Second order elliptic operators in L^2 with first order degeneration at the boundary and outward pointing drift

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Abstract

We study second order elliptic operators whose diffusion coefficients degenerate at the boundary in first order and whose drift term strongly point outward. It is shown that these operators generate analytic semigroups in L^2 where they are equipped with their natural domain without boundary conditions. Hence, the corresponding parabolic problem can be solved with optimal regularity. In a previous work we had treated the case of inward pointing drift terms.

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1 Introduction

In this paper we study wellposedness and regularity of elliptic and parabolic partial differential equations on the half space and on bounded domains assuming that the second order coefficients degenerate at the boundary of first order. Since we are looking at second order problems, first order degeneration is a borderline case where the drift term in normal direction is (roughly speaking) of the same 'order' as the diffusion part. Thus size and direction of the drift term can influence the generation result in a crucial way. In this sense, first order degeneration is the most interesting case in this context.

Locally, there are essentially two cases of first order degeneration at the boundary. Either the diffusion coefficients behave as the distance to the boundary or only the tangential component of the coefficients behave as the distance. (All other cases can be reduced to these two.) For the case of tangential degeneration, in [6] we have recently developed a wellposedness theory in L^p–spaces and spaces of continuous functions, and established various properties of the generated semigroups. (See also [9].) In the tangential case, the size or direction of the drift coefficients have no effect on the generation result. This is different in the case of full degeneration of first order. We explain the effects of the drift term on the level of the model operator

$$A = -y\Delta + a \cdot \nabla_x + bD_y$$

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with constant drift coefficients $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ acting on the half space

$$\mathbb{R}_+^{N+1} = \{z = (x, y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N, y > 0\}.$$

In the paper [5] (co-authored by three of the present authors), it was proved that the operator $-A$ with the domain

$$D_p^0 = \{u \in W_0^{1,p}(\mathbb{R}_+^{N+1}) \cap W_{\text{loc}}^{2,p}(\mathbb{R}_+^{N+1}) : \sqrt{y}\,|\nabla u|, \; y|D^2u| \in L^p(\mathbb{R}_+^{N+1})\}$$

generates an analytic C_0-semigroup of positive contractions on $L^p(\mathbb{R}_+^{N+1})$ **if** $b > -1/p$ and $p \in (1, \infty)$. In this case the drift points inward at the boundary, or only mildly outward. Correspondingly, one has to impose Dirichlet boundary conditions. It was also shown by a one dimensional example that $-A$ with domain D_p^0 is not a generator if $b \leq -1/p$.

In the paper [10] parabolic problems with full degeneration at the boundary were studied in a more general framework, but assuming that the drift coefficients vanish at $\partial\Omega$ (which means $b = 0$ in the model operator above). We also refer to e.g. [11], [15], [16] and [17] for other contributions to degenerate problems, which however do not deal with the interplay of diffusion and drift in the case of first order degeneration at the boundary.

To understand the situation if $b \leq -1/p$, we investigated in detail the one dimensional case $\Omega = (0, 1)$ in [7]. It turned out that then $A = -yD_{yy} + bD_y$ exhibits a surprisingly complicated behavior. In Section 2 we recall the corresponding results, which have been the starting point for the study in higher dimensions.

In the present paper, we establish that $-A$ generates an analytic C_0-semigroup on L^2 for each $b < -1/2$. Here the model operator $A = -y\Delta + a \cdot \nabla_x + bD_y$ on $L^2(\mathbb{R}_+^{N+1})$ has the domain

$$D_2 = \{u \in W^{1,2}(\mathbb{R}_+^{N+1}) \cap W_{\text{loc}}^{2,2}(\mathbb{R}_+^{N+1}) : \sqrt{y}\,|\nabla u|, \; y|D^2u| \in L^2(\mathbb{R}_+^{N+1})\}$$

which possesses optimal regularity, but imposes no boundary condition because the drift points outward and is large enough. In addition, the operator (A, D_2) is accretive for $b \leq -1$ and $a = 0$, see Proposition 3.6, but it fails to be (quasi) accretive for $b \in (-1, -1/2)$ and $a = 0$, see Remark 3.8. This indicates that one cannot use form methods here.

Observe that our results complement those of [5] for $p = 2$ where the opposite condition $b > -1/2$ was assumed. The approach of [5] relies on Hardy's inequality which only works with the Dirichlet boundary condition and under the restriction $b > -1/2$. We thus have to proceed differently in the present paper.

In previous our works [5] or [6] we have approximated the model operator A on $\mathbb{R}_+^{N+1}$ by its realization on the strip $\{(x, y) : x \in \mathbb{R}^N, \, \varepsilon < y < 1/\varepsilon\}$ with Dirichlet boundary conditions. In contrast, following the analysis in [7], in the present paper we impose Neumann boundary conditions at $y = \varepsilon$. The resolvent equation $\lambda u + Au = f$ for $u \in D_2$ is then solved by letting $\varepsilon \to 0^+$. The crucial step of our arguments are the gradient estimates in Proposition 3.3 which ensure that $D_2 \subset W^{1,2}(\mathbb{R}_+^{N+1})$. They are valid for all $b < -1/2$, but we need $a = 0$ here. So far we do not know how to extend these estimates to the case $p \neq 2$ which is the main reason for the restriction to $p = 2$ in this paper. As a by-product of these estimates we derive an inequality leading to analyticity in Proposition 3.5. The result for $b \leq -1$ and $a = 0$ can then be derived in Proposition 3.6. The cases $b \in (-1, -1/2)$ and $a \neq 0$ are treated in Proposition 3.7 and Theorem 3.9, respectively, by means of perturbation arguments. In Proposition 3.7 we perturb the operator A_0 for $b = -1$ and $a = 0$ by the drift term $(b+1)D_y$ which is relatively bounded w.r.t. A_0 with precisely the constants needed to construct the perturbed resolvent by a Neumann series. In Theorem 3.9 we use the Kalton-Weis theorem on sums of resolvent commuting operators to finally add the tangential drift term $a \cdot \nabla_x$.

