FABER-KRAHN INEQUALITIES FOR THE ROBIN-LAPLACIAN: A FREE DISCONTINUITY APPROACH

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Abstract. We introduce a new method to prove the isoperimetric property of the ball for the first eigenvalue of the Robin-Laplacian. Our technique applies to a full range of Faber-Krahn inequalities in a nonlinear setting and for non smooth domains, including the open case of the torsional rigidity. The analysis is based on regularity issues for free discontinuity problems in spaces of functions of bounded variation. As a byproduct, we obtain the best constants for a class of Poincaré inequalities with trace terms in $\mathbb{R}^N$.

Keywords: Faber-Krahn inequality, Robin boundary conditions, free discontinuity problems, functions of bounded variation

Contents

1. Introduction

The isoperimetric property of the ball concerning the first eigenvalue of the Dirichlet-Laplacian, conjectured for plane domains by Lord Rayleigh in 1877, and proved independently by Faber and Krahn in the 1920's, states that if $\Omega \subseteq \mathbb{R}^N$ is open and bounded, then

$$\lambda^D(B) \le \lambda^D(\Omega), \tag{1.1}$$

where B is a ball such that $|B| = |\Omega|$. Here $\lambda^D(\Omega)$ is defined as the lowest value for which the problem

$$\begin{cases} -\Delta u = \lambda^D(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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admits a non trivial solution. Following the review paper [25] (see also [3]), Lord Rayleigh was motivated in his conjecture by the study of the principal frequency of vibration of a plane elastic membrane fixed at its boundary, stating that the circular shape has the lowest mode of vibration (and giving some evidence of it).

Inequality (1.1) is usually referred to as the Faber-Krahn inequality for the first eigenvalue of the Dirichlet-Laplacian. When Ω has irregular boundary, the eigenvalue problem should be interpreted in the weak sense of Sobolev functions $W_0^{1,2}(\Omega)$ vanishing at the boundary.

The modern approach to the proof of the Faber-Krahn inequality is due to Pólya and Szegö and it is described in their book [26]. It relies on the *spherically symmetric decreasing rearrangement* technique applied to the expression of $\lambda^D(\Omega)$ as the Rayleigh quotient

$$\lambda^D(\Omega) = \min_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2\, dx}.$$

Considering the first eigenfunction $u \in W_0^{1,2}(\Omega)$, one obtains a radial symmetric decreasing function $u_* \in W_0^{1,2}(B)$ equimeasurable with u (so that L^p-norms are preserved) such that

$$\int_B |\nabla u_*|^2\, dx \le \int_\Omega |\nabla u|^2\, dx,$$

so that inequality (1.1) readily follows since $\lambda^D(B)$ is lower than the Rayleigh quotient of u^*. The properties of the spherically symmetric decreasing rearrangement show moreover that equality holds in (1.1) if and only if Ω is equivalent to a ball up to negligible sets.

Such an approach provides easily the validity of a whole family of Faber-Krahn inequalities: setting for $1 \le q < \frac{2N}{N-2}$

$$\lambda_q^D(\Omega) := \min_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega |u|^q\, dx\right)^{\frac{2}{q}}}, \tag{1.2}$$

then again

$$\lambda_q^D(B) \le \lambda_q^D(\Omega)$$

where B is a ball such that $|\Omega| = |B|$, and equality holds if and only if Ω is a ball up to negligible sets.

For plane domains, the case $q = 1$ is relevant in the elasticity theory of beams, and goes under the name of *torsion rigidity problem* (see e.g. [27, Section 35]): the inverse of $\lambda_1^D(\Omega)$ is proportional to the *torsional rigidity* of a beam with cross section Ω (here u has the meaning of a *stress function*, its derivatives being connected with the elastic forces inside the beam). That the shape of the cross section which provides the greatest torsional rigidity (under an area constraint) should be a circle was conjectured by Saint Venant in 1856.

The problem is completely different if we consider *Robin boundary conditions* (the case of Neumann conditions being trivial, with infimum equal to 0). Given $\beta > 0$ and $\Omega \subseteq \mathbb{R}^N$ open, bounded and with a sufficiently smooth boundary, $\lambda_\beta^R(\Omega) = \lambda_\beta(\Omega)$ (we omit the superscript R, since this case is the main concern of the paper) is defined as the lowest value for which the following problem

$$\begin{cases} -\Delta u = \lambda_\beta(\Omega) u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a non trivial solution, ν being the exterior normal. In the plane case, λ_β is proportional to the square of the principal frequency of vibration of an *elastically supported membrane*, since the constraint of vanishing displacement at the boundary is replaced by that of a restoring force of elastic type (with elastic constant β).

In terms of Rayleigh quotients, we can write

$$\lambda_\beta(\Omega) := \min_{u \in W^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2\, dx + \beta \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}}{\int_\Omega u^2\, dx}, \tag{1.3}$$

where $\mathcal{H}^{N-1}$ denotes the Hausdorff $(N-1)$-dimensional measure (which coincides with the usual area measure on $\partial\Omega$ in view of the regularity of the domain).

The presence of the boundary term in (1.3) entails that λ_β has a completely different behaviour with respect to λ^D. For instance, being the term of area type, rescaling properties under dilations are not available; moreover, since the competing functions are not necessarily zero at the boundary, λ_β does not enjoy the natural monotonicity properties of λ^D (see [19]).

The Faber-Krahn inequality

$$\lambda_\beta(B) \le \lambda_\beta(\Omega) \tag{1.4}$$

where B is a ball with $|B| = |\Omega|$, conjectured by Pólya in 1951, has been established only quite recently by Bossel in 1986 for two dimensional smooth domains [5], and by Daners for N-dimensional Lipschitz regular domains [12]. Equality still holds if and only if Ω is a ball, as proved by Daners and Kennedy [13], at least for domains of class C^2.

The validity of (1.4) cannot be established using the arguments of the Dirichlet case. Indeed the symmetric decreasing rearrangement technique requires the functions involved to vanish at the boundary, while for $\lambda_\beta(\Omega)$ the trace term plays an essential role. Up to our knowledge, the only result in the literature concerning rearrangements in presence of trace terms (treated as jumps) is due to Cianchi and Fusco [8], and requires a linear growth in the gradient term of the functional, which is not the case for (1.3).

A direct comparison between Ω and B is obtained in the method of proof by Bossel and Daners by means of a level set representation for λ_β together with a dearrangement of the ball's eigenfunction onto the domain Ω. Unfortunately such an approach is specifically tailored on the linear case, so that an adaptation of the method in order to study the analogue of (1.2) under Robin conditions

$$\lambda_{\beta,q}(\Omega) := \min_{\substack{u\in W^{1,2}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2\,dx + \beta\int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q\,dx\right)^{\frac{2}{q}}}, \qquad 1 \le q < \frac{2N}{N-2} \tag{1.5}$$

seems prohibitive. In terms of differential equations, $\lambda_{\beta,q}(\Omega)$ and the associated (non trivial) function u are such that

$$\begin{cases} -\Delta u = \lambda_{\beta,q}(\Omega)\left(\int_\Omega u^q\,dx\right)^{\frac{2-q}{q}} u^{q-1} & \text{in } \Omega \\ \frac{\partial u}{\partial\nu} + \beta u = 0 & \text{on } \partial\Omega \\ u \ge 0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

Starting from the results in [6], the aim of the present paper is to formulate a different approach to the proof of the Faber-Krahn inequality (1.4) which is completely variational in character, and which is not confined to the linear case.

Our analysis is composed of two parts.

(A) Firstly we identify a class of domains $\mathcal{A}(\mathbb{R}^N)$ (see (1.9) below) containing the Lipschitz regular ones for which the map

$$\Omega \mapsto \lambda_{\beta,q}(\Omega)$$

achieves a minimum under a volume constraint ($\lambda_{\beta,q}$ is suitably defined if Ω is not regular, see (1.10)). The class $\mathcal{A}(\mathbb{R}^N)$ is sufficiently large to be stable under intersections and reflections across hyperplanes.

(B) Given an optimal domain, we use optimality to show that it is necessarily equivalent to a ball.

Our approach parallels under several aspects the De Giorgi's proof of the classical *isoperimetric inequality* [14]. In that paper, De Giorgi shows that the isoperimetric property of the ball still holds among the family of *sets with finite perimeter* (named also *Caccioppoli sets*). This class provides a natural framework in order to get the existence of sets with minimal perimeter under a volume constraint, thanks to the compactness and lower semicontinuity properties of the perimeter. Given an optimal domain, De Giorgi uses the classical (and intuitive) symmetrization argument due to Steiner in order to show that the minimizers coincide up to negligible sets with a ball. Consequently, the class of sets with finite perimeter is a natural framework in which Steiner's arguments, emended from the critic he received by his contemporaries concerning the unclear assumption of the existence of an optimal domain, provide indeed a stronger minimality property of the ball.

Something similar occurs in our problem, with the choice of the family $\mathcal{A}(\mathbb{R}^N)$ of domains and the extension of the notion of the first eigenvalue, leading to a stronger form of the minimality property of the ball.

The first main result of our paper is the following, restricting to the case $q \in [1,2]$.

Theorem 1.1 (The Faber-Krahn inequality). *Let $q \in [1,2]$. Then for every domain $\Omega \in \mathcal{A}(\mathbb{R}^N)$*

$$\lambda_{\beta,q}(B) \le \lambda_{\beta,q}(\Omega), \tag{1.7}$$

where B is a ball such that $|B| = |\Omega|$. Moreover, equality holds if and only if Ω is a ball up to a $\mathcal{H}^{N-1}$-negligible set.

In particular the Faber-Krahn inequality is established for the torsion rigidity case $q = 1$ also for Robin boundary conditions, substantiating several hints present in the literature concerning the optimality of the ball (see e.g. the paper by Bandle and Wagner [4], where the second order volume preserving shape derivative of $\lambda_{\beta,1}$ is shown to be strictly positive on a ball).

Our approach provides also some results when $q > 2$, replacing the volume constraint by a volume penalization.

Theorem 1.2 (Volume-penalized Faber-Krahn inequality). *Let $k > 0$ and $1 \le q < \frac{2N}{N-1}$. Then there exists a ball B such that for every $\Omega \in \mathcal{A}(\mathbb{R}^N)$*

$$\lambda_{\beta,q}(B) + k|B| \le \lambda_{\beta,q}(\Omega) + k|\Omega|.$$

Moreover, equality holds if and only if Ω coincides with a ball up to a $\mathcal{H}^{N-1}$-negligible set.

Here the threshold $2N/(N-1)$, strictly lower than the Sobolev critical exponent, arises naturally in connection with monotonicity properties of the eigenvalue under dilations, see Lemma 3.1.

While point (A) of our analysis, involving the definition of $\mathcal{A}(\mathbb{R}^N)$ and the existence of an optimal domain in this class of domains is very delicate, point (B), i.e. to show that optimal domains are equivalent to balls is *relatively* simple.

In order to grasp the main ideas, it suffices to consider for the moment $\mathcal{A}(\mathbb{R}^N)$ as a family of domains with possibly irregular boundary (for example admitting cusps) but for which a trace operator is defined, in such a way that formula (1.5) for $\lambda_{\beta,q}$ is still available (with associated function satisfying (1.6)). Referring to Theorem 1.1, let us assume that $\lambda_{\beta,q}$ can be minimized among the domains of $\mathcal{A}(\mathbb{R}^N)$ under a volume constraint, and that the optimal domains are connected (for regular disconnected domains, it is not difficult to see that one connected component is more convenient than the whole set).

Let Ω_{opt} be an optimal domain. In order to see that Ω_{opt} is a ball, the first step consists in proving that the associated function u is *radial* (Theorem 4.4). This can be obtained by means of symmetrization arguments of the following type. By considering an hyperplane π which divides Ω_{opt} in two parts $\Omega^{\pm}_{opt}$ with equal volume, and employing the inequality

$$\min\left\{\frac{2a}{(2c)^{2/q}}, \frac{2b}{(2d)^{2/q}},\right\} \le \frac{a+b}{(c+d)^{2/q}} \qquad a,b,c,d > 0, \quad 1 \le q \le 2, \tag{1.8}$$

we can construct by reflection of one of the two parts an optimal domain symmetric with respect to π. By symmetrizing in succession with respect to hyperplanes parallel to the coordinate axis, we end up with a domain $\tilde{\Omega}_{opt} \in \mathcal{A}(\mathbb{R}^N)$ with a center of symmetry, which we may assume as the new origin of our coordinate system.

The domain $\tilde{\Omega}_{opt}$ is thus connected (by optimality), with $\lambda_{\beta,q}(\tilde{\Omega})$ achieved on an function which is given by successive reflections of the original u associated to Ω. Now, every hyperplane π through the origin divides $\tilde{\Omega}_{opt}$ in two parts with equal volume, so that the symmetrization of at least one of them leads again to a new minimizer of the problem, with associated function given by the reflection of $\tilde{u}$ across π. Since this function should satisfy (1.6), we get that for every $x \in \tilde{\Omega}_{opt}$ and $n \in \mathbb{R}^N$ with $n \cdot x = 0$

$$\frac{\partial \tilde{u}}{\partial n}(x) = 0.$$

Being analytic on the connected domain $\tilde{\Omega}_{opt}$, we conclude that $\tilde{u}$ is radial. The same holds also for u which is itself analytic and coincides with $\tilde{u}$ on a portion of the connected domain Ω.

In view of the radiality of u, we can write $u(x) = \psi(|x|)$ with $\psi : I \to]0, +\infty[$ maximal positive solution of the ordinary differential equation

$$-\psi'' - \frac{N-1}{r}\psi' = \lambda_{\beta,q}(\Omega)\psi^{q-1},$$

and derive the *Robin boundary condition* (Theorem 4.7)

$$\psi'(|x|) \cdot e_r(x) + \beta\psi(|x|) = 0 \qquad x \in \partial\Omega,$$

where $e_r(x) := x/|x|$. This condition imposes severe restrictions on the form of Ω, entailing for example that the boundary of Ω is contained in the region

$$\left\{ x \in \mathbb{R}^N \,:\, \left| \frac{\psi'(|x|)}{\psi(|x|)} \right| \geq \beta \right\}.$$

This represents a great simplification of the problem: the proof that the domain is a ball is nearly straightforward.

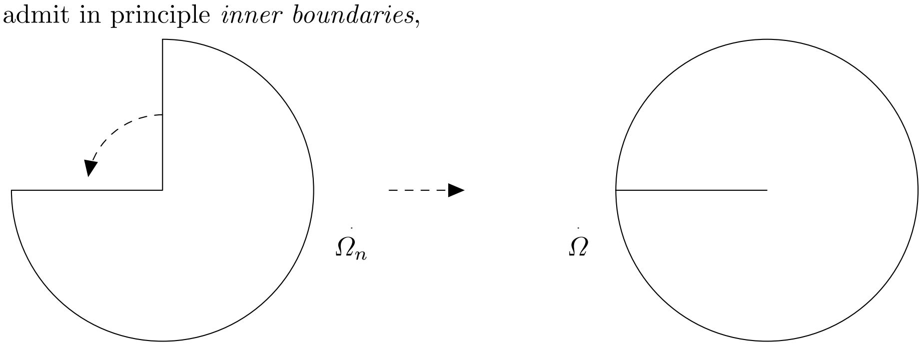
Indeed, if $0 \in \overline{\Omega}$, one sees that ψ is defined up to the origin with $\psi' \leq 0$ on I. This entails that Ω contains a ball centered at the origin. Denoting by B' the maximal ball contained in Ω, one shows (Proposition 4.10) that if Ω does not coincides with B', then the ball itself (if $q = 2$) or the ball B'' having the same volume of Ω are more convenient for $\lambda_{\beta,q}$: the eigenvalues $\lambda_{\beta,q}(B')$ and $\lambda_{\beta,q}(B'')$ are easily estimated by restricting $\psi(|x|)$ on these balls.

The case $0 \notin \overline{\Omega}$ leads with similar arguments (Proposition 4.12) to the conclusion that Ω coincides with an annulus.

Finally, a direct comparison shows that the ball is more convenient than an annulus (the comparison being much simpler in view of the particular geometry of the sets).

Let us now come back to the precise definition of the class $\mathcal{A}(\mathbb{R}^N)$ and to the extension of the notion of $\lambda_{\beta,q}$, i.e., to point (A) of our analysis.

In order to highlight the main geometrical properties we should expect for this class in order to be suitable for a variational analysis, let us consider a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}}$ for $\lambda_{\beta,q}$ given by Lipschitz domains of bounded volume. If we assume that they *converge* to a limit domain Ω, and that the associated first eigenfunctions *converge* to a function u, we immediately realize that Ω could admit in principle *inner boundaries*,



and that the limit of the eigenvalues is connected to an expression of the type

$$\frac{\int_\Omega |\nabla u|^2\,dx + \beta\int_{\partial\Omega}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q\,dx\right)^{\frac{2}{q}}}.$$

The values $u^{\pm}$ should be the traces of u from both sides of $\partial\Omega$: on the *external boundary*, one of the two traces will be zero, while on the *inner boundaries* they could be different, in general. In view of this simple observation, and trying to preserve some weak regularity for the boundary, at least in the sense of *geometric measure theory*, we are led to set

$$\begin{aligned}(1.9)\quad \mathcal{A}(\mathbb{R}^N) := \{\Omega \subseteq \mathbb{R}^N \,:\, \Omega \text{ is open with } |\Omega| < +\infty,\\ \text{and } \partial\Omega \text{ is rectifiable with } \mathcal{H}^{N-1}(\partial\Omega) < +\infty\}\end{aligned}$$

and define

$$\lambda_{\beta,q}(\Omega) := \inf_{\substack{u \in W^{1,2}(\Omega)\cap L^\infty(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2\,dx + \beta\int_{\partial\Omega}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q\,dx\right)^{\frac{2}{q}}}. \tag{1.10}$$

The *rectifiability* of $\partial\Omega$ (see Subection 2.3) means that it is contained, up to $\mathcal{H}^{N-1}$-negligible sets, into the union of a countable family of C^1-regular manifolds. This weak regularity requirement is readily seen to be stable under intersections and reflections. Moreover a normal vector field ν on $\partial\Omega$ can be defined (and this is important for the Robin condition). Finally domains in $\mathcal{A}(\mathbb{R}^N)$ have finite perimeter, so that a weak form of the integration by parts is still available (see Subsection 2.4).

The traces in (1.10) are well defined since, after extending by zero outside Ω, u belongs to the space of functions of *bounded variation* $BV(\mathbb{R}^N)$ (see Subsection 2.5). Recall that $v \in BV(\mathbb{R}^N)$ if $v \in L^1(\mathbb{R}^N)$ and the integration by parts formula

$$\forall\varphi \in C_c^\infty(\mathbb{R}^N) \,:\, -\int_{\mathbb{R}^N} v\,div(\varphi)\,dx = \int_{\mathbb{R}^N} \varphi\,dDv$$

holds for a suitable finite measure Dv with values in $\mathbb{R}^N$. In our case, viewing u as a BV function on $\mathbb{R}^N$, Du is composed of a part supported on Ω, absolutely continuous with respect to the volume Lebesgue measure with density ∇u, and of a part of “jump type” supported on the jump set $J_u \subseteq \partial\Omega$ and absolutely continuous with respect to $\mathcal{H}^{N-1}$.

In view of the fine properties of BV functions, at $\mathcal{H}^{N-1}$-a.e. point of $x \in \partial\Omega$ with normal $\nu(x)$, the two values

$$u^\pm(x) = \lim_{r\to 0^+} \frac{1}{|B_r^\pm(x,\nu(x))|}\int_{B_r^\pm(x,\nu(x))\cap\Omega} u(y)\,dy$$

are well defined, where $B_r^\pm(x,\nu(x)) := \{y \in B_r(x) \,:\, (y-x)\cdot\nu(x) \gtrless 0\}$.

To deal variationally with the Faber-Krahn inequality, we are thus naturally led to the shape optimization problem

$$\min_{\Omega\in\mathcal{A}(\mathbb{R}^N),|\Omega|\le\gamma} \lambda_{\beta,q}(\Omega). \tag{1.11}$$

Unfortunately, the existence of a minimizer is not clear, because compactness properties for a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}}$ seem difficult to be derived: for example, a bound on the perimeter cannot be obtained by controlling the surface term

$$\beta\int_{\partial\Omega_n} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1},$$

since no uniform bound from below is available on the associated (*almost first eigenfunctions*) u_n.

On the contrary, as pointed out in [6], some compactness is available for u_n. Indeed, at least if Ω_n is regular, extending by zero outside the domain, it turns out that $u_n^2 \in BV(\mathbb{R}^N)$ with uniformly bounded total variation. More precisely, in view of the preceding considerations, u_n^2 belongs to the space $SBV(\mathbb{R}^N)$ of *special functions of bounded variation* introduced by De Giorgi and Ambrosio [15] and defined as

$$SBV(\mathbb{R}^N) := \{u \in BV(\mathbb{R}^N) \,:\, Du \text{ is absolutely continuous w.r.t. } dx + \mathcal{H}^{N-1}\lfloor J_u\}.$$

In addition, as u_n is bounded from below on Ω_n by a strictly positive constant in view of the Robin condition and of Hopf lemma, we have $J_{u_n} = \partial\Omega_n$ and

$$\lambda_{\beta,q}(\Omega_n) = R_{\beta,q}(u_n),$$

where

$$R_{\beta,q}(u) := \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}}. \tag{1.12}$$

So instead of studying the shape optimization problem (1.11) directly, we turn our attention to the *free discontinuity problem*

$$\min_{u\in SBV^{\frac{1}{2}}(\mathbb{R}^N),|supp(u)|\le\gamma} R_{\beta,q}(u), \tag{1.13}$$

where the space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$, introduced in [6], is given by

$$SBV^{\frac{1}{2}}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \,:\, u \ge 0 \text{ a.e. in } \mathbb{R}^N \text{ and } u^2 \in SBV(\mathbb{R}^N)\}.$$

The existence of a minimizer u for (1.13) can be derived in a standard way from the compactness and lower semicontinuity properties of the space $SBV^{\frac{1}{2}}$. The link with problem (1.11) is an issue of *regularity*: if the support of u is an open set Ω belonging to $\mathcal{A}(\mathbb{R}^N)$, then the optimality of u entails the optimality Ω (Subsection 6.5). We prove the regularity of the support by showing that

$$\mathcal{H}^{N-1}(J_u) < +\infty \qquad \text{and} \qquad \mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0, \tag{1.14}$$

i.e., the jump set of u has finite length and is *essentially closed*.

Essential closedness of the jump set of SBV-minimizers is known to hold for the Mumford-Shah functional

$$MS(u) := \int_A |\nabla u|^2\,dx + \mathcal{H}^{N-1}(J_u) + \int_A |u-g|^2\,dx, \qquad g \in L^\infty(A), \tag{1.15}$$

thanks to the result of De Giorgi, Carriero and Leaci [16]. In view of this fact, the minimizers of (1.15) provide solutions to the original problem proposed by Mumford and Shah [24] in the context of image segmentation (in dimension two, see also the paper by Dal Maso, Morel and Solimini [10]).

Our free discontinuity problem (1.13) can be seen as a weak formulation of the shape optimization problem (1.11) in much the same way (1.15) is a weak form of the original Mumford-Shah image segmentation functional. The key difference between the two functionals is that the surface term depends on the traces of u. However, thanks to its particular dependence (involving the *sum* of the squares of $u^\pm$), we prove that minimizers of (1.13) belong to $L^\infty(\mathbb{R}^N)$ (Theorem 6.11), and more importantly (Theorem 6.13), using a key result of [7], that

$$u > \alpha > 0 \qquad \text{a.e. on } supp(u).$$

This entails immediately that $\mathcal{H}^{N-1}(J_u) < +\infty$, and that the surface term of our functional can be estimated from above and below by the area of the jump set. Then it turns out (Theorem 6.14) that minimizers of (1.13) are indeed in $SBV(\mathbb{R}^N)$, and are *almost local quasi-minimizers* of the Mumford Shah functional (see Definition 6.2), so that their jump set is essentially closed thanks to the results of [7]. Then (1.14) is established, and the support of u provides an optimal domain for (1.11).

