Regularity for minimal boundaries in $\mathbf{R}^n$ with mean curvature in L^n

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Let $E_0 \subseteq \mathbf{R}^n$ be a minimal set with mean curvature in L^n that is a minimum of the functional $E \mapsto P(E,\Omega) + \int_{E\cap\Omega} H$, where $\Omega \subseteq \mathbf{R}^n$ is open and $H \in L^n(\Omega)$. We prove that if $2 \leq n \leq 7$ then ∂E_0 can be parametrized over the $(n-1)$-dimensional disk with a $C^{0,\alpha}$ mapping with $C^{0,\alpha}$ inverse.

Introduction

We are concerned with the study of the functional

$$F_H(E) = P(E,\Omega) + \int_{E\cap\Omega} H$$

where Ω is an open set of $\mathbf{R}^n$, $H \in L^1(\Omega)$, E is a *Caccioppoli set* (i.e. the distributional derivative $D\chi_E$ of the characteristic function of E is a Radon vector measure) and $P(E,\Omega) = |D\chi_E|$ (the total variation of $D\chi_E$) is the *perimeter* of E in Ω. If E is a local minimizer of F_H (i.e. if $F_H(E) \leq F_H(X)$ for all X such that $E\triangle X$[1] has compact closure in Ω), then we will say that *E is a minimal set with mean curvature H in Ω*. As a matter of fact, if H and ∂E are sufficiently regular near a point $x \in \partial E \cap \Omega$, then $-H(x)/(n-1)$ is the usual mean curvature of ∂E in x.

In 1974 U. Massari [5, 6] proved that if E is a minimal set with mean curvature $H \in L^p$ and $p > n$ then, up to a set of dimension not greater than $n-8$, ∂E is a hypersurface of class $C^{1,\alpha}$ with

[1] Here $E\triangle X = (E\setminus X)\cup(X\setminus E)$ denotes the symmetric difference between E and X.

$\alpha = (p-n)/4p$. Then, in 1987 E. Barozzi, E. H. A. Gonzales and I. Tamanini [1, 2] proved that given any Caccioppoli set E it is possible to find a suitable $H \in L^1$ such that E is a minimal set with mean curvature H. Moreover they noticed that if E is a minimal set with curvature $H \in L^p$ with $1 \le p < n$ then ∂E may contain many singular points. Nothing was said about the case $H \in L^n$ until 1992 when E. De Giorgi conjectured the existence of a singular curve in $\mathbf{R}^2$ which is the boundary of a Caccioppoli set and which is a minimal set with mean curvature $H \in L^2$. This conjecture was proved by E. H. A. Gonzales, U. Massari and I. Tamanini in [4].

A still open conjecture by De Giorgi is that if E is a minimal set with mean curvature $H \in L^n(\Omega)$ then (up to a set of dimension not greater than $n-8$) ∂E can be parametrized by a bi-lipschitz map. The aim of this paper is to give a partial answer to this conjecture proving the following

Theorem 0.1 *Let Ω be an open set of $\mathbf{R}^n$. If $H \in L^n(\Omega)$, E_0 is a minimal set with mean curvature H in Ω and $2 \le n \le 7$, then given any $x_0 \in \partial E_0$ and given $0 < \alpha < 1$ there exists a neighbourhood of x_0 in ∂E in which ∂E_0 may be parametrized by an α-Hölder map with α-Hölder inverse.*

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1 Fundamental tools

To prove this theorem we follow the techniques used by Reifenberg [8] in the study of minimal surfaces. In his work Reifenberg represented surfaces simply as closed sets whose area is the Hausdorff measure and whose boundaries are given by homological properties. A fundamental tool used by Reifenberg is the topological-disk theorem (Theorem 1.2), which gives a sufficient condition for parametric $C^{0,\alpha}$-regularity for a closed set.

To apply Reifenberg's techniques, we take advantage of the theory of Caccioppoli sets. It's well known that a minimum of F_H does exist but, as we have seen, regularity results hold only if $H \in L^p$ with $p > n$. Anyway in [4] many properties for minima are given even in the case $H \in L^n$. In Theorem 1.3 we summarize the properties we need in this paper.

Given X a subset of $\mathbf{R}^n$ and $\rho > 0$ we denote by

$$(X)_\rho = \{y \in \mathbf{R}^n : \exists x \in X \ |x - y| \leq \rho\}$$

the closed ρ-neighbourhood of X.

Definition 1.1 *Let S be a closed set of $\mathbf{R}^n$, $x_0 \in S$, $R > 0$, $\varepsilon > 0$, and $m \leq n$ an integer. We say that S satisfies* (ε, R, m)-Reifenberg property *in x_0 if given any ball $B(x,r) \subseteq B(x_0, R)$ with $x \in S$ there exists an affine m-plane Σ through x such that*

$$\begin{aligned} S \cap B(x,r) &\subseteq (\Sigma \cap B(x,r))_{\varepsilon r}, \\ \Sigma \cap B(x,r) &\subseteq (S \cap B(x,r))_{\varepsilon r}. \end{aligned}$$

The following theorem can be found in [8, Chapter 4, page 64].

Theorem 1.2 (topological disk) *Given $0 < \alpha < 1$ there exists $\varepsilon_0 > 0$ such that if S is a closed set in $\mathbf{R}^n$ satisfying the (ε, R, m)-Reifenberg property in $x_0 \in S$ for some $R > 0$, some integer $0 \leq m \leq n$ and some $0 < \varepsilon < \varepsilon_0$ then there exists a neighbourhood of x_0 in S which is homeomorphic to the m-dimensional disk through an homeomorphism τ such that both τ and τ^{-1} are α-Hölder maps.*

Theorem 1.3 *Let $E \subseteq \mathbf{R}^n$ be a minimal set with mean curvature H in Ω, with $H \in L^n(\Omega)$. Then we can find a closed set E_0 which differs from E by a set of Lebesgue measure zero, which is a minimal set with mean curvature H in $\mathbf{R}^n$ and satisfies the following properties:*

1. *∂E_0 is $(n-1)$-rectifiable and $P(E_0, A) = \mathcal{H}^{n-1}(A \cap \partial E_0)$ for any open set $A \subseteq \mathbf{R}^n$;*

2. *given a sequence of points $x_k \to x_0$ with $x_0, x_k \in \partial E_0 \cap \Omega$, and a sequence $\lambda_k \to \infty$ ($\lambda_k > 0$) there exist a subsequence k_j and a minimal set E_∞ with mean curvature 0 in $\mathbf{R}^n$ such that*

$$\lim_{j \to \infty} \lambda_{k_j}(E_0 - x_{k_j}) = E_\infty$$

in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$, and

$$\lim_{j \to \infty} P(\lambda_{k_j}(E_0 - x_{k_j}), B(0,1)) = P(E_\infty, B(0,1)).$$

Moreover if $n \leq 7$ then E_∞ is an half-space.