Based on the properties of the model operator, we also treat the problem on a bounded domain Ω in $\mathbb{R}^{N+1}$. We study an operator A in nondivergence form given in (4.1) with continuous diffusion and drift coefficients on $\overline{\Omega}$, where the normal component of the drift is strictly less than $-1/2$ times the normal component of the matrix of the diffusion coefficients, see (H3) in Section 4. We then show that the negative of this operator generates an analytic semigroup on $L^2(\Omega)$ when equipped with the domain

$$D_2^\Omega = \left\{u \in W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,2}(\Omega) : \varrho |D^2 u| \in L^2(\Omega)\right\}$$

having optimal regularity and no boundary conditions. (Here, ϱ is a smooth extension of the distance function to the boundary.) By standard semigroup theory, this generation result allows to solve the corresponding inhomogeneous parabolic partial differential equation in optimal regularity, see Corollary 4.2.

2 One dimensional operators

In this section we recall the basic results of the paper [7] concerning the one dimensional operator $A = -yD_{yy} + bD_y$ in $L^p(0,1)$ with $b \in \mathbb{R}$.

First, we constructed an operator $(-A, D^D_{p,b})$ by Dirichlet approximation, i.e., we solved the resolvent equation $\lambda u + Au = f$ on $(\varepsilon, 1)$, where A is endowed with the domain $W^{2,p}(\varepsilon,1) \cap W^{1,p}_0(\varepsilon,1)$, and then let $\varepsilon \to 0^+$. We have shown that $(-A, D^D_{p,b})$ generates an analytic semigroup for all $b \in \mathbb{R}$ and $p \in (1,\infty)$. However, the domain $D^D_{p,b}$ heavily depends on b: If $b \le -1$, then $u \in D^D_{p,b}$ is contained in $W^{1,p}(0,1)$ and satisfies $yu'' \in L^p(0,1)$, but *no* boundary condition at $y = 0$ is imposed. If $b \in (-1, -1/p]$, then $D^D_{p,b}$ is *not* contained in $W^{1,p}(0,1)$, but one imposes $u(0) = 0$ for $u \in D^D_{p,b}$.

We have further seen that the Dirichlet approximation is unstable in the sense that for the solutions $u_\varepsilon \in W^{2,p}(\varepsilon,1) \cap W^{1,p}_0(\varepsilon,1)$ of $Au_\varepsilon = f$ the norms $\|u'_\varepsilon\|_p$ blow up as $\varepsilon \to 0^+$ for certain $f \in L^p(0,1)$ and each $b \le -1/p$ (even though the limit function belongs to $W^{1,p}(0,1)$ if $b \le -1$). We thus also employed Neumann approximations of A with the domains

$$D^N_{p,\varepsilon} = \{u \in W^{2,p}(\varepsilon,1) :\, u'(\varepsilon) = 0,\, u(1) = 0\}.$$

This approximation turned out to be stable in $W^{1,p}$ for all $b < -1/p$. Moreover, the limit operator possesses the (optimal) domain

$$D_p = \{u \in W^{1,p}(0,1) :\, yu'' \in L^p(0,1),\, u(1) = 0\}$$

and generates an analytic semigroup on $L^p(0,1)$ for every $p \in (1,\infty)$ and $b < -1/p$. The Neumann boundary condition at $y = \varepsilon$ is lost in the limit, as we impose no boundary condition at $y = 0$ in D_p. We checked that the two approximations yield the same operator for $b \le -1$, but different ones for $b \in (-1, -1/p)$. Here the Neumann approximation gives the better regularity without any boundary condition. In the case $b = -1/p$ the Neumann approximation does not work and is unstable in $W^{1,p}$. This borderline case is excluded in our further investigations.

These one dimensional results crucially depend on properties which are not available in higher dimensions. In particular, the full description of the domain of the generator relies on the possibility of writing explicitly the solutions of the ordinary differential equation $Au = f$; the proof of analyticity uses generation theorems from [1] and [14] in sup-norm spaces which are based on Feller's theory of diffusion processes on intervals, see [3] and [4].

3 Generation on the half space

In this section we establish the generation result for the model operator

$$A = -y\Delta + a \cdot \nabla_x + bD_y$$

with constant drift coefficients $a \in \mathbb{R}^N$ and $b < -1/2$ acting on the half space

$$\mathbb{R}^{N+1}_+ = \{z = (x, y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N, y > 0\}.$$

This operator will be endowed with the domain

$$D_2 = \left\{u \in W^{1,2}(\mathbb{R}^{N+1}_+) \cap W^{2,2}_{\text{loc}}(\mathbb{R}^{N+1}_+) : \sqrt{y}\,|\nabla u|,\ y|D^2 u| \in L^2(\mathbb{R}^{N+1}_+)\right\}$$

in $L^2(\mathbb{R}^{N+1}_+)$. Let $\varepsilon \in (0, 1/2]$. To construct the resolvent of A, we use approximating problems on the strip

$$S_\varepsilon := \{(x, y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N,\ \varepsilon < y < \varepsilon^{-1}\},$$

where we equip A with the domains

$$D^N_{2,\varepsilon} = \{u \in W^{2,2}(S_\varepsilon) : u(\cdot, 1/\varepsilon) = 0,\ D_y u(\cdot, \varepsilon) = 0\}.$$

To unify the notation, we set $S_0 := \mathbb{R}^{N+1}_+$ and $D^N_{2,0} := D_2$. Lemma 2.1 of [5] provides us with the following density result.

