SCATTERING IN TWISTED WAVEGUIDES

PHILIPPE BRIET, HYNEK KOVAŘÍK, AND GEORGI RAIKOV

Abstract. We consider a twisted quantum waveguide, i.e., a domain of the form $\Omega_\theta := r_\theta\omega \times \mathbb{R}$ where $\omega \subset \mathbb{R}^2$ is a bounded domain, and $r_\theta = r_\theta(x_3)$ is a rotation by the angle $\theta(x_3)$ depending on the longitudinal variable x_3. We investigate the nature of the essential spectrum of the Dirichlet Laplacian $\mathcal{H}_\theta$, self-adjoint in $\mathrm{L}^2(\Omega_\theta)$, and consider related scattering problems. First, we show that if the derivative of the difference $\theta_1 - \theta_2$ decays fast enough as $|x_3| \to \infty$, then the wave operators for the operator pair $(\mathcal{H}_{\theta_1}, \mathcal{H}_{\theta_2})$ exist and are complete. Further, we concentrate on appropriate perturbations of constant twisting, i.e. $\theta' = \beta - \varepsilon$ with constant $\beta \in \mathbb{R}$, and ε which decays fast enough at infinity together with its first derivative. In that case the unperturbed operator corresponding to ε is an analytically fibered Hamiltonian with purely absolutely continuous spectrum. Obtaining Mourre estimates with a suitable conjugate operator, we prove, in particular, that the singular continuous spectrum of $\mathcal{H}_\theta$ is empty.

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1. Introduction

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\omega \in C^2$. Denote by $\Omega := \omega \times \mathbb{R}$ the straight tube in $\mathbb{R}^3$. For a given $\theta \in C^1(\mathbb{R}, \mathbb{R})$ we define the twisted tube Ω_θ by

$$\Omega_\theta = \left\{ r_\theta(x_3)\, x \in \mathbb{R}^3 \,|\, x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ x_\omega := (x_1, x_2) \in \omega \right\},$$

where

$$r_\theta(x_3) = \begin{pmatrix} \cos\theta(x_3) & \sin\theta(x_3) & 0 \\ -\sin\theta(x_3) & \cos\theta(x_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define the Dirichlet Laplacian $\mathcal{H}_\theta$ as the unique self-adjoint operator generated in $\mathrm{L}^2(\Omega_\theta)$ by the closed quadratic form

$$\mathcal{Q}_\theta[u] := \int_{\Omega_\theta} |\nabla u|^2 \, d\mathbf{x}, \qquad u \in \mathrm{D}(\mathcal{Q}_\theta) := \mathrm{H}_0^1(\Omega_\theta). \tag{1.1}$$

In fact, we do not work directly with $\mathcal{H}_\theta$, but rather with a unitarily equivalent operator $H_{\theta'}$ acting in the straight tube Ω, see (2.4). The related unitary transformation is generated by a change of variables which maps the twisted tube Ω_θ onto the straight tube Ω, see equation (2.3).

The goal of the present article is to study the nature of the essential spectrum of the operator $\mathcal{H}_\theta$ under appropriate assumptions about the twisting angle θ. Although the spectral properties of a twisted waveguide have been intensively studied in recent years, attention has been paid mostly to the discrete spectrum of $\mathcal{H}_\theta$, [5, 10, 14], or to the Hardy inequality for $\mathcal{H}_\theta$, [9].

In this article we discuss the influence of twisting on the nature of the essential spectrum of $\mathcal{H}_\theta$. First, we show that if the difference $\theta_1' - \theta_2'$ decays fast enough as $|x_3| \to \infty$, then the wave operators for the operator pair $(H_{\theta_1'}, H_{\theta_2'})$ exist and are complete, and in particular, the absolutely continuous spectra of $H_{\theta_1'}$ and $H_{\theta_2'}$ coincide. Further, we observe that if $\theta' = \beta$ is constant, then the operator H_β is analytically fibered, cf. (2.9), and therefore its singular continuous spectrum is empty, [11, 13]. Assuming that $\theta'(x_3) = \beta - \varepsilon(x_3)$ with $\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$, we then show that if ε decays fast enough at infinity, then $H_{\theta'}$ has no singular continuous spectrum, see Theorem 2.7. The proof of Theorem 2.7 is based on the Mourre commutator method, [19, 20, 1]. We construct a suitable conjugate operator A and show that the commutator $[H_{\theta'}, iA]$ satisfies a Mourre estimate on sufficiently small intervals outside a discrete subset of $\mathbb{R}$, Theorem 8.2. The construction of the conjugate operator is based on a careful analysis of the band functions $E_n(k)$ of the unperturbed operator H_β, $k \in \mathbb{R}$ being the Fourier variable dual to x_3. A similar strategy was used in [12, 2, 6, 17], where the generator of dilations in the longitudinal direction of the waveguide was used as a conjugate operator. However, in the situations studied in these works the associated band functions have a non zero derivative everywhere except for the origin. In our model, contrary to [12, 2, 6, 17], the band functions E_n may have many stationary points. In addition, we have to take into account possible crossing points between different band functions. The generator of dilations therefore cannot be used as a conjugate operator in our case, and a different approach is needed. Our conjugate operator acts in the fibered space as

$$\frac{i}{2}\left(\gamma(k)\,\partial_k + \partial_k\,\gamma(k)\right) \tag{1.2}$$

where $\gamma \in C_0^\infty(\mathbb{R}; \mathbb{R})$ is a suitably chosen function, whose particular form depends on the interval on which the Mourre estimate is established, see Theorem 7.2.

We would like to mention that a general theory of Mourre estimates for analytically fibered operators and their appropriate perturbations was developed in [13]. The situation with the twisted waveguide analyzed in the present article is much more specific than the general abstract scheme studied in [13]. Hence, although the construction in (1.2) is influenced in some extent by [13], our conjugate operator is essentially different from the one used in [13], and is considerably more useful for our purposes. In particular, the construction of this quite explicit conjugate operator allows us to handle the specific second-order differential perturbation which arises in the context of the twisted waveguide, and to verify all the regularity conditions for e^{itA}, $[H_{\theta'}, iA]$ and $[[H_{\theta'}, iA], iA]$ needed for the passage from the Mourre estimate to the absence of the singular continuous spectrum, see Proposition 8.3. We have thus been able to apply the Mourre theory to the perturbed operator $H_{\theta'}$, and to find simple and efficient sufficient conditions on ε under which the singular continuous spectrum of $H_{\theta'}$ is empty. We therefore believe that our construction of the conjugate operator might be of independent interest.

The article is organized as follows. In Section 2 we state our main results. In Section 3 we prove Proposition 2.1 describing the domain of the operator $\mathcal{H}_\theta$. In Section 4 we prove Theorem 2.3 which entails the existence and the completeness of the wave operators for the operator pair $(H_{\theta_1'}, H_{\theta_2'})$ for appropriate $\theta_1' - \theta_2'$, and hence the coincidence of $\sigma_{\rm ac}(H_{\theta_1'})$ and $\sigma_{\rm ac}(H_{\theta_2'})$. In Section 5 we assume that the twisting is constant, i.e. $\theta' = \beta$ and examine the spectral and analytical properties of the fiber family $h_\beta(k)$, $k \in \mathbb{R}$. In Section 6 we construct the conjugate operator needed for the subsequent Mourre estimates. In Section 7 we obtain Mourre estimates for the case of a constant twisting. Finally, in Section 8 we extend these estimates to the case of $\theta' = \beta - \varepsilon$ where $\beta \in \mathbb{R}$, and ε decays fast enough together with its first derivative.

2. Main results

2.1. **Notation.** Let us fix some notation. Given a measure space $(M, \mathcal{A}, \mu)$, we denote by $\mathbb{1}_M$ the identity operator in $\mathrm{L}^2(M) = \mathrm{L}^2(M; d\mu)$. Further, we will denote by $(u, v)_{\mathrm{L}^2(M)} = \int_M \bar{u}\, v d\mu$ the scalar product in $\mathrm{L}^2(M)$ and by $\|u\|_{\mathrm{L}^p(M)}$, $p \in [1, \infty]$, the L^p-norm of u. If there is no risk of confusion we will drop the indication to the set M and write (u, v) and $\|u\|_p$ instead in order to simplify the notation. Given a set M and two functions f_1, $f_2 : M \to \mathbb{R}$, we write $f_1(m) \asymp f_2(m)$, $m \in M$, if there exists a constant $c \in (0, \infty)$ such that for each $m \in M$ we have

$$c^{-1} f_1(m) \leq f_2(m) \leq c\, f_1(m).$$

Given a separable Hilbert space X, we denote by $\mathcal{L}(X)$ (resp., $S_\infty(X)$) the class of bounded (resp., compact) linear operators acting in X. Similarly, by $S_p(X)$, $p \in [1, \infty)$, we denote the Schatten-von Neumann classes of compact operators acting in X; we recall that the norm in $S_p(X)$ is defined as $\|T\|_{S_p} := \left(\mathrm{Tr}\, (T^*T)^{p/2}\right)^{1/p}$, $T \in S_p(X)$. In particular, S_1 is the trace class, and S_2 is the Hilbert-Schmidt class. Moreover, if T is a self-adjoint operator acting in X, we denote by $\mathrm{D}(T)$ the operator domain of T. Finally, for $\alpha \in \mathbb{R}$ define the function

$$\phi_\alpha(s) := (1 + s^2)^{-\alpha/2}, \quad s \in \mathbb{R}. \tag{2.1}$$

2.2. **Domain issues.** Our first result shows that if both θ' and θ'' are continuous and bounded, then the domain of the operator $\mathcal{H}_\theta$ coincides with $\mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta)$.

Proposition 2.1. *Assume that $\omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\omega \in C^2$, and $\theta \in C^2(\mathbb{R})$ with $\theta', \theta'' \in \mathrm{L}^\infty(\mathbb{R})$. Then*

$$\mathrm{D}\,(\mathcal{H}_\theta) = \mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta). \tag{2.2}$$

Proposition 2.1 could be considered a fairly standard result but since we have not been able to find in the literature a version suitable for our purposes (most of the references available treat bounded domains or the complements of compact sets), we include a detailed sketch of the proof in Section 3.

Next, we introduce the operator $U_\theta : \mathrm{L}^2(\Omega_\theta) \to \mathrm{L}^2(\Omega)$ generated by the change of variables

$$\Omega \ni x \mapsto r_\theta(x_3)\, x \in \Omega_\theta. \tag{2.3}$$

Namely, for $w \in \mathrm{L}^2(\Omega_\theta)$ set

$$(U_\theta\, w)(x) = w\left(r_\theta(x_3)\, x\right), \quad x \in \Omega.$$

Evidently, $U_\theta : \mathrm{L}^2(\Omega_\theta) \to \mathrm{L}^2(\Omega)$ is unitary since (2.3) defines a diffeomorphism whose Jacobian is identically equal to one. Now assume $g \in C(\mathbb{R}; \mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ and introduce the quadratic form

$$Q_g[u] = \int_\Omega \left(|\nabla_\omega u|^2 + |\partial_3 u + g\, \partial_\tau u|^2\right) \mathrm{d}x, \quad u \in \mathrm{D}(Q_g) = \mathrm{H}^1_0(\Omega),$$

where $\nabla_\omega := (\partial_1, \partial_2)^T$, and $\partial_\tau := x_1\partial_2 - x_2\partial_1$. Denote by H_g the self-adjoint operator generated in $\mathrm{L}^2(\Omega)$ by the closed quadratic form Q_g. The transformation U_θ also maps $\mathrm{H}^1_0(\Omega_\theta)$ bijectively onto $\mathrm{H}^1_0(\Omega)$. Hence, for $g = \theta'$ we get

$$\mathcal{Q}[w] = Q_{\theta'}[U_\theta\, w], \quad w \in \mathrm{H}^1_0(\Omega_\theta),$$

which implies

$$H_{\theta'} = U_\theta \, \mathcal{H}_\theta \, U_\theta^{-1}. \tag{2.4}$$

Assume now that $g \in C^1(\mathbb{R})$ with $g, g' \in \mathrm{L}^\infty(\mathbb{R})$. Set $G(x_3) := \int_0^{x_3} g(s)ds$, $x_3 \in \mathbb{R}$. Then U_G maps bijectively $\mathrm{H}^2(\Omega_\mathrm{G})$ onto $\mathrm{H}^2(\Omega)$. Therefore, Proposition 2.1 and the unitarity $U_G : \mathrm{L}^2(\Omega_G) \to \mathrm{L}^2(\Omega)$ implies the following

Corollary 2.2. *Assume that $\omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\omega \in C^2$, and $g \in C^1(\mathbb{R})$ with $g, g' \in \mathrm{L}^\infty(\mathbb{R})$. Then the domain of the operator H_g coincides with $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$.*

Furthermore, if $g \in C^1(\mathbb{R})$ with $g, g' \in \mathrm{L}^\infty(\mathbb{R})$ we have

$$H_g \, u = \left(-\partial_1^2 - \partial_2^2 - (\partial_3 + g \, \partial_\tau)^2\right) u, \quad u \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega), \tag{2.5}$$

since $\mathcal{H}_G \, \varphi = -\Delta\varphi$, $\varphi \in \mathrm{H}^2(\Omega_\mathrm{G}) \cap \mathrm{H}^1_0(\Omega_\mathrm{G})$.

2.3. **Existence and completeness of the wave operators.** Next we show that under appropriate assumptions on the difference $g_1 - g_2$, the wave operators for the operator pair (H_{g_1}, H_{g_2}) exist and are complete, and hence the absolutely continuous spectra of the operators H_{g_1} and H_{g_2} coincide.

Theorem 2.3. *Assume that $\omega \subset \mathbb{R}^2$ is a bounded domain with C^2-boundary. Let $g_j \in C^1(\mathbb{R};\mathbb{R})$ with $g_j, g_j' \in \mathrm{L}^\infty(\mathbb{R})$, $j = 1, 2$. Suppose that there exists $\alpha > 1$ such that*

$$\|\phi_{-\alpha}(g_1 - g_2)\|_{\mathrm{L}^\infty(\mathbb{R})} < \infty, \tag{2.6}$$

the function ϕ_α being defined in (2.1)*. Then we have*

$$H_{g_1}^{-2} - H_{g_2}^{-2} \in S_1(\mathrm{L}^2(\Omega)). \tag{2.7}$$

Theorem 2.3 is proven in Section 4. By a classical result from the stationary scattering theory (see the original work [4] or [22, Corollary 3, Section 3, Chapter XI], [27, Chapter 6, Section 2, Theorem 6]), this theorem implies the following

Corollary 2.4. *Under the assumptions of Theorem 2.3 the wave operators*

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{itH_{g_1}} e^{-itH_{g_2}} P_{\mathrm{ac}}(H_{g_2})$$

for the operator pair (H_{g_1}, H_{g_2}) exist and are complete. Therefore, the absolutely continuous parts of H_{g_1} and H_{g_2} are unitarily equivalent, and, in particular,

$$\sigma_{\mathrm{ac}}(H_{g_1}) = \sigma_{\mathrm{ac}}(H_{g_2}). \tag{2.8}$$

Corollary 2.4 admits an equivalent formulation in terms of the operator pair $(\mathcal{H}_{\theta_1}, \mathcal{H}_{\theta_2})$:

Corollary 2.5. *Assume that $\omega \subset \mathbb{R}^2$ is a bounded domain with C^2-boundary. Let $\theta_j \in C^2(\mathbb{R};\mathbb{R})$ with $\theta_j, \theta_j', \theta_j'' \in \mathrm{L}^\infty(\mathbb{R})$, $j = 1, 2$. Suppose that there exists $\alpha > 1$ such that*

$$\|\phi_{-\alpha}(\theta_1' - \theta_2')\|_{\mathrm{L}^\infty(\mathbb{R})} < \infty,$$

Then the wave operators

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\mathcal{H}_{\theta_1}} \mathcal{J} e^{-it\mathcal{H}_{\theta_2}} P_{\mathrm{ac}}(\mathcal{H}_{\theta_2}), \quad \mathcal{J} := U_{\theta_1}^{-1} U_{\theta_2},$$

for the operator pair $(\mathcal{H}_{\theta_1}, \mathcal{H}_{\theta_2})$ exist and are complete. Therefore, the absolutely continuous parts of $\mathcal{H}_{\theta_1}$ and $\mathcal{H}_{\theta_2}$ are unitarily equivalent, and, in particular, $\sigma_{\mathrm{ac}}(\mathcal{H}_{\theta_1}) = \sigma_{\mathrm{ac}}(\mathcal{H}_{\theta_2})$.

