

IMPROVED BEREZIN-LI-YAU INEQUALITIES WITH MAGNETIC FIELD

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ABSTRACT. In this paper we study the eigenvalue sums of Dirichlet Laplacians on bounded domains. Among our results we establish an improvement of the Berezin bound and of the Li-Yau bound in the presence of a constant magnetic field previously obtained in [5] and [20].

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. We consider the Dirichlet Laplacian $-\Delta_\Omega$ on $L^2(\Omega)$ defined in the quadratic form sense. Since the embedding $H_0^1 \hookrightarrow L^2(\Omega)$ is compact, the spectrum of the non-negative operator $-\Delta_\Omega$ is discrete and accumulates to infinity only. Denote by $\{\lambda_j(\Omega)\}_{j \in \mathbb{N}} = \{\lambda_j(\Omega)\}_{j \in \mathbb{N}}$ the increasing sequence of the eigenvalues of $-\Delta_\Omega$, where we repeat entries according to their multiplicity.

In particular, we shall study the so-called Riesz means of these eigenvalues, given by ¹

$$\mathrm{tr}(-\Delta_\Omega - \Lambda)_-^\gamma = \sum_k (\Lambda - \lambda_k(\Omega))_+^\gamma, \quad \gamma \geq 0.$$

Here and below we use the notation $x_\pm = (|x| \pm x)/2$. It is well-known that these Riesz means satisfy the Weyl asymptotics [24]

$$(1.1) \quad \sum_k (\Lambda - \lambda_k(\Omega))_+^\gamma = L_{\gamma,d}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma + \frac{d}{2}} + o(\Lambda^{\gamma + \frac{d}{2}}), \quad \Lambda \rightarrow \infty,$$

where

$$L_{\gamma,d}^{\mathrm{cl}} = \frac{\Gamma(\gamma + 1)}{(4\pi)^{\frac{d}{2}} \Gamma(\gamma + 1 + d/2)}.$$

In 1972 Berezin [2] showed that for $\gamma \geq 1$ the leading term in (1.1) gives actually a uniform upper bound on the Riesz means, namely for any $\gamma \geq 1$ it holds

$$(1.2) \quad \sum_k (\Lambda - \lambda_k(\Omega))_+^\gamma \leq L_{\gamma,d}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma + \frac{d}{2}}.$$

In view of the asymptotics (1.1) the constant on the r.h.s. of (1.2) is optimal. The bound (1.2) is assumed to hold for all $0 \leq \gamma < 1$ as well. However, so far this has been shown for tiling domains [21] and cartesian products with tiling domains [14] only. On the other hand, it follows from (1.2) that a similar inequality holds for arbitrary domains and for all $0 \leq \gamma < 1$ with some probably non-sharp excess factor on the r.h.s. [14]

$$(1.3) \quad \sum_k (\Lambda - \lambda_k(\Omega))_+^\gamma \leq 2 \left(\frac{\gamma}{\gamma + 1} \right)^\gamma L_{\gamma,d}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma + \frac{d}{2}}, \quad 0 \leq \gamma < 1.$$

¹For $\gamma = 0$ this is simply the counting function of all eigenvalues $\lambda_j(\Omega) < \Lambda$.

Here we are going to focus on the border-line case $\gamma = 1$, in which the inequality (1.2) is equivalent, via Legendre transformation, to the lower bound

$$(1.4) \quad \sum_{j=1}^N \lambda_j(\Omega) \geq C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}}, \quad C_d = \frac{4\pi d}{d+2} \Gamma(d/2 + 1)^{\frac{2}{d}}.$$

The above estimate was proved in [17] independently on [2] and it is known as the Li-Yau inequality. Similarly as in the case of Berezin inequality, the constant C_d cannot be improved, since the right hand side of (1.4) gives the leading term of the Weyl asymptotic formula, see (3.4) below.

However, the bounds (1.2) and (1.4) can be improved by adding to its right hand side reminder term of a lower order in Λ or in N , respectively. Several results in this direction were obtained recently both for the Berezin inequality [9, 23] (for $\gamma \geq \frac{3}{2}$) and for the Li-Yau estimate [8, 20, 13, 25, 26]. In particular, Melas proved in [20] that there exists a positive constant M_d such that

$$(1.5) \quad \sum_{j=1}^N \lambda_j(\Omega) \geq C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}} + M_d \frac{|\Omega|}{I(\Omega)} N, \quad I(\Omega) = \min_{a \in \mathbb{R}^d} \int_{\Omega} |x - a|^2 dx,$$

where $M_d \geq \frac{1}{24(d+2)}$. Note that by the Legendre transform (1.5) is equivalent to

$$(1.6) \quad \sum_k (\Lambda - \lambda_k(\Omega))_+ \leq L_{1,d}^{cl} |\Omega| \left(\Lambda - M_d \frac{|\Omega|}{I(\Omega)} \right)^{1+\frac{d}{2}}.$$

Alongside with the ordinary Dirichlet Laplacian we shall also consider its magnetic version $H(A) = (i\nabla + A(x))^2$ on $L^2(\Omega)$ generated by the closed quadratic form

$$(1.7) \quad \|(i\nabla + A)u\|_{L^2(\Omega)}^2, \quad u \in H_0^1(\Omega),$$

where A is a real-valued vector potential satisfying mild regularity conditions. Moreover, the magnetic Sobolev norm on the bounded domain Ω is equivalent to the non-magnetic one and the operator $H(A)$ has discrete spectrum as well. We notate its eigenvalues by $\lambda_k = \lambda_k(\Omega; A)$, repeating eigenvalues according to their multiplicities. Note that the magnetic Riesz means satisfy the very same Weyl asymptotics (1.1).

