

ON THE LOWEST EIGENVALUE OF LAPLACE OPERATORS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. In this paper we consider a Robin-type Laplace operator on bounded domains. We study the dependence of its lowest eigenvalue on the boundary conditions and its asymptotic behaviour in shrinking and expanding domains. For convex domains we establish two-sided estimates on the lowest eigenvalues in terms of the inradius and of the boundary conditions.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Given a measurable function $\sigma : \partial\Omega \rightarrow \mathbb{R}$, we consider the quadratic form

$$\int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} \sigma |u|^2 d\nu, \quad (1.1)$$

where $d\nu$ denotes the $(N-1)$ dimensional surface measure on $\partial\Omega$. If Ω is regular enough and if $\sigma \in L^{N-1}(\partial\Omega)$, then by the boundary trace imbedding theorems, equation (2.1) below, it follows that the quadratic form (1.1) is closed on $H^1(\Omega)$ and generates in $L^2(\Omega)$ a unique self-adjoint operator, the so-called Robin-Laplacian. The case $\sigma = +\infty$ then corresponds to the Dirichlet boundary conditions while by choosing $\sigma = 0$ we get the Neumann boundary conditions.

The lowest eigenvalue of the Robin-Laplacian, which we denote by $\lambda_1(\sigma, \Omega)$, is the main object of our interest. Problems related to the Robin-Laplacian have been intensively studied in the literature. Among other questions, various problems such as Faber-Krahn inequalities, Hardy inequalities, monotonicity properties of the lowest eigenvalue, and comparison between Robin and Dirichlet or Neumann eigenvalues were considered in the literature, see [Bo1, Bo2, BG, CU, Da1, Da2, GS, KL, LP, PW, Ph, Sp1, Sp2].

The purpose of the present paper is twofold. First we will study the dependence of the lowest eigenvalue of the Robin-Laplacian on the function σ . We start by addressing the following question: which functions σ maximise (or minimise) $\lambda_1(\sigma, \Omega)$ among all positive functions from $L^1(\partial\Omega)$ with a fixed integral mean and with a support contained in a prescribed subset of the boundary? It turns out that while the minimising σ generically does not exist, the maximising function exists and is unique. An explicit description is given in Theorem 3.3. Next one would like to know how big the resulting maximum is. Sharp two-sided estimates on the corresponding maximal eigenvalue are given in Propositions 3.7, 3.8 and Corollary 3.11.

In the second part of the paper we will study the properties of $\lambda_1(\sigma, \Omega)$ for a fixed σ . In Theorem 4.1 it will be shown that, contrary to the lowest eigenvalue of the Dirichlet-Laplacian, $\lambda_1(\sigma, \Omega)$ scales in a different way when the domain Ω shrinks to zero respectively when Ω blows up to infinity. We

then prove a two-sided bound for $\lambda_1(\sigma, \Omega)$ on convex domains with constant σ . In particular, we will show that

$$\lambda_1(\sigma, \Omega) \asymp \frac{\sigma}{R_\Omega(1 + \sigma R_\Omega)} \quad [\Omega \text{ convex, } \sigma \text{ constant}],$$

where R_Ω is the inradius of Ω , see Theorem 4.4.

2. Preliminaries

Throughout the paper will always assume that the following condition is satisfied:

Assumption 2.1. Ω is an open bounded and connected set with a boundary which satisfies the strong local Lipschitz condition, see e.g. [Ad, Chap.4].

Under the above assumption a trace operator is well defined on $H^1(\Omega)$. More precisely, we have

$$\|u\|_{L^q(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}, \quad \forall q \begin{cases} \leq 2(N-1)/(N-2) & \text{if } N > 2, \\ < +\infty & \text{if } N = 2 \end{cases} \quad (2.1)$$

with a compact imbedding. This follows from standard Sobolev imbedding theorems and trace inequalities, see e.g. [Ad, Thm.5.22]. For a given $\sigma \in L^1(\partial\Omega)$ we then consider the functional $Q[\sigma, \cdot]$ on $H^1(\Omega)$ defined by

$$Q[\sigma, u] = \frac{\int_\Omega |\nabla u(x)|^2 dx + \int_{\partial\Omega} \sigma(s) |u(s)|^2 d\nu(s)}{\|u\|_{L^2(\Omega)}^2} \quad (2.2)$$

if the right hand side is finite and by $Q[\sigma, u] = +\infty$ otherwise. Let

$$\lambda_1(\sigma, \Omega) = \inf_{u \in H^1(\Omega)} Q[\sigma, u]. \quad (2.3)$$

Lemma 2.2. *Let $\sigma \in L^1(\partial\Omega)$ and assume that $\sigma \geq 0$. Then the functional $Q[\sigma, \cdot]$ admits a positive minimiser $\psi \in H^1(\Omega)$ which satisfies*

$$-\Delta\psi = \lambda_1(\sigma, \Omega) \psi \quad \text{in } \Omega, \quad \partial_n \psi + \sigma \psi = 0 \quad \text{a.e. on } \partial\Omega, \quad (2.4)$$

where ∂_n denotes the outer normal derivative.

Proof. Let $\{u_j\}_{j \in \mathbb{N}}$ be a minimising sequence for $Q[\sigma, \cdot]$. Without loss of generality we assume that $\|u_j\|_{L^2(\Omega)} = 1$ for all $j \in \mathbb{N}$. Since $\{u_j\}$ is bounded in $H^1(\Omega)$, there exists a subsequence, which we still denote by u_j and a function $\psi \in H^1(\Omega)$ such that $u_j \rightarrow \psi$ weakly in $H^1(\Omega)$. Next, from the compactness of the imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, see (2.1), it follows that the trace of u_j converges strongly in $L^2(\partial\Omega)$ to the trace of ψ . Therefore we can find a subsequence $\{v_j\} \subset \{u_j\}$ such that $v_j|_{\partial\Omega} \rightarrow \psi|_{\partial\Omega}$ almost everywhere on $\partial\Omega$. By the weak lower semicontinuity of $\int_\Omega |\nabla u|^2$ and the Fatou Lemma we thus obtain

$$\liminf_{j \rightarrow \infty} Q[\sigma, v_j] \geq Q[\sigma, \psi].$$

Hence ψ is a minimiser of $Q[\sigma, \cdot]$. From the fact that $Q[\sigma, \psi] \geq Q[\sigma, |\psi|]$ it follows that $\psi \geq 0$ in Ω . Therefore, by Harnack inequality $\psi > 0$ in Ω . The Euler-Lagrange equation for $Q[\sigma, \cdot]$ then gives (2.4). \square