The arguments of point (B) of our approach explained above can be adapted to the optimal domains in $\mathcal{A}(\mathbb{R}^N)$, showing that they coincide with balls up to $\mathcal{H}^{N-1}$-negligible sets: this notion of equivalence is natural in our context, since if we remove a point from a ball, we remain in $\mathcal{A}(\mathbb{R}^N)$ and not modify the eigenvalue.

A byproduct of our analysis of the free discontinuity problem (1.13) is the following Poincaré inequality in $SBV(\mathbb{R}^N)$.

Corollary 1.3 (Poincaré inequality with trace term in $SBV(\mathbb{R}^N)$). *Let $q \in [1,2]$ and $\beta, m > 0$. For every $u \in SBV(\mathbb{R}^N)$ such that $|\{u \neq 0\}| \le m$*

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \ge \lambda_{\beta,q}(B) \left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{\frac{2}{q}},$$

where B is a ball of measure m, the constant $\lambda_{\beta,q}(B)$ being optimal. Equality holds if and only if u is the first eigenfunction associated to B according to (1.5).

In fact, this corollary encodes the Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian on arbitrary domains, including the non smooth setting of the class $\mathcal{A}(\mathbb{R}^N)$ (and so the family of Lipschitz sets) and even the case of arbitrary open sets for which the eigenvalues are defined through the Mazja space [11, 6].

The reflection argument which is essential for the analysis of point (B) cannot be used in the case $q > 2$, since inequality (1.8) does not hold true. However, replacing the volume constraint by a volume penalization term, the method can be adapted to yield the inequality of Theorem 1.2 concerning the case $1 \le q < \frac{2N}{N-1}$.

The volume penalized inequality can lead to a classical Faber-Krahn inequality as (1.7) provided that, given a domain Ω, we can tune k in such a way that the associated optimal ball has volume precisely given by $|\Omega|$. We show (Theorem 5.3) that this can be done for a family of volumes which is infinite, accumulating at zero and at infinity. In general, a sufficient condition if order to get rid of the penalization term is that the function $r \mapsto \lambda_{\beta,q}(B_r)$ is convex (Theorem 5.4): this seems difficult to prove, but it holds true for the most relevant cases $q = 2$ and $q = 1$, i.e., for the linear eigenvalue and the torsion rigidity problem.

The paper is organized as follows. In Section 2 we fix the notation and recall some basic facts concerning sets with finite perimeter and functions of bounded variation. Section 3 contains the definition of the admissible class of domains $\mathcal{A}(\mathbb{R}^N)$ together with the associated eigenvalue. The proof of the Faber-Krahn inequality for the cases $1 \le q \le 2$ and $1 \le q < 2N/(N-1)$ (in its volume penalized version) are contained in Section 4 and Section 5 respectively. They are based on the existence of an optimal domain for $\lambda_{\beta,q}$ in $\mathcal{A}(R^N)$, which is established by means of a free discontinuity approach in Section 6.

2. Notation and preliminaries

In this section we fix the basic notation employed throughout the paper, and recall some notions concerning sets with finite perimeter and functions of bounded variation. The main references are [18, 2, 17].

2.1. Basic notation.

Given $E \subseteq \mathbb{R}^N$, we will denote by $|E|$ its Lebesgue measure, and by $\bar{E}$ its topological closure. Sometimes the Lebesgue measure will be denoted by $\mathcal{L}^N$, while the associate integration will be indicated by dx. We will use also the Hausdorff $(N-1)$-dimensional measure $\mathcal{H}^{N-1}$ (see [17, Chapter 2]), which coincides on piecewise regular hypersurfaces with the usual area measure. If μ is a Borel measure on $\mathbb{R}^N$ and $A \subseteq \mathbb{R}^N$ is Borel regular, we will denote by $\mu\lfloor A$ the restriction of μ to A.

For $A, B \subseteq \mathbb{R}^N$, we will write $A \subset\subset B$ if $\bar{A}$ is compact and $\bar{A} \subset B$. For $t > 0$, we set $tA := \{tx \,:\, x \in A\}$.

For $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x)$ stands for the ball of center x and radius r. If $x = 0$, we will write simply B_r. The volume of the unit ball is denoted by ω_N. Given $\nu \in S^{N-1}$, where S^{N-1} denotes the unit sphere in $\mathbb{R}^N$, we set

$$B_r^{\pm}(x, \nu) = \{y \in B_r(x) \,:\, (y - x) \cdot \nu \gtrless 0\}.$$

We say that an open set $\Omega \subseteq \mathbb{R}^N$ has a Lipschitz boundary if for every $x \in \partial\Omega$, there exist a neighborhood U of x and a Lipschitz function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, up to a rotation,

$$\Omega \cap U = \{y = (y', y_N) \in U \,:\, y_N > f(y')\}.$$

Given $\Omega \subseteq \mathbb{R}^N$ open and $1 \le p \le +\infty$, $L^p(\Omega; \mathbb{R}^k)$ stands for the usual space of (classes of) p-summable $\mathbb{R}^k$-valued functions on Ω, while $W^{1,p}(\Omega)$ will denote the Sobolev space of p-summable functions whose gradient in the sense of distributions is also p-summable. The localized version of the previous spaces will be indicated by $L^p_{loc}(\Omega; \mathbb{R}^k)$ and $W^{1,p}_{loc}(\Omega)$.

If $u \in L^1_{loc}(\mathbb{R}^N)$, we will denote by $supp(u)$ its support: this set turns out to be well defined up to negligible sets by the relation $u = 0$ almost everywhere on $\mathbb{R}^N \setminus supp(u)$. We will say that $supp(u) \subseteq \Omega$ with $\Omega \subseteq \mathbb{R}^N$ open if $u = 0$ a.e. on $\mathbb{R}^N \setminus \Omega$. We will write $supp(u) \subset\subset \Omega$ if u is zero a.e. outside a compact subset of Ω.

2.2. **Densities and approximate limits.** For $E \subseteq \mathbb{R}^N$ and $x \in \mathbb{R}^N$ we denote by $\underline{D}(E,x)$ and $\overline{D}(E,x)$ the *lower* and *upper densities* of the set E at the point x (see [18, Section 2.9.12])

$$\underline{D}(E,x) := \liminf_{r\to 0^+} \frac{|E\cap B_r(x)|}{|B_r(x)|} \qquad \text{and} \qquad \overline{D}(E,x) := \limsup_{r\to 0^+} \frac{|E\cap B_r(x)|}{|B_r(x)|}.$$

We say that E has density $\alpha \in [0,1]$ at x if

$$D(x,E) = \underline{D}(E,x) = \overline{D}(E,x) = \alpha$$

and set

$$E^{(\alpha)} := \{x \in \mathbb{R}^N \,:\, D(x,E) = \alpha\}.$$

The essential boundary of E is defined as

$$\partial^e E := \mathbb{R}^N \setminus \{E^{(1)} \cup (\mathbb{R}^N \setminus E)^{(1)}\}.$$

Given $f : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$ we denote by $f^{\pm}(x)$ the *approximate lower and upper limits* of f at x

$$f^-(x) := \sup\{D(\{f<t\},x) = 0\} \qquad \text{and} \qquad f^+(x) := \inf\{t \in \mathbb{R} \,:\, D(\{f>t\},x) = 0\}.$$

When $f^+(x) = f^-(x) = \tilde{f}(x)$, we say that f is *approximately continuous* at x with value $\tilde{f}(x)$.

2.3. **Rectifiable sets.** We say that $E \subseteq \mathbb{R}^N$ is ($\mathcal{H}^{N-1}$-countably) rectifiable (see [2, Section 2.9]) if there exists a sequence of C^1-regular submanifolds $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^N$ such that

$$\mathcal{H}^{N-1}\left(E \setminus \bigcup_{n\in\mathbb{N}} \mathcal{M}_n\right) = 0,$$

or equivalently

$$E = E_0 \cup \bigcup_{n\in\mathbb{N}} K_n$$

where $\mathcal{H}^{N-1}(E_0) = 0$, and the sets K_n are disjoint and such that $K_n \subseteq \mathcal{M}_n$. A Borel measurable normal vector field on E (up to a $\mathcal{H}^{N-1}$-negligible set) is defined by considering the normals to $\mathcal{M}_n$ on the disjoint sets K_n.

If $E_1, E_2 \subseteq \mathbb{R}^N$ are rectifiable, with ν_1, ν_2 associated normal vector fields, then $\nu_1(x) = \pm\nu_2(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in E_1 \cap E_2$.

An important class of rectifiable sets is given by the image of Lipschitz functions: if $A \subseteq \mathbb{R}^{N-1}$ is open, and $f : A \to \mathbb{R}^N$ is Lipschitz continuous, then $f(A)$ is rectifiable in $\mathbb{R}^N$. In particular, if $\Omega \subseteq \mathbb{R}^N$ has Lipschitz boundary, then $\partial\Omega$ is rectifiable.

2.4. **Sets with finite perimeter.** For the general theory of sets with finite perimeter, we refer the reader to [2, Section 3.3]. Here we recall some basic facts in a form which is suitable to our analysis.

Given $E \subseteq \mathbb{R}^N$ measurable and $\Omega \subseteq \mathbb{R}^N$ open, the *perimeter* of E in Ω is defined as

$$P(E,\Omega) := \sup\left\{\int_E div(\varphi)\,dx \,:\, \varphi \in C_c^\infty(\Omega;\mathbb{R}^N), \|\varphi\|_\infty \le 1\right\},$$

and E is said to have *finite perimeter* in Ω if $P(E,\Omega) < +\infty$. When $\Omega = \mathbb{R}^N$, we write simply $P(E)$.

If $E \subseteq \mathbb{R}^N$ has finite perimeter, then $\partial^e E$ turns out to be rectifiable with

$$P(E) = \mathcal{H}^{N-1}(\partial^e E) \qquad \text{and} \qquad \mathcal{H}^{N-1}(\partial^e E \setminus E^{(1/2)}) = 0.$$

Moreover there exists a Borel *exterior normal vector field* $\nu_E : \partial^e E \to S^{N-1}$ such that the following integration by parts formula holds true:

$$\forall \varphi \in C_c^\infty(\mathbb{R}^N) \,:\, \int_E div(\varphi)\,dx = \int_{\partial^e E} \varphi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

In terms of densities, $\nu_E(x)$ is characterized for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^e E$ as

$$\lim_{r\to 0^+} \frac{|E \cap B_r^+(x,\nu_E(x))|}{|B_r(x)|} = 0 \qquad \text{and} \qquad \lim_{r\to 0^+} \frac{|E \cap B_r^-(x,\nu_E(x))|}{|B_r(x)|} = \frac{1}{2}.$$

2.5. **Functions of bounded variation.** Let us briefly recall the definition of functions of bounded variation: we refer the reader to [2, Chapter 3] or to [17, Chapter 5] for more details.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We say that $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and its derivative in the sense of distributions is a finite Radon measure on Ω. $BV(\Omega)$ is called the space of *functions of bounded variation* on Ω and it is a Banach space under the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|,$$

the last term denoting the mass of the measure Du. By Sobolev inequality it turns out that $BV(\mathbb{R}^N)$ is continuously embedded in $L^{N/N-1}(\mathbb{R}^N)$.

The measure Du admits the following representation for every Borel set $B \subseteq \Omega$:

$$Du(B) = \int_B \nabla u \, dx + \int_{J_u \cap B} (u^+ - u^-)\nu_u \, d\mathcal{H}^{N-1} + D^c u(B). \tag{2.1}$$

Here $\nabla u \in L^1(\Omega; \mathbb{R}^N)$ is the density of the absolutely continuous part of Du, and it is called the *absolutely continuous gradient* of u.

The set J_u is the *jump set* of u and is defined as

$$J_u := \{x \in \Omega \, : \, u^-(x) < u^+(x)\},$$

where $u^\pm(x)$ are the approximate upper and lower limits of u at x. It turns out that J_u is rectifiable and that $\nu_u : J_u \to S^{N-1}$ is a Borel normal vector field such that for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$

$$\lim_{r \to 0^+} \frac{1}{|B_r^\pm(x, \nu_u(x))|} \int_{B_r^\pm(x,\nu_u(x)) \cap \Omega} |u(y) - u^\pm(x)|^{\frac{N}{N-1}} \, dy = 0.$$

In particular for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$ we have

$$u^\pm(x) = \lim_{r \to 0^+} \frac{1}{|B_r^\pm(x, \nu_u(x))|} \int_{B_r^\pm(x,\nu_u(x)) \cap \Omega} u(y) \, dy,$$

so that $u^\pm$ can be considered as the traces of u on J_u.

For $x \notin J_u$, we have $u^+(x) = u^-(x) = \tilde{u}(x)$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus J_u$

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x) \cap \Omega} |u(y) - \tilde{u}(x)|^{\frac{N}{N-1}} \, dy = 0,$$

so that $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ turns out to be a Lebesgue point for u.

The measure $D^c u$ in (2.1) is singular with respect to the Lebesgue measure with

$$D^c u(E) = 0$$

for every rectifiable set $E \subseteq \Omega$. So $D^c u$ is in a certain sense intermediate between $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, and is referred to as the *Cantor part* of Du (the one dimensional Cantor-Vitali function has bounded variation, its derivative being of Cantor type).

The link between sets with finite perimeter and functions of bounded variation is the following: if $E \subseteq \mathbb{R}^N$ has finite perimeter and $|E| < +\infty$, it turns out that $1_E \in BV(\mathbb{R}^N)$ with associated measure of jump type given by

$$D1_E = -\nu_E \mathcal{H}^{N-1} \lfloor \partial^e E,$$

where ν_E is the exterior normal.

Finally we will use several times the following result (see [2, Theorem 3.96]).

Theorem 2.1 (**Chain rule in** BV)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $u \in BV(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz and piecewise C^1 (with $f(0) = 0$ if $|\Omega| = +\infty$). Then $f \circ u \in BV(\Omega)$ with*

$$D(f \circ u) = f'(u)\nabla u \, dx + [f(u^+) - f(u^-)]\nu_u \, \mathcal{H}^{N-1} \lfloor J_u + f'(\tilde{u}) D^c u.$$

3. Nonlinear eigenvalue problems for the Robin-Laplacian

Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Given $1 \le q < \frac{2N}{N-2}$ and $\beta > 0$, let us consider the nonlinear eigenvalue problem

$$\lambda_{\beta,q}(\Omega) := \min_{\substack{u \in W^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q \, dx\right)^{\frac{2}{q}}}, \tag{3.1}$$

where the integral on the boundary involves the trace of the function u in the sense of Sobolev spaces.

For $q = 2$ the eigenvalue $\lambda_{\beta,2}(\Omega)$ coincides with the first eigenvalue of the Laplace operator under Robin boundary conditions. The case $q = 1$ is usually referred to as the *torsion rigidity problem* with Robin conditions.

For the sequel, it will be important to extend the definition of the *principal frequency* $\lambda_{\beta,q}$ to a larger class of domains, with possibly irregular boundary. Let us consider the class

$$\begin{aligned} \mathcal{A}(\mathbb{R}^N) := \{\Omega \subseteq \mathbb{R}^N \, : \, \Omega \text{ is open with } |\Omega| < +\infty, \\ \text{and } \partial\Omega \text{ is rectifiable with } \mathcal{H}^{N-1}(\partial\Omega) < +\infty\}, \end{aligned} \tag{3.2}$$

which contains that of Lipschitz regular domains. Rectifiable sets are defined in Subsection 2.3.

In order to extend the notion of $\lambda_{\beta,q}$ to domains in $\mathcal{A}(\mathbb{R}^N)$, we proceed in the following way. Since domains in $\mathcal{A}(\mathbb{R}^N)$ have boundary of finite $\mathcal{H}^{N-1}$-measure, by [2, Proposition 4.4] every $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ is such that

$$u 1_\Omega \in BV(\mathbb{R}^N).$$

In particular, thanks to the fine properties of functions of bounded variation (see Subsection 2.5), for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ the function u admits two "traces" $u^-(x) \le u^+(x)$ in the following sense: if $\nu(x) \in \mathbb{R}^N$ is a normal vector to $\partial\Omega$, up to a change in sign for ν, the values

$$u^\pm(x) = \lim_{r \to 0^+} \frac{1}{|B_r^\pm(x, \nu(x))|} \int_{B_r^\pm(x,\nu(x)) \cap \Omega} u(y) \, dy \tag{3.3}$$

are well defined and finite.

If Ω is Lipschitz regular, then $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ one trace is zero and the other coincides with the usual trace in the sense of Sobolev spaces. If Ω is not regular, for example it admits inner cracks, it could be the case that both traces are non zero and different.

In view of the preceding arguments, the definition of $\lambda_{\beta,q}$ can be generalized to a domain $\Omega \in \mathcal{A}(\mathbb{R}^N)$ by setting

$$\lambda_{\beta,q}(\Omega) := \inf_{\substack{u \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} (u^+)^2 + (u^-)^2 \, d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q \, dx\right)^{\frac{2}{q}}}. \tag{3.4}$$

Using a simple truncation argument, it is readily seen that $\lambda_{\beta,q}(\Omega)$ coincides with the value given in (3.1) when Ω is Lipschitz regular. Clearly, in the Rayleigh quotient above, one can replace the minimizer u by its absolute value. As a consequence, we can assume that the minimization is carried in the family of nonnegative functions. Throughout the paper, this is implicitly assumed.

The following rescaling property will be useful.

Lemma 3.1. *Let $\Omega \in \mathcal{A}(\mathbb{R}^N)$. Then for every $t \ge 1$*

$$\lambda_{\beta,q}(t\Omega) \le t^{N - \frac{2N}{q} - 1} \lambda_{\beta,q}(\Omega).$$

Proof. Let $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ be admissible for the computation of $\lambda_{\beta,q}(\Omega)$. Then

$$v(x) := u\left(\frac{x}{t}\right)$$

is an admissible function for the computation of $\lambda_{\beta,q}(t\Omega)$. We get

$$\begin{aligned}
\lambda_{\beta,q}(t\Omega) &\leq \frac{\int_{t\Omega} |\nabla v|^2\,dx + \beta \int_{\partial(t\Omega)} [v^+]^2 + [v^-]^2\,d\mathcal{H}^{N-1}}{\left(\int_{t\Omega} |v|^q\,dx\right)^{2/q}} \\
&= \frac{t^{-2}\int_{t\Omega} |\nabla u|^2\left(\frac{x}{t}\right)\,dx + \beta \int_{\partial(t\Omega)} [u^+]^2\left(\frac{x}{t}\right) + [u^-]^2\left(\frac{x}{t}\right)\,d\mathcal{H}^{N-1}}{\left(\int_{t\Omega} |u\left(\frac{x}{t}\right)|^q\,dx\right)^{2/q}} \\
&= \frac{t^{-2+N}\int_{\Omega} |\nabla u|^2\,dx + \beta t^{N-1} \int_{\partial\Omega} [u^+]^2 + [u^-]^2\,d\mathcal{H}^{N-1}}{t^{\frac{2N}{q}}\left(\int_{\Omega} |u|^q\,dx\right)^{2/q}} \\
&= t^{N-\frac{2N}{q}-1}\frac{t^{-1}\int_{\Omega} |\nabla u|^2\,dx + \beta \int_{\partial\Omega} [u^+]^2 + [u^-]^2\,d\mathcal{H}^{N-1}}{\left(\int_{\Omega} |u|^q\,dx\right)^{2/q}} \\
&\leq t^{N-\frac{2N}{q}-1}\frac{\int_{\Omega} |\nabla u|^2\,dx + \beta \int_{\partial\Omega} [u^+]^2 + [u^-]^2\,d\mathcal{H}^{N-1}}{\left(\int_{\Omega} |u|^q\,dx\right)^{2/q}}
\end{aligned}$$

so that the result follows by taking the infimum on u. □

The rescaling property entails the following result.

Corollary 3.2. *For every $1 \leq q < \frac{2N}{N-1}$ and $\beta > 0$, the map*

$$r \mapsto \lambda_{\beta,q}(B_r)$$

is strictly decreasing on $]0, +\infty[$.

4. The Faber-Krahn inequality

The present section is devoted to the proof of the Faber-Krahn inequality for the principal frequency $\lambda_{\beta,q}$ given in (3.4) when $1 \leq q \leq 2$ among the class $\mathcal{A}(\mathbb{R}^N)$ defined in (3.2). The result entails the validity of the inequality among the class of Lipschitz regular domains, on which $\lambda_{\beta,q}$ reduces to the usual eigenvalue given in (3.1).

Theorem 4.1 (Faber-Krahn inequality: the case $1 \leq q \leq 2$). *Let $q \in [1, 2]$ and $\beta > 0$. Then for every domain $\Omega \in \mathcal{A}(\mathbb{R}^N)$*

$$\lambda_{\beta,q}(\Omega) \geq \lambda_{\beta,q}(B), \tag{4.1}$$

where B is a ball such that $|B| = |\Omega|$. Moreover, equality holds if and only if Ω is a ball up to a $\mathcal{H}^{N-1}$-negligible set.

The starting point of our analysis is given by the following result.

Theorem 4.2 (Existence of an optimal domain). *Let $\beta, \gamma > 0$ and $1 \leq q < \frac{2N}{N-1}$. Then the minimum problem*

$$\min_{\Omega \in \mathcal{A}(\mathbb{R}^N), |\Omega| \leq \gamma} \lambda_{\beta,q}(\Omega) \tag{4.2}$$

admits a solution. Moreover every minimizer $\Omega \in \mathcal{A}(\mathbb{R}^N)$ coincides up to a $\mathcal{H}^{N-1}$-negligible set with an open connected set such that $\lambda_{\beta,q}(\Omega)$ is achieved on an analytic and positive function $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ with

$$-\Delta u = \lambda_{\beta,q}(\Omega) u^{q-1} \qquad \text{on } \Omega, \qquad \int_\Omega u^q\,dx = 1, \tag{4.3}$$

and such that

$$u > \alpha \qquad \text{on } \Omega \tag{4.4}$$

for some $\alpha > 0$. Finally if we extend u to zero outside Ω (still denoted by u), we have

$$u \in BV(\mathbb{R}^N), \qquad \partial\Omega = \overline{J_u}, \qquad \text{and} \qquad \mathcal{H}^{N-1}(\partial\Omega \setminus J_u) = 0. \tag{4.5}$$

The previous theorem is a particular case of Theorem 6.1 of Section 6: its proof is obtained by studying a free discontinuity problem for a suitable space of functions of bounded variation, and exploiting the regularity properties of the associated minimizers.

Remark 4.3. Note that the upper bound for q in Theorem 4.2 is not given by the Sobolev critical exponent $2N/(N-2)$ which naturally arises in the study of $\lambda_{\beta,q}$ on Lipschitz domains : this is related to the monotonicity of the eigenvlaue under dilations (see Lemma 3.1) and also to our technique which is based on the interpretation of u^2 as a BV function on $\mathbb{R}^N$, so that the associated critical exponent is given precisely by $2N/(N-1)$.

Theorem 4.1 follows readily if we show that every minimizer of problem (4.2) coincides up to a $\mathcal{H}^{N-1}$-negligible set with a ball. Indeed, given $\Omega \in \mathcal{A}(\mathbb{R}^N)$ and setting $\gamma := |\Omega|$, if a ball B is optimal for (4.2), then $|B| = \gamma$ in view of Corollary 3.2, so that (4.1) follows.

