GENERALIZED QUASIDISKS AND CONFORMALITY II

CHANG-YU GUO

Abstract. We introduce a weaker variant of the concept of three point property, which is equivalent to a non-linear local connectivity condition introduced in [12], sufficient to guarantee the extendability of a conformal map $f : \mathbb{D} \to \Omega$ to the entire plane as a homeomorphism of locally exponentially integrable distortion. Sufficient conditions for extendability to a homeomorphism of locally p-integrable distortion are also given.

1. Introduction

One calls a Jordan domain $\Omega \subset \mathbb{R}^2$ a quasidisk if it is the image of the unit disk $\mathbb{D}$ under a quasiconformal mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ of the entire plane. If f is K-quasiconformal, we say that Ω is a K-quasidisk. Another possibility is to require that f is additionally conformal in the unit disk $\mathbb{D}$. It is essentially due to Kühnau [17] that Ω is a K-quasidisk if and only if Ω is the image of $\mathbb{D}$ under a K^2-quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is conformal in $\mathbb{D}$, see [9]. The concept of a quasidisk is central in the theory of planar quasiconformal mappings; see, for example, [2, 4, 8, 21].

A substantial part of the theory of quasiconformal mappings has recently been shown to extend in a natural form to the setting of mappings of locally exponentially integrable distortion [3, 4, 7, 13, 14, 16, 23, 26]. See Section 2 below for the definition of this class of mappings. Thus one could say that $\Omega \subset \mathbb{R}^2$ is a generalized quasidisk if it is the image of the unit disk $\mathbb{D}$ under a homeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ of the entire plane with locally exponentially integrable distortion. However, requiring that f is additionally conformal in the unit disk $\mathbb{D}$ leads to different classes of domains, see [12, Theorem 1.1]. In this paper, a Jordan domain $\Omega \subset \mathbb{R}^2$ is termed a generalized quasidisk if the additional conformality requirement is satisfied.

For a quasidisk $\Omega \subset \mathbb{R}^2$, there are several equivalent characterizations. One of the simplest is Ahlfors [1] three point property. Recall that a Jordan domain $\Omega \subset \mathbb{R}^2$ has the three point property if there

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exists a constant $C \geq 1$ such that for each pair of distinct points $P_1, P_2 \in \partial\Omega$,

$$\min_{i=1,2} \operatorname{diam}(\gamma_i) \leq C|P_1 - P_2|, \tag{1.1}$$

where γ_1, γ_2 are components of $\partial\Omega \backslash \{P_1, P_2\}$. In order to understand the geometry of generalized quasidisks, one naturally has to weaken the three point property. A Jordan domain $\Omega \subset \mathbb{R}^2$ is said to have the three point property with a control function ψ if there exists a constant $C \geq 1$ and an increasing function $\psi : [0, \infty) \to [0, \infty)$ such that for each pair of distinct points $P_1, P_2 \in \partial\Omega$,

$$\min_{i=1,2} \operatorname{diam}(\gamma_i) \leq \psi\Big(C|P_1 - P_2|\Big). \tag{1.2}$$

A closely related concept is the following ψ-local connectivity, which was introduced in [12]. A domain $\Omega \subset \mathbb{R}^2$ is said to be ψ-locally connected if for each x and all $r > 0$,

- each pair of points in $B(x, r) \cap \Omega$ can be joined by an arc in $B(x, \psi^{-1}(r)) \cap \Omega$, and
- each pair of points in $\Omega \backslash B(x, r)$ can be joined by an arc in $\Omega \setminus B(x, \psi(r))$.

If we were to choose $\psi(t) = Ct$, then this would reduce to the usual linear local connectivity condition. In Lemma 3.1 below, we show that that a Jordan domain $\Omega \subset \mathbb{R}^2$ has the three point property with a control function ψ if and only if Ω is ψ^{-1}-locally connected.

In [12, Theorem 1.2], it was proved that if a Jordan domain $\Omega \subset \mathbb{R}^2$ is ψ-locally connected with $\psi(t) = \frac{Ct}{\log^s \log \frac{1}{t}}$ for some positive constant C and some $s \in (0, \frac{1}{4})$, then Ω is a generalized quasidisk. However, the result is not sharp regarding well-studied examples, see [12]. In fact, the previous studies in [12, 19, 20, 24] suggest that the critical case should be $\psi(t) = \frac{Ct}{\log \frac{1}{t}}$. By the equivalence of local connectivity and generalized three point property mentioned in the end of the previous paragraph, for a domain satisfying the three point property with a control function ψ, the critical case (essentially) becomes $\psi(t) = Ct \log \frac{1}{t}$. Our first main result approaches this critical case and improves on [12, Theorem 1.2].

Theorem 1.1. *If a Jordan domain $\Omega \subset \mathbb{R}^2$ has the three point property with the control function $\psi(t) = Ct \log^{\frac{1}{2}} \frac{1}{t}$ for some positive constant C, then Ω is a generalized quasidisk.*

Equivalently, Theorem 1.1 provides a general sufficient condition for extendability of a conformal mapping $f : \mathbb{D} \to \Omega$ to a homeomorphism of locally exponentially integrable distortion. It was pointed out in [12] that this is essentially equivalent to extending the corresponding conformal welding to the whole plane as a homeomorphism of locally exponentially integrable distortion, see also Section 4 below.