From the pointwise diamagnetic inequality (see e.g. [18, Thm.7.21])

$$(1.8) \quad |\nabla |u(x)|| \leq |(i\nabla + A)u(x)| \quad \text{a. e. } x \in \Omega,$$

it follows that $\lambda_1(\Omega; A) \geq \lambda_1(\Omega; 0) = \lambda_1(\Omega)$. However, the estimate $\lambda_j(\Omega; A) \geq \lambda_j(\Omega; 0) = \lambda_j(\Omega)$ fails in general if $j \geq 2$. Therefore, it is a priori not clear whether bounds similar to (1.2)-(1.5) remain true when the eigenvalues $\lambda_j(\Omega)$ are replaced by their magnetic counterparts $\lambda_j(\Omega; A)$.

By now it has been shown that

- the sharp bound (1.2) holds true for arbitrary magnetic fields if $\gamma \geq \frac{3}{2}$ ([15]),
- the sharp bound (1.2) holds true for constant magnetic fields if $\gamma \geq 1$ ([5]),
- in the dimension $d = 2$ the bound (1.3) holds true for constant magnetic fields if $0 \leq \gamma < 1$ and the constant on the r.h.s. of (1.3) cannot be improved ([7]) even in the class of constant magnetic fields and tiling domains Ω .

So far it is not known, whether the bound (1.2) holds true for arbitrary magnetic fields if $1 \leq \gamma \leq \frac{3}{2}$.

For $\gamma = 1$ and constant magnetic field the magnetic version of (1.2) is again dual to the magnetic version of the Li-Yau bound (1.4). Since (1.2) fails without excess factor for all $\gamma < 1$, the case $\gamma = 1$ is the threshold case, in which the Berezin bound with the classical constant remains true.

Therefore it is of a particular interest to study, whether either the magnetic Berezin bound for $\gamma = 1$ or equivalently the magnetic Li-Yau bound admits any further improvement by lower order remainder terms. It should be mentioned that the method of Melas cannot be applied in the presence of a magnetic field.

The purpose of this paper is twofold. First, we establish an improved Li-Yau bound with an additional term of the Melas order for magnetic Dirichlet Laplacians on planar domains $\Omega \subset \mathbb{R}^2$ with constant magnetic field. For this end we prove a different version of the Melas result (1.5) in the non-magnetic case. We obtain a remainder term of the same order as in (1.5), but with a different geometrical factor, see Theorem 3.1 and Corollary 3.2. Our proof is based on a new approach and, in contrast to the classical Melas proof, extends to a lower bound for the magnetic eigenvalues $\lambda_k(\Omega; A)$ as well, see Theorem 3.7 and Corollary 3.8.

Secondly, we will prove a Berezin inequality with a remainder term which is, with as well as without a magnetic field, of a better order than the Berezin-type equivalent (1.6) of the Melas bound, see Theorems 3.5 and 3.9.

2. Preliminaries

Given a set $\Omega \subset \mathbb{R}^d$ we denote its volume by $|\Omega|$. Moreover, we denote by

$$(2.1) \quad \delta(x) = \text{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|$$

the distance between a given $x \in \Omega$ and the boundary of Ω , and by

$$R_i(\Omega) = \sup_{x \in \Omega} \delta(x)$$

the in-radius of Ω . Given $\beta > 0$ we introduce

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \quad \beta > 0,$$

and define the quantity

$$(2.2) \quad \sigma(\Omega) := \inf_{0 < \beta < R_i(\Omega)} \frac{|\Omega_\beta|}{\beta}.$$

Note that $\sigma(\Omega) > 0$ since the right hand side of (2.2) is a positive continuous function of β and

$$\liminf_{\beta \rightarrow 0} \frac{|\Omega_\beta|}{\beta} > 0.$$

The quantity $\sigma(\Omega)$, which depends only on the geometry of Ω , will play an important role in the sequel. Throughout the paper we will suppose that Ω satisfies the following condition:

Assumption 2.1. *The domain $\Omega \subset \mathbb{R}^d$ is open bounded and such that*

$$(2.3) \quad \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2 / \delta^2} =: c_h^{-1}(\Omega) > 0.$$

Note that $c_h(\Omega)$ is the best constant in the Hardy inequality

$$(2.4) \quad \int_\Omega \frac{|u(x)|^2}{\delta(x)^2} dx \leq c_h(\Omega) \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in H_0^1(\Omega).$$

Remark 2.2. Assumption 2.1 is satisfied, for example, for all open bounded domains with Lipschitz boundary, see [1]. It is known that for simply connected planar domains $c_h(\Omega) \leq 16$, [1], and for convex domains $c_h(\Omega) = 4$, see e.g. [3, 19],

3. Main results

3.1. Li-Yau inequalities for Dirichlet Laplacian.

Theorem 3.1. *For any $N \in \mathbb{N}$ we have*

$$(3.1) \quad \sum_{j=1}^N \lambda_j(\Omega) \geq C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}} + \frac{1}{16 c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^2} N.$$

For convex domains, in particular, we have

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^d$ satisfy assumption 2.1 and suppose moreover that Ω is convex. Then for any $N \in \mathbb{N}$ it holds*

$$(3.2) \quad \sum_{j=1}^N \lambda_j(\Omega) \geq C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}} + \frac{N}{64 R_i^2(\Omega)}.$$

Remark 3.3. Let us compare the lower bound (3.2) with (1.5). Assume that $\mathfrak{a} \in \mathbb{R}^d$ is such that $I(\Omega) = \int_{\Omega} |x - \mathfrak{a}|^2 dx$ and let $B(\mathfrak{a}, R)$ be the ball centered in \mathfrak{a} with radius R chosen such that $|B(\mathfrak{a}, R)| = |\Omega|$. Then it is easily seen that

$$(3.3) \quad I(\Omega) \geq I(B(\mathfrak{a}, R)) = \frac{d}{d+2} |\Omega| R^2.$$

By using the fact that $R \geq R_i(\Omega)$ we thus obtain

$$\frac{1}{R_i^2(\Omega)} \geq \frac{d}{d+2} \frac{|\Omega|}{I(\Omega)}.$$

Hence, for convex Ω , inequality (3.2) implies (1.5) with $M_d = \frac{d}{64(d+2)}$. For $d \geq 3$ this is better than the lower bound $M_d \geq \frac{1}{24(d+2)}$ obtained in [20].