2.4. **Constant twisting.** In our remaining results, we concentrate on the case of appropriate perturbations of a constant twisting, i.e. the case where θ' is equal to a constant $\beta \in \mathbb{R}$. First, we consider the unperturbed operator H_β. We define the partial Fourier transform $\mathcal{F}$, unitary in $\mathrm{L}^2(\Omega)$, by

$$(\mathcal{F}\, u)(x_\omega, k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ikx_3}\, u(x_\omega, x_3)\, dx_3, \qquad k \in \mathbb{R}, \quad x_\omega \in \omega.$$

Then we have

$$\hat{H}_\beta = \mathcal{F}\, H_\beta\, \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h_\beta(k)\, dk, \tag{2.9}$$

where, by (2.5) with $g = \beta$, the operator $h_\beta(k)$ acts on its domain $\mathrm{D}\left(h_\beta(k)\right) = \mathrm{H}^2(\omega) \cap \mathrm{H}^1_0(\omega)$ as

$$h_\beta(k) = -\Delta_\omega + (\beta i \partial_\tau - k)^2,$$

$-\Delta_\omega$ being the self-adjoint operator generated in $\mathrm{L}^2(\omega)$ by the closed quadratic form

$$\int_\omega |\nabla v|^2 dx_\omega, \quad v \in \mathrm{H}^1_0(\omega).$$

Note that for all $k \in \mathbb{R}$ the resolvent $h_\beta(k)^{-1}$ is compact, and $h_\beta(k)$ has a purely discrete spectrum. Let

$$0 < E_1(k) \le E_2(k) \le \dots \le E_n(k) \le \dots, \qquad k \in \mathbb{R}, \tag{2.10}$$

be the non-decreasing sequence of the eigenvalues of $h_\beta(k)$. Denote by $p_n(k)$ the orthogonal projection onto $\mathrm{Ker}(h_\beta(k) - E_n(k))$, $k \in \mathbb{R}$ and $n \in \mathbb{N}$. By [5, 10] we have

$$\sigma(H_\beta) = \sigma_{\mathrm{ac}}(H_\beta) = [E_1(0), \infty). \tag{2.11}$$

A detailed discussion of the properties of $E_n(k)$ is given in Section 5. It turns out that the functions $E_n(k)$ are piecewise analytic, and that for any given $k_0 \in \mathbb{R}$, the function $E_n(k)$ can be analytically extended into an open neighborhood of k_0. We denote such an extension by $\tilde{E}_{n,k_0}(k)$. If k_0 is a point where $E_n(k)$ is analytic, then of course $\tilde{E}_{n,k_0}(\cdot) = E_n(\cdot)$. With this notation at hand, we introduce the following subsets of $\mathbb{R}$:

$$\begin{aligned} \mathcal{E}_1 &:= \big\{ E \in \mathbb{R} \,:\, \exists\, n \in \mathbb{N},\, \exists\, k_0 \in \mathbb{R} \,:\, E_n(k_0) = E \;\wedge\; \partial_k \tilde{E}_{n,k_0}(k_0) = 0 \big\}, \\ \mathcal{E}_2 &:= \big\{ E \in \mathbb{R} \,:\, \exists\, k_0 \in \mathbb{R},\, \exists\, n, m \in \mathbb{N},\, n \neq m \,:\, E_n(k_0) = E_m(k_0) = E \;\wedge \\ &\qquad \wedge\; \partial_k \tilde{E}_{n,k_0}(k_0)\, \partial_k \tilde{E}_{m,k_0}(k_0) < 0 \big\}. \end{aligned}$$

We then define the set $\mathcal{E}$ of critical levels as follows:

$$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2. \tag{2.12}$$

Lemma 2.6. *The set $\mathcal{E}$ is locally finite.*

The proof of Lemma 2.6 is given in Section 5, immediately after Lemma 5.4.

2.5. **Absence of singular continuous spectrum of $H_{\beta-\varepsilon}$.**

Theorem 2.7. *Let $\theta'(x_3) = \beta - \varepsilon(x_3)$, where $\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ is such that*

$$\|\varepsilon\, \phi_{-2}\|_\infty + \|\varepsilon'\, \phi_{-2}\|_\infty < \infty, \tag{2.13}$$

the function ϕ_α being defined in (2.1). Then:

(a) *Any compact subinterval of $\mathbb{R} \setminus \mathcal{E}$ contains at most finitely many eigenvalues of $H_{\theta'}$, each having finite multiplicity;*

(b) *The point spectrum of* $H_{\theta'}$ *has no accumulation points in* $\mathbb{R} \setminus \mathcal{E}$*;*

(c) *The singular continuous spectrum of* $H_{\theta'}$ *is empty.*

Theorem 2.7 is proven in Subsection 8.2.

Remark 2.8. If $\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ is such that ε' is bounded and $\|\varepsilon\, \phi_{-\alpha}\|_\infty < \infty$ for some $\alpha > 1$, then Corollary 2.4 and equation (2.11) imply

$$\sigma_{\rm ac}(H_{\theta'}) = \sigma_{\rm ac}(H_\beta) = [E_1(0), \infty).$$

Note that in order to prove the absence of singular continuous spectrum of $H_{\theta'}$ we need stronger hypothesis on ε and ε', see equation (2.13).

By [22, Section XI.3], Corollary 2.4 and Theorem 2.7 part (c) imply

Corollary 2.9. *Under the assumptions of Theorem 2.7 the wave operators for the operator pair* $(H_\beta, H_{\theta'})$ *exist and are asymptotically complete.*

3. Proof of Proposition 2.1

Denote by $C_0^\infty(\overline{\Omega_\theta})$ the class of functions $u \in C^\infty(\overline{\Omega_\theta})$, compactly supported in $\overline{\Omega_\theta}$. Set

$$\dot{C}^\infty(\Omega_\theta) := \left\{ u \in C_0^\infty(\overline{\Omega_\theta}) \,|\, u_{|\partial\Omega_\theta} = 0 \right\}.$$

Lemma 3.1. *Under the assumptions of Proposition 2.1 there exists a constant* $c \in (0, \infty)$ *such that*

$$\|u\|^2_{{\rm H}^2(\Omega_\theta)} \le c \int_{\Omega_\theta} (|\Delta u|^2 + |u|^2) dx \tag{3.1}$$

for any $u \in \dot{C}^\infty(\Omega_\theta)$.

Proof. Our argument will follow closely the proof of [18, Chapter 3, Lemma 8.1]. We have

$$\int_{\Omega_\theta} (|\Delta u|^2 + c_0|u|^2) dx = \int_{\Omega_\theta} \left(\sum_{j,k=1}^3 |\partial_j \partial_k u|^2 + c_0 |u|^2 \right) dx + 2 \int_{\partial\Omega_\theta} K \left| \frac{\partial u}{\partial \nu} \right|^2 dS, \quad u \in \dot{C}^\infty(\Omega_\theta), \tag{3.2}$$

(see [18] or [25, Chapter 5, Section 5, Problem 6]) where $c_0 \in (0, \infty)$ is an arbitrary constant which is to be specified later, K is the mean curvature, and ν is the exterior normal unit vector at $\partial\Omega_\theta$. Our assumptions on $\partial\omega$ and θ imply that for any $u \in \dot{C}^\infty(\Omega_\theta)$ we have

$$2 \int_{\partial\Omega_\theta} K \left| \frac{\partial u}{\partial \nu} \right|^2 dS \ge -c_1 \int_{\partial\Omega_\theta} |\nabla u|^2 dS. \tag{3.3}$$

with

$$c_1 := 2 \sup_{x \in \partial\Omega_\theta} |K(x)| \le 2 \sup_{(x_\omega, x_3) \in \partial\omega \times \mathbb{R}} \left\{ (|\theta''(x_3)| + \theta'(x_3)^2)|x_\omega| + (1 + \theta'(x_3)^2 |x_\omega|^2) |\kappa(x_\omega)| \right\},$$

where $\kappa(x_\omega)$ is the curvature of $\partial\omega$ at the point $x_\omega \in \partial\omega$. Let us check that for any $\varepsilon > 0$ there exists a constant $c_2(\varepsilon)$ such that for any $v \in C_0^\infty(\overline{\Omega_\theta})$ we have

$$\int_{\partial\Omega_\theta} |v|^2 dS \le \int_{\Omega_\theta} \left(\varepsilon |\nabla_\omega v|^2 + c_2(\varepsilon) |v|^2 \right) dx \tag{3.4}$$

where, as above, $\nabla_\omega := (\partial_1, \partial_2)^T$. In order to prove this, we note the inequality

$$\int_{\partial\Omega_\theta} |v|^2 dS \le c_3 \int_{\mathbb{R}} \left(\int_{\partial\omega_{\theta(x_3)}} |v|^2 ds \right) dx_3 \tag{3.5}$$

where

$$c_3 := \sup_{(x_\omega, x_3)\in\partial\omega\times\mathbb{R}} \left(1 + \theta'(x_3)^2 |x_\omega|^2\right)^{1/2},$$

and $\omega_{\theta(a)}$ is the cross-section of Ω_θ with the plane $\{x_3 = a\}$, $a \in \mathbb{R}$.

Next, since ω is a bounded domain with sufficiently regular boundary, we find that for any $\varepsilon > 0$ there exists a constant $c_4(\varepsilon)$ such that for any $x_3 \in \mathbb{R}$ and any $w \in C^\infty(\overline{\omega_{\theta(x_3)}})$ we have

$$\int_{\partial\omega_{\theta(x_3)}} |w|^2 ds \le \int_{\omega_{\theta(x_3)}} \left(\varepsilon |\nabla w|^2 + c_4(\varepsilon)|w|^2\right) dx_\omega \tag{3.6}$$

(see e.g. [18, Chapter 2, Eq. (2.25)]). Choosing $w = v(\cdot, x_3)$ in (3.6), integrating with respect to x_3, and bearing in mind (3.5), we get

$$\int_{\partial\Omega_\theta} |v|^2 dS \le \int_{\Omega_\theta} \left(c_3\varepsilon |\nabla_\omega v|^2 + c_3 c_4(\varepsilon)|v|^2\right) dx$$

which implies (3.4) with $c_2(\varepsilon) = c_3 c_4(\varepsilon/c_3)$. Now the combination of (3.3) and (3.4) implies

$$2\int_{\partial\Omega_\theta} K \left|\frac{\partial u}{\partial \nu}\right|^2 dS \ge -c_1 \int_{\Omega_\theta} \left(\varepsilon \sum_{j,k=1}^{3} |\partial_j\partial_k u|^2 + c_2(\varepsilon)|\nabla u|^2 \right) dx. \tag{3.7}$$

Further, we have

$$\int_{\Omega_\theta} |\nabla u|^2 dx = -\mathrm{Re} \int_{\Omega_\theta} \Delta u \overline{u} dx \le \frac{1}{2} \int_{\Omega_\theta} \left(|\Delta u|^2 + |u|^2\right) dx. \tag{3.8}$$

Combining (3.2), (3.7), and (3.8), we find that for any $\varepsilon > 0$ we have

$$\int_{\Omega_\theta} \left((1 + c_1 c_2(\varepsilon)/2)\, |\Delta u|^2 + c_0 |u|^2 \right) dx \ge$$

$$\int_{\Omega_\theta} \left((1 - c_1\varepsilon) \sum_{j,k=1}^{3} |\partial_j\partial_k u|^2 + (c_0 - c_1 c_2(\varepsilon)/2)\, |u|^2 \right) dx$$

which yields (3.1) under appropriate choice of c_0, c and ε. □

Denote by $\tilde{\mathrm{H}}^2(\Omega_\theta)$ the Hilbert space $\left\{u \in \mathrm{H}_0^1(\Omega_\theta) \,|\, \Delta \mathrm{u} \in \mathrm{L}^2(\Omega_\theta)\right\}$ with scalar product generated by the quadratic form $\int_{\Omega_\theta} (|\Delta u|^2 + |u|^2) dx$.

Lemma 3.2. *Under the assumptions of Proposition 2.1 we have* $u \in \tilde{\mathrm{H}}^2(\Omega_\theta)$ *if and only if* $u \in \mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}_0^1(\Omega_\theta)$.

Proof. By

$$\int_{\Omega_\theta} (|\Delta u|^2 + |u|^2) dx \le 3\|u\|^2_{\mathrm{H}^2(\Omega_\theta)}, \quad u \in \mathrm{H}^2(\Omega_\theta), \tag{3.9}$$

and (3.1), we have

$$\int_{\Omega_\theta} (|\Delta u|^2 + |u|^2) dx \asymp \|u\|^2_{\mathrm{H}^2(\Omega_\theta)}, \quad u \in \dot{C}^\infty(\Omega_\theta). \tag{3.10}$$

Evidently, the class $\dot{C}^\infty(\Omega_\theta)$ is dense in $\mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta)$. Then (3.9) easily implies that $\dot{C}^\infty(\Omega_\theta)$ is dense in $\tilde{\mathrm{H}}^2(\Omega_\theta)$ as well. Now the claim of the lemma follows from (3.10). □

Proof of Proposition 2.1. Let L be the operator $-\Delta$ with domain $C_0^\infty(\Omega_\theta)$, and L^* be the adjoint of L in $\mathrm{L}^2(\Omega_\theta)$. If $v \in \mathrm{D}(L^*)$, then a standard argument from the theory of distributions over $C_0^\infty(\Omega_\theta)$ shows that $L^* v = -\Delta v \in \mathrm{L}^2(\Omega_\theta)$. Since $\mathcal{H}_\theta$ is a restriction of L^*, we find that $u \in \mathrm{D}(\mathcal{H}_\theta)$ implies that $\mathcal{H}_\theta u = -\Delta u \in \mathrm{L}^2(\Omega_\theta)$. On the other hand, $u \in \mathrm{D}(\mathcal{H}_\theta)$ implies $u \in \mathrm{D}(\mathcal{Q}_\theta) = \mathrm{H}^1_0(\Omega_\theta)$. By Lemma 3.2 we have $u \in \mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta)$, i.e.

$$\mathrm{D}(\mathcal{H}_\theta) \subseteq \mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta). \tag{3.11}$$

If we now suppose that

$$\mathrm{D}(\mathcal{H}_\theta) \neq \mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta), \tag{3.12}$$

then (3.11) and (3.12) would imply that the operator $\mathcal{H}_\theta$ has a proper symmetric extension, namely the operator $-\Delta$ with domain $\mathrm{H}^2(\Omega_\theta) \cap \mathrm{H}^1_0(\Omega_\theta)$, which contradicts the self-adjointness of $\mathcal{H}_\theta$. Therefore, (2.2) holds true, and the proof of Proposition 2.1 is complete. □

4. Proof of Theorem 2.3

For the proof of Theorem 2.3 we need an auxiliary result, Lemma 4.1, preceded by some necessary notation.

