Brunn-Minkowski inequality for the 1-Riesz capacity and level set convexity for the 1/2-Laplacian

M. Novaga [*], B. Ruffini [†]

Abstract

We prove that that the 1-Riesz capacity satisfies a Brunn-Minkowski inequality, and that the capacitary function of the 1/2-Laplacian is level set convex.

Keywords: fractional Laplacian; Brunn-Minkowski inequality; level set convexity; Riesz capacity.

1 Introduction

In this paper we consider the following problem

$$\begin{cases} (-\Delta)^s u = 0 & \text{on } \mathbb{R}^N \setminus E \\ u = 1 & \text{on } E \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{1}$$

where $N \geq 2$, $s \in (0, N/2)$, and $(-\Delta)^s$ stands for the s-fractional Laplacian, defined as the unique pseudo-differential operator $(-\Delta)^s : \mathcal{S} \mapsto L^2(\mathbb{R}^N)$, being $\mathcal{S}$ the Schwartz space of functions with fast decay to 0 at infinity, such that

$$\mathcal{F}\left((-\Delta)^s f\right)(\xi) = |\xi|^{2s}\mathcal{F}(f)(\xi) \qquad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform. We refer to the guide [12, Section 3] for more details on the subject. A quantity strictly related to Problem (1) is the so-called *Riesz potential energy* of a set E, defined as

$$I_\alpha(E) = \inf_{\mu(E)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} \qquad \alpha \in (0, N). \tag{2}$$

[*]Dipartmento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
email: novaga@dm.unipi.it

[†]Institut Fourier, 100 rue des maths, BP 74, 38402 St Martin d'Hères cedex, France
email: berardo.ruffini@ujf-grenoble.fr

It is possible to prove (see [19]) that if E is a compact set, then the infimum in the definition of $I_\alpha(E)$ is achieved by a unique Radon measure μ supported on the boundary of E if $\alpha \leq N-2$, and with support equal to the whole E if $\alpha \in (N-2, N)$. If μ is the optimal measure for the set E, we define the *Riesz potential* of E as

$$u(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\alpha}}, \tag{3}$$

so that

$$I_\alpha(E) = \int_{\mathbb{R}^N} u(x) d\mu(x).$$

It is not difficult to check (see [19, 16]) that the potential u satisfies, in distributional sense, the equation

$$(-\Delta)^{\frac{\alpha}{2}} u = c(\alpha, N)\, \mu,$$

where $c(\alpha, N)$ is a positive constant, and that $u = I_\alpha(E)$ on E. In particular, if $s = \alpha/2$, then $u_E = u/I_{2s}(E)$ is the unique solution of Problem (1).
Following [19], we define the *α-Riesz capacity* of a set E as

$$\mathrm{Cap}_\alpha(E) = \frac{1}{I_\alpha(E)}. \tag{4}$$

We point out that this is not the only concept of capacity present in literature. Indeed, another one is given by the p-capacity of a set E, which for $N \geq 3$ and $p > 1$ is defined as

$$\mathcal{C}_p(E) = \min \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^p : \varphi \in C^1_c(\mathbb{R}^N, [0,1]),\ \varphi \geq \chi_E \right\} \tag{5}$$

where χ_E is the characteristic function of the set E. It is possible to prove that, if E is a compact set, then the minimum in (5) is achieved by a function u satisfying

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 & \text{on } \mathbb{R}^N \setminus E \\ u = 1 & \text{on } E \\ \lim_{|x| \to +\infty} u(x) = 0. \end{cases} \tag{6}$$

Such a function u is usually called *capacitary function* of the set E. It is worth noticing that 2-Riesz capacity and the 2-capacity coincide on compact sets. We refer to [20, Section 11.15] for a discussion on this topic.

In a series of works (see for instance [7, 11, 18] and the monograph [17]) it has been proved that the solutions of (6) are level set convex provided E is a convex body, that is, a compact convex set with non-empty interior. Moreover, in [2] (and in [10] in a more general setting) it has been proved that the 2-capacity satisfies a suitable version of the Brunn-Minkowski inequality: given two convex bodies K_0 and K_1 in $\mathbb{R}^N$, for any $\lambda \in [0,1]$ it holds

$$\mathcal{C}_2(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-2}} \geq \lambda\, \mathcal{C}_2(K_1)^{\frac{1}{N-2}} + (1-\lambda)\, \mathcal{C}_2(K_0)^{\frac{1}{N-2}}.$$

We refer to [21, 15] for comprehensive surveys on the Brunn-Minkowski inequality.