In the following we thus fix $\Omega \in \mathcal{A}(\mathbb{R}^N)$ minimum of problem (4.2) and we let u be the associated function on which $\lambda_{\beta,q}(\Omega)$ is achieved according to Theorem 4.2. Our aim is to show that Ω coincides up to a $\mathcal{H}^{N-1}$-negligible set with a ball: this will be obtained in Theorem 4.14 after an analysis based on the optimality of Ω.

The first step consists in proving that the function u is radial: we use some reflection techniques considered in [21] and [22].

Theorem 4.4 (**The function** u **is radial).** *Up to a translation, the function u is the restriction on Ω of an analytic positive radial function (still denoted by u) defined on an open neighborhood A of $\overline{\Omega}$ which satisfies*

$$-\Delta u = \lambda_{\beta,q}(\Omega) u^{q-1} \qquad \text{on } A. \tag{4.6}$$

Proof. Let us consider an hyperplane π_1 parallel to $x_1 = 0$ which splits Ω in two parts $\Omega^{\pm}$ such that

$$|\Omega^+| = |\Omega^-|.$$

Note that the term

$$\int_{\partial\Omega\cap\pi_1} (u^+)^2 + (u^-)^2 \, d\mathcal{H}^{N-1}$$

which (eventually) appears in the surface part of the Rayleigh quotient defining $\lambda_{\beta,q}(\Omega)$ can be reinterpreted, since the normal ν involved in the definition of $u^{\pm}$ coincides $\mathcal{H}^{N-1}$-a.e. with that of π_1, as

$$\int_{\partial\Omega\cap\pi_1} (u_+)^2 + (u_-)^2 \, d\mathcal{H}^{N-1},$$

where $u_{\pm}$ are the traces of $u_{\Omega^{\pm}}$ on π_1 (defined as in (3.3)).

Up to a switch between the two open sets, we can assume (denoting with $\pi_1^{\pm}$ the two half-spaces determined by π_1)

$$\frac{\int_{\Omega^+} |\nabla u|^2 \, dx + \beta \int_{\partial\Omega\cap\pi_1^+} (u^+)^2 + (u^-)^2 \, d\mathcal{H}^{N-1} + \beta \int_{\partial\Omega\cap\pi_1} (u_+)^2 \, d\mathcal{H}^{N-1}}{\left(\int_{\Omega^+} u^q \, dx\right)^{2/q}}$$
$$\le \frac{\int_{\Omega^-} |\nabla u|^2 \, dx + \beta \int_{\partial\Omega\cap\pi_1^-} (u^+)^2 + (u^-)^2 \, d\mathcal{H}^{N-1} + \beta \int_{\partial\Omega\cap\pi_1} (u_+)^2 \, d\mathcal{H}^{N-1}}{\left(\int_{\Omega^-} u^q \, dx\right)^{2/q}}.$$

Since $1 \le q \le 2$, the convexity of $x \mapsto x^{2/q}$ yields that for every $a, b, c, d > 0$

$$\min\left\{ \frac{2a}{(2c)^{2/q}}, \frac{2b}{(2d)^{2/q}}, \right\} \le \frac{a+b}{(c+d)^{2/q}}. \tag{4.7}$$

Then, denoting with $\tilde{\Omega}^+$ the reflection of Ω^+ across π_1, let

$$\Omega_1 := int\left(\Omega^+ \cup \pi_1 \cup \tilde{\Omega}^+\right) \in \mathcal{A}(\mathbb{R}^N). \tag{4.8}$$

Since $|\Omega_1| = |\Omega|$, and in view of inequality (4.7), Ω_1 is a minimizer for problem (4.2) which is symmetric with respect to π_1: the associated $\lambda_{\beta,q}(\Omega_1)$ is achieved on the function u_1 given by the reflection of u, which is thus symmetric with respect to π_1.

If we consider now a hyperplane π_2 parallel to $x_2 = 0$, and proceed as before reasoning on Ω_1, we obtain a minimizer $\Omega_2 \in \mathcal{A}(\mathbb{R}^N)$ which is symmetric with respect to π_1 and π_2 together with the associated function u_2. Proceeding in this way using hyperplanes parallel to $x_i = 0$ with $3 \le i \le N$, and putting the origin at their intersection, we end up with a minimizer Ω_N which is symmetric with respect to 0 together with its associated function u_N. By Theorem 4.2 we have that Ω_N is connected.

If $x \in \Omega_N$, and π is a hyperplane through x and the origin, we can reflect Ω_N across π (see (4.8)) and get a solution $\tilde{\Omega}_N$ of the problem which symmetric with respect to π together with associated function $\tilde{u}_N$: indeed the previous arguments can be applied since by the symmetry properties of Ω_N, the hyperplane π splits Ω_N in two parts $\Omega_N^\pm$ such that

$$|\Omega_N^+| = |\Omega_N^-|.$$

Notice that $B_r(x) \subseteq \tilde{\Omega}_N \cap \Omega_N$ with $r > 0$ suitably small. Since $\tilde{u}_N$ is analytic in $\tilde{\Omega}_N$ (since it realizes $\lambda_{\beta,q}(\tilde{\Omega}_N)$ and so it satisfies the Euler Lagrange equation (4.3)), hence smooth, we get

$$D_\nu \tilde{u}_N(x) = 0,$$

where ν denotes the normal to π at x. But also u_N is analytic on Ω_N, so that $u_N = \tilde{u}_N$ on $B_r(x)$, which yields

$$D_\nu u_N(x) = 0.$$

Since Ω_N is connected, this means that u_N depends only on r. Recall that u_N coincides with our original u on a part of $\Omega \cap \Omega_N$: since also u is analytic, and Ω is connected, we get immediately that there exists $\psi : I \to \mathbb{R}^+$ where $I \subset]0, +\infty[$ such that

$$\forall x \in \Omega \setminus \{0\} \, : \, u(x) = \psi(|x|). \tag{4.9}$$

ψ is a maximal positive solution of the ordinary differential equation

$$-\psi'' - \frac{N-1}{r}\psi' = \lambda_{\beta,q}(\Omega)\psi^{q-1} \tag{4.10}$$

which follows readily from (4.3).

In order to conclude, we have to show that u is defined on an open set containing $\overline{\Omega}$. If $0 \notin \overline{\Omega}$, the conclusion follows from (4.9) and the fact that $u > \alpha > 0$ on $\overline{\Omega}$.

Let $0 \in \overline{\Omega}$. Then I is of the form $]0, a[$ with $a > 0$. Since u is bounded on Ω, it turns out that ψ is bounded near 0 and $v(x) := \psi(|x|)$ is smooth, bounded and satisfies

$$-\Delta v = \lambda_{\beta,q}(\Omega)v^{q-1} \tag{4.11}$$

on $B_{r_1} \setminus \{0\}$ for some $r_1 > 0$. Testing the equation with v on $B_{r_1} \setminus \overline{B_\varepsilon}$ we get

$$\int_{B_{r_1}\setminus\overline{B_\varepsilon}} |\nabla v|^2 \, dx - \psi'(r_1)\psi(r_1)\omega_N r_1^{N-1} + \psi'(\varepsilon)\psi(\varepsilon)\omega_N \varepsilon^{N-1} = \lambda_{\beta,q}(\Omega) \int_{B_{r_1}\setminus\overline{B_\varepsilon}} v^q \, dx. \tag{4.12}$$

From equation (4.10) we immediately deduce that for $r \in I$

$$-(\psi'(r)r^{N-1})' = \lambda_{\beta,q}(\Omega)\psi^{q-1}(r)r^{N-1}$$

which entails that $\psi'(r)r^{N-1}$ is bounded near $r = 0$. From (4.12), letting $\varepsilon \to 0$ we infer $v \in W^{1,2}(B_{r_1})$. As a consequence v can be extended smoothly to 0 satisfying again (4.11). The conclusion follows again from (4.13) and the fact that $u > \alpha > 0$ on $\overline{\Omega}$. □

According to the previous theorem, we perform (eventually) a translation of the axis in order to write

$$\forall x \in \Omega \setminus \{0\} \, : \, u(x) = \psi(|x|) \tag{4.13}$$

where $\psi : I \to \mathbb{R}^+$.

The following result shows that Ω does not admit *inner boundaries*.

Proposition 4.5 (**Ω does not have inner boundaries).** *We have*

$$\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^e\Omega) = 0,$$

where $\partial^e\Omega$ denotes the essential boundary of Ω (see Subsection 2.4). In particular, $\mathcal{H}^{N-1}$-a.e. point of $\partial\Omega$ has density $1/2$ with respect to Ω.

Proof. By Theorem 4.4, u is the restriction to Ω of a radial bounded positive analytic function φ defined on an open set A such that $\overline{\Omega} \subseteq A$. In view of Theorem 4.2, extending u to zero outside Ω, we obtain a function in $BV(\mathbb{R}^N)$ with $\partial\Omega = \overline{J_u}$ and $\mathcal{H}^{N-1}(\partial\Omega \setminus J_u) = 0$. Since this function coincides on A with $\varphi 1_\Omega$, by the chain rule in BV (see Theorem 2.1) we get

$$D^s u = \varphi D^s 1_\Omega \qquad \text{on } A.$$

Since $D^s 1_\Omega$ is supported on $\partial^e\Omega$, this entails that up to $\mathcal{H}^{N-1}$-negligible sets

$$J_u \subseteq \partial^e\Omega \subseteq \partial\Omega,$$

so that the result follows. □

Remark 4.6. Since $\mathcal{H}^{N-1}$-a.e. point of $\partial\Omega$ has density $1/2$ with respect to Ω, for every $v \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ we have

$$v^-(x) = 0 \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega.$$

Thanks to the absence of inner boundaries, we can derive a weak form of the Robin condition on $\partial^e\Omega$.

Theorem 4.7 (**The Robin condition).** *The following Robin condition holds true (recall that u is smooth on a neighborhood of $\overline{\Omega}$):*

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial^e\Omega, \tag{4.14}$$

where ν is an external normal vector field to Ω on $\partial^e\Omega$. Equivalently, recalling (4.13)

$$\psi'(|x|)e_r(x) \cdot \nu(x) + \beta\psi(|x|) = 0 \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial^e\Omega, \tag{4.15}$$

where $e_r(x) := x/|x|$.

Proof. It suffices to work out the Euler-Lagrange equation satisfied by u. If $\varphi \in C_c^\infty(\mathbb{R}^N)$, for $\varepsilon > 0$ small enough the function $u + \varepsilon\varphi$ is an admissible function for the computation of $\lambda_{\beta,q}(\Omega)$. Since u is bounded from below on Ω by a strictly positive constant, taking into account that $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^e\Omega) = 0$ and in view also of Remark 4.6, we deduce that

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx + \beta \int_{\partial^e\Omega} u\varphi \, d\mathcal{H}^{N-1} = \lambda_{\beta,q}(\Omega) \int_\Omega u^{q-1}\varphi \, dx.$$

Since u is smooth on an open set containing $\overline{\Omega}$, integration by parts on Ω, together with equation (4.3) yields

$$\int_{\partial^e\Omega} \left(\frac{\partial u}{\partial \nu} + \beta u \right) \varphi \, d\mathcal{H}^{N-1} = 0.$$

Since φ is arbitrary, the conclusion follows. □

Remark 4.8. The Robin condition (4.15) imposes severe restrictions on the domain Ω: indeed

$$\left| \frac{\psi'(|x|)}{\psi(|x|)} \right| \geq \beta \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial^e\Omega,$$

so that points for which $|\psi'/\psi| < \beta$ are, up to $\mathcal{H}^{N-1}$ negligible sets, points of density 1 or 0 for Ω.

Remark 4.9. In the sequel, we will make use of the following property. Let $\varepsilon > 0$, and let $E \subset\subset \mathbb{R}^N \setminus B_\varepsilon$ be a set with finite perimeter and finite volume. Then

$$\mathcal{H}^{N-1}\left(\{x \in \partial^e E \,:\, e_r(x) \cdot \nu(x) < 0\}\right) > 0 \quad \text{and} \quad \mathcal{H}^{N-1}\left(\{x \in \partial^e E \,:\, e_r(x) \cdot \nu(x) > 0\}\right) > 0,$$

where ν denotes the exterior normal to E, while $e_r(x) := x/|x|$. Indeed, by considering the divergence free vector field

$$V(x) := \frac{x}{|x|^N} \qquad x \neq 0,$$

we can write (multiply firstly V by a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$ and then let $\eta \nearrow 1$)

$$0 = \int_E div(V)\, dx = \int_{\partial^e E} V \cdot \nu \, d\mathcal{H}^{N-1} = \int_{\partial^e E} \frac{1}{|x|^{N-1}} e_r(x) \cdot \nu(x)\, d\mathcal{H}^{N-1},$$

from which the result follows.

The Robin condition of Theorem 4.7 is the key property to show that the optimal domain Ω is either a ball or an annulus.

Proposition 4.10. *Assume that $0 \in \overline{\Omega}$. Then Ω coincides up to a $\mathcal{H}^{N-1}$-negligible set with B_r for some $r > 0$.*

Proof. If $0 \in \overline{\Omega}$, then by Theorem 4.4

$$u(x) = \psi(|x|)$$

where $\psi : I \to [0, +\infty[$ is smooth with $0 \in I$ and $\psi'(0) = 0$. From (4.10) we get

$$-(r^{N-1}\psi'(r))' = \lambda_{\beta,q}(\Omega) r^{N-1} \psi^{q-1}(r)$$

which entails that $r \mapsto r^{N-1}\psi'(r)$ is strictly decreasing on I. In particular we get that $\psi' < 0$, so that ψ is strictly decreasing on I. In particular $\psi(0) > 0$.

We claim that

$$B_\varepsilon \subseteq \Omega \qquad \text{up to a } \mathcal{H}^{N-1}\text{-negligible set} \tag{4.16}$$

for some $\varepsilon > 0$. Indeed let $\varepsilon > 0$ small enough be such that

$$-\beta < \frac{\psi'}{\psi} \leq 0 \qquad \text{on } [0, \varepsilon[.$$

In view of Remark 4.8 we deduce that

$$\mathcal{H}^{N-1}(\partial^e \Omega \cap B_\varepsilon) = 0 \tag{4.17}$$

which entails that

$$D1_{\Omega \cap B_\varepsilon} = 0 \qquad \text{on } B_\varepsilon.$$

Then $1_{\Omega \cap B_\varepsilon}$ is constant on B_ε, that is

$$1_{\Omega \cap B_\varepsilon} = 1 \qquad \text{or} \qquad 1_{\Omega \cap B_\varepsilon} = 0$$

almost everywhere on B_ε.

In the second case we have $\Omega \cap B_\varepsilon = \emptyset$ (Ω is open), so that by Remark 4.9 we deduce that

$$\mathcal{H}^{N-1}\left(\{x \in \partial^e \Omega \,:\, e_r(x) \cdot \nu(x) < 0\}\right) > 0.$$

Since $\psi' < 0$ on I, while $\psi(|x|)| > 0$ for $x \in \partial^e \Omega$, we conclude that (4.15) is violated on a set of positive $\mathcal{H}^{N-1}$–measure, a contradiction.

In the first case, we get $\Omega^c \cap B_\varepsilon = \partial\Omega \cap B_\varepsilon$, so that, since $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^e \Omega) = 0$ and in view of (4.17), claim (4.16) follows.

Let $\varepsilon \leq r_{min} < +\infty$ be the last radius such that $B_{r_{min}} \subseteq \Omega$ up to a $\mathcal{H}^{N-1}$-negligible set. By the Robin condition (4.14) we have

$$\frac{\psi'(r_{min})}{\psi(r_{min})} \leq -\beta. \tag{4.18}$$

Moreover by construction, for $\bar{r} > r_{min}$ we have

$$|\Omega^c \cap B_{\bar{r}}| > 0. \tag{4.19}$$

Assume by contradiction that

$$\Omega \setminus B_{r_{min}} \neq \emptyset.$$

Let us treat firstly the case $1 \le q < 2$. Select $r > r_{min}$ such that $|B_r| = |\Omega|$. Notice that

$$|\Omega \setminus B_r| > 0. \tag{4.20}$$

We claim

$$\frac{\psi'(r)}{\psi(r)} = -\beta' \le -\beta. \tag{4.21}$$

Then integrating by parts and using the equation (4.6) we get in view of (4.18)

$$\begin{aligned}
\lambda_{\beta,q}(B_r) &\le \frac{\int_{B_r} |\nabla u|^2\,dx + \beta \int_{\partial B_r} u^2\,d\mathcal{H}^{N-1}}{\left(\int_{B_r} u^q\,dx\right)^{\frac{2}{q}}} \\
&= \frac{\int_{B_r} (-\Delta u \cdot u)\,dx + (-\beta' + \beta) \int_{\partial B_r} u^2\,d\mathcal{H}^{N-1}}{\left(\int_{B_r} u^q\,dx\right)^{\frac{2}{q}}} \\
&\le \frac{\lambda_{\beta,q}(\Omega) \int_{B_r} u^q\,dx}{\left(\int_{B_r} u^q\,dx\right)^{\frac{2}{q}}} = \lambda_{\beta,q}(\Omega) \frac{1}{\left(\int_{B_r} u^q\,dx\right)^{\frac{2}{q}-1}}.
\end{aligned}$$

Since ψ is strictly decreasing on I and taking into account (4.20), we get

$$\int_{B_r} u^q\,dx > \int_{\Omega} u^q\,dx = 1.$$

Since $1 \le q < 2$ we deduce $\lambda_{\beta,q}(B_r) < \lambda_{\beta,q}(\Omega)$, a contradiction.

The proof of claim (4.21) is as follows. If

$$\frac{\psi'(r)}{\psi(r)} > -\beta,$$

then thanks to Remark 4.8 and since Ω is connected we deduce that

$$B_{r+\delta} \setminus \overline{B_{r-\delta}} \subseteq \Omega \qquad \text{up to a } \mathcal{H}^{N-1}\text{-negligible set}$$

for some $\delta > 0$. Let us consider

$$E := \Omega^c \cap [B_{r+\delta} \setminus \overline{B_{\varepsilon/2}}].$$

We have that $E \subset\subset \mathbb{R}^N \setminus \{0\}$ has finite perimeter and $|E| > 0$ thanks to (4.19). In view of Remark 4.9 we deduce

$$\mathcal{H}^{N-1}\left(\{x \in \partial^e E \,:\, e_r(x) \cdot \nu_E(x) > 0\}\right) > 0,$$

which contradicts the Robin condition (4.15) since $\partial^e E \subseteq \partial^e \Omega$ and $\nu_E = -\nu$ (recall that $\psi' \le 0$ on I).

Let us come to the case $q = 2$. We have in view of (4.18) and integrating by parts

$$\lambda_{\beta,2}(B_{r_{min}}) \le \frac{\int_{B_{r_{min}}} |\nabla u|^2\,dx + \beta \int_{\partial B_{r_{min}}} u^2\,d\mathcal{H}^{N-1}}{\int_{B_{r_{min}}} u^2\,dx} \le \frac{\int_{B_{r_{min}}} (-\Delta u \cdot u)\,dx}{\int_{B_{r_{min}}} u^2\,dx} = \lambda_{\beta,2}(\Omega).$$

Since $|B_{r_{min}}| < |\Omega|$, this is against the optimality of Ω in view of the rescaling property for $\lambda_{\beta,q}$ of Lemma 3.1. $\square$

Remark 4.11. For future reference, we note that the computation done for $q = 2$ can indeed be extended to $q > 2$ as follows:

$$\begin{aligned}
\lambda_{\beta,q}(B_{r_{min}}) &\le \frac{\int_{B_{r_{min}}} |\nabla u|^2\,dx + \beta \int_{\partial B_{r_{min}}} u^2\,d\mathcal{H}^{N-1}}{\left(\int_{B_{r_{min}}} u^q\,dx\right)^{\frac{2}{q}}} \le \frac{\int_{B_{r_{min}}} (-\Delta u \cdot u)\,dx}{\left(\int_{B_{r_{min}}} u^q\,dx\right)^{\frac{2}{q}}} \\
&= \lambda_{\beta,q}(\Omega) \left(\int_{B_{r_{min}}} u^q\,dx\right)^{1-\frac{2}{q}}.
\end{aligned}$$

If $q > 2$ and since

$$\int_{B_{r_{min}}} u^q\,dx < \int_\Omega u^q\,dx = 1,$$

we deduce $\lambda_{\beta,q}(B_{r_{min}}) < \lambda_{\beta,q}(\Omega)$, against the optimality of Ω.

Similar arguments show that Ω can be an annulus.

Proposition 4.12. *Assume that $0 \notin \overline{\Omega}$. Then up to a $\mathcal{H}^{N-1}$-negligible set, Ω is an annulus of the form*

$$\Omega = B_{r_2} \setminus \overline{B_{r_1}}$$

for some $0 < r_1 < r_2$. Moreover $u(x) = \psi(|x|)$ with ψ strictly increasing on $[r_1, r_0]$ and strictly decreasing on $[r_0, r_2]$, where $r_0 \in]r_1, r_2[$.

Proof. Notice that ψ' has to change sign on the interval associated to the points of Ω: this is a consequence of the Robin condition (4.15) and of Remark 4.9, since $0 \notin \overline{\Omega}$.

Let $r_0 > 0$ be such that $\psi'(r_0) = 0$. From (4.10) we get

$$-(r^{N-1}\psi'(r))' = \lambda_{\beta,q}(\Omega) r^{N-1}\psi^{q-1}(r),$$

which entails that $r \mapsto r^{N-1}\psi'(r)$ is strictly decreasing on I. Then we deduce

$$\psi'(r) \geq 0 \quad \text{for } r \leq r_0 \qquad \text{and} \qquad \psi'(r) \leq 0 \quad \text{for } r \geq r_0.$$

Let $\delta > 0$ be so small that

$$\left|\frac{\psi'}{\psi}\right| < \beta \qquad \text{on } [r_0 - \delta, r_0 + \delta].$$

The Robin condition (4.15) together with the connectedness of Ω entails

$$B_{r_0+\delta} \setminus \overline{B_{r_0-\delta}} \subseteq \Omega \qquad \text{up to a } \mathcal{H}^{N-1}\text{-negligible set.}$$

Let $B_{r_2} \setminus \overline{B_{r_1}}$ be the maximal annulus with $0 < r_1 < r_0 < r_2$ such that

$$B_{r_2} \setminus \overline{B_{r_1}} \subseteq \Omega \qquad \text{up to a } \mathcal{H}^{N-1}\text{-negligible set.}$$

We have

$$\frac{\psi'(r_1)}{\psi(r_1)} = \beta \qquad \text{and} \qquad \frac{\psi'(r_2)}{\psi(r_2)} = -\beta. \tag{4.22}$$

By construction, for every $\bar{r} > r_2$

$$|\Omega^c \cap [B_{\bar{r}} \setminus \overline{B_{r_2}}]| > 0, \tag{4.23}$$

and similarly for $\hat{r} < r_1$

$$|\Omega^c \cap [B_{r_1} \setminus \overline{B_{\hat{r}}}]| > 0.$$

Assume by contradiction that

$$\Omega \setminus [B_{r_2} \setminus \overline{B_{r_1}}] \neq \emptyset.$$

Let us treat firstly the case $1 \leq q < 2$. Choose $\rho_1 \leq r_1$ and $\rho_2 \geq r_2$ such that

$$|B_{r_1} \setminus \overline{B_{\rho_1}}| = |\Omega \cap B_{r_1}| \qquad \text{and} \qquad |B_{\rho_2} \setminus \overline{B_{r_2}}| = |\Omega \setminus B_{r_2}|.$$

Notice that one of the previous inequality is strict: let us assume without loss of generality that $\rho_1 \leq r_1$ and $\rho_2 > r_2$. Then by (4.23) we have

$$|\Omega \setminus B_{\rho_2}| > 0. \tag{4.24}$$

We claim that

$$\frac{\psi'(\rho_1)}{\psi(\rho_1)} = \beta' \geq \beta \qquad \text{and} \qquad \frac{\psi'(\rho_2)}{\psi(\rho_2)} = -\beta'' \leq -\beta. \tag{4.25}$$

Let us consider the annulus $C_{\rho_1,\rho_2} := B_{\rho_2} \setminus \overline{B_{\rho_1}}$. By construction we have $|C_{\rho_1,\rho_2}| = |\Omega|$. Integrating by parts and using equation (4.6) we get in view of (4.25)

$$\lambda_{\beta,q}(C_{\rho_1,\rho_2}) \leq \frac{\int_{C_{\rho_1,\rho_2}} |\nabla u|^2\,dx + \beta \int_{\partial B_{\rho_2}\cup\partial B_{\rho_1}} u^2\,d\mathcal{H}^{N-1}}{\left(\int_{C_{\rho_1,\rho_2}} u^q\,dx\right)^{2/q}}$$

$$= \frac{\int_{C_{\rho_1,\rho_2}} -\Delta u\cdot u\,dx + (-\beta'' + \beta)\int_{\partial B_{\rho_2}} u^2\,d\mathcal{H}^{N-1} + (-\beta' + \beta)\int_{\partial B_{\rho_1}} u^2\,d\mathcal{H}^{N-1}}{\left(\int_{C_{\rho_1,\rho_2}} u^q\,dx\right)^{2/q}}$$

$$\leq \frac{\lambda_{\beta,q}(\Omega)}{\left(\int_{C_{\rho_1,\rho_2}} u^q\,dx\right)^{\frac{2}{q}-1}}.$$

Since ψ is strictly increasing on $I\cap]0, r_0[$ and strictly decreasing on $I\cap]r_0, +\infty[$, taking into account (4.24) and (4.23) we get

$$\int_{C_{\rho_1,\rho_2}} u^q\,dx > \int_\Omega u^q\,dx = 1.$$

Since $1 \leq q < 2$ we deduce $\lambda_{\beta,q}(C_{\rho_1,\rho_2}) < \lambda_{\beta,q}(\Omega)$, a contradiction.