Our second main result asserts that if we relax the control function ψ to be a root in Theorem 1.1, then we end up with a homeomorphism of the whole plane with locally p-integrable distortion.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain that has the three point property with the control function $\psi(t) = t^s$, $0 < s < 1$. Then any conformal mapping $f\colon \mathbb{D} \to \Omega$ can be extended to the entire plane as a homeomorphism of locally p-integrable distortion for all $p \in (0, \frac{s^2}{2(1-s^2)})$.*

As pointed out in [12], (polynomial) interior cusps are more dangerous than (polynomial) exterior cusps in the locally exponentially integrable distortion case. Thus one expects that this is still the case for extensions with locally p-integrable distortion. Our next result confirms this expectation.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a LLC-1 Jordan domain. Then any conformal mapping $f\colon \mathbb{D} \to \Omega$ can be extended to the entire plane as a homeomorphism of locally p-integrable distortion for some $p > 0$.*

This paper is organized as follows. Section 2 contains the basic definitions and Section 3 some auxiliary results. In Section 4, we study the relation of extending a Riemann mapping and the corresponding conformal welding. We prove our main results in Section 5. In the final section, Section 6, we make some concluding remarks.

2. Notation and Definitions

We sometimes associate the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ for convenience and denote by $\hat{\mathbb{C}}$ the extended complex plane. The closure of a set $U \subset \mathbb{R}^2$ is denoted $\overline{U}$ and the boundary ∂U. The open disk of radius $r > 0$ centered at $x \in \mathbb{R}^2$ is denoted by $B(x, r)$ and we simply write $\mathbb{D}$ for the unit disk. The boundary of $B(x, r)$ will be denoted by $S(x, r)$ and the boundary of the unit disk $\mathbb{D}$ is written as $\partial\mathbb{D}$. The symbol Ω always refers to a domain, i.e. a connected and open subset of $\mathbb{R}^2$. We call a homeomorphism $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^2$ a homeomorphism of finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ and

$$\|Df(x)\|^2 \leq K(x) J_f(x) \text{ a.e. in } \Omega, \tag{2.1}$$

for some measurable function $K(x) \geq 1$ that is finite almost everywhere. Recall here that $J_f \in L^1_{loc}(\Omega)$ for each homeomorphism $f \in W^{1,1}_{loc}(\Omega; \mathbb{R}^2)$(cf. [4]). In the distortion inequality (2.1), $Df(x)$ is the formal differential of f at the point x and $J_f(x) := \det Df(x)$ is the Jacobian. The norm of $Df(x)$ is defined as

$$\|Df(x)\| := \max_{e \in \partial\mathbb{D}} |Df(x)e|.$$

For a homeomorphism of finite distortion it is convenient to write K_f for the optimal distortion function. This is obtained by setting $K_f(x) = \|Df(x)\|^2 / J_f(x)$ when $Df(x)$ exists and $J_f(x) > 0$, and $K_f(x) = 1$

otherwise. The distortion of f is said to be locally λ-exponentially integrable if $\exp(\lambda K_f(x)) \in L^1_{\text{loc}}(\Omega)$, for some $\lambda > 0$. Note that if we assume $K_f(x)$ to be bounded, $K_f \leq K$, we recover the class of K-quasiconformal mappings, see [4] for the theory of quasiconformal mappings.

Recall that a domain Ω is said to be linearly locally connected (LLC) if there is a constant $C \geq 1$ so that

- (LLC-1) each pair of points in $B(x,r) \cap \Omega$ can be joined by an arc in $B(x,Cr) \cap \Omega$, and
- (LLC-2) each pair of points in $\Omega \backslash B(x,r)$ can be joined by an arc in $\Omega \setminus B(x, C^{-1}r)$.

We need a weaker version of this condition, defined as follows. We say that Ω is (φ,ψ)-locally connected ((φ,ψ)-LC) if

- (φ-LC-1) each pair of points in $B(x,r) \cap \Omega$ can be joined by an arc in $B(x,\varphi(r)) \cap \Omega$, and
- (ψ-LC-2) each pair of points in $\Omega \backslash B(x,r)$ can be joined by an arc in $\Omega \setminus B(x,\psi(r))$,

where $\varphi,\psi : [0,\infty) \to [0,\infty)$ are smooth increasing functions such that $\varphi(0) = \psi(0) = 0$, $\varphi(r) \geq r$ and $\psi(r) \leq r$ for all $r > 0$. For technical reasons, we assume that the function $t \mapsto \frac{t}{\varphi^{-1}(t)^2}$ is decreasing and that there exist constants C_1, C_2 so that $C_1\varphi(t) \leq \varphi(2t) \leq C_2\varphi(t)$ and $C_1\psi(t) \leq \psi(2t) \leq C_2\psi(t)$ for all $t > 0$. If $\varphi^{-1} = \psi$ above, as in the introduction, Ω will simply be called ψ-LC. One could relax joinability by an arc above to joinability by a continuum, but this leads to the same concept; see [15, Theorem 3-17]. Notice that if Ω is simply connected and bounded, then φ-LC-1 guarantees that Ω is a Jordan domain.

