}}^\varepsilon := \left\{x\in\tilde{\mathcal{O}} : \operatorname{dist}(x,\partial\tilde{\mathcal{O}}) > \varepsilon\right\}$ for every $\varepsilon > 0$. This can be done in such a way that $|D\zeta_\varepsilon| \le \frac{2}{\varepsilon}$ holds for every $\varepsilon > 0$. Since we have assumed, that $\tilde{\mathcal{O}}$ is a Lipschitz regular domain, get in the same fashion of (5.28) that every $\varphi \in C^0([t_1,t_2];L^2(\mathcal{O}))\cap W_0^{p(\cdot)}(\mathcal{O}_I)$ satisfies the following version of the Poincaré type inequality (2.4):

$$\int_{(\mathcal{O}\setminus\tilde{\mathcal{O}}^\varepsilon)\times I} |\varphi|^{p(\cdot)}\,\mathrm{d}z \le c\varepsilon^{\gamma_1}\left(\sup_{t_1\le t\le t_2}\|\varphi(\cdot,t)\|^{\frac{4\gamma_2}{n+2}}_{L^2(\mathcal{O}\setminus\tilde{\mathcal{O}}^\varepsilon)} + 1\right)\int_{(\mathcal{O}\setminus\tilde{\mathcal{O}}^\varepsilon)\times I} |D\varphi|^{p(\cdot)} + 1\,\mathrm{d}z$$

for any $\varepsilon\in(0,1]$, where $c = c(n,\gamma_1,\gamma_2,\nu,L,L_1)$. This implies in particular, since $v \in W_u^{p(\cdot)}(\mathcal{O}_I)$, that there holds

$$\begin{aligned}\|D\left[\left(1-\zeta_\varepsilon^2\right)(v-u)\right]\|_{L^{p(\cdot)}((\mathcal{O}\setminus\tilde{\mathcal{O}}^\varepsilon)\times I)} \le& c\varepsilon^{\gamma_1}\left(\sup_{t_1\le t\le t_2}\|(v-u)(\cdot,t)\|^{\frac{2\gamma_2}{n+2}}_{L^2(\mathcal{O}\setminus\tilde{\mathcal{O}}^\varepsilon)} + 1\right)\\ &\times\int_{(\mathcal{O}\setminus\tilde{\mathcal{O}}^\varepsilon)\times I} |D(v-u)|^{p(\cdot)} + 1\,\mathrm{d}z \to 0\end{aligned}$$

as $\varepsilon\downarrow 0$ and consequently, $\zeta_\varepsilon^2(v-u)\to v-u$ in $W_0^{p(\cdot)}(\mathcal{O}_I)$. Choosing $\zeta = \zeta_\varepsilon$ in (5.37) and letting $\varepsilon\downarrow 0$, we can conclude

$$\begin{aligned}\langle\!\langle \partial_t v, v-u\rangle\!\rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} a(z,Du)\cdot D(v-u)\,\mathrm{d}z \;+\; \frac{1}{2}\|(v-u)(\cdot,t_1)\|^2_{L^2(\mathcal{O})}\\ \ge \int_{\mathcal{O}_I} f(v-u) + |F|^{p(\cdot)-2}F\cdot D(v-u)\,\mathrm{d}z,\end{aligned}$$

which is the desired local version of the variational inequality.

Step 7: Uniqueness. For the proof of uniqueness, we consider an arbitrary solution $u_* \in \mathcal{K}_{\psi,g}(\Omega_T)$ with $u_*(\cdot,0) = g(\cdot,0)$ of the variational inequality

$$\langle\!\langle \partial_t v, v - u_* \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_*) \cdot D(v - u_*)\, \mathrm{d}z \;+\; \frac{1}{2}\|v(\cdot,0) - g(\cdot,0)\|^2_{L^2(\Omega)}$$
$$\geq \int_{\Omega_T} f(v - u_*) + |F|^{p(\cdot)-2} F \cdot D(v - u_*)\, \mathrm{d}z$$

for all comparison maps $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$, and wish to show that u_* agrees with the solution u from above. Next, we use the abbreviations $\breve{u}_i := u_i - \psi_i$ and introduce the analogous notation $\breve{u}_* := u_* - \psi$. These functions satisfy the initial and boundary conditions $\breve{u}_i = \tilde{g} - \psi = \breve{u}_*$ on $\partial_{\mathcal{P}}\Omega_T$. Now, we may choose $v := u_i - \psi_i + \psi = \breve{u}_i - \breve{u}_* + u_*$ as comparison function in the above variational inequality, which gives

$$\langle\!\langle \partial_t u_i - \partial_t \psi_i + \partial_t \psi, \breve{u}_i - \breve{u}_* \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_*) \cdot D(\breve{u}_i - \breve{u}_*)\, \mathrm{d}z$$
$$\geq \int_{\Omega_T} f(\breve{u}_i - \breve{u}_*) + |F|^{p(\cdot)-2} F \cdot D(\breve{u}_i - \breve{u}_*)\, \mathrm{d}z.$$

Obversely, we plug $v_i := u_* - \psi - \psi_i = \breve{u}_* - \breve{u}_i + u_i$ into the variational inequality (5.5) for u_i, with the result

$$\langle\!\langle \partial_t u_i, \breve{u}_* - \breve{u}_i \rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_i) \cdot D(\breve{u}_* - \breve{u}_i)\, \mathrm{d}z \geq \int_{\Omega_T} f_i \cdot (\breve{u}_* - \breve{u}_i)\, \mathrm{d}z.$$