The main purpose of this paper is to show analogous results in the fractional setting $\alpha = 1$, that is, $s = 1/2$ in Problem (1). More precisely, we shall prove the following result.

Theorem 1.1. *Let $K \subset \mathbb{R}^N$ be a convex body and let u be the solution of Problem* (1) *with $s = 1/2$ and $E = K$. Then*

(i) u is continuous and level set convex, that is, for every $c \in \mathbb{R}$ the level set $\{u > c\}$ is convex;

(ii) the 1-Riesz capacity $\mathrm{Cap}_1(K)$ satisfies the following Brunn-Minkowski inequality: for any couple of convex bodies K_0 and K_1 and for any $\lambda \in [0,1]$ we have

$$\mathrm{Cap}_1(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-1}} \geq \lambda \mathrm{Cap}_1(K_1)^{\frac{1}{N-1}} + (1-\lambda)\mathrm{Cap}_1(K_0)^{\frac{1}{N-1}}. \tag{7}$$

The strategy of the proof of Theorem 1.1 is the following. First, for a continuous and bounded function $u : \mathbb{R}^N \to \mathbb{R}$, we consider the (unique) bounded solution (see [14]) of the problem

$$\begin{cases} -\Delta_{(x,t)} U &= 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty) \\ U(\cdot,0) &= u(\cdot) \qquad \text{in } \mathbb{R}^N. \end{cases} \tag{8}$$

It holds true the following classical result (see for instance [9]).

Lemma 1.2. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be smooth and bounded, and let $U : \mathbb{R}^N \times [0,+\infty)$ be the solution of Problem* (8)*. There holds*

$$\lim_{t\to 0^+} \partial_t U(x,t) = (-\Delta)^{\frac{1}{2}} u(x) \qquad \textit{for any } x \in \mathbb{R}^N. \tag{9}$$

Thanks to Lemma 1.2 we are able to show that the (unique) solution of Problem (1) with $s = 1/2$ is the trace of the capacitary function U in $\mathbb{R}^{N+1}$ of $K \subset \mathbb{R}^{N+1}$. This allows us to exploit results which hold for capacitary functions of a convex set contained in [10, 11]. Eventually we show that the results obtained in the $(N+1)$-dimensional setting for U hold true as well for u.

We point out that our results do not extend straightfowardly to solutions of (1) with a general s. Indeed, for $s \neq 1/2$ the extension function U in Lemma 1.2 satisfies a more complicated equation (see [9]), to which the results in [10, 11] do not directly apply.

Eventually, in Section 3 we provide an application of Theorem 1.1 and we state some open problems.

2 Proof of the main result

This section is devoted to the proof of Theorem 1.1. We start by an approximation result, which extends most of the results in [11, 10], originally aimed to convex bodies, to general convex sets with positive capacity.

Lemma 2.1. *Let K be a compact convex set with positive 2-capacity and let, for $\varepsilon > 0$, $K_\varepsilon = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) \le \varepsilon\}$. Letting u_ε and u be the capacitary functions of K_ε and K respectively, we have that u_ε converges uniformly to u as $\varepsilon \to 0$. Moreover we have that the level sets $\{u_\varepsilon > s\}$ converge to $\{u > s\}$ for any $1 > s > 0$, with respect to the Hausdorff distance.*

Proof. Let us prove that $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$. Since $u_\varepsilon - u$ is a harmonic function on $\mathbb{R}^N \setminus K_\varepsilon$, we have that

$$\sup_{\mathbb{R}^N \setminus K_\varepsilon} |u_\varepsilon - u| \le \sup_{\partial K_\varepsilon} |u_\varepsilon - u| \le 1 - \min_{\partial K_\varepsilon} u. \tag{10}$$