Finally we define the central tool for us – the modulus of a path family. A Borel function $\rho\colon \mathbb{R}^2 \to [0,\infty]$ is said to be admissible for a path family Γ if $\int_\gamma \rho\, ds \geq 1$ for each locally rectifiable $\gamma \in \Gamma$. The modulus of the path family Γ is then

$$\operatorname{mod}(\Gamma) := \inf\Big\{ \int_\Omega \rho^2(x)\, dx : \rho \text{ is admissible for } \Gamma \Big\}.$$

For subsets E and F of $\overline{\Omega}$ we write $\Gamma(E,F,\Omega)$ for the path family consisting of all locally rectifiable paths joining E to F in Ω and abbreviate $\operatorname{mod}(\Gamma(E,F,\Omega))$ to $\operatorname{mod}(E,F,\Omega)$. In what follows, $\gamma(x,y)$ refers to a curve or an arc from x to y.

When we write $f(x) \lesssim g(x)$, we mean that $f(x) \leq Cg(x)$ is satisfied for all x with some fixed constant $C \geq 1$. Similarly, the expression $f(x) \gtrsim g(x)$ means that $f(x) \geq C^{-1}g(x)$ is satisfied for all x with some fixed constant $C \geq 1$. We write $f(x) \approx g(x)$ whenever $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$.

3. Auxiliary results

We begin this section by showing the equivalence of the generalized three point property and generalized local connectivity mentioned in the introduction.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then Ω has the three point property with the control function ψ if and only if Ω is ψ^{-1}-locally connected.*

Proof. For the proof we need the following duality result from [11, Theorem 1.3]: for a Jordan domain $\Omega \subset \mathbb{R}^2$, Ω is ψ-LC-2 if and only if $\mathbb{R}^2 \backslash \overline{\Omega}$ is ψ-LC-1.

First, suppose that Ω has the three point property with the control function ψ. We want to show that Ω is ψ-LC-1. To this end, let $x, y \in B(z,r) \cap \Omega$. We may assume that there exist $x', y' \in B(z,r) \cap \partial\Omega$ such that

$$d(x,x') = d(x,\partial\Omega), d(y,y') = d(y,\partial\Omega)$$

and that x can be connected to x' by an arc β_1 in $\overline{\Omega} \cap B(z,r)$ and y can be connected to y' by an arc β_2 in $\overline{\Omega} \cap B(z,r)$. Let α_1 and α_2 be the components of $\partial\Omega \backslash \{x', y'\}$. We may assume that $\alpha_1 \leq \alpha_2$. Then

$$\operatorname{diam}(\alpha_1) \leq \psi(|x' - y'|) \leq \psi(2r).$$

Hence, $\gamma = \beta_1 \cup \alpha_1 \cup \beta_2$ is a curve that connects x and y in $\overline{\Omega}$ with diameter less than $2\psi(2r)$. Then the Jordan assumption for Ω implies that we may connect x to y in Ω by a curve with diameter no more than $3\psi(2r)$. This together with Lemma 3.2 below implies that Ω is ψ-LC-1. Similarly, one can prove that $\mathbb{R}^2 \backslash \overline{\Omega}$ is ψ-LC-1. Then the duality result recalled in the beginning of the proof implies that Ω is ψ-LC.

Next, we assume that Ω is ψ^{-1}-LC. Then, again by the duality result, we know that both Ω and $\mathbb{R}^2 \backslash \overline{\Omega}$ are ψ-LC-1. Let $x, y \in \partial\Omega$ and let α_1, α_2 be the components of $\partial\Omega \backslash \{x, y\}$. We may assume that $\operatorname{diam}(\alpha_1) \leq \operatorname{diam}(\alpha_2)$. Let $z = \frac{x+y}{2}$ and $r = |x - y|$. Then $x, y \in B(z,r)$. We may choose two points x' and y' in $\Omega \cap B(z,r)$ such that x can be connected to x' by an arc β_1 in $\overline{\Omega} \cap B(z,r)$ and y can be connected to y' by an arc β_2 in $\overline{\Omega} \cap B(z,r)$. Then we may connect x' to y' by an arc γ in $\Omega \cap B(z, 2\psi(r))$. Then the curve $\eta = \beta_1 \cup \gamma \cup \beta_2$ forms a crosscut of Ω with diameter no more than $4\psi(r)$. Similarly, we may form a crosscut η' of $\mathbb{R}^2 \backslash \overline{\Omega}$ with diameter no more than $4\psi(r)$. Thus $\eta \cup \eta'$ is a Jordan curve which contains the Jordan arc α_1. Therefore, the diameter of α_1 is no more than $8\psi(r)$. This together with Lemma 3.2 below implies that Ω has the three point property with the control function ψ. □

Lemma 3.2 (Lemma 3.5, [11])**.** *Let $C_1 \geq 1$, $C_2 \geq 1$, and $C_3 \geq 1$ be given. There exists a constant C, depending only on C_0, C_1, C_2 and C_3, such that*

$$C_1\varphi(C_2 t) + C_3 t \leq \varphi(Ct) \tag{3.1}$$

for all $t > 0$. Above, C_0 is the doubling constant of φ^{-1}.

The following two modulus estimates are standard, see e.g. [25].

Lemma 3.3. *Let E, F be disjoint nondegenerate continua in $B(x, R)$. Then*

$$C_0^{-1}\frac{1}{\log(1+t)} \geq \operatorname{mod}(E, F, B(x, R)) \geq C_0\frac{1}{\log(1+t)}, \tag{3.2}$$

where $t = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}$ and C_0 is an absolute constant.