We point out, that for the last step, it is crucial that (5.5) holds for every comparison map $v \in W^{p(\cdot)}_{u_i}(\Omega_T)$ with $v \geq \psi_i$. Subtracting the two preceding inequalities, we have

$$\int_{\Omega_T} (a(z, Du_*) - a(z, Du_i)) \cdot D(\breve{u}_* - \breve{u}_i)\, \mathrm{d}z \leq \langle\!\langle \partial_t \psi - \partial_t \psi_i, \breve{u}_i - \breve{u}_* \rangle\!\rangle_{\Omega_T}$$
$$+ \int_{\Omega_T} |F|^{p(\cdot)-2} F \cdot D(\breve{u}_* - \breve{u}_i) + (f - f_i)(\breve{u}_* - \breve{u}_i)\, \mathrm{d}z.$$

Because of the strong convergence $f_i \to f - \operatorname{div}(|F|^{p(\cdot)-2}F)$ in $L^{\gamma_1'}(\Omega_T)$ according to (5.2) and since the sequence $\{\breve{u}_i\}_{i\in\mathbb{N}}$ is bounded in $W^{p(\cdot)}(\Omega_T)$, the last integral vanishes in the limit $i \to \infty$. Analogously, the first integral on the right-hand side vanishes, since $\partial_t \psi_i \to \partial_t \psi$ strongly in $L^{\gamma_1'}(\Omega_T)$, as $i \to \infty$. Consequently, the preceding inequality implies

$$\limsup_{i\to\infty} \int_{\Omega_T} (a(z, Du_*) - a(z, Du_i)) \cdot D(\breve{u}_* - \breve{u}_i)\, \mathrm{d}z \leq 0,$$

and recalling the definition of $\breve{u}_*$ and $\breve{u}_i$, the strong convergences (5.17) and (5.18) and the strong convergence $D\psi_i \to D\psi$ in $L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)$, this implies

$$\int_{\Omega_T} (a(z, Du_*) - a(z, Du)) \cdot D(u_* - u)\, \mathrm{d}z \leq 0.$$

But in the view of the monotonicity (1.3) of $a(\cdot)$, this can only hold if $Du_* = Du$, and since u_* agrees with u on the lateral boundary of Ω_T, we have the desired identity $u_* = u$. This completes the proof of the theorem. □

6. Proof of Theorem 1.8: Existence result to degenerate parabolic obstacle problems on irregular domains

Finally, we consider general bounded domains and general obstacle functions ψ. Since in this general situation, we can approximate ψ only locally by functions with better regularity and integrability properties, we can show strong convergence to a solution only on every compactly contained subdomain $\Omega' \Subset \Omega$. More precisely, we give the

Proof of Theorem 1.8. First, we may replace the boundary data g by a function $\hat{g} \in C^0([0,T]; L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T)$ with $\partial_t \hat{g} \in L^{\gamma_1'}(\Omega_T)$, which satisfies the obstacle constraint $\hat{g} \geq \psi$ a.e. on Ω_T and attains the initial values assumption. Therefore, we define $\hat{g} := \max\{g, \psi\}$.

Step 1: Regularization. In the general case, that is without any regularity condition on the boundary of the domain Ω, we can only approximate the obstacle function locally by more regular functions. For this regularization, we follow classical ideas by Meyers and Serrin [40], see also [43, 44]. Therefore, we define a countable open cover of Ω by letting $U_\ell := \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \in \left(\frac{1}{\ell+1}, \frac{1}{\ell-1}\right)\right\}$ for every $\ell \in \mathbb{N}$ and choose a partition of unity $\{\zeta_\ell\}_{\ell\in\mathbb{N}} \subset C_0^\infty(U_\ell)$, where $0 \leq \zeta_\ell \leq 1$, subordinate to the cover $\{U_\ell\}_{\ell\in\mathbb{N}}$. By $\phi \in C_0^\infty(B_1)$, we denote a standard, radially symmetric smoothing kernel with $\int_{\mathbb{R}^n} \phi \,\mathrm{d}x = 1$ and we write $\phi_\rho(x) := \rho^{-n}\phi(x/\rho)$ for the rescaled versions. Then, we define $\psi_k^{(\ell)}(\cdot,t) := [\zeta_\ell \psi(\cdot,t)] * \phi_{\rho_{k,\ell}}$ for all $t \in (0,T)$ and $k, \ell \in \mathbb{N}$, where the radii $\rho_{k,\ell} \in \left(0, \frac{1}{\ell+1}\right)$ are chosen so small, such that

$$\begin{aligned} \|\psi_k^{(\ell)} - \zeta_\ell\psi\|_{W^{p(\cdot)}(\Omega_T)} &+ \|\psi_k^{(\ell)} - \zeta_\ell\psi\|_{L^\infty(0,T;L^2(\Omega))} \\ &+ \|\partial_t \psi_k^{(\ell)} - \partial_t \zeta_\ell \psi\|_{L^{\gamma_1'}(\Omega_T)} \leq \frac{1}{k2^\ell}. \end{aligned} \tag{6.1}$$

Then, let $\psi_k := \sum_{\ell\in\mathbb{N}} \psi_k^{(\ell)}$. By the choice of the smoothing radii in (6.1) and since $\{\zeta_\ell\}_{\ell\in\mathbb{N}}$ is a partition of unity, we have

$$\|\psi_k - \psi\|_{W^{p(\cdot)}(\Omega_T)} \leq \sum_{\ell\in\mathbb{N}} \|\psi_k^{(\ell)} - \zeta_\ell\psi\|_{W^{p(\cdot)}(\Omega_T)} \leq \frac{1}{k}$$

and similarly, $\|\psi_k - \psi\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{1}{k}$ and $\|\partial_t\psi_k - \partial_t\psi\|_{L^{\gamma_1'}(\Omega_T)} \leq \frac{1}{k}$. We have thereby shown

$$\begin{cases} \psi_k \to \psi & \text{in } W^{p(\cdot)}(\Omega_T) \text{ and in } L^\infty(0,T;L^2(\Omega)), \\ \partial_t\psi_k \to \partial_t\psi & \text{in } L^{\gamma_1'}(\Omega_T), \end{cases} \tag{6.2}$$