Proof:
See Theorem 1.1 and Theorem 1.2 of [4]. □

The following lemma gives a link between minimal surfaces and surfaces with prescribed mean curvature, specifing in which sense the latter may be called *quasi-minimal surfaces.*

Lemma 1.4 *Let $H \in L^n(\Omega)$ and E_0 be a minimal set with mean curvature H in Ω. Then, there exists an increasing function $\omega: \mathbf{R}^+ \to \mathbf{R}^+$ with*

$$\lim_{r \to 0^+} \omega(r) = 0$$

such that if E is a Caccioppoli set and $\overline{E \triangle E_0} \subseteq B(x,r) \subseteq \Omega$ then

$$P(E_0, B(x,r)) \leq P(E, B(x,r)) + r^{n-1}\omega(r).$$

If only $E \triangle E_0 \subseteq B(x,r) \subseteq \Omega$ then

$$P(E_0, \overline{B(x,r)}) \leq P(E, \overline{B(x,r)}) + r^{n-1}\omega(r).$$

Proof:
If $\overline{E \triangle E_0} \subseteq B(x,r)$, in view of the locality of perimeter, one can find an open set which contains $\Omega \setminus B(x,r)$ such that the measures $P(E_0, \cdot)$ and $P(E, \cdot)$ coincide on it. So $P(E_0, \Omega \setminus B(x,r)) = P(E, \Omega \setminus B(x,r))$. From the minimality of E_0 we obtain

$$P(E_0, B(x,r)) + \int_{B(x,r) \cap E_0} H \leq P(E, B(x,r)) + \int_{B(x,r) \cap E} H$$

hence

$$P(E_0, B(x,r)) \leq P(E, B(x,r)) + \int_{B(x,r)} |H|$$

and from Hölder inequality we get

$$\int_{B(x,r)} |H| \leq ||H||_{L^n(B(x,r))} |B(x,r)|^{\frac{n-1}{n}} = \gamma_n^{\frac{n-1}{n}} r^{n-1} ||H||_{L^n(B(x,r))}$$

(in the whole paper we denote by γ_k the measure of the k-dimensional unit ball and use the notation $|X| = \mathcal{H}^n(X)$). So if we set

$$\omega(r) = \sup\{\gamma_n^{\frac{n-1}{n}} ||H||_{L^n(B(x,\rho))} : B(x,\rho) \subseteq \Omega,\ \rho \leq r\} \qquad (1)$$

using $H \in L^n$ we obtain, as needed, that ω is increasing and with limit 0 as $r \to 0$.

If $E \triangle E_0 \subseteq B(x,r)$ we know that $P(E_0, \cdot)$ and $P(E, \cdot)$ coincide on the open set $\Omega \setminus \overline{B(x,r)}$. With similar considerations as before, we obtain the second statement. □

2 Technical Lemmas

Let us fix some notations. Let $\Omega\subseteq\mathbf{R}^n$, $H \in L^n(\Omega)$, E_0 be a minimum of F_H satisfying the properties given in Theorem 1.3 and ω the function described in Lemma 1.4. Let $2 \le n \le 7$ and $m = n-1$. We adopt the following notations

$$\begin{aligned} S_0 &= \partial E_0; \\ \varphi(x,r) &= \mathcal{H}^m(S_0 \cap B(x,r)); \\ \psi(x,r) &= \int_0^r \mathcal{H}^{m-1}(S_0 \cap \partial B(x,t))\,\mathrm{d}t. \end{aligned}$$

Here by $\mathcal{H}^k$ we mean the k-dimensional spherical Hausdorff measure that is

$$\mathcal{H}^k(X) = \lim_{\delta\to 0} \mathcal{H}^k_\delta(X)$$

where

$$\mathcal{H}^k_\delta(X) = \inf\left\{\sum_i \gamma_m r_i^m \colon \{B(x_i,r_i)\}_i \text{ is a covering of } X,\ r_i < \delta\right\}.$$

Notice that Theorem 1.3 was proved for Hausdorff measure, but it also holds for spherical Hausdorff measure as these two measures coincide on rectifiable sets (see [3, 3.2.26]). On the other hand, for a general Caccioppoli set E we will only use the following property

$$P(E,A) \le \mathcal{H}^m(\partial E \cap A) \tag{2}$$

which holds for Hausdorff measure and then also for spherical Hausdorff measure, the second being bigger than the first (see [3, 2.10.2]).

Lemma 2.1 *Let S be a closed set of $\mathbf{R}^n$ and denote by S_t the set of the points of S which have distance t from a given k-dimensional plane Σ where $0 \le k \le n$. Then, for every integer $m \le n$,*

$$\int_0^\infty \mathcal{H}^{m-1}(S_t)\,\mathrm{d}t \le \mathcal{H}^m(S) \tag{3}$$

Proof:
See [7, Theorem 10.2.3]. □

Lemma 2.2 *Given $x \in \Omega$, for almost every r with $B(x,r)\subseteq\Omega$ and every $z \in B(x,r)$ there exists a set E such that*

$$E \triangle E_0 \subseteq B(x,r)$$

and such that for every open set $A \subseteq \mathbf{R}^n$ the following estimate holds

$$\begin{aligned} P(E, \overline{B(x,r)}) &\leq \frac{r_1}{m}\mathcal{H}^{m-1}(S_0 \cap \partial B(x,r) \cap A) \\ &\quad + \frac{r_2}{m}\mathcal{H}^{m-1}(S_0 \cap \partial B(x,r) \setminus A) \end{aligned}$$

where

$$\begin{aligned} r_1 &= \sup\{|y-z| : y \in S_0 \cap \partial B(x,r) \cap A\}; \\ r_2 &= \sup\{|y-z| : y \in (S_0 \cap \partial B(x,r)) \setminus A\}. \end{aligned}$$

In particular, if we choose $A = \mathbf{R}^n$ and $z = x$ we obtain

$$P(E, \overline{B(x,r)}) \leq \frac{r}{m}\mathcal{H}^{m-1}(S_0 \cap \partial B(x,r)).$$

Proof:
As S_0 is an m-rectifible set, for almost every r the set $S_0 \cap \partial B(x,r)$ is $(m-1)$-rectifiable. Let $C = C(z, E_0 \cap \partial B(x,r))$ be the cone over $E_0 \cap \partial B(x,r)$ with vertex z and let $E = (E_0 \setminus B(x,r)) \cup C$. We can easily find that

$$\partial E = (S_0 \setminus B(x,r)) \cup C(z, S_0 \cap \partial B(x,r))$$

so ∂E is an m-rectifiable set with finite m-dimensional measure, that is E is a Caccioppoli set.

Moreover, from (2),

$$\begin{aligned} &P(E, \overline{B(x,r)}) \\ &\quad \leq \mathcal{H}^m(\partial E \cap \overline{B(x,r)}) \\ &\quad = \mathcal{H}^m\left(C(z, S_0 \cap \partial B(x,r))\right) \\ &\quad = \mathcal{H}^m\left(C(z, S_0 \cap \partial B(x,r) \cap A)\right) \\ &\qquad + \mathcal{H}^m\left(C(z, (S_0 \cap \partial B(x,r)) \setminus A)\right) \\ &\quad \leq \frac{r_1}{m}\mathcal{H}^{m-1}(S_0 \cap \partial B(x,r) \cap A) + \frac{r_2}{m}\mathcal{H}^{m-1}((S_0 \cap \partial B(x,r)) \setminus A) \end{aligned}$$

where in the last inequality we use the fact that if $X \subseteq B(z,r)$ is an $(m-1)$-rectifiable set then

$$\mathcal{H}^m\left(C(z,X)\right) \leq \frac{r}{m}\mathcal{H}^{m-1}(X).$$

□

Lemma 2.3 *If $x \in \Omega$ and $B(x,r) \subseteq \Omega$ then*

$$\psi(x,r) \leq \varphi(x,r)$$

and for almost every r such that $B(x,r) \subseteq \Omega$

$$\varphi(x,r) \leq \frac{r\mathcal{H}^{m-1}(S_0 \cap \partial B(x,r))}{m} + r^m \omega(r).$$

Proof:
First inequality is given by Lemma 2.1. To prove the second inequality we consider the set E given by the second part of Lemma 2.2. Then

$$P(E, \overline{B(x,r)}) \leq \frac{r}{m} \mathcal{H}^{m-1}(S_0 \cap \partial B(x,r))$$

and, on the other hand, from Lemma 1.4 we get

$$\begin{aligned} \varphi(x,r) &= \mathcal{H}^m(S_0 \cap B(x,r)) = P(E_0, B(x,r)) \\ &\leq P(E, \overline{B(x,r)}) + r^m \omega(r). \end{aligned}$$

□

Lemma 2.4 *If $B(x,r) \subseteq \Omega$ and $E = E_0 \setminus B(x,r)$ then, for almost every r,*

$$P(E_0, \overline{B(x,r)}) \leq \frac{1 + c_n \omega(r)}{1 - c_n \omega(r)} P(E, \overline{B(x,r)}).$$

where c_n is a constant depending only on n.