Lemma 3.1. *The set $C^\infty_c(\mathbb{R}^{N+1})$ is dense in D_2 endowed with its canonical norm.*

We first show that the operator A is accretive on $D^N_{2,\varepsilon}$ if $b \le -1$.

Proposition 3.2. *Assume that $b \le -1$. Let $\operatorname{Re}\lambda \ge 0$, $u \in D^N_{2,\varepsilon}$, and $0 \le \varepsilon \le 1/2$. Set $f = \lambda u + Au$. We then have*

$$(\operatorname{Re}\lambda)\|u\|_{L^2(S_\varepsilon)} \le \|f\|_{L^2(S_\varepsilon)}$$

In particular, the operator $(A, D^N_{2,\varepsilon})$ is accretive in $L^2(S_\varepsilon)$.

Proof. Let first $\varepsilon > 0$ and fix $u \in D^N_{2,\varepsilon}$. We multiply the equation $\lambda u + Au = f$ by $\bar{u}$ and integrate by parts on S_ε. It follows

$$\int_{S_\varepsilon} f\bar{u} = \lambda\|u\|^2_{L^2(S_\varepsilon)} + \int_{S_\varepsilon} y|\nabla u|^2 + \int_{S_\varepsilon} (a \cdot \nabla_x u)\bar{u} + (b+1)\int_{S_\varepsilon} (D_y u)\bar{u}. \tag{3.1}$$

Since $\operatorname{Re}((\nabla u)\bar{u}) = \frac{1}{2}\nabla|u|^2$, we can evaluate the last two integrals and deduce

$$\begin{aligned}\operatorname{Re}\int_{S_\varepsilon} f\bar{u} &= (\operatorname{Re}\lambda)\|u\|^2_{L^2(S_\varepsilon)} + \int_{S_\varepsilon} y|\nabla u|^2 - \frac{(b+1)}{2}\int_{\mathbb{R}^N} |u(x,\varepsilon)|^2 dx \\ &\ge (\operatorname{Re}\lambda)\|u\|^2_{L^2(S_\varepsilon)}\end{aligned}$$

using $b \le -1$. On $\mathbb{R}^{N+1}_+$ we obtain the corresponding estimate in the same way for $u \in C^\infty_c(\mathbb{R}^{N+1})$. Due to Lemma 3.1, approximation yields the result for $u \in D_2$. $\square$

Our approach relies on the following gradient estimates for A with quite explicit constants depending only on b. For technical reasons we first restrict ourselves to the case $a = 0$. This restriction will be removed at the end of the section by a perturbation argument.

Proposition 3.3. *Assume that $a = 0$ and $b < -\frac{1}{2}$. Let $\lambda \geq 0$, $u \in D^N_{2,\varepsilon}$, and $0 \leq \varepsilon \leq 1/2$. Set $f = \lambda u + Au$. We then have*

$$\|\nabla_x u\|_{L^2(S_\varepsilon)} \leq \frac{2}{\sqrt{-2b-1}} \|f\|_{L^2(S_\varepsilon)}, \tag{3.2}$$

$$\|D_y u\|_{L^2(S_\varepsilon)} \leq \frac{1}{-b-\frac{1}{2}} \|f\|_{L^2(S_\varepsilon)}. \tag{3.3}$$

Proof. Let first $\varepsilon > 0$. Take $u \in D^N_{2,\varepsilon}$ and $\lambda \in \mathbb{C}$. Multiplying the equation $\lambda u + Au = f$ by $D_y \bar{u}$ and integrating by parts in x on S_ε, we obtain

$$\lambda \int_{S_\varepsilon} u D_y \bar{u} - \int_{S_\varepsilon} y D_{yy} u \, D_y \bar{u} + \int_{S_\varepsilon} y \nabla_x u \cdot \nabla_x D_y \bar{u} + b \int_{S_\varepsilon} |D_y u|^2 = \int_{S_\varepsilon} f D_y \bar{u}.$$

The real parts thus satisfy

$$\begin{aligned} \int_{S_\varepsilon} \mathrm{Re}\,(f D_y \bar{u}) = \frac{\mathrm{Re}\,\lambda}{2} \int_{S_\varepsilon} D_y |u|^2 - \mathrm{Im}\,\lambda \int_{S_\varepsilon} \mathrm{Im}\,(u D_y \bar{u}) - \frac{1}{2} \int_{S_\varepsilon} y D_y |D_y u|^2 \\ + \frac{1}{2} \int_{S_\varepsilon} y \, D_y |\nabla_x u|^2 + b \int_{S_\varepsilon} |D_y u|^2. \end{aligned}$$

Integrating by parts in y, we then compute

$$\begin{aligned} \int_{S_\varepsilon} \mathrm{Re}\,(f D_y \bar{u}) = -\frac{\mathrm{Re}\,\lambda}{2} \int_{\mathbb{R}^N} |u(x,\varepsilon)|^2 - \mathrm{Im}\,\lambda \int_{S_\varepsilon} \mathrm{Im}\,(u D_y \bar{u}) - \frac{1}{2\varepsilon} \int_{\mathbb{R}^N} |D_y u\left(x, \tfrac{1}{\varepsilon}\right)|^2 \\ - \frac{\varepsilon}{2} \int_{\mathbb{R}^N} |\nabla_x u(x,\varepsilon)|^2 - \frac{1}{2} \int_{S_\varepsilon} |\nabla_x u|^2 + \left(b + \frac{1}{2}\right) \int_{S_\varepsilon} |D_y u|^2. \end{aligned}$$