as $k \to \infty$. Moreover, the regularized obstacle functions $\partial_t\psi_k \in L^{\gamma_1'}(\Omega_T)$ satisfy obviously $\partial_t\psi_k \in L^{\gamma_1'}(\Omega_T')$ for every subdomain $\Omega' \Subset \Omega$ and

$$\sup_{\Omega_T'}(|D^2\psi_k| + |D\psi_k|) \leq c(n, \Omega, \Omega', \ell, k, \psi) < \infty, \tag{6.3}$$

since $\psi \in C^0([0,T]; L^2(\Omega))$. Next, we define boundary data adapted to the regularized obstacle functions by letting $\hat{g}_k := \hat{g} - \psi + \psi_k$ for all $k \in \mathbb{N}$. This defines functions that satisfy the obstacle constraint $\hat{g}_k \geq \psi_k$ a.e. on Ω_T, since $\hat{g} \geq \psi$ a.e. on Ω_T, and they converge to $\hat{g}$ in the sense

$$\begin{cases} \hat{g}_k \to \hat{g} & \text{in } W^{p(\cdot)}(\Omega_T) \text{ and in } L^\infty(0,T;L^2(\Omega)), \\ \partial_t\hat{g}_k \to \partial_t\hat{g} & \text{in } L^{\gamma_1'}(\Omega_T), \end{cases} \tag{6.4}$$

as $k \to \infty$. Moreover, we extended f and F by zero outside of Ω_T and define mollifications $F_k(\cdot,t) := F(\cdot,t) * \phi_{\delta_k}$ for every $t \in (0,T)$, with an arbitrary sequence $\delta_k \downarrow 0$, and let $f_k(\cdot,t) := f(\cdot,t) - \operatorname{div}\left((\delta_k^2 + |F_k(\cdot,t)|^2)^{\frac{p(\cdot)-2}{2}} F_k(\cdot,t)\right)$. Thus $f_k \in L^{\gamma'}(\Omega_T)$ and these functions converge in the sense

$$f_k \to f - \operatorname{div}(|F|^{p(\cdot)-2}F) \text{ strongly in } W^{p(\cdot)}(\Omega_T)', \tag{6.5}$$

as $k \to \infty$. Moreover, by standard results on mollifications, we have the convergence

$$F_k(\cdot,t) \to F \text{ strongly in } L^{p(\cdot)}(\Omega_T, \mathbb{R}^n), \tag{6.6}$$

since $\delta_i \downarrow 0$, as $i \to \infty$ and therefore, $(\delta_i^2 + |F_k|^2)^{\frac{p(\cdot)-2}{2}} F_k \to |F|^{p(\cdot)-2}F$ strongly in $L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$. Moreover, since $|D\psi_k|$, $|D^2\psi_k| \in L^\infty(0,T; L^\infty(\Omega'))$ for every $\Omega' \Subset \Omega$ according to (6.3), we can conclude that

$$|\operatorname{div} a(\cdot, D\psi_k)| \in L^\infty(0,T; L^\infty(\Omega')) \text{ for every } \Omega' \Subset \Omega, \tag{6.7}$$

cf. the approach of (5.4).

Step 2: Construction of the solution. Next, we will construct the solution as the limit of a regularized variational inequality. Therefore, notice that the properties $\partial_t \psi_k \in L^{\gamma_1'}(\Omega_T')$ and (6.7) permit us to conclude the strong existence Lemma 4.1 on the subsets $\Omega_T^k := \Omega^k \times (0,T)$, where $\Omega^k := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta_k\}$. More precisely, we will construct the solution as the limit of solution as the limit of solutions to the regularized variational inequalities

$$\langle\!\langle \partial_t u_k, v - u_k \rangle\!\rangle_{\Omega_T^k} + \int_{\Omega_T^k} a(z, Du_k) \cdot D(v - u_k)\, \mathrm{d}z \;\geq\; \int_{\Omega_T^k} f_k (v - u_k)\, \mathrm{d}z \tag{6.8}$$

for every $v \in W^{p(\cdot)}_{\hat{g}_k}(\Omega_T)$ with $v \geq \psi_k$ a.e. on Ω_T^k, since $\mathcal{K}'_{\psi_k,\hat{g}_k}(\Omega_T^k)$ is dense in $\mathcal{K}_{\psi_k,\hat{g}_k}(\Omega_T^k)$ and where we prescribe the initial and boundary values

$$u_k = \hat{g}_k \text{ on } \partial_{\mathcal{P}} \Omega_T^k. \tag{6.9}$$

Due to $f_k, \partial_t \psi_k \in L^{\gamma_1'}(\Omega_T^k)$ and (6.7), we can utilize Lemma 4.1 in order to find solutions $u_k \in C^0([0,T]; L^2(\Omega^k)) \cap W^{p(\cdot)}(\Omega_T^k)$ with $\partial_t u_k \in W^{p(\cdot)}(\Omega_T^k)$, satisfying $u_k \geq \psi_k$ a.e. on Ω_T^k, (6.8) and (6.9) for every $k \in \mathbb{N}$. Notice that by Lemma 4.2, the solution also satisfies the weak formulation of the variational inequality, that is

$$\begin{aligned}\langle\!\langle \partial_t v, v - u_k \rangle\!\rangle_{\Omega_T^k} + \int_{\Omega_T^k} a(z, Du_k) \cdot D(v - u_k)\, \mathrm{d}z + \frac{1}{2}\|v(\cdot,0) - g(\cdot,0)\|^2_{L^2(\Omega^k)} \\ \geq \int_{\Omega_T^k} f_k (v - u_k)\, \mathrm{d}z\end{aligned} \tag{6.10}$$

