A singular radial connection over $\mathbb{B}^5$ minimizing the Yang-Mills energy

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Abstract

We prove that the pullback of the $SU(n)$-soliton of Chern class $c_2 = 1$ over $\mathbb{S}^4$ via the radial projection $\pi : \mathbb{B}^5\setminus\{0\} \to \mathbb{S}^4$ minimizes the Yang-Mills energy under the fixed boundary trace constraint. In particular this shows that stationary Yang-Mills connections in high dimension can have singular sets of codimension 5.

1 Introduction

Let G be a compact connected Lie group and $E \to M$ be a vector bundle associated to the adjoint representation of a principal G-bundle $P \to M$ over a compact Riemannian n-manifold M. Following [1] let $\nabla = d + A$ locally represent a connection over E in a given trivialization and let the Lie algebra valued 2-form representing the curvature of ∇ be given by $F = dA + A \wedge A$. We recall that the Yang-Mills functional is defined in term of the ad-invariant norm $|\cdot|$ on $\mathfrak{g}$ by

$$\mathcal{YM}(\nabla) = \int_M |F|^2 d\mathrm{vol}_M \ .$$

Consider now the case $n = 4, M = \mathbb{S}^4$ with the standard metric and $G = SU(n)$. We denote by

$$F_{\mathbb{S}^4} = dA_{\mathbb{S}^4} + A_{\mathbb{S}^4} \wedge A_{\mathbb{S}^4}$$

the instanton on $\mathbb{S}^4$ minimizing the Yang-Mills energy on associated $SU(n)$-bundles $E \to \mathbb{S}^4$ for the adjoint representation, under fixed second Chern number constraint $c_2(E) = 1$:

$$A \in \operatorname{argmin}\left\{ \int_{\mathbb{S}^4} |F_A|^2 d\mathrm{vol}_{\mathbb{S}^4} \ \middle|\ \begin{array}{c} A \text{ is loc. } W^{1,2}, \\ \frac{1}{8\pi^2}\int_{\mathbb{S}^4} \mathrm{tr}(F_A \wedge F_A) = 1 \end{array} \right\} .$$

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The underlying minimization was studied in [10], where it was proved that the minimizer exists and the Chern class constraint is preserved under the underlying weak convergence of connections. It is well known (see [1], [3]) that in this case any minimizing curvature must be **anti-self-dual**. Indeed by Chern-Weil theory we may write

$$c_2(E) = \frac{1}{8\pi^2}\int_{\mathbb{S}^4} \mathrm{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\int_{\mathbb{S}^4} \left(|F_A^-|^2 - |F_A^+|^2\right) d\mathrm{vol}_{\mathbb{S}^4} \ . \tag{1.1}$$

Recall that the space of $\mathfrak{su}(n)$-valued 2-forms Ω^2 splits into the L^2-orthogonal eigenspaces $\Omega^\pm$ of the Hodge star operator $: \Omega^2 \to \Omega^2$ and thus F_A splits as $F_A = F_A^+ + F_A^-$ with $*F_A^\pm = \pm F_A^\pm$. It follows then from equation (1.1) that minimizers of $\mathcal{YM}$ are anti-self-dual and we have

$$\mathrm{tr}(F \wedge F) = |F|^2 \ . \tag{1.2}$$

1.1 Spaces of weak connections

In [9],[10] the analytic study of Yang-Mills connections on bundles $E \to M^4$ over 4-dimensional compact Riemannian manifolds $\mathbb{M}^4$ individuated the following atural space of $\mathfrak{su}(n)$-valued connection forms:

$$\mathcal{A}_{SU(n)}(M^4) := \left\{ \begin{array}{c} A \in L^2,\ F_A \stackrel{\mathcal{D}'}{=} dA + A \wedge A \in L^2 \in L^2 \ , \\ \text{and loc. } \exists\ g \in W^{1,2}(M^4, SU(n)) \text{ s.t. } A^g \in W^{1,2}_{loc} \end{array} \right\} ,$$

where $A^g := g^{-1}dg + g^{-1}Ag$ is the formula representing the change of a connection form A under a change of trivialization g.

For two L^2 connection forms A, A' over $\mathbb{B}^5$ we write $A \sim A'$ if there exists a gauge change $g \in W^{1,2}(\mathbb{B}^5, SU(n))$ such that $A' = A^g$. The class of all such L^2 connection forms A' is denoted $[A]$. In [6] a class suited to the direct minimization of $\mathcal{YM}$ in 5 dimensions was defined as follows:

$$\mathcal{A}_{SU(n)}(\mathbb{B}^5) := \left\{ \begin{array}{c} [A]:\ A \in L^2,\ F_A \stackrel{\mathcal{D}'}{=} dA + A \wedge A \in L^2 \\ \forall p \in \mathbb{B}^5 \text{ a.e. } r > 0,\ \exists A(r) \in \mathcal{A}_{SU(n)}(\partial B_r(p)) \\ i^*_{\partial B_r(p)} A \sim A(r) \end{array} \right\} .$$