On the other hand, for domains which are wide in one direction and thin in another the estimate (3.2) is much sharper than (1.5) due to the fact that $\lambda_1(\Omega)$ is proportional to $R_i(\Omega)^{-2}$. Indeed, consider for example the rectangle $\Omega_\varepsilon = (0, \varepsilon^{-1}) \times (0, \varepsilon)$ in \mathbb{R}^2 . Then as $\varepsilon \rightarrow 0$ we find $|\Omega_\varepsilon|/I(\Omega_\varepsilon) \sim 3\varepsilon^2$, while on the right hand side of (3.2) we have $R_i^{-2}(\Omega_\varepsilon) = \varepsilon^{-2}$ which is of the same order of ε as the left hand side.

Remark 3.4. The remainder terms in both bounds (3.2) and (1.5) are not sharp in the order of N . This follows from the refined Weyl asymptotic

$$(3.4) \quad \sum_{j=1}^N \lambda_j(\Omega) = C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}} + K_d \frac{|\partial\Omega|}{|\Omega|^{1+\frac{1}{d}}} N^{1+\frac{1}{d}} (1 + o(1)) \quad N \rightarrow \infty,$$

with a positive constant K_d depending only d . The asymptotic equation (3.4) was first proven by Ivrii [11, 12] for smooth domains under an additional assumption on the set of all periodic geodesic billiards in Ω , see also [22]. Recently, (3.4) was extended to all domains with $C^{1,\alpha}$ boundary (with $\alpha > 0$) by Frank and Geisinger [6].

3.2. Berezin inequalities for Dirichlet Laplacian. We put $\mu = \mu(\Omega) = \sqrt{c_h(\Omega)}$ and introduce the constant

$$K(\Omega) := \frac{2 + \mu}{\mu} (4 + 4\mu)^{-\frac{2+2\mu}{2+\mu}}.$$

We then have

Theorem 3.5. *For any $\Lambda \geq \lambda_1(\Omega)$ it holds*

$$(3.5) \quad \sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq L_{1,d}^{cl} |\Omega| \Lambda^{1+\frac{d}{2}} - L_{1,d}^{cl} K(\Omega) \sigma(\Omega) \left(\frac{\sigma(\Omega)}{|\Omega|} \right)^{\frac{\mu}{2+\mu}} \Lambda^{\frac{d}{2} + \frac{1}{2+\mu}}.$$

In particular, if Ω is convex, then

$$(3.6) \quad \sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq L_{1,d}^{cl} |\Omega| \Lambda^{1+\frac{d}{2}} - \frac{L_{1,d}^{cl}}{72} R_i(\Omega)^{-\frac{3}{2}} |\Omega| \Lambda^{\frac{d}{2} + \frac{1}{4}}.$$

Remark 3.6. The order $\Lambda^{\frac{d}{2} + \frac{1}{2+\mu}}$ of the reminder term in (3.5) is larger than in (1.6) by additional factor $\frac{1}{2+\mu}$. Note also that $\mu \geq 2$ and that the second term in the Weyl asymptotics (1.1) is of the order $\Lambda^{\frac{d}{2} + \frac{1}{2}}$.

3.3. Li-Yau inequalities for magnetic Dirichlet Laplacian. As already mentioned in the introduction, our approach enables us to extend the bound (3.1) to the magnetic Dirichlet Laplacian. Let $B \in \mathbb{R}$ be a non-zero constant define vector potential $A(x) = \frac{1}{2}(-Bx_2, Bx_1)$ so that $\text{curl } A = B$. We then have

Theorem 3.7. *Let $d = 2$. Then for any $N \in \mathbb{N}$ it holds*

$$(3.7) \quad \sum_{j=1}^N \lambda_j(\Omega; A) \geq \frac{2\pi N^2}{|\Omega|} + \frac{1}{16 c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^2} N.$$

Corollary 3.8. *Let $\Omega \subset \mathbb{R}^2$ be bounded and convex. Then*

$$(3.8) \quad \sum_{j=1}^N \lambda_j(\Omega; A) \geq \frac{2\pi N^2}{|\Omega|} + \frac{N}{64 R_i^2(\Omega)}.$$

3.4. Berezin inequalities for magnetic Dirichlet Laplacian. As above we denote $\mu = \sqrt{c_h(\Omega)}$. Moreover, we put

$$\tilde{K}(\Omega) := \frac{2 + \mu}{16 \pi \mu} (2 + 2\mu)^{-\frac{2+3\mu}{2+\mu}}.$$

With this notation we have

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^2$ and let $A = \frac{1}{2}(-Bx_2, Bx_1)$. Then for any $\Lambda \geq \lambda_1(\Omega; A)$ it holds*

$$(3.9) \quad \sum_{j: \lambda_j(\Omega; A) < \Lambda} (\Lambda - \lambda_j(\Omega; A)) \leq \frac{|\Omega|}{8\pi} \Lambda^2 - \tilde{K}(\Omega) \sigma(\Omega) \left(\frac{\sigma(\Omega)}{|\Omega|} \right)^{\frac{\mu}{2+\mu}} \Lambda^{\frac{3+\mu}{2+\mu}}.$$

In particular, if Ω is convex, then

$$(3.10) \quad \sum_{j:\lambda_j(\Omega;\Lambda) < \Lambda} (\Lambda - \lambda_j(\Omega;\Lambda)) \leq \frac{|\Omega|}{8\pi} \left(\Lambda^2 - \frac{\Lambda^{5/4}}{36 R_i(\Omega)^{3/2}} \right).$$