for all comparison functions $v \in \mathcal{K}'_{\psi_k,\hat{g}_k}(\Omega_T^k)$. Similar to Step 2 of Theorem 1.7, we get the following energy estimate

$$\sup_{t\in(0,T)} \int_{\Omega^k} |u_k(\cdot,t)|^2\, \mathrm{d}x \;+ \int_{\Omega_T^k} |Du_k|^{p(\cdot)}\, \mathrm{d}z \;\leq\; cM, \tag{6.11}$$

for all sufficiently large $k \in \mathbb{N}$ with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$ and an uniform $L^{p(\cdot)}$-bound $\|u_k\|_{L^{p(\cdot)}(\Omega_T^k)} \leq c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega), M)$. Recalling (6.4), we deduce from the above estimates that the extended functions

$$\tilde{u}_k := \begin{cases} u_k & \text{on } \Omega_T^k \\ \hat{g}_k & \text{on } \Omega \backslash \Omega_T^k \end{cases}$$

are uniformly bounded in $L^\infty(0,T; L^2(\Omega))$ and in $W^{p(\cdot)}(\Omega_T)$. Therefore, we may find a function $u \in L^\infty(0,T; L^2(\Omega)) \cap W^{p(\cdot)}_{\hat{g}}(\Omega_T)$, such that - possibly after passing to a subsequence - there holds

$$\begin{cases} \tilde{u}_k \rightharpoonup u & \text{weakly in } L^{p(\cdot)}(\Omega_T), \\ D\tilde{u}_k \rightharpoonup Du & \text{weakly in } L^{p(\cdot)}(\Omega_T, \mathbb{R}^n), \\ \tilde{u}_k \rightharpoonup^* u & \text{weakly in } L^\infty(0,T; L^2(\Omega)), \end{cases} \tag{6.12}$$

as $k \to \infty$. Finally, we can infer - with respect to the weak convergence - that

$$\sup_{t\in(0,T)} \int_{\Omega} |u(\cdot,t)|^2\, \mathrm{d}x \;+ \int_{\Omega_T} |Du|^{p(\cdot)}\, \mathrm{d}z \;\leq\; cM,$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, \operatorname{diam}(\Omega))$. This implies the claimed estimate (1.28). Furthermore, the growth assumption (1.2) of $a(z,\cdot)$ and (6.11) imply that the sequence $\{a(z, D\tilde{u}_k)\}_{k\in\mathbb{N}}$ is bounded in $L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$. Consequently, after passing to a subsequence once more, we can find a limit map $A_0 \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$ with

$$a(z, D\tilde{u}_k) \rightharpoonup A_0 \text{ weakly in } L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n) \text{ as } k \to \infty. \tag{6.13}$$

Step 3: Construction of extensions and convergence of boundary values. Notice that this step is similar to the first part of Step 5 in the proof of Theorem 1.7 - see page 26 - with the difference that at this stage, we are not yet able to show the second part, even that the extensions satisfy $w \in C^0([0,T]; L^2(\mathcal{O}))$.

This will be proved in the last step of this proof. For this aim we observe a subdomains $\mathcal{O} \Subset \Omega$, which are compactly contained in Ω and contained in a ball with radius $\rho_0 = \rho_0(\theta, \omega(\cdot)) \in (0,1]$ with $\theta \le \theta_0 = \theta_0(n, \gamma_1) \in (0,1)$ [cf. Lemma 2.5], while the times $t_1 < t_2$ can be arbitrary with $t_1, t_2 \in (0,T)$. Moreover, we define $\mathcal{O}_I := \mathcal{O} \times I$. Then, we observe the boundary value problems

$$\begin{cases} \partial_t w_k - \operatorname{div} a(z, Dw_k) &= \partial_t \psi_k - \operatorname{div} a(z, D\psi_k) \text{ in } W^{p(\cdot)}(\mathcal{O}_I)' \\ w_k &= u_k \text{ on } \partial\mathcal{O} \times I \\ w_k(\cdot, t_1) &= g(\cdot, 0) \text{ on } \mathcal{O} \times \{t_1\}, \end{cases} \tag{6.14}$$

where we defined w_k as the solutions to (6.14). By Corollary 3.4 we know that, there exists a solution $w_k \in C^0(I; L^2(\mathcal{O})) \cap W^{p(\cdot)}(\mathcal{O}_I)$ with $\partial_t w_k \in W^{p(\cdot)}(\mathcal{O}_I)'$. In addition, the comparison principle in Lemma 2.8 yields the obstacle constraint $w_k \ge \psi_k$ a.e. on $\mathcal{O}_I$. Furthermore, we have the energy estimate (2.11)

$$\begin{aligned} \int_{\mathcal{O}_I} |Dw_k - Du_k|^{p(\cdot)} \,\mathrm{d}z &\le \delta \int_{\mathcal{O}_I} (\mu + |Du_k|)^{p(\cdot)} \,\mathrm{d}z + c_\delta (\|\partial_t \psi_k\|_{L^{\gamma_1'}(\mathcal{O}_I)} + \|f\|_{L^{\gamma_1'}(\mathcal{O}_I)})^{\gamma_1'} \\ &\quad + c_\delta \left(\left(\int_{\mathcal{O}_I} |D\psi_k|^{p(\cdot)} + |(\delta_k^2 + |F_k|^2)|^{\frac{p(\cdot)-2}{2}} F_k|^{\frac{p(\cdot)}{p(\cdot)-1}} \,\mathrm{d}z \right)^{\frac{1}{\gamma_1}} + 1 \right)^{\gamma_1'}, \end{aligned} \tag{6.15}$$