Lemma 3.4. *Let Γ be a curve family such that for all $t \in (r, R)$, the circle $|z - z_1| = t$ contains a curve $\gamma \in \Gamma$. Then*

$$\operatorname{mod}(\Gamma) \geq \frac{1}{2\pi}\log\frac{R}{r}. \tag{3.3}$$

Next, we recall the following result on the modulus of continuity of a quasiconformal mapping. The proof can be found in [18]; also see [10].

Lemma 3.5. *Suppose $g\colon \Omega \to \mathbb{D}$ is a K-quasiconformal mapping from a simply connected domain Ω onto the unit disk. Then there exists a positive constant C, (depending on f), such that for any $\omega, \xi \in \Omega$,*

$$|g(\omega) - g(\xi)| \leq C d_I(\omega, \xi)^{\frac{1}{2K}}, \tag{3.4}$$

where $d_I(\omega, \xi)$ is defined as $\inf_{\gamma(\omega,\xi)\subset\Omega} \operatorname{diam}(\gamma(\omega, \xi))$. In particular, if Ω above is φ-LC-1, then

$$|g(\omega) - g(\xi)| \leq C\varphi(|\omega - \xi|)^{\frac{1}{2K}}. \tag{3.5}$$

Finally, we need the following key estimate.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain that has the three point property with the control function ψ. Let α_1 and α_2 be two disjoint arcs in $\partial\Omega$ and let Γ and Γ' be the family of curves which join α_1 and α_2 in Ω and $\mathbb{R}^2\backslash\overline{\Omega}$, respectively. If $\operatorname{mod}(\Gamma) \leq C$, then*

$$\min\{\operatorname{diam}(\alpha_1), \operatorname{diam}(\alpha_2)\} \leq \psi \circ \psi(d(\alpha_1, \alpha_2)) \tag{3.6}$$

and hence

$$\operatorname{mod}(\Gamma') \leq C_0^{-1}\log^{-1}\Big(1 + \frac{\psi^{-1}\circ\psi^{-1}(\min\{\operatorname{diam}(\alpha_1), \operatorname{diam}(\alpha_2)\})}{\min\{\operatorname{diam}(\alpha_1), \operatorname{diam}(\alpha_2)\}}\Big). \tag{3.7}$$

Proof. The idea of the proof is similar to that of the proof of Theorem 5.1 in [12]. Let α_1 and α_2 be two disjoint arcs in $\partial\Omega$. Choose $z_1 \in \alpha_1$, $z_2 \in \alpha_2$ so that

$$|z_1 - z_2| = d(\alpha_1, \alpha_2) := d.$$

Without loss of generality, we may assume that

$$r := \operatorname{diam}(\alpha_1) \leq \operatorname{diam}(\alpha_2).$$

Our aim is to show that $r \leq 2\psi \circ \psi(d)$. Thus we may clearly assume that $r > 2\psi \circ \psi(d)$. Note that our assumption on ψ implies that $r >$

$\psi(d)$. Since Ω has the three point property with the control function ψ,

$$\min_{i=1,2} \operatorname{diam}(\gamma_i) \leq \psi(d),$$

where γ_1, γ_2 are the components of $\partial\Omega\backslash\{z_1, z_2\}$. Again, we may assume that

$$\operatorname{diam}(\gamma_1) \leq \psi(d).$$

Let β_1, β_2 be the components of $\partial\Omega\backslash(\alpha_1 \cup \alpha_2)$, labeled so that $\beta_i \subset \gamma_i$. Then $\beta_1 \subset \gamma_1 \subset \overline{B}(z_1, \psi(d))$. Choose $z_0 \in \beta_2$ and let δ_1, δ_2 denote the components of $\partial\Omega\backslash\{z_0, z_1\}$ labeled so that $\alpha_2 \subset \delta_2$. Then the fact Ω has the three point property with the control function ψ implies that

$$\min_{i=1,2} \operatorname{diam}(\delta_i) \leq \psi(|z_1 - z_0|).$$

Choose $\omega_1, \omega_2 \in \alpha_1$ so that

$$r = |\omega_1 - \omega_2| = \operatorname{diam}(\alpha_1).$$

Then $\omega_i \in \gamma_1 \cup \delta_1$, and the fact that $\operatorname{diam}(\gamma_1) \leq \psi(d) < r$ implies that not both of these points can lie in γ_1. If $\omega_1 \in \gamma_1$, then

$$\begin{aligned} \operatorname{diam}(\delta_1) &\geq |\omega_2 - z_1| \geq |\omega_1 - \omega_2| - |z_1 - \omega_1| \\ &\geq r - \operatorname{diam}(\gamma_1) \geq r - \psi(d) \geq \frac{r}{2}. \end{aligned}$$

If both lie in δ_1, then

$$\operatorname{diam}(\delta_1) \geq |\omega_1 - \omega_2| = r.$$

Thus

$$\frac{r}{2} \leq \min_{i=1,2} \operatorname{diam}(\delta_i) \leq \psi(|z_1 - z_0|).$$

It follows that

$$\beta_2 \cap B(z_1, \psi^{-1}(\frac{r}{2})) = \emptyset.$$

In particular, the circle $|z - z_1| = t$ separates β_1 and β_2 for $t \in (\psi(d), \psi^{-1}(\frac{r}{2}))$ and hence must contain an arc γ joining α_1 and α_2 in Ω. Thus Lemma 3.4 implies that

$$\frac{1}{2\pi} \log \frac{\psi^{-1}(r/2)}{\psi(d)} \leq \operatorname{mod}(\Gamma) \leq C$$

from which the claim follows. The desired inequality (3.7) follows from Lemma 3.3 directly.