Remark 2.3. The assumption $\sigma \geq 0$ in the above Lemma is necessary. Indeed, if $\sigma \in L^1(\partial\Omega)$ changes sign, then the functional $Q[\sigma, \cdot]$ might not even be bounded from below

3. Optimising problem for $\lambda_1(\sigma, \Omega)$

In this section we will only assume that $0 \leq \sigma \in L^1(\partial\Omega)$. Note that although this condition does not guarantee the finiteness of $Q[\sigma, u]$ for all $u \in H^1(\Omega)$, the quantity $\lambda_1(\sigma, \Omega)$ is well defined. Let $\Gamma \subseteq \partial\Omega$ be a closed subset of the boundary (which might coincide with $\partial\Omega$). For a given $m > 0$ we define

$$\Sigma_m := \left\{ \sigma \in L^1(\partial\Omega) : \sigma \geq 0, \int_{\partial\Omega} \sigma = m, \text{supp } \sigma \subseteq \Gamma \right\}.$$

Our goal is to study $\lambda_1(\sigma, \Omega)$ as a functional of σ on Σ_m . To start with we show that the functional $\lambda_1(\cdot, \Omega)$ admits no minimum on Σ_m when $N \geq 2$.

3.1. The infimum.

Proposition 3.1. *Let $m > 0$ and suppose that $N \geq 2$. Then $\lambda_1(\cdot, \Omega)$ has no minimiser on Σ_m and $\inf_{\sigma \in \Sigma_m} \lambda_1(\sigma, \Omega) = 0$.*

Proof. In the sequel we denote by $B(x, r)$ the open ball of radius r centred in $x \in \mathbb{R}^N$. Let $s_0 \in \partial\Omega$ and let $\sigma_n \geq 0$ be given by

$$\sigma_n(s) = \begin{cases} \alpha_n & \text{if } s \in B(s_0, 2^{-n}) \cap \partial\Omega, \\ 0 & \text{elsewhere,} \end{cases}$$

where α_n is a positive constant chosen so that $\sigma_n \in \Sigma_m$ for all $n \in \mathbb{N}$. Depending on the dimension we construct a family of test functions u_n as follows:

$$u_n(x) = \frac{-\log n}{\log(|x - s_0|)} \quad \text{on } B(s_0, \frac{1}{n}) \cap \Omega, \quad u_n \equiv 1 \quad \text{on } \Omega \setminus B(s_0, \frac{1}{n}), \quad N = 2.$$

$$u_n(x) = n|x - s_0| \quad \text{on } B(s_0, \frac{1}{n}) \cap \Omega, \quad u_n \equiv 1 \quad \text{on } \Omega \setminus B(s_0, \frac{1}{n}), \quad N \geq 3.$$

Then $u_n \in H^1(\Omega)$ for all $n \in \mathbb{N}$ and a direct calculation shows that

$$\lim_{n \rightarrow \infty} Q[\sigma_n, u_n] = 0.$$

This proves that $\inf_{\sigma \in \Sigma_m} \lambda_1(\sigma, \Omega) = 0$. To show that the infimum is not attained, assume that $\sigma \in \Sigma_m$, $m > 0$. Then there exists an $\varepsilon > 0$ and $\gamma_\varepsilon \subset \Gamma$ such that $\sigma \geq \varepsilon$ on γ_ε and $\nu(\gamma_\varepsilon) > 0$. By the Poincaré inequality

$$\int_{\Omega} |\nabla u(x)|^2 dx + \varepsilon \int_{\gamma_\varepsilon} |u(s)|^2 d\nu(s) \geq c_\varepsilon \|u\|_{L^2(\Omega)}^2$$

for some $c_\varepsilon > 0$ and all $u \in H^1(\Omega)$. Hence $\lambda_1(\sigma, \Omega) > 0$ for any $\sigma \in \Sigma_m$ with $m > 0$. \square

The assumption $N \geq 2$ in Proposition 3.1 is crucial, see Section 3.3 for related results in dimension one.

3.2. The supremum. The main object of our interest here is the quantity

$$\Lambda_1(m, \Omega) = \sup_{\sigma \in \Sigma_m} \lambda_1(\sigma, \Omega), \tag{3.1}$$

and the function $\sigma \in \Sigma_m$ which realises the above supremum. It will be shown that, contrary to $\inf_{\sigma \in \Sigma_m} \lambda_1(\sigma, \Omega)$, the supremum $\Lambda_1(m, \Omega)$ is achieved on Σ_m . We will give an explicit characterisation of the maximising σ , and prove sharp two-sided estimates for the related maximal eigenvalue in terms of m and the volume of Ω . The existence of the maximising σ in (3.1) is related to the following simple observation:

If $\hat{\sigma} \in \Sigma_m$ is such that the corresponding minimiser \hat{u} of $Q[\hat{\sigma}, \cdot]$ is *constant on* Γ , then $\Lambda_1(m, \Omega) = \lambda_1(\hat{\sigma}, \Omega)$. Indeed, for any $\sigma \in \Sigma_m$ we then have

$$\inf_{u \in H^1(\Omega)} Q[\sigma, u] \leq Q[\sigma, \hat{u}] = Q[\hat{\sigma}, \hat{u}] = \lambda_1(\hat{\sigma}, \Omega). \quad (3.2)$$

It thus suffices to find a suitable candidate for $\hat{\sigma}$. To do so, we consider the corresponding limiting problem for $\sigma \rightarrow \infty$, which is associated with the Laplace operator $-\Delta_D^\Gamma$ in $L^2(\Omega)$ subject to Dirichlet boundary conditions on Γ and to Neumann boundary conditions on $\partial\Omega \setminus \Gamma$. More precisely, $-\Delta_D^\Gamma$ is generated by the closed quadratic form

$$\mathcal{Q}_\Gamma[u] = \int_\Omega |\nabla u(x)|^2 dx, \quad D(\mathcal{Q}_\Gamma) = \{u \in H^1(\Omega) : u|_\Gamma = 0\}. \quad (3.3)$$