After multiplying by -1, for $\mathrm{Im}\,\lambda = 0$ and $\lambda \geq 0$ we derive

$$\frac{1}{2} \int_{S_\varepsilon} |\nabla_x u|^2 - \left(b + \frac{1}{2}\right) \int_{S_\varepsilon} |D_y u|^2 \leq \|f\|_{L^2(S_\varepsilon)} \|D_y u\|_{L^2(S_\varepsilon)}.$$

It follows that

$$\|D_y u\|_{L^2(S_\varepsilon)} \leq \frac{1}{-b-\frac{1}{2}} \|f\|_{L^2(S_\varepsilon)}$$

and consequently

$$\frac{1}{2} \int_{S_\varepsilon} |\nabla_x u|^2 \leq \|D_y u\|_{L^2(S_\varepsilon)} \|f\|_{L^2(S_\varepsilon)} \leq \frac{1}{-b-\frac{1}{2}} \|f\|^2_{L^2(S_\varepsilon)},$$

as asserted. If $\varepsilon = 0$, the previous estimates can be performed for $u \in C_c^\infty(\mathbb{R}^{N+1})$. By density (see Lemma 3.1), the inequalities (3.2) and (3.3) then also hold in D_2. $\square$

Remark 3.4. Inspecting the above proof, one sees that the estimates in Proposition 3.3 also hold for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$ and $(\mathrm{Im}\,\lambda) \int_{S_\varepsilon} \mathrm{Im}\,(u D_y \bar{u}) \geq 0$.

Again for $b \le -1$, we next establish a sectoriality estimate for $(-A, D_2)$.

Proposition 3.5. *Assume that $a = 0$ and $b \le -1$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$, $u \in D^N_{2,\varepsilon}$, and $0 \le \varepsilon \le 1/2$. Set $c_b = \frac{4b+3}{2b+1}$. We then have*

$$\|u\|_{L^2(S_\varepsilon)} \le \frac{c_b}{|\operatorname{Im}\lambda|} \|\lambda u + Au\|_{L^2(S_\varepsilon)}.$$

Proof. We use the equation (3.1) with $a = 0$ that was shown in the proof of Proposition 3.2. (If $\varepsilon = 0$, as before we first take $u \in C_c^\infty(\mathbb{R}^{N+1})$ and then derive the assertion by approximation.) Taking the imaginary parts, we obtain

$$\operatorname{Im}\int_{S_\varepsilon} f\bar{u} = (\operatorname{Im}\lambda)\, \|u\|^2_{L^2(S_\varepsilon)} + (b+1)\operatorname{Im}\int_{S_\varepsilon} (D_y u)\bar{u}. \tag{3.4}$$

If $(\operatorname{Im}\lambda)\displaystyle\int_{S_\varepsilon} \operatorname{Im}(uD_y\bar{u}) \ge 0$, then (3.4) and Remark 3.4 yield

$$\begin{aligned}|\operatorname{Im}\lambda|\, \|u\|^2_{L^2(S_\varepsilon)} &\le \|f\|_{L^2(S_\varepsilon)}\, \|u\|_{L^2(S_\varepsilon)} - (b+1)\|D_y u\|_{L^2(S_\varepsilon)}\, \|u\|_{L^2(S_\varepsilon)} \\ &\le \|f\|_{L^2(S_\varepsilon)}\, \|u\|_{L^2(S_\varepsilon)} + \frac{2b+2}{2b+1}\|f\|_{L^2(S_\varepsilon)}\, \|u\|_{L^2(S_\varepsilon)} = c_b\, \|f\|_{L^2(S_\varepsilon)}\, \|u\|_{L^2(S_\varepsilon)},\end{aligned}$$

which gives the asserted estimate. If $(\operatorname{Im}\lambda)\displaystyle\int_{S_\varepsilon} \operatorname{Im}(uD_y\bar{u}) < 0$, we derive from (3.4) and the assumption $b + 1 \le 0$ that

$$\|u\|^2_{L^2(S_\varepsilon)} = -(b+1)\,\frac{\operatorname{Im}\int_{S_\varepsilon}(D_y u)\bar{u}}{\operatorname{Im}\lambda} + \frac{\operatorname{Im}\int_{S_\varepsilon} f\bar{u}}{\operatorname{Im}\lambda} \le \frac{\operatorname{Im}\int_{S_\varepsilon} f\bar{u}}{\operatorname{Im}\lambda} \le \frac{\|f\|_{L^2(S_\varepsilon)}\|u\|_{L^2(S_\varepsilon)}}{|\operatorname{Im}\lambda|}.$$

Again the asserted estimate follows. ☐

We can now derive our basic generation result for the case $b \le -1$ and $a = 0$.