Let ϕ be a smooth $\mathfrak{su}(n)$-valued connection 1-form over $\partial\mathbb{B}^5$. Recall from [6] that $A_{SU(n)}(\mathbb{B}^5)$ is the strong L^2-closure of the following space of more regular connections:

$$\mathcal{R}^\infty(\mathbb{B}^5) := \left\{ \begin{array}{c} F \text{ corresponding to some } [A] \in \mathcal{A}_{SU(n)}(\mathbb{B}^5) \text{ s.t.} \\ \exists k, \exists a_1, \ldots, a_k \in \mathbb{B}^5, \quad F = F_\nabla \text{ for a smooth connection} \nabla \\ \text{on some smooth } SU(n)\text{-bundle } E \to \mathbb{B}^5 \setminus \{a_1, \ldots, a_k\} \end{array} \right\} .$$

In [6] it was proved that the trace condition $i^*_{\partial\mathbb{B}^5}A \sim \phi$ can be formalized e.g. and the class $\mathcal{A}^{\phi}_{SU(n)}(\mathbb{B}^5)$ of weak connections with trace ϕ was introduced. A characterization of such class is as the strong closure of connection classes $[A] \in \mathcal{R}^\infty(\mathbb{B}^5)$ which satisfy $i^*_{\partial\mathbb{B}^5}A \sim \phi$. The main results of [6] can be combined into the following theorem:

Theorem 1.1 (Main results of [6]). *The minimizer of*

$$\inf\left\{\|F_A\|_{L^2(\mathbb{B}^5)} : [A] \in \mathcal{A}^{\phi}_{G}(\mathbb{B}^5)\right\}$$

exists and is smooth outside a set of isolated singular points.

A question which arised naturally is whether such result is optimal.

Indeed in [8] a conjecture was formulated, according to which the singular set of F would have Hausdorff dimension smaller than $n-6$ in the case of so-called admissible Ω-**anti-self-dual** curvatures F, i.e. curvatures satisfying $F = -\Omega \wedge F$ for a smooth closed $(n-4)$-form Ω in dimension n under the further requirement of F being admissible i.e. that the underlying connection be locally smooth outside of a $(n-4)$-dimensional rectifiable set. A natural question is to ask for examples of stationary or energy-minimizing connection classes which show that the Ω-anti-self-dual requirement is necessary.

To the author's knowledge, in the literature no proof is available that stationary curvatures F having a singular set of Hausdorff codimension greater or equal than 5 exist. This situation is similar to the one taking place in the theory of harmonic maps precedently to R. Hardt, F.H. Lin and C.Y. Wang's celebrated paper [4] where it was proved that the map $x/|x| : \mathbb{B}^3 \to \mathbb{S}^2$ minimizes the p-th norm of the gradient among maps whose boundary trace is equal to the identity.

1.2 Main result of the paper

The main goal of this paper is to show that a result similar in spirit to [4] holds for the case of Yang-Mills minimization in dimension 5.

Theorem 1.2 (Main result). *Consider the connection form $A_{\mathbb{S}^4}$ and its pullback $A_{rad} := \left(\frac{x}{|x|}\right)^* A_{\mathbb{S}^4}$. Then A_{rad} is a minimzer for the problem*

$$\min\left\{\int_{\mathbb{B}^5} |F_A|^2 dvol_{\mathbb{B}^5} : [A] \in \mathcal{A}_{SU(n)}(\mathbb{B}^5), [i^*_{\partial B^5}A] = [A_{\mathbb{S}^4}]\right\} .$$

Note that for the above minimizer $[A]$ there holds

$$d\,(\mathrm{tr}(F_A \wedge F_A)) = 8\pi^2 \delta_0 \ ,$$

i.e. the minimizer presents a topological singularity.

We note that the curvature form $F_{rad} := F_{A_{rad}}$ of Theorem 1.2 is Ω-anti-self-dual with respect to the radial 1-form

$$\Omega = \sum_{k=1}^{5} \frac{x_k}{|x|} dx_k = dr$$

outside the origin. In other words we have $F_{rad} \wedge \Omega = - * F_{rad}$. The form Ω is closed in the sense of distributions, however it is not smooth. Therefore it does not fully enter the setting presented in [8] and Conjecture 2 of [8] remains still open.

1.3 Minimizing L^1 vector fields with defects

We note that to an L^2-integrable $\mathfrak{su}(n)$-values 2-form F defined on $\mathbb{B}^5$ we may associate an L^1 vector field X by requiring the duality formula

$$\langle \phi, X \rangle = \langle \mathrm{tr}(F \wedge F), *\phi \rangle = \int_{\mathbb{B}^5} \mathrm{tr}(F \wedge F) \wedge \phi$$

to hold for all smooth 1-forms on $\overline{\mathbb{B}^5}$. Through the pointwise inequality

$$|\mathrm{tr}(F \wedge F)| \leq |F|^2 \tag{1.3}$$

we also deduce that

$$\|X\|_{L^1(\mathbb{B}^5)} = \|\mathrm{tr}(F \wedge F)\|_{L^1(\mathbb{B}^5)} \leq \|F\|^2_{L^2(\mathbb{B}^5)} \ . \tag{1.4}$$

Note that the curvature form A_{rad} of Theorem 1.2 realizes the pointwise equality in (1.3) and thus also in (1.4). We will deduce our main theorem from the following result, which is of independent interest.