4. Proofs of the main results: Li-Yau inequalities

4.1. **Dirichlet Laplacian.** Given $\Lambda > 0$ we denote by

$$n(\Lambda) = \text{card}\{\lambda_j(\Omega) : \lambda_j(\Omega) < \Lambda\}$$

the counting function. Let $\{u_j\}_{j \in \mathbb{N}}$ be the set of eigenfunctions of $-\Delta_\Omega$ corresponding to the eigenvalues $\lambda_j(\Omega)$. We assume that the eigenfunctions are normalised in $L^2(\Omega)$ and denote by $\hat{u}_j(\xi)$ the Fourier transform of u_j extended by zero to \mathbb{R}^d ;

$$(4.1) \quad \hat{u}_j(\xi) = (2\pi)^{-d/2} \int_{\Omega} e^{-ix \cdot \xi} u_j(x) dx.$$

Then

$$(4.2) \quad \begin{aligned} \sum_{j:\lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) &= \sum_{j \leq n(\Lambda)} \int_{\Omega} (\Lambda |u_j(x)|^2 - |\nabla u_j(x)|^2) dx = \sum_{j \leq n(\Lambda)} \int_{\mathbb{R}^d} (\Lambda - |\xi|^2) |\hat{u}_j(\xi)|^2 d\xi \\ &= \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ |\hat{u}_j(\xi)|^2 d\xi \\ &\quad - \int_{\mathbb{R}^d} (|\xi|^2 - \Lambda)_+ R_1(\Lambda, \xi) d\xi - \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ R_2(\Lambda, \xi) d\xi, \end{aligned}$$

where

$$R_1(\Lambda, \xi) = \sum_{j \leq n(\Lambda)} |\hat{u}_j(\xi)|^2, \quad R_2(\Lambda, \xi) = \sum_{j > n(\Lambda)} |\hat{u}_j(\xi)|^2.$$

Since $\{u_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $\|e^{-ix \cdot \xi}\|_{L^2(\Omega)}^2 = |\Omega|$, the Parseval identity implies

$$(4.3) \quad R_1(\Lambda, \xi) + R_2(\Lambda, \xi) = \sum_{j \in \mathbb{N}} |\hat{u}_j(\xi)|^2 = (2\pi)^{-d} |\Omega| \quad \forall \xi \in \mathbb{R}^d.$$

Note also that, by the Pythagoras theorem, we have

$$(4.4) \quad R_2(\Lambda, \xi) = (2\pi)^{-d} \int_{\Omega} |e^{-ix \cdot \xi} - (2\pi)^{d/2} \sum_{j \leq n(\Lambda)} \hat{u}_j(\xi) u_j(x)|^2 dx.$$

Our aim is to estimate $R_2(\Lambda, \xi)$ from below by a function of Λ , uniformly in ξ . Since $|a - b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$ for all $a, b \in \mathbb{C}$, from (4.4) it follows that for any $\beta > 0$

$$(4.5) \quad \begin{aligned} R_2(\Lambda, \xi) &\geq (2\pi)^{-d} \int_{\Omega_\beta} |e^{-ix \cdot \xi} - (2\pi)^{d/2} \sum_{j \leq n(\Lambda)} \hat{u}_j(\xi) u_j(x)|^2 dx \\ &\geq \frac{1}{2} (2\pi)^{-d} |\Omega_\beta| - \int_{\Omega_\beta} |F_\Lambda(\xi, x)|^2 dx, \end{aligned}$$

where we used the shorthand

$$(4.6) \quad F_\Lambda(\xi, x) = \sum_{j \leq n(\Lambda)} \hat{u}_j(\xi) u_j(x).$$

Since $F_\Lambda(\xi, \cdot) \in H_0^1(\Omega)$ for each $\Lambda > 0$ and each $\xi \in \mathbb{R}^d$, the Hardy inequality (2.4) in combination with (4.3) gives

$$\begin{aligned} \int_{\Omega_\beta} |F_\Lambda(\xi, x)|^2 dx &\leq \beta^2 \int_{\Omega_\beta} \frac{|F_\Lambda(\xi, x)|^2}{\delta^2(x)} dx \leq \beta^2 \int_{\Omega} \frac{|F_\Lambda(\xi, x)|^2}{\delta^2(x)} dx \\ &\leq \beta^2 c_h(\Omega) \int_{\Omega} |\nabla_x F_\Lambda(\xi, x)|^2 dx = \beta^2 c_h(\Omega) \sum_{j \leq n(\Lambda)} \lambda_j(\Omega) |\hat{u}_j(\xi)|^2 \\ (4.7) \quad &\leq \beta^2 \Lambda c_h(\Omega) (2\pi)^{-d} |\Omega|. \end{aligned}$$

Hence in view of (4.5) and (4.7) we get

$$(4.8) \quad R_2(\Lambda, \xi) \geq (2\pi)^{-d} \left(\frac{1}{2} \frac{|\Omega_\beta|}{\beta} - \Lambda \beta c_h(\Omega) |\Omega| \right) \beta.$$

Now we choose

$$(4.9) \quad \beta = \frac{\sigma(\Omega)}{4 c_h(\Omega) |\Omega|} \Lambda^{-1},$$

where $c_h(\Omega)$ is the constant from the Hardy inequality (2.4). Note that the latter implies

$$(4.10) \quad \lambda_1(\Omega) \geq \frac{1}{c_h(\Omega) R_i^2(\Omega)}.$$

Using the definition of $\sigma(\Omega)$ we then find that for any $\Lambda \geq \lambda_1(\Omega)$ it holds

$$(4.11) \quad \beta \leq \frac{\sigma(\Omega)}{4 c_h(\Omega) |\Omega|} \lambda_1^{-1}(\Omega) \leq \frac{1}{4 c_h(\Omega) R_i(\Omega)} \lambda_1^{-1}(\Omega) \leq \frac{R_i(\Omega)}{4}.$$