Proposition 3.6. *Assume that $b \le -1$ and $a = 0$. The operator $(-A, D_2)$ then generates a bounded analytic C_0–semigroup of positive contractions on $L^2(\mathbb{R}^{N+1}_+)$.*

Proof. Let $\lambda > 0$ and $f \in L^2(\mathbb{R}^{N+1}_+)$ be fixed. For every $\varepsilon \in (0, 1/2)$, there is a unique solution $u_\varepsilon \in D^N_{2,\varepsilon}$ of the equation $\lambda u + Au = f$. Propositions 3.2 and 3.3 yield

$$\|u_\varepsilon\|_{L^2(S_\varepsilon)} \le \lambda^{-1}\|f\|_{L^2(\mathbb{R}^{N+1}_+)}, \qquad \|\nabla u_\varepsilon\|_{L^2(S_\varepsilon)} \le K\|f\|_{L^2(\mathbb{R}^{N+1}_+)}, \tag{3.5}$$

where the constant K only depends on b. By local elliptic regularity and (weak) compactness, there exists a sequence $\varepsilon_n \to 0$ such that the corresponding functions u_{ε_n} converge weakly in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ to some $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$. Moreover, $\lambda u + Au = f$ in $\mathbb{R}^{N+1}_+$. Estimate (3.5) implies that $u \in W^{1,2}(\mathbb{R}^{N+1}_+)$ and

$$\|u\|_{L^2(\mathbb{R}^{N+1}_+)} \le \lambda^{-1}\|f\|_{L^2(\mathbb{R}^{N+1}_+)}, \qquad \|\nabla u\|_{L^2(\mathbb{R}^{N+1}_+)} \le K\|f\|_{L^2(\mathbb{R}^{N+1}_+)}. \tag{3.6}$$

It follows that $Au \in L^2(\mathbb{R}^{N+1}_+)$ and therefore $y\Delta u \in L^2(\mathbb{R}^{N+1}_+)$. To control yD^2u, for each $k \in \mathbb{N}$ we take $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 1$ in $[0, k]$, $\eta = 0$ in $[2k, +\infty)$, $0 \le \eta \le 1$, $\|\eta'\|_\infty \le ck^{-1}$ and $\|\eta''\|_\infty \le ck^{-2}$. Then $v = y\eta u \in W^{1,2}_0(\mathbb{R}^{N+1}_+)$ and $\Delta v \in L^2(\mathbb{R}^{N+1}_+)$. Set

$\Omega_k = \mathbb{R}^N \times (0, k)$. Applying the Calderón-Zygmund estimate to v (see e.g. Lemma 9.12 in [8]), we derive

$$\begin{aligned}
&\|yD_x^2 u\|_{L^2(\Omega_k)} + \|y\nabla_x D_y u + \nabla_x u\|_{L^2(\Omega_k)} + \|yD_y^2 u + 2D_y u\|_{L^2(\Omega_k)} \\
&\quad \le \sqrt{3}\,\|D^2 v\|_{L^2(\Omega_k)} \le \sqrt{3}\,\|D^2 v\|_{L^2(\mathbb{R}^{N+1}_+)} \le C\,\|\Delta v\|_{L^2(\mathbb{R}^{N+1}_+)} \\
&\quad \le C\big(\|\eta y \Delta u\|_{L^2(\mathbb{R}^{N+1}_+)} + \|\eta' y D_y u\|_{L^2(\mathbb{R}^{N+1}_+)} + \|\eta' u\|_{L^2(\mathbb{R}^{N+1}_+)} + \|\eta D_y u\|_{L^2(\mathbb{R}^{N+1}_+)} \\
&\qquad + \|\eta'' y\, u\|_{L^2(\mathbb{R}^{N+1}_+)}\big)
\end{aligned}$$

for a positive constant C depending only on N. In the sequel, C may change from line to line. Since both η' and η'' are supported in $[k, 2k]$, we conclude

$$\begin{aligned}
&\|yD_x^2 u\|_{L^2(\Omega_k)} + \|y\nabla_x D_y u + \nabla_x u\|_{L^2(\Omega_k)} + \|yD_y^2 u + 2D_y u\|_{L^2(\Omega_k)} \\
&\quad \le C\big(\|y\Delta u\|_{L^2(\mathbb{R}^{N+1}_+)} + \|D_y u\|_{L^2(\mathbb{R}^{N+1}_+)} + k^{-1}\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\big).
\end{aligned}$$

The estimate (3.6) then yields

$$\begin{aligned}
\|yD^2 u\|_{L^2(\Omega_k)} &\le \|yD_x^2 u\|_{L^2(\Omega_k)} + \|y\nabla_x D_y u\|_{L^2(\Omega_k)} + \|yD_y^2 u\|_{L^2(\Omega_k)} \\
&\le C\big(\|f\|_{L^2(\mathbb{R}^{N+1}_+)} + \|y\Delta u\|_{L^2(\mathbb{R}^{N+1}_+)} + k^{-1}\|u\|_{L^2(\mathbb{R}^{N+1}_+)}\big).
\end{aligned}$$

Observe that $y\Delta u = \lambda u + bD_y u - f$. Letting $k \to +\infty$ and using (3.6), we thus infer

$$\|yD^2 u\|_{L^2(\mathbb{R}^{N+1}_+)} \le C\|f\|_{L^2(\mathbb{R}^{N+1}_+)}.$$

To conclude that $u \in D_2$, it remains to show that $\sqrt{y}\,|\nabla u| \in L^2(\mathbb{R}^{N+1}_+)$. We apply the interpolative estimates (iii) and (iv) of Lemma 2.7 in [5] to the truncated functions $u_k = \eta u \in D_2$. As above, we deduce $\sqrt{y}\,|\nabla u| \in L^2(\mathbb{R}^{N+1}_+)$ letting $k \to +\infty$, and hence $u \in D_2$.