Since ∂K_ε Hausdorff converge to ∂K, we get that the right-hand side of (10) converges to 0 as $\varepsilon \to 0$. To prove that the level sets $\{u_\varepsilon > s\}$ converge to $\{u > s\}$ for any $1 > s > 0$, with respect to the Hausdorff distance, we begin by showing that the following equality

$$\overline{\{v > s\}} = \{v \ge s\} \tag{11}$$

holds true for $v = u$ or $v = u_\varepsilon$, $\varepsilon > 0$. Indeed, let $x \in \overline{\{v > s\}}$, with $v = u$ or $v = u_\varepsilon$. Then there exists $x_k \to x$ such that $v(x_k) > s$. Thus

$$v(x) = \lim_{k \to \infty} v(x_k) \ge s,$$

and so $x \in \{v \ge s\}$.
Suppose now that $v(x) \ge s$ but $x \notin \overline{\{v > s\}}$. Notice that in this case $v(x) = s$. Suppose moreover that there exists an open neighborhood A of x such that $A \cap \{v > s\} = \emptyset$, so that $v \le s$ on A. In this case we get that x is a local maximum in A, and v is harmonic in a neighborhood of A. This leads to a contradiction thanks to the maximum principle for hamonic functions. To conclude the proof of (11), we just notice that if such an A does not exist, then x is an adjacency point for $\{v > s\}$ so that it belongs to $\overline{\{v > s\}}$.

Suppose by contradiction that there exist $c > 0$ and a sequence $x_\varepsilon \in \{u_\varepsilon > s\}$ such that $\operatorname{dist}(x_n, \{u > s\}) \ge c > 0$. Recalling that $K_{\varepsilon_0} \subset K_{\varepsilon_1}$ if $\varepsilon_0 < \varepsilon_1$, and applying again the maximum principle, it is easy to show that $\{u_\varepsilon \ge s\}$ is a family of uniformly bounded compact sets. Thus, there exists an $x \in \mathbb{R}^N$ such that x_ε converges to x, up to extracting a (not relabeled) subsequence. By uniform convergence we have that

$$|u_\varepsilon(x_\varepsilon) - u(x)| \le |u_\varepsilon(x_\varepsilon) - u(x_\varepsilon)| + |u(x_\varepsilon) - u(x)| \to 0$$

as $\varepsilon \to 0$. Thus, for any $\delta > 0$ there exists ε such that

$$u(x) \geq u_\varepsilon(x_\varepsilon) - \delta \geq s - \delta,$$

whence $u(x) \geq s$. But thanks to (11) we have that

$$0 < c \leq \operatorname{dist}(x, \{u > s\}) = \operatorname{dist}(x, \overline{\{u > s\}}) = \operatorname{dist}(x, \{u \geq s\}) = 0.$$

This is a contradiction and thus the proof is concluded.

□

Remark 2.2. Notice that a compact convex set has positive 2-capacity if and only if its $\mathcal{H}^{N-1}$-measure is non-zero (see for instance [13]). In particular if K is a convex body of $\mathbb{R}^N$, then, although its $(N+1)$-Lebesgue measure is 0, K has positive capacity in $\mathbb{R}^{N+1}$.

To prove Theorem 1.1, we wish to apply Lemma 1.2 to the function $u = u_K$. Since u_K is not a smooth function (being non-regular on the boundary of K) for our purposes we need a weaker version of Lemma 1.2.

Lemma 2.3. *Let $u : \mathbb{R}^N \to [0, +\infty)$ be a continuous, bounded function and let U be the extension of u in the sense of Lemma 1.2. Suppose that u is of class C^1 in a neighbourhood of $x \in \mathbb{R}^N$. Then the partial derivative $\partial_t U$ of U with respect to the last coordinate is well defined at the point $(x, 0)$, and it holds*

$$\partial_t U(x, 0) = (-\Delta)^{1/2} u(x).$$

Proof. For $\varepsilon > 0$ let $u_\varepsilon = u * \rho_\varepsilon$ where ρ_ε is a mollifying smooth kernel and U_ε is the extension of u_ε in the sense of Lemma 1.2. Then, since u_ε is a regular bounded function, by Lemma 1.2 we have $\partial_t U_\varepsilon(x, 0) = (-\Delta)^{1/2} u_\varepsilon(x)$ for every $x \in \mathbb{R}^N$. If u is C^1-regular in a neighbourhood A of x, then so is U on $A \times [0, \infty)$ and it holds $\partial_t U_\varepsilon(x, 0) \to \partial_t U(x, 0)$ as $\varepsilon \to 0$. Hence, in order to conclude, we only need to check that $(-\Delta)^{1/2} u_\varepsilon(x)$ converges to $(-\Delta)^{1/2} u(x)$ as $\varepsilon \to 0$. To do this, it is sufficient to show that it holds $(-\Delta)^{1/2}(u * \rho_\varepsilon)(x) = ((-\Delta)^{1/2} u) * \rho_\varepsilon(x)$. This latter fact is true since, recalling that u is bounded, we have