where $c_\delta = c(\delta, n, \gamma_1, \gamma_2, \nu, L, L_1)$, for every $k \in \mathbb{N}$. Next, we choose in particular $\delta = 1$. Then, combining this with the bounds (6.11), the convergences of ψ_k in (6.2) and of F_k in (6.6), and the growth condition on $a(z, \cdot)$ in (1.2) we infer

$$\limsup_{k \to \infty} \int_{\mathcal{O}_I} |Dw_k|^{p(\cdot)} \,\mathrm{d}z \le cM \tag{6.16}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$. Next, we utilize (1.9), the local Poincaré type inequality (2.6), (6.11) and (6.12) to infer that

$$\limsup_{k \to \infty} \|w_k\|_{L^{p(\cdot)}(\mathcal{O}_I)} \le cM^{\frac{2\gamma_2}{n+2}+1}, \tag{6.17}$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$. Finally, the equation (6.14) gives similar to (5.23) that

$$\limsup_{k \to \infty} \|\partial_t w_k\|_{W^{p(\cdot)}(\mathcal{O}_I)'} \le \; c \tag{6.18}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1, M, \|\partial_t \psi\|_{L^{p'(\cdot)}})$. Due to the bounds (6.16), (6.17) and (6.18) and the compactness argument from Theorem 2.2, we can find a limit map $w \in W_u^{p(\cdot)}(\mathcal{O}_I)$ with $\partial_t w \in W^{p(\cdot)}(\mathcal{O}_I)'$, such that

$$\begin{cases} w_k \to w & \text{strongly in } L^{\hat{p}(\cdot)}(\mathcal{O}_I, \mathbb{R}), \\ Dw_k \to Dw & \text{weakly in } L^{p(\cdot)}(\mathcal{O}_I, \mathbb{R}^n), \\ \partial_t w_k \to \partial_t w & \text{weakly in } W^{p(\cdot)}(\mathcal{O}_I)', \end{cases} \tag{6.19}$$

as $k \to \infty$, possibly after extraction of a suitable subsequence. In addition, we may assume $w_k \to w$ and $\psi_k \to \psi$ a.e. on $\mathcal{O}_I$ as $k \to \infty$. Therefore, we can conclude that w holds the obstacle constraint $w \ge \psi$ a.e. on $\mathcal{O}_I$. Finally, we have to remove again the smallness condition on $\mathcal{O}$. Using the same argument as in the preceding proof, cf. page 28 with j replaced by k, we get the conclusions from above on a domain $\mathcal{O}$ without smallness condition.

Step 4: Locally strong convergence of the gradient. Our next aim, is to prove the strong convergence of the gradient $Du_k \to Du$, locally in Ω_T. In addition, we consider a cut-off function $\zeta \in C^\infty_{\mathrm{cpt}}(\Omega)$, $0 \le \zeta \le 1$ that will be specified later. For i, $k \in \mathbb{N}$ that are large enough to guarantee $\Omega^i \cap \Omega^k \supset \operatorname{spt}\zeta$, we test the variational inequality (6.8) for u_i with the comparison map $v := \zeta^2(u_k - \psi_k + \psi_i) + (1 - \zeta^2)u_i \in W_{u_i}^{p(\cdot)}(\Omega_T^i)$, since $\Omega^i \supset \operatorname{spt}\zeta$. This comparison function respects the obstacle constraint, $v \ge \psi_i$ because it is a convex combination of the functions $u_k - \psi_k + \psi_i$ and u_i, where both of which respect the same obstacle condition. For this reason, the function v is an admissible comparison map for the solution u_i of (6.8). Therefore, we have the following estimate

$$\langle\!\langle \partial_t u_i, \zeta^2(u_i - \psi_i + \psi_k - u_k)\rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_i)\cdot D(\zeta^2(u_i - \psi_i + \psi_k - u_k))\,\mathrm{d}z \leq \int_{\Omega_T} f_i \cdot \zeta^2(u_i - \psi_i + \psi_k - u_k)\,\mathrm{d}z.$$

Next, we define $\check{u}_i := u_i - \psi_i$, so we can rewrite the last estimate

$$\begin{aligned}\langle\!\langle \partial_t \check{u}_i, \zeta^2(\check{u}_i - \check{u}_k)\rangle\!\rangle_{\Omega_T} &+ \int_{\Omega_T} a(z, Du_i)\cdot D(\zeta^2(u_i - u))\,\mathrm{d}z \\ &\leq \int_{\Omega_T} a(z, Du_i) D(\zeta^2(u_k - u + \psi_i - \psi_k)) + f_i\zeta^2(\check{u}_i - \check{u}_k)\,\mathrm{d}z \\ &\quad + \langle\!\langle \partial_t \psi_i, \check{u}_i - \check{u}_k\rangle\!\rangle_{\Omega_T}.\end{aligned}$$

Now, we exchange the roles of i and k and adding the resulting estimate to the previous one, we get

$$\begin{aligned}&\langle\!\langle \partial_t \check{u}_i - \partial_t \check{u}_k, \zeta^2(\check{u}_i - \check{u}_k)\rangle\!\rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du_i)\cdot D(\zeta^2(u_i - u))\,\mathrm{d}z \\ &+ \int_{\Omega_T} a(z, Du_k)\cdot D(\zeta^2(u_k - u))\,\mathrm{d}z \\ &\leq \int_{\Omega_T} a(z, Du_i)\cdot D(\zeta^2(u_k - u)) \; + \; a(z, Du_k)\cdot D(\zeta^2(u_i - u))\,\mathrm{d}z \\ &+ \int_{\Omega_T} [a(z, Du_i) - a(z, Du_k)]\cdot D(\zeta^2(\psi_i - \psi_k)) + (f_i - f_k)\zeta^2(\check{u}_i - \check{u}_k)\,\mathrm{d}z \\ &+ \langle\!\langle \partial_t \psi_i - \partial_t \psi_k, \zeta^2(\check{u}_i - \check{u}_k)\rangle\!\rangle_{\Omega_T}.\end{aligned} \tag{6.20}$$