Proof:
As in Lemma 1.4 we have (being $E = E_0 \setminus B(x,r)$)

$$P(E_0, \overline{B(x,r)}) \leq P(E, \overline{B(x,r)}) + ||H||_{L^n(B(x,r))} |E_0 \cap B(x,r)|^{\frac{n-1}{n}}.$$

From the isoperimetric inequality we obtain

$$\begin{aligned} |E_0 \cap B(x,r)| &\leq c'_n P(E_0 \cap B(x,r), \mathbf{R}^n)^{\frac{n}{n-1}} \\ &\leq c'_n \left(P(E_0, \overline{B(x,r)}) + P(E, \overline{B(x,r)}) \right)^{\frac{n}{n-1}}. \end{aligned}$$

So

$$\begin{aligned} &||H||_{L^n(B(x,r))} |E_0 \cap B(x,r)|^{\frac{n-1}{n}} \\ &\leq \ ||H||_{L^n(B(x,r))} c''_n \left(P(E_0, \overline{B(x,r)}) + P(E, \overline{B(x,r)}) \right) \end{aligned}$$

and by the definition of ω we have

$$\begin{aligned} P(E_0,\overline{B(x,r)}) &\leq P(E,\overline{B(x,r)}) \\ &\quad +c_n\omega(r)\left(P(E_0,\overline{B(x,r)})+P(E,\overline{B(x,r)})\right) \end{aligned}$$

which is the claim of the lemma. □

Lemma 2.5 *Let $n\leq 7$ and $x_0\in S_0\cap\Omega$. Given $\alpha>0$ there exists $r_0=r_0(\alpha)>0$ such that if $x\in S_0$, $B(x,r)\subseteq B(x_0,r_0)$ then*

$$1-\alpha\leq\frac{\varphi(x,r)}{\gamma_m r^m}\leq 1+\alpha.$$

In particular every point of $S_0\cap\Omega$ has m-density 1 in S_0.

Proof:
Suppose there exist $\alpha>0$, a sequence of points $x_k\to x_0$, $x_k\in S_0$ and a sequence of radii $r_k\to 0$ such that

$$\left|\frac{\mathcal{H}^m(\partial E_0\cap B(x_k,r_k))}{\gamma_m r_k^m}-1\right|\geq\alpha. \tag{4}$$

By Theorem 1.3, up to a subsequence, the sequence $(1/r_k)(E_0-x_k)$ converges to an halfspace E_∞. So we obtain

$$\begin{aligned} \gamma_m &= P(E_\infty,B(0,1))=\lim_{k\to\infty}P((1/r_k)(E_0-x_k),B(0,1)) \\ &= \lim_{k\to\infty}\frac{1}{r_k^m}P(E_0,B(x_k,r_k))=\lim_{k\to\infty}\frac{\mathcal{H}^m(\partial E_0\cap B(x_k,r_k))}{r_k^m} \end{aligned}$$

which is a contradiction with (4). □

Lemma 2.6 *Let E be a closed Caccioppoli set in Ω, $B(x,r)\subseteq\Omega$, Σ an affine hyperplane through x, π_Σ the orthogonal projection onto Σ and $0<\varepsilon<\frac{1}{2}$. If $\partial E\cap\overline{B(x,r)}$ is contained in the interior of $(\Sigma)_{\varepsilon r}$, then either*

$$\pi_\Sigma(\partial E\cap B(x,r))\supseteq\Sigma\cap B(x,(1-\varepsilon)r) \tag{5}$$

or, if $E'=E\setminus B(x,r)$,

$$P(E',\overline{B(x,r)})\leq 4\varepsilon r\mathcal{H}^{m-1}(\partial E\cap\partial B(x,r)). \tag{6}$$

Proof:
Let $x_1, \dots, x_n$ be a system of coordinates such that $\Sigma = \{x_n = 0\}$ and define $E^+ = \{x_n \geq \varepsilon r\} \cap \overline{B(x,r)}$ and $E^- = \{x_n \leq -\varepsilon r\} \cap \overline{B(x,r)}$. From the assumptions on E we know that ∂E itersects neither E^+ nor E^-. So, being ∂E closed and E^+, E^- compact sets, we conclude that ∂E has positive distance from E^+ and E^-. So one of the following statements must hold:

1. $E \supseteq E^-$, $E \cap E^+ = \emptyset$;
2. $E \supseteq E^+$, $E \cap E^- = \emptyset$;
3. $E \cap E^+ = \emptyset$, $E \cap E^- = \emptyset$;
4. $E \supseteq E^+ \cup E^-$.

Suppose that 1 is true, than we claim that (5) holds. Let $\Sigma^+ = \{x_n = \varepsilon r\} \cap \overline{B(x,r)}$ and $\Sigma^- = \{x_n = -\varepsilon r\} \cap \overline{B(x,r)}$. Given $y \in \Sigma \cap B(x,(1-\varepsilon)r)$ denote by y^+ and y^- respectively the intersections of $\pi_\Sigma^{-1}(x)$ with Σ^+ and Σ^-. Since $y^- \in E$ while $y^+ \notin E$ there must certainly be a point in the segment $[x^-, x^+]$ which belongs to ∂E (and that is contained in $B(x,r)$) so $\pi_\Sigma(\partial E \cap B(x,r)) \ni y$.

If statement 2 holds with a symmetry with respect to Σ we obtain (5) again.

Suppose that 3 is true. This time we claim that (6) holds. Let $E' = E \setminus B(x,r)$. Obviously E' is a closed Caccioppoli set, $E \triangle E' \subseteq B(x,r)$ and we get

$$\partial E' \cap \overline{B(x,r)} = \partial E' \cap \partial B(x,r) = E \cap \partial B(x,r). \tag{7}$$

Consider now the projection

$$\pi \colon \partial B(x,r) \cap (\Sigma)_{\varepsilon r} \to \partial B(x,r) \cap \Sigma$$

which maps a point z to the point of $\partial B(x,r) \cap \Sigma$ with minimal distance from z. Then we have

$$\pi(E \cap \partial B(x,r)) \subseteq \pi(\partial E \cap \partial B(x,r))$$

so that

$$\begin{aligned} \mathcal{H}^m(E \cap \partial B(x,r)) &\leq \mathcal{H}^m(\pi^{-1}(\pi(\partial E \cap \partial B(x,r)))) \\ &\leq 2\varepsilon r \mathcal{H}^{m-1}(\pi(\partial E \cap \partial B(x,r))) \\ &\leq 4\varepsilon r \mathcal{H}^{m-1}(\partial E \cap \partial B(x,r)). \end{aligned} \tag{8}$$