$$\mathcal{F}^{-1}\left(\mathcal{F}((-\Delta)^{1/2}(u * \rho_\varepsilon))\right)(x) = \mathcal{F}^{-1}\left(|\xi|^{1/2} \mathcal{F}(u) \mathcal{F}(\rho_\varepsilon)\right)(x) = (-\Delta)^{1/2} u * \rho_\varepsilon(x).$$

□

We recall the well known fact that the capacitary function of a compact set $K \subset \mathbb{R}^N$ of positive capacity is continuous on $\mathbb{R}^N$. We offer a simple proof of this fact for the reader's convenience.

Lemma 2.4. *Let $K \subset \mathbb{R}^N$ be a compact set of strictly positive capacity and let u be its capacitary function. Then u is continuous on $\mathbb{R}^N$.*

Proof. Since u is harmonic on $\mathbb{R}^N \setminus K$ and it is constantly equal to 1 on K, we only have to show that if $x \in \partial K$ then $u(y) \to 1$ as $y \to x$. To do this, we recall that u is a lower semicontinuous function, which follows, for instance, from the fact that u is the convolution of a positive kernel and a non-negative Radon measure (see for instance [19, pag. 59]). Hence, since K is closed, we get

$$1 = u(x) \leq \liminf_{y\to x} u(y) \leq \limsup_{y\to x} u(y) \leq 1,$$

where the last inequality is due to the fact that, thanks to the maximum principle, $0 \leq u \leq 1$. □

Proof of Theorem 1.1. Let us prove claim (i). We begin by showing that u is a continuous function. Indeed, let U be the capacitary function of K in $\mathbb{R}^{N+1}$ (in this setting, K is contained in the hyperspace $\{x_{N+1} = 0\}$), that is, let U be the solution of

$$\begin{cases} -\Delta_{(x,t)} U = 0 & \text{in } \mathbb{R}^{N+1} \\ U(x,0) = 1 & x \in K \\ \lim_{|(x,t)|\to+\infty} U(x,t) = 0. \end{cases} \tag{12}$$

Then, since K is symmetric with respect to the hyperplane $\{x_{N+1} = 0\} = \mathbb{R}^N$, also U is symmetric with respect to the same hyperplane, as can be shown by applying a suitable version of the Pólya-Szegö inequality for the Steiner symmetrization (see for instance [3, 5]). Let $v(x) = U(x,0)$. Notice that v is a continuous function, since U is continuous, being the capacitary function of a compact set of positive capacity (and thanks to Lemma 2.4).

Let us prove that v is the solution of (1). It is clear that $\lim_{|x|\to\infty} v(x) = 0$ and that $v(x) = 1$ if $x \in K$. Moreover we have that v is bounded and regular on $\mathbb{R}^N \setminus K$ (being so U) thus we can apply Lemma 2.3 to get that for every $x \in \mathbb{R}^N \setminus K$ we have $(-\Delta)^{s/2} v(x) = \partial_t U(x,0) = 0$, and thus v solves (1). By uniqueness it then follows that $u = v$ is a continuous function.

We now prove that u is level set convex. Notice first that, for any $c \in \mathbb{R}$ we have

$$\{u \geq c\} = \{(x,t) : U(x,t) \geq c\} \cap \{t = 0\}.$$

In particular, the claim is proved if we show that U is level set convex.

We recall from [10] that the capacitary function of a convex body is always level set convex. Let now $K_\varepsilon = \{x : \mathrm{dist}(x,K) \leq \varepsilon\}$ and let u_ε be the capacitary function of K_ε. From Lemma 2.1 we know that, for any $s \in (0,1)$ the level set $\{U > s\}$ is the Hausdorff limit of the level sets $\{u_\varepsilon > s\}$, which are convex by the result in [10]. It follows that U is level set convex, and this concludes the proof of (i).