Let $\{\mu_j\}_{j\in\mathbb{N}}$ be the the non-decreasing sequence of the eigenvalues of the operator $-\Delta_\omega$. Since $H_g \geq \mu_1 \mathbb{1}_\Omega$, and $\mu_1 > 0$, the operator H_g is invertible.

Lemma 4.1. *Let $g \in C^1(\mathbb{R};\mathbb{R})$ with $g, g' \in \mathrm{L}^\infty(\mathbb{R})$.*
(i) Assume $f \in \mathrm{L}^2(\mathbb{R})$. Then we have

$$f(x_3) H_g^{-1} \in S_2(\mathrm{L}^2(\Omega)). \tag{4.1}$$

(ii) Assume $h \in \mathrm{L}^4(\mathbb{R})$. Then we have

$$h(x_3)\partial_j H_g^{-1} \in S_4(\mathrm{L}^2(\Omega)), \quad j = 1, 2, 3. \tag{4.2}$$

Proof. By Corollary 2.2 the operator $H_0 H_g^{-1}$ is bounded, so that it suffices to prove (4.1) – (4.2) for $g = 0$. Evidently,

$$\|f H_0^{-1}\|^2_{S_2(\mathrm{L}^2(\Omega))} = \sum_{j\in\mathbb{N}} \|f(-\partial_3^2 + \mu_j)^{-1}\|^2_{S_2(\mathrm{L}^2(\mathbb{R}))} =$$

$$= (2\pi)^{-1} \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} |f(s)|^2 ds \int_{\mathbb{R}} \frac{d\xi}{(\xi^2 + \mu_j)^2} = (2\pi)^{-1} \sum_{j\in\mathbb{N}} \mu_j^{-3/2} \int_{\mathbb{R}} |f(s)|^2 ds \int_{\mathbb{R}} \frac{d\xi}{(\xi^2+1)^2}. \tag{4.3}$$

Set $\mathcal{N}(\lambda) := \#\{j \in \mathbb{N} \,|\, \mu_j < \lambda\}$, $\lambda > 0$. By the celebrated Weyl law, we have $\mathcal{N}(\lambda) = \frac{|\omega|}{4\pi}\lambda(1 + o(1))$ as $\lambda \to \infty$ where $|\omega|$ is the area of ω (see the original work [26] or [23, Theorem XIII.78]). Therefore, the series $\sum_{j\in\mathbb{N}} \mu_j^{-\gamma} = \gamma \int_{\mu_1}^\infty \lambda^{-\gamma-1}\mathcal{N}(\lambda) d\lambda$ is convergent if and only if $\gamma > 1$. In particular,

$$\sum_{j\in\mathbb{N}} \mu_j^{-3/2} < \infty, \tag{4.4}$$

so that the r.h.s. of (4.3) is finite which implies (4.1) with $g = 0$.

Let us now prove (4.2) with $g = 0$ and $j = 1, 2$. We have

$$h\partial_j H_0^{-1} = \partial_j((-\Delta_\omega) \otimes \mathbb{1}_{\mathbb{R}})^{-1/2} h((-\Delta_\omega) \otimes \mathbb{1}_{\mathbb{R}})^{1/2} H_0^{-1}.$$

Since the operators $\partial_j((-\Delta_\omega)\otimes \mathbb{1}_{\mathbb{R}})^{-1/2}$, $j=1,2$, are bounded, it suffices to show that

$$h((-\Delta_\omega)\otimes \mathbb{1}_{\mathbb{R}})^{1/2}H_0^{-1} \in S_4(\mathrm{L}^2(\Omega)). \tag{4.5}$$

Applying a standard interpolation result (see e.g. [24, Theorem 4.1] or [3, Section 4.4]), and bearing in mind (4.4), we get

$$\|h((-\Delta_\omega)\otimes \mathbb{1}_{\mathbb{R}})^{1/2}H_0^{-1}\|^4_{S_4(\mathrm{L}^2(\Omega))} = \sum_{j\in\mathbb{N}} \mu_j^2 \|h(-\partial_3^2+\mu_j)^{-1}\|^4_{S_4(\mathrm{L}^2(\mathbb{R}))} \leq$$

$$\leq (2\pi)^{-1}\sum_{j\in\mathbb{N}}\mu_j^2\int_{\mathbb{R}}|h(s)|^4ds\int_{\mathbb{R}}\frac{d\xi}{(\xi^2+\mu_j)^4} = (2\pi)^{-1}\sum_{j\in\mathbb{N}}\mu_j^{-3/2}\int_{\mathbb{R}}|h(s)|^4ds\int_{\mathbb{R}}\frac{d\xi}{(\xi^2+1)^4} < \infty$$

which implies (4.5). Finally, we prove (4.2) with $g=0$ and $j=3$. To this end it suffices to apply again [24, Theorem 4.1] and (4.4), and get

$$\|h\partial_3 H_0^{-1}\|^4_{S_4(\mathrm{L}^2(\Omega))} = \sum_{j\in\mathbb{N}} \|h\partial_3(-\partial_3^2+\mu_j)^{-1}\|^4_{S_4(\mathrm{L}^2(\mathbb{R}))} \leq$$

$$\leq (2\pi)^{-1}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}}|h(s)|^4ds\int_{\mathbb{R}}\frac{\xi^4\,d\xi}{(\xi^2+\mu_j)^4} = (2\pi)^{-1}\sum_{j\in\mathbb{N}}\mu_j^{-3/2}\int_{\mathbb{R}}|h(s)|^4ds\int_{\mathbb{R}}\frac{\xi^4\,d\xi}{(\xi^2+1)^4} < \infty.$$

□

Proof of Theorem 2.3. For $z \in \rho(H_{g_1})\cap\rho(H_{g_2})$ we have

$$(H_{g_1}-z)^{-2}-(H_{g_2}-z)^{-2} = \frac{\partial}{\partial z}(H_{g_1}-z)^{-1}W(H_{g_2}-z)^{-1} =$$
$$(H_{g_1}-z)^{-2}W(H_{g_2}-z)^{-1}+(H_{g_1}-z)^{-1}W(H_{g_2}-z)^{-2} \tag{4.6}$$

with

$$W := \partial_\tau(g_1^2-g_2^2)\partial_\tau + \partial_3(g_1-g_2)\partial_\tau + \partial_\tau(g_1-g_2)\partial_3.$$

Choosing $z=0$, we obtain

$$H_{g_1}^{-2}-H_{g_2}^{-2} =$$
$$-\left(\phi_{3\alpha/4}\partial_\tau H_{g_1}^{-2}\right)^*\left((g_1-g_2)\phi_{-\alpha}\phi_{\alpha/4}\partial_3 H_{g_2}^{-1}+(g_1^2-g_2^2)\phi_{-\alpha}\phi_{\alpha/4}\partial_\tau H_{g_2}^{-1}\right)-$$
$$\left(\phi_{3\alpha/4}\partial_3 H_{g_1}^{-2}\right)^*(g_1-g_2)\phi_{-\alpha}\phi_{\alpha/4}\partial_\tau H_{g_2}^{-1}-$$
$$\left(\phi_{\alpha/4}\partial_\tau H_{g_1}^{-1}\right)^*\left((g_1-g_2)\phi_{-\alpha}\phi_{3\alpha/4}\partial_3 H_{g_2}^{-2}+(g_1^2-g_2^2)\phi_{-\alpha}\phi_{3\alpha/4}\partial_\tau H_{g_2}^{-2}\right)-$$
$$\left(\phi_{\alpha/4}\partial_3 H_{g_1}^{-1}\right)^*(g_1-g_2)\phi_{-\alpha}\phi_{3\alpha/4}\partial_\tau H_{g_2}^{-2}.$$

Since the multipliers by $(g_1-g_2)\phi_{-\alpha}$ and $(g_1^2-g_2^2)\phi_{-\alpha}$ are bounded operators by (2.6) and $g_j\in\mathrm{L}^\infty(\mathbb{R})$, $j=1,2$, while

$$\phi_{\alpha/4}\partial_\ell H_{g_j}^{-1} \in S_4(\mathrm{L}^2(\Omega)), \quad \ell=\tau,3, \quad j=1,2,$$

by Lemma 4.1 (ii), it suffices to show that

$$\phi_{3\alpha/4}\partial_\ell H_{g_j}^{-2} \in S_{4/3}(\mathrm{L}^2(\Omega)), \quad \ell=\tau,3, \quad j=1,2. \tag{4.7}$$

In what follows we write g instead of g_j, $j=1,2$. Commuting multipliers by functions ϕ which depend only on x_3 and belong to appropriate Hörmander classes, with the resolvent H_g^{-1}, and bearing in mind that

$$[\phi, H_g^{-1}] = -H_g^{-1}(\phi''+2\phi'(\partial_3+g\partial_\tau))H_g^{-1},$$

we obtain

$$\phi_{3\alpha/4}\partial_\tau H_g^{-2} = \phi_{\alpha/4}\partial_\tau H_g^{-1}\phi_{\alpha/2}H_g^{-1} - \phi_{\alpha/4}\partial_\tau H_g^{-1}\phi''_{\alpha/2}H_g^{-2} + 2\phi_{\alpha/4}\partial_\tau H_g^{-1}(\phi'_{\alpha/2})^2\phi_{-\alpha/2}H_g^{-2} -$$
$$2\phi_{\alpha/4}\partial_\tau H_g^{-1}\phi'_{\alpha/2}\phi_{-\alpha/2}\left(\partial_3 + g\partial_\tau\right)H_g^{-1}\left(\phi_{\alpha/2}H_g^{-1} - 2\left(\phi''_{\alpha/2} + \phi'_{\alpha/2}\left(\partial_3 + g\partial_\tau\right)\right)H_g^{-2}\right), \quad (4.8)$$

$$\phi_{3\alpha/4}\partial_3 H_g^{-2} = \phi_{\alpha/4}\partial_3 H_g^{-1}\phi_{\alpha/2}H_g^{-1} - \phi_{\alpha/4}\partial_3 H_g^{-1}\phi''_{\alpha/2}H_g^{-2} + 2\phi_{\alpha/4}\partial_3 H_g^{-1}(\phi'_{\alpha/2})^2\phi_{-\alpha/2}H_g^{-2}$$
$$-2\phi_{\alpha/4}\partial_3 H_g^{-1}\phi'_{\alpha/2}\phi_{-\alpha/2}\left(\partial_3 + g\partial_\tau\right)H_g^{-1}\left(\phi_{\alpha/2}H_g^{-1} - 2\left(\phi''_{\alpha/2} + \phi'_{\alpha/2}\left(\partial_3 + g\partial_\tau\right)\right)H_g^{-2}\right)$$
$$+\phi'_{\alpha/2}\phi_{-\alpha/4}H_g^{-1}\left(\phi_{\alpha/2}H_g^{-1} - \left(\phi''_{\alpha/2} + 2\left(\partial_3 + g\partial_\tau\right)\right)H_g^{-2}\right). \quad (4.9)$$

Bearing in mind that $S_p \subset S_q$ if $p < q$, and that H_g^{-1} is a bounded operator, we find that Lemma 4.1 implies that all the terms at the r.h.s. of (4.8) and (4.9) can be presented either as a product of an operator in S_2 and an operator in S_4, or as a product of three operators in S_4, which yields (4.7), and the proof of Theorem 2.3 is complete. ☐

5. Kato theory for a constant twisting

In this section we assume that $\theta' = \beta$ is constant. Then by (2.9) the operator H_β is unitarily equivalent to $\int_{\mathbb{R}}^{\oplus} h_\beta(k)\, dk$ with $h_\beta(k) = -\Delta_\omega + (i\beta\partial_\tau + k)^2$, $k \in \mathbb{R}$. The goal of the section is to establish various properties of the fiber operator $h_\beta(k)$, which will be used later in Section 7 for the Mourre estimates involving the commutator $[H_\beta, iA]$ with a suitable conjugate operator A described in Section 6.

Lemma 5.1. *The operators $h_\beta(k)$, $k \in \mathbb{R}$, with common domain $\mathrm{H}^2(\omega) \cap \mathrm{H}^1_0(\omega)$, form a self-adjoint holomorphic family of type (A) in the sense of Kato.*

Proof. Note that

$$h_\beta(k) = h_\beta(0) - 2\,\mathrm{Re}\,\beta\, i\, k\partial_\tau + k^2,$$

and that $h_\beta(0)$ is self-adjoint on $\mathrm{H}^2(\omega) \cap \mathrm{H}^1_0(\omega)$. Let $u \in \mathrm{H}^1_0(\omega)$. Then for any $\varepsilon > 0$ we have

$$\|\beta\, i\partial_\tau\, u\|_2^2 \le (u, h_\beta(0)\, u)_{\mathrm{L}^2(\omega)} \le \|u\|_2\, \|h_\beta(0)\, u\|_2 \le \varepsilon^{-1}\|u\|_2^2 + \varepsilon\, \|h_\beta(0)\, u\|_2^2.$$

Hence $\beta i\partial_\tau$ is relatively bounded with respect to $h_\beta(0)$ with relative bound zero and the assertion follows from [16, Theorem VII.2.6]. ☐

From Lemma 5.1 and the Rellich Theorem, [16, Theorem VII.3.9], it follows that all the eigenvalues of $h_\beta(k)$ can be represented by a family of functions

$$\{\lambda_\ell(k)\}_{\ell\in\mathcal{L}}, \qquad \mathcal{L} \subset \mathbb{N}, \quad k \in \mathbb{R}, \quad (5.1)$$

which are analytic on $\mathbb{R}$. Each eigenvalue $\lambda_\ell(k)$ has a finite multiplicity which is constant in $k \in \mathbb{R}$. Moreover, if $\ell \ne \ell'$, then $\lambda_\ell(k) = \lambda_{\ell'}(k)$ may hold only on a discrete subset of $\mathbb{R}$.