We have thus proved that $\lambda + A : D_2 \to L^2(\mathbb{R}^{N+1}_+)$ is surjective. Since $(-A, D_2)$ is dissipative by Proposition 3.2, this operator generates a contractive C_0–semigroup on $L^2(\mathbb{R}^{N+1}_+)$. This semigroup is bounded analytic due to Proposition 3.5 and e.g. Theorem II.4.6 in [2]. Finally, if $f \ge 0$, then $u_\varepsilon \ge 0$ and thus $u \ge 0$. Hence, the resolvent of $-A$ is positive for $\lambda > 0$ which implies the positivity of the semigroup by e.g. Theorem VI.1.8 in [2]. □

As in [7] we use a perturbation argument to extend the generation result to the range $b \in (-1, -1/2)$. We point out that the gradient estimate (3.3) precisely gives the needed smallness condition.

Proposition 3.7. *Assume that $b \in (-1, -1/2)$ and $a = 0$. The operator $(-A, D_2)$ then generates a positive bounded analytic C_0–semigroup on $L^2(\mathbb{R}^{N+1}_+)$.*

Proof. We first show that $-A = (-A, D_2)$ generates a bounded analytic C_0–semigroup. We write the operator A as $A = A_0 + (b+1)D_y$, where $A_0 = -y\Delta - D_y$ is endowed with the domain D_2 and corresponds to $b = -1$. Due to the previous result, A_0 generates a bounded analytic C_0–semigroup. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Then $\lambda \in \rho(-A_0)$, and estimate (3.3) with $\varepsilon = 0$ and $b = -1$ yields

$$\|D_y(\lambda + A_0)^{-1} f\|_{L^2(\mathbb{R}^{N+1}_+)} \le 2\,\|f\|_{L^2(\mathbb{R}^{N+1}_+)},$$

for every $f \in L^2(\mathbb{R}^{N+1}_+)$. Since $b \in (-1, -1/2)$, it follows

$$\|(b+1)D_y(\lambda + A_0)^{-1}\| \le 2(b+1) =: \beta < 1, \tag{3.7}$$

and hence the operator $I + (b+1)D_y(\lambda + A_0)^{-1}$ is invertible. From the identity

$$\lambda + A = \Big(I + (b+1)D_y(\lambda + A_0)^{-1}\Big)(\lambda + A_0) \tag{3.8}$$

we infer that $\lambda \in \rho(-A)$ and $\|(\lambda + A)^{-1}\| \le \frac{1}{1-\beta}\|(\lambda + A_0)^{-1}\| \le \frac{1}{1-\beta}\frac{M}{|\lambda|}$ for some $M > 0$. Therefore $-A = (-A, D_2)$ generates a bounded analytic C_0–semigroup.

To show the positivity, we again approximate the resolvent. Let $0 \le f \in L^2(\mathbb{R}^{N+1}_+)$ and $\lambda > 0$. For every $\varepsilon \in (0, 1/2)$ there is a unique solution $u_\varepsilon \in D^N_{2,\varepsilon}$ of $\lambda u + Au = f$. The maximum principle yields that $u_\varepsilon \ge 0$. Note that we cannot use Proposition 3.2 to obtain a uniform bound on $\|u_\varepsilon\|_{L^2(S_\varepsilon)}$ since $b > -1$. It is straightforward to check that $A_{0,\varepsilon} = (A_0, D^N_{2,\varepsilon})$ is symmetric, and thus selfadjoint, on $L^2(S_\varepsilon)$. Hence, the resolvents $(\lambda + A_{0,\varepsilon})^{-1}$ are symmetric for $\lambda > 0$. It follows that

$$\|(\lambda + A_{0,\varepsilon})^{-1}\| = r((\lambda + A_{0,\varepsilon})^{-1}) \le 1/\lambda,$$

where $r(\cdot)$ denots the spectral radius. Moreover, the estimate (3.7) holds with A_0 replaced with $A_{0,\varepsilon}$. Setting $A_\varepsilon = (A, D^N_{2,\varepsilon})$, the identity (3.8) is true for A_ε and $A_{0,\varepsilon}$. These relations imply

$$\|(\lambda + A_\varepsilon)^{-1}\| \le \frac{1}{1-\beta}\|(\lambda + A_{0,\varepsilon})^{-1}\| \le \frac{1}{1-\beta}\frac{1}{|\lambda|},$$

which means that

$$\|u_\varepsilon\|_{L^2(S_\varepsilon)} \le \frac{1}{1-\beta}\frac{1}{|\lambda|}\|f\|_{L^2(\mathbb{R}^{N+1}_+)}.$$

Proposition 3.3 further yields a constant K such that

$$\|\nabla u_\varepsilon\|_{L^2(S_\varepsilon)} \le K\,\|f\|_{L^2(\mathbb{R}^{N+1}_+)}.$$

There thus exists a sequence $\varepsilon_n \to 0$ and a positive function $u \in W^{1,2}(\mathbb{R}^{N+1}_+) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ such that u_{ε_n} converges to u weakly in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$. Moreover, $\lambda u + Au = f$. As in the proof of Proposition 3.6 one can see that $u \in D_2$. As a result, $0 \le u = (\lambda + A)^{-1}f$, and hence the semigroup generated by $-A$ is positive. $\square$

Remark 3.8. *If $b \in (-1, -1/2)$, then the operator (A, D_2) is not quasi–accretive (i.e., $A+\omega$ is not accretive for any $\omega \in \mathbb{R}$).*