Claim (4.25) can be proved as follows. Let us show the second inequality, the first one being similar. If $\psi'(\rho_2)/\psi(\rho_2) \in]-\beta, 0[$, then the Robin condition (4.15), relation (4.24) and the connectedness of Ω entail that there exists $\delta > 0$ such that

$$B_{\rho_2+\delta} \setminus \overline{B_{\rho_2-\delta}} \subseteq \Omega \qquad \text{up to a } \mathcal{H}^{N-1}\text{-negligible set.}$$

Let us consider

$$E := \Omega^c \cap [B_{\rho_2+\delta} \setminus \overline{B_{r_0}}].$$

We have that $E \subset\subset \mathbb{R}^N \setminus \{0\}$ has finite perimeter and $|E| > 0$ thanks to (4.23). In view of Remark 4.9 we deduce

$$\mathcal{H}^{N-1}\left(\{x \in \partial^e E \,:\, e_r(x)\cdot\nu_E(x) > 0\}\right) > 0, \qquad e_r(x) := x/|x|,$$

which contradicts the Robin condition (4.15) since $\partial^e E \subseteq \partial^e\Omega$ and $\nu_E = -\nu$ (recall that $\psi'(r) \leq 0$ for $r \geq r_0$).

Let us consider the case $q = 2$. Thanks to (4.22) we have setting $C_{r_1,r_2} := B_{r_2} \setminus \overline{B_{r_1}}$

$$\lambda_{\beta,2}(C_{r_1,r_2}) \leq \frac{\int_{C_{r_1,r_2}} |\nabla u|^2\,dx + \beta\int_{\partial B_{r_2}\cup\partial B_{r_1}} u^2\,d\mathcal{H}^{N-1}}{\int_{C_{r_1,r_2}} u^2\,dx} = \frac{\int_{C_{r_1,r_2}} -\Delta u\cdot u\,dx}{\int_{C_{r_1,r_2}} u^2\,dx} \leq \lambda_{\beta,2}(\Omega).$$

Since $|C_{r_1,r_2}| < |\Omega|$, this is against the optimality of Ω in view of the rescaling property for $\lambda_{\beta,q}$ of Lemma 3.1. □

Remark 4.13. For future reference, we note that the computation done for $q = 2$ can indeed be extended to $q > 2$ as follows:

$$\lambda_{\beta,q}(C_{r_1,r_2}) \leq \frac{\int_{C_{r_1,r_2}} |\nabla u|^2\,dx + \beta\int_{\partial B_{r_2}\cup\partial B_{r_1}} u^2\,d\mathcal{H}^{N-1}}{\left(\int_{C_{r_1,r_2}} u^q\,dx\right)^{2/q}}$$

$$= \frac{\int_{C_{r_1,r_2}} -\Delta u\cdot u\,dx}{\left(\int_{C_{r_1,r_2}} u^q\,dx\right)^{2/q}} \leq \lambda_{\beta,q}(\Omega)\left(\int_{C_{r_1,r_2}} u^q\,dx\right)^{1-\frac{2}{q}}.$$

If $q > 2$ and since

$$\int_{C_{r_1,r_2}} u^q\,dx < \int_\Omega u^q\,dx = 1,$$

we deduce $\lambda_{\beta,q}(C_{r_1,r_2}) < \lambda_{\beta,q}(\Omega)$, against the optimality of Ω.

The following result concludes the analysis of the optimal set Ω, proving the validity of the Faber-Krahn inequality of Theorem 4.1

Theorem 4.14 (*Ω* **is a ball).** *Ω coincides with a ball up to a $\mathcal{H}^{N-1}$-negligible set.*

Proof. In view of Propositions 4.10 and 4.12, it suffices to exclude that Ω is an annulus.

Assume by contradiction that according to Proposition 4.12.

$$\Omega = B_{r_2} \setminus \overline{B_{r_1}} \qquad \text{up to a } \mathcal{H}^{N-1}\text{-negligible set}$$

for some $0 < r_1 < r_2$, and $u(x) = \psi(|x|)$ with ψ strictly increasing on $[r_1, r_0]$ and strictly decreasing on $[r_0, r_2]$, where $r_0 \in]r_1, r_2[$.

We claim that

$$\psi(r_1) < \psi(r_2). \tag{4.26}$$

Let $r^* \in [r_1, r_0[$ be such that $\psi(r^*) = \psi(r_2)$. We perform a spherically decreasing rearrangement of the restriction of u to the annulus $B_{r_2} \setminus \overline{B_{r^*}}$, and then extend the function with the value $\psi(r_2)$ to the entire ball B_r with $|B_r| = |\Omega|$. Let us denote by v this new function. We have in view of the properties of the spherically decreasing rearrangement

$$\int_{B_r} |\nabla v|^2 \, dx \le \int_\Omega |\nabla u|^2 \, dx,$$

while concerning the surface part, since $r < r_2$,

$$\int_{\partial B_r} v^2 \, dx < \int_{\partial \Omega} u^2 \, dx.$$

Finally

$$\int_{B_r} v^q \, dx \ge \int_\Omega u^q \, dx$$

so that

$$\frac{\int_{B_r} |\nabla v|^2 \, dx + \beta \int_{\partial B_r} v^2 \, d\mathcal{H}^{N-1}}{\left(\int_{B_r} v^q \, dx\right)^{\frac{2}{q}}} < \lambda_{\beta,q}(\Omega),$$

against the optimality of Ω.

Claim (4.26) follows in view of the optimality of Ω. Let us consider indeed the function

$$g(\rho_1, \rho_2) := \frac{\int_{\rho_1}^{\rho_2} \psi'(\rho)^2 N\omega_N \rho^{N-1} \, d\rho + \beta\psi^2(\rho_2) N\omega_N \rho_2^{N-1} + \beta\psi^2(\rho_1) N\omega_N \rho_1^{N-1}}{\left(\int_{\rho_1}^{\rho_2} \psi^q(\rho) N\omega_N \rho^{N-1} \, d\rho\right)^{\frac{2}{q}}}.$$

Let us vary ρ_2 and consequently ρ_1 in such a way to preserve the volume of the annulus, i.e., such that

$$\rho_2^N - \rho_1^N = r_2^N - r_1^N.$$

The optimality of Ω entails that

$$\frac{\partial g}{\partial \rho_2}(r_1, r_2) + \frac{\partial g}{\partial \rho_1}(r_1, r_2) \left(\frac{r_2}{r_1}\right)^{N-1} = 0. \tag{4.27}$$

Since

$$\int_{r_1}^{r_2} \psi^q(\rho) N\omega_N \rho^{N-1} \, d\rho = 1,$$

and taking into account the Robin conditions

$$\psi'(r_2) = -\beta\psi(r_2) \qquad \text{and} \qquad \psi'(r_1) = \beta\psi(r_1),$$

a straight-forward computation shows that (4.27) amounts in requiring that

$$\begin{aligned} &- \beta^2\psi^2(r_2) N\omega_N r_2^{N-1} + \beta\psi^2(r_2) N(N-1)\omega_N r_2^{N-2} - \frac{2}{q}\lambda_{\beta,q}(\Omega)\psi^q(r_2) N\omega_N r_2^{N-1} \\ &+ \beta^2\psi^2(r_1) N\omega_N r_2^{N-1} + \beta\psi^2(r_1) N(N-1)\omega_N r_1^{N-2} \left(\frac{r_2}{r_1}\right)^{N-1} + \frac{2}{q}\lambda_{\beta,q}(\Omega)\psi^q(r_1) N\omega_N r_2^{N-1} = 0, \end{aligned}$$

so that

$$\beta^2 N\omega_N r_2^{N-1}[\psi^2(r_2)-\psi^2(r_1)]+\frac{2}{q}\lambda_{\beta,q}(\Omega)N\omega_N r_2^{N-1}[\psi^q(r_2)-\psi^q(r_1)]$$
$$=\beta\psi^2(r_2)N(N-1)\omega_N r_2^{N-2}+\beta\psi^2(r_1)N(N-1)\omega_N r_1^{N-2}\left(\frac{r_2}{r_1}\right)^{N-1}>0.$$

This entails $\psi(r_2)>\psi(r_1)$, and claim (4.26) follows. □

Remark 4.15. We can summarize the proof of the Faber-Krahn inequality of Theorem 4.1 in the following way.

(a) Theorem 4.2 provides the existence of an optimal domain Ω with associated optimal function u.
(b) Theorem 4.4 shows that up to a translation, u is the restriction to Ω of a radial and analytic function. The proof is based on a reflection technique which requires $q\in[1,2]$ together with the analysis of the Euler Lagrange equation satisfied by u.
(c) As a consequence of the radiality of u and of the properties (4.3), (4.4) and (4.5) given by Theorem 4.2, it is shown in Theorem 4.7 that a Robin boundary condition is satisfied on the essential boundary $\partial^e\Omega$.
(d) Proposition 4.10 and Proposition 4.12 show that Ω is either a ball or an annulus up to $\mathcal{H}^{N-1}$-negligible sets: the proof is obtained by comparing Ω with suitable balls and annuli with lower or equal volume. In view of Remark 4.11 and of Remark 4.13, the computations can be formally extended to cover also the case $q>2$.
(e) Finally, the possibility that Ω coincides with an annulus is excluded in Theorem 4.14 by showing that the ball with equal volume is more convenient (also a comparison with suitable annuli with the same volume is used).

5. Volume penalized Faber-Krahn inequalities

In this section we establish a Faber-Krahn inequality for the eigenvalue $\lambda_{\beta,q}$ with a volume penalization. The class of domains under consideration is again given by $\mathcal{A}(\mathbb{R}^N)$ defined in (3.2) which contains the family of Lipschitz regular domains. The volume penalized inequality provides new information about the minimality of balls for the parameter q belonging to the interval $[1,2N/(N-1)[$.

The following result holds.

Theorem 5.1 (Volume-penalized Faber-Krahn inequality). *Let $k,\beta>0$ and $1\le q<\frac{2N}{N-1}$. Then there exists a ball B such that for every $\Omega\in\mathcal{A}(\mathbb{R}^N)$*

$$\lambda_{\beta,q}(B)+k|B|\le\lambda_{\beta,q}(\Omega)+k|\Omega|. \tag{5.1}$$

Moreover, equality holds if and only if Ω coincides with a ball up to a $\mathcal{H}^{N-1}$-negligible set.

Proof. The starting point is again the existence of an optimal domain. Thanks to Theorem 6.1 in Section 6, the minimum problem

$$\min_{\Omega\in\mathcal{A}(\mathbb{R}^N)}\lambda_{\beta,q}(\Omega)+k|\Omega| \tag{5.2}$$

admits a solution. Moroever every minimizer $\Omega\in\mathcal{A}(\mathbb{R}^N)$ coincides up to a $\mathcal{H}^{N-1}$-negligible set with an open connected set such that $\lambda_{\beta,q}(\Omega)$ is achieved on an analytic and positive function $u\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ with

$$-\Delta u=\lambda_{\beta,q}(\Omega)u^{q-1}\qquad\text{on }\Omega,\qquad\int_\Omega u^q\,dx=1,$$

and such that

$$u>\alpha\qquad\text{on }\Omega$$

for some $\alpha>0$. Finally if we extend u to zero outside Ω, we have

$$u\in BV(\mathbb{R}^N),\qquad\partial\Omega=\overline{J_u},\qquad\text{and}\qquad\mathcal{H}^{N-1}(\partial\Omega\setminus J_u)=0.$$

In order to prove (5.1), we need simply to show that an optimal domain Ω is necessarily a ball up to $\mathcal{H}^{N-1}$-negligible sets.

We will follow the strategy of the proof of Theorem 4.1 developed in Section 4 and summarized in Remark 4.15. In order to cover the new setting, we need just to adapt the proof of Theorem 4.4 concerning the radiality of the function u, which employed a reflection technique specifically tailored to the case $q \in [1, 2]$. This can be done as follows.

Let us consider an hyperplane π_1 parallel to $x_1 = 0$ which splits Ω in two parts $\Omega^\pm$ such that

$$\int_{\Omega^+} u^q\,dx = \int_{\Omega^-} u^q\,dx = \frac{1}{2}.$$

Note that the term

$$\int_{\partial\Omega\cap\pi_1} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}$$

which (eventually) appears in the surface part of the Rayleigh quotient defining $\lambda_{\beta,q}(\Omega)$ can be reinterpreted, since the normal ν involved in the definition of $u^\pm$ coincides $\mathcal{H}^{N-1}$-a.e. with that of π_1, as

$$\int_{\partial\Omega\cap\pi_1} (u_+)^2 + (u_-)^2\,d\mathcal{H}^{N-1},$$

where $u_\pm$ are the traces of $u_{\Omega^\pm}$ on π_1 (defined as in (3.3))

Up to a switch between the two open sets, we can assume (denoting with $\pi_1^\pm$ the two half-spaces determined by π_1)

$$\begin{aligned}\int_{\Omega^+} |\nabla u|^2\,dx + \beta\int_{\partial\Omega\cap\pi_1^+} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} + \beta\int_{\partial\Omega\cap\pi_1} (u_+)^2\,d\mathcal{H}^{N-1} + k|\Omega^+| \\ \le \int_{\Omega^-} |\nabla u|^2\,dx + \beta\int_{\partial\Omega\cap\pi_1^-} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} + \beta\int_{\partial\Omega\cap\pi_1} (u_-)^2\,d\mathcal{H}^{N-1} + k|\Omega^-|.\end{aligned}$$

Denoting with $\tilde{\Omega}^+$ the reflection of Ω^+ across π_1, let us consider

$$\Omega_1 := int\left(\Omega^+ \cup \pi \cup \tilde{\Omega}^+\right) \in \mathcal{A}(\mathbb{R}^N).$$

Ω_1 is a minimizer for problem (5.2) which is symmetric with respect to π_1: the associated $\lambda_{\beta,q}(\Omega_1)$ is achieved on the function u_1 given by the reflection of u, which is thus symmetric with respect to π_1.

The rest of the proof of Theorem 4.4, based on reflections with respect to the other coordinate hyperplanes, can be adapted analogously, yielding the radiality of u. The proof is thus concluded. □

In the rest of the section, as mentioned above, we derive some consequences of the volume penalized Faber-Krahn inequality (5.1). The following technical result will be useful.

Lemma 5.2. *For $k, \beta > 0$ and $1 \le q < 2N/(N-1)$, let $B = B_{r(k)}$ be an optimal ball given by Theorem 5.1. Then*

$$\lim_{k\to+\infty} r(k) = 0 \qquad \text{and} \qquad \lim_{k\to 0^+} r(k) = +\infty. \tag{5.3}$$

Proof. Since

$$\lambda_{\beta,q}(B_{r(k)}) + k|B_{r(k)}| \le \lambda_{\beta,q}(\Omega) + k|\Omega| \tag{5.4}$$

for every $\Omega \in \mathcal{A}(\mathbb{R}^N)$, by letting $k \to 0^+$ we have

$$\limsup_{k\to 0^+} \lambda_{\beta,q}(B_{r(k)}) = 0.$$

The second relation in (5.3) follows recalling that $r \mapsto \lambda_{\beta,q}(B_r)$ is decreasing in view of Corollary 3.2. Dividing by k in (5.4) and sending $k \to +\infty$ we deduce the first relation in (5.3), so that the proof is concluded. □

A first consequence of inequality (5.1) is the fact that the classical Faber-Krahn inequality for $\lambda_{\beta,q}$ holds in the case $q \in]2, \frac{2N}{N-1}[$ at least among domains whose volume can range in a set accumulating at zero and at infinity.

Theorem 5.3. *Let $\beta > 0$ and $q \in]2, 2N/(N-1)[$. There exists a set $\mathcal{M} \subseteq]0, +\infty[$ with*

$$\inf \mathcal{M} = 0 \qquad \text{and} \qquad \sup \mathcal{M} = +\infty$$

such that following holds: for every domain $\Omega \in \mathcal{A}(\mathbb{R}^N)$ with $|\Omega| \in \mathcal{M}$, we have

$$\lambda_{\beta,q}(\Omega) \geq \lambda_{\beta,q}(B),$$

where B is a ball such that $|B| = |\Omega|$. Moreover equality holds if and only if Ω coincides up to a $\mathcal{H}^{N-1}$ negligible set with a ball.

Proof. According to Theorem 5.1, for every $k > 0$ let $B = B_{r(k)}$ be a ball such that

$$\forall \Omega \in \mathcal{A}(\mathbb{R}^N) \,:\, \lambda_{\beta,q}(B_{r(k)}) + k|B_{r(k)}| \leq \lambda_{\beta,q}(\Omega) + k|\Omega|.$$

By Lemma 5.2 we know that

$$\lim_{k \to +\infty} r(k) = 0 \qquad \text{and} \qquad \lim_{k \to 0^+} r(k) = +\infty.$$

The conclusion follows by setting $\mathcal{M} := \{\omega_N r(k)^N\}_{k>0}$. □

The following result shows that we can get rid of the volume term in (5.1), obtaining thus a classical Faber-Krahn inequality for $\lambda_{\beta,q}$ provided that the map

$$r \mapsto \lambda_{\beta,q}(B_r)$$

is convex on $]0, +\infty[$. This is the case for example when $q = 2$ (the linear case) and $q = 1$ (the torsion rigidity case) as shown in Remark 5.5 below, and it could suggest a unified approach to the proof of the Faber-Krahn inequality for the whole range of parameters q in $[1, 2N/(N-1)[$. Unfortunately, despite of the simplicity of the property (it is a one dimensional problem), an analytical justification seems at the moment out of reach.

Theorem 5.4 (Sufficient condition for the Faber-Krahn inequality). *Let $\beta > 0$ and $1 \leq q < 2N/(N-1)$. If the decreasing map $r \mapsto \lambda_{\beta,q}(B_r)$ is also convex on $]0, +\infty[$, then for every domain $\Omega \in \mathcal{A}(\mathbb{R}^N)$*

$$\lambda_{\beta,q}(\Omega) \geq \lambda_{\beta,q}(B),$$

where B is a ball such that $|B| = |\Omega|$. Moreover, equality holds if and only if Ω coincides with a ball up to $\mathcal{H}^{N-1}$-negligible sets.

Proof. According to Theorem 5.1, given $k > 0$ let $r(k) > 0$ be the radius of the optimal ball $B_{r(k)}$ such that

$$\forall \Omega \in \mathcal{A}(\mathbb{R}^N) \,:\, \lambda_{\beta,q}(B_{r(k)}) + k|B_{r(k)}| \leq \lambda_{\beta,q}(\Omega) + k|\Omega|. \tag{5.5}$$

Equality holds if and only if Ω coincides with a ball up to a $\mathcal{H}^{N-1}$-negligible set. By assumption on $r \mapsto \lambda_{\beta,q}(B_r)$, the map

$$r \mapsto \lambda_{\beta,q}(B_r) + k|B_r|$$

is strictly convex, so that the optimal radius $r(k)$ is uniquely determined by k. This entails also that $k \mapsto r(k)$ is continuous on $]0, +\infty[$. By Lemma 5.2 we know that

$$\lim_{k \to +\infty} r(k) = 0 \qquad \text{and} \qquad \lim_{k \to 0^+} r(k) = +\infty.$$

Consequently for every $\Omega \in \mathcal{A}(\mathbb{R}^N)$ we can tune k in such a way that $|B_{r(k)}| = |\Omega|$, so that (5.5) entails

$$\lambda_{\beta,q}(B_{r(k)}) \leq \lambda_{\beta,q}(\Omega)$$

and the proof is concluded. □

Remark 5.5 (The linear and the torsion rigidity cases). As mentioned above, the requirement

$$r \mapsto \lambda_{\beta,q}(B_r) \text{ is convex}$$

is verified in the most relevant cases $q = 1$ (torsion rigidity) and $q = 2$ (linear eigenvalue). For $q = 1$, a direct computation shows that

$$\lambda_{\beta,1}(B_r) = \frac{1}{\frac{\omega_N}{\beta N} r^{N+1} + \frac{\omega_N}{N(N+2)} r^{N+2}},$$

which is readily seen to be convex.

When $q = 2$, the explicit form of $\lambda_{\beta,2}(B_r)$ involves Bessel functions, and so it is not easy to handle. The convexity follows by adapting the arguments of [9, Section 5] to the case of Robin boundary conditions.

6. The shape optimization problems

This section is devoted to the proof of the shape optimization problems on the class $\mathcal{A}(\mathbb{R}^N)$ defined in (3.2) which were pivotal in the analysis of Section 4 and Section 5.

We will say that a pair (k, γ) is admissible if

$$k \in]0, +\infty[\text{ and } \gamma = +\infty \qquad \text{or} \qquad k = 0, \gamma \in]0, +\infty[. \tag{6.1}$$

We can reformulate Theorem 4.2 and Theorem 5.1 in the following unified form.

Theorem 6.1 (The shape optimization problem). *Let $\beta > 0$ and $1 \le q < \frac{2N}{N-1}$. For every admissible pair (k, γ) the minimum problem*

$$\min_{\Omega \in \mathcal{A}(\mathbb{R}^N), |\Omega| \le \gamma} \lambda_{\beta,q}(\Omega) + k|\Omega| \tag{6.2}$$

admits a solution. Moreover every minimizer $\Omega \in \mathcal{A}(\mathbb{R}^N)$ coincides up to a $\mathcal{H}^{N-1}$-negligible set with an open connected set such that $\lambda_{\beta,q}(\Omega)$ is achieved on an analytic and positive function $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ with

$$-\Delta u = \lambda_{\beta,q}(\Omega) u^{q-1} \qquad \text{on } \Omega, \qquad \int_\Omega u^q \, dx = 1,$$

and such that

$$u > \alpha \qquad \text{on } \Omega$$

for some $\alpha > 0$. Finally if we extend u to zero outside Ω, we have

$$u \in BV(\mathbb{R}^N), \qquad \partial\Omega = \overline{J_u}, \qquad \text{and} \qquad \mathcal{H}^{N-1}(\partial\Omega \setminus J_u) = 0.$$

The strategy to prove Theorem 6.1 is the following. In Subsection 6.3 we relax the problem on $\mathcal{A}(\mathbb{R}^N)$ to a free discontinuity problem on a class of functions of bounded variation, namely the space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ introduced in [6]: we recall some results concerning special functions of bounded variation and the space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ in Subsection 6.1.