Lemma 5.2. *Let $\lambda_\ell(k)$ be one of the analytic eigenvalues* (5.1). *Let $k_0 \in \mathbb{R}$ be given. Then*

$$\left|\sqrt{\lambda_\ell(k)} - \sqrt{\lambda_\ell(k_0)}\,\right| \le |k - k_0|, \qquad k \in \mathbb{R}. \quad (5.2)$$

Proof. By [16, Theorem VII.3.9] there exists an analytic normalized eigenvector $\psi_\ell(k)$ associated to $\lambda_\ell(k)$. From the Feynman-Hellmann formula, see e.g. [16, Section VII.3.4], we obtain

$$|\partial_k\lambda_\ell(k)|^2 = 4\,|((k - i\beta\partial_\tau)\,\psi_\ell(k),\, \psi_\ell(k))_{\mathrm{L}^2(\omega)}|^2 \le 4\,\|(k - i\beta\partial_\tau)\,\psi_\ell(k)\|^2_{\mathrm{L}^2(\omega)}$$
$$\le 4\,(\psi_\ell(k), h_\beta(k)\,\psi_\ell(k))_{\mathrm{L}^2(\omega)} = 4\,\lambda_\ell(k), \qquad k \in \mathbb{R}.$$

Hence

$$|\partial_k \lambda_\ell(k)| \leq 2\sqrt{\lambda_\ell(k)} \qquad k \in \mathbb{R}. \tag{5.3}$$

By integrating this differential inequality we arrive at (5.2). □

Remark 5.3. The eigenvalues $E_n(k)$ given in (2.10) might be degenerate. For example if $\beta = 0$ and if the operator $-\Delta_\omega$ has a degenerate eigenvalue $\mu_n = \mu_m = \mu$, then $E_n(k) = E_m(k) = \mu^2 + k^2$, $\forall k \in \mathbb{R}$.
On the other hand, since every $E_n(k)$ coincides with one of the functions $\lambda_\ell(k)$ locally on intervals between the crossing points of $\{\lambda_\ell(k)\}_\ell$, its multiplicity on these intervals is constant.

Let us define the set

$$\begin{aligned}\mathcal{E}_c :=& \{E \in \mathbb{R} : \exists\, k \in \mathbb{R}, \exists\, \ell, \ell' \in \mathcal{L},\ \ell \neq \ell' : \lambda_\ell(k) = \lambda_{\ell'}(k) = E\} \\ & \cup \{E \in \mathbb{R} : \exists\, k \in \mathbb{R}, \exists\, \ell \in \mathcal{L} : \lambda_\ell(k) = E \ \wedge\ \partial_k \lambda_\ell(k) = 0\}.\end{aligned}$$

Lemma 5.4. *Let $R \in \mathbb{R}$. Then the set $(-\infty, R] \cap \mathcal{E}_c$ is finite. Moreover, there exists an $N_R \in \mathbb{N}$ such that for all $n > N_R$ and all $k \in \mathbb{R}$ we have $E_n(k) > R$.*

Proof. We know that

$$\inf \sigma(h_\beta(k)) = E_1(k) \geq E_1(0) + c\, k^2, \qquad k \in \mathbb{R}, \tag{5.4}$$

for some $c \in (0,1)$, see [5, Theorem 3.1]. This means that there exists some $k_R > 0$ such that

$$E_1(k) > R, \qquad k : |k| > k_R. \tag{5.5}$$

Let us denote $I_R = [-k_R, k_R]$. Hence for any $\ell \in \mathcal{L}$ we have $\lambda_\ell(k) \geq E_1(k) > R$ on $\mathbb{R} \setminus I_R$. We claim that the set

$$\mathcal{L}_R := \{\ell \in \mathcal{L} : \exists\, k \in I_R : \lambda_\ell(k) \leq R\}$$

is finite. Indeed, if $\#\mathcal{L}_R = \infty$, then, in view of (5.5), there is an infinite sequence $\{k_j\} \subset I_R$ such that $\lambda_j(k_j) = R$ for all $j \in \mathcal{L}_R$. By inequalities (5.2) and (5.3) it follows that

$$\sup_{j \in \mathcal{L}_R} \max_{k \in I_R} |\partial_k \lambda_j(k)| \leq 4\, k_R + 2\sqrt{R}.$$

Let $k_\infty \in I_R$ be an accumulation point of the sequence $\{k_j\}$. Hence, for any $\varepsilon > 0$ there exists an infinite set $\mathcal{J}_\varepsilon \subset \mathcal{L}_R$ such that $|\lambda_j(k_\infty) - R| \leq \varepsilon$ for all $j \in \mathcal{J}_\varepsilon$. This means that R is an accumulation point of the spectrum of $h_\beta(k_\infty)$ which contradicts the fact that $\sigma(h_\beta(k_\infty))$ is discrete. We thus conclude that the set $\mathcal{L}_R$ is finite.

Since $\lambda_\ell(k) - \lambda_{\ell'}(k)$ is an analytic function for any $\ell, \ell' \in \mathcal{L}_R$, it has finitely many zeros in the interval I_R. Next, by (5.4) it follows that none of the eigenvalues $\lambda_\ell(k), \ell \in \mathcal{L}$, is constant and therefore, by analyticity, every $\partial_k \lambda_\ell(k)$ has finitely many zeros in I_R. Hence the sets

$$\cup_{\ell \neq \ell', \ell, \ell' \in \mathcal{L}_R} \{k \in I_R : \lambda_\ell(k) = \lambda_{\ell'}(k)\} \quad \text{and} \quad \cup_{\ell \in \mathcal{L}_R} \{k \in I_R : \partial_k \lambda_\ell(k) = 0\}$$

are finite and therefore $(-\infty, R] \cap \mathcal{E}_c$ is finite too. As for the second statement of the Lemma, note that, by (5.5), $E_n(k) > R$ for all $k \notin I_R$ and for all $n \in \mathbb{N}$. If we now set $N_R = \#\mathcal{L}_R + 1$, then N_R satisfies the claim. □

Proof of Lemma 2.6. Let $-\infty < a < b < \infty$ be given. By Lemma 5.4 we know that $\mathcal{E}_c \cap (a, b)$ is a finite set. Since the functions $E_n(k)$ are analytic away from the crossing points of the functions (5.1), it follows that $\mathcal{E} \subset \mathcal{E}_c$. Hence $\mathcal{E} \cap (a, b)$ is finite too. □

Lemma 5.5. *Let $I \subset \mathbb{R}$ be an open interval. Assume that $E_n(k)$ is analytic on I and let $p_n(k)$ be the associated eigenprojection. Then*

$$p_n(k)\,\partial_k E_n(k) = 2\,p_n(k)\,(k - i\beta\partial_\tau)\,p_n(k), \qquad k \in I. \tag{5.6}$$

Proof. Since $E_n(k)$ is analytic on I, it coincides there with one of the analytic functions (5.1). Hence by the Rellich Theorem, [16, Theorem VII.3.9], there exists a family of orthonormal eigenvectors $\phi_n^j(k), j = 1, \dots, q(n,I)$, analytic on I, associated with $E_n(k)$. Here $q(n,I)$ denotes the multiplicity of $E_n(k)$ on I. Since $(\phi_n^j(k),\, \phi_n^i(k))_{\mathrm{L}^2(\omega)} = \delta_{i,j}$ for all $k \in I$, where $\delta_{i,j}$ is the Kronecker symbol, we have

$$(h_\beta(k)\,\phi_n^j(k),\, \phi_n^i(k))_{\mathrm{L}^2(\omega)} = E_n(k)\,\delta_{i,j} \qquad k \in I.$$

By differentiating this identity with respect to k, we easily obtain

$$2\,((k - i\beta\partial_\tau)\,\phi_n^j(k), \phi_n^i(k))_{\mathrm{L}^2(\omega)} = \partial_k E_n(k)\,\delta_{i,j} \qquad k \in \mathbb{R}, \tag{5.7}$$

Hence for any $u \in \mathrm{L}^2(\omega)$

$$\begin{aligned} 2\,p_n(k)\,(k - i\beta\partial_\tau)\,p_n(k)\,u &= 2\sum_{i,j=1}^{q(n,I)} \phi_n^i(k)\,(\phi_n^i(k), (k - i\beta\partial_\tau)\,\phi_n^j(k))_{\mathrm{L}^2(\omega)}\,(\phi_n^j(k), u)_{\mathrm{L}^2(\omega)} \\ &= \partial_k E_n(k) \sum_{j=1}^{q(n,I)} \phi_n^j(k)\,(\phi_n^j(k), u)_{\mathrm{L}^2(\omega)} = \partial_k E_n(k)\,p_n(k)\,u. \end{aligned}$$

□

For the next lemma we need the following definition. Let $\mathcal{I} \subset \mathbb{R}$ be an open interval. Fix $0 < \eta < |\mathcal{I}|/2$ and define the interval

$$\mathcal{I}(\eta) := \{r \in \mathcal{I} \ :\ \mathrm{dist}(\mathrm{r}, \mathbb{R} \setminus \mathcal{I}) \geq \eta\}. \tag{5.8}$$

Let $\chi_{\mathcal{I}}$ be a C^∞ smooth function such that

$$\chi_{\mathcal{I}}(r) = 1 \quad \text{if} \quad r \in \mathcal{I}(\eta) \qquad \text{and} \qquad \chi_{\mathcal{I}}(r) = 0 \quad \text{if} \quad r \notin \mathcal{I}. \tag{5.9}$$

Lemma 5.6. *Suppose that $I \subset \mathbb{R}$ is an open interval. Let $\lambda(k)$ and $\mu(k)$ be two analytic functions from the family (5.1) and assume that there is exactly one point $k_0 \in I$ such that $\lambda(k_0) = \mu(k_0)$, and $\lambda(k) \neq \mu(k)$ for $k_0 \neq k \in I$. Let $\pi_\lambda(k)$ and $\pi_\mu(k)$ be the eigenprojections associated with $\lambda(k)$ and $\mu(k)$. Then in the sense of quadratic forms on $\mathrm{L}^2(\omega)$ we have*

$$\begin{aligned} &\chi_{\mathcal{I}}(\lambda(k))\,\pi_\lambda(k)\,(k - i\beta\partial_\tau)\,\pi_\mu(k)\,\chi_{\mathcal{I}}(\mu(k)) \ \leq \\ &\qquad \leq b_{\lambda,\mu}\,|\lambda(k) - \mu(k)|\,(\chi^2_{\mathcal{I}}(\lambda(k))\,\pi_\lambda(k) + \chi^2_{\mathcal{I}}(\mu(k))\,\pi_\mu(k)) \end{aligned} \tag{5.10}$$

for all $k \in I, k \neq k_0$, where $b_{\lambda,\mu} > 0$ is a constant which depends only on λ, μ and I.

Proof. Let $q(\lambda)$ and $q(\mu)$ denote the multiplicities of $\lambda(k)$ and $\mu(k)$. Let $\psi_\lambda^i(k), i = 1, \dots, q(\lambda)$ and $\psi_\mu^j(k), i = 1, \dots, q(\mu)$ be sets of mutually orthonormal eigenvectors associated to $\lambda(k)$ and $\mu(k)$. By the Rellich Theorem, [16, Theorem VII.3.9], these vectors can be chosen analytic in k. Hence, by differentiating the equation

$$(h_\beta(k)\,\psi_\lambda^i(k),\, \psi_\mu^j(k))_{\mathrm{L}^2(\omega)} = 0 \qquad k \in I,\ k \neq k_0$$

with respect to k we arrive at

$$2((k - i\beta\partial_\tau)\,\psi_\lambda^i(k),\, \psi_\mu^j(k))_{\mathrm{L}^2(\omega)} = (\lambda(k) - \mu(k))(\partial_k\psi_\lambda^i(k), \psi_\mu^j(k))_{\mathrm{L}^2(\omega)} \qquad k \in I,\ k \neq k_0. \tag{5.11}$$

Note that for all $k \neq k_0, k \in I$ we have

$$\pi_\lambda(k) = \sum_{j=1}^{q(\lambda)} \psi_\lambda^j(k)\, (\psi_\lambda^j(k), \cdot\,)_{\mathrm{L}^2(\omega)}, \quad \pi_\mu(k) = \sum_{i=1}^{q(\mu)} \psi_\mu^i(k)\, (\psi_\mu^i(k), \cdot\,)_{\mathrm{L}^2(\omega)}.$$

Let $u \in \mathrm{L}^2(\omega)$ and let

$$\max_{1\leq j\leq q(\mu)} \max_{1\leq i\leq q(\lambda)} \sup_{k\in I} |(\partial_k \psi_\lambda^i(k), \psi_\mu^j(k))_{\mathrm{L}^2(\omega)}| =: \tilde{b}_{\lambda,\mu}. \tag{5.12}$$

From (5.11) we obtain

$$\begin{aligned}
&(u,\, \chi_{\mathcal{I}}(\lambda(k))\, \pi_\lambda(k)\, (k - i\beta\partial_\tau)\, \pi_\mu(k)\, \chi_{\mathcal{I}}(\mu(k))\, u)_{\mathrm{L}^2(\omega)} = \\
&= \frac{1}{2} \sum_{j=1}^{q(\lambda)} \sum_{i=1}^{q(\mu)} \chi_{\mathcal{I}}(\lambda(k))\, \chi_{\mathcal{I}}(\mu(k))(u, \psi_\lambda^i(k))\, (\psi_\mu^j(k), u)\, (\lambda(k) - \mu(k))(\partial_k \psi_\lambda^j(k), \psi_\mu^i(k)) \\
&\leq \tilde{b}_{\lambda,\mu}\, |\lambda(k) - \mu(k)| \sum_{j=1}^{q(\lambda)} \sum_{i=1}^{q(\mu)} \big(\chi_{\mathcal{I}}^2(\lambda(k))|(u, \psi_\lambda^j(k))|^2 + \chi_{\mathcal{I}}^2(\mu(k))\, |(\psi_\mu^i(k), u)|^2\big) \\
&\leq b_{\lambda,\mu}\, |\lambda(k) - \mu(k)| \big(\chi_{\mathcal{I}}^2(\lambda(k))(u, \pi_\lambda(k)\, u) + \chi_{\mathcal{I}}^2(\mu(k))\, (u, \pi_\mu(k)\, u)\big),
\end{aligned}$$

for all $k \neq k_0, k \in I$, where $b_{\lambda,\mu} = \tilde{b}_{\lambda,\mu}\, \max\{q(\lambda), q(\mu)\}$. ☐

6. The conjugate operator

This section is devoted to the construction of the conjugate operator A occurring in the Mourre estimates obtained in the subsequent two sections.

Pick $\gamma \in C_0^\infty(\mathbb{R};\mathbb{R})$, and introduce the operator

$$\hat{A}_0 = \frac{i}{2}(\gamma\, \partial_k + \partial_k\, \gamma), \quad \mathrm{D}(\hat{A}_0) = \mathcal{S}(\mathbb{R}), \tag{6.1}$$

with $\mathcal{S}(\mathbb{R})$ being the Schwartz class on $\mathbb{R}$.

Proposition 6.1. *Let $\gamma \in C_0^\infty(\mathbb{R};\mathbb{R})$. Then the operator $\hat{A}_0$ defined in* (6.1) *is essentially self-adjoint in* $\mathrm{L}^2(\mathbb{R})$.