Here, the first integral on the left-hand side is non-negative because of Lemma 2.1, since $\check{u}_i(\cdot, 0) = \hat{g}_i(\cdot, 0) - \psi_i(\cdot, 0)$ on $\operatorname{spt}\zeta$ and consequently, the initial values of $\check{u}_i$ are independent from the index i. Now, we plug the above estimate into (6.20) and let $k \to \infty$. Moreover, we utilize the strong convergence (6.2) and (6.5) of the data f_k and ψ_k. Therefore and together with the weak convergence (6.12) and (6.13) of Du_k, we can conclude

$$\begin{aligned}\int_{\Omega_T} a(z, Du_i)\cdot D(\zeta^2(u_i - u))\,\mathrm{d}z \; &+ \; \limsup_{k\to\infty} \int_{\Omega_T} a(z, Du_k)\cdot D(\zeta^2(u_k - u))\,\mathrm{d}z \\ \leq \int_{\Omega_T} &A_0 \cdot D(\zeta^2(u_i - u)) + [a(z, Du_i) - A_0]\cdot D(\zeta^2(\psi_i - \psi))\,\mathrm{d}z \\ + \int_{\Omega_T} &(f_i - f)\zeta^2(\check{u}_i - \check{u})\,\mathrm{d}z - |F|^{p(\cdot)-2}F\cdot D(\zeta^2(\check{u}_i - \check{u}))\,\mathrm{d}z \\ + \langle\!\langle &\partial_t \psi_i - \partial_t \psi, \zeta^2(\check{u}_i - \check{u})\rangle\!\rangle_{\Omega_T},\end{aligned}$$

where $\check{u} := u - \psi$. In the next step, we let $i \to \infty$ and apply the same convergences stated above in oder to check that the right-hand side of the preceding estimate vanishes in the limit. Hence, we have shown that

$$\limsup_{k\to\infty} \int_{\Omega_T} a(z, Du_k)\cdot D(\zeta^2(u_k - u))\,\mathrm{d}z \leq 0.$$

This together with the growth condition (1.2) of $a(z, \cdot)$, the bound (6.11), the generalized Hölder's inequality (1.8) and (1.9), we can conclude

$$\limsup_{k\to\infty} \int_{\Omega_T} \zeta^2 a(z, Du_k)\cdot D(u_k - u)\,\mathrm{d}z \leq cM \limsup_{k\to\infty} \|D\zeta(u_k - u)\|_{L^{p(\cdot)}(\Omega_T)}. \tag{6.21}$$

At this stage, we specify the cut-off function ζ. For this aim, we first fix an arbitrary ball $B_R(x_0) \Subset \Omega$ and let $\varepsilon \in (0, R/2)$ be arbitrary. We employ the notations $A^\varepsilon := B_R(x_0)\backslash B_{R-\varepsilon}(x_0)$ for an annulus in $B_R(x_0)$ of width ε and write $A^\varepsilon_T := A^\varepsilon \times (0, T)$ for the corresponding space-time cylinder. Then, we choose $\zeta_\varepsilon \in C^\infty_0(B_R(x_0))$, $0 \leq \zeta_\varepsilon \leq 1$, such that $\zeta_\varepsilon \equiv 1$ on $B_{R-\varepsilon}(x_0)$, $\zeta_\varepsilon \equiv 0$ on $\partial B_R(x_0)$ with $\|D\zeta_\varepsilon\|_{L^\infty(B_R(x_0))} \leq$

$\frac{\varepsilon}{\varepsilon}$. Moreover, we consider the extension maps constructed in Step 3 for the domain $\mathcal{O}_I = A^\varepsilon_T$, i.e. $w_k \in W^{p(\cdot)}_{u_k}(A^\varepsilon_T)$, and denote by w their limit in the sense of (6.19). For sufficiently large values of $k \in \mathbb{N}$, we estimate

$$\|D\zeta(u_k-u)\|_{L^{p(\cdot)}(A^\varepsilon_T)} \le \frac{c}{\varepsilon}\Big(\|u_k-w_k\|_{L^{p(\cdot)}(A^\varepsilon_T)} + \|w_k-w\|_{L^{p(\cdot)}(A^\varepsilon_T)} + \|w-u\|_{L^{p(\cdot)}(A^\varepsilon_T)}\Big). \tag{6.22}$$

Now, we apply a version of the Poincaré type inequality (2.5) to the functions $(w_k-u_k) \in C^0([0,T];L^2(A^\varepsilon)) \cap W^{p(\cdot)}_0(A^\varepsilon_T)$. This yields similar to (5.28) the bound

$$\frac{c}{\varepsilon}\|w_k-u_k\|_{L^{p(\cdot)}(A^\varepsilon_T)} \le c\left[M^{\frac{2\gamma_2}{n+2}}\left(\int_{U^\varepsilon_I}|Dw_i-Du_i|^{p(\cdot)}+1\,\mathrm{d}z\right)\right]^{\frac{1}{\gamma_1}}.$$