We prove the existence of minimizers of the free discontinuity problem through a concentration-compactness argument, for which some results collected in Subsection 6.2 will be useful.

Regularity properties of the minimizers are exploited in Subsection 6.4. We show firstly that they are in L^∞, and then that they are bounded from below on their support by a strictly positive constant. These facts entail that the minimizers are suitable local minimizers of the Mumford-Shah functional [24] (see Subsection 6.1.1), so that, thanks to the regularity result of [7], their jump set is essentially closed. It turns then out that the support of a minimizer is a connected domain $\Omega \in \mathcal{A}(\mathbb{R}^N)$. In Subsection 6.5 we prove that such an Ω provides a solution to problem (6.2).

6.1. **Some preliminaries on free discontinuity problems.**

6.1.1. **Special functions of bounded variation and the Mumford-Shah functional.** The space SBV of *special functions of bounded variation* was introduced in [15] and studied in detail in [1] in order to deal variationally with the Mumford-Shah functional arising in image segmentation [24].

Given $\Omega \subseteq \mathbb{R}^N$ open, we set

$$SBV(\Omega) := \{u \in BV(\Omega) \, : \, D^s u \text{ is concentrated on } J_u\}.$$

Here J_u denotes the jump set of u and $D^s u$ stands for the singular part with respect to the Lebesgue measure of the finite Radon measure Du. The localized version of SBV will be denoted by $SBV_{loc}(\Omega)$.

The Mumford-Shah functional on SBV assumes the following form

$$MS(u) := \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(J_u) + \int_\Omega |u-g|^2 \, dx,$$

where $g \in L^\infty(\Omega)$ is a given function.

As a consequence of Ambrosio's compactness and lower semicontinuity theorem [1], the Mumford-Shah functional admits minimizers on SBV. Following De Giorgi, Carriero and Leaci [16], such minimizers turn out to enjoy regularity properties, namely the jump set J_u is *essentially closed* and the function u is *regular* (in dependence of g) outside $\overline{J_u}$.

For the purposes of our analysis, it is convenient to introduce the following local minimality property formulated in [7].

Definition 6.2 (**Almost quasi-minimality**). *We say that $u \in SBV_{loc}(\mathbb{R}^N)$ is an almost-quasi minimizer for the Mumford-Shah functional if there exist $\Lambda \geq 1$ and $\alpha, c_\alpha, r_0 > 0$ such that for every ball $B_r(y) \subseteq \mathbb{R}^N$ with center $y \in \mathbb{R}^N$ and radius $0 < r < r_0$ and for every $v \in SBV_{loc}(\mathbb{R}^N)$ with $\{u \neq v\} \subseteq B_r(y)$*

$$\int_{B_r(y)} |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(J_u \cap \overline{B_r(y)}) \leq \int_{B_r(y)} |\nabla v|^2 \, dx + \Lambda \mathcal{H}^{N-1}(J_v \cap \overline{B_r(y)}) + c_\alpha r^{N-1+\alpha}.$$

Minimizers of the Mumford-Shah fuctional are easily shown to be almost quasi-minimizers in the sense above. The following result has been proved in [7, Theorem 3.1].

Theorem 6.3 (**Essential closedness of the jump set**). *Let $u \in SBV_{loc}(\mathbb{R}^N)$ be an almost quasi-minimizer for the Mumford-Shah functional. Then the jump set of u is essentially closed, i.e.,*

$$\mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0.$$

6.1.2. **The space $SBV^{\frac{1}{2}}$.** In order to deal with the shape optimization problem (6.2), we resort to the following space of functions introduced in [6]

$$SBV^{1/2}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \, : \, u \geq 0 \text{ a.e. in } \mathbb{R}^N \text{ and } u^2 \in SBV(\mathbb{R}^N)\}. \tag{6.3}$$

The first eigenfunction of a regular domain Ω, extended to zero outside Ω, belongs naturally to $SBV^{\frac{1}{2}}(\mathbb{R}^N)$.

Fine properties of functions in $SBV^{1/2}(\mathbb{R}^N)$ are detailed below (see [6, Lemma 1]).

Lemma 6.4. *Let $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$. Then the following facts hold.*

(a) *u is a.e. approximately differentiable (see* [2, Definition 3.70]*) with approximate gradient ∇u such that*
$$\nabla u^2 = 2u\nabla u \qquad a.e. \text{ in } \mathbb{R}^N.$$
(b) *The jump set J_u is $\mathcal{H}^{N-1}$-rectifiable and a normal ν_u can be chosen in such a way that*
$$D^j(u^2) = [(u^+)^2 - (u^-)^2] \, \nu_u \, d\mathcal{H}^{N-1} \llcorner J_u.$$
(c) *For every $\varepsilon > 0$ we have $u \vee \varepsilon := \max\{u, \varepsilon\} \in SBV(\Omega)$ for every bounded open set $\Omega \subset \mathbb{R}^N$.*

The main compactness and lower semicontinuity properties of $SBV^{\frac{1}{2}}$ are contained in the following result (see [6, Theorem 2]).

Theorem 6.5. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ and let $C>0$ be such that for every $n\in\mathbb{N}$*

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} u_n^2\,dx \le C.$$

Then there exist $u\in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that the following facts hold.

(a) *Compactness: $u_{n_k}\to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$.*
(b) *Lower semicontinuity: for every open set $A\subseteq\mathbb{R}^N$ we have*

$$\int_A |\nabla u|^2\,dx \le \liminf_{k\to\infty}\int_A |\nabla u_{n_k}|^2\,dx$$

and

$$\int_{J_u\cap A} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \le \liminf_{k\to\infty}\int_{J_{u_{n_k}}\cap A} (u_{n_k}^+)^2 + (u_{n_k}^-)^2\,d\mathcal{H}^{N-1}.$$

Finally, the following result proved in [7, Theorem 4.1] will be crucial in our analysis.

Theorem 6.6. *Let $\beta,k>0$ and $u\in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $|supp(u)|<+\infty$. Assume that there exist $\varepsilon_0>0$ and $C>0$ such that for a.e. $0<\delta<\varepsilon<\varepsilon_0$ the following inequality holds:*

$$\int_{\{\delta<u<\varepsilon\}} |\nabla u|^2\,dx + \beta\delta^2\mathcal{H}^{N-1}(\partial^e\{\delta<u<\varepsilon\}\cap J_u) + k|\{u<\varepsilon\}| \le C\beta\varepsilon^2\mathcal{H}^{N-1}(\partial^e\{u\ge\varepsilon\}\setminus J_u).$$

Then

$$u\ge\alpha \qquad \text{a.e. on } supp(u)$$

for some $\alpha>0$.

6.2. **Principal frequencies on $SBV^{\frac{1}{2}}(\mathbb{R}^N)$.** In this section we deal with the main properties of the Rayleigh quotient

$$R_{\beta,q}(u) := \frac{\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2+(u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} \tag{6.4}$$

on the set

$$\{u\in SBV^{\frac{1}{2}}(\mathbb{R}^N)\,:\,u\ne 0,\ |supp(u)|\le m\},$$

where $SBV^{\frac{1}{2}}$ is defined in (6.3). The quotient is well defined (eventually with value $+\infty$) in view of the fine properties of functions in $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ recalled in Lemma 6.4, and since $u\in L^{2N/N-1}(\mathbb{R}^N)$.

The following properties hold true.

Lemma 6.7. *Given $1\le q<\frac{2N}{N-1}$ and $\beta,m>0$, let us set*

$$\lambda_{\beta,q}(m) := \inf\{R_{\beta,q}(u)\,:\,u\in SBV^{\frac{1}{2}}(\mathbb{R}^N), u\ne 0, |supp(u)|\le m\}. \tag{6.5}$$

Then following items hold true.

(a) $\lambda_{\beta,q}(m)>0$.
(b) *For every $t>0$*

$$\lambda_{\beta,q}(m) = t^{N-2-\frac{2N}{q}}\lambda_{\beta t,q}\left(\frac{m}{t^N}\right). \tag{6.6}$$

(c) *For every $t\ge 1$*

$$\lambda_{\beta,q}(tm) \le t^{N-\frac{2N}{q}-1}\lambda_{\beta,q}(m). \tag{6.7}$$

(d) *We have*

$$\liminf_{t\to 0^+}\frac{\lambda_{t\beta,q}(1)}{t} > 0.$$

Proof. Assume that there exists $u_n \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$, $u_n \neq 0$, with $|supp(u_n)| \le m$ and such that

$$R_{\beta,q}(u_n) \to 0.$$

We may assume that

$$\int_{\mathbb{R}^N} u_n^q \, dx = 1, \tag{6.8}$$

so that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \beta \int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2 \, d\mathcal{H}^{N-1} \to 0.$$

Using the embedding of $BV(\mathbb{R}^N)$ into $L^{N/N-1}(\mathbb{R}^N)$ applied to u_n^2, taking into account that $|supp(u_n)| \le m$, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^N} u_n^{\frac{2N}{N-1}} \, dx\right)^{\frac{N-1}{N}} &\le C\left[\int_{\mathbb{R}^N} |u_n \nabla u_n| \, dx + \int_{J_{u_n}} |(u_n^+)^2 - (u_n^-)^2| \, d\mathcal{H}^{N-1}\right] \\ &\le \varepsilon \int_{\mathbb{R}^N} u_n^2 \, dx + \frac{C_\varepsilon}{\varepsilon} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + C \int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2 \, d\mathcal{H}^{N-1} \\ &\le \varepsilon |m|^{\frac{1}{N}} \left(\int_{\mathbb{R}^N} u_n^{\frac{2N}{N-1}} \, dx\right)^{\frac{N-1}{N}} + \frac{C_\varepsilon}{\varepsilon} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + C \int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2 \, d\mathcal{H}^{N-1}.\end{aligned}$$

We thus infer

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} u_n^{\frac{2N}{N-1}} \, dx = 0$$

which is against (6.8). Point (a) is thus proved.

Point (b) follows by simple rescaling arguments. If $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $|supp(u)| \le m$, setting $v(x) := u(tx)$ we obtain

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx = t^{2-N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,$$

$$\int_{J_v} (v^+)^2 + (v^-)^2 \, d\mathcal{H}^{N-1} = t^{1-N} \int_{J_u} (u^+)^2 + (u^-)^2 \, d\mathcal{H}^{N-1}$$

and

$$\int_{\mathbb{R}^N} v^q \, dx = t^{-N} \int_{\mathbb{R}^N} u^q \, dx$$

so that

$$\begin{aligned}R_{\beta,q}(u) &= \frac{t^{N-2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + t^{N-1} \beta \int_{J_u} (v^+)^2 + (v^-)^2 \, d\mathcal{H}^{N-1}}{\left(t^N \int_{\mathbb{R}^N} v^q \, dx\right)^{\frac{2}{q}}} \\ &= t^{N-2-\frac{2N}{q}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \beta t \int_{J_u} (v^+)^2 + (v^-)^2 \, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v^q \, dx\right)^{\frac{2}{q}}} = t^{N-2-\frac{2N}{q}} R_{\beta t,q}(v).\end{aligned}$$

Since $|supp(v)| = |supp(u)|/t^N$, the result easily follows.

The proof of point (c) is similar (compare also with the proof Lemma 3.1). Let us come to point (d). For every $t > 0$ there exists $u_t \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $|supp(u_t)| \le 1$, $\int_{\mathbb{R}^N} u_t^q \, dx = 1$ and such that

$$t^2 + \lambda_{t\beta,q}(1) > \int_{\mathbb{R}^N} |\nabla u_t|^2 \, dx + \beta t \int_{J_{u_t}} (u_t^+)^2 + (u_t^-)^2 \, d\mathcal{H}^{N-1}.$$

As a consequence

$$\liminf_{t \to 0^+} \frac{\lambda_{t\beta,q}(1)}{t} \ge \liminf_{t \to 0^+} \left[\frac{1}{t} \int_{\mathbb{R}^N} |\nabla u_t|^2 \, dx + \beta \int_{J_{u_t}} (u_t^+)^2 + (u_t^-)^2 \, d\mathcal{H}^{N-1}\right].$$

Assume by contradiction that the right-hand side is zero. Then along a suitable $t_n \searrow 0$ we would get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_{t_n}|^2\, dx + \beta \int_{J_{u_{t_n}}} (u_{t_n}^+)^2 + (u_{t_n}^-)^2\, d\mathcal{H}^{N-1} = 0.$$

Following the arguments used to prove point (a) above, we get a contradiction. □

The following lemma will be useful in order to deal with exponents q ranging in $[1,2[$. For $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $|supp(u)| < +\infty$ let us set

$$T_{\beta,q}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{\beta}{2}\int_{J_u} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1} - \frac{1}{q}\int_{\mathbb{R}^N} u^q\, dx. \tag{6.9}$$

Lemma 6.8. *For $1 \le q < 2$ and $\beta, m > 0$ let*

$$t_{\beta,q}(m) := \inf\{T_{\beta,q}(u) \,:\, u \in SBV^{\frac{1}{2}}(\mathbb{R}^N), |supp(u)| \le m\}.$$

The following items hold true.

(a) *If $(u_n)_{n\in\mathbb{N}}$ is minimizing sequence for $R_{\beta,q}$ (see (6.4)) on*

$$\left\{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N) \,:\, u \neq 0, |supp(u)| \le m\right\},$$

then $(c_n u_n)_{n\in\mathbb{N}}$ with

$$c_n := \frac{[R_{\beta,q}(u_n)]^{\frac{1}{q-2}}}{\left(\int_{\mathbb{R}^N} u_n^q\, dx\right)^{\frac{1}{q}}}$$

is a minimizing sequence for $T_{\beta,q}$ on the same set with

$$T_{\beta,q}(c_n u_n) = \left(\frac{1}{2} - \frac{1}{q}\right)(R_{\beta,q}(u_n))^{\frac{q}{q-2}}.$$

In particular $t_{\beta,q}(m) < 0$.

(b) *If $0 < m_1 < m_2$, then*

$$\frac{-t_{\beta,q}(m_1)}{m_1^{1+\frac{q}{(2-q)N}}} \le \frac{-t_{\beta,q}(m_2)}{m_2^{1+\frac{q}{(2-q)N}}}. \tag{6.10}$$

Proof. For $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $u \neq 0$ and $|supp(u)| \le m$, we claim that

$$T_{\beta,q}(u) \ge \left(\frac{1}{2} - \frac{1}{q}\right)(R_{\beta,q}(u))^{\frac{q}{q-2}}, \tag{6.11}$$

and

$$T_{\beta,q}\left(\frac{[R_{\beta,q}(u)]^{\frac{1}{q-2}}}{\left(\int_{\mathbb{R}^N} u^q\, dx\right)^{1/q}} u\right) = \left(\frac{1}{2} - \frac{1}{q}\right)(R_{\beta,q}(u))^{\frac{q}{q-2}}. \tag{6.12}$$

Since $1 \le q < 2$, from (6.11) we get (recall that $\inf R_{\beta,q} > 0$ thanks to Lemma 6.7)

$$T_{\beta,q}(u) \ge \left(\frac{1}{2} - \frac{1}{q}\right)(\inf R_{\beta,q})^{\frac{q}{q-2}},$$

while from (6.12),

$$T_{\beta,q}(c_n u_n) = \left(\frac{1}{2} - \frac{1}{q}\right)[R_{\beta,q}(u_n)]^{\frac{q}{q-2}} \to \left(\frac{1}{2} - \frac{1}{q}\right)(\inf R_{\beta,q})^{\frac{q}{q-2}}.$$

In order to conclude the proof of point (a), we need to show claims (6.11) and (6.12). Notice that

$$\min_{t>0} T_{\beta,q}(tu) = T_{\beta,q}(t_* u),$$

where

$$t_* := \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta\int_{J_u} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^q\, dx}\right)^{\frac{1}{q-2}} = \frac{[R_{\beta,q}(u)]^{\frac{1}{q-2}}}{\left(\int_{\mathbb{R}^N} u^q\, dx\right)^{1/q}}.$$

We thus get

$$T_{\beta,q}(u) \geq T_{\beta,q}(t_* u) = \frac{t_*^2}{2}\left[\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}\right] - \frac{t_*^q}{q}\int_{\mathbb{R}^N} u^q\,dx$$

$$= \left(\frac{1}{2} - \frac{1}{q}\right)\left[\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}}\right]^{\frac{q}{q-2}} = \left(\frac{1}{2} - \frac{1}{q}\right)[R_{\beta,q}(u)]^{\frac{q}{q-2}}$$

which proves (6.11). Equality (6.12) is now straight-forward.

Let us come to point (b). Let $0 < t < 1$, and let us consider for $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $u \neq 0$ and $|supp(u)| \leq m$ the function

$$v(x) := t^\alpha u(tx),$$

where $\alpha \in \mathbb{R}$. We get

$$\int_{\mathbb{R}^N} |\nabla v|^2\,dx = t^{2\alpha+2-N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

$$\int_{J_v} (v^+)^2 + (v^-)^2\,d\mathcal{H}^{N-1} = t^{2\alpha+1-N}\int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1},$$

and

$$\int_{\mathbb{R}^N} v^q\,dx = t^{\alpha q - N}\int_{\mathbb{R}^N} u^q\,dx.$$

By imposing $2\alpha + 1 - N = \alpha q - N$, we get $\alpha = \frac{1}{q-2}$ and (since $t < 1$)

$$T_{\beta,q}(v) = t^{\frac{q}{q-2}-N}\left[\frac{t}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{\beta}{2}\int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} - \frac{1}{q}\int_{\mathbb{R}^N} u^q\,dx\right]$$
$$\leq t^{\frac{q}{q-2}-N} T_{\beta,q}(u).$$

Since $|supp(v)| = \frac{|supp(u)|}{t^N}$, we infer

$$t_{\beta,q}\left(\frac{m}{t^N}\right) \leq t^{\frac{q}{q-2}-N} t_{\beta,q}(m).$$

The result now follows by putting $m = m_1$ and $t = \left(\frac{m_1}{m_2}\right)^{1/N}$. □

6.3. **The free discontinuity problem: existence of a solution.** In this section we consider the following free discontinuity problem on $SBV^{\frac{1}{2}}(\mathbb{R}^N)$

$$\inf_{\substack{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)\\ u\neq 0, |supp(u)|\leq\gamma}} \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} + k|supp(u)|\right), \tag{6.13}$$

where

$$\beta > 0, \qquad 1 \leq q < \frac{2N}{N-1}, \qquad \text{and } (k,\gamma) \text{ satisfies (6.1).} \tag{6.14}$$

Such a problem is a *relaxed version* of the shape optimization problem (6.2), the attention being now on the *principal function* rather than on the domain.

Proposition 6.9. *Assuming* (6.14)*, there exists a minimizing sequence* $(u_n)_{n\in\mathbb{N}}$ *for the free discontinuity problem* (6.13) *bounded in* $L^{2N/N-1}(\mathbb{R}^N)$*, and such that*

$$|supp(u_n)| = m > 0, \qquad \textit{and} \qquad \int_{\mathbb{R}^N} u_n^q\,dx = 1. \tag{6.15}$$

In particular the infimum is strictly positive.

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a minimizing sequence. It is not restrictive to assume

$$\int_{\mathbb{R}^N} v_n^q\,dx = 1. \tag{6.16}$$

Comparing with 1_B, where B denotes the unit ball in $\mathbb{R}^N$ centred at the origin, we obtain

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1} + k|supp(v_n)| \le C \tag{6.17}$$

for some $C > 0$. In particular we deduce that up to a subsequence

$$|supp(v_n)| \to m < +\infty. \tag{6.18}$$

Indeed either $k > 0$, and we can invoke (6.17), or $k = 0$, but then $|supp(v_n)| \le \gamma < +\infty$.

We claim that $m > 0$. If by contradiction $m = 0$, by the Sobolev embedding of $BV(\mathbb{R}^N)$ into $L^{N/N-1}(\mathbb{R}^N)$ applied to v_n^2 we get for every $\varepsilon > 0$

$$\left(\int_{\mathbb{R}^N} v_n^{\frac{2N}{N-1}}\,dx\right)^{\frac{N-1}{N}} \le \varepsilon \int_{\mathbb{R}^N} v_n^2 + C_\varepsilon \left[|\nabla v_n|^2\,dx + \int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}\right]$$
$$\le \varepsilon \left(\int_{\mathbb{R}^N} v_n^{\frac{2N}{N-1}}\,dx\right)^{\frac{N-1}{N}} |supp(v_n)|^{\frac{1}{N}} + C_\varepsilon \left[\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}\right]$$

where $C_\varepsilon > 0$ is a suitable constant, so that in view of (6.17)

$$\left(\int_{\mathbb{R}^N} v_n^{\frac{2N}{N-1}}\,dx\right)^{\frac{N-1}{N}} \le C, \tag{6.19}$$

where $C > 0$. Then by Hölder inequality we get easily

$$\int_{\mathbb{R}^N} v_n^q\,dx \to 0,$$

against (6.16).

Let us consider t_n such that

$$\frac{|supp(v_n)|}{t_n^N} = m$$

and set

$$u_n(x) := v_n(t_n x).$$

A straight-forward calculation shows that

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{\frac{2}{q}}} + k|supp(u_n)|$$
$$= \frac{t_n^{2-N} \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + t_n^{1-N} \beta \int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{t_n^{-2N/q} \left(\int_{\mathbb{R}^N} v_n^q\right)^{2/q}} + km,$$

and since $t_n \to 1$, in view of (6.16), (6.17) and (6.18), we get

$$\liminf_{n\to\infty} \left(\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{\frac{2}{q}}} + k|supp(u_n)| \right)$$
$$= \liminf_{n\to\infty} \left(\frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta \int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{\frac{2}{q}}} + k|supp(v_n)| \right).$$

Renormalizing in L^q, we get that $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence satisfying (6.15). The bound in $L^{2N/N-1}(\mathbb{R}^N)$ follows from (6.19).

Finally the infimum is strictly positive. Indeed, if $k > 0$, the result follows since $m > 0$. If $k = 0$, then thanks to Lemma 6.7

$$\inf_{\substack{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)\\ u\neq 0, |supp(u)|\leq\gamma}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2+(u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}}$$
$$= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}}(u_n^+)^2+(u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{\frac{2}{q}}} \geq \lambda_{\beta,q}(m) > 0.$$

□

We are now in a position to prove the main result of the Subsection.

Theorem 6.10 (Existence of a solution). *Assuming* (6.14)*, the free discontinuity problem* (6.13) *admits a solution.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (6.13) according to Proposition 6.9, i.e.,

$$|supp(u_n)| = m > 0, \qquad \int_{\mathbb{R}^N} u_n^q\,dx = 1, \qquad \text{and} \qquad \int_{\mathbb{R}^N} u_n^{2N/N-1}\,dx \leq C, \tag{6.20}$$

for some $C > 0$. Note that in particular $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence for the Rayleigh quotient $R_{\beta,q}$ defined in (6.4) on $\{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N) : u \neq 0, |supp(u)| \leq m\}$, so that in the notation of Lemma 6.7

$$\lim_n \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}}(u_n^+)^2+(u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{2/q}} = \lambda_{\beta,q}(m). \tag{6.21}$$

We resort to a concentration-vanishing-compactness alternative. For every $R \in [0, +\infty[$ let us set

$$f_n(R) := \sup_{y\in\mathbb{R}^N} \int_{y+Q_R} u_n^q\,dx,$$

where $Q_R :=]-R/2, R/2[^N$. In view of Helly's theorem on sequences of increasing functions, up to a subsequence we may assume that

$$f_n \to f \qquad \text{pointwise on } [0, +\infty[$$

for some increasing function $f : [0, +\infty[\to [0, 1]$. Let

$$\alpha := \lim_{R\to+\infty} f(R).$$

The various alternatives are connected to the different possible values of α: vanishing ($\alpha = 0$), dichotomy ($0 < \alpha < 1$), and compactness ($\alpha = 1$).