From (2.2) it thus follows that with our choice of β we have

$$\frac{|\Omega_\beta|}{\beta} \geq \sigma(\Omega).$$

Inserting the above estimate together with (4.9) into (4.8) we obtain

$$(4.12) \quad R_2(\Lambda, \xi) \geq \frac{1}{16 c_h(\Omega)} (2\pi)^{-d} \frac{\sigma^2(\Omega)}{|\Omega|} \Lambda^{-1}.$$

Proposition 4.1. *For any $\Lambda \geq \lambda_1(\Omega)$ it holds*

$$(4.13) \quad \sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq L_{1,d}^{cl} |\Omega| \Lambda^{1+\frac{d}{2}} - \frac{L_{1,d}^{cl}}{16 c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} \Lambda^{\frac{d}{2}},$$

where

$$(4.14) \quad L_{1,d}^{cl} = \frac{1}{2^d \pi^{d/2} \Gamma(2 + d/2)}.$$

Proof. Since $R_1(\Lambda, \xi) \geq 0$, equations (4.2) and (4.3) imply

$$\sum_{j: \lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) \leq (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ d\xi - \int_{\mathbb{R}^d} (\Lambda - |\xi|^2)_+ R_2(\Lambda, \xi) d\xi.$$

The claim now follows by inserting the lower bound (4.12) and integrating with respect to ξ . ■

Note that the right hand side of (4.13) is positive for all $\Lambda \geq \lambda_1(\Omega)$ in view of inequality (4.10) and $\sigma(\Omega) \leq |\Omega|/R_i(\Omega)$.

Proof of Theorem 3.1. From (4.13) it follows that

$$\begin{aligned} \sum_{j:\lambda_j(\Omega) < \Lambda} (\Lambda - \lambda_j(\Omega)) &\leq L_{1,d}^{\text{cl}} |\Omega| \Lambda^{1+\frac{d}{2}} \left(1 - \frac{1}{16 c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^2 \Lambda}\right) \\ &\leq L_{1,d}^{\text{cl}} |\Omega| \Lambda^{1+\frac{d}{2}} \left(1 - \frac{1}{16 c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^2 \Lambda}\right)^{1+\frac{d}{2}} \\ &= L_{1,d}^{\text{cl}} |\Omega| \left(\Lambda - \frac{1}{16 c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|^2}\right)^{1+\frac{d}{2}}. \end{aligned}$$

Since both sides of the above inequality are convex functions of Λ , we can apply the Legendre transform. This yields (3.1). ■

Convex domains.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^d$ be bounded and convex. Then*

$$(4.15) \quad \sigma(\Omega) = \frac{|\Omega|}{R_i(\Omega)}.$$

Proof. Let us first prove the statement for domains with C^1 boundary. We are going to show that

$$f(\beta) = \frac{|\Omega_\beta|}{\beta}$$

is a decreasing function of β on $(0, R_i(\Omega))$. To this end let $\beta_0 \in (0, R_i(\Omega))$ and consider the sets

$$E_0 = \{x \in \Omega : \delta(x) \geq \beta_0\}, \quad \text{and} \quad E_t = \{x \in \Omega \setminus E_0 : \text{dist}(x, E_0) \leq t\}, \quad t > 0.$$

From the convexity of Ω it follows that δ is concave and therefore E_0 is a compact convex set. Hence by the Steiner formula, see e.g. [10], it holds

$$(4.16) \quad |E_t| = \sum_{j=0}^d K_j(E_0) t^j,$$

where $K_j(E_0)$ are non-negative coefficients depending on the geometry of E_0 . We claim that

$$(4.17) \quad E_{\beta_0-\beta} \cup E_0 = \Omega_\beta^c, \quad 0 < \beta < \beta_0,$$

where $\Omega_\beta^c = \Omega \setminus \Omega_\beta$ is the complement of Ω_β in Ω . Indeed, let $y \in \partial E_0$ and denote by r_y the half-line emanating from y perpendicularly to the tangent plane of ∂E_0 at y . Let $z_y \in \partial \Omega$ be given by the intersection of $\partial \Omega$ and r_y . Since $\delta(y) = \beta_0$ we have

$$(4.18) \quad \text{dist}(y, z_y) = \delta(y) = \beta_0, \quad y \in \partial E_0.$$

Now let $x \in \Omega_\beta^c$. Then there exists an $y(x) \in \partial E_0$ such that $x \in r_{y(x)}$. Hence

$$\text{dist}(y(x), x) = \delta(y(x)) - \text{dist}(x, z_{y(x)}) = \beta_0 - \text{dist}(x, z_{y(x)}) \leq \beta_0 - \delta(x) \leq \beta_0 - \beta.$$

This implies that $\Omega_\beta^c \subseteq E_{\beta_0-\beta} \cup E_0$. To prove the opposite inclusion, let $x \in (E_{\beta_0-\beta} \cup E_0)$. By the triangle inequality and (4.18)

$$\beta_0 \leq \text{dist}(x, E_0) + \delta(x) \leq \beta_0 - \beta + \delta(x),$$

which shows that $x \in \Omega_\beta^c$. Therefore (4.17) holds true and consequently

$$(4.19) \quad |\Omega_\beta| = |\Omega| - |E_{\beta_0-\beta} \cup E_0|.$$

In view of (4.16) it follows that $|E_{\beta_0-\beta} \cup E_0|$ is a convex function of β . Hence $|\Omega_\beta|$ is a concave function of β on $(0, \beta_0)$, see (4.19), and since $|\Omega_0| = 0$, we easily verify that $f(\beta) = |\Omega_\beta|/\beta$ is decreasing on $(0, \beta_0)$ for any $\beta_0 < R_i(\Omega)$. This proves the statement of the Lemma for C^1 smooth domains.