□

4. Extension of a conformal welding

Before stating the main result of this section, let us describe the standard way of extending a conformal map $f\colon \mathbb{D} \to \Omega$, where Ω is a Jordan domain, to a mapping of the entire plane. First of all, f can be extended to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. For simplicity, we denote this extended homeomorphism also by f. It follows from the Riemann Mapping Theorem that there exists a conformal mapping $g\colon \mathbb{R}^2\backslash\overline{\mathbb{D}} \to \mathbb{R}^2\backslash\overline{\Omega}$ such that the complement of the closed unit disk gets mapped to the complement of $\overline{\Omega}$. In this correspondence the boundary curve $\Gamma = \partial\Omega$ is mapped homeomorphically onto the boundary circle $\partial\mathbb{D}$ and hence the composed mapping $G = g^{-1} \circ f$ is a well-defined circle homeomorphism, called conformal welding. Suppose we are able to extend G to the exterior of the unit disk, with the extension still denoted by G. Then the mapping $G' = g \circ G$ will be well-defined outside the unit disk and it coincides with f on the boundary circle $\partial\mathbb{D}$. Finally, if we define

$$F(x) = \begin{cases} G'(x) & \text{if } |x| \geq 1 \\ f(x) & \text{if } |x| \leq 1, \end{cases}$$

then we obtain an extension of f to the entire plane. In the case of a quasidisk, that is when Ω is linearly locally connected (LLC), the extension G can be chosen to be quasiconformal and hence the obtained map F is also quasiconformal.

On the other hand, the extendability of a conformal mapping $f : \mathbb{D} \to \Omega$ to a homeomorphism $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ of locally integrable distortion is essentially equivalent to being able to extend the conformal welding G' above to this class. Indeed, if $\hat{f}$ extends f, then $g^{-1} \circ \hat{f}$ extends G to the exterior of $\mathbb{D}$ and has the same distortion as $\hat{f}$. Reflecting (twice) with respect to the unit circle one then further obtains an extension to $\mathbb{D} \setminus \{0\}$. Hence, one obtains an extension $\hat{G}'$ of G' to $\mathbb{R}^2 \setminus \{0\}$ with distortion that has the same local integrability degree as the distortion of $\hat{f}$. If the latter distortion is sufficiently nice in a neighborhood of infinity (e.g. bounded), then this holds in all of $\mathbb{R}^2$ as well.

Given a sense-preserving homeomorphism $f\colon \partial\mathbb{D} \to \partial\mathbb{D}$ and $0 < t < \frac{\pi}{2}$, set

$$\delta_f(\theta, t) = \max\left\{\frac{|f(e^{i(\theta+t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta-t)})|}, \frac{|f(e^{i(\theta-t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta+t)})|}\right\}. \tag{4.1}$$

Clearly δ_f is continuous in both variables, $\delta_f \geq 1$ and $\delta_f(\theta + 2\pi, t) = \delta_f(\theta, t)$. The scalewise distortion of f is defined as $\rho_f(t) = \sup_\theta \delta_f(\theta, t)$.

In the following, we discuss a standard way of extending a conformal welding $G : \partial\mathbb{D} :\to \partial\mathbb{D}$ to a global homeomorphism of the whole plane with controlled distortion. More precisely, we want to present the following result, which is implicitly contained in [26, Section 2 and Section 3].

Proposition 4.1. *Given a conformal welding $G : \partial\mathbb{D} :\rightarrow \partial\mathbb{D}$, there exists a homeomorphism $\hat{G} : \mathbb{R}^2 \to \mathbb{R}^2$ with the following property:*

- *For some $\delta \in (0, \frac{1}{2})$, $\hat{G}(z) = z$ if $|z| < \delta$ or $|z| > \frac{1}{\delta}$.*
- *The distortion of $\hat{G}$ is bounded above by the scalewise distortion of G, i.e.*

$$K_{\hat{G}}(z) \le C\rho_G(\log|z|) = C \sup_{\theta\in[0,2\pi]} \delta_G(\theta, \log|z|), \tag{4.2}$$

if $\delta \le |z| \le \frac{1}{\delta}$ for some absolute constant $C > 0$.

Let us describe below the argument leading to Proposition 4.1. Given a conformal welding $G : \partial\mathbb{D} \to \partial\mathbb{D}$, we first want to extend G to a homeomorphism $\tilde{G} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. We may represent G in the form

$$G(e^{2\pi ix}) = e^{2\pi ih(x)},$$

where $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism of the real line which commutes with the unit translation $x \mapsto x + 1$. For simplicity, we may assume that $G(1) = 1$ and hence $h(0) = 0$.