Step 1: Vanishing cannot occur. By contradiction let $\alpha = 0$. In view of (6.20), by interpolation we get up to a subsequence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n^2\,dx > 0.$$

By [6, Lemma 4] we can find $y_n \in \mathbb{R}^N$ such that

$$\mathcal{H}^{N-1}(J_{u_n} \cap \partial Q_1(y_n)) = 0 \qquad \text{and} \qquad |supp(u_n) \cap Q_1(y_n)| \geq c \tag{6.22}$$

for some $c \in]0, m[$ independent of n. Since

$$\int_{Q_1(y_n)} u_n^q\,dx \leq f_n(1) \to 0, \tag{6.23}$$

so that by interpolation thanks to (6.20)

$$\int_{Q_1(y_n)} u_n^2\,dx \to 0,$$

it is not restrictive, up to reducing the side of $Q_1(y_n)$, to assume that

$$\lim_{n\to\infty}\int_{\partial Q_1(y_n)} \gamma(u_n^2)\,d\mathcal{H}^{N-1}=0, \tag{6.24}$$

where $\gamma(u_n^2)$ is the trace on $\partial Q_1(y_n)$ in the sense of BV-functions. Let us consider

$$v_n := u_n 1_{\mathbb{R}^N\setminus Q_1(y_n)} \in SBV^{\frac{1}{2}}(\mathbb{R}^N).$$

Notice that in view of (6.22), (6.23) and (6.24) we have

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx+\beta\int_{J_{v_n}}(v_n^+)^2+(v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{2/q}}\\
&=\frac{\int_{\mathbb{R}^N\setminus Q_1(y_n)}|\nabla u_n|^2\,dx+\beta\int_{J_{u_n}\setminus Q_1(y_n)}(u_n^+)^2+(u_n^-)^2\,d\mathcal{H}^{N-1}+\beta\int_{\partial Q_1(y_n)}\gamma(u_n^2)\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N\setminus Q_1(y_n)} u_n^q\,dx\right)^{2/q}}\\
&\qquad\le\frac{\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx+\beta\int_{J_{u_n}}(u_n^+)^2+(u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{2/q}}+e_n,
\end{aligned}$$

where $e_n\to 0$. Moreover

$$|supp(v_n)|\le m-c<m.$$

In view of (6.21) we deduce

$$\lambda_{\beta,q}(m-c)\le\lambda_{\beta,q}(m),$$

which is against the rescaling property (6.7) for the frequency $\lambda_{\beta,q}$. The proof of the step is now complete.

Step 2: Dichotomy cannot occur. By contradiction, let us assume that $0<\alpha<1$. We can find two radial cut off functions $\varphi_n,\psi_n:\mathbb{R}^N\to[0,1]$ with $\|\nabla\varphi_n\|_\infty\to 0$, $\|\nabla\psi_n\|_\infty\to 0$,

$$\lim_{n\to\infty}\text{dist}(\text{supp}(\varphi_n),\text{supp}(1-\psi_n))=+\infty,$$

and we can find $y_n\in\mathbb{R}^N$ such that by setting

$$v_n:=\varphi_n(\cdot+y_n)u_n\qquad\text{and}\qquad w_n:=(1-\psi_n)(\cdot+y_n)u_n$$

we have

$$0\le v_n+w_n\le u_n,\qquad \|u_n^q-v_n^q-w_n^q\|_{L^1(\mathbb{R}^N)}\to 0$$

and

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} v_n^q\,dx-\alpha\right|=0\qquad\text{and}\qquad\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} w_n^q\,dx-(1-\alpha)\right|=0. \tag{6.25}$$

Moreover we have

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2-|\nabla v_n|^2-|\nabla w_n|^2\,dx\ge 0 \tag{6.26}$$

and

$$\int_{J_{u_n}}(u_n^+)^2+(u_n^-)^2\,d\mathcal{H}^{N-1}\ge\int_{J_{v_n}}(v_n^+)^2+(v_n^-)^2\,d\mathcal{H}^{N-1}+\int_{J_{w_n}}(w_n^+)^2+(w_n^-)^2\,d\mathcal{H}^{N-1}. \tag{6.27}$$

Clearly

$$|supp(v_n)|+|supp(w_n)|\le|supp(u_n)|.$$

Moreover, thanks to (6.25) and (6.20), there exists $\eta>0$ such that

$$|supp(v_n)|\ge\eta\qquad\text{and}\qquad|supp(w_n)|\ge\eta. \tag{6.28}$$

Let us assume that $2\le q<\frac{2N}{N-1}$. By employing the numerical inequality

$$\frac{a+b}{(c+d)^{2/q}}\ge\min\left\{\frac{a}{c^{2/q}},\frac{b}{d^{2/q}}\right\},\qquad a,b\ge 0,\ c,d>0 \tag{6.29}$$

we deduce that there exists $e_n \searrow 0$ such that

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{2/q}} \geq \min\left\{ \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta\int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\right)^{2/q}}, \frac{\int_{\mathbb{R}^N} |\nabla w_n|^2\,dx + \beta\int_{J_{w_n}} (w_n^+)^2 + (w_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} w_n^q\right)^{2/q}} \right\} - e_n.$$

We can assume using the notation of Lemma 6.7

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{2/q}} \geq \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta\int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{2/q}} - e_n \geq \lambda_{\beta,q}(m-\eta) - e_n,$$

so that passing to the limit and recalling (6.21) we obtain

$$\lambda_{\beta,q}(m) \geq \lambda_{\beta,q}(m-\eta).$$

This is against the rescaling property (6.7) of $\lambda_{\beta,q}$.

Let us come to the case $1 \leq q < 2$ for which the numerical inequality (6.29) is false. Let us consider

$$\tilde{u}_n := c_n u_n, \qquad \tilde{v}_n := c_n v_n, \qquad \tilde{w}_n := c_n w_n$$

where

$$c_n := \left[\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}\right]^{\frac{1}{q-2}}.$$

By considering the functional $T_{\beta,q}$ defined in (6.9), and taking into account (6.25), (6.26) and (6.27), we get

$$T_{\beta,q}(\tilde{u}_n) \geq T_{\beta,q}(\tilde{v}_n) + T_{\beta,q}(\tilde{w}_n) + e_n$$

where $e_n \to 0$. According to point (a) of Lemma 6.8 we thus obtain

$$t_{\beta,q}(m) \geq t_{\beta,q}(m_{1,n}) + t_{\beta,q}(m_{2,n}) + \tilde{e}_n$$

with $\tilde{e}_n \to 0$, where

$$m_{1,n} := |supp(v_n)| \qquad \text{and} \qquad m_{2,n} := |supp(w_n)|.$$

We deduce (recall that $t_{\beta,q}(m) \leq 0$)

$$\frac{-t_{\beta,q}(m)}{m^{1+\frac{q}{(2-q)N}}} \leq \frac{-t_{\beta,q}(m_{1,n}) - t_{\beta,q}(m_{2,n})}{(m_{1,n}+m_{2,n})^{1+\frac{q}{(2-q)N}}} + \hat{e}_n$$

with $\hat{e}_n \to 0$. Now recall that given $\varepsilon, \eta_1, \eta_2 > 0$, there exists $\delta > 0$ such that for every $a, b > 0$ and $\eta_1 \leq c, d \leq \eta_2$

$$\frac{a+b}{(c+d)^{1+\varepsilon}} \leq (1-\delta)\max\left\{\frac{a}{c^{1+\varepsilon}}, \frac{b}{d^{1+\varepsilon}}\right\}.$$

Since thanks to (6.28)

$$\eta \leq \min\{m_{1,n}, m_{2,n}\} \qquad \text{and} \qquad \max\{m_{1,n}, m_{2,n}\} \leq m-\eta,$$

in view of point (b) of Lemma 6.8 we deduce that

$$\frac{-t_{\beta,q}(m)}{m^{1+\frac{q}{(2-q)N}}} \leq (1-\delta)\max\left\{\frac{-t_{\beta,q}(m_{1,n})}{m_{1,n}^{1+\frac{q}{(2-q)N}}}, \frac{-t_{\beta,q}(m_{2,n})}{m_{2,n}^{1+\frac{q}{(2-q)N}}}\right\} + \hat{e}_n \leq (1-\delta)\frac{-t_{\beta,q}(m-\eta)}{(m-\eta)^{1+\frac{q}{(2-q)N}}} + \hat{e}_n,$$

so that letting $n \to \infty$
$$\frac{-t_{\beta,q}(m)}{m^{1+\frac{q}{(2-q)N}}} \le (1-\delta)\frac{-t_{\beta,q}(m-\eta)}{(m-\eta)^{1+\frac{q}{(2-q)N}}}.$$
This is against inequality (6.10), and the proof of the step is thus concluded.

Step 3: Compactness and conclusion. Let us assume that $\alpha = 1$, *i.e.*,
$$\lim_{R\to+\infty} f(R) = 1.$$
There exists $y_n \in \mathbb{R}^N$ such that by setting
$$v_n(x) := u_n(x + y_n),$$
up to a subsequence (not relabeled)
$$v_n \rightharpoonup u \qquad \text{weakly in } L^q(\mathbb{R}^N)$$
with
$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N\setminus Q_R} v_n^q\,dx \le \varepsilon(R),$$
where $\varepsilon(R) \to 0$ as $R \to +\infty$. Thanks to (6.20), by interpolation v_n is also bounded in $L^2(\mathbb{R}^N)$, so that using the compactness property given in Theorem 6.5, we infer that $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$. Moreover we can assume that the convergence $v_n \to u$ is also strong in $L^q_{loc}(\mathbb{R}^N)$ and pointwise almost everywhere. Then for every $R > 0$
$$\int_{Q_R} u^q\,dx = \lim_{n\to+\infty} \int_{Q_R} v_n^q\,dx \ge 1 - \varepsilon(R),$$
so that we conclude
$$\int_{\mathbb{R}^N} u^q\,dx = 1$$
which entails
$$v_n \to u \qquad \text{strongly in } L^q(\mathbb{R}^N).$$
In particular $|supp(u)| \le m$. By lower semicontinuity (see Theorem 6.5) we finally get
$$\begin{aligned}
\liminf_{n\to\infty} &\left(\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_n^q\,dx\right)^{2/q}} + k|supp(u_n)| \right) \\
&= \liminf_{n\to\infty} \left(\frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta\int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{2/q}} + k|supp(v_n)| \right) \\
&\ge \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{2/q}} + k|supp(u)|,
\end{aligned}$$
so that u is a minimum for problem (6.13). The proof is thus concluded. □

6.4. **Regularity of the minimizers.** This subsection is devoted to prove regularity results for minimizers of (6.13). We start with the following L^∞-bound.

Theorem 6.11 (L^∞**-bound).** *Assuming* (6.14)*, let* $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *be a minimizer of problem* (6.13) *according to Theorem 6.10. Then* $u \in L^\infty(\mathbb{R}^N)$.

Proof. Let us assume by contradiction that $u \notin L^\infty(\mathbb{R}^N)$. Notice that for every $M > 0$
$$u \wedge M := \min\{u, M\} \in SBV^{\frac{1}{2}}(\mathbb{R}^N).$$
Since
$$|supp(u \wedge M)| \le |supp(u)|,$$

by comparing u with $u \wedge M$ we obtain

$$\begin{aligned}(6.30)\quad & \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \\ & \le \left(\frac{\int_{\mathbb{R}^N} u^q\,dx}{\int_{\mathbb{R}^N} (u \wedge M)^q\,dx} \right)^{2/q} \left[\int_{\{u \le M\}} |\nabla u|^2\,dx + \beta \int_{\{u^- < u^+ \le M\}} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \right. \\ & \qquad \left. + \beta \int_{\{u^- < M < u^+\}} M^2 + (u^-)^2\,d\mathcal{H}^{N-1} \right].\end{aligned}$$

By renormalization we may assume

$$\int_{\mathbb{R}^N} u^q\,dx = 1,$$

so that we can write

$$\begin{aligned}\left(\frac{\int_{\mathbb{R}^N} u^q\,dx}{\int_{\mathbb{R}^N} (u \wedge M)^q\,dx} \right)^{2/q} &= \frac{1}{\left(1 - \left(\int_{\mathbb{R}^N} u^q\,dx - \int_{\mathbb{R}^N} (u \wedge M)^q\,dx\right)\right)^{2/q}} \\ &= 1 + \frac{2}{q} \left(\int_{\mathbb{R}^N} u^q\,dx - \int_{\mathbb{R}^N} (u \wedge M)^q\,dx \right) + e(M),\end{aligned}$$

with

$$\lim_{M \to +\infty} \frac{e(M)}{\int_{\mathbb{R}^N} u^q\,dx - \int_{\mathbb{R}^N} (u \wedge M)^q\,dx} = 0. \tag{6.31}$$

Since

$$\begin{aligned}& \int_{\{u \le M\}} |\nabla u|^2\,dx + \beta \int_{\{u^- < u^+ \le M\}} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} + \beta \int_{\{u^- < M < u^+\}} M^2 + (u^-)^2\,d\mathcal{H}^{N-1} \\ & \qquad \le \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \le C,\end{aligned}$$

inequality (6.30) together with (6.31) entails that for $M \ge M_0$ large enough

$$\begin{aligned}(6.32)\quad & \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \\ & \le \int_{\{u \le M\}} |\nabla u|^2\,dx + \beta \int_{\{u^- < u^+ \le M\}} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} + \beta \int_{\{u^- < M < u^+\}} M^2 + (u^-)^2\,d\mathcal{H}^{N-1} \\ & \qquad + \hat{C} \left(\int_{\mathbb{R}^N} u^q\,dx - \int_{\mathbb{R}^N} (u \wedge M)^q\,dx \right)\end{aligned}$$

for some $\hat{C} > 0$.

Let us consider for a.e. $M > 0$

$$u_M := (u - M)_+ \in SBV^{\frac{1}{2}}(\mathbb{R}^N).$$

Notice that

$$\int_{\mathbb{R}^N} |\nabla u_M|^2\,dx = \int_{\{u > M\}} |\nabla u|^2\,dx \tag{6.33}$$

and

$$\begin{aligned}(6.34)\quad \int_{J_{u_M}} (u_M^+)^2 + (u_M^-)^2\,d\mathcal{H}^{N-1} &= \int_{M \le u^- < u^+} (u^+ - M)^2 + (u^- - M)^2\,d\mathcal{H}^{N-1} \\ &\quad + \int_{\{u^- < M < u^+\}} (u^+ - M)^2\,d\mathcal{H}^{N-1}.\end{aligned}$$

Since

$$\int_{\{M\le u^-<u^+\}}(u^+)^2+(u^-)^2\,d\mathcal{H}^{N-1}+\int_{\{u^-<M<u^+\}}(u^+)^2-M^2\,d\mathcal{H}^{N-1}$$
$$\ge\int_{\{M\le u^-<u^+\}}(u^+-M)^2+(u^--M)^2\,d\mathcal{H}^{N-1}+\int_{\{u^-<M<u^+\}}(u^+-M)^2\,d\mathcal{H}^{N-1},$$

taking into account (6.33) and (6.34), we get from (6.32)

$$\int_{\mathbb{R}^N}|\nabla u_M|^2\,dx+\beta\int_{J_{u_M}}(u_M^+)^2+(u_M^-)^2\,d\mathcal{H}^{N-1}\le\hat{C}\left(\int_{\mathbb{R}^N}u^q\,dx-\int_{\mathbb{R}^N}(u\wedge M)^q\,dx\right).$$

Setting

$$\alpha(M):=|supp(u_M)|>0,$$

in view of Lemma 6.7 we have for every $1\le p<2N/N-1$

$$\int_{\mathbb{R}^N}|\nabla u_M|^2\,dx+\beta\int_{J_{u_M}}(u_M^+)^2+(u_M^-)^2\,d\mathcal{H}^{N-1}\ge\lambda_{\beta,p}(\alpha(M))\left(\int_{\mathbb{R}^N}u_M^p\,dx\right)^{\frac{2}{p}},$$

where $\lambda_{\beta,p}(\alpha(M))>0$ is defined by (6.5), so that we obtain

$$\lambda_{\beta,p}(\alpha(M))\left(\int_{\mathbb{R}^N}u_M^p\,dx\right)^{\frac{2}{p}}\le\hat{C}\left(\int_{\mathbb{R}^N}u^q\,dx-\int_{\mathbb{R}^N}(u\wedge M)^q\,dx\right).$$

Thanks to the rescaling property (6.6) of Lemma 6.7 we obtain the inequality

$$\lambda_{\alpha(M)^{1/N}\beta,p}(1)\alpha(M)^{1-\frac{2}{N}-\frac{2}{p}}\left(\int_{\mathbb{R}^N}(u-M)_+^p\,dx\right)^{\frac{2}{p}}\le\hat{C}\left(\int_{\mathbb{R}^N}u^q\,dx-\int_{\mathbb{R}^N}(u\wedge M)^q\,dx\right)$$

which can be rewritten in the form

$$g(M)\alpha(M)^{1-\frac{1}{N}-\frac{2}{p}}\left(\int_{\mathbb{R}^N}(u-M)_+^p\,dx\right)^{\frac{2}{p}}\le\hat{C}\int_{\{u\ge M\}}u^q-M^q\,dx,\tag{6.35}$$

where

$$g(M):=\frac{\lambda_{\alpha(M)^{1/N}\beta,p}(1)}{\alpha(M)^{1/N}}$$

and $M\ge M_0\ge 1$. Recall that in view of point (d) of Lemma 6.7

$$\liminf_{M\to+\infty}g(M)>0.\tag{6.36}$$

In order to reach a contradiction, we will manipulate the right-hand side of (6.35) and choose conveniently the exponent p. Let us consider the cases $1\le q\le 2$ and $2<q<2N/N-1$ separately.

Let us assume that $1\le q\le 2$. Since for $C>q$ and $a\ge M\ge 1$

$$a^q-M^q\le C\left[(a-M)^2+M(a-M)\right],$$

we may write choosing $p=2$ in (6.35) and by setting $\tilde{C}:=\hat{C}\cdot C$

$$g(M)\alpha(M)^{-\frac{1}{N}}\int_{\mathbb{R}^N}(u-M)_+^2\,dx\le\tilde{C}\int_{\mathbb{R}^N}(u-M)_+^2+M(u-M)_+\,dx,$$

so that

$$(g(M)-\tilde{C}\alpha(M)^{1/N})\int_{\mathbb{R}^N}(u-M)_+^2\,dx\le\tilde{C}M\alpha(M)^{1/N}\int_{\mathbb{R}^N}(u-M)_+\,dx.$$

Since

$$\int_{\mathbb{R}^N}(u-M)_+\,dx\le\left(\int_{\mathbb{R}^N}(u-M)_+^2\,dx\right)^{\frac{1}{2}}\alpha(M)^{1/2},$$

we obtain

$$g(M)-\tilde{C}\alpha(M)^{1/N}\le\frac{\tilde{C}M\alpha(M)^{1+\frac{1}{N}}}{\int_{\mathbb{R}^N}(u-M)_+\,dx},$$

which can be rearranged as

$$\frac{\left(g(M) - \tilde{C}\alpha(M)^{1/N}\right)^{\frac{N}{N+1}}}{M^{\frac{N}{N+1}}} \le \tilde{C}^{\frac{N}{N+1}}\alpha(M)\left(\int_{\mathbb{R}^N}(u-M)_+\,dx\right)^{-\frac{N}{N+1}}.$$

Since $\alpha(M) \to 0$, thanks to (6.36) we get

$$\int_{M_0}^{+\infty} \frac{\left(g(M) - \tilde{C}\alpha(M)^{1/N}\right)^{\frac{N}{N+1}}}{M^{\frac{N}{N+1}}}\,dM = +\infty. \tag{6.37}$$

On the other hand, setting

$$f(M) := \int_{\mathbb{R}^N}(u-M)_+\,dx,$$

and using the fact that f is absolutely continuous with $f'(M) - \alpha(M)$, we obtain

$$\begin{aligned}\int_{M_0}^{+\infty}\alpha(M)\left(\int_{\mathbb{R}^N}(u-M)_+\,dx\right)^{-\frac{N}{N+1}}\,dM &= -\int_{M_0}^{+\infty} f'(M)f(M)^{-\frac{N}{N+1}}\,dM\\ &= (N+1)f(M_0)^{\frac{1}{N+1}} < +\infty\end{aligned}$$

which is against (6.37).

Let us consider now the case $2 < q < 2N/N-1$. Since there exists $C > q$ such that for $a \ge M$

$$a^q - M^q \le C\left[(a-M)^q + M^{q-1}(a-M)\right],$$

choosing $p = q$ in (6.35) we deduce for $\tilde{C} := \hat{C}\cdot C$

$$g(M)\alpha(M)^{1-\frac{1}{N}-\frac{2}{q}}\left(\int_{\mathbb{R}^N}(u-M)_+^q\,dx\right)^{\frac{2}{q}} \le \tilde{C}\int_{\mathbb{R}^N}(u-M)_+^q\,dx + \tilde{C}M^{q-1}\int_{\mathbb{R}^N}(u-M)_+\,dx.$$

Notice that

$$1 - \frac{1}{N} - \frac{2}{q} < 0.$$

Since $\lim_{M\to+\infty}\int_{\mathbb{R}^N}(u-M)_+^q\,dx = 0$ and $2/q < 1$, taking into account (6.36) we obtain that for $M \ge M_1 \ge M_0$ with M_1 large enough

$$g(M)\left(\int_{\mathbb{R}^N}(u-M)_+^q\,dx\right)^{\frac{2}{q}} \le 2\tilde{C}M^{q-1}\alpha(M)^{-\left(1-\frac{1}{N}-\frac{2}{q}\right)}\int_{\mathbb{R}^N}(u-M)_+\,dx.$$

Since

$$\int_{\mathbb{R}^N}(u-M)_+\,dx \le \left(\int_{\mathbb{R}^N}(u-M)_+^q\,dx\right)^{\frac{1}{q}}\alpha(M)^{\frac{q-1}{q}},$$

we finally obtain

$$g(M) \le 2\tilde{C}\frac{M^{q-1}\alpha(M)^{1+\frac{1}{N}}}{\int_{\mathbb{R}^N}(u-M)_+\,dx}.$$

For $\varepsilon > 0$ we may write

$$g(M) \le 2\tilde{C}\frac{M^{q-1}\alpha(M)^{\frac{1}{N}-\varepsilon}\alpha(M)^{1+\varepsilon}}{\int_{\mathbb{R}^N}(u-M)_+\,dx}$$

so that

$$\frac{g(M)^{\frac{1}{1+\varepsilon}}}{M} \le (2\tilde{C})^{\frac{1}{1+\varepsilon}}\left(M^{q-2-\varepsilon}\alpha(M)^{\frac{1}{N}-\varepsilon}\right)^{\frac{1}{1+\varepsilon}}\alpha(M)\left(\int_{\mathbb{R}^N}(u-M)_+\,dx\right)^{-\frac{1}{1+\varepsilon}}. \tag{6.38}$$

If we choose $\varepsilon < \frac{1}{N}$ such that also

$$\frac{q-2-\varepsilon}{\frac{1}{N}-\varepsilon} < \frac{2N}{N-1},$$

i.e.,

$$\varepsilon < \frac{N-1}{N+1}\left(\frac{2N}{N-1} - q\right),$$

we deduce

$$M^{q-2-\varepsilon}\alpha(M)^{\frac{1}{N}-\varepsilon} \le \left(\int_{\{u\ge M\}} u^{2N/N-1}\,dx\right)^{\frac{1}{N}-\varepsilon} \le \left(\int_{\mathbb{R}^N} u^{2N/N-1}\,dx\right)^{\frac{1}{N}-\varepsilon} < +\infty$$

so that

$$\sup_{M\ge 1} M^{q-2-\varepsilon}\alpha(M)^{\frac{1}{N}-\varepsilon} < +\infty.$$

Integrating now from M_1 to $+\infty$ both sides in (6.38) and reasoning as in the case $1 \le q \le 2$, we get a contradiction. □

In order to prove a positive bound from below on the support for the minimizer, we need the following lemma which proves that, also in the case $k = 0$ and $\gamma < +\infty$, the minimality property can be modified in order to involve directly the measure of the support of the competing functions.