The compactness of the imbedding $D(\mathcal{Q}_\Gamma) \hookrightarrow L^2(\Omega)$ implies that the spectrum of $-\Delta_D^\Gamma$ is purely discrete. Let $E_j(\Gamma)$ be the non-decreasing sequence of its eigenvalues and let φ_j be the associated normalised eigenfunctions. Hence

$$E_1(\Gamma) = \min_{u \in D(\mathcal{Q}_\Gamma)} \frac{\mathcal{Q}_\Gamma[u]}{\|u\|_{L^2(\Omega)}^2}. \quad (3.4)$$

If Γ has a positive measure, then in view of the Poincaré inequality we have $E_1(\Gamma) > 0$. Recall that the operator domain $D(-\Delta_D^\Gamma)$ of $-\Delta_D^\Gamma$ satisfies

$$D(-\Delta_D^\Gamma) \subseteq \{u \in H^2(\Omega) : u|_\Gamma = 0 \wedge \partial_n u|_{\partial\Omega \setminus \Gamma} = 0 \text{ a. e.}\}. \quad (3.5)$$

Now for $\xi \in (0, E_1(\Gamma))$ we define

$$U_\xi = (-\Delta_D^\Gamma - \xi)^{-1} \mathbb{1},$$

where $\mathbb{1}$ denotes the function identically equal to 1 on Ω . Since $(-\Delta_D^\Gamma - \xi)^{-1}$ is positivity preserving, we have $U_\xi > 0$ in Ω . Hence from the strong maximum principle and the fact that $(-\Delta_D^\Gamma - \xi)^{-1}$ maps $L^2(\Omega)$ into $D(-\Delta_D^\Gamma)$ it follows that

$$\partial_n U_\xi|_\Gamma < 0 \quad \text{and} \quad \partial_n U_\xi|_{\partial\Omega \setminus \Gamma} = 0 \quad \text{a.e.} \quad (3.6)$$

Together with U_ξ we introduce the function $F : (0, E_1(\Gamma)) \rightarrow \mathbb{R}$ given by

$$F(\xi) := \xi^2 \int_\Omega U_\xi dx + \xi |\Omega|, \quad \xi \in (0, E_1(\Gamma)), \quad (3.7)$$

where $|\Omega|$ denotes the volume of Ω .

Lemma 3.2. *The function F is a C^2 increasing convex bijection from $(0, E_1(\Gamma))$ onto $(0, \infty)$.*

Proof. Let

$$g(\xi) := \int_\Omega U_\xi dx = (\mathbb{1}, (-\Delta_D^\Gamma - \xi)^{-1} \mathbb{1})_{L^2(\Omega)},$$

so that $F(\xi) = \xi^2 g(\xi) + \xi |\Omega|$. From the resolvent identity

$$(-\Delta_D^\Gamma - \xi)^{-1} - (-\Delta_D^\Gamma - \xi')^{-1} = (\xi - \xi') (-\Delta_D^\Gamma - \xi)^{-1} (-\Delta_D^\Gamma - \xi')^{-1} \quad (3.8)$$

we easily find out that $g'(\xi) = \|U_\xi\|_{L^2(\Omega)}^2 > 0$. This shows that F is increasing. Moreover, using (3.8) again we get

$$g''(\xi) = 2 (U_\xi, (-\Delta_D^\Gamma - \xi)^{-1} U_\xi)_{L^2(\Omega)} > 0,$$

which implies that F is convex. It remains to show that $F : (0, E_1(\Gamma)) \rightarrow (0, \infty)$ is surjective. Obviously, $F(\xi) \rightarrow 0$ as $\xi \rightarrow 0$. On the other hand, using the explicit expression for the integral kernel of $(-\Delta_D^\Gamma - \xi)^{-1}$ we obtain

$$g(\xi) = \frac{1}{E_1(\Gamma) - \xi} \left(\int_{\Omega} \varphi_1 dx \right)^2 + \mathcal{O}(1), \quad \xi \nearrow E_1(\Gamma). \quad (3.9)$$

Since F is continuous, it follows that F maps $(0, E_1(\Gamma))$ onto $(0, \infty)$. \square

The above Lemma allows us to introduce the function ξ on $(0, \infty)$ given by

$$\xi(m) = F^{-1}(m), \quad m > 0. \quad (3.10)$$

In view of Lemma 3.2 we easily see that ξ is concave and maps $(0, \infty)$ onto $(0, E_1(\Gamma))$.

Theorem 3.3. *The supremum $\Lambda_1(m, \Omega) = \sup_{\sigma \in \Sigma_m} \lambda_1(\sigma, \Omega)$ is attained for any $m > 0$ and satisfies*

$$\Lambda_1(m, \Omega) = \lambda_1(\sigma_m, \Omega) = \xi(m), \quad \text{where } \sigma_m = -\xi(m) \partial_n U_{\xi(m)}|_{\partial\Omega}. \quad (3.11)$$

Moreover, the maximiser σ_m is unique in Σ_m .

Proof. As mentioned above, to prove that σ_m is a maximiser it suffices to show that the minimiser of the functional $Q[\sigma_m, \cdot]$ is constant on Γ . Recall that $U_\xi \in H^2(\Omega)$. From the imbedding (2.1) we find that $\sigma_m \in L^1(\partial\Omega)$. Moreover, since $-\Delta U_{\xi(m)} = \xi(m)U_{\xi(m)} + 1$, the Green formula and equations (3.6), (3.10) yield

$$\int_{\partial\Omega} \sigma_m d\nu = - \int_{\partial\Omega} \partial_n U_{\xi(m)} d\nu = -\xi(m) \int_{\Omega} \Delta U_{\xi(m)} dx = F(\xi(m)) = m.$$

This in combination with (3.6) shows that $\sigma_m \in \Sigma_m$. Next we define

$$u_m := \xi(m) U_{\xi(m)} + 1,$$

so that

$$-\Delta u_m = \xi(m) u_m \quad \text{in } \Omega, \quad \partial_n u_m + \sigma_m = 0 \quad \text{a.e. on } \partial\Omega. \quad (3.12)$$

We claim that u_m is a minimiser of $Q[\sigma_m, \cdot]$. Indeed, by (3.12) we have $Q[\sigma_m, u_m] = \xi(m)$. Assume that $\lambda_1(\sigma_m, \Omega) < \xi(m)$. Then in view of Lemma 2.2 there exists a positive minimiser ψ of $Q[\sigma_m, \cdot]$ which satisfies equation (2.4) with $\sigma = \sigma_m$. This in combination with (3.12) and integration by parts implies that $(\psi, u_m)_{L^2(\Omega)} = 0$, which is in contradiction with the positivity of ψ and u_m . We thus conclude that $\lambda_1(\sigma_m, \Omega) = \xi(m)$.