To prove (ii) we start by noticing that the 1-Riesz capacity is a $(1-N)$-homogeneous functional, hence inequality (7) can be equivalently stated (see for instance [2]) by requiring that, for any couple of convex sets K_0 and K_1 and for any $\lambda \in [0,1]$, the inequality

$$\mathrm{Cap}_1(\lambda K_1 + (1-\lambda)K_0) \geq \min\{\mathrm{Cap}_1(K_0), \mathrm{Cap}_1(K_1)\} \tag{13}$$

holds true.

We divide the proof of (13) into two steps.

Step 1.
We characterize the 1-Riesz capacity of a convex set K as the behaviour at infinity of the solution of the following PDE

$$\begin{cases} (-\Delta)^{1/2} u_K = 0 & \text{in } \mathbb{R}^N \setminus K \\ u_K = 1 & \text{in } K \\ \lim_{|x|\to\infty} |x|^{N-1} u_K(x) = \mathrm{Cap}_1(K). \end{cases}$$

We recall that, if μ_K is the optimal measure for the minimum problem in (2), then the function

$$u(x) = \int_{\mathbb{R}^N} \frac{d\mu_K(y)}{|x-y|^{N-1}}$$

is harmonic on $\mathbb{R}^N \setminus K$ and is constantly equal to $I_1(K)$ on K (see for instance [16]). Moreover the optimal measure μ_K is supported on K, so that $|x|^{N-1}u(x) \to \mu_K(K) = 1$ as $|x| \to \infty$. The claim follows by letting $u_K = u/I_1(K)$.

Step 2.
Let $K_\lambda = \lambda K_1 + (1-\lambda)K_0$ and $u_\lambda = u_{K_\lambda}$. We want to prove that

$$u_\lambda(x) \geq \min\{u_0(x), u_1(x)\}$$

for any $x \in \mathbb{R}^N$. To this aim we define the auxiliary function (first introduced in [2])

$$\widetilde{u}_\lambda(x) = \sup\big\{\min\{u_0(x_0), u_1(x_1)\} : x_0, x_1 \in \mathbb{R}^N,\ x = \lambda x_1 + (1-\lambda)x_0\big\},$$

and we notice that the claim follows if we show that $u_\lambda \geq \widetilde{u}_\lambda$. An equivalent formulation of this statement is to require that for any $s > 0$ we have

$$\{\widetilde{u}_\lambda > s\} \subseteq \{v_\lambda > s\}. \tag{14}$$

A direct consequence of the definition of $\widetilde{u}_\lambda$ is that

$$\{\widetilde{u}_\lambda > s\} = \lambda\{u_1 > s\} + (1-\lambda)\{u_0 > s\}.$$

For all $\lambda \in [0,1]$, we let U_λ be the harmonic extension of u_λ on $\mathbb{R}^N \times [0,\infty)$, which solves

$$\begin{cases} -\Delta_{(x,t)} U_\lambda = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ U_\lambda(x,0) = u_\lambda(x) & \text{in } \mathbb{R}^N \times \{0\} \\ \lim_{|(x,t)|\to\infty} U_\lambda(x,t) = 0. \end{cases} \tag{15}$$

Notice that U_λ is the capacitary function of K_λ in $\mathbb{R}^{N+1}$, restricted to $\mathbb{R}^N \times [0,+\infty)$. Letting $H = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} : t = 0\}$, for any $\lambda \in [0,1]$ and $s \in \mathbb{R}$ we have

$$\{U_\lambda > s\} \cap H = \{u_\lambda > s\}.$$

Letting also

$$\widetilde{U}_\lambda(x,t) = \sup\{\min\{U_0(x_0,t_0), U_1(x_1,t_1)\} : (x,t) = \lambda(x_1,t_1) + (1-\lambda)(x_0,t_0)\}, \tag{16}$$

as above we have that

$$\{\widetilde{U}_\lambda > s\} = \lambda\{U_1 > s\} + (1-\lambda)\{U_0 > s\}.$$

By applying again Lemma 2.1 to the sequences $K_0^\varepsilon = K_0 + B(\varepsilon)$ and $K_1^\varepsilon = K_1 + B(\varepsilon)$, we get that the corresponding capacitary functions, denoted respectively as U_0^ε and U_1^ε, converge uniformly to U_0 and U_1 in $\mathbb{R}^N$, and that $\widetilde{U}_\lambda^\varepsilon$, defined as in (16), converges uniformly to $\widetilde{U}_\lambda$ on $\mathbb{R}^N \times [0,+\infty)$.