Next, we extend our mapping h to a homeomorphism $H : \mathbb{H} \to \mathbb{H}$. This can be done via the well-known Beurling-Ahlfors extension. To be more precise, for $0 < y < 1$, set

(4.3)

$$H(x+iy) = \frac{1}{2}\int_0^1 (h(x+ty)+h(x-ty))dt + i\int_0^1 (h(x+ty)-h(x-ty))dt.$$

Then $H = h$ on the real axis and H is a C^1-smooth homeomorphism of $\mathbb{H}$. Since $h(x + 1) = h(x) + 1$, for $y = 1$

$$H(x + i) = x + i + C_0,$$

where $C_0 = \int_0^1 h(t)dt - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2}]$. For $1 \le y \le 2$ we extend H linearly by setting

$$H(z) = z + (2 - y)C_0, \quad z = x + iy.$$

Finally, we set $H(z) = z$ if $y = Im(z) \ge 2$. It is easy to check that $H(z + k) = H(z) + k$ for $k \in \mathbb{Z}$. We set

$$\tilde{G} = \mathbf{e} \circ H \circ L, \tag{4.4}$$

where $\mathbf{e} : z \mapsto e^{2\pi iz}$ is the lifting mapping and $L : z \mapsto \frac{\log z}{2\pi i}$ is the logarithmic mapping. We claim that $\tilde{G} : \mathbb{D}\backslash\{0\} \to \mathbb{D}\backslash\{0\}$ is a well-defined homeomorphism. To see thisu], let $z = re^{0i} = re^{2\pi i}$ be as in Figure 1. We need to show that L is well-defined on the segment $P := \{z : r \le |z| \le 1\}$. Note that in Figure 1, the vertical line $[0, L(z)]$ corresponds to the image of P with argument 0 and the vertical line $[1, L(z) + 1]$ corresponds to the image of P with argument 2π under the mapping L. Note also that $L(re^{0i}) = \frac{\log r}{2\pi i}$ and $L(re^{2\pi i}) = \frac{\log r + 2\pi i}{2\pi i}$. Since H satisfies that $H(z+1) = H(z) + 1$ and $\mathbf{e}$ is 1-periodic, the mapping $\tilde{G}$ is a homeomorphism in the annulus $\mathbb{D}\backslash B(0,r)$. Moreover,

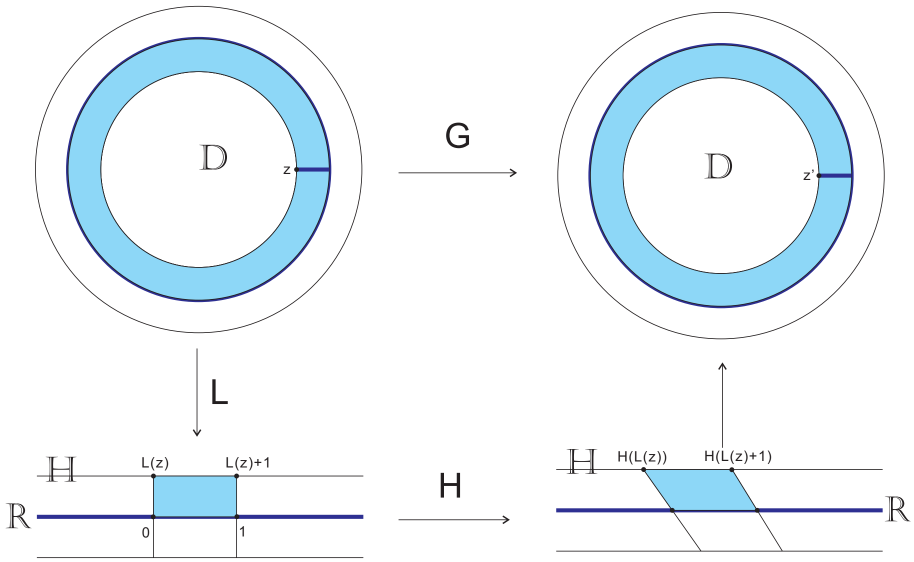

FIGURE 1. The homeomorphism $\tilde{G}$

$\tilde{G} = G$ on $\partial\mathbb{D}$ and $\tilde{G}(z) = z$ for $0 < |z| \leq \delta := e^{-4\pi}$. Thus $\tilde{G}$ is well-defined homeomorphism of the unit disk if we additionally set $\tilde{G}(0) = 0$.

Finally, we may define our mapping $\hat{G} : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$\hat{G}(z) = \begin{cases} \tilde{G}(z) & \text{if } |z| \geq 1 \\ R \circ \tilde{G} \circ R(z) & \text{if } |z| \leq 1, \end{cases}$$

where $R(z) = \frac{1}{\bar{z}}$ is the inversion with respect to the unit circle. To complete the proof of Proposition 4.1, we need to estimate the distortion of $\hat{G}$.

It is clear that we only need to estimate the distortion of $\tilde{G}$. Since $\mathbf{e}$ and L are conformal mappings, it follows that

$$K_{\tilde{G}}(z) = K_H(\omega), \quad z = e^{2\pi i\omega},\ \omega \in \mathbb{H}.$$

So we reduce all distortion estimates for $\tilde{G}$ to the corresponding ones for H. Since H is conformal for $y > 2$ and linear for $y \in [1,2]$, it suffices to estimate K_H in the strip $S = \mathbb{R} \times [0,1]$. The desired estimate

$$K_H(x+iy) \leq C_0\rho_h(y), \quad x + iy \in S$$

follows from the calculation in [6], where

$$\begin{aligned}\rho_h(t) &= \sup_{\theta\in\mathbb{R}} \delta_h(\theta,t)\\ &:= \sup_{\theta\in\mathbb{R}} \max\left\{\frac{|h(\theta+t)-h(\theta)|}{|h(\theta)-h(\theta-t)|}, \frac{|h(\theta-t)-h(\theta)|}{|h(\theta)-h(\theta+t)|}\right\}.\end{aligned}$$

Note that if $t \in [0,1]$, then

$$\delta_G(\theta,t) \approx \delta_h(\theta,t)$$

and hence

$$\rho_G(t) \approx \rho_h(t).$$

The proof of Proposition 4.1 is complete.