To show that $\Lambda_1(m, \Omega) = \lambda_1(\sigma_m, \Omega)$ pick an arbitrary $\sigma \in \Sigma_m$. As already pointed out in (3.2), using (2.3) and the fact that $u_m = 1$ on Γ we obtain

$$\lambda_1(\sigma, \Omega) \leq Q[\sigma, u_m] = Q[\sigma_m, u_m] = \lambda_1(\sigma_m, \Omega). \quad (3.13)$$

It remains to show the uniqueness of σ_m . To this end suppose that $\lambda_1(\bar{\sigma}, \Omega) = \lambda_1(\sigma_m, \Omega)$ for some $\bar{\sigma} \in \Sigma_m$. By the same argument used in (3.13) we find out that u_m is a minimiser of $Q[\bar{\sigma}, \cdot]$:

$$\lambda_1(\sigma_m, \Omega) = \lambda_1(\bar{\sigma}, \Omega) \leq Q[\bar{\sigma}, u_m] = Q[\sigma_m, u_m] = \lambda_1(\sigma_m, \Omega).$$

By Lemma 2.2 it follows that u_m satisfies the Euler-Lagrange equation (3.12) with σ_m replaced by $\bar{\sigma}$. Hence $\bar{\sigma} = -\partial_n u_m|_{\Gamma} = \sigma_m$ almost everywhere on Γ . \square

Remark 3.4. A slightly different optimising problem for two-dimensional domains was studied in [CU], where the authors addressed the question on which part of boundary one has to impose Dirichlet boundary conditions to minimise or maximise the lowest eigenvalue of a mixed Dirichlet-Neumann boundary value problem.

3.3. **The case $N = 1$.** In the case of dimension one we have $\Omega = (a, b)$ and

$$\Sigma_m = \{\sigma = (\sigma(a), \sigma(b)) : \sigma(a), \sigma(b) \geq 0 \wedge \sigma(a) + \sigma(b) = m\}.$$

Since Theorem 3.3 holds true in any dimension, the maximiser of $\lambda_1(\cdot, \Omega)$ is given by that σ for which the associated minimiser u in (2.3) satisfies $u(a) = u(b)$. In other words

$$\sigma_m = \left(\frac{m}{2}, \frac{m}{2}\right).$$

On the other hand, the claim of Proposition 3.1 fails if $N = 1$ since the capacity of a point on one-dimensional bounded intervals is positive. Consequently, the functional $\lambda_1(\sigma, \Omega)$ admits minimisers on Σ_m and the resulting minimum is positive for any $m > 0$.

Proposition 3.5. *Let $\Omega = (a, b)$. Then for any $m > 0$ it holds*

$$\inf_{\sigma \in \Sigma_m} \lambda_1(\sigma, \Omega) = \lambda_1(\sigma_1, \Omega) = \lambda_1(\sigma_2, \Omega) \geq \frac{1}{4} \left(b - a + \frac{1}{2m}\right)^{-2}, \quad (3.14)$$

where $\sigma_1 = (m, 0)$ and $\sigma_2 = (0, m)$.

Proof. We start by proving that σ_1 and σ_2 are minimisers of $\lambda_1(\cdot, \Omega)$ on Σ_m . Given $u \in H^1(a, b)$ we set $\hat{u}(x) = u(a + b - x)$. Let $\sigma \in \Sigma_m$ and denote by u_σ the positive normalised minimiser of $Q[\sigma, \cdot]$. It is easily seen that

$$\sigma(a) \leq \sigma(b) \Rightarrow u_\sigma(a) \geq u_\sigma(b), \quad \sigma(a) > \sigma(b) \Rightarrow u_\sigma(a) \leq u_\sigma(b) \quad (3.15)$$

This follows from the fact that if $\sigma(a) \leq \sigma(b)$ and u is such that $0 \leq u(a) < u(b)$, then $Q[\sigma, u] > Q[\sigma, \hat{u}]$. The same argument proves the second implication in (3.15).

Assume first that $\sigma(a) > \sigma(b)$. Then in view of (3.15) and the fact that $\sigma(a) + \sigma(b) = m$ we get

$$\lambda_1(\sigma_1, \Omega) \leq Q[\sigma_1, u_\sigma] = \lambda_1(\sigma, \Omega) + m u_\sigma^2(a) - \sigma(a) u_\sigma^2(a) - \sigma(b) u_\sigma^2(b) \leq \lambda_1(\sigma, \Omega).$$

On the other hand, if $\sigma(a) \leq \sigma(b)$, then again with the help of (3.15) it follows that

$$\lambda_1(\sigma_1, \Omega) \leq Q[\sigma_1, \hat{u}_\sigma] = \lambda_1(\sigma, \Omega) + m u_\sigma^2(b) - \sigma(a) u_\sigma^2(a) - \sigma(b) u_\sigma^2(b) \leq \lambda_1(\sigma, \Omega).$$

Hence σ_1 is a minimiser of $\lambda_1(\cdot, \Omega)$. The proof for σ_2 is completely analogous. Obviously, $\lambda_1(\sigma_1, \Omega) = \lambda_1(\sigma_2, \Omega)$. To prove the inequality in (3.14) we note that for any $u \in H^1(a, b)$ it holds

$$\begin{aligned} \int_a^b \left(u'(x) - \frac{u(x)}{2(x - a + \frac{1}{2m})}\right)^2 dx &= \int_a^b |u'(x)|^2 dx + m u^2(a) - \frac{u^2(b)}{2(b - a + \frac{1}{2m})} \\ &\quad - \frac{1}{4} \int_a^b \frac{u^2(x)}{(x - a + \frac{1}{2m})^2} dx, \end{aligned}$$

where we have integrated by parts to evaluate the mixed term. It follows that

$$\int_a^b |u'(x)|^2 dx + m u^2(a) \geq \frac{1}{4} \int_a^b \frac{u^2(x)}{(x - a + \frac{1}{2m})^2} dx \quad \forall u \in H^1(a, b),$$

which yields the sought lower bound in (3.14). \square

Remark 3.6. It is clear from the proof of Proposition 3.5 that σ_1 and σ_2 are the only minimisers of $\lambda_1(\cdot, \Omega)$. Note also that the explicit form of σ_1 and σ_2 is reminiscent of the properties of the sequence σ_n used in the proof of Proposition 3.1.