Since $\widetilde{U}_\lambda^\varepsilon(x,t) \le U_\lambda^\varepsilon(x,t)$ for any $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, as shown in [10, pages $474-476$], we have that $\widetilde{U}_\lambda(x,t) \le U_\lambda(x,t)$. As a consequence, we get

$$\begin{aligned}\{u_\lambda > s\} = \{U_\lambda > s\} \cap H \supseteq \{\widetilde{U}_\lambda > s\} \cap H = \Big[\lambda\{U_1 > s\} + (1-\lambda)\{U_0 > s\}\Big] \cap H \\ \supseteq \lambda\{U_1 > s\} \cap H + (1-\lambda)\{U_0 > s\} \cap H = \lambda\{u_1 > s\} + (1-\lambda)\{u_0 > s\}\end{aligned}$$

for any $s > 0$, which is the claim of *Step 2*.

We conclude by observing that inequality (13) follows immediately, by putting together *Step* 1 and *Step* 2. This concludes the proof of (ii), and of the theorem. □

Remark 2.5. The fact that the solution of (1) is a continuous function can be proved without using the extension problem thanks to the formulation (3) which entails that u is a superharmonic function and the Evans Theorem (see for instance [6, Theorem 1]). We used a less direct approach to show at once the fact that u can be seen as the trace of the capacitary function U of K in $\mathbb{R}^{N+1}$.

Remark 2.6. The equality case in the Brunn-Minkowski inequality (7) is not easy to address by means of our techniques. The problem is not immediate even in the case of the 2-capacity. In that case it has been studied in [8, 10].

3 Applications and open problems

In this section we state a corollary of Theorem 1.1. To do this we introduce some tools which arise in the study of convex bodies. The *support function* of a convex body $K \subset \mathbb{R}^N$ is defined on the unit sphere centred at the origin $\partial B(1)$ as

$$h_K(\nu) = \sup_{x \in \partial K} \langle x, \nu \rangle.$$

The *mean width* of a convex body K is

$$M(K) = \frac{2}{\mathcal{H}^{N-1}(\partial B(1))} \int_{\partial B(1)} h_K(\nu)\, d\mathcal{H}^{N-1}(\nu).$$

We refer to [21] for a complete reference on the subject. We observe that, if $N = 2$, then $M(K)$ coincides up to a constant with the perimeter $P(K)$ of K (see [4]).

We denote by $\mathcal{K}_N$ the set of convex bodies of $\mathbb{R}^N$ and we set

$$\mathcal{K}_{N,c} = \{K \in \mathcal{K}_N,\, M(K) = c\}.$$

The following result has been proved in [4, 1].

Theorem 3.1. *Let $F : \mathcal{K}_N \to [0, \infty)$ be a q-homogeneous functional which satisfies the Brunn-Minkowski inequality, that is, such that $F(K + L)^{1/q} \geq F(K)^{1/q} + F(L)^{1/q}$ for any $K, L \in \mathcal{K}_N$. Then the ball is the unique solution of the problem*

$$\min_{K \in \mathcal{K}_N} \frac{M(K)}{F^{1/q}(K)}. \tag{17}$$

An immediate consequence of Theorem 3.1, Theorem 1.1 and Definition (4) is the following result.