Proof. Without loss of generality we may assume that there exist $a < b$ such that $\gamma(a) = \gamma(b) = 0$ and $\gamma(k) > 0$ for $k \in (a, b)$. Consider solutions $u_\pm$ to the equations

$$(\hat{A}_0^*\, u)(k) = \frac{i}{2}\, (\gamma(k)\partial_k + \partial_k\, \gamma(k))\, u(k) = \pm\, i\, u(k). \tag{6.2}$$

A direct calculation gives

$$u_\pm(k) = \exp\Big(\int_{k_0}^k \frac{\pm 2 - \gamma'(r)}{2\gamma(r)}\, dr\Big), \quad k \in (a, b) \tag{6.3}$$

for some $k_0 \in (a, b)$. The positivity of γ in (a, b) implies that $\gamma'(a) \geq 0$ and $\gamma'(b) \leq 0$. Hence by the Taylor expansion there exists an $\varepsilon > 0$ and positive constants d_a, d_b such that

$$\gamma(r) \leq d_a\, (r - a) \quad \text{for} \quad r \in (a, a + \varepsilon), \qquad \gamma(r) \leq d_b\, (b - r) \quad \text{for} \quad r \in (b - \varepsilon, b).$$

This, combined with (6.3), yields

$$u_+(k) = \Big(\frac{\gamma(k_0)}{\gamma(k)}\Big)^{1/2} \exp\Big(\int_{k_0}^k \frac{dr}{\gamma(r)}\Big) \geq \Big(\frac{\gamma(k_0)}{\gamma(k)}\Big)^{1/2} \exp\Big(\int_{b-\varepsilon}^k \frac{dr}{d_b(b-r)}\Big)$$
$$\geq c_\varepsilon\,(b-k)^{-\frac12-\frac{1}{d_b}}, \qquad k \in (b-\varepsilon, b),$$

for some $c_\varepsilon > 0$. Hence $u_+ \notin \mathrm{L}^2(\mathbb{R})$. The same argument shows that

$$u_-(k) \geq \tilde c_\varepsilon\,(k-a)^{-\frac12-\frac{1}{d_a}}, \qquad \forall\, k \in (a, a+\varepsilon), \quad \tilde c_\varepsilon > 0,$$

which implies $u_- \notin \mathrm{L}^2(\mathbb{R})$. We thus conclude that $\hat A_0$ has deficiency indices $(0,0)$ and therefore is essentially self-adjoint. □

We define the self-adjoint operator $\hat A$ as the closure of $\hat A_0$ in $\mathrm{L}^2(\mathbb{R})$.

Further, we describe explicitly the action of the unitary group generated by $\hat A$. Given a $k \in \mathbb{R}$ and a function $\gamma \in C_0^\infty(\mathbb{R})$, we consider the initial value problem

$$\frac{d}{dt}\,\varphi(t,k) = -\gamma(\varphi(t,k)), \qquad \varphi(0,k) = k. \tag{6.4}$$

Proposition 6.2. *The mapping*

$$(W(t)f)(k) = |\partial_k \varphi(t,k)|^{1/2}\, f(\varphi(t,k)) \tag{6.5}$$

defines a strongly continuous one-parameter unitary group on $\mathrm{L}^2(\mathbb{R})$*. Moreover,* $\hat A$ *is the generator of* $W(t)$*.*

Proof. Since γ is globally Lipschitz, the Cauchy problem (6.4) has a unique global solution. By the regularity of γ and [15, Corollary V.4.1], it follows that $\varphi \in C^\infty(\mathbb{R}^2)$. Moreover,

$$\partial_k \varphi(t,k) = \exp\Big(-\int_0^t \gamma'(\varphi(s,k))\,ds\Big) \qquad \forall\, t \geq 0,\ \forall\, k \in \mathbb{R}, \tag{6.6}$$

[15, Corollary V.3.1]. Hence $\partial_k \varphi(t,k) > 0$. Since $\varphi(t+t',k) = \varphi(t, \varphi(t',k))$, we have

$$W(t)\,W(t') = W(t+t').$$

Next, from (6.4) and (6.6) we deduce that for $k \notin \operatorname{supp}\gamma$ we have $\varphi(t,k) = k$ for all $t \geq 0$. In order to verify that $W(t)$ is strongly continuous on $\mathrm{L}^2(\mathbb{R})$, let that $f \in \mathrm{L}^2(\mathbb{R})$. We then have

$$\|W(t)f - f\|^2_{\mathrm{L}^2(\mathbb{R})} \leq 2\,\|\partial_k\varphi(t,k)^{1/2}(f\circ\varphi(t,k) - f)\|^2_{\mathrm{L}^2(\mathbb{R})} + 2\,\|(\partial_k\varphi(t,k)^{1/2} - 1)\,f\|^2_{\mathrm{L}^2(\mathbb{R})} \tag{6.7}$$
$$\leq c\int_{\operatorname{supp}\gamma}\Big(|f(\varphi(t,k)) - f(k)|^2 + |\partial_k\varphi(t,k)^{1/2} - 1|^2\,|f(k)|^2\Big)\,dk.$$

From (6.6) and from the fact that $\gamma' \in \mathrm{L}^\infty(\mathbb{R})$ it is easily seen that $\varphi(t,k) \to k$ and $\partial_k\varphi(t,k) \to 1$ as $t \to 0$ uniformly in k on compact subsets of $\mathbb{R}$. Since $\operatorname{supp}\gamma$ is compact, (6.7) implies that

$$\|W(t)f - f\|_{\mathrm{L}^2(\mathbb{R})} \ \to\ 0, \qquad t \to 0.$$

Moreover, using (6.4), a direct calculation gives

$$\frac{d}{dt}(W(t)f)(k)\,\Big|_{t=0} = -\frac12\,\gamma'(k)f(k) - \gamma(k)f'(k) = (i\,\hat A\,f)(k), \quad f \in \mathcal{S}(\mathbb{R}).$$

Hence by [21, Theorem VIII.10] it follows that $\hat A$ generates the unitary group $W(t)$. □

Let γ be as in Theorem 7.2. By Proposition 6.1 and [21, Theorem VIII.33] it follows that the operator $\mathbb{1}_\omega \otimes \hat{A}$ is essentially self-adjoint on $C_0^\infty(\omega) \otimes \mathcal{S}(\mathbb{R})$. The same is true for the operator $\mathcal{F}^* \, (\mathbb{1}_\omega \otimes \hat{A}) \, \mathcal{F}$. We define the conjugate operator A in $\mathrm{L}^2(\Omega)$ as its closure:

$$A = \bar{A}_0, \qquad A_0 = \mathcal{F}^* \, (\mathbb{1}_\omega \otimes \hat{A}) \, \mathcal{F}, \quad \mathrm{D}(A_0) = C_0^\infty(\omega) \otimes \mathcal{S}(\mathbb{R}). \tag{6.8}$$

Let Γ be the operator in $\mathrm{L}^2(\mathbb{R})$ acting as

$$(\Gamma \, \psi)(x_3) := (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{\gamma}(x_3 - t) \, \psi(t) \, dt, \qquad \hat{\gamma} := \mathcal{F}_1 \gamma, \tag{6.9}$$

where $\mathcal{F}_1$ denotes the Fourier transform from $\mathrm{L}^2(\mathbb{R})$ onto $\mathrm{L}^2(\mathbb{R})$:

$$(\mathcal{F}_1 \, f)(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iks} f(s) \, ds, \qquad f \in \mathrm{L}^2(\mathbb{R}).$$

A direct calculation then shows that

$$A_0 = -\frac{1}{2} \, \mathbb{1}_\omega \otimes (\Gamma \, x_3 + x_3 \, \Gamma).$$

7. Mourre estimates for a constant twisting

In this section we establish a Mourre estimate for the commutator $[H_\beta, iA]$ with β constant and A defined in (6.8).

In the sequel, we use the following notation. Given a self-adjoint positive operator S, invertible in $\mathrm{L}^2(\Omega)$, we denote by $\mathrm{D}(S^\nu)^*$, $\nu > 0$, the completion of $\mathrm{L}^2(\Omega)$ with respect to the norm $\|S^{-\nu} \, u\|_{\mathrm{L}^2(\Omega)}$.

Lemma 7.1. *The commutator $[\hat{H}_\beta, \, i(\mathbb{1}_\omega \otimes \hat{A})]$ defined as a quadratic form on $C_0^\infty(\Omega)$ extends to a bounded operator from $\mathrm{D}(\hat{H}_\beta)$ into $\mathrm{D}(\hat{H}_\beta)^*$. Moreover,*

$$[\hat{H}_\beta, \, i(\mathbb{1}_\omega \otimes \hat{A})] = 2\gamma(k)(k - i\beta\partial_\tau). \tag{7.1}$$

Proof. Let $u \in C_0^\infty(\Omega)$. A simple calculation then gives

$$(\hat{H}_\beta \, u, \, i(\mathbb{1}_\omega \otimes \hat{A}) \, u)_{\mathrm{L}^2(\Omega)} - (i(\mathbb{1}_\omega \otimes \hat{A}) \, u, \, \hat{H}_\beta \, u)_{\mathrm{L}^2(\Omega)} = 2(u, \gamma \, (k - i\beta\partial_\tau) \, u)_{\mathrm{L}^2(\Omega)}.$$

Hence (7.1) follows. Moreover, from the above equation we easily obtain

$$|([\hat{H}_\beta, \, i(\mathbb{1}_\omega \otimes \hat{A})] \, u, \, u)_{\mathrm{L}^2(\Omega)}| \le \, C \left(\|\hat{H}_\beta \, u\|^2_{\mathrm{L}^2(\Omega)} + \|u\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{7.2}$$

So $[\hat{H}_\beta, i(\mathbb{1}_\omega \otimes \hat{A})]$ is a bounded operator from $\mathrm{D}(\hat{H}_\beta)$ into $\mathrm{L}^2(\Omega)$, and hence it is also bounded from $\mathrm{D}(\hat{H}_\beta)$ into $\mathrm{D}(\hat{H}_\beta)^*$. □

Theorem 7.2. *Let $E \in \mathbb{R} \setminus \mathcal{E}$. Then there exist $\delta > 0$, a function $\gamma \in C_0^\infty(\mathbb{R})$ and a positive constant $c = c(E, \delta)$ such that in the form sense on $\mathrm{L}^2(\Omega)$ we have*

$$\chi_{\mathcal{I}}(\hat{H}_\beta) \, [\hat{H}_\beta, i(\mathbb{1}_\omega \otimes \hat{A})] \, \chi_{\mathcal{I}}(\hat{H}_\beta) \ge c \, \chi^2_{\mathcal{I}}(\hat{H}_\beta), \tag{7.3}$$

where $\mathcal{I} = (E - \delta, E + \delta)$, $\chi_{\mathcal{I}}$ is given by (5.9) and the commutator $[\hat{H}_\beta, i(\mathbb{1}_\omega \otimes \hat{A})]$ is understood as a bounded operator from $\mathrm{D}(\hat{H}_\beta)$ into $\mathrm{D}(\hat{H}_\beta)^*$.

Proof. First of all we chose δ small enough such that

$$\mathrm{dist}(E, (\mathcal{E}_c \setminus E)) > \delta, \tag{7.4}$$

which is possible in view of Lemma 5.4. Recall that $\mathcal{E} \subset \mathcal{E}_c$. Next we define

$$\mathcal{K}(n, E) = \big\{ k \in \mathbb{R} \, : \, E_n(k) = E \big\}.$$

Note that by Lemma 5.4 $\mathcal{K}(n, E)$ is finite for every n and $\mathcal{K}(n, E) = \emptyset$ for all $n > N_{E+\delta}$. In the rest of the proof we use the notation $N = N_{E+\delta}$. Let

$$\mathcal{K}(E) = \cup_{n=1}^{\infty} \, \mathcal{K}(n, E) = \cup_{n=1}^{N} \, \mathcal{K}(n, E),$$

and define

$$\begin{aligned} &\mathcal{K}_0(E) = \{ k \in \mathbb{R} : \text{there exists a unique } n \text{ such that } E_n(k) = E \}, \\ &\mathcal{K}_1(E) = \mathcal{K}(E) \setminus \mathcal{K}_0(E). \end{aligned}$$

Now we introduce the sets

$$B(n, E) = \{ k \in \mathbb{R} \, : \, E_n(k) \in (E - \delta, E + \delta) \}.$$

By Lemma 5.4 we have $B(n, E) = \emptyset$ for all $n > N$. From (7.4) it follows that each $B(n, E)$ is given by a union of finitely many non-degenerate disjoint open intervals:

$$B(n, E) = \cup_{j=1}^{G_n} Q(j, n), \qquad Q(j, n) \cap Q(i, n) = \emptyset \quad \text{if} \quad i \neq j.$$

Moreover, every $Q(j, n)$ contains exactly one element of $\mathcal{K}(n, E)$. We will label the intervals $Q(n, E)$ as follows:

$$\begin{aligned} Q_0(j, n) &:= Q(j, n) \quad \text{if} \quad Q(j, n) \cap \mathcal{K}(n, E) \subset \mathcal{K}_0(E) \\ Q_1(j, n) &:= Q(j, n) \quad \text{if} \quad Q(j, n) \cap \mathcal{K}(n, E) \subset \mathcal{K}_1(E). \end{aligned}$$

By the hypothesis on E we can take δ small enough so that

$$Q_0(j, n) \cap Q_0(j, m) = \emptyset \qquad n \neq m,$$

and at the same time

$$Q_1(j, n) \cap \mathcal{K}_1(E) \neq Q_1(i, m) \cap \mathcal{K}_1(E)$$

implies

$$Q_1(j, n) \cap Q_1(i, m) = \emptyset.$$

Hence, for δ sufficiently small, we can construct intervals $J_{0,l}$ with $l = 1, \dots, L(E)$, and $J_{1,p}$ with $p = 1, \dots, P(E)$, such that

$$J_{0,l} \cap J_{0,l'} = \emptyset \quad l \neq l', \quad J_{1,p} \cap J_{1,p'} = \emptyset \quad p \neq p', \quad J_{0,l} \cap J_{1,p} = \emptyset \quad \forall \; l, \, p, \tag{7.5}$$

and such that

$$\mathcal{M}_0(E) := \bigcup_{n=1}^{N} \Big(\cup_j \; Q_0(j, n) \Big) = \bigcup_{l=1}^{L(E)} \; J_{0,l}, \quad \mathcal{M}_1(E) := \bigcup_{n=1}^{N} \Big(\cup_j \; Q_1(j, n) \Big) = \bigcup_{p=1}^{P(E)} \; J_{1,p}.$$

Moreover, each $J_{0,l}$ contains exactly one element $k_{0,l}$ of $\mathcal{K}_0(E)$ and each $J_{1,p}$ contains exactly one element $k_{1,p}$ of $\mathcal{K}_1(E)$. By construction, we have

$$\mathcal{M}(E) := \mathcal{M}_0(E) \cup \mathcal{M}_1(E) = \cup_{n=1}^{N} B(n, E), \qquad \mathcal{M}_0(E) \cap \mathcal{M}_1(E) = \emptyset.$$

With these preliminaries we can proceed with the estimation of the commutator. From Lemma 7.1 we find that

$$
\begin{aligned}
\chi_{\mathcal{I}}(\hat{H}_\beta)\,[\hat{H}_\beta, i(\mathbb{1}_\omega\otimes\hat{A})]\,\chi_{\mathcal{I}}(\hat{H}_\beta) = \qquad (7.6)\\
&= 2\sum_{n,m=1}^{\infty}\int_{\mathbb{R}}^{\oplus}\chi_{\mathcal{I}}(E_n(k))p_n(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk\\
&= 2\sum_{n,m=1}^{N}\int_{\mathcal{M}_0(E)}^{\oplus}\chi_{\mathcal{I}}(E_n(k))p_n(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk\\
&\quad+2\sum_{n,m=1}^{N}\int_{\mathcal{M}_1(E)}^{\oplus}\chi_{\mathcal{I}}(E_n(k))p_n(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk.
\end{aligned}
$$

To estimate the first term on the right hand side of (7.6) we note that by construction of $\mathcal{M}_0(E)$, for each $l=1,\dots,L(E)$ there exists exactly one $n(l)\le N$ such that

$$
\begin{aligned}
&\sum_{n,m=1}^{N}\int_{J_{0,l}}^{\oplus}\chi_{\mathcal{I}}(E_n(k))p_n(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk=\\
&\qquad=\int_{J_{0,l}}^{\oplus}\chi_{\mathcal{I}}(E_{n(l)}(k))p_{n(l)}(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_{n(l)}(k))p_{n(l)}(k)\,dk.
\end{aligned}
$$