3.4. The maximal eigenvalue $\Lambda_1(m, \Omega)$. Theorem 3.3 gives us information about the asymptotic behaviour of $\Lambda_1(m, \Omega)$ for $m \rightarrow 0$ as well as for $m \rightarrow \infty$. Indeed by (3.9)

$$\lim_{m \rightarrow \infty} \Lambda_1(m, \Omega) = E_1(\Gamma). \quad (3.16)$$

Moreover, by Lemma 3.2 it follows that $\Lambda_1(m, \Omega)$ is a concave increasing function of m . As for the behaviour of $\Lambda_1(m)$ for small values of m , by using a test function equal to a constant we see that $\Lambda_1(m, \Omega) \rightarrow 0$ when $m \rightarrow 0$. Moreover, by the resolvent equation (3.8) and (3.7) we get

$$F(\xi) = \xi |\Omega| + \xi^2 \int_{\Omega} U_0(x) dx + o(\xi^2) \quad \xi \rightarrow 0.$$

In view of (3.10) and (3.11) we then get

$$\Lambda_1(m, \Omega) = m |\Omega|^{-1} + o(m), \quad m \rightarrow 0+. \quad (3.17)$$

A natural question is how to estimate $\Lambda_1(m, \Omega)$ for a fixed value of m . It turns out that to this end it is not convenient to use directly the equation for $\Lambda_1(m, \Omega)$ given by Theorem 3.3, because we have very little information about the function U_{ξ} . Instead, we are going to employ merely the fact that the corresponding minimiser is constant on Γ .

Proposition 3.7. *For any $m > 0$ it holds*

$$\Lambda_1(m, \Omega) \geq \frac{m E_1(\Gamma)}{m + |\Omega| E_1(\Gamma)}. \quad (3.18)$$

Proof. Since $\Lambda_1(m, \Omega) = \lambda_1(\sigma_m, \Omega)$ admits a normalised eigenfunction which is constant on Γ , by Theorem 3.3, we have

$$\Lambda_1(m, \Omega) = \inf_{u \in \mathcal{F}} Q[\sigma_m, u],$$

where

$$\mathcal{F} = \{u \in H^1(\Omega) : \|u\|_{L^2(\Omega)} = 1, \exists k \geq 0 : u|_{\Gamma} = k\}. \quad (3.19)$$

Now let $u \in \mathcal{F}$ and let k be the corresponding constant in (3.19). Then

$$\begin{aligned} Q[\sigma_m, u] &= \int_{\Omega} |\nabla u|^2 dx + m k^2 = \int_{\Omega} |\nabla(u - k)|^2 dx + m k^2 \\ &\geq E_1(\Gamma) \int_{\Omega} |u - k|^2 dx + m k^2 = E_1(\Gamma) (1 - 2k \int_{\Omega} u dx + k^2 |\Omega|) + m k^2, \end{aligned}$$

where we have used the fact that the function $u - k$ belongs to the form domain $D(Q_{\Gamma})$ of the operator $-\Delta_D^{\Gamma}$, see (3.3). By the Cauchy-Schwarz inequality we have $|\int_{\Omega} u dx| \leq \sqrt{|\Omega|}$. Consequently

$$Q[\sigma_m, u] \geq E_1(\Gamma) (1 - k \sqrt{|\Omega|})^2 + m k^2.$$

Minimising the right hand side with respect to k then gives

$$Q[\sigma_m, u] \geq \frac{m E_1(\Gamma)}{m + |\Omega| E_1(\Gamma)} \quad \forall u \in \mathcal{F}.$$

This yields (3.18). □

In order to estimate $\Lambda_1(m, \Omega)$ from above by a quantity comparable with the lower bound (3.18) we employ a test function which results from an "interpolation" between a constant function and the eigenfunction φ_1 of $-\Delta_D^\Gamma$ relative to $E_1(\Gamma)$.

Proposition 3.8. *For any $m > 0$ it holds*

$$\Lambda_1(m, \Omega) \leq \frac{2m E_1(\Gamma)}{m + |\Omega| E_1(\Gamma) + \sqrt{(|\Omega| E_1(\Gamma) - m)^2 + 4\gamma_1^2 m E_1(\Gamma)}}, \quad (3.20)$$

where $\gamma_1 = \int_\Omega \varphi_1$.

Proof. Let $\sigma \in \Sigma_m$. We consider a family of test functions given by

$$f_t(x) = |\Omega| (1 - t) \varphi_1(x) + \gamma_1 t, \quad t \geq 0. \quad (3.21)$$

Then $f_t \in H^1(\Omega)$ for all $t \geq 0$. A direct calculation shows that

$$Q[\sigma, f_t] = \frac{E_1(\Gamma) |\Omega| \gamma_1^{-2} (1 - t)^2 + m |\Omega|^{-1} t^2}{1 + (|\Omega| \gamma_1^{-2} - 1)(1 - t)^2}$$

attains its minimum at

$$t_0 = \frac{E_1(\Gamma) |\Omega| + m - \sqrt{(|\Omega| E_1(\Gamma) - m)^2 + 4\gamma_1^2 m E_1(\Gamma)}}{2(|\Omega| - \gamma_1^2) m |\Omega|^{-1}}.$$

Since t_0 solves the equation

$$|\Omega| E_1(\Gamma) (1 - t)^2 = t m |\Omega|^{-1} (|\Omega| - t (|\Omega| - \gamma_1^2)) (1 - t),$$

we find out that

$$\lambda_1(\sigma, \Omega) \leq Q[\sigma, f_{t_0}] = t_0 m |\Omega|^{-1} = \frac{2m E_1(\Gamma)}{m + |\Omega| E_1(\Gamma) + \sqrt{(|\Omega| E_1(\Gamma) - m)^2 + 4\gamma_1^2 m E_1(\Gamma)}}.$$

□

Remark 3.9. The right hand side of (3.20) is obviously larger than the right hand side of (3.18) since

$$\gamma_1^2 = |\Omega| - \frac{1}{2} \int_\Omega \int_\Omega |\varphi_1(x) - \varphi_1(x')|^2 dx dx'.$$

Hence the upper bound (3.20) coincides with the lower bound (3.18) if and only if φ_1 is constant, in other words if and only if $\Gamma = \emptyset$, in which case we have $E_1(\Gamma) = \Lambda_1(m, \Omega) = 0$.