As an application of Proposition 4.1, we easily obtain the following corollary.

Let δ be as in Proposition 4.1 and let $\varepsilon < \delta$ is sufficiently small such that

$$\log|z| \le 2|r-1|$$

for $z = re^{i\theta} \in A := \overline{B(0,1+\varepsilon)}\backslash B(0,1-\varepsilon)$. Proposition 4.1 implies that

$$K_{\hat{G}}(z) \le C\rho_G(\log|z|) \le C\rho_G(2|r-1|)$$

for $z \in A$. If $\rho_G(t) \le C' t^{-\alpha}$ as $t \to 0$, then

$$K_{\hat{G}}(z) \le C|r-1|^{-\alpha},$$

for $z \in A$. Integrating in polar coordinates, we immediately obtain the following corollary.

Corollary 4.2. *Let $G : \partial\mathbb{D} :\to \partial\mathbb{D}$ be a conformal welding. If*

$$\rho_G(t) = O(\log\frac{1}{t}) \quad as \quad t \to 0,$$

then G extends to a homeomorphism of the entire plane of locally exponentially integrable distortion. Moreover, if

$$\rho_G(t) = O(t^{-\alpha}) \quad as \quad t \to 0$$

for some $\alpha > 0$, then G extends to a homeomorphism of the entire plane of locally p-integrable distortion with any $p \in (0, \frac{1}{\alpha})$.

5. Main proofs

Theorem 1.1 follows from the following more general result.

Theorem 5.1. *If $\Omega \subset \mathbb{R}^2$ is a Jordan domain that has the three point property with a control function ψ such that*

$$\limsup_{r\to 0} \frac{r}{\psi^{-1}\circ\psi^{-1}(r)\log\frac{1}{r}} \le C' \tag{5.1}$$

for some constant C', then Ω is a generalized quasidisk.

Proof of Theorem 5.1. The idea is similar to that used in [12, Theorem 5.1]. Since Ω is a Jordan domain, f extends to a homeomorphism between $\mathbb{D}$ and $\overline{\Omega}$ and we denote also this extension by f. Let $e^{i(\theta-t)}$, $e^{i\theta}$ and $e^{i(\theta+t)}$ be three points on S. Since f is a sense-preserving homeomorphism, $f(e^{i(\theta-t)})$, $f(e^{i\theta})$ and $f(e^{i(\theta+t)})$ will be on the boundary of Ω in order. Let $g\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ be a conformal mapping from the Riemann Mapping Theorem. Then g extends to a homemorphism between $\mathbb{R}^2 \setminus \mathbb{D}$ and $\mathbb{R}^2 \setminus \Omega$. As before, we still denote this extension by g. Based on the discussion in the previous section, we only need to estimate the scale-wise distortion of the conformal welding $G := g^{-1} \circ f$.

Let $P = e^{i(\theta+\pi)}$ be the anti-polar point of $e^{i\theta}$ on $\partial\mathbb{D}$ and let $\gamma_f(P, \theta - t)$ denote the arc from $f(P)$ to $f(e^{i(\theta-t)})$ on $\partial\Omega$. There exists a t_0 small enough such that $\mathrm{diam}(\gamma_f(-1, \theta - t)) \geq \mathrm{diam}(\gamma_f(\theta, \theta + t))$ when $t \in [0, t_0)$. Let Γ_1 be the family of curves in $\mathbb{D}$ joining $\gamma(P, e^{i(\theta-t)})$ and $\gamma(e^{i\theta}, e^{i(\theta+t)})$. Then Lemma 3.3 implies that

$$\mathrm{mod}(\Gamma_1) \leq C_1 \tag{5.2}$$

for some absolute constant $C_1 > 0$. The conformal invariance of modulus gives us that

$$\mathrm{mod}(\Gamma) := \mathrm{mod}(f(\Gamma_1)) \leq C_2. \tag{5.3}$$

Thus, we may use Lemma 3.6 for $\alpha_1 = \gamma_f(\theta, \theta+t)$ and $\alpha_2 = \gamma_f(P, \theta-t)$ and conclude that

$$\mathrm{diam}(\gamma_f(\theta, \theta + t)) \leq \psi \circ \psi(d),$$

where $d = d(\alpha_1, \alpha_2)$ is the distance between these two arcs. Moreover,

$$\mathrm{mod}(\Gamma') \leq C \log^{-1}\Big(1 + \frac{\psi^{-1} \circ \psi^{-1}(\mathrm{diam}(\alpha_1))}{\mathrm{diam}(\alpha_1)}\Big), \tag{5.4}$$

where Γ' is the family of curves joining α_1 and α_2 in $\mathbb{R}^2 \setminus \overline{\Omega}$. Again by conformal invariance of modulus, we obtain that

$$\log^{-1}(1 + \frac{1}{\delta_G(\theta, t)}) \leq C_0^{-1}\,\mathrm{mod}(\Gamma'), \tag{5.5}$$

where C_0 is the constant from Lemma 3.3. Note that

$$\frac{1}{\log(1+t)} \approx \frac{1}{t} \quad \text{as} \quad t \to 0.$$

Combining (5.4) with (5.5) gives us the estimate

$$\delta_G(\theta, t) \leq \frac{C\,\mathrm{diam}(\alpha_1)}{\psi^{-1} \circ \psi^{-1}(\mathrm{diam}(\alpha_1))}.$$