Lemma 6.12. *Assuming* (6.14)*, let* $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *be a minimizer of problem* (6.13) *according to Theorem 6.10. Let moreover* $k = 0$ *and* $\gamma < +\infty$*. Then the following items hold true.*

(a) *There exist* $\varepsilon > 0$ *and* $\tilde{k} > 0$ *such that for every* $v \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *with*

$$|supp(u)| < |supp(v)| < |supp(u)| + \varepsilon, \tag{6.39}$$

then

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} + \tilde{k}|supp(u)| \le \frac{\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \beta\int_{J_v}(v^+)^2 + (v^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v^q\,dx\right)^{\frac{2}{q}}} + \tilde{k}|supp(v)|. \tag{6.40}$$

(b) *There exist* $\varepsilon > 0$ *and* $\hat{k} > 0$ *such that for every* $v \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *with*

$$|supp(u)| - \varepsilon < |supp(v)| < |supp(u)|, \tag{6.41}$$

then

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} + \hat{k}|supp(u)| \le \frac{\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \beta\int_{J_v}(v^+)^2 + (v^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v^q\,dx\right)^{\frac{2}{q}}} + \hat{k}|supp(v)|. \tag{6.42}$$

Proof. Let us start with point (a). By contradiction, let us assume that for every $\varepsilon > 0$ and $\tilde{k} > 0$ there exists $v \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ satisfying (6.39) but for which (6.40) is violated. Let us consider $\varepsilon_n \searrow 0$ and $\tilde{k}_n \to +\infty$, and let us denote by v_n the associated function such that

$$|supp(u)| < |supp(v_n)| < |supp(u)| + \varepsilon_n$$

and

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} + \tilde{k}_n|supp(u)| > \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta\int_{J_{v_n}}(v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{\frac{2}{q}}} + \tilde{k}_n|supp(v_n)|.$$

Let us set

$$t_n := \left(\frac{|supp(v_n)|}{|supp(u)|}\right)^{\frac{1}{N}}.$$

Then $t_n > 1$ and $t_n \to 1$ as $n \to \infty$. If we set

$$w_n(x) := v_n(t_n x)$$

we obtain since $|supp(w_n)| = |supp(u)|$

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} &\le \frac{\int_{\mathbb{R}^N} |\nabla w_n|^2\,dx + \beta\int_{J_{w_n}}(w_n^+)^2 + (w_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} w_n^q\,dx\right)^{\frac{2}{q}}}\\ &= \left(\frac{1}{t_n}\right)^{N-2-\frac{2N}{q}} \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + t_n^{-1}\beta\int_{J_{v_n}}(v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{\frac{2}{q}}}\\ &\le \left(\frac{1}{t_n}\right)^{N-2-\frac{2N}{q}} \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta\int_{J_{v_n}}(v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v_n^q\,dx\right)^{\frac{2}{q}}}\end{aligned}$$

so that

$$\begin{aligned}t_n^{N-2-\frac{2N}{q}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} &< \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} + \tilde{k}_n(|supp(u)| - |supp(v_n)|).\end{aligned}$$

Since $|supp(v_n)| = t_n^N|supp(u)|$, we infer

$$\tilde{k}_n|supp(u)| \le \frac{1 - t_n^{N-2-\frac{2N}{q}}}{t_n^N - 1} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}}.$$

Note that the right hand side is bounded as $n \to +\infty$: this is against $\tilde{k}_n \to +\infty$, a contradiction. The proof of point (a) is thus concluded.

Let us pass to the proof of point (b), proceeding again by contradiction but considering this time $\varepsilon_n \to 0$, $\hat{k}_n \to 0$, and the associated $v_n \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ satisfying (6.41) but violating (6.42). Reasoning as above we get for $t_n < 1$

$$\begin{aligned}t_n^{N-1-\frac{2N}{q}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} &< \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} + \hat{k}_n(1 - t_n^N)|supp(u)|\end{aligned}$$

so that

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{2}{q}}} \le \hat{k}_n \frac{1 - t_n^N}{t_n^{N-1-\frac{2N}{q}} - 1}|supp(u)|.$$

As $n \to \infty$, we have $t_n \to 1$, and the right-hand side of the previous inequality tends to 0, a contradiction. The proof is now complete. □

We are now in a position to derive the following bound from below on the support for the minimizers.

Theorem 6.13 (**Bound from below on the support**). *Assuming* (6.14)*, let* $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *be a minimizer of problem* (6.13) *according to Theorem 6.10. Then there exists* $\alpha > 0$ *such that*

$$u \ge \alpha \qquad \text{a.e. on } supp(u).$$

Proof. Let us consider firstly the case $k > 0$ and $\gamma = +\infty$. It is not restrictive to assume that $\int_{\mathbb{R}^N} u^q\,dx = 1$. Let $\varepsilon > 0$ be such that

$$v_\varepsilon := u1_{\{u\ge\varepsilon\}} \in SBV^{\frac{1}{2}}(\mathbb{R}^N).$$

Comparing u and v_ε we get

$$
\begin{aligned}
(6.43)\quad & \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} + k|\{u<\varepsilon\}| \\
&\le \left(\frac{1}{\int_{\{u\ge\varepsilon\}} u^q\,dx}\right)^{\frac{2}{q}} \left[\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2\,dx + \beta \int_{J_{v_\varepsilon}} (v_\varepsilon^+)^2 + (v_\varepsilon^-)^2\,d\mathcal{H}^{N-1}\right] \\
&= \left(\frac{1}{\int_{\{u\ge\varepsilon\}} u^q\,dx}\right)^{\frac{2}{q}} \left[\int_{\{u\ge\varepsilon\}} |\nabla u|^2\,dx + \beta \int_{\{u^-<\varepsilon\le u^+\}} (u^+)^2\,d\mathcal{H}^{N-1}\right. \\
&\qquad \left. +\beta\varepsilon^2 \mathcal{H}^{N-1}(\partial^e\{u\ge\varepsilon\}\setminus J_u) + \beta \int_{\{\varepsilon\le u^-<u^+\}} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}\right].
\end{aligned}
$$

Notice that there exist $0<\varepsilon_0<1$ such that for every $\varepsilon<\varepsilon_0$

$$
\left(\frac{1}{\int_{\{u\ge\varepsilon\}} u^q\,dx}\right)^{\frac{2}{q}} = \left(\frac{1}{1-\int_{\{u<\varepsilon\}} u^q\,dx}\right)^{\frac{2}{q}} \le 1 + \frac{3}{q}\varepsilon^q|\{u<\varepsilon\}|.
$$

Moreover

$$
\begin{aligned}
\int_{\{u\ge\varepsilon\}} |\nabla u|^2\,dx + \beta \int_{\{u^-<\varepsilon\le u^+\}} (u^+)^2\,d\mathcal{H}^{N-1} + \beta \int_{\{\varepsilon\le u^-<u^+\}} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \\
\le \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}.
\end{aligned}
$$

We can thus write for a.e. $0<\delta<\varepsilon<\varepsilon_0$

$$
\begin{aligned}
\int_{\{\delta<u<\varepsilon\}} |\nabla u|^2\,dx + \beta\delta^2\mathcal{H}^{N-1}(\partial^e\{\delta<u<\varepsilon\}\cap J_u) + k|\{u<\varepsilon\}| \\
\le \left(1+\frac{3}{q}\varepsilon^2|\{u<\varepsilon\}|\right)\beta\varepsilon^2\mathcal{H}^{N-1}(\partial^e\{u\ge\varepsilon\}\setminus J_u) + \tilde{C}\varepsilon^q|\{u<\varepsilon\}|
\end{aligned}
$$

for some $\tilde{C}>0$ independent of ε_0. Up to reducing ε_0, we get for a.e. $0<\delta<\varepsilon<\varepsilon_0$

$$
\begin{aligned}
\int_{\{\delta<u<\varepsilon\}} |\nabla u|^2\,dx + \beta\delta^2\mathcal{H}^{N-1}(\partial^e\{\delta<u<\varepsilon\}\cap J_u) + \frac{k}{2}|\{u<\varepsilon\}| \\
\le 2\beta\varepsilon^2\mathcal{H}^{N-1}(\partial^e\{u\ge\varepsilon\}\setminus J_u).
\end{aligned}
$$

The result now follows from Theorem 6.6.

The proof can be adapted to the case $k=0$. Indeed in view of point (b) of Lemma 6.12, for ε small enough inequality (6.43) holds with $\hat{k}$ in place of k. The proof is now concluded. ☐

As a consequence of the previous results, we obtain that minimizers are indeed in SBV with jump set with finite measure and essentially closed.

Theorem 6.14 (**The jump set of minimizers).** *Assuming* (6.14)*, let* $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *be a minimizer of problem* (6.13) *according to Theorem 6.10. Then the following items hold true.*

(a) $u \in SBV(\mathbb{R}^N)\cap L^\infty(\Omega)$ *with* $\mathcal{H}^{N-1}(J_u) < +\infty$.
(b) J_u *is essentially closed, i.e.,* $\mathcal{H}^{N-1}(\overline{J_u}\setminus J_u) = 0$.

Proof. By Theorem 6.11 we deduce that $u \in L^\infty(\mathbb{R}^N)$, while by Theorem 6.13 there exists $\alpha>0$ such that

$$
u \ge \alpha \qquad \text{a.e. on } supp(u). \tag{6.44}
$$

This entails

$$
\alpha^2\mathcal{H}^{N-1}(J_u) \le \int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} < +\infty
$$

so that $\mathcal{H}^{N-1}(J_u) < +\infty$.

Let us prove that $u \in SBV(\mathbb{R}^N)$. If A is an open bounded set in $\mathbb{R}^N$, by the chain rule in BV (see Theorem 2.1), we get that for every $\varepsilon > 0$

$$u_\varepsilon := (u^2+\varepsilon)^{\frac{1}{2}} \in SBV(A)$$

and for $\varepsilon \to 0$

$$u_\varepsilon \to u \qquad \text{strongly in } L^1(A).$$

Since $u \in L^\infty(\mathbb{R}^N)$ and

$$\int_A |\nabla u_\varepsilon|^2\,dx + \mathcal{H}^{N-1}(J_{u_\varepsilon}\cap A) \le \int_A |\nabla u|^2\,dx + \mathcal{H}^{N-1}(J_u \cap A),$$

by Ambrosio's theorem [1] we deduce that $u \in SBV(A)$. Since

$$\begin{aligned}|Du|(A) \le \int_A |\nabla u|\,dx + 2\|u\|_\infty \mathcal{H}^{N-1}(J_u\cap A)\\ \le |supp(u)|^{1/2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^{1/2} + 2\|u\|_\infty \mathcal{H}^{N-1}(J_u),\end{aligned}$$

we deduce that $|Du|(\mathbb{R}^N) < +\infty$ so that $u \in SBV(\mathbb{R}^N)$. The proof of point (a) is concluded.

Let us come to point (b). It is not restrictive to assume that

$$\int_{\mathbb{R}^N} u^q\,dx = 1.$$

We claim that there exists $c > 0$ such that for every $y \in \mathbb{R}^N$, $r > 0$, and $v \in SBV_{loc}(\mathbb{R}^N)$ with

$$\{u \neq v\} \subseteq B_r(y),$$

then

$$\begin{aligned}\int_{B_r(y)} |\nabla u|^2\,dx + \beta\alpha^2\mathcal{H}^{N-1}(J_u\cap \overline{B_r(y)})\\ \le \int_{B_r(y)}|\nabla v|^2\,dx + 2\beta\|u\|_\infty^2\mathcal{H}^{N-1}(J_v\cap\overline{B_r(y)}) + cr^N.\end{aligned} \tag{6.45}$$

Then a suitable multiple of u (more precisely $(\beta\alpha^2)^{-\frac{1}{2}}u$) is an almost-quasi minimizer for the Mumford-Shah functional according to Definition 6.2, and point (b) follows by Theorem 6.3.

In order to complete the proof, we need to prove claim (6.45). Let us consider firstly the case $k > 0$ and $\gamma = +\infty$. It is not restrictive to assume that

$$\int_{B_r(y)}|\nabla v|^2\,dx + 2\beta\|u\|_\infty^2\mathcal{H}^{N-1}(J_v\cap\overline{B_r(y)}) \le \int_{B_r(y)}|\nabla u|^2\,dx + \beta\alpha^2\mathcal{H}^{N-1}(J_u\cap\overline{B_r(y)}). \tag{6.46}$$

By comparing u with

$$\tilde v := |v| \wedge \|u\|_\infty := \min\{|v|, \|u\|_\infty\} \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$$

we get

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \beta\int_{J_u}(u^+)^2+(u^-)^2\,d\mathcal{H}^{N-1} + k|supp(u)|\\ \le \left(\frac{1}{1+\int_{B_r(y)}(\tilde v^q - u^q)\,dx}\right)^{2/q}\left[\int_{\mathbb{R}^N}|\nabla\tilde v|^2\,dx + \beta\int_{J_{\tilde v}}(\tilde v^+)^2+(\tilde v^-)^2\,d\mathcal{H}^{N-1}\right] + k|supp(\tilde v)|.\end{aligned} \tag{6.47}$$

In view of (6.46) and of the definition of $\tilde v$ we have

$$\int_{\mathbb{R}^N}|\nabla\tilde v|^2\,dx + \beta\int_{J_{\tilde v}}(\tilde v^+)^2+(\tilde v^-)^2\,d\mathcal{H}^{N-1} \le C$$

where C depends only on u and k. Since

$$|supp(\tilde v)| \le |supp(u)| + \omega_N r^N$$

and

$$\left| \int_{B_r(y)} (\tilde{v}^q - u^q)\, dx \right| \le 2\|u\|_\infty^q \omega_N r^N,$$

for r small enough (depending only on u and k) we deduce from (6.47)

$$\int_{B_r(y)} |\nabla u|^2\, dx + \beta \int_{J_u \cap \overline{B_r(y)}} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1} \\ \le \int_{B_r(y)} |\nabla \tilde{v}|^2\, dx + \beta \int_{J_{\tilde{v}} \cap \overline{B_r(y)}} (\tilde{v}^+)^2 + (\tilde{v}^-)^2\, d\mathcal{H}^{N-1} + cr^N$$

for some $c > 0$ depending on u and k. Recalling (6.44) and the very definition of $\tilde{v}$, we can write

$$\int_{B_r(y)} |\nabla u|^2\, dx + \beta\alpha^2 \mathcal{H}^{N-1}(J_u \cap \overline{B_r(y)}) \\ \le \int_{B_r(y)} |\nabla v|^2\, dx + 2\beta\|u\|_\infty^2 \mathcal{H}^{N-1}(J_v \cap \overline{B_r(y)}) + cr^N. \tag{6.48}$$

Since the left-hand side of the previous inequality is bounded in r (since $\mathcal{H}^{N-1}(J_u) < +\infty$), then up to increasing c (still depending only on u and k), we obtain that the inequality holds for every $r > 0$. Claim (6.45) is thus proved.

The previous arguments can be used to prove claim (6.45) also the case $k = 0$ and $\gamma < +\infty$. Indeed, if $\varepsilon, \hat{k}, \tilde{k} > 0$ are as in Lemma 6.12, we start considering the case $\omega_N r^N < \varepsilon$. It suffices then to distinguish between the cases

$$|supp(v)| < |supp(u)| \qquad \text{and} \qquad |supp(u)| \le |supp(v)|,$$

for which inequality (6.47) holds with the constant k replaced by $\hat{k}$ and $\tilde{k}$ respectively. Using the previous arguments we thus obtain again inequality (6.48), with c depending on u, $\hat{k}$ and $\tilde{k}$. Up to increasing c, the inequality holds for every $r > 0$, and claim (6.45) follows. □

Thanks to the preceding results, we can now turn our attention towards the support of minimizers.

Theorem 6.15 (The support of minimizers). *Assuming* (6.14), *let* $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ *be a minimizer of problem* (6.13) *according to Theorem 6.10. Then there exists an open connected set* $\Omega \in \mathcal{A}(\mathbb{R}^N)$ *such that the following items hold true.*

(a) $\partial\Omega = \overline{J_u}$, $\mathcal{H}^{N-1}(\partial\Omega \setminus J_u) = 0$, *and*

$$u = 0 \qquad \text{a.e. on } \mathbb{R}^N \setminus \Omega.$$

(b) $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ *admits a smooth representative of* Ω *such that*

$$u > \alpha > 0 \qquad \text{on } \Omega$$

and

$$-\Delta u = \lambda_u u^{q-1} \qquad \text{on } \Omega, \tag{6.49}$$

where

$$\lambda_u := \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\, dx\right)^{\frac{2}{q}}}.$$

In particular u *is analytic on* Ω *and* $|supp(u)| = |\Omega|$.

Proof. By Theorem 6.14, we know that $u \in SBV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with J_u such that $\mathcal{H}^{N-1}(J_u) < +\infty$ and $\mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0$. Moreover, by Theorem 6.13 we have

$$u > \alpha > 0 \qquad \text{a.e. on } supp(u) \tag{6.50}$$

for some $\alpha > 0$.

Decompose $\mathbb{R}^N \setminus \overline{J_u}$ into its connected components, select those on which u is not identically zero, and let Ω denote their union. Since $\partial\Omega = \overline{J_u}$ and $\mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0$, we get that $\partial\Omega$ is rectifiable with

$$\mathcal{H}^{N-1}(\partial\Omega) = \mathcal{H}^{N-1}(\overline{J_u}) = \mathcal{H}^{N-1}(J_u) < +\infty \qquad \text{and} \qquad \mathcal{H}^{N-1}(\partial\Omega \setminus J_u) = 0,$$

so that point (a) is proved.

Clearly the restriction of u to Ω belongs to $W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Since u is not identically zero on the connected components of Ω, from (6.50) we deduce that

$$u > \alpha \qquad \text{a.e. on } \Omega.$$

As a consequence $|\Omega| = |supp(u)| < +\infty$, so that in particular $\Omega \in \mathcal{A}(\mathbb{R}^N)$.

By comparing u with $u + t\varphi$ with $\varphi \in C_c^\infty(\Omega)$ and $t \in \mathbb{R}$ small enough, we get that u is a weak solution of the elliptic equation (6.49). Thanks to elliptic regularity we infer that u has a smooth representative on Ω, and that the equation is satisfied in a classical sense. The analyticity of u follows from [23, Theorem 5.8.6] or [20]. Point (b) is thus proved.

In order to conclude the proof of the theorem, we need to show that Ω is connected. By contradiction, let us assume that

$$\Omega = \Omega_1 \cup \Omega_2$$

with Ω_1, Ω_2 open sets such that $\Omega_1 \neq \emptyset$, $\Omega_2 \neq \emptyset$, and $\Omega_1 \cap \Omega_2 = \emptyset$. Note that Ω_i has finite perimeter for $i = 1, 2$, since $\partial\Omega_i \subseteq \partial\Omega$. Let us set

$$u_i := u 1_{\Omega_i} \qquad i = 1, 2.$$

Since $u \in L^\infty(\mathbb{R}^N)$, by [2, Theorem 3.84] we get $u_i \in SBV(\mathbb{R}^N)$ with

$$Du_i = Du\lfloor \Omega_i^{(1)} - u_{\partial^e \Omega_i} \otimes \nu_{\Omega_i}\, \mathcal{H}^{N-1}\lfloor \partial^e \Omega_i, \tag{6.51}$$

where $\Omega_i^{(1)}$ denotes the point of density 1 of Ω_i, while for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^e \Omega_i$

$$u_{\partial^e \Omega_i}(x) = \lim_{r \to 0^+} \frac{1}{|B_r^-(x, \nu_{\Omega_i})|} \int_{B_r^-(x, \nu_{\Omega_i}(x))} u(y)\, dy,$$

where ν_{Ω_i} denotes the exterior normal to Ω_i (see Subsection 2.2). Notice that $supp(u_i) = \Omega_i$.

We claim that the following additivity relation concerning the surface energy holds true:

$$\int_{J_u} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1} = \int_{J_{u_1}} (u_1^+)^2 + (u_1^-)^2\, d\mathcal{H}^{N-1} + \int_{J_{u_2}} (u_2^+)^2 + (u_2^-)^2\, d\mathcal{H}^{N-1}. \tag{6.52}$$

Let us consider firstly the case $2 \le q < \frac{2N}{N-1}$. Using the inequality

$$\frac{a+b}{(c+d)^{2/q}} \ge \min\left\{ \frac{a}{c^{2/q}}, \frac{b}{d^{2/q}} \right\}, \qquad a, b \ge 0,\ c, d > 0,$$

we may thus assume that

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\, dx\right)^{\frac{2}{q}}} \\
&= \frac{\int_{\mathbb{R}^N} |\nabla u_1|^2\, dx + \beta \int_{J_{u_1}} (u_1^+)^2 + (u_1^-)^2\, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} |\nabla u_2|^2\, dx + \beta \int_{J_{u_2}} (u_2^+)^2 + (u_2^-)^2\, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_1^q\, dx + \int_{\mathbb{R}^N} u_2^q\, dx\right)^{\frac{2}{q}}} \\
&\qquad \ge \frac{\int_{\mathbb{R}^N} |\nabla u_1|^2\, dx + \beta \int_{J_{u_1}} (u_1^+)^2 + (u_1^-)^2\, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_1^q\, dx\right)^{\frac{2}{q}}},
\end{aligned}$$

so that, using the notation of Lemma 6.7, and setting $|\Omega| = m$ and $|\Omega_1| = m_1$

$$\lambda_{\beta,q}(m) = \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u^q\, dx\right)^{\frac{2}{q}}}$$
$$\geq \frac{\int_{\mathbb{R}^N} |\nabla u_1|^2\, dx + \beta \int_{J_{u_1}} (u_1^+)^2 + (u_1^-)^2\, d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u_1^q\, dx\right)^{\frac{2}{q}}} \geq \lambda_{\beta,q}(m_1),$$

against the rescaling property (6.7).

Let us consider the case $1 \leq q < 2$. By Lemma 6.8 we know that a suitable multiple cu of u is a minimizer of the functional $T_{\beta,q}$ defined in (6.9) on

$$\{v \in SBV^{\frac{1}{2}}(\mathbb{R}^N) \,:\, |supp(v)| \leq m\},$$

where $m = |\Omega|$. For simplicity of notation, let us assume that $c = 1$. In view of (6.52) we get

$$T_{\beta,q}(u) = T_{\beta,q}(u_1) + T_{\beta,q}(u_2).$$

Now recall that given $\varepsilon, \eta > 0$, there exists $\delta > 0$ such that for every $a, b \geq 0$ and $\eta \leq c, d \leq m$

$$\frac{a+b}{(c+d)^{1+\varepsilon}} \leq (1-\delta) \max\left\{\frac{a}{c^{1+\varepsilon}}, \frac{b}{d^{1+\varepsilon}}\right\}.$$

We thus obtain, if $\eta = \min\{|\Omega_1|, |\Omega_2|\}$ and $\varepsilon := 1 + \frac{q}{(2-q)N}$

$$\frac{-T_{\beta,q}(u)}{|\Omega|^{1+\frac{q}{(2-q)N}}} = \frac{-T_{\beta,q}(u_1) - T_{\beta,q}(u_2)}{(|\Omega_1| + |\Omega_2|)^{1+\frac{q}{(2-q)N}}} \leq (1-\delta) \max\left\{\frac{-T_{\beta,q}(u_1)}{|\Omega_1|^{1+\frac{q}{(2-q)N}}}, \frac{-T_{\beta,q}(u_2)}{|\Omega_2|^{1+\frac{q}{(2-q)N}}}\right\}$$

for some $\delta > 0$. If we assume that the maximum is realized by u_1, we obtain using the notation of Lemma 6.8 and letting $m_1 = |\Omega_1|$

$$\frac{-t_{\beta,q}(m)}{|m|^{1-\frac{q}{(q-2)N}}} \leq (1-\delta) \frac{-t_{\beta,q}(m_1)}{|m_1|^{1-\frac{q}{(q-2)N}}},$$

against inequality (6.10).