At this stage, it is crucial that we gain a factor ε in the Poincaré type inequality (2.5), which is true since w_k-u_k vanishes on ∂A^ε and the annulus A^ε has width ε. The right-hand side of the above estimate can be bounded further by the comparison estimate (6.15) in the form

$$\int_{A^\varepsilon_T}|Dw_k-Du_k|^{p(\cdot)}\,\mathrm{d}z \le \delta\cdot M + c_\delta\Big(\|(\delta_k^2+|F_k|^2)|^{\frac{p(\cdot)-2}{2}}F_k\|^{\gamma_1'}_{L^{p'(\cdot)}(A^\varepsilon_T)} + \|(1+|D\psi_k|)\|^{\gamma_1'}_{L^{p(\cdot)}(A^\varepsilon_T)} + \|\partial_t\psi_k\|^{\gamma_1'}_{L^{\gamma_1'}(A^\varepsilon_T)} + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(A^\varepsilon_T)}\Big)$$

for any $\delta \in (0,1)$, where $c_\delta = c(\delta,n,\gamma_1,\gamma_2,\nu,L,L_1)$. Combining the last two estimates, letting $k\to\infty$ and using the convergences (6.2) and (6.5), we can conclude that

$$\limsup_{k\to\infty}\frac{c}{\varepsilon}\|w_k-u_k\|_{L^{p(\cdot)}(A^\varepsilon_T)} \le cM^{\frac{1}{\gamma_1}}\left(\delta M + c_\delta\Psi_\varepsilon\right), \tag{6.23}$$

where we used (6.11), (3.7) for w_i (modulus the covering argument from above) and

$$\Psi_\varepsilon := \|(\delta_k^2+|F_k|^2)|^{\frac{p(\cdot)-2}{2}}F_k\|^{\gamma_1'}_{L^{p'(\cdot)}(A^\varepsilon_T)} + \|(1+|D\psi_k|)\|^{\gamma_1'}_{L^{p(\cdot)}(A^\varepsilon_T)} + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(A^\varepsilon_T)} + \|\partial_t\psi\|^{\gamma_1'}_{L^{\gamma_1'}(A^\varepsilon_T)}.$$

Then, the absolute continuity of the integral implies that Ψ_ε vanish in the limit, so that $\Psi_\varepsilon \to 0$ as $\varepsilon\downarrow 0$. Next, we use the lower semi continuity of the norm with respect to the weak convergence. Therefore, we conclude from (6.23) that

$$\frac{c}{\varepsilon}\|w-u\|_{L^{p(\cdot)}(A^\varepsilon_T)} \le cM^{\frac{1}{\gamma_1}}\left(\delta M + c_\delta\Psi_\varepsilon\right) \tag{6.24}$$

for every $\delta>0$ and $\varepsilon\in(0,R/2)$. Finally, the convergence (6.19) implies

$$\lim_{k\to\infty}\frac{c}{\varepsilon}\|w_k-w\|_{L^{p(\cdot)}(A^\varepsilon_T)} = 0. \tag{6.25}$$

Plugging (6.23), (6.24) and (6.25) into (6.22), we deduce $\limsup_{k\to\infty}\|D\zeta_\varepsilon(u_k-u)\|_{L^{p(\cdot)}(A^\varepsilon_T)} \le cM^{\frac{1}{\gamma_1}}\left(\delta M + c_\delta\Psi_\varepsilon\right)$. At this stage, we utilize the estimate (6.21), where we use the cut-off function $\zeta=\zeta_\varepsilon$ from above and combine it with the preceding estimate. This yields

$$\limsup_{k\to\infty}\int_{\Omega_T}\zeta_\varepsilon^2 a(z,Du_k)\cdot D(u_k-u)\,\mathrm{d}z \le cM^{\frac{1}{\gamma_1}}\left(\delta M + c_\delta\Psi_\varepsilon\right), \tag{6.26}$$

where $\delta\in(0,1)$ and $\varepsilon\in(0,R/2)$ can be chosen arbitrarily. Therefore and together with the weak convergence $Du_k \rightharpoonup Du$ in $L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)$ by (6.12), we can conclude that

$$\limsup_{k\to\infty}\int_{\Omega_T}\zeta_\varepsilon^2 a(z,Du_k)\cdot D(u_k-u)\,\mathrm{d}z \;=\; 0 \tag{6.27}$$

for every $\varepsilon \in (0, R/2)$. Joining (6.26) with (6.27) and applying the monotonicity condition (1.3) of $a(z, \cdot)$, we arrive at

$$\limsup_{k\to\infty} \int_{\Omega_T} \zeta_\varepsilon^2 (\tilde{\mu}^2 + |Du_k|^2 + |Du|^2)^{\frac{p(\cdot)-2}{2}} |Du_k - Du|^2 \,\mathrm{d}z \le c\delta M + c_\delta \Psi_\varepsilon,$$

where $\delta \in (0, 1)$ and $\varepsilon \in (0, R/2)$ can be chosen arbitrarily. By the choice of ζ_ε, this implies in particular

$$\limsup_{k\to\infty} \int_0^T \int_{B_{\frac{R}{2}}(x_0)} (\tilde{\mu}^2 + |Du_k|^2 + |Du|^2)^{\frac{p(\cdot)-2}{2}} |Du_k - Du|^2 \,\mathrm{d}z \le cM^{\frac{1}{\gamma_1}} (\delta M + c_\delta \Psi_\varepsilon).$$

Notice, the left-hand side does not depends on ε. Therefore, we can first choose $\delta \in (0, 1)$ and then $\varepsilon \in (0, R/2)$ so small that the right-hand side becomes arbitrarily small. So, we can infer

$$\limsup_{k\to\infty} \int_0^T \int_{B_{\frac{R}{2}}(x_0)} (\tilde{\mu}^2 + |Du_k|^2 + |Du|^2)^{\frac{p(\cdot)-2}{2}} |Du_k - Du|^2 \,\mathrm{d}x \,\mathrm{d}z = 0.$$

Since the ball $B_R(x_0) \Subset \Omega$ is arbitrary, this implies in a standard way the desired convergence

$$Du_k \to Du \text{ strongly in } L^{p(\cdot)}(\Omega' \times (0, T), \mathbb{R}^n) \tag{6.28}$$

for every subdomain $\Omega' \Subset \Omega$, as $k \to \infty$.