Corollary 3.2. *The minimum of I_1 on the set $\mathcal{K}_{N,c}$ is achieved by the ball of mean width c. In particular, if $N = 2$, the ball of radius r solves the isoperimetric type problem*

$$\min_{K \in \mathcal{K}_2, P(K) = 2\pi r} I_1(K). \tag{18}$$

Motivated by Theorem 1.1 and Corollary 3.2 we conclude the paper with the following conjecture:

Conjecture 3.3. *For any $N \geq 2$ and $\alpha \in (0, N)$, the α-Riesz capacity $\mathrm{Cap}_\alpha(K)$ satisfies the following Brunn-Minkowski inequality:*
for any couple of convex bodies K_0 and K_1 and for any $\lambda \in [0, 1]$ we have

$$\mathrm{Cap}_\alpha(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-\alpha}} \geq \lambda \mathrm{Cap}_\alpha(K_1)^{\frac{1}{N-\alpha}} + (1-\lambda)\mathrm{Cap}_\alpha(K_0)^{\frac{1}{N-\alpha}}. \tag{19}$$

In particular, for any $\alpha \in (0, 2)$ the ball of radius r is the unique solution of the isoperimetric type problem

$$\min_{K \in \mathcal{K}_2, P(K) = 2\pi r} I_\alpha(K). \tag{20}$$

Acknowledgements

The authors wish to thank G. Buttazzo and D. Bucur for useful discussions on the subject of this paper. The authors were partially supported by the Project ANR-12-BS01-0014-01 *Geometrya*, and by the GNAMPA Project 2014 *Analisi puntuale ed asintotica di energie di tipo non locale collegate a modelli della fisica.*

References

[1] C. Bianchini, P. Salani: *Concavity properties for elliptic free boundary problems*, Nonlinear Anal., 71 n.10 (2009), 4461-4470; 72 n. 7-8 (2010), 3551.

[2] C. Borell: *Capacitary inequalities of the Brunn–Minkowski type*, Math. Ann., **263** (1984), 179–184.

[3] F. Brock: *Weighted Dirichlet-type inequalities for Steiner Symmetrization*, Calc. Var. Partial Differential Equations, **8** (1999), 15–25.

[4] D. Bucur, I. Fragalà, J. Lamboley: *Optimal convex shapes for concave functionals*, ESAIM Control Optim. Calc. Var., **18** (2012), 693–711.

[5] A. Burchard: *Steiner symmetrization is continuous in* $W^{1,p}$, Geom. Funct. Anal., **7** (1997), 823–860.

[6] L. A. Caffarelli: *The obstacle problem revisited*, Journal of Fourier Analysis and Applications, 4 (4), 383–402.

[7] L.A. Caffarelli, J. Spruck: *Convexity of Solutions to Some Classical Variational Problems*, Comm. P.D.E., **7** (1982), 1337–1379.

[8] L.A. Caffarelli, D. Jerison, E.H. Lieb: *On the Case of Equality in the Brunn–Minkowski Inequality for Capacity*, Adv. Math., **117** (1996), 193–207.

[9] L.A. Caffarelli, L. Silvestre: *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, **32** (2007), 1245–1260.

[10] A. Colesanti, P. Salani: *The Brunn-Minkowski inequality for p-capacity of convex bodies*, Math. Ann., **327** (2003), 459–479.

[11] A. Colesanti, P. Salani *Quasi-concave Envelope of a Function and Convexity of Level Sets of Solutions to Elliptic Equations*, Math. Nach., **258** (2003), 3–15.

[12] E. Di Nezza, G. Palatucci, E. Valdinoci *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136** (2012), 521–573.

[13] L.C. Evans, R.F. Gariepy: *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[14] G. Folland: *Lectures on partial differential equations*, Princeton Univ. Press, Princeton, 1976.

[15] R. Gardner: *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc., **353** (2002), 355–405.

[16] M. Goldman, M. Novaga, B. Ruffini: *Existence and stability for a non-local isoperimetric model of charged liquid drops*, Preprint (2013), available at http://cvgmt.sns.it/paper/2267/

[17] B. Kawohl: *Rearrangements and Convexity of Level Sets in P.D.E.*, Lecture Notes in Mathematics, 1150, Springer, Berlin, 1985.

[18] N. Korevaar, *Convexity of Level Sets for Solutions to Elliptic Ring Problems*, Comm. Partial Differential Equations, **15** (1990), 541–556.

[19] N.S. Landkof: *Foundations of Modern Potential Theory*, Springer-Verlag, Heidelberg 1972.

[20] E.H. Lieb, M. Loss: *Analysis*, Graduate Studies in Mathematics, AMS, 2000.

[21] R. Schneider: *Convex bodies: the Brunn-Minkowski theory*, Cambridge Univ. Press, 1993.