On the other hand, by applying Lemma 3.5 and noticing that our technical assumptions on ψ implies that $\psi(t) \geq Ct^\alpha$ for some $\alpha > 0$, we obtain that

$$\mathrm{diam}(\alpha_1) \geq C\psi(t^2) \geq Ct^{2\alpha}.$$

Since $\frac{t}{\psi^{-1}\circ\psi^{-1}(t)}$ is non-increasing, we obtain that

$$\delta_G(\theta,t) \le \frac{Ct^{2\alpha}}{\psi^{-1}\circ\psi^{-1}(t^{2\alpha})}. \tag{5.6}$$

Theorem 5.1 follows immediately from (5.1), (5.6) and Corollary 4.2. □

Proof of Theorem 1.2. This is basically contained in the proof of Theorem 5.1. In this case, the desired bound (5.6) reads as follows:

$$\delta_G(\theta,t) \le Ct^{2(1-\frac{1}{s^2})}.$$

The claim follows directly from Corollary 4.2 with $\alpha = 2(\frac{1}{s^2}-1)$. □

Proof of Theorem 1.3. If Ω is LLC-1, then Lemma 3.5 implies that f^{-1} is uniformly Hölder continuous. On the other hand, the duality result implies that $\mathbb{R}^2\backslash\overline{\Omega}$ is LLC-2, which is further equivalent to $\mathbb{R}^2\backslash\overline{\Omega}$ being John by the results in [22]. Then by the results in [18], g is also Hölder continuous. Hence G^{-1} is uniformly Hölder continuous with some exponent α. Therefore, for t sufficiently small, we have

$$\begin{aligned}\delta_G(\theta,t) &\le \max\left\{\frac{|G(e^{i(\theta+t)})-G(e^{i\theta})|}{|G(e^{i\theta})-G(e^{i(\theta-t)})|}, \frac{|G(e^{i\theta})-G(e^{i(\theta-t)})|}{|G(e^{i(\theta+t)})-G(e^{i\theta})|}\right\}\\ &\lesssim t^{-1/\alpha}.\end{aligned}$$

The claim follows from Corollary 4.2. □

6. Concluding remarks

6.1. **Definition of generalized quasidisks.** This was previously discussed briefly in the introduction. Recall that $\Omega\subset\mathbb{R}^2$ is a generalized quasidisk if it is the image of the unit disk $\mathbb{D}$ under a homeomorphism $f\colon\mathbb{R}^2\to\mathbb{R}^2$ of the entire plane with locally exponentially integrable distortion and f is conformal in the unit disk $\mathbb{D}$. However, this is not natural from the technical point of view since our extended mapping $\hat{f}$ is the identity outside a compact disk.

On the other hand, from the point view of conformal welding, requiring that f is identity at infinity is reasonable since it makes the two extension problems equivalent as discussed in Section 4.

From the point view of finding a geometric characterization of generalized quasidisks, this additional requirement is also natural. More precisely, the geometry of a generalized quasidisk $\Omega\subset\mathbb{R}^n$ should be determined by the geometry of its boundary (at least this is the case if Ω is a quasidisk). Intuitively the geometry of $\partial\Omega$ should have nothing to do with the behavior of the global homeomorphism f at infinity.

These observations suggest that it is better to require the global homeomorphism f to be identity at ∞ in the definition of a generalized quasidisk.

6.2. **Inward pointing and outward pointing cusps.** As we already observed, (polynomial) interior cusps are more dangerous than (polynomial) exterior cusps for our extension problems. This is not a big surprise from the technical point of view since our aim is to estimate the scalewise distortion of our conformal welding G. It is fairly easy to observe that this is closely related to the modulus of continuity of G^{-1}. On the other hand, combining the duality results in [11] with the global Hölder continuity estimates of conformal mappings in [10, 18], one can immediately see how the role of Ω being φ-LC-1 or ψ-LC-2 is related the modulus of continuity of G and G^{-1}. In fact, this is exactly the way we proved [12, Theorem 4.4].

6.3. **Open problems.** To end the article, we put forward some open problems, which appear reasonable.

Problem 6.1. *In Theorem 1.1, can we further relax the control function ψ to be of the form $\psi(t) = Ct\log\frac{1}{t}$? By the result in* [12]*, we know that the result fails for $\psi(t) = Ct\log^{1+\delta}\frac{1}{t}$ for any $\delta > 0$.*

Problem 6.2. *In Theorem 1.3, can we conclude that the extension has better integrability for the distortion, say locally exponentially integrable distortion, if we additionally assume that Ω is ψ-LC-2 for $\psi(t) = t^s$ with $s > 1$?*

Problem 6.3. *If we require reasonable good moduli of continuity for both f, g and their inverses, say both f and g are bi-Hölder continuous up to boundary, can we conclude that Ω is a generalized quasidisk ?*

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(Chang-Yu Guo) Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, FI-40014 University of Jyväskylä, Finland

E-mail address: changyu.c.guo@jyu.fi