Moreover, since $E_{n(l)}(k)$ does not cross any other eigenvalue of $h_\beta(k)$ on $J_{0,l}$, it is analytic on $J_{0,l}$. Hence by Lemma 5.5 we obtain

$$
\begin{aligned}
&\int_{J_{0,l}}^{\oplus}\chi_{\mathcal{I}}(E_{n(l)}(k))p_{n(l)}(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_{n(l)}(k))p_{n(l)}(k)\,dk=\\
&\qquad=\int_{J_{0,l}}^{\oplus}\chi_{\mathcal{I}}^2(E_{n(l)}(k))\,p_{n(l)}(k)\,\gamma(k)\,\partial_k E_{n(l)}dk.
\end{aligned}
$$

In view of (7.5) we can choose the function γ such that

$$
\gamma(k)\,\partial_k E_{n(l)}(k)=|\partial_k E_{n(l)}(k)|\qquad\forall\ k\in J_{0,l}\,,\quad\forall\ l=1,\dots,L(E). \tag{7.7}
$$

Note that $|\partial_k E_{n(l)}(k)|$ is strictly positive on $J_{0,l}$. Therefore we have

$$
d_0:=\min_{1\le l\le L(E)}\ \inf_{k\in J_{0,l}}|\partial_k E_{n(l)}(k)|>0.
$$

Hence,

$$
\begin{aligned}
&\sum_{n,m=1}^{N}\int_{\mathcal{M}_0(E)}^{\oplus}\chi_{\mathcal{I}}(E_n(k))\,p_n(k)\,\gamma(k)(k-i\beta\partial_\tau)\,\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk\ge \qquad (7.8)\\
&\qquad\ge d_0\sum_{n=1}^{N}\int_{\mathcal{M}_0(E)}^{\oplus}\chi_{\mathcal{I}}^2(E_n(k))p_n(k)\,dk.
\end{aligned}
$$

Let us now estimate the second term on the right hand side of (7.6). On every interval $J_{1,p}$ we have

$$\sum_{n,m=1}^{N} \int_{J_{1,p}}^{\oplus} \chi_{\mathcal{I}}(E_n(k)) p_n(k)\gamma(k)(k - i\beta\partial_\tau)\chi_{\mathcal{I}}(E_m(k))p_m(k)\, dk =$$
$$= \int_{J_{1,p}}^{\oplus} \sum_{r,r'\in R(p)} \chi_{\mathcal{I}}(E_r(k))\, p_r(k)\, \gamma(k)(k - i\beta\partial_\tau)\chi_{\mathcal{I}}(E_{r'}(k))\, p_{r'}(k)\, dk$$

for some $R(p) \subset \{1, \dots, N\}$. Moreover, from the construction of the intervals $J_{1,p}$ it follows that there exists a family of analytic eigenfunctions $\lambda_s(k), s \in S(p)$, with $S(p)$ being a finite subset of $\mathbb{N}$, such that each $E_r(k)$ coincides with some $\lambda_s(k)$ on $J_{1,p} \cap (-\infty, k_{1,p})$ and with some $\lambda_{s'}(k)$ on $J_{1,p} \cap (k_{1,p}, \infty)$, where $k_{1,p}$ is the only element of $\mathcal{K}_1(E)$ contained in $J_{1,p}$. Let $\pi_s(k)$ be the eigenprojection associated with $\lambda_s(k)$. With the help of Lemma 5.6 we obtain

$$\begin{aligned}
&\int_{J_{1,p}}^{\oplus} \sum_{r,r'\in R(p)} \chi_{\mathcal{I}}(E_r(k))\, p_r(k)\gamma(k)(k - i\beta\partial_\tau)\chi_{\mathcal{I}}(E_{r'}(k))p_{r'}(k)\, dk = \\
&\quad = \int_{J_{1,p}}^{\oplus} \sum_{s,s'\in S(p)} \chi_{\mathcal{I}}(\lambda_s(k))\, \pi_s(k)\gamma(k)(k - i\beta\partial_\tau)\chi_{\mathcal{I}}(\lambda_{s'}(k))\, \pi_{s'}(k)\, dk = \\
&\quad = \int_{J_{1,p}}^{\oplus} \sum_{s\in S(p)} \chi^2_{\mathcal{I}}(\lambda_s(k))\, \gamma(k)\, \partial_k\lambda_s(k)\, \pi_s(k)\, dk + \\
&\quad\quad + \int_{J_{1,p}}^{\oplus} \sum_{s\neq s'\in S(p)} \chi_{\mathcal{I}}(\lambda_s(k))\, \pi_s(k)\gamma(k)(k - i\beta\partial_\tau)\chi_{\mathcal{I}}(\lambda_{s'}(k))\, \pi_{s'}(k)\, dk.
\end{aligned} \tag{7.9}$$

Since the intervals $J_{1,p}$ are mutually disjoint and also disjoint from the set $\mathcal{M}_0(E)$, see (7.5), and since the functions $\partial_k\lambda_s(k)$ are either all strictly negative or all strictly positive on every interval $J_{1,p}$, by the construction of $J_{1,p}$, we can choose γ such that, in addition to (7.7), it holds

$$\gamma(k)\, \partial_k\lambda_s(k) = |\partial_k\lambda_s(k)| \qquad \forall\, k \in J_{1,p}\,,\ \forall\, s \in S(p),\ \forall\, p = 1, \dots, P(E). \tag{7.10}$$

Moreover,

$$d_1 := \min_{1\leq p\leq P(E)} \min_{s\in S(p)} \inf_{k\in J_{1,p}} |\partial_k\lambda_s(k)| > 0.$$

Now, to control the last term in (7.9) assume that $s \neq s'$ and let $b_{\lambda_s,\lambda_{s'}}$ be the constant given in Lemma 5.6 with $I = J_{1,p}$. Note that $|J_{1,p}|$ decreases as $\delta \to 0$. From the explicit expression for $b_{\lambda_s,\lambda_{s'}}$, see (5.12), it is then easily seen that there exists $b_p > 0$, independent of δ, such that

$$\max_{s,s'\in S(p), s\neq s'} b_{\lambda_s,\lambda_{s'}} \leq b_p.$$

Hence, (7.9), in combination with Lemmata 5.5 and 5.6, yields

$$\int_{J_{1,p}}^{\oplus} \sum_{r,r'\in R(p)} \chi_{\mathcal{I}}(E_r(k))\, p_r(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_{r'}(k))p_{r'}(k)\,dk \geq$$

$$\geq (d_1 - c_p\, b_p\, \delta)\int_{J_{1,p}}^{\oplus} \sum_{s\in S(p)} \chi^2_{\mathcal{I}}(\lambda_s(k))\,\pi_s(k)\,dk =$$

$$= (d_1 - c_p\, b_p\, \delta)\int_{J_{1,p}}^{\oplus} \sum_{r\in R(p)} \chi^2_{\mathcal{I}}(E_r(k))\,p_r(k)\,dk,$$

where $c_p > 0$ depends only on p. Therefore we obtain

$$\sum_{n,m=1}^{N}\int_{\mathcal{M}_1(E)}^{\oplus} \chi_{\mathcal{I}}(E_n(k))\,p_n(k)\,\gamma(k)(k-i\beta\partial_\tau)\,\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk \geq \tag{7.11}$$

$$\geq (d_1 - C_E\,\delta)\sum_{n=1}^{N}\int_{\mathcal{M}_1(E)}^{\oplus} \chi^2_{\mathcal{I}}(E_n(k))p_n(k)\,dk,$$

with $C_E = \max_{1\leq p\leq P(E)} c_p\, b_p$. Taking into account (7.8), we thus conclude that for δ small enough there exists some $c > 0$ such that

$$\sum_{n,m=1}^{N}\int_{\mathcal{M}(E)}^{\oplus} \chi_{\mathcal{I}}(E_n(k))p_n(k)\gamma(k)(k-i\beta\partial_\tau)\chi_{\mathcal{I}}(E_m(k))p_m(k)\,dk$$

$$\geq c\sum_{n=1}^{N}\int_{\mathcal{M}(E)}^{\oplus} \chi^2_{\mathcal{I}}(E_n(k))p_n(k)\,dk = c\sum_{n=1}^{\infty}\int_{\mathbb{R}}^{\oplus} \chi^2_{\mathcal{I}}(E_n(k))p_n(k)\,dk = c\,\chi^2_{\mathcal{I}}(\hat{H}_\beta).$$

In view of (7.6) this proves the theorem.

□

Corollary 7.3. *Let $E \in \mathbb{R}\setminus\mathcal{E}$ and $\mathcal{I} = (E-\delta, E+\delta)$ be given as in Theorem 7.2. Then*

$$\chi_{\mathcal{I}}(H_\beta)\,[H_\beta, iA]\,\chi_{\mathcal{I}}(H_\beta) \geq c\,\chi^2_{\mathcal{I}}(H_\beta), \tag{7.12}$$

where $[H_\beta, iA]$ is understood as a bounded operator from $\mathrm{D}(H_\beta)$ into $\mathrm{D}(H_\beta)^*$, and the conjugate operator is defined by (6.1) *and* (6.8)*.*

Proof. This follows from (2.9), (6.8) and Theorem 7.2. □

8. Perturbation of the constant twisting

8.1. **Mourre estimate for $[H_{\theta'}, iA]$.** In the sequel we will suppose that

$$\theta'(x_3) = \beta - \varepsilon(x_3).$$

In this section we will prove a Mourre estimate for the commutator $[H_{\theta'}, iA]$, see below Theorem 8.2. Notice that $H_{\theta'}$ acts as

$$H_{\theta'} = H_\beta + W, \quad W = (2\varepsilon\beta - \varepsilon^2)\partial_\tau^2 + 2\,\varepsilon\,\partial_\tau\partial_3 + +\varepsilon'\,\partial_\tau \tag{8.1}$$

on $\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$, cf. Corollary 2.2. Together with (8.1) we will also use the decomposition

$$H_{\theta'} = H_0 + U, \quad U = W - \beta^2\partial_\tau^2 - 2\,\beta\,\partial_\tau\partial_3. \tag{8.2}$$

Lemma 8.1. *Let $\chi_{\mathcal{I}} \in C_0^\infty(\mathbb{R})$ be given by (5.9). Then the operator $\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta)$ is compact in* $\mathrm{L}^2(\Omega)$.

Proof. The Helffer-Sjöstrand formula, [7, 8], gives

$$\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial\tilde{\chi}}{\partial\bar{z}}\,(H_{\theta'}-z)^{-1}W(H_\beta-z)^{-1}\,dx\,dy, \tag{8.3}$$

where $z = x+iy$, and $\tilde{\chi}$ is a compactly supported quasi-analytic extension of $\chi_{\mathcal{I}} I$ in $\mathbb{R}^2$ which satisfies

$$\sup_{x\in\mathbb{R}} \Big|\frac{\partial\tilde{\chi}}{\partial\bar{z}}(x+iy)\Big| \leq \text{ const}\, y^4\;, \qquad |y|\leq 1. \tag{8.4}$$

Since $(H_{\theta'}-z)^{-1}W(H_0-z)^{-1}$ is compact whenever $y\neq 0$, see [5], it follows that $\frac{\partial\tilde{\chi}}{\partial\bar{z}}(H_{\theta'} - z)^{-1}W(H_\beta-z)^{-1}$ is compact for all $(x,y)\in\mathbb{R}^2$ with $y\neq 0$. Moreover, by the resolvent equation the norm of $(H_{\theta'}-z)^{-1}W(H_0-z)^{-1}$ is bounded by a constant times y^{-2}. In view of (8.4) the integrand on the right hand side of (8.3) is then uniformly norm-bounded in $\mathbb{R}^2$ and hence $\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta)$ is compact. □

Theorem 8.2. *Let $E\in\mathbb{R}\setminus\mathcal{E}$ and let ε satisfy (2.13). Choose $\delta>0$ and $\gamma\in C_0^\infty(\mathbb{R})$ as in Theorem 7.2. Then there exists a positive constant c and a compact operator K in $\mathrm{L}^2(\Omega)$ such that*

$$P_{\mathcal{I}(E,\delta)}\,[H_{\theta'}, iA]\,P_{\mathcal{I}(E,\delta)} \;\geq\; c\;P^2_{\mathcal{I}(E,\delta)} + P_{\mathcal{I}(E,\delta)}\,K\,P_{\mathcal{I}(E,\delta)}, \tag{8.5}$$

where $P_{\mathcal{I}(E,\delta)}$ is the spectral projection for the interval

$$\mathcal{I}(E,\delta) := (E-\delta/2, E+\delta/2),$$

associated to $H_{\theta'}$.

Proof. Let $\mathcal{I} = (E-\delta, E+\delta)$. We proceed in several steps. First we show that there exists $c>0$ and a compact operator K_1 in $\mathrm{L}^2(\Omega)$ such that

$$\chi_{\mathcal{I}}(H_{\theta'})[H_\beta, iA]\chi_{\mathcal{I}}(H_{\theta'}) \geq c\,\chi^2_{\mathcal{I}}(H_\beta) + K_1. \tag{8.6}$$

We write

$$\begin{aligned}\chi_{\mathcal{I}}(H_{\theta'})[H_\beta, iA]\chi_{\mathcal{I}}(H_{\theta'}) &= \chi_{\mathcal{I}}(H_\beta)[H_\beta, iA]\chi_{\mathcal{I}}(H_\beta) + \chi_{\mathcal{I}}(H_\beta)[H_\beta, iA](\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta))\\ &\quad + (\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta))[H_\beta, iA]\chi_{\mathcal{I}}(H_{\theta'}).\end{aligned}$$

By Corollary 7.3 there exist $c>0$ such that

$$\chi_{\mathcal{I}}(H_\beta)[H_\beta, iA]\chi_{\mathcal{I}}(H_\beta) \geq c\chi^2_{\mathcal{I}}(H_\beta).$$

It can be verified by a simple calculation that the operator Γ defined in (6.9) commutes with H_β. Hence

$$\chi_{\mathcal{I}}(H_\beta)[H_\beta, iA](\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta)) = 2\chi_{\mathcal{I}}(H_\beta)(i\partial_3 + \beta i\partial_\tau)\Gamma(\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta)).$$

We know that $\Gamma(\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta))$ is compact (see e.g. Lemma 8.1). The operators $(H_{\theta'} + 1)^{-1}(i\partial_3+\beta i\partial_\tau)$ and $\chi_{\mathcal{I}}(H_{\theta'})(H_{\theta'}+1)$ are bounded so $\chi_{\mathcal{I}}(H_\beta)(i\partial_3+\beta i\partial_\tau)$ is bounded too and $K_{11} := \chi_{\mathcal{I}}(H_\beta)(i\partial_3+\beta i\partial_\tau)\Gamma(\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta))$ is compact. The same arguments show that

$$K_{12} := (\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta))[H_\beta, iA]\chi_{\mathcal{I}}(H_{\theta'}) := 2(\chi_{\mathcal{I}}(H_{\theta'}) - \chi_{\mathcal{I}}(H_\beta))\Gamma(i\partial_3 - \beta i\partial_\tau)\chi_{\mathcal{I}}(H_{\theta'})$$

is compact. Putting $K_1 = K_{11} + K_{12}$ concludes the first step of the proof.