We conclude, using (7) and (8),

$$\begin{aligned} P(E', \overline{B(x,r)}) &\le \mathcal{H}^m(\partial E' \cap \overline{B(x,r)}) \\ &= \mathcal{H}^m(E \cap \partial B(x,r)) \le 4\varepsilon r \mathcal{H}^{m-1}(\partial E \cap \partial B(x,r)). \end{aligned}$$

If statement 4 holds the proof is that of case 3 where instead of E we consider its complementary set. □

Lemma 2.7 *Let S_0 be a closed set, $\theta > 0$, $e > 0$, $\alpha > 0$ and $0 < \varepsilon < 1 - \cos\theta$. Let $B(x,r)$ be a ball such that given any arbitrarily small $\delta > 0$ there exist a family of disjoint balls $\{B(x_i,r_i)\}$ and a family of m-dimensional affine planes $\{\Sigma_i\}$ with $x_i \in \Sigma_i$ such that every plane Σ_i makes an angle greater than θ with the segment $[x_i;x]$ and, moreover, the following properties hold*

$$\begin{aligned} B(x_i,r_i) &\subseteq B(x,r) \\ r_i &< \delta \\ \varphi(x_i,r_i) &\ge (1-\alpha)\gamma_m r_i^m \\ \sum_i \gamma_m r_i^m &> e \\ S_0 \cap B(x_i,r_i) &\subseteq (\Sigma_i)_{\varepsilon r_i}. \end{aligned}$$

Then

$$\psi(x,r) \le \frac{\varphi(x,r)}{1-\alpha} - e\left(\frac{1-\cos\theta-\varepsilon}{2}\right)^m.$$

Proof:
(See [8, Lemma 7*]). Suppose Σ is an m-dimensional plane and that y_1 and y_2 are two points of Σ at a distance l apart: define $d(l,\rho,t)$ and $\sigma(l,\rho,t)$ to be respectively the diameter and the $\mathcal{H}^{m-1}$ measure of the set $\Sigma \cap B(y_1,\rho) \cap \partial B(y_2,t)$. It follows immediately from the definition of the symbols concerned that

$$\int_0^r \sigma(l,\rho,t)\,\mathrm{d}t = \mathcal{H}^m(\Sigma \cap B(y_1,\rho) \cap B(y_2,r)), \tag{9}$$

and that for $\delta > \rho$

$$H_\delta^{m-1} B(y_1,\rho) \cap \partial B(y_2,t) \le \gamma_{m-1}(\frac{1}{2}d(l,\rho,t))^{m-1} \le \sigma(l,\rho,t). \tag{10}$$

From the definition of spherical Hausdorff measure we can choose a set of balls $B(x_j',r_j')$ such that for all j

$$r_j' < \delta$$

$$
\begin{aligned}
S_0 \cap B(x,r) \setminus \bigcup_i (S_0 \cap B(x_i,r_i)) &\subseteq \bigcup_i B(x'_j, r'_j) \\
\mathcal{H}^m\Big((S_0 \cap B(x,r)) \setminus \bigcup_i (S_0 \cap B(x_i,r_i))\Big) + \delta &\geq \sum_j \gamma_m {r'_j}^m.
\end{aligned}
$$

By (10) and (9) for any m-plane $\Sigma \supseteq \{x, x'_j\}$

$$
\begin{aligned}
&\int_0^r \mathcal{H}^{m-1}_\delta(S_0 \cap \partial B(x,t) \cap B(x'_j, r'_j))\,\mathrm{d}t \qquad (11)\\
&\leq \int_0^r \mathcal{H}^{m-1}_\delta(\partial B(x,t) \cap B(x_j, r'_j))\,\mathrm{d}t \\
&\leq \int_0^r \sigma(|x - x'_j|, r'_j, t)\,\mathrm{d}t \\
&= \mathcal{H}^m(\Sigma \cap B(x'_j, r'_j) \cap B(x,r)) \\
&\leq \mathcal{H}^m(\Sigma \cap B(x'_j, r'_j)) = \gamma_m {r'_j}^m.
\end{aligned}
$$

Consider now any i such that $|x_i - x| \geq r_i$. If we let $R = |x - x_i| - r_i \cos\theta - r_i \varepsilon$ we obtain that $S_0 \cap B(x_i, r_i) \cap B(x, R) = \emptyset$ so, for any m-plane $\Sigma \supseteq \{x, x_i\}$ and by (10) and (9), we obtain

$$
\begin{aligned}
&\int_0^r \mathcal{H}^{m-1}_\delta(S_0 \cap \partial B(x,t) \cap B(x_i, r_i))\,\mathrm{d}t \\
&\leq \int_R^r \mathcal{H}^{m-1}_\delta(\partial B(x,t) \cap B(x_i, r_i))\,\mathrm{d}t \\
&\leq \int_R^r \mathcal{H}^{m-1}(\partial B(x,t) \cap B(x_i, r_i))\,\mathrm{d}t \\
&= \int_R^r \sigma(|x - x_i|, r_i, t)\,\mathrm{d}t.
\end{aligned}
$$

As the set $B(x,R) \cap B(x_i, r_i)$ contains a ball of radius $\frac{1}{2}(1 - \cos\theta - \varepsilon) r_i$ we get

$$
\begin{aligned}
\int_0^R \sigma(|x - x_i|, r_i, t)\,\mathrm{d}t &= \mathcal{H}^m(\Sigma \cap B(x_i, r_i) \cap B(x, R)) \\
&\geq \gamma_m \left(\frac{1 - \cos\theta - \varepsilon}{2}\right)^m r_i^m
\end{aligned}
$$

and, on the other hand,

$$
\int_0^r \sigma(|x - x_i, r_i, t)\,\mathrm{d}t = \mathcal{H}^m(\Sigma \cap B(x_i, r_i) \cap B(x,t)) \leq \gamma_m r_i^m
$$

so that

$$
\begin{aligned}
&\int_0^r \mathcal{H}^{m-1}_\delta(S_0 \cap \partial B(x,t) \cap B(x_i, r_i))\,\mathrm{d}t \\
&\leq \gamma_m r_i^m - \gamma_m \left(\frac{1 - \cos\theta - \varepsilon}{2}\right)^m r_i^m.
\end{aligned}
$$

Since the balls $B(x_i, r_i)$ are disjoint, there is at most one $i = i_0$ such that $|x - x_i| < r_i \leq \delta$ and we obtain, as in (11)

$$\int_0^r \mathcal{H}_\delta^{m-1}(S_0 \cap \partial B(x,t) \cap B(x_{i_0}, r_{i_0}))\,\mathrm{d}t \leq \gamma_m r_{i_0}^m \leq \gamma_m \delta^m.$$

Thus

$$\begin{aligned}
&\int_0^r \mathcal{H}_\delta^{m-1}(S_0 \cap \partial B(x,t))\,\mathrm{d}t \\
&= \int_0^r \mathcal{H}_\delta^{m-1}\Big((\bigcup_j B(x'_j, r'_j) \cup \bigcup_i B(x_i, r_i)) \cap S_0 \cap \partial B(x,t)\Big)\,\mathrm{d}t \\
&\leq \sum_j \int_0^r \mathcal{H}_\delta^{m-1}(S_0 \cap \partial B(x,t) \cap B(x'_j, r'_j))\,\mathrm{d}t \\
&\quad + \sum_i \int_0^r \mathcal{H}_\delta^{m-1}(S_0 \cap \partial B(x,t) \cap B(x_i, r_i))\,\mathrm{d}t \\
&\leq \sum_j \gamma_m {r'_j}^m + \sum_i \gamma_m r_i^m \left(1 - \left(\frac{1 - \cos\theta - \varepsilon}{2}\right)^m\right) + \gamma_m \delta^m \\
&\leq \mathcal{H}^m(S_0 \cap B(x,r) \setminus \bigcup_i (S_0 \cap B(x_i, r_i))) + \delta \\
&\quad + \frac{1}{1-\alpha}\mathcal{H}^m(\bigcup_i (S_0 \cap B(x_i, r_i))) - e\left(\frac{1 - \cos\theta - \varepsilon}{2}\right)^m + \gamma_m \delta^m \\
&\leq \frac{\mathcal{H}^m(S_0 \cap B(x,r))}{1-\alpha} - e\left(\frac{1 - \cos\theta - \varepsilon}{2}\right)^m + \delta + \gamma_m \delta^m
\end{aligned}$$

and letting $\delta \to 0$ we conclude the proof.