In order to complete the proof, we need to show the additivity relation (6.52). Notice that for $i = 1, 2$

$$J_{u_i} = \partial^e \Omega_i \qquad \text{up to } \mathcal{H}^{N-1}\text{-negligible sets.} \tag{6.53}$$

Indeed this is a consequence of (6.51), of the regularity of u, and of the lower bound of point (b). Since $J_{u_i} \subseteq \partial\Omega = \overline{J_u}$, and $\mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0$, we obtain that up to $\mathcal{H}^{N-1}$-negligible sets

$$J_u = J_{u_1} \cup J_{u_2} = (J_{u_1} \setminus J_{u_2}) \cup (J_{u_2} \setminus J_{u_1}) \cup (J_{u_1} \cap J_{u_2}).$$

In view of (6.53), for $\mathcal{H}^{N-1}$-a.e. $x \in J_{u_1} \setminus J_{u_2}$

$$x \in \partial^e \Omega_1 \setminus \partial^e \Omega_2.$$

Since $\Omega_1 \cap \Omega_2 = \emptyset$, x has necessarily zero density with respect to Ω_2, so that $u_2^+(x) = u_2^-(x) = 0$ and

$$(u^+(x))^2 + (u^-(x))^2 = (u_1^+(x))^2 + (u_1^-(x))^2. \tag{6.54}$$

Similarly, for $\mathcal{H}^{N-1}$-a.e. $x \in J_{u_2} \setminus J_{u_1}$ we obtain $u_1^+(x) = u_1^-(x) = 0$ and

$$(u^+(x))^2 + (u^-(x))^2 = (u_2^+(x))^2 + (u_2^-(x))^2. \tag{6.55}$$

Let now $x \in J_{u_1} \cap J_{u_2}$. In view of (6.53), up to $\mathcal{H}^{N-1}$-negligible sets, we have

$$x \in \Omega_1^{(1/2)} \cap \Omega_2^{(1/2)}. \tag{6.56}$$

Then

$$u_1^-(x) = u_2^-(x) = 0.$$

Moreover, by the properties of rectifiable sets, it is not restrictive (again up to $\mathcal{H}^{N-1}$ negligible sets) to assume that J_{u_1} and J_{u_2} have the same normal $\nu_{\Omega_1} = -\nu_{\Omega_2}$ at x. Then using (6.56) we get that

$$u_1^+(x) = \lim_{r\to 0^+} \frac{1}{|B_r^-(x,\nu_{\Omega_1}(x))|}\int_{B_r^-(x,\nu_{\Omega_1}(x))\cap\Omega_1} u(y)\,dy = \lim_{r\to 0^+} \frac{1}{|B_r^-(x,\nu_{\Omega_1}(x))|}\int_{B_r^-(x,\nu_{\Omega_1}(x))} u(y)\,dy$$

and

$$u_2^+(x) = \lim_{r\to 0^+} \frac{1}{|B_r^+(x,\nu_{\Omega_1}(x))|}\int_{B_r^+(x,\nu_{\Omega_1}(x))\cap\Omega_2} u(y)\,dy = \lim_{r\to 0^+} \frac{1}{|B_r^+(x,\nu_{\Omega_1}(x))|}\int_{B_r^+(x,\nu_{\Omega_1}(x))} u(y)\,dy.$$

Then we deduce $\nu_u(x) = \pm\nu_{\Omega_1}(x)$ and

$$(u^+(x), u^-(x)) = (u_1^+(x), u_2^+(x)) \qquad \text{or} \qquad (u^+(x), u^-(x)) = (u_2^+(x), u_1^+(x)).$$

In any case

$$(u^+(x))^2 + (u^-(x))^2 = (u_1^+(x))^2 + (u_2^+(x))^2 = (u_1^+(x))^2 + (u_1^-(x))^2 + (u_2^+(x))^2 + (u_2^-(x))^2. \tag{6.57}$$

By collecting (6.54), (6.55) and (6.57), claim (6.52) follows. □

6.5. **Existence of optimal domains.** This subsection contains the proof of Theorem 6.1. Recall that for $\Omega \in \mathcal{A}(\mathbb{R}^N)$ and $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, the extension of u to $\mathbb{R}^N$ by zero outside Ω, still denoted by u, is such that $u \in SBV(\mathbb{R}^N)$ with

$$D^a u = \nabla u\,dx\lfloor\Omega \qquad \text{and} \qquad D^j u \text{ absolutely continuous w.r.t. } \mathcal{H}^{N-1}\lfloor\partial\Omega.$$

In particular $J_u \subseteq \partial\Omega$ up to $\mathcal{H}^{N-1}$-negligible sets. The following lemma will be useful.

Lemma 6.16. *Let $\Omega \in \mathcal{A}(\mathbb{R}^N)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive functions in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that*

$$\int_\Omega |\nabla u_n|^2\,dx + \int_{\partial\Omega} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1} \le C$$

for some $C > 0$. Then there exists $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $u = 0$ a.e. on $\mathbb{R}^N \setminus \Omega$, $u_{|\Omega} \in W^{1,2}(\Omega)$, and such that up to a subsequence

$$\nabla u_n \rightharpoonup \nabla u \qquad \text{weakly in } L^2(\Omega), \tag{6.58}$$

$$\int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} \le \liminf_n \int_{\partial\Omega} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1}, \tag{6.59}$$

with

$$u_n \to u \qquad \text{strongly in } L^q(\Omega) \qquad \text{for every } 1 \le q < \frac{2N}{N-1}. \tag{6.60}$$

Proof. Since $u_n \in SBV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, by the chain rule in BV (see Theorem 2.1) we get that $u_n^2 \in SBV(\mathbb{R}^N)$. Thanks to the embedding of $BV(\mathbb{R}^N)$ into $L^{N/N-1}(\mathbb{R}^N)$, it is easily seen that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^{2N/N-1}(\mathbb{R}^N)$ and consequently also in $L^2(\mathbb{R}^N)$ since $|\Omega| < +\infty$. We can now apply Theorem 6.5 to get the existence of $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ such that, up to a subsequence, (6.60) holds true. Clearly $u_{|\Omega} \in W^{1,2}(\Omega)$ and (6.58) is satisfied.

Let us prove (6.59). We proceed using a slicing technique, taking advantage of the fact that $\partial\Omega$ is rectifiable with $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$. Let ν denote a normal vector field on $\partial\Omega$.

For every $\xi \in \mathbb{R}^N$ with $|\xi| = 1$, let

$$\pi^\xi := \{x \in \mathbb{R}^N \,:\, x\cdot\xi = 0\},$$

and for $E \subseteq \mathbb{R}^N$ let

$$E^\xi := \{y \in \pi^\xi \,:\, y + t\xi \in E \text{ for some } t \in \mathbb{R}\} \qquad \text{and} \qquad E_y^\xi := \{t \in \mathbb{R} \,:\, y + t\xi \in E\}.$$

In view of the rectifiability of $\partial\Omega$, the following area formula holds true (see [2, Theorem 2.71]): for every positive Borel function $g : \partial\Omega \to [0, +\infty]$

$$\int_{\partial\Omega} g(x)|\nu(x)\cdot\xi|\, d\mathcal{H}^{N-1}(x) = \int_{(\partial\Omega)^\xi} \left[\int_{(\partial\Omega)^\xi_y} g(y+t\xi)\, d\mathcal{H}^0(t)\right] d\mathcal{H}^{N-1}(y). \tag{6.61}$$

Notice that since $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$, this formula entails that for $\mathcal{H}^{N-1}$-a.e. $y \in \pi^\xi$ we have

$$\mathcal{H}^0\left((\partial\Omega)^y_\xi\right) < +\infty. \tag{6.62}$$

Since $u_n \in SBV(\mathbb{R}^N)$, thanks to the slicing theory of BV functions (see [2, Section 3.11]), we have that for $\mathcal{H}^{N-1}$-a.e. $y \in \pi^\xi$

$$(u_n)^\xi_y(t) := u_n(y+t\xi) \in SBV(\mathbb{R})$$

with

$$[(u_n)^\xi_y]'(t) = \nabla u_n(y+t\xi) \qquad \text{and} \qquad J_{(u_n)^\xi_y} = (J_{u_n})^\xi_y. \tag{6.63}$$

Moreover for $t \in J_{(u_n)^\xi_y}$

$$(u_n^+)^2(y+t\xi) + (u_n^-)^2(y+t\xi) = [(u_n)^\xi_y]^2(t-) + [(u_n)^\xi_y]^2(t+). \tag{6.64}$$

Since $J_{u_n} \subseteq \partial\Omega$ up to $\mathcal{H}^{N-1}$-negligible sets, in view of (6.63) and (6.60), for $\mathcal{H}^{N-1}$-a.e. $y \in \pi^\xi$ we infer that

$$(u_n)^\xi_y \in W^{1,2}(\mathbb{R} \setminus (\partial\Omega)^y_\xi)$$

with

$$(u_n)^\xi_y \rightharpoonup u^\xi_y \qquad \text{weakly in } W^{1,2}(\mathbb{R} \setminus (\partial\Omega)^y_\xi).$$

Thanks to (6.62) and (6.64), by the trace theory of one-dimensional Sobolev functions we deduce that for every $A \subseteq \mathbb{R}^N$ open

$$\begin{aligned}\lim_n \sum_{t\in(\partial\Omega\cap A)^\xi_y} (u_n^+)^2(y+t\xi) + (u_n^-)^2(y+t\xi) &= \lim_n \sum_{t\in(\partial\Omega\cap A)^\xi_y} [(u_n)^\xi_y]^2(t-) + [(u_n)^\xi_y]^2(t+)\\ = \sum_{t\in(\partial\Omega\cap A)^\xi_y} [(u)^\xi_y]^2(t-) + [(u)^\xi_y]^2(t+) &= \sum_{t\in(\partial\Omega\cap A)^\xi_y} (u^+)^2(y+t\xi) + (u^-)^2(y+t\xi).\end{aligned}$$

Using the area formula (6.61) we infer

$$\begin{aligned}\liminf_n \int_{\partial\Omega\cap A} (u_n^+)^2 + (u_n^-)^2\, d\mathcal{H}^{N-1} &\geq \liminf_n \int_{\partial\Omega\cap A} [(u_n^+)^2 + (u_n^-)^2]|\nu\cdot\xi|\, d\mathcal{H}^{N-1}\\ &\geq \int_{(\partial\Omega\cap A)^\xi}\left[\liminf_n \int_{(\partial\Omega\cap A)^\xi_y} (u_n^+)^2(y+t\xi) + (u_n^-)^2(y+t\xi)\, d\mathcal{H}^0(t)\right] d\mathcal{H}^{N-1}(y)\\ &= \int_{(\partial\Omega\cap A)^\xi}\left[\int_{(\partial\Omega\cap A)^\xi_y} (u^+)^2(y+t\xi) + (u^-)^2(y+t\xi)\, d\mathcal{H}^0(t)\right] d\mathcal{H}^{N-1}(y)\\ &= \int_{\partial\Omega\cap A} [(u^+)^2 + (u^-)^2]|\nu\cdot\xi|\, d\mathcal{H}^{N-1}.\end{aligned}$$

Since A and ξ are arbitrary, we deduce that

$$\begin{aligned}\liminf_n \int_{\partial\Omega} (u_n^+)^2 + (u_n^-)^2\, d\mathcal{H}^{N-1} &\geq \int_{\partial\Omega} [(u^+)^2 + (u^-)^2] \sup_{|\xi|=1} |\nu\cdot\xi|\, d\mathcal{H}^{N-1}\\ &= \int_{\partial\Omega} (u^+)^2 + (u^-)^2\, d\mathcal{H}^{N-1},\end{aligned}$$

so that (6.59) follows, and the proof is concluded. □

We are now in a position to prove the main result of this section.

Proof of Theorem 6.1. In view of Theorem 6.15, every minimizer of problem (6.13) is of the form $u1_\Omega$ where $\Omega \in \mathcal{A}(\mathbb{R}^N)$ is connected, while $u \in C^\infty(\Omega) \cap L^\infty(\Omega)$ with $u > \alpha > 0$ on Ω solves (6.49). In particular u belongs to $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and is analytic on Ω.

Since $\partial\Omega = \overline{J_u}$ and $\mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0$, the minimal value of (6.13) associated to u is given by

$$\frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega u^q\,dx\right)^{\frac{2}{q}}} + k|\Omega|.$$

Since u is admissible for the evaluation of $\lambda_{\beta,q}(\Omega)$ we get

$$\lambda_{\beta,q}(\Omega) \le \frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega u^q\,dx\right)^{\frac{2}{q}}}. \tag{6.65}$$

Let $A \in \mathcal{A}(\mathbb{R}^N)$ be admissible for problem (6.2), and let $w \in W^{1,2}(A) \cap L^\infty(A)$. Since $\tilde{w} := |w|1_A \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with $J_{\tilde{w}} \subseteq \partial A$ up to $\mathcal{H}^{N-1}$-negligible sets, we have

$$\begin{aligned}
&\frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega u^q\,dx\right)^{\frac{2}{q}}} + k|\Omega| \\
&\qquad \le \frac{\int_{\mathbb{R}^N} |\nabla \tilde{w}|^2\,dx + \beta \int_{J_{\tilde{w}}} (\tilde{w}^+)^2 + (\tilde{w}^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} \tilde{w}^q\,dx\right)^{\frac{2}{q}}} + k|A| \\
&\qquad\qquad \le \frac{\int_A |\nabla w|^2\,dx + \beta \int_{\partial A} (w^+)^2 + (w^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_A |w|^q\,dx\right)^{\frac{2}{q}}} + k|A|,
\end{aligned}$$

where in the last line $w^\pm$ denote the traces of w on ∂A as defined in (3.3). By taking the infimum over w we obtain in view of (6.65)

$$\lambda_{\beta,q}(\Omega) + k|\Omega| \le \frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega u^q\,dx\right)^{\frac{2}{q}}} + k|\Omega| \le \lambda_{\beta,q}(A) + k|A|.$$

We conclude that the shape optimization problem (6.2) admits Ω as a solution. If $A = \Omega$, we immediately deduce that

$$\lambda_{\beta,q}(\Omega) = \frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega u^q\,dx\right)^{\frac{2}{q}}}$$

so that $\lambda_{\beta,q}(\Omega)$ is achieved on a bounded, analytic and positive function u satisfying the properties of Theorem 6.1. Moreover, the minimum values of the shape optimization problem (6.2) and of the free discontinuity problem (6.13) are equal.

In order to conclude the proof, we have to show that any minimizer $\tilde{\Omega} \in \mathcal{A}(\mathbb{R}^N)$ of problem (6.2) coincides up to a $\mathcal{H}^{N-1}$-negligible set with a domain Ω of the type above. In view of the preceding considerations we get

$$\lambda_{\beta,q}(\tilde{\Omega}) + k|\tilde{\Omega}| \le \frac{\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \beta \int_{J_v} (v^+)^2 + (v^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v^q\,dx\right)^{\frac{2}{q}}} + k|supp(v)|$$

for every $v \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ admissible for the free discontinuity problem (6.13).

If $\tilde{u}_n \in W^{1,2}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega})$ with $\tilde{u}_n \ge 0$ is a minimizing sequence for the principal frequency $\lambda_{\beta,q}(\tilde{\Omega})$ with

$$\int_{\tilde{\Omega}} \tilde{u}_n^q\,dx = 1,$$

by Lemma 6.16 there exists $\tilde u \in SBV^{\frac12}(\mathbb{R}^N)$ with $\tilde u = 0$ a.e. on $\mathbb{R}^N \setminus \tilde\Omega$, $\tilde u_{|\tilde\Omega} \in W^{1,2}(\tilde\Omega)$, and such that

$$\frac{\int_{\tilde\Omega} |\nabla \tilde u|^2\,dx + \beta \int_{\partial\tilde\Omega} (\tilde u^+)^2 + (\tilde u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\tilde\Omega} \tilde u^q\,dx\right)^{\frac2q}} \le \liminf_n \frac{\int_{\tilde\Omega} |\nabla \tilde u_n|^2\,dx + \beta \int_{\partial\tilde\Omega} (\tilde u_n^+)^2 + (\tilde u_n^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\tilde\Omega} \tilde u_n^q\,dx\right)^{\frac2q}} = \lambda_{\beta,q}(\tilde\Omega).$$

We infer that for every $v \in SBV^{\frac12}(\mathbb{R}^N)$ admissible for the free discontinuity problem (6.13)

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |\nabla \tilde u|^2\,dx + \beta \int_{J_{\tilde u}} (\tilde u^+)^2 + (\tilde u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} \tilde u^q\,dx\right)^{\frac2q}} + k|supp(\tilde u)| \\ \le \frac{\int_{\tilde\Omega} |\nabla \tilde u|^2\,dx + \beta \int_{\partial\tilde\Omega} (\tilde u^+)^2 + (\tilde u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\tilde\Omega} \tilde u^q\,dx\right)^{\frac2q}} + k|\tilde\Omega| \le \lambda_{\beta,q}(\tilde\Omega) + k|\tilde\Omega| \\ \le \frac{\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \beta \int_{J_v} (v^+)^2 + (v^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} v^q\,dx\right)^{\frac2q}} + k|supp(v)|,\end{aligned}$$

which entails

$$\frac{\int_{\mathbb{R}^N} |\nabla \tilde u|^2\,dx + \beta \int_{J_{\tilde u}} (\tilde u^+)^2 + (\tilde u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} \tilde u^q\,dx\right)^{\frac2q}} = \frac{\int_{\tilde\Omega} |\nabla \tilde u|^2\,dx + \beta \int_{\partial\tilde\Omega} (\tilde u^+)^2 + (\tilde u^-)^2\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} \tilde u^q\,dx\right)^{\frac2q}} = \lambda_{\beta,q}(\tilde\Omega) \tag{6.66}$$

and

$$|supp(\tilde u)| = |\tilde\Omega|. \tag{6.67}$$

Notice that $\tilde u$ is a minimizer for the free discontinuity problem (6.13). Let therefore $\tilde\Omega_0 \in \mathcal{A}(\mathbb{R}^N)$ be the open and connected set associated to $\tilde u$ according to Theorem 6.15.

Since $|\tilde\Omega_0| = |\tilde\Omega|$, $\tilde u$ is bounded a.e. from below by a strictly positive constant also on $\tilde\Omega$. Since $\tilde u$ realizes $\lambda_{\beta,q}(\tilde\Omega)$, we deduce that $\tilde u$ has a smooth representative on $\tilde\Omega$, so that thanks to (6.66)

$$\partial\tilde\Omega_0 = \overline{J_{\tilde u}} \subseteq \partial\tilde\Omega \qquad \text{and} \qquad \mathcal{H}^{N-1}(\partial\tilde\Omega \setminus J_{\tilde u}) = 0.$$

We conclude that $\tilde\Omega \subseteq \tilde\Omega_0$: indeed, if $x \in \tilde\Omega \setminus \tilde\Omega_0$, then $x \notin \overline{\tilde\Omega_0}$ so that $\tilde u = 0$ a.e. on $B_r(x) \subseteq \tilde\Omega$ for some $r > 0$, which is against (6.67). Finally, if $x \in \tilde\Omega_0 \setminus \tilde\Omega$, we get $x \in \partial\tilde\Omega$ so that

$$\mathcal{H}^{N-1}(\tilde\Omega_0 \setminus \tilde\Omega) \le \mathcal{H}^{N-1}(\partial\tilde\Omega \cap \tilde\Omega_0) \le \mathcal{H}^{N-1}(J_{\tilde u} \cap \tilde\Omega_0) = 0,$$

and the proof is thus concluded. □

References

[1] L. Ambrosio. Existence theory for a new class of variational problems. *Arch. Rational Mech. Anal.*, 111(4):291–322, 1990.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[3] M. S. Ashbaugh and R. D. Benguria. Isoperimetric inequalities for eigenvalues of the Laplacian. In *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, pages 105–139. Amer. Math. Soc., Providence, RI, 2007.

[4] C. Bandle and A. Wagner. Second variation of domain functionals and applications to problems with Robin boundary conditions. *Report No. 65, Institute for Mathematics, RWTH Aachen University*, 2013.

[5] M.-H. Bossel. Membranes élastiquement liées: extension du théorème de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(1):47–50, 1986.

[6] D. Bucur and A. Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. *Arch. Ration. Mech. Anal.*, 198(3):927–961, 2010.

[7] D. Bucur and S. Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, DOI 10.1007/s00205-013-0671-3, 2013.
[8] A. Cianchi and N. Fusco. Functions of bounded variation and rearrangements. *Arch. Ration. Mech. Anal.*, 165(1):1–40, 2002.
[9] A. Colesanti. Brunn-Minkowski inequalities for variational functionals and related problems. *Adv. Math.*, 194(1):105–140, 2005.
[10] G. Dal Maso, J.-M. Morel, and S. Solimini. A variational method in image segmentation: existence and approximation results. *Acta Mathematica*, 168(1):89–151, 1992.
[11] D. Daners. Robin boundary value problems on arbitrary domains. *Trans. Amer. Math. Soc.*, 352(9):4207–4236, 2000.
[12] D. Daners. A Faber-Krahn inequality for Robin problems in any space dimension. *Math. Ann.*, 335(4):767–785, 2006.
[13] D. Daners and J. Kennedy. Uniqueness in the Faber-Krahn inequality for Robin problems. *SIAM J. Math. Anal.*, 39(4):1191–1207, 2007/08.
[14] E. De Giorgi. Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8)*, 5:33–44, 1958.
[15] E. De Giorgi and L. Ambrosio. New functionals in the calculus of variations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 82(2):199–210 (1989), 1988.
[16] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.
[17] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
[18] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
[19] T. Giorgi and R. G. Smits. Monotonicity results for the principal eigenvalue of the generalized Robin problem. *Illinois J. Math.*, 49(4):1133–1143 (electronic), 2005.
[20] Y. Hashimoto. A remark on the analyticity of the solutions for non-linear elliptic partial differential equations. *Tokyo J. Math.*, 29(2):271–281, 2006.
[21] O. Lopes. Radial and nonradial minimizers for some radially symmetric functionals. *Electron. J. Differential Equations*, pages No. 03, approx. 14 pp. 1996.
[22] M. Mariş. On the symmetry of minimizers. *Arch. Ration. Mech. Anal.*, 192(2):311–330, 2009.
[23] C. B. Morrey, Jr. *Multiple integrals in the calculus of variations*. Die Grundlehren der mathematischen Wissenschaften, Band 130. Springer-Verlag New York, Inc., New York, 1966.
[24] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.*, 42(5):577–685, 1989.
[25] L. E. Payne. Isoperimetric inequalities and their applications. *SIAM Rev.*, 9:453–488, 1967.
[26] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics*. Annals of Mathematics Studies, no. 27. Princeton University Press, Princeton, N. J., 1951.
[27] I. S. Sokolnikoff. *Mathematical theory of elasticity*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. 2d ed.

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