Next we consider $\chi_{\mathcal{I}}(H_{\theta'})[W, iA]\chi_{\mathcal{I}}(H_{\theta'})$. For the sake of simplicity we now write s instead of x_3. Defining

$$\eta(s) := 2\varepsilon(s)\beta - \varepsilon(s)^2$$

we get

$$[W, iA] = [\eta\,, iA]\,\partial_\tau^2 + [\varepsilon\,\partial_s, iA]\,\partial_\tau + [\partial_s\,\varepsilon,\, iA]\,\partial_\tau. \tag{8.7}$$

We first deal with the term $[\eta\,\partial_\tau^2, iA] = i[\eta, A]\partial_\tau^2 = -\frac{i}{2}[\eta, \Gamma s + s\Gamma]\,\partial_\tau^2$. For an appropriate test function ϕ we obtain

$$\begin{aligned}\sqrt{2\pi}\;([\eta, \Gamma s + s\Gamma]\phi)(s) =: (T\phi)(s) &= \eta(s)\int_{\mathbb{R}} \hat\gamma(s-s')s'\phi(s')\,ds' + \eta(s)\int_{\mathbb{R}} s\hat\gamma(s-s')\phi(s')\,ds' \\ &\quad - \int_{\mathbb{R}} \hat\gamma(s-s')\eta(s')s'\phi(s')\,ds' - \int_{\mathbb{R}} s\hat\gamma(s-s')\eta(s')\phi(s')\,ds'.\end{aligned}$$

Hence T is an integral operator on $\mathrm{L}^2(\mathbb{R})$ with the kernel

$$T(s, s') = \eta(s)\hat\gamma(s-s')s' + \eta(s)s\hat\gamma(s-s') - \hat\gamma(s-s')\eta(s')s' - s\hat\gamma(s-s')\eta(s').$$

To control the s-dependence we rewrite the kernel as

$$\begin{aligned}T(s, s') &= \eta(s)\hat\gamma(s-s')(s'-s) + 2\eta(s)\,s\,\hat\gamma(s-s') \\ &\quad - 2\hat\gamma(s-s')\eta(s')\,s' - (s-s')\hat\gamma(s-s')\eta(s').\end{aligned} \tag{8.8}$$

Next we recall that if $f \in \mathrm{L}^q(\mathbb{R})$, $g \in \mathrm{L}^p(\mathbb{R})$, $q \in [2, \infty)$, $1/q + 1/p = 1$, then the Hausdorff–Young inequality $\|\hat g\|_{\mathrm{L}^q(\mathbb{R})} \le (2\pi)^{\frac{1}{2}-\frac{1}{p}}\|g\|_{\mathrm{L}^p(\mathbb{R})}$ and the interpolation result which we already used in the proof of Lemma 4.1 (see [24, Theorem 4.1] or [3, Section 4.4]), imply that the integral operator with a kernel of the form $f(s)\,g(s-s')$, $s, s' \in \mathbb{R}$, belongs to the class S_q, and hence is compact on $\mathrm{L}^2(\mathbb{R})$. By (2.13), both functions $\eta(s)$ and $s\eta(s)$ are in $\mathrm{L}^q(\mathbb{R})$ for q large enough. Since $\gamma \in C_0^\infty(\mathbb{R})$, its Fourier transform $\hat\gamma$ is in the Schwartz class on $\mathbb{R}$ and therefore in any $\mathrm{L}^p(\mathbb{R})$ with $p \ge 1$. Therefore, the operator $[\eta, \Gamma s + s\Gamma]$ is compact on $\mathrm{L}^2(\mathbb{R})$. In order to ensure the compactness of $\chi_{\mathcal{I}}(H_{\theta'})\,T\,\partial_\tau^2\,\chi_{\mathcal{I}}(H_{\theta'})$ on $\mathrm{L}^2(\Omega)$, we note that by Corollary 2.2 and the closed graph theorem the operators $H_\beta^{-1}\,H_{\theta'}$ and $H_{\theta'}^{-1}\,H_\beta$ are bounded on $\mathrm{L}^2(\Omega)$. Since $H_{\theta'}\,\chi_{\mathcal{I}}(H_{\theta'})$ is bounded too, it suffices to prove that

$$H_\beta^{-1}\,T\,\partial_\tau^2\,H_\beta^{-1} \tag{8.9}$$

is compact on $\mathrm{L}^2(\Omega)$. To this end we point out that $H_\beta \ge \mathbb{1}_\omega \otimes (-\Delta_\omega)$ and that the operators $\partial_\tau^2\,(-\Delta_\omega)^{-1}$ and $(-\Delta_\omega)^{-1}$ are respectively bounded and compact on $\mathrm{L}^2(\omega)$. Hence $(\mathbb{1}_\omega \otimes (-\Delta_\omega))^{-1}\,T\,\partial_\tau^2\,(\mathbb{1}_\omega \otimes (-\Delta_\omega))^{-1}$ is a product of a bounded and a compact operator and hence is compact on $\mathrm{L}^2(\Omega)$. This yields the compactness of the operator (8.9).

In the same way we deal with the remaining terms on the right hand side of (8.7). As for the the operator

$$[\partial_\tau\varepsilon\partial_s,\, iA] = -\frac{i}{2}\;\partial_\tau\,[\varepsilon\partial_s, \Gamma s + s\Gamma],$$

with the help of the integration by parts we find that

$$\begin{aligned}([\varepsilon\partial_s, \Gamma s + s\Gamma]\phi)(s) &= (2\pi)^{-1/2}(R_1\,\phi)(s) + (2\pi)^{-1/2}(R_2\,\phi)(s) \\ &= (2\pi)^{-1/2}\int_{\mathbb{R}} R_1(s, s')\,\phi(s')\,ds' + (2\pi)^{-1/2}\int_{\mathbb{R}} R_2(s, s')\,\phi(s')\,ds',\end{aligned}$$

where the integral kernels $R_1(s,s')$ and $R_2(s,s')$ of the operators R_1 and R_2 are given by

$$R_1(s,s') = \varepsilon(s)\left(\hat\gamma'(s-s')(s'-s) + \hat\gamma(s-s') + 2\,s\,\hat\gamma'(s-s')\right) \tag{8.10}$$

$$R_2(s,s') = \varepsilon(s')\left(-\hat\gamma'(s-s')s' + \hat\gamma(s-s') - s\,\hat\gamma(s-s')\right) + \varepsilon'(s')\left(\hat\gamma(s-s')s' + s\hat\gamma(s-s')\right),$$

and $\hat\gamma'$ denotes the derivative of $\hat\gamma$. As above we need to write also the kernel $T_2(s,s')$ as a sum of the terms of the form $f(s)\,g(s-s')$ and $f(s')\,g(s-s')$:

$$\begin{aligned} R_2(s,s') = {} & \varepsilon(s')\left(-2\hat\gamma'(s-s')s' + \hat\gamma(s-s') - (s-s')\,\hat\gamma(s-s')\right) \\ & + \varepsilon'(s')\left(2\hat\gamma(s-s')s' + (s-s')\hat\gamma(s-s')\right). \end{aligned} \tag{8.11}$$

Using the assumptions of Theorem 2.7 and the fact that $\hat\gamma'$ is the Schwartz class on $\mathbb{R}$, we conclude as before that R_1 and R_2 are compact on $\mathrm{L}^2(\mathbb{R})$ and therefore

$$\chi_{\mathcal{I}}(H_{\theta'})\,\partial_\tau\,[\varepsilon\,\partial_s, \Gamma s + s\Gamma]\,\chi_{\mathcal{I}}(H_{\theta'})$$

is compact on $\mathrm{L}^2(\Omega)$. The compactness of

$$\chi_{\mathcal{I}}(H_{\theta'})\,[\partial_s\,\varepsilon\,\partial_\tau, iA]\,\chi_{\mathcal{I}}(H_{\theta'})$$

follows in a completely analogous way. Hence we obtain

$$\chi_{\mathcal{I}}(H_{\theta'})[H_{\theta'}, iA]\,\chi_{\mathcal{I}}(H_{\theta'}) \geq c\,\chi^2_{\mathcal{I}}(H_{\theta'}) + K \tag{8.12}$$

where $\mathcal{I} = (E-\delta, E+\delta)$ and K is compact. Now we fix $\eta = \delta/2$ in (5.9). The statement then follows by multiplying the last inequality from the left and from the right by $P_{\mathcal{I}(E,\delta)}$. □

8.2. **Proof of Theorem 2.7.** In order to prove Theorem 2.7, we will need, in addition to the Mourre estimate established in Theorem 8.2, a couple of technical results. We introduce the norm

$$\|u\|_{+2,\theta} := \left(\|H_{\theta'}\,u\|^2_{\mathrm{L}^2(\Omega)} + \|u\|^2_{\mathrm{L}^2(\Omega)}\right)^{1/2}, \qquad u \in \mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega),$$

and recall that if ε satisfies (2.13), then in view of Corollary 2.2

$$\|u\|_{+2,\theta} \asymp \|u\|_{+2,0} \asymp \|u\|_{\mathrm{H}^2(\Omega)}, \qquad u \in \mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega). \tag{8.13}$$

Proposition 8.3. *Let ε satisfy the assumptions of Theorem 2.7. Then*

(a) *The unitary group e^{itA} leaves $\mathrm{D}(H_{\theta'})$ invariant. Moreover, for each $u \in \mathrm{D}(H_{\theta'})$, $\sup_{|t|\leq 1}\|e^{itA}u\|_{+2,\theta} < \infty$.*

(b) *The operator $B_0 = [H_0, iA]$ defined as a quadratic form on $\mathrm{D}(A)\cap\mathrm{D}(H_0)$ is bounded on $\mathrm{L}^2(\Omega)$.*

(c) *The operator $B = [H_{\theta'}, iA]$ defined as a quadratic form on $\mathrm{D}(A)\cap\mathrm{D}(H_{\theta'})$ is bounded from $\mathrm{D}(H_{\theta'})$ into $\mathrm{D}(H_{\theta'}^{1/2})^*$.*

(d) *There is a common core C for A and H_0 so that A maps C into $\mathrm{H}^1_0(\Omega)$.*

Proof. Note that $H_0 = -\Delta$ and that $\mathrm{D}(H_{\theta'}) = \mathrm{D}(H_0) = \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$, in view of Corollary 2.2. To prove assertion (a), pick $f \in \mathrm{D}(H_{\theta'})$ and denote $g = \mathcal{F}f$. By Lemma 6.2,

$$(e^{itA}f)(x) = \mathcal{F}^*\left[(\partial_k\varphi(t,k))^{1/2}\,g(x_\omega, \varphi(t,k))\right]. \tag{8.14}$$

Hence,

$$\begin{aligned} \|\Delta\,(e^{itA}f)\|^2_{\mathrm{L}^2(\Omega)} &= \|e^{itA}(\Delta_\omega f) + \partial_3^2(e^{itA}f)\|^2_{\mathrm{L}^2(\Omega)} \\ &\leq \|\Delta_\omega f\|^2_{\mathrm{L}^2(\Omega)} + \|k^2\,(\partial_k\varphi(t,k))^{1/2}\,g(x_\omega,\varphi(t,k))\|^2_{\mathrm{L}^2(\Omega)}, \end{aligned} \tag{8.15}$$

where we have used the fact that $e^{itA}: \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ and $\mathcal{F}^*: \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ are unitary. Assume that $\operatorname{supp}\gamma \subset [-k_c, k_c]$ for some $k_c > 0$. Then $\varphi(t,k) = k$ and $\partial_k\varphi(t,k) = 1$ for all k with $|k| > k_c$ and all $t \geq 0$, see the proof of Lemma 6.2. We thus obtain

$$\|k^2\,\partial_k\varphi(t,k)\,g(x_\omega,\varphi(t,k))\|^2_{\mathrm{L}^2(\Omega)} \leq k_c^4 \int_{\omega\times[-k_c,k_c]} \partial_k\varphi(t,k)\,|g(x_\omega,\varphi(t,k))|^2\,dk\,dx_\omega$$
$$+ \int_\Omega k^4\,|g(x_\omega,k)|^2\,dk\,dx_\omega \;\leq\; k_c^4 \int_\Omega |g(x_\omega,z)|^2\,dz\,dx_\omega + \int_\Omega k^4\,|g(x_\omega,k)|^2\,dk\,dx_\omega$$
$$= k_c^4\,\|f\|^2_{\mathrm{L}^2(\Omega)} + \|\partial_3^2 f\|^2_{\mathrm{L}^2(\Omega)},$$

where in the first integral on right hand side we have used the change of variables $z = \varphi(t,k)$ taking into account that $\partial_k\varphi(t,k) > 0$, see (6.6). In view of (8.13) and (8.15) we have

$$\|e^{itA} f\|^2_{+2,0} = \|\Delta\,(e^{itA} f)\|^2_{\mathrm{L}^2(\Omega)} + \|f\|^2_{\mathrm{L}^2(\Omega)} \;\leq\; \text{const}\,\|f\|^2_{\mathrm{H}^2(\Omega)}.$$

This implies that $\sup_{|t|\leq 1}\|e^{itA} f\|_{+2,\theta} < \infty$, see (8.13). Moreover, since $e^{itA} f = 0$ on $\partial\Omega$, see (8.14), we find that $e^{itA} f \in \mathrm{D}(H_{\theta'})$. This proves (a). Next we note that by Lemma 7.1

$$[H_0, iA] = \mathbb{1}_\omega \otimes \mathcal{F}_1^*(2\,\gamma(k)\,k)\,\mathcal{F}_1$$

which is a bounded operator on $\mathrm{L}^2(\Omega)$. This proves (b).