Remark 3.10. Note that in view of (3.16) and (3.17) the estimates (3.18), (3.20) are sharp in the limit $m \rightarrow \infty$ as well as in the limit $m \rightarrow 0$.

Corollary 3.11. *We have*

$$\frac{m E_1(\Gamma)}{m + |\Omega| E_1(\Gamma)} \leq \Lambda_1(m, \Omega) \leq \frac{2m E_1(\Gamma)}{m + |\Omega| E_1(\Gamma)} \quad \forall m > 0.$$

Proof. This follows immediately from Propositions 3.7 and 3.8. □

4. Estimates on $\lambda_1(\sigma, \Omega)$

In this section we are going to study the properties of $\lambda_1(\sigma, \Omega)$ for a fixed σ . This problem has attracted a considerable attention mainly in the case when σ is constant. An extension of the Faber-Krahn inequality, well-known for the Dirichlet-Laplacian, was established first in [Bo1] in dimension two and later in [Da2] for any dimension, see also [BG]. Monotonicity properties of $\lambda_1(\sigma, \Omega)$ with respect to the domain shrinking were studied in [PW, GS]. Various bounds on $\lambda_1(\sigma, \Omega)$ in terms of eigenvalues of Dirichlet and (or) Neumann Laplacian were found in [Ph, Sp1, Sp2].

Our aim is to estimate $\lambda_1(\sigma, \Omega)$ only in terms of σ and the geometric properties of Ω . For this purpose we introduce some notation. Let

$$\delta(x) = \min_{y \in \partial\Omega} |x - y|, \quad x \in \Omega$$

be the distance between a point x and the boundary of Ω , and let

$$R_\Omega = \sup_{x \in \Omega} \delta(x)$$

be the inradius of Ω . Finally, let K_N denote the lowest eigenvalue of the Dirichlet Laplacian on a unit ball in \mathbb{R}^N . It is known that the lowest eigenvalue $\lambda_1^D(\Omega)$ of the Dirichlet-Laplacian on a convex domain Ω can be estimated in terms of the inradius as follows:

$$\frac{1}{4} R_\Omega^{-2} \leq \lambda_1^D(\Omega) \leq K_N R_\Omega^{-2}. \quad (4.1)$$

Here the upper bound follows by scaling and monotonicity of $\lambda_1^D(\Omega)$ with respect to the domain enlarging, while the lower bound is a consequence of the Hardy inequality for Dirichlet-Laplacians on convex domains

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{\delta^2} dx \quad \forall u \in H_0^1(\Omega), \quad (4.2)$$

see e.g. [D, Sect.5.3]. It is well-known that the constant 1/4 on the right hand side of (4.2) is sharp. In order to get an idea how (4.1) should be modified when $\lambda_1^D(\Omega)$ is replaced by $\lambda_1(\sigma, \Omega)$ we will first study the scaling properties of the latter.

Theorem 4.1. *Assume that $\sigma \in L^\infty(\partial\Omega)$ is non-negative. Let $\varepsilon > 0$ and let $\lambda_1(\sigma_\varepsilon, \varepsilon\Omega)$ be the lowest eigenvalue of the Robin Laplacian on the rescaled domain $\varepsilon\Omega$ with $\sigma_\varepsilon(s) = \sigma(s/\varepsilon)$. Then*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) = |\Omega|^{-1} \int_{\partial\Omega} \sigma d\nu. \quad (4.3)$$

Moreover,

$$\lim_{\varepsilon \rightarrow \infty} \varepsilon^2 \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) = E_1(\Gamma), \quad (4.4)$$

where $\Gamma = \text{supp } \sigma$ and $E_1(\Gamma)$ is given by (3.4).

Proof. By a change of variables we obtain

$$\lambda_1(\sigma_\varepsilon, \varepsilon\Omega) = \inf_{u \in H^1(\Omega)} \frac{\varepsilon^{N-2} \int_\Omega |\nabla u|^2 dx + \varepsilon^{N-1} \int_{\partial\Omega} \sigma |u|^2 d\nu}{\varepsilon^N \int_\Omega |u|^2 dx}. \quad (4.5)$$

Lemma 2.2 implies that there exists a sequence of positive minimisers $u_\varepsilon \in H^1(\Omega)$ of problem (4.5). We may suppose that $\|u_\varepsilon\|_{L^2(\Omega)} = 1$ for all $\varepsilon > 0$. Hence

$$\lambda_1(\sigma_\varepsilon, \varepsilon\Omega) = \varepsilon^{-2} \int_\Omega |\nabla u_\varepsilon|^2 dx + \varepsilon^{-1} \int_{\partial\Omega} \sigma |u_\varepsilon|^2 d\nu. \quad (4.6)$$

Consider first the limit $\varepsilon \rightarrow 0$. A simple test function argument with a constant function shows that

$$\varepsilon \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) \leq \frac{\int_{\partial\Omega} \sigma \, d\nu}{|\Omega|} \quad \forall \varepsilon > 0. \quad (4.7)$$

In view of (4.7) and (4.6)

$$\varepsilon \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) = \varepsilon^{-1} \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx + \int_{\partial\Omega} \sigma |u_\varepsilon|^2 \, d\nu \leq \frac{\int_{\partial\Omega} \sigma \, d\nu}{|\Omega|}.$$