□

3 A fundamental Lemma

Lemma 3.1 *Given $\varepsilon > 0$, there exist positive numbers $R_0 = R_0(\omega, \varepsilon, n)$, $\lambda = \lambda(n, \varepsilon)$, $\alpha = \alpha(n, \varepsilon)$ such that for all $0 < r_0 < R_0$, $x_0 \in S_0$ and $B(x_0, r_0) \subseteq \Omega$ the following assertion holds: if the inequality*

$$1 - \alpha \leq \frac{\psi(x,r)}{\gamma_m r^m} \leq \frac{\varphi(x,r)}{\gamma_m r^m} \leq 1 + \alpha, \tag{12}$$

is true for each ball $B(x,r) \subseteq B(x_0, r_0)$ with center $x \in S_0$, then there exist a point $x^* \in S_0 \cap B(x_0, r_0)$ and a hyperplane Σ through x^* such that $B(x^*, \lambda r_0) \subseteq B(x_0, r_0)$ and

$$S_0 \cap B(x^*, \lambda r_0) \subseteq (\Sigma)_{\varepsilon \lambda r_0}.$$

Proof:
Suppose $\varepsilon < 1$ and choose

$$\alpha = \left(\frac{\varepsilon^m}{64 \cdot 8^m}\right)^6 \tag{13}$$

and R_0 such that $\omega(R_0) \leq \gamma_m \alpha$.

Step one. Let $r_1 = \lambda_1 r_0$ with $\lambda_1 = \frac{4^m}{2(16\sqrt{n})^n}$. We will estimate how many disjoint balls of radius $2r_1$ can fit into $B(x_0, r_0/4)$. First of all note that $B(x_0, r_0/4)$ contains an n-cube K with side $\frac{r_0}{2\sqrt{n}}$ while a ball of radius $2r_1$ is contained in an n-cube of side $4r_1$. So, splitting the cube K into small cubes of side $4r_1$, and putting one ball into every small cube we find at least

$$\left\lfloor \frac{\frac{r_0}{\sqrt{n}}}{8r_1} \right\rfloor^n \geq \left(\frac{\frac{r_0}{\sqrt{n}}}{16r_1}\right)^n = (16\sqrt{n}\frac{r_1}{r_0})^{-n}$$

disjoint balls of radius $2r_1$ that fit into $B(x_0, r_0/4)$. Let $\{x_i\}$ be the set of the centers of these balls. We have determined the radius r_1 such that at least one of the balls $B(x_i, r_1)$ does not intersect S_0. In fact, suppose that for every i there exists $y_i \in B(x_i, r_1) \cap S_0$. $B(y_i, r_1)$ are disjoint balls all contained in $B(x_0, r_0/4)$, so we get (using (12))

$$\begin{aligned}
(1+\alpha)\gamma_m \left(\frac{r_0}{4}\right)^m &\geq \mathcal{H}^m(S_0 \cap B(x_0, \frac{1}{4}r_0)) \\
&\geq \sum_i \mathcal{H}^m(S_0 \cap B(y_i, r_1)) \\
&\geq \sum_i (1-\alpha)\gamma_m r_1^m \\
&\geq \left(16\sqrt{n}\frac{r_1}{r_0}\right)^{-n} (1-\alpha)\gamma_m r_1^m.
\end{aligned}$$

which is a contradiction being $\alpha < 1/3$.

So, we have found a ball $B(y, r_1) \subseteq B(x_0, r_0/2)$ which does not intersect S_0. Let

$$r_2 = \max\{r \in \mathbf{R} \quad : \quad B(y, r) \cap S_0 = \emptyset\}$$

so that $B(y, r_2) \cap S_0 = \emptyset$ while there exists a point $x^* \in S_0 \cap \partial B(y, r_2)$. Since $x_0 \in S_0$ we have $r_2 \leq |x_0 - y| \leq r_0/4$ so $B(y, r_2) \subseteq B(x_0, r_0/2)$.

Step two. Define

$$r_3 = \alpha^{\frac{1}{3}} r_2$$

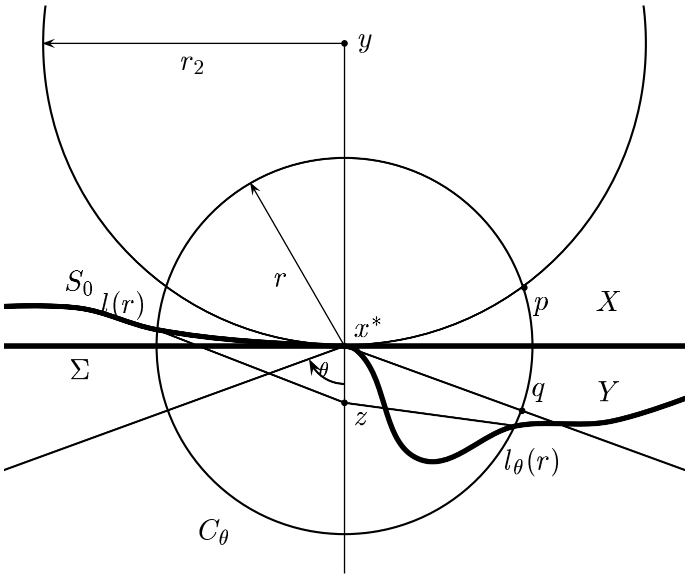

Figure 1: The surface S_0, the plane Σ and the Cone C_θ.

$$\begin{aligned} r_4 &= \alpha^{\frac{1}{6}} r_3 \\ d &= \alpha^{\frac{2}{3}} r_3 \\ \theta &= \arccos(\alpha^{\frac{1}{6}}). \end{aligned}$$

Let Σ be the m-dimensional plane tangent to $B(y, r_2)$ in x^* (see Figure 1), consider the cone C_θ with vertex x^* and amplitude θ defined as follows

$$C_\theta = \{v \in \mathbf{R}^n \colon (v - x^*) \cdot (x^* - y) \geq |v - x^*||x^* - y| \cos\theta\}$$

and take a point $z \in C_\theta$ lying on the line x^*y and at a distance d from x^*. For a radius $r \in [r_4, r_3]$ define

$$\begin{aligned} l(r) &= S_0 \cap \partial B(x^*, r); \\ l_\theta(r) &= S_0 \cap \partial B(x^*, r) \cap C_\theta; \\ d_1(r) &= \sup\{|z - x| \colon x \in l(r) \setminus l_\theta(r)\}; \\ d_2(r) &= \sup\{|z - x| \colon x \in l_\theta(r)\}. \end{aligned}$$