As for assertion (c), note that $B = [H_\beta, iA] + [W, iA]$. By inequality (7.2) we know that $(H_\beta + 1)^{-1}[H_\beta, iA]$ is bounded on $\mathrm{L}^2(\Omega)$. On the other hand, from the proof of Theorem 8.2 it follows that the same is true for the operator $(H_\beta + 1)^{-1}[W, iA]$. Since $(H_\beta + 1)(H_{\theta'} + 1)^{-1}$ is bounded, by Corollary 2.2, we conclude that $(H_{\theta'} + 1)^{-1}B$ is bounded on $\mathrm{L}^2(\Omega)$ and (c) follows. To prove (d) we define $C := \mathrm{D}(-\Delta_\omega) \otimes \mathcal{S}(\mathbb{R})$. By definition of A, C is a core for A. On the other hand, C is also a core for H_0. Since $\mathcal{F}: C \to C$ is a bijection and since $\hat{A}: \mathcal{S} \to \mathcal{S}$, it follows that $A: C \to C \subset \mathrm{H}^1_0(\Omega)$. □

Lemma 8.4. *Let ε satisfy assumptions of Theorem 2.7. Then $(H_{\theta'} + 1)^{-1}[B, A]\,(H_{\theta'} + 1)^{-1}$ is bounded as an operator on $\mathrm{L}^2(\Omega)$.*

Proof. Recall that $B = [H_{\theta'}, iA]$. We write

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2, \quad \text{where} \quad \mathcal{B}_1 = [[H_\beta, iA], iA], \quad \mathcal{B}_2 = [[W, iA], iA].$$

As for the term $\mathcal{B}_1$, a direct calculation gives

$$\mathcal{B}_1 = \mathcal{F}^*\,[[\hat{H}_\beta, i(\mathbb{1}_\omega \otimes \hat{A})], i(\mathbb{1}_\omega \otimes \hat{A})]\,\mathcal{F} = \mathcal{F}^*\left(\gamma(k)^2 + \gamma(k)\gamma'(k)(k - i\beta\,\partial_\tau)\right)\mathcal{F}. \tag{8.16}$$

Let $u \in \mathrm{L}^2(\Omega)$. Similarly as in (7.2) we find out that

$$|(u,\,(k - i\beta\,\partial_\tau)\,u)_{\mathrm{L}^2(\Omega)}| \;\leq\; (u, (\hat{H}_\beta + 1)\,u)_{\mathrm{L}^2(\Omega)}.$$

Since γ and γ' are bounded, the last inequality implies that also $(H_\beta + 1)^{-1}\mathcal{B}_1(H_\beta + 1)^{-1}$ is bounded. From Proposition 2.1 it then follows that

$$(H_{\theta'} + 1)^{-1}\mathcal{B}_1(H_{\theta'} + 1)^{-1}$$

is bounded too. As for the remaining part of the double commutator, we first note that in view of (8.7) and of the fact that the operators $\partial_\tau(H_{\theta'} + 1)^{-1}$ and $\partial_\tau^2(H_{\theta'} + 1)^{-1}$ are bounded, it suffices to show that

$$\Big[[\eta\,, \mathcal{F}_1^*\hat{A}\,\mathcal{F}_1] + [\varepsilon',\, \mathcal{F}_1^*\hat{A}\,\mathcal{F}_1] + [\varepsilon\,\partial_s, \mathcal{F}_1^*\hat{A}\,\mathcal{F}_1],\, \mathcal{F}_1^*\hat{A}\,\mathcal{F}_1\Big] \tag{8.17}$$

is a bounded operator on $\mathrm{L}^2(\mathbb{R})$. Let $u \in \mathrm{L}^2(\mathbb{R})$ and recall that

$$(\mathcal{F}_1^* \hat{A}\, \mathcal{F}_1\, u)(s) = -\frac{1}{2\sqrt{2\pi}} \Big(\int_{\mathbb{R}} \hat{\gamma}(s-s')s'\, u(s')\, ds' + s \int_{\mathbb{R}} \hat{\gamma}(s-s')\, u(s')\, ds' \Big).$$

It will be useful to introduce the shorthands

$$\hat{\gamma}_j(r) = r^j\, \hat{\gamma}(r).$$

Note that $\hat{\gamma}_j \in \mathcal{S}(\mathbb{R})$ for all $j \in \mathbb{N}$. We have

$$\begin{aligned} -2\sqrt{2\pi}\, [\eta\,, \mathcal{F}_1^* \hat{A}\, \mathcal{F}_1]u &= 2\eta(s) \int_{\mathbb{R}} \hat{\gamma}(s-s')s' u(s')\, ds' + \eta(s) \int_{\mathbb{R}} \hat{\gamma}_1(s-s')u(s')\, ds' \\ &\quad - \int_{\mathbb{R}} \hat{\gamma}(s-s')s'\eta(s')u(s')\, ds' - s \int_{\mathbb{R}} \hat{\gamma}(s-s')\eta(s')u(s')\, ds' \\ &=: \sum_{j=1}^{4} T_j\, u. \end{aligned} \tag{8.18}$$

Accordingly,

$$\begin{aligned} -\sqrt{2\pi}\, [T_1,\, \mathcal{F}_1^* \hat{A}\, \mathcal{F}_1]u &= \eta(s) \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\gamma}(s-s')\, s'^2 \hat{\gamma}(s'-s'')\, u(s'')\, ds'' ds' \\ &\quad + \eta(s) \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\gamma}(s-s')s'\, s''\, \hat{\gamma}(s'-s'')\, u(s'')\, ds'' ds' \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\gamma}(s-s')\, \eta(s')\, \hat{\gamma}(s'-s'')\, s'\, s''\, u(s'')\, ds'' ds' \\ &\quad - s \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\gamma}(s-s')\, \eta(s')\, \hat{\gamma}(s'-s'')\, u(s'')\, ds'' ds' \\ &=: \sum_{j=1}^{4} T_{1,j}\, u. \end{aligned}$$

Note that

$$s'^2\, \hat{\gamma}(s-s')\hat{\gamma}(s'-s'') = s^2\, \hat{\gamma}(s-s')\hat{\gamma}(s'-s'') + \hat{\gamma}_2(s-s')\hat{\gamma}(s'-s'') - 2s\, \hat{\gamma}_1(s-s')\hat{\gamma}(s'-s''),$$

which implies

$$T_{1,1}\, u(s) = s^2 \eta(s)\, \hat{\gamma} * (\hat{\gamma} * u) + \eta(s)\, \hat{\gamma}_2 * (\hat{\gamma} * u) - 2s\, \eta(s)\, \hat{\gamma} * (\hat{\gamma}_1 * u).$$

Hence, by a repeated use of the Young inequality

$$\|g * h\|_p \le C\, \|g\|_q\, \|h\|_r, \quad \frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}, \tag{8.19}$$

with $p = q = 2$ and $r = 1$, we get

$$\|T_{1,1}\, u\|_2 \le C_{1,1} \left(\|s^2 \eta\|_\infty\, \|\hat{\gamma}\|_1^2 + \|\eta\|_\infty \|\hat{\gamma}\|_1\, \|\hat{\gamma}_2\|_1 + \|s\, \eta\|_\infty\, \|\hat{\gamma}\|_1\, \|\hat{\gamma}_1\|_1 \right)\, \|u\|_2,$$

for some $C_{1,1} < \infty$. Moreover, since

$$s' s'' \hat{\gamma}(s-s')\hat{\gamma}(s'-s'') = s'^2\, \hat{\gamma}(s-s')\hat{\gamma}(s'-s'') - s\, \hat{\gamma}(s-s')\hat{\gamma}_1(s'-s'') + \hat{\gamma}_1(s-s')\hat{\gamma}_1(s'-s''),$$

with the help of (8.19) we obtain

$$\|T_{1,2}\, u\|_2 \le \|T_{1,1}\, u\|_2 + C_{1,2} \left(\|\eta\|_\infty \|\hat{\gamma}_1\|_1^2 + \|s\, \eta\|_\infty\, \|\hat{\gamma}\|_1\, \|\hat{\gamma}_1\|_1 \right)\, \|u\|_2.$$

As for $T_{1,3}$, we note that

$$T_{1,3}\, u = \hat\gamma * (s\eta(\hat\gamma_1 * u)) - \hat\gamma * (s^2\eta(\hat\gamma * u)),$$

which, in combination with (8.19), implies

$$\|T_{1,3}\, u\|_2 \le C_{1,3} \left(\|s\eta\|_\infty \|\hat\gamma_1\|_1\, \|\hat\gamma\|_1 + \|s^2\eta\|_\infty \|\hat\gamma\|_1\, \|\hat\gamma\|_1\right)\, \|u\|_2.$$

Next, for $T_{1,4}\, u$ we find

$$T_{1,4}\, u = -\hat\gamma_1 * (\eta(\hat\gamma_1 * u)) + \hat\gamma_1 * (s\eta(\hat\gamma * u)) - \hat\gamma * (s\eta(\hat\gamma_1 * u)) + \hat\gamma * (s^2\eta(\hat\gamma * u)).$$

By using again (8.19) we get

$$\|T_{1,4}\, u\|_2 \le C_{1,4} \left(\|\eta\|_\infty \|\hat\gamma_1\|_1^2 + 2\|s\eta\|_\infty \|\hat\gamma_1\|_1 \|\hat\gamma\|_1 + \|s^2\eta\|_\infty \|\hat\gamma\|_1^2\right)\, \|u\|_2.$$

This implies that

$$\|[T_1, \mathcal{F}_1^* \hat A\, \mathcal{F}_1]\, u\|_2 = \frac{1}{2} \sum_{j=1}^{4} \|T_{1,j}\, u\|_2 \le C_1\, \|u\|_2,$$

for some $C_1 < \infty$. As for the term $[T_2, \mathcal{F}_1^* \hat A\, \mathcal{F}_1]\, u$, we find out that

$$\begin{aligned} -2\sqrt{2\pi}\, [T_2,\, \mathcal{F}_1^* \hat A\, \mathcal{F}_1] u &= -\eta\, \hat\gamma_1 * (\hat\gamma_1 * u) - 2\, \eta\, \hat\gamma_2 * (\hat\gamma * u) + 2s\, \eta\, \hat\gamma_1 * (\hat\gamma * u) \\ &\quad - 2\hat\gamma * (s\eta(\hat\gamma_1 * u)) - \hat\gamma_1 * (\eta(\hat\gamma_1 * u)). \end{aligned}$$

Hence by (8.19)

$$\|[T_2, \mathcal{F}_1^* \hat A\, \mathcal{F}_1]\, u\|_2 \le C_2\, \left(\|\eta\|_\infty (\|\hat\gamma_1\|_1^2 + \|\hat\gamma_2\|_1 \|\hat\gamma\|_1) + \|s\eta\|_\infty\, \|\hat\gamma_1\|_1 \|\hat\gamma\|_1\right)\, \|u\|_2.$$

Next we consider the last term on the right hand side of (8.18). A direct calculation gives

$$\begin{aligned} -2\sqrt{2\pi}\, [T_4, \mathcal{F}_1^* \hat A\, \mathcal{F}_1] u &= 2\, \hat\gamma_1 * (s\eta\, (\hat\gamma * u)) - 2\, \hat\gamma_1 * (\eta\, (\hat\gamma_1 * u)) \\ &\quad + 2\, \hat\gamma * (s^2\eta\, (\hat\gamma * u)) - \hat\gamma * (s\eta\, (\hat\gamma_1 * u)). \end{aligned}$$

By the Young inequality,

$$\|[T_4, \mathcal{F}_1^* \hat A\, \mathcal{F}_1] u\|_2 \le C_4 \left(\|\hat\gamma_1\|_1\, \|\hat\gamma\|_1\, \|s\eta\|_\infty + \|\hat\gamma_1\|_1^2\, \|\eta\|_\infty + \|\hat\gamma\|_1^2\, \|s^2\eta\|_\infty\, \right) \|u\|_2,$$

with some $C_4 < \infty$. The same argument applies to $[T_3, \mathcal{F}_1^* \hat A\, \mathcal{F}_1]$. We thus conclude that the first term in (8.17) defines a bounded operator in $\mathrm{L}^2(\mathbb{R})$. The same arguments apply to the second term in (8.17) replacing η by ε'. As for the last term in (8.17), integration by parts shows that

$$\begin{aligned} -\, 2\sqrt{2\pi}\, [\varepsilon\partial_s, \mathcal{F}_1^* \hat A\, \mathcal{F}_1] u &= \varepsilon(s) \int_{\mathbb{R}} \hat\gamma'(s-s') s' u(s') ds' + \varepsilon(s) \int_{\mathbb{R}} \hat\gamma(s-s') u(s') ds' \\ &+ \varepsilon(s) s \int_{\mathbb{R}} \hat\gamma'(s-s') u(s') ds' + \int_{\mathbb{R}} \left[\hat\gamma(s-s')(\varepsilon(s') + s'\varepsilon'(s') - \hat\gamma'(s-s') s' \varepsilon(s')\right] u(s') ds' \\ &+ s \int_{\mathbb{R}} \left[\hat\gamma(s-s')\varepsilon'(s') - \hat\gamma'(s-s')\varepsilon(s')\right] u(s') ds'. \end{aligned}$$

Note that each term on the right hand side of the above equation is of the same type as one of the terms that we have already treated above, with $\hat\gamma$ replaced by $\hat\gamma'$ when necessary. Since $r^j\, \hat\gamma' \in \mathcal{S}(\mathbb{R})$ for all $j \in \mathbb{N}$, by following the same line of arguments as above we obtain

$$\left\| \left[[\varepsilon\, \partial_s, \mathcal{F}_1^*\, \hat A\, \mathcal{F}_1],\, \mathcal{F}_1^* \hat A\, \mathcal{F}_1\right] u \right\|_2 \le \tilde C \left(\|\varepsilon\, (1+s^2)\|_\infty + \|\varepsilon'(1+s^2)\|_\infty\right) \|u\|_2.$$

for some constant $\tilde C < \infty$. This together with the previous estimates implies that (8.17) defines a bounded operator in $\mathrm{L}^2(\mathbb{R})$. □

With these prerequisites, we can finally state the result about the nature of the essential spectrum of $H_{\theta'}$:

Corollary 8.5. *Let ε satisfy the assumptions of Theorem 2.7. Let $E \in \mathbb{R} \setminus \mathcal{E}$ be given and define the interval $\mathcal{I}(E, \delta) = (E - \delta/2, E + \delta/2)$ as in Theorem 8.2. Then:*

(a) *$\mathcal{I}(E, \delta)$ contains at most finitely many eigenvalues of $H_{\theta'}$, each having finite multiplicity;*

(b) *The point spectrum of $H_{\theta'}$ has no accumulation point in $\mathcal{I}(E, \delta)$;*

(c) *$H_{\theta'}$ has no singular continuous spectrum in $\mathcal{I}(E, \delta)$.*

Proof. Since A is self-adjoint, the statement follows from Proposition 8.3, Lemma 8.4, Theorem 8.2 and [20, Theorem 1.2]. □

Proof of Theorem 2.7. Let $J \subset \mathbb{R} \setminus \mathcal{E}$ be a compact interval. For each $E \in J$ choose $\mathcal{I}(E, \delta)$ as in Theorem 8.2. Then $J \subset \cup_{E \in J} \mathcal{I}(E, \delta)$ and since J is compact, there exists a finite subcovering:

$$J \subset \cup_{n=1}^{N} \mathcal{I}(E_n, \delta_n). \tag{8.20}$$

By Corollary 8.5(a), each interval $\mathcal{I}(E_n, \delta_n)$ contains at most finitely many eigenvalues of $H_{\theta'}$, each of them having finite multiplicity. This proves assertion (a). Part (b) follows immediately from (a). To prove (c) assume that $\sigma_{\rm sc}(H_{\theta'}) \cap (\mathbb{R} \setminus \mathcal{E}) \neq \emptyset$. Since the set $\mathcal{E}$ is locally finite, see Lemma 2.6, it follows that there exists a compact interval $J \subset \mathbb{R} \setminus \mathcal{E}$ such that $\sigma_{\rm sc}(H_{\theta'}) \cap J \neq \emptyset$. This is in contradiction with (8.20) and Corollary 8.5(c). Hence, $\sigma_{\rm sc}(H_{\theta'}) \cap (\mathbb{R} \setminus \mathcal{E}) = \emptyset$. Since $\mathcal{E}$ is discrete, this implies that $\sigma_{\rm sc}(H_{\theta'}) = \emptyset$. □

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Ph. Briet
Université du Sud Toulon Var
Centre de Physique Théorique
CNRS-Luminy, Case 907
13288 Marseille, France
E-mail: briet@cpt.univ-mrs.fr

H. Kovařík
Dipartimento di Matematica
Università degli studi di Brescia
Via Branze, 38
25123 Brescia, Italy
E-mail: hynek.kovarik@ing.unibs.it

G. Raikov
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Av. Vicuña Mackenna 4860
Santiago de Chile
E-mail: graikov@mat.puc.cl