Step 5: Continuity in time and initial values. Our next aim is to show that $u \in C^0([0, T]; L^2(\Omega))$ for any subdomain $\mathcal{O} \Subset \Omega$ and that u attains the initial values $g(\cdot, 0)$ at the time $t = 0$. We consider $i,\ k \in \mathbb{N}$ so large that $\mathcal{O} \Subset \Omega^i \cap \Omega^k$ and fix a time $\tau \in (0, T)$. Moreover, we choose a cut-off function $\zeta \in C_0^\infty(\Omega)$, $0 \le \zeta \le 1$, such that $\zeta \equiv 1$ on $\mathcal{O}$, $\zeta \equiv 0$ on $\mathcal{O} \backslash (\Omega^i \cap \Omega^k)$ and continue to use the notation $\check{u}_i := u_i - \psi_i$. Now, we test the variational inequality (6.8) for u_i with the comparison map $v := u_i + (\check{u}_k - \check{u}_i)\zeta^2(x)\mathbb{1}_{(0,\tau)}(t) \in W^{p(\cdot)}_{u_i}(\Omega_T)$. This map respects the obstacle constraint $v \ge \psi_i$ since it can be written as a convex combination of the functions u_i and $u_k - \psi_k + \psi_i$, both of which satisfy the mentioned obstacle condition. Therefore, the variational inequality (6.8), considering the growth property (1.2) of $a(z, \cdot)$ and using the generalized Hölder's inequality (1.8) and (1.9), yields the bound

$$\begin{aligned}\langle\langle \partial_t \check{u}_i, \zeta^2(\check{u}_i - \check{u}_k)\rangle\rangle_{\Omega_T} \le & c \left(\int_{\Omega_T} |Du_i|^{p(\cdot)} \,\mathrm{d}z + \|f_i\|_{L^{p'(\cdot)}(\Omega_T)} + \|\partial_t \psi_i\|_{L^{p'(\cdot)}(\Omega_T)} + 1 \right) \\ & \times \|\zeta^2(\check{u}_k - \check{u}_i)\|_{W^{p(\cdot)}(\Omega_T)}\end{aligned}$$

with a constant $c = c(n, \gamma_1, \gamma_1, L, \mathrm{diam}(\Omega))$. Taking into account the energy bound (6.11), the convergences (6.2) and (6.5), and the locally strong convergence (6.28) of Du_i, we observe that the right-hand side of the above estimate tends to zero as $i,\ k \to \infty$, uniformly in $\tau \in (0, T)$. This yields

$$\limsup_{i,\ k\to\infty} \sup_{\tau\in(0,T)} \langle\langle \partial_t \check{u}_i, \zeta^2(\check{u}_i - \check{u}_k)\rangle\rangle_{\Omega_T} \le 0.$$

Adding the same inequality with exchanged roles of i and k and then applying Lemma 2.1, we deduce

$$\frac{1}{2} \limsup_{i,\ k\to\infty} \sup_{\tau\in(0,T)} \int_{\Omega\times\{\tau\}} |\zeta(\check{u}_i - \check{u}_k)|^2 \,\mathrm{d}x \le 0.$$

Moreover, since $\zeta \equiv 1$ on $\mathcal{O}$, we infer that $\{\check{u}_i \mathbb{1}_{\mathcal{O}_T}\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0, T]; L^2(\Omega))$. Since $\psi_i \to \psi$ strongly in $C^0([0, T]; L^2(\Omega))$ according to (1.24), we deduce

$$u_i \to u \text{ strongly in } C^0([0, T]; L^2(\mathcal{O})), \tag{6.29}$$

as $i \to \infty$. This yields on the one hand the claimed regularity $u \in C^0([0, T]; L^2(\mathcal{O}))$ and on the other hand, we can calculate the initial values of u by

$$u(\cdot, 0)_{|_{\mathcal{O}}} = \lim_{i\to\infty} u_i(\cdot, 0)_{|_{\mathcal{O}}} = \lim_{i\to\infty} \hat{g}_i(\cdot, 0)_{|_{\mathcal{O}}} = \hat{g}(\cdot, 0)_{|_{\mathcal{O}}} = g(\cdot, 0)_{|_{\mathcal{O}}},$$

where we used the convergence (6.4) of $\hat{g}_i$ and the limits have to be understood with respect to the norm in $L^2(\mathcal{O})$. Since $\mathcal{O} \Subset \Omega$ was arbitrary, we infer the claimed values $u(\cdot,0) = g(\cdot,0)$.

Step 6: Proof of the extension property and the variational inequality. Since at this stage, we have established the strong convergence $u_i \to u$ in the spaces $W^{p(\cdot)}(\Omega'_T)$ and $C^0([0,T];L^2(\Omega'))$ for every subdomain $\Omega' \Subset \Omega$, the remainder of the proof is analogous to the Step 5 and Step 6 of the preceding proof - see page 26 and page 30. The only difference is that since the mentioned convergence holds only locally, we have to restrict ourselves to subdomains $\mathcal{O} \Subset \Omega$ that are compactly contained in Ω. For such domains, extension maps w to u were already constructed in Step 3. For the proof of $w \in \mathcal{K}'_{\psi,u}(\mathcal{O}_I)$ it only remains to show that $w \in C^0([0,T];L^2(\mathcal{O}))$. This follows by repeating the arguments after (5.25) from the proof of Theorem 1.7, which completes the proof of the extension property $\mathcal{K}'_{\psi,u}(\mathcal{O}_I) \neq \emptyset$ of the limit map u.

The derivation of the variational inequality (1.20) on the domain $\mathcal{O}_I$ now follows with the same arguments as developed in Step 6 of the proof of Theorem 1.7 - see page 30 - using the strong convergence $u_i \to u$ in $W^{p(\cdot)}(\mathcal{O}_I)$ and $C^0([t_1,t_2];L^2(\mathcal{O}))$ that hold according (6.28) and (6.29). This concludes the proof of the Theorem. □

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