Proof. We only look at the one dimensional operator $A = -yD^2 + bD$ on the half line $(0, +\infty)$. (For the general case, consider functions of the form $u(x)v(y)$ with $u \in C^\infty_c(\mathbb{R}^N)$.) If A were quasi–accretive in $L^2(0, +\infty)$, then there would exist a constant $\omega \in \mathbb{R}$ such that

$$\mathrm{Re}\ (Au \cdot u) \ge \omega\|u\|^2_{L^2(0,+\infty)} \tag{3.9}$$

for every $u \in D_2$. Fix $\eta \in C^2(\mathbb{R})$ with $\eta = 1$ in $(-\infty, (4e^2)^{-1}]$, $\eta = 0$ in $[(2e^2)^{-1}, +\infty)$ and $0 \le \eta \le 1$. For small $\delta > 0$ and $\alpha \in (0, 1/2)$, we define

$$u_\delta(y) = \eta(y)(-\log(y + \delta))^\alpha.$$

Then $u_\delta \in D_2$. Integrating by parts, (3.9) yields

$$\frac{b+1}{2}\big(u_\delta(0)\big)^2 \le -\omega \int_0^{+\infty} u_\delta^2 + \int_0^{+\infty} y(u_\delta')^2. \tag{3.10}$$

The functions u_δ converge pointwise to $u_0 = \eta(-\log)^\alpha$ as $\delta \to 0$ and $u_\delta^2 \le u_0^2 \in L^1(0,+\infty)$. Hence, u_δ tend to u_0 in $L^2(0,+\infty)$. Moreover, $y(u_\delta')^2$ converge pointwise to $y(u_0')^2$. We estimate

$$\begin{aligned} y(u_\delta'(y))^2 &\le 2\alpha^2\eta(y)^2 \frac{y}{y+\delta}\frac{(-\log(y+\delta))^{2\alpha-2}}{y+\delta} + 2y\big(\eta'(y)\big)^2(-\log(y+\delta))^{2\alpha} \\ &\le 2\alpha^2\eta(y)^2\frac{(-\log(y))^{2\alpha-2}}{y} + 2y\big(\eta'(y)\big)^2(-\log(y))^{2\alpha} =: v(y), \end{aligned}$$

using that the function $H(t) = t^{-1}(-\log(t))^{2\alpha-2}$ is decreasing in $(0,e^{-2})$ and that η vanishes on $[(2e^2)^{-1},+\infty)$. Since $v \in L^1(0,+\infty)$, the norms $\|\sqrt{y}\,u_\delta'\|_{L^2(0,+\infty)}$ tend to $\|\sqrt{y}\,u_0'\|_{L^2(0,+\infty)}$ as $\delta \to 0$. Letting $\delta \to 0$, we get a contradiction in (3.10). □

We conclude the section by proving the generation result in the case $a \neq 0$.

Theorem 3.9. *Assume that $b < -1/2$ and $a \in \mathbb{R}^N$. The operator $(-A, D_2)$ then generates an analytic C_0–semigroup on $L^2(\mathbb{R}_+^{N+1})$. This semigroup is positive and bounded.*

Proof. We write $A = B+C$, where $B = -y\Delta + bD_y$, $C = a\cdot\nabla_x$ and $D(C) = \{u \in L^2(\mathbb{R}_+^{N+1}) : Cu \in L^2(\mathbb{R}_+^{N+1})\} \supset D_2$. Propositions 3.6 and 3.7 show that $(-B, D_2)$ generates a positive, bounded, analytic C_0-semigroup $T(\cdot)$ on $L^2(\mathbb{R}_+^{N+1})$. It is known that $(C, D(C))$ generates the positive, contractive C_0–group $S(\cdot)$ on $L^2(\mathbb{R}_+^{N+1})$ given by $(S(t)f)(x,y) = f(x+at,y)$.

We want to check that these semigroups commute. Take $v \in C_c^\infty(\mathbb{R}_+^{N+1})$ and $t \ge 0$. Note that $S(s)v \in C_c^\infty(\mathbb{R}_+^{N+1})$ and $C_c^\infty(\mathbb{R}_+^{N+1}) \subset D(B^2)$. Hence, $T(t)C_c^\infty(\mathbb{R}_+^{N+1}) \subset D(B^2) \subset D_2 \subset D(C)$. We can thus differentiate

$$\partial_s\left(S(t-s)T(t)S(s)v\right) = S(t-s)[T(t)C - CT(t)]S(s)v$$

for $s \in [0,t]$. Since $C : D(B^2) \to D(B)$ is bounded, we similarly obtain

$$\partial_r\left(T(t-r)CT(r)S(s)v\right) = T(t-r)(CB - BC)T(r)S(s)v = 0,$$

also using that B and C commute on $D(B^2)$. Integrating in $r \in [0,t]$, it follows $(CT(t) - T(t)C)S(s)v = 0$, and hence $T(t)S(t)v = S(t)T(t)v$. By density, the semigroups commute.

As result, the closure of $A = B+C$ (initially defined on D_2) generates the C_0–semigroup given by $U(t) = T(t)S(t)$, $t \ge 0$, see Paragraph II.2.7 in [2]. Observe that $U(t)$ is positive and bounded. Moreover, the resolvents of B and C commute.

In a next step we show that A is actually closed on D_2 using a theorem on operator sums by Kalton and Weis. We refer to [12] for the relevant background information. Due to e.g. Theorem 11.5 in [12], the m–accretive operator $-C$ has a bounded H^∞–calculus of any angle $\omega_C > \pi/2$. Since $-B$ generates a bounded analytic semigroup on a Hilbert space, it is R–sectorial of an angle $\omega_B < \pi/2$, cf. p.75 and 76 of [12]. Theorem 12.13 of [12] now shows that $A = B + C$ is closed on D_2. Hence, the graph norm of A is equivalent to the norm of D_2 which in turn is equivalent to the graph norm of B. The analyticity of $U(\cdot)$ then follows from that of $T(\cdot)$ because of