If $\partial\Omega$ is not C^1 , then we approximate Ω by a sequence of domains Ω^n with C^1 smooth boundary and such that the Hausdorff distance between Ω and Ω^n tends to zero as $n \rightarrow \infty$. Then

$$f(\beta) = \lim_{n \rightarrow \infty} \frac{|\Omega_\beta^n|}{\beta}.$$

Since a pointwise limit of a sequence of decreasing functions is a decreasing function, we again conclude that $f(\beta)$ is decreasing. This completes the proof. ■

Proof of Corollary 3.2. The claim follows from Theorem 3.1, Lemma 4.2 and the fact that for convex domains $c_h(\Omega) = 4$ independently of Ω , [3, 19]. ■

4.2. Magnetic Dirichlet Laplacian. Let P_k be the orthogonal projection onto the k th Landau level $B(2k-1)$ of the Landau Hamiltonian with constant magnetic field B in $L^2(\mathbb{R}^2)$. Denote by $P_k(x, y)$ the integral kernel of P_k . Note that

$$(4.20) \quad P_k(x, x) = \frac{1}{2\pi} B,$$

$$(4.21) \quad \int_{\mathbb{R}^2} \left(\int_{\Omega} |P_k(y, x)|^2 dx \right) dy = \int_{\Omega} \left(\int_{\mathbb{R}^2} P_k(y, x) \overline{P_k(x, y)} dy \right) dx \\ = \int_{\Omega} P_k(x, x) dx = \frac{B}{2\pi} |\Omega|.$$

Let ϕ_j be the normalised eigenfunctions of $H_\Omega(A)$ corresponding to the eigenvalues $\lambda_j(\Omega; A)$. Put

$$f_{k,j}(y) = \int_{\Omega} P_k(y, x) \phi_j(x) dx, \quad y \in \mathbb{R}^2.$$

Our goal is to establish an analog of Proposition 4.1 for magnetic Dirichlet Laplacians on planar domains. Let $\Lambda > 0$. We have

$$\sum_{j: \lambda_j(\Omega; A) < \Lambda} (\Lambda - \lambda_j(\Omega; A)) = \sum_{j: \lambda_j(\Omega; A) < \Lambda} (\Lambda \|\phi_j\|_{L^2(\Omega)}^2 - \|(\mathbf{i}\nabla_x + A)\phi_j\|_{L^2(\Omega)}^2) \\ = \sum_{j: \lambda_j(\Omega; A) < \Lambda} \sum_{k \in \mathbb{N}} (\Lambda \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 - \|(\mathbf{i}\nabla_x + A)f_{k,j}\|_{L^2(\mathbb{R}^2)}^2) \\ = \sum_{j: \lambda_j(\Omega; A) < \Lambda} \sum_{k \in \mathbb{N}} (\Lambda - B(2k-1)) \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2.$$

In analogy with the procedure in the non-magnetic case we split

$$\begin{aligned}
(4.22) \quad \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} (\Lambda - \lambda_j(\Omega; \mathbf{A})) &= \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} \sum_{k \in \mathbb{N}} (\Lambda - B(2k-1)) \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 \\
&= \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} \sum_{k:\Lambda > B(2k-1)} (\Lambda - B(2k-1)) \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 \\
&+ \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} \sum_{k:\Lambda \leq B(2k-1)} (\Lambda - B(2k-1)) \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 \\
&= \sum_{k:\Lambda > B(2k-1)} (\Lambda - B(2k-1)) \sum_{j \in \mathbb{N}} \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 \\
&- \sum_{k:\Lambda \leq B(2k-1)} (B(2k-1) - \Lambda) R_1(\Lambda, k) \\
&- \sum_{k:\Lambda > B(2k-1)} (\Lambda - B(2k-1)) R_2(\Lambda, k),
\end{aligned}$$

where

$$R_1(\Lambda, k) = \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2, \quad R_2(\Lambda, k) = \sum_{j:\lambda_j(\Omega; \mathbf{A}) \geq \Lambda} \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2.$$

By Parseval's identity and equation (4.21) it follows that for all $\Lambda > 0$ and all $k \in \mathbb{N}$ we have

$$\begin{aligned}
(4.23) \quad \sum_{j \in \mathbb{N}} \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 &= R_1(\Lambda, k) + R_2(\Lambda, k) = \int_{\mathbb{R}^2} \left| \sum_{j \in \mathbb{N}} \int_{\Omega} P_k(\mathbf{y}, \mathbf{x}) \phi_j(\mathbf{x}) d\mathbf{x} \right|^2 d\mathbf{y} \\
&= \int_{\mathbb{R}^2} \int_{\Omega} |P_k(\mathbf{y}, \mathbf{x})|^2 d\mathbf{x} d\mathbf{y} = \frac{B}{2\pi} |\Omega|.
\end{aligned}$$

Let

$$(4.24) \quad Q_k(\mathbf{x}, \mathbf{y}; \Lambda) = \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} f_{k,j}(\mathbf{y}) \bar{\phi}_j(\mathbf{x}).$$

We now use identities (4.20)-(4.21) to find that, similarly as in section 4, for any $\beta \leq R_i(\Omega)$ it holds

$$\begin{aligned}
(4.25) \quad R_2(\Lambda, k) &= \int_{\mathbb{R}^2} \left(\int_{\Omega} |P_k(\mathbf{x}, \mathbf{y}) - Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2 d\mathbf{x} \right) d\mathbf{y} \\
&\geq \frac{1}{2} \int_{\mathbb{R}^2} \int_{\Omega_\beta} |P_k(\mathbf{x}, \mathbf{y})|^2 d\mathbf{x} d\mathbf{y} - \int_{\mathbb{R}^2} \int_{\Omega_\beta} |Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2 d\mathbf{x} d\mathbf{y} \\
&= \frac{B}{4\pi} |\Omega_\beta| - \int_{\mathbb{R}^2} \int_{\Omega_\beta} |Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2 d\mathbf{x} d\mathbf{y}.
\end{aligned}$$