For almost every $r \in [r_4, r_3]$, in view of Lemma 2.2 (with $A = \mathbf{R}^n \setminus C_\theta$), we can find a Caccioppoli set E such that

$$P(E, \overline{B(x^*, r)}) \leq \frac{d_1(r)}{m} \mathcal{H}^{m-1}(l(r) \setminus l_\theta(r)) + \frac{d_2(r)}{m} \mathcal{H}^{m-1}(l_\theta(r))$$

and from Lemma 1.4 we obtain

$$\begin{aligned} m\mathcal{H}^m(S_0 \cap B(x^*, r)) &= mP(E_0, B(x^*, r)) \\ &\leq mP(E, B(x^*, r)) + mr^m\omega(r) \\ &\leq d_1(r)\mathcal{H}^{m-1}(l(r) \setminus l_\theta(r)) + d_2(r)\mathcal{H}^{m-1}(l_\theta(r)) + mr^m\omega(r). \end{aligned}$$

Given any point $p \in \partial B(y, r_2) \cap \partial B(x^*, r)$ and $q \in \partial C_\theta \cap \partial B(x^*, r)$, with simple geometric considerations we deduce

$$\begin{aligned} d_1(r) &\leq |p - z| = \sqrt{r^2 + d^2 + d\frac{r^2}{r_2}} \leq r\sqrt{1 + \frac{d^2}{r_4^2} + \frac{d}{r_2}} \\ &= r\sqrt{1 + 2\alpha} \leq r(1 + \alpha), \\ d_2(r) &\leq |q - z| = \sqrt{r^2 - 2dr\cos\theta + d^2} \leq r\sqrt{1 - 2\frac{d}{r_3}\cos\theta + \frac{d^2}{r_4^2}} \\ &= r\sqrt{1 - 2\alpha^{\frac{5}{6}} + \alpha} \leq r\sqrt{1 - \alpha^{\frac{5}{6}}} \leq r(1 - \frac{1}{2}\alpha^{\frac{5}{6}}), \end{aligned}$$

so that we obtain

$$\begin{aligned} & m\mathcal{H}^m(S_0 \cap B(x^*, r)) - mr^m\omega(r) \\ &\leq r(1 + \alpha)\mathcal{H}^{m-1}(l(r) \setminus l_\theta(r)) + r(1 - \frac{1}{2}\alpha^{\frac{5}{6}})\mathcal{H}^{m-1}(l_\theta(r)) \\ &= r(1 + \alpha)\mathcal{H}^{m-1}(l(r)) - r(1 + \alpha - 1 + \frac{1}{2}\alpha^{\frac{5}{6}})\mathcal{H}^{m-1}(l_\theta(r)) \\ &\leq r(1 + \alpha)\mathcal{H}^{m-1}(l(r)) - \frac{1}{2}r\alpha^{\frac{5}{6}}\mathcal{H}^{m-1}(l_\theta(r)). \end{aligned}$$

Dividing by r, integrating and using hypothesis (12) we obtain

$$\begin{aligned} & \frac{1}{2}\alpha^{\frac{5}{6}} \int_{r_4}^{r_3} \mathcal{H}^{m-1}(l_\theta(r))\,\mathrm{d}r \\ \leq\ & (1 + \alpha) \int_{r_4}^{r_3} \mathcal{H}^{m-1}(l(r))\,\mathrm{d}r \\ & - \int_{r_4}^{r_3} \frac{m}{r}\Big(\mathcal{H}^m(S_0 \cap B(x^*, r)) - r^m\omega(r)\Big)\,\mathrm{d}r \\ \leq\ & (1 + \alpha)\Big((1 + \alpha)\gamma_m r_3^m - (1 - \alpha)\gamma_m r_4^m\Big) \\ & - (1 - \alpha - \frac{\omega(R_0)}{\gamma_m})\gamma_m(r_3^m - r_4^m) \\ =\ & (1 + \alpha)(1 + \alpha - \alpha^{\frac{m}{6}} + \alpha^{1+\frac{m}{6}})\gamma_m r_3^m \\ & - (1 - \alpha - \frac{\omega(R_0)}{\gamma_m} - \alpha^{\frac{m}{6}} + \alpha^{1+\frac{m}{6}} + \alpha^{\frac{m}{6}}\frac{\omega(R_0)}{\gamma_m})\gamma_m r_3^m \\ =\ & (3\alpha + \alpha^2 + \alpha^{2+\frac{m}{6}} + \frac{\omega(R_0)}{\gamma_m}(1 - \alpha^{\frac{m}{6}}) - \alpha^{1+\frac{m}{6}})\gamma_m r_3^m \leq 6\alpha\gamma_m r_3^m \end{aligned}$$

that is

$$\int_{r_4}^{r_3} \mathcal{H}^{m-1}(l_\theta(r))\,\mathrm{d}r \leq 12\alpha^{\frac{1}{6}}\gamma_m r_3^m.$$

So, by Lemma 2.1 and (12)

$$\begin{aligned}
&\mathcal{H}^m(S_0 \cap B(x^*, r_3) \cap C_\theta) \qquad (14)\\
&= \mathcal{H}^m(S_0 \cap B(x^*, r_3)) - \mathcal{H}^m((S_0 \cap B(x^*, r_3)) \setminus C_\theta)\\
&\leq (1+\alpha)\gamma_m r_3^m - \int_0^{r_3} \mathcal{H}^{m-1}(l(r) \setminus l_\theta(r))\,\mathrm{d}r\\
&= (1+\alpha)\gamma_m r_3^m - \int_0^{r_3} \mathcal{H}^{m-1}(l(r))\,\mathrm{d}r\\
&\quad + \int_{r_4}^{r_3} \mathcal{H}^{m-1}(l_\theta(r))\,\mathrm{d}r + \int_0^{r_4} \mathcal{H}^{m-1}(l_\theta(r))\,\mathrm{d}r\\
&\leq (1+\alpha)\gamma_m r_3^m - (1-\alpha)\gamma_m r_3^m + 12\alpha^{\frac{1}{6}}\gamma_m r_3^m + \alpha^{\frac{m}{6}}(1+\alpha)\gamma_m r_3^m\\
&\leq 16\alpha^{\frac{1}{6}}\gamma_m r_3^m.
\end{aligned}$$

Step three. Now we claim that every point $w \in S_0 \cap B(x^*, r_3/2)$ is not further than $\frac{1}{4}\varepsilon r_3$ from Σ. First of all note that Σ divides the space $\mathbf{R}^n$ into two halfspaces X and Y, one of which (suppose X) contains the ball $B(y, r_2)$ while the other (Y) contains the cone C_θ.