$$\|AU(t)f\|_{L^2(\mathbb{R}_+^{N+1})} \le c\left(\|BT(t)S(t)f\|_{L^2(\mathbb{R}_+^{N+1})} + \|T(t)S(t)f\|_{L^2(\mathbb{R}_+^{N+1})}\right)$$

$$
\begin{aligned}
&\le c\left(t^{-1}\|S(t)f\|_{L^2(\mathbb{R}^{N+1}_+)} + \|T(t)S(t)f\|_{L^2(\mathbb{R}^{N+1}_+)}\right)\\
&\le c\,t^{-1}\|f\|_{L^2(\mathbb{R}^{N+1}_+)}
\end{aligned}
$$

for $t \in (0,1]$, $f \in L^2(\mathbb{R}^{N+1}_+)$ and some constants $c > 0$. □

4 Generation on bounded domains

Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ with C^2 boundary and let ϱ be a function in $C^2(\overline{\Omega})$ such that $\varrho > 0$ in Ω, $\varrho = 0$ on $\partial\Omega$ and $\nabla\varrho(\xi) = \nu(\xi)$, for every $\xi \in \partial\Omega$. Here, $\nu(\xi)$ is the inward unitary normal vector to $\partial\Omega$ at ξ. We consider the operator

$$
A = -\varrho \sum_{i,j=1}^{N+1} a_{ij} D_{ij} + \sum_{i=1}^{N+1} b_i D_i, \tag{4.1}
$$

and set $a(\xi) = \big(a_{ij}(\xi)\big)_{i,j}$ and

$$
\kappa = \max_{\xi\in\partial\Omega} \frac{\langle b(\xi), \nu(\xi)\rangle}{\langle a(\xi)\nu(\xi), \nu(\xi)\rangle}\,.
$$

Assume that

(H1) a_{ij} are real continuous functions on $\overline{\Omega}$, $a_{ij} = a_{ji}$, and satisfy the ellipticity condition $\langle a(\xi)\zeta, \zeta\rangle \ge \alpha|\zeta|^2$ for every $\xi \in \overline{\Omega}$, $\zeta \in \mathbb{R}^{N+1}$ and some $\alpha > 0$.

(H2) b_i are real continuous functions on $\overline{\Omega}$.

(H3) $\kappa < -1/2$.

We endow A with the domain

$$
D_2^{\Omega} = \big\{u \in W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,2}(\Omega) : \varrho|D^2u| \in L^2(\Omega)\big\}.
$$

Theorem 4.1. *Under assumptions (H1), (H2) and (H3) the operator $(-A, D_2^{\Omega})$ generates an analytic C_0–semigroup on $L^2(\Omega)$.*

The proof is based on Theorem 3.9. It follows the lines of the arguments in Lemma 2.13, Corollary 2.14 and Section 3 of [5]. We thus omit the proof, but briefly indicate the main ideas. One first extends Theorem 3.9 to operators on $\mathbb{R}^{N+1}_+$ where one replaces $y\Delta$ by a term $y\sum_{ij} a_{ij}D_{ij}$ with constant coefficients. Then one localises the operator A on Ω around suitably chosen points $\xi_1, \cdots, \xi_m \in \partial\Omega$ and $\xi_0 \in \Omega$ and for $j \ge 1$ one transforms the localised operators to the half space $\mathbb{R}^{N+1}_+$ in such a way that the normal is preserved at ξ_j. In particular, the factor ϱ transforms into functions ϕ_j that behave like y. One freezes the coefficients of the transformed operators and replaces ϕ_j by y, thus obtaining operators as in the indicated extension of Theorem 3.9. Condition (H3) then yields that the resulting normal drift coefficient is strictly less than $-1/2$. (In [5] we had the opposite sign.) For these operators with frozen coefficients one has a resolvent in $L^2(\mathbb{R}^{N+1}_+)$ with the regularity properties established in the previous section. Using this regularity, the backward transformation, perturbation and partitions of unity, one can now construct the resolvent of A on Ω that satisfies the appropriate estimates.

The above theorem enables to solve the parabolic problem on Ω corresponding to A in optimal regularity. We thus consider the evolution equation

$$\begin{aligned} \partial_t u(t) + Au(t) &= f(t) \qquad \text{on } \ \Omega, \quad t>0, \\ u(0) &= u_0 \qquad \text{on } \ \Omega. \end{aligned} \tag{4.2}$$

The next result follows from standard theory of analytic semigroups. We also refer to e.g. Corollary 1.7 in [12] for the needed result about $W^{1,2}$–regularity and to e.g. Proposition 6.2 and Corollary 1.14 of [13] for the regularity of semigroup orbits starting in the real interpolation space $(L^2(\Omega), D_2^\Omega)_{1/2,2}$.

Corollary 4.2. *Assume that (H1), (H2), (H3) hold. Let $u_0 \in L^2(\Omega)$ and $f \in C(\mathbb{R}_+, L^2(\Omega))$. Then the problem* (4.2) *has a unique solution $u \in C^1((0,\infty); L^2(\Omega)) \cap C((0,\infty); D_2^\Omega) \cap C(\mathbb{R}_+, L^2(\Omega))$, which belongs to $C^1(\mathbb{R}_+; L^2(\Omega)) \cap C(\mathbb{R}_+; D_2^\Omega)$ if $u_0 \in D_2^\Omega$. Let $u_0 \in (L^2(\Omega), D_2^\Omega)_{1/2,2} =: V$ and $f \in L^2((0,T); L^2(\Omega))$ for any $T > 0$. Then the evolution equation* (4.2) *has a unique solution $u \in W^{1,2}((0,T); L^2(\Omega)) \cap L^2((0,T); D_2^\Omega) \cap C([0,T]; V)$.*

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