We thus have $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let v_ε be a subsequence of u_ε . Since v_ε is bounded in $H^1(\Omega)$, it contains another subsequence (which we still denote by v_ε), such that v_ε converges weakly in $H^1(\Omega)$ to some v . Hence $\|v\|_{L^2(\Omega)}^2 = 1$. Moreover, the weak lower semicontinuity of $\int_{\Omega} |\nabla u|^2$ implies that $\|\nabla v\|_{L^2(\Omega)} = 0$ and therefore $v = 1/\sqrt{|\Omega|}$ almost everywhere in Ω . We thus conclude that $v_\varepsilon \rightarrow v$ in $H^1(\Omega)$. Since this holds for any subsequence of u_ε , we conclude that $u_\varepsilon \rightarrow 1/\sqrt{|\Omega|}$ in $H^1(\Omega)$. By (2.1) it follows that

$$\lim_{\varepsilon \rightarrow 0} \|u - u_\varepsilon\|_{L^2(\partial\Omega)} = 0.$$

Since $\sigma \in L^\infty(\partial\Omega)$, in view of equation (4.6) we then have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) \geq \liminf_{\varepsilon \rightarrow 0} \int_{\partial\Omega} \sigma |u_\varepsilon|^2 \, d\nu = \int_{\partial\Omega} \sigma |u|^2 \, d\nu = \frac{\int_{\partial\Omega} \sigma \, d\nu}{|\Omega|}.$$

This in combination with (4.7) proves (4.3). To prove (4.4) we first note that

$$\varepsilon^2 \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) \leq E_1(\Gamma) \quad \forall \varepsilon > 0, \quad (4.8)$$

which follows by choosing the first eigenfunction φ_1 of the operator $-\Delta_D^\Gamma$ in $L^2(\Omega)$ as a test function in (4.5). Now let w_ε be a subsequence of u_ε . The sequence w_ε is then bounded in $H^1(\Omega)$ as $\varepsilon \rightarrow \infty$, see (4.6). Let w be a weak limit of w_ε (or a suitable subsequence which we still denote by w_ε) in $H^1(\Omega)$. Thus $\|w\|_{L^2(\Omega)} = 1$. From (4.6) and (4.8) we conclude that $\int_{\partial\Omega} \sigma |w_\varepsilon|^2 \, d\nu \rightarrow 0$ as $\varepsilon \rightarrow \infty$. Since $w_\varepsilon \rightarrow w$ strongly in $L^2(\partial\Omega)$, see (2.1), it follows that $\int_{\partial\Omega} \sigma |w|^2 \, d\nu = 0$. Consequently, $w(s) = 0$ for almost every $s \in \Gamma$ which implies that w belongs to the form domain $D(Q_\Gamma)$ of the operator $-\Delta_D^\Gamma$, see (3.3). By the weak lower semicontinuity of $\int_{\Omega} |\nabla u|^2$ and (3.4) we thus conclude that

$$\liminf_{\varepsilon \rightarrow \infty} \left(\int_{\Omega} |\nabla w_\varepsilon|^2 \, dx + \varepsilon \int_{\partial\Omega} \sigma |w_\varepsilon|^2 \, d\nu \right) \geq \int_{\Omega} |\nabla w|^2 \, dx \geq E_1(\Gamma).$$

On the other hand, from (4.6) and (4.8) we get

$$\limsup_{\varepsilon \rightarrow \infty} \left(\int_{\Omega} |\nabla w_\varepsilon|^2 \, dx + \varepsilon \int_{\partial\Omega} \sigma |w_\varepsilon|^2 \, d\nu \right) \leq E_1(\Gamma).$$

Hence $w = \varphi_1$. Since w_ε was arbitrary, we conclude that $u_\varepsilon \rightarrow \varphi_1$ weakly in $H^1(\Omega)$, which implies

$$\liminf_{\varepsilon \rightarrow \infty} \varepsilon^2 \lambda_1(\sigma_\varepsilon, \varepsilon\Omega) \geq \int_{\Omega} |\nabla \varphi_1|^2 \, dx = E_1(\Gamma).$$

In view of (4.8) this yields (4.4). \square

Remark 4.2. The asymptotic behaviour (4.3) appears only when we deal with the first eigenvalue $\lambda_1(\sigma, \Omega)$. In fact, for any $\sigma \geq 0$ we have by the variational principle $\lambda_j^N(\Omega) \leq \lambda_j(\sigma, \Omega) \leq \lambda_j^D(\Omega)$, where $\lambda_j^N(\Omega)$, $\lambda_j^D(\Omega)$ and $\lambda_j(\sigma, \Omega)$ denote the j th eigenvalues of the Neumann, Dirichlet and Robin Laplacian respectively. By scaling

$$\varepsilon^{-2} \lambda_j^N(\Omega) = \lambda_j^N(\varepsilon\Omega) \leq \lambda_j(\sigma_\varepsilon, \varepsilon\Omega) \leq \lambda_j^D(\varepsilon\Omega) = \varepsilon^{-2} \lambda_j^D(\Omega).$$

Since $\lambda_j^N(\Omega) > 0$ whenever $j \geq 2$, it follows that $\lambda_j(\sigma_\varepsilon, \varepsilon\Omega) \asymp \varepsilon^{-2}$ for all $j \geq 2$. This shows that the Robin Laplacian differs from both Dirichlet and Neumann Laplacians in the sense that its lowest eigenvalue scales, when $\varepsilon \rightarrow 0$, in a different way than all the other eigenvalues.

Theorem 4.1 says that $\lambda_1(\sigma, \Omega) \sim R_\Omega^{-1}$ as $R_\Omega \rightarrow 0$ and therefore inequality (4.1) must fail if we replace $\lambda_1^D(\Omega)$ by $\lambda_1(\sigma, \Omega)$.

We are going to prove an analogue of (4.1) for $\lambda_1(\sigma, \Omega)$ in the case when σ is *constant* and Ω is *convex*. For an upper bound we will use the results of the previous section. In order to find an appropriate lower bound we start by proving a modified version of Hardy inequality (4.2).

Lemma 4.3. *Let $\sigma \geq 0$ and assume that Ω is convex. Then the inequality*

$$\int_{\Omega} |\nabla u(x)|^2 dx + \sigma \int_{\partial\Omega} |u(s)|^2 d\nu(s) \geq \alpha\sigma(1 - \alpha\sigma) \int_{\Omega} \frac{|u(x)|^2}{(\delta(x) + \alpha)^2} dx. \quad (4.9)$$

holds true for all $u \in H^1(\Omega)$ and any $\alpha > 0$.