Suppose $w \in X$. Since the ball $B(y, r_2)$ doesn't intersect S_0, if we consider a point $p \in \partial B(x^*, r_3/2) \cap \partial B(y, r_2)$, with simple geometrical considerations we obtain

$$d(w, \Sigma) \leq d(p, \Sigma) = \frac{1}{2}\frac{(\frac{r_3}{2})^2}{r_2} = \frac{1}{8}\alpha^{\frac{1}{3}} r_3 \leq \frac{1}{4}\varepsilon r_3.$$

Consider now $w \in Y$ and suppose $d(w, \Sigma) > \frac{1}{4}\varepsilon r_3$. Choose $q \in \partial B(x^*, r_3/2) \cap \partial C_\theta$ and denote by Σ' the hyperplane parallel to Σ through q. Again with a geometrical argument we obtain

$$\begin{aligned}
d(w, \partial C_\theta) &\geq d(w, \Sigma') \geq \frac{1}{4}\varepsilon r_3 - \frac{1}{2} r_3 \cos\theta\\
&= (\frac{1}{4}\varepsilon - \frac{1}{2}\cos\theta) r_3
\end{aligned}$$

so that

$$B(w, (\frac{1}{4}\varepsilon - \frac{1}{2}\cos\theta) r_3) \subseteq B(x^*, r_3) \cap C_\theta.$$

From inequality (14) we obtain (using also (12))

$$\gamma_m (1-\alpha) r_3^m (\frac{1}{4}\varepsilon - \frac{1}{2}\cos\theta)^m$$

$$\begin{aligned}
&\leq \mathcal{H}^m\left(S_0 \cap B(w, (\frac{1}{4}\varepsilon - \frac{1}{2}\cos\theta) r_3)\right)\\
&\leq \mathcal{H}^m(S_0 \cap B(x^*, r_3) \cap C_\theta)\\
&\leq 16\alpha^{\frac{1}{6}}\gamma_m r_3^m
\end{aligned}$$

so that (recall (13))

$$32\alpha^{\frac{1}{6}} = \frac{\varepsilon^m}{2\cdot 8^m} \leq (1-\alpha)(\frac{1}{4}\varepsilon - \frac{1}{2}\alpha^{\frac{1}{6}})^m \leq 16\alpha^{\frac{1}{6}}$$

which is a contradiction.

Thus, we've proved that $S_0 \cap B(x^*, r_3/2)$ is within a distance of $\frac{1}{4}\varepsilon r_3$ from Σ that is the lemma for $\lambda = \frac{1}{2}\lambda_1\alpha^{\frac{1}{3}}$. □

4 Main theorem

In view of Theorem 1.2 to prove Theorem 0.1 we only need the following

Theorem 4.1 *If $x_0 \in S_0 = \partial E_0$ and $\varepsilon_0 > 0$ then there exist $R > 0$, $\varepsilon' < \varepsilon_0$ such that $(\varepsilon', R, n-1)$-Reifenberg property holds in x_0 for the set S_0.*

Proof:
Recall that $m = n-1$ and choose positive numbers ε', ε, λ, α and r_0 such that the following hypotheses hold:

$$\begin{array}{rcl}
\varepsilon' &<& \frac{1}{32\cdot 2^m}\\
\varepsilon' &<& \varepsilon_0\\
\varepsilon &=& \left(\frac{\varepsilon'}{320(m+1)(m+2)}\right)^2\\
\lambda &=& \lambda(n,\varepsilon)\\
\alpha &<& \alpha(n,\varepsilon)\\
\alpha &<& \frac{1}{12(m+3)}\left(\frac{\lambda\varepsilon}{4}\right)^m
\end{array}
\qquad
\begin{array}{rcl}
\alpha &<& \frac{1}{5}\\
r_0 &<& r_0(\alpha/2)\\
\omega(r_0) &<& \gamma_m\frac{\alpha}{2}\\
\omega(r_0) &<& \frac{1}{3c_n}\\
r_0 &<& R_0(\omega,\varepsilon,n)
\end{array}$$

where $\lambda(n,\varepsilon)$, $\alpha(n,\varepsilon)$ and $R_0(\omega,\varepsilon,n)$ are the constants given by Lemma 3.1, c_n is the constant given by Lemma 2.4 and $r_0(\alpha/2)$ is the constant given in Lemma 2.5.

By Lemma 2.5 for any ball $B(x,r) \subseteq B(x_0, r_0)$ with center $x \in S_0$ we have

$$1 - \frac{\alpha}{2} \leq \frac{\varphi(x,r)}{\gamma_m r^m} \leq 1 + \frac{\alpha}{2}.$$

On the other hand, by Lemma 2.3

$$
\begin{aligned}
\psi(x,r) &= \int_0^r \mathcal{H}^{m-1}(S_0 \cap \partial B(x,t))\,\mathrm{d}t \\
&\geq \int_0^r \frac{m}{t}\mathcal{H}^m(S_0 \cap B(x,t)) - m\omega(t)t^{m-1}\,\mathrm{d}t \\
&\geq (1-\frac{\alpha}{2})\gamma_m r^m - \omega(r_0) r^m \\
&= \left(1 - \frac{\alpha}{2} - \frac{\omega(r_0)}{\gamma_m}\right)\gamma_m r^m
\end{aligned}
$$

and hence the following holds:

$$
1-\alpha \leq \frac{\psi(x,r)}{\gamma_m r^m} \leq \frac{\varphi(x,r)}{\gamma_m r^m} \leq 1+\alpha. \tag{15}
$$

By Lemma 3.1, given any ball $B(x,r) \subseteq B(x_0,r_0)$ with center $x \in S_0$ there exist a point $x^* \in S_0$ and an m-dimensional plane Σ through x^* such that

$$
\begin{aligned}
S_0 \cap B(x^*,\lambda r) &\subseteq (\Sigma)_{\varepsilon\lambda r} \\
B(x^*,\lambda r) &\subseteq B(x,r).
\end{aligned}
$$

Now take any ball $B(x',r') \subseteq B(x_0,r_0)$ with $x' \in S_0$ and consider $m+2$ points $y_1, \ldots,\ y_{m+2}$ contained in $S_0 \cap B(x',r'/4)$. By well known convering theorems, given $\delta > 0$ we can find a finite family $\{B(x_{i\delta},\rho_{i\delta})\}_{i\in I_\delta}$ of disjoint balls with center on S_0, contained in $B(x',r'/4)$ and with radius $\rho_{i\delta} < \delta$, such that

$$
\sum_{i\in I_\delta} \mathcal{H}^m(S_0 \cap B(x_{i\delta},\rho_{i\delta})) \geq \frac{1}{2}\mathcal{H}^m(S_0 \cap B(x',r'/8)).
$$

Thus, by (15)

$$
(1+\alpha)\sum_i \gamma_m \rho_{i\delta}^m \geq \frac{1}{2}\mathcal{H}^m(S_0 \cap B(x',r'/8)) \geq \frac{(1-\alpha)\gamma_m}{2}\left(\frac{r'}{8}\right)^m. \tag{16}
$$

Obviously for each $j \leq m+2$ and for each $i \in I_\delta$

$$
B(x_{i\delta},\rho_{i\delta}) \subseteq B(x',r'/4) \subseteq B(y_j,r'/2) \subseteq B(x',r')
$$

so by Lemma 3.1 we can find points $x_{i\delta}^* \in S_0$ and m-dimensional planes $\Sigma_{i\delta}$ such that

$$
\begin{aligned}
S_0 \cap B(x_{i\delta}^*,\lambda\rho_{i\delta}) &\subseteq (\Sigma_{i\delta})_{\varepsilon\lambda\rho_{i\delta}} \\
B(x_{i\delta}^*,\lambda\rho_{i\delta}) &\subseteq B(x_{i\delta},\rho_{i\delta}).
\end{aligned}
$$

Choose θ so that $1-\cos\theta = 3\varepsilon$, and let $I_{j\delta}$ be the set of indices i such that the segment $[x^*_{i\delta}, y_j]$ makes an angle greater than θ with $\Sigma_{i\delta}$. Let

$$e = \frac{\gamma_m \lambda^m}{4(m+3)(1-\alpha)} \left(\frac{r'}{8}\right)^m .$$