Since $Q_k(\cdot, \mathbf{y}; \Lambda) \in H_0^1(\Omega)$ for all $k \in \mathbb{N}, \mathbf{y} \in \mathbb{R}^2$ and $\Lambda > 0$, the Hardy inequality (2.4) in combination with (1.8) yield

$$\begin{aligned}
\int_{\Omega_\beta} |Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2 d\mathbf{x} &\leq \beta^2 \int_{\Omega_\beta} \frac{|Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2}{\delta^2(\mathbf{x})} d\mathbf{x} \leq \beta^2 \int_{\Omega} \frac{|Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2}{\delta^2(\mathbf{x})} d\mathbf{x} \\
&\leq \beta^2 c_h(\Omega) \int_{\Omega} |(i\nabla_{\mathbf{x}} + \Lambda) Q_k(\mathbf{x}, \mathbf{y}; \Lambda)|^2 d\mathbf{x} \\
&= \beta^2 c_h(\Omega) \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} |f_{k,j}(\mathbf{y})|^2 \lambda_j(\Omega; \mathbf{A}) \leq \beta^2 c_h(\Omega) \Lambda \sum_{j:\lambda_j(\Omega; \mathbf{A}) < \Lambda} |f_{k,j}(\mathbf{y})|^2.
\end{aligned}$$

By inserting the above estimate into (4.25) and using (4.23) again we obtain

$$\begin{aligned} R_2(\Lambda, k) &\geq \frac{B}{4\pi} |\Omega_\beta| - \beta^2 c_h(\Omega) \wedge \sum_{j: \lambda_j(\Omega; \Lambda) < \Lambda} \|f_{k,j}\|_{L^2(\mathbb{R}^2)}^2 \\ &\geq \frac{B}{4\pi} \left(\frac{|\Omega_\beta|}{\beta} - 2\beta \wedge c_h(\Omega) |\Omega| \right) \beta. \end{aligned}$$

Note that in view of (1.8) we have

$$(4.26) \quad \lambda_1(\Omega; \Lambda) \geq \lambda_1(\Omega).$$

Hence choosing β as in (4.9) and following the reasoning in (4.11) we conclude that $\beta \leq R_i(\Omega)/4$ and therefore $\frac{|\Omega_\beta|}{\beta} \geq \sigma(\Omega)$. This implies

$$(4.27) \quad R_2(\Lambda, k) \geq \frac{B}{32\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} \Lambda^{-1} \quad \forall k \in \mathbb{N}.$$

Proposition 4.3. *Let $d = 2$. For any $\Lambda \geq \lambda_1(\Omega; \Lambda)$ it holds*

$$(4.28) \quad \sum_{j: \lambda_j(\Omega; \Lambda) < \Lambda} (\Lambda - \lambda_j(\Omega; \Lambda)) \leq \frac{|\Omega|}{8\pi} \Lambda^2 - \frac{1}{128\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} \Lambda.$$

Proof. Put $M = \lfloor \frac{\Lambda}{2B} + \frac{1}{2} \rfloor$ and $m = \{ \frac{\Lambda}{2B} + \frac{1}{2} \}$ and thus $M + m = \frac{\Lambda}{2B} + \frac{1}{2}$. Then

$$\begin{aligned} \sum_{k: \Lambda > B(2k-1)} (\Lambda - B(2k-1)) &= M\Lambda - BM^2 = BM \left(\frac{\Lambda}{B} - M \right) \\ &= B \left(\frac{\Lambda}{2B} + \frac{1}{2} - m \right) \left(\frac{\Lambda}{2B} - \frac{1}{2} + m \right) \\ &= B \left(\frac{\Lambda^2}{4B^2} - \left(\frac{1}{2} - m \right)^2 \right). \end{aligned}$$

Since $R_1(\Lambda, k) \geq 0$, the above identity together with (4.22) and (4.23) implies

$$\begin{aligned} \sum_{j: \lambda_j(\Omega; \Lambda) < \Lambda} (\Lambda - \lambda_j(\Omega; \Lambda)) &\leq \frac{|\Omega|}{8\pi} \Lambda^2 - \frac{1}{128\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} \Lambda \\ &\quad - B^2 \left(\frac{1}{2} - m \right)^2 \left(\frac{|\Omega|}{2\pi} - \frac{1}{32\pi c_h(\Omega)} \frac{\sigma^2(\Omega)}{|\Omega|} \Lambda \right). \end{aligned}$$

The last term on the right hand side of the last inequality is negative since $\Lambda c_h(\Omega) \geq \lambda_1(\Omega) c_h(\Omega) \geq R_i^{-2}(\Omega)$, by (2.4) and (4.26), and $\sigma(\Omega) \leq |\Omega|/R_i(\Omega)$. The claim now follows. ■

Proof of Theorem 3.7. Inequality (3.7) now follows from Proposition 4.3 by the Legendre transformation in the same as in the case of the Dirichlet Laplacian. ■

Corollary 3.8 is a consequence of Theorem 3.7 and Lemma 4.2.

5. Proofs of the main results: Berezin inequalities

The order of the reminder term in (4.28) can further be improved applying a straightforward generalization of a result by Davies, [4]. We are grateful to Rupert Frank who pointed this fact out to us. The proof relies on the following result.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Let $A \in C(\overline{\Omega}, \mathbb{R}^2)$ and let $H(A)$ be the associated magnetic Dirichlet Laplacian in $L^2(\Omega)$. Assume that the Hardy inequality*

$$(5.1) \quad \int_{\Omega} |i\nabla u + Au|^2 dx \geq c^{-2} \int_{\Omega} \frac{|u|^2}{\delta^2} dx, \quad u \in C_0^\infty(\Omega),$$

holds for some $c \geq 2$. Then for every $\beta > 0$,

$$(5.2) \quad \int_{\Omega_\beta} |u|^2 dx \leq c^{2+2/c} \beta^{2+2/c} \|H(A)u\| \|H(A)^{1/c}u\|$$

for any u in the operator domain of $H(A)$.