Proof. The inequality is obvious for $\sigma = 0$. Hence we may assume that $\sigma > 0$. In view of the regularity of Ω it suffices to prove (4.9) for all $u \in C^1(\overline{\Omega})$. Since $|\nabla\delta| = 1$ almost everywhere, we have

$$\int_{\Omega} \left| \nabla u - \frac{\alpha\sigma u}{\delta + \alpha} \nabla\delta \right|^2 dx = \int_{\Omega} \left(|\nabla u|^2 + \frac{\alpha^2\sigma^2 u^2}{(\delta + \alpha)^2} - \frac{2\alpha\sigma u}{\delta + \alpha} \nabla u \cdot \nabla\delta \right) dx. \quad (4.10)$$

Moreover, integration by parts gives

$$2 \int_{\Omega} \frac{u}{\delta + \alpha} \nabla u \cdot \nabla\delta dx = \int_{\partial\Omega} \partial_n\delta(s) \frac{u^2(s)}{\alpha} d\nu(s) - \int_{\Omega} \frac{u^2 \Delta\delta}{\delta + \alpha} dx + \int_{\Omega} \frac{u^2}{(\delta + \alpha)^2} dx. \quad (4.11)$$

Recall that $|\partial_n\delta(s)| = 1$. Moreover, from the convexity of Ω follows that δ is concave and therefore $\Delta\delta \leq 0$ in the sense of distributions. Hence inserting (4.11) into (4.10) we get

$$\int_{\Omega} |\nabla u|^2 dx + \sigma \int_{\partial\Omega} u^2(s) d\nu(s) \geq \alpha\sigma(1 - \alpha\sigma) \int_{\Omega} \frac{u^2}{(\delta(x) + \alpha)^2} dx + \int_{\Omega} \left| \nabla u - \frac{\alpha\sigma u}{\delta + \alpha} \nabla\delta \right|^2 dx,$$

which proves the statement. \square

Armed with Lemma 4.3 we can state the following

Theorem 4.4. *Assume that Ω is convex and that $\sigma > 0$ is constant. Then*

$$\frac{1}{4} \frac{\sigma}{R_\Omega(1 + \sigma R_\Omega)} \leq \lambda_1(\sigma, \Omega) \leq 2 K_N \frac{\sigma}{R_\Omega(1 + \sigma R_\Omega)}. \quad (4.12)$$

Remark 4.5. The expression

$$\frac{\sigma}{R_\Omega(1 + \sigma R_\Omega)} \quad (4.13)$$

which appears on both sides of inequality (4.12) is proportional to R_Ω^{-1} for $R_\Omega \rightarrow 0$ and to R_Ω^{-2} for $R_\Omega \rightarrow \infty$. This is in agreement with Theorem 4.1. It is also worth noticing that (4.13) is, just like $\lambda_1(\sigma, \Omega)$, an increasing function of σ , and that in the limit $\sigma \rightarrow \infty$ the two-sided inequality (4.12) turns, up to the multiplicative factor 2, into (4.1).

Proof of Theorem 4.4. By inequality (4.9) we have

$$\lambda_1(\sigma, \Omega) \geq \frac{\alpha\sigma(1 - \alpha\sigma)}{(R_\Omega + \alpha)^2} \quad \forall \alpha > 0.$$

The lower bound in (4.12) then follows by maximising the right hand side of the above inequality with respect to α . As for the upper bound, we apply Theorem 3.3 and Proposition 3.8 with $\Gamma = \partial\Omega$ to obtain

$$\lambda_1(\sigma, \Omega) \leq \Lambda_1(\sigma |\partial\Omega|, \Omega) \leq 2\sigma \left(\frac{|\Omega|}{|\partial\Omega|} + \frac{\sigma}{\lambda_1^D(\Omega)} \right)^{-1}. \quad (4.14)$$

Let us fix a system of coordinates in such a way that the ball $B(o, R_\Omega)$ centred in the origin o satisfies $B(o, R_\Omega) \subseteq \Omega$. As mentioned above, the function $\delta(x)$ is concave on Ω . Hence

$$\nabla\delta(x) \cdot (y - x) \geq \delta(y) - \delta(x)$$

for all $x, y \in \Omega$ for which $\nabla\delta(x)$ exists. Since $\nabla\delta(s) = -n(s)$, we can insert $x = s \in \partial\Omega$ and $y = o$ in the above inequality to find out that $s \cdot n(s) \geq R_\Omega$ almost everywhere on $\partial\Omega$. Consequently, by the Gauss Theorem

$$|\Omega| = \frac{1}{N} \int_\Omega \operatorname{div} x \, dx = \frac{1}{N} \int_{\partial\Omega} s \cdot n(s) \, d\nu(s) \geq |\partial\Omega| \frac{R_\Omega}{N}.$$

This in combination with (4.1) and (4.14) gives

$$\lambda_1(\sigma, \Omega) \leq C_N \frac{\sigma}{R_\Omega(1 + \sigma R_\Omega)}, \quad C_N = 2 \max\{N, K_N\}.$$

Moreover, from the Li-Yau inequality, see [LY] or [LL, p.305], it follows that

$$K_N \geq \frac{4N}{N+2} \Gamma\left(1 + \frac{N}{2}\right)^{\frac{4}{N}},$$

where $\Gamma(\cdot)$ is the Euler gamma functions. By induction we then find out that $K_N \geq N$ for all $N \in \mathbb{N}$, which shows that $C_N = 2K_N$. This completes the proof of the upper bound in (4.12). \square

Remark 4.6. By setting $\alpha = 1/2\sigma$ in (4.9) we obtain

$$\int_\Omega |\nabla u(x)|^2 \, dx + \sigma \int_{\partial\Omega} |u(s)|^2 \, d\nu(s) \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{(\delta(x) + \frac{1}{2\sigma})^2} \, dx, \quad (4.15)$$

which is a special case of [KL, Thm3.1], where a Hardy inequality for Robin-Laplacians with general (not necessarily constant) σ was established. However, inequality (4.15) would not allow us to arrive at the desired lower bound on $\lambda_1(\sigma, \Omega)$. For this reason we need the family of inequalities (4.9) parametrized by α .

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