If for fixed j there were arbitrarily small δ such that

$$\sum_{i\in I_{j\delta}} \gamma_m \rho_{i\delta}^m \lambda^m > e$$

by Lemma 2.7 we had

$$\psi(y_j, r'/2) \le \frac{\varphi(y_j, r'/2)}{1-\alpha} - e\left(\frac{1-\cos\theta-\varepsilon}{2}\right)^m$$

hence by (15)

$$\begin{aligned}(1-\alpha)\gamma_m\left(\frac{r'}{2}\right)^m &\le \psi(y_j, r'/2) \le \frac{\varphi(y_j, r'/2)}{1-\alpha} - e\varepsilon^m \\ &\le \frac{1+\alpha}{1-\alpha}\left(\frac{r'}{2}\right)^m \gamma_m - e\varepsilon^m\end{aligned}$$

that is

$$12\alpha(m+3) \ge \left(\frac{\lambda\varepsilon}{4}\right)^m .$$

But by hypotheses on α the previous inequality is false so we deduce that for a sufficiently small δ (using also (16))

$$\begin{aligned}\sum_{i\in I_{j\delta}} \gamma_m \rho_{i\delta}^m \lambda^m &\le \frac{\gamma_m \lambda^m}{4(m+3)(1-\alpha)}\left(\frac{r'}{8}\right)^m \\ &\le \frac{\lambda^m (1+\alpha)}{2(1-\alpha)^2(m+3)} \sum_{i\in I_\delta} \gamma_m \rho_{i\delta}^m\end{aligned}$$

so that (since $\alpha < \frac{1}{5}$)

$$\sum_{i\in I_{j\delta}} \rho_{i\delta}^m \le \frac{1}{m+3}\sum_{i\in I_\delta} \rho_{i\delta}^m$$

and

$$\sum_{j=1}^{m+2}\sum_{i\in I_{j\delta}} \rho_{i\delta}^m < \sum_{i\in I_\delta} \rho_{i\delta}^m .$$

Therefore there exists an index $i_0 \in I_\delta$ contained in none of the sets $I_{j\delta}$. So we have proved that given any $m+2$ points y_j of $S_0 \cap B(x', r'/4)$ it is possible to find an m-dimensional plane $\Sigma = \Sigma_{i_0\delta}$ such that every y_j lies within a distance of $\frac{1}{2} r' \sin\theta$ from Σ. Being $1 - \cos\theta = 3\varepsilon$ we have

$$\begin{aligned} \frac{1}{2} r' \sin\theta &= \frac{1}{2} r' \sqrt{1-\cos^2\theta} = \frac{1}{2} r' \sqrt{3\varepsilon(1+\cos\theta)} \\ &< \frac{5}{4} r' \sqrt{\varepsilon}. \end{aligned}$$

Let's consider all m-simplexes[2] with each vertex in $S_0 \cap B(x', r'/4)$. Let then $y_1, \dots, y_{m+1}$ be the vertices of the m-simplex of greatest m-area among all the simplexes considered, and call a its area. Let y_{m+2} be any other point of $S_0 \cap B(x', r'/4)$ and let Σ be the m-dimensional plane, corresponding to the points $y_1, \dots, y_{m+2}$ as we have seen before. If we denote by Δ the $(m+1)$-simplex with vertices $y_1, \dots, y_{m+2}$, we find that the m-area of the faces of Δ could be at most a, so the area of the projection of Δ onto Σ cannot be greater than $(m+2)a$ and the $(m+1)$-area of Δ would be less than $\frac{5}{2}(m+2) a r' \sqrt{\varepsilon}$. Thus y_{m+2} is within a distance of $\frac{5}{2}(m+1)(m+2) r' \sqrt{\varepsilon} = \varepsilon' r'/128$ from the m-dimensional plane through $y_1, \dots, y_{m+1}$. If Σ' is the m-dimensional plane through x' parallel to the plane through $y_1, \dots, y_{m+1}$, we know that every point of $S_0 \cap B(x', r'/4)$ is whithin a distance of $\varepsilon' r'/64$ from Σ'.

So we have proved that given any $B(x', r') \subseteq B(x_0, r_0)$ with $x' \in S_0$ it is possible to find an m-dimensional plane Σ' through x' such that

$$S_0 \cap B\left(x', \frac{r'}{4}\right) \subseteq (\Sigma')_{\varepsilon' \frac{r'}{64}}. \tag{17}$$

Since

$$\int_{\frac{1}{8}r'}^{\frac{1}{4}r'} \mathcal{H}^{m-1}(S_0 \cap \partial B(x', t))\, \mathrm{d}t \le \psi(x', r'/4) \le (1+\alpha) \left(\frac{r'}{4}\right)^m \gamma_m$$

we are able to find a set of positive measure of radii r with $\frac{1}{8} r' < r < \frac{1}{4} r'$ such that

$$\mathcal{H}^{m-1}(S_0 \cap \partial B(x', r)) \le (1+\alpha)\gamma_m \frac{\left(\frac{r'}{4}\right)^m}{\frac{1}{8} r'}$$

[2]Here an m-simplex is meant to be the convex envelope of $m+1$ points, which are the vertices of the simplex. The m-area of the simplex is its m-dimensional measure.

and by (15)

$$\begin{aligned}\mathcal{H}^m(S_0 \cap B(x',r)) &\geq (1-\alpha)\gamma_m r^m = \gamma_m \left(\frac{r'}{4}\right)^m 4^m (1-\alpha)\left(\frac{r}{r'}\right)^m \\ &\geq \mathcal{H}^{m-1}(S_0 \cap \partial B(x',r))\frac{1}{4\cdot 2^m} r.\end{aligned}$$

So, by the hypotheses on ε' and r_0 we have

$$\mathcal{H}^m(S_0 \cap B(x',r)) > 4\varepsilon' r \frac{1+c_n\omega(r)}{1-c_n\omega(r)} \mathcal{H}^{m-1}(S_0 \cap \partial B(x',r)).$$

If now we let $E = E_0 \setminus B(x',r)$ by Lemma 2.4 there exist r such that

$$\begin{aligned}P(E, \overline{B(x',r)}) &\geq \frac{1-c_n\omega(r)}{1+c_n\omega(r)} P(E_0, B(x',r)) \\ &= \frac{1-c_n\omega(r)}{1+c_n\omega(r)} \mathcal{H}^m(S_0 \cap B(x',r)) \\ &> 4\varepsilon' r \mathcal{H}^{m-1}(S_0 \cap \partial B(x',r))\end{aligned}$$

and then, by Lemma 2.6, we get that the projection of $S_0 \cap B(x',r)$ on Σ' must contain $\Sigma' \cap B(x', (1-\frac{1}{4}\varepsilon')r)$.

We claim that

$$\Sigma' \cap B\left(x', \frac{1}{16}r'\right) \subseteq \left(S_0 \cap B\left(x', \frac{1}{16}r'\right)\right)_{\frac{\varepsilon' r'}{32}}. \tag{18}$$

Given any point $p \in \Sigma' \cap B(x', \frac{1}{16}r')$ let $p' \in \Sigma' \cap B(x', \frac{1}{16}r' - \frac{\varepsilon' r'}{64})$ be a point with minimal distance from p. Obviously $|p-p'| \leq \frac{\varepsilon' r'}{64}$. As we have seen previously, being $(1-\frac{1}{4}\varepsilon')r > \frac{r'}{16}$, we know that there exists a point $p'' \in S_0 \cap B(x',r)$ whose projection is p'. Equation (17) gives $|p'-p''| \leq \frac{\varepsilon' r'}{64}$ so that $p'' \in B(x', \frac{r'}{16})$. Thus we've found a point $p'' \in S_0 \cap B(x', \frac{r'}{16})$ whose distance from p is less than $\frac{\varepsilon' r'}{32}$ as desired.

By (17) and (18) we have proved that the (ε', R, m)-Reifenberg property holds in x_0 if we let $R = \frac{r_0}{16}$. □

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