Proposition 5.1 was proved in [4] for the case $A = 0$. However, a detailed inspection of the proof of [4, Thm. 4] shows that the same method applies also to the magnetic Dirichlet Laplacian.

5.1. Dirichlet Laplacian.

Proof of Theorem 3.5. Consider the function $F_\Lambda(\xi, x)$ given by (4.6). Since $F_\Lambda(\xi, \cdot)$ belongs to the operator domain of $-\Delta$, we can apply Proposition 5.1 with $A = 0$ and $c = \mu = \sqrt{c_h(\Omega)}$. This yields

$$\begin{aligned} \int_{\Omega_\beta} |F_\Lambda(\xi, x)|^2 dx &\leq (\mu \beta)^{2+\frac{2}{\mu}} \left\| \sum_{j \leq n(\Lambda)} \lambda_j(\Omega) \hat{u}_j(\xi) u_j \right\| \left\| \sum_{j \leq n(\Lambda)} \lambda_j^{\frac{1}{\mu}}(\Omega) \hat{u}_j(\xi) u_j \right\| \\ &= (\mu \beta)^{2+\frac{2}{\mu}} \left(\sum_{j \leq n(\Lambda)} \lambda_j^2(\Omega) |\hat{u}_j(\xi)|^2 \right)^{1/2} \left(\sum_{j \leq n(\Lambda)} \lambda_j^{\frac{2}{\mu}}(\Omega) |\hat{u}_j(\xi)|^2 \right)^{1/2} \\ &\leq (\mu \beta)^{2+\frac{2}{\mu}} \Lambda^{1+\frac{1}{\mu}} (2\pi)^{-d} |\Omega|, \end{aligned}$$

where we have taken into account (4.3). This together with (4.5) implies the bound

$$(5.3) \quad R_2(\Lambda, \xi) \geq (2\pi)^{-d} \left(\frac{|\Omega_\beta|}{2\beta} - \Lambda^{1+\frac{1}{\mu}} \beta^{1+\frac{2}{\mu}} \mu^{2+\frac{2}{\mu}} |\Omega| \right) \beta.$$

We now put

$$(5.4) \quad \beta = \frac{1}{\mu} (4 + 4\mu)^{-\frac{\mu}{\mu+2}} \left(\frac{\sigma(\Omega)}{|\Omega|} \right)^{\frac{\mu}{\mu+2}} \Lambda^{-\frac{1+\mu}{2+\mu}}.$$

Since $\mu \geq 2$, $\sigma(\Omega)/|\Omega| \leq R_i(\Omega)^{-1}$, by definition, and $\Lambda \geq \lambda_1(\Omega) \geq \mu^{-2} R_i(\Omega)^{-2}$, by (2.4), it is easily seen that the right hand side of (5.4) is less than $R_i(\Omega)$. Hence $|\Omega_\beta|/\beta \geq \sigma(\Omega)$ for β given by (5.4). By using this bound in (5.3) and inserting there (5.4) we find that

$$R_2(\Lambda, \xi) \geq (2\pi)^{-d} K(\Omega) \sigma(\Omega) \left(\frac{\sigma(\Omega)}{|\Omega|} \right)^{\frac{\mu}{\mu+2}} \Lambda^{-\frac{1+\mu}{2+\mu}}.$$

In view of (4.2) this proves (3.5). Inequality (3.6) then follows from Lemma 4.2. ■

5.2. Magnetic Dirichlet Laplacian.

Proof of Theorem 3.9. Let us fix $k \in \mathbb{N}$ and $y \in \mathbb{R}^2$. Since $Q_k(\cdot, y; \Lambda)$ belongs to the domain of $H(\Lambda)$ for any $\Lambda > 0$, see (4.24), we can apply inequality (5.2), with $c = \mu = \sqrt{c_h(\Omega)}$, to the function $u = Q_k(\cdot, y; \Lambda)$. This yields

$$\int_{\Omega_\beta} |Q_k(x, y; \Lambda)|^2 dx \leq (\mu \beta)^{2+\frac{2}{\mu}} \Lambda^{1+\frac{1}{\mu}} \sum_{j: \lambda_j(\Omega; \Lambda) < \Lambda} |f_{k,j}(y)|^2 \leq (\mu \beta)^{2+\frac{2}{\mu}} \Lambda^{1+\frac{1}{\mu}} \frac{B}{2\pi} |\Omega|,$$

where we have also used (4.23). From (4.25) we thus obtain the estimate

$$R_2(\Lambda, k) \geq \frac{B}{4\pi} \left(\frac{|\Omega_\beta|}{2\beta} - \mu^{2+\frac{2}{\mu}} \beta^{1+\frac{2}{\mu}} \Lambda^{1+\frac{1}{\mu}} |\Omega| \right) \beta.$$

Inserting (5.4) into the right hand side together with the estimate $|\Omega_\beta|/\beta \geq \sigma(\Omega)$ then gives

$$(5.5) \quad R_2(\Lambda, k) \geq B \tilde{K}(\Omega) \sigma(\Omega) \left(\frac{\sigma(\Omega)}{|\Omega|} \right)^{\frac{\mu}{2+\mu}} \Lambda^{-\frac{1+\mu}{2+\mu}}.$$

We now follow the arguments of the proof of Proposition 4.3 with the lower bound (4.27) replaced by (5.5) and arrive at (3.9). The upper bound (3.10) then follows again by Lemma 4.2. ■

Acknowledgements

The work on this paper was initiated by the programme Hamiltonians in Magnetic Fields in fall 2012 at the Institut Mittag- Leffler, Djursholm, Sweden. H.K. has been partially supported by the MIUR-PRIN 2010-11 grant for the project "Calcolo delle Variazioni". T.W. has been partially supported by the DFG project WE 1964/4-1.

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