Theorem 1.3. *The vector field $X_{rad}(x) = |\mathbb{S}^4|^{-1} \frac{x}{|x|^5}$ minimizes the L^1-norm among vector fields $X \in L^1(\mathbb{B}^5, \mathbb{R}^5)$ which are locally smooth outside some finite subset $\Sigma \subset \mathbb{B}^5$, satisfy*

$$div X \llcorner \mathbb{B}^5 = \sum_{x \in \Sigma} d_x \delta_x \ , \tag{1.5}$$

for some integers d_x and $X \cdot \nu_{\mathbb{S}^4} \equiv 1$ where ν is the interior normal vector field to $\mathbb{S}^4$.

This result is proved using similar tools as in [5], i.e. Smirnov's decomposition for 1-currents [7] and a combinatorial result based on the maxflow-mincut theorem.

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2 Proof of Theorem 1.3

2.1 Smirnov's decomposition and combinatorial reduction

We start by recalling a version of Smirnov's result [7], which allows to reduce the larger step of the prof of Theorem 1.3 to a combinatorial argument. We formulate the result in the case of vector fields with divergence of special form as in Theorem 1.3.

We recall the following definitions and notations:

- An **arc** in $\overline{\mathbb{B}^5}$ is a rectifiable curve which has an injective parameterization $\gamma : [0,1] \to \overline{\mathbb{B}^5}$. To an arc we may associate a continuous linear functional on smooth 1-forms α given via $\langle [\gamma], \alpha \rangle := \int_\gamma \omega$. We also call an arc the functional $[\gamma]$.

- The space of all arcs $[\gamma]$ is topologized with the weak topology. Note that the functions $s([\gamma]), e([\gamma])$ which to an arc assign its starting and ending point respectively, are Borel measurable. The variation measure of such $[\gamma]$ is denoted by $\|\gamma\|$ and its total variation is the lenght $\|\gamma\|(\overline{\mathbb{B}^5}) = \ell(\gamma)$. The boundary $\partial[\gamma]$ is given by the difference of Dirac masses $\delta_{e([\gamma])} - \delta_{s([\gamma]}$. and its variation measure is $\delta_{e([\gamma])} + \delta_{s([\gamma]}$.

- We say that two vector fields B, C with divergences of finite total variation **decompose** a vector field A if $A = B + C$, $|A| = |B| + |C|$. We say that a vector field X is **acyclic** if for each such decomposition with $\partial C = 0$ there holds $C = 0$. Note that any minimizer as in Theorem 1.3 must be acyclic since if we had a decomposition $X = B + C$ as above with $\partial C = 0, C \neq 0$ then B would be a competitor to X of strictly smaller L^1 norm.

Theorem 2.1 (Decomposition of vector fields, [7]). *Assume X is an acyclic L^1 vector field over $\overline{\mathbb{B}^5}$ such that $divX$ is a measure of finite total variation. We may then find a finite Borel measure over the space of arcs such that the* *following hold for all smooth 1-forms α and for all smooth functions f over $\overline{\mathbb{B}^5}$:*

$$\langle X, \alpha \rangle = \int \langle [\gamma], \alpha \rangle d\mu(\gamma) \ , \tag{2.1}$$

$$\langle |X|, f \rangle = \int \langle \|\gamma\|, f \rangle d\mu(\gamma) \ , \tag{2.2}$$

$$\langle divX, f \rangle = \int \langle \delta_{e([\gamma])} - \delta_{s([\gamma]}, f \rangle d\mu(\gamma) \ , \tag{2.3}$$

$$\langle \| divX \|, f \rangle = \int \langle \delta_{e([\gamma])} + \delta_{s([\gamma]}, f \rangle d\mu(\gamma) \ . \tag{2.4}$$

In other words the vector field X decomposes as a superposition of arcs without cancellations whose boundaries decompose div$X$ without cancellations. See Figure 1.

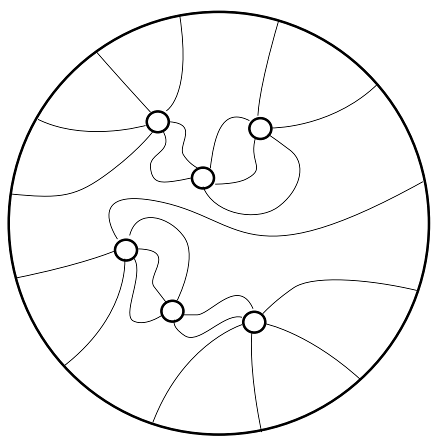

Figure 1: *We present here some of the arcs in the Smirnov decomposition of a vector field X as in Theorem 1.3, including some of the charges and some of the curves in the support of μ as in Theorem 2.1. Note that the arcs are actually oriented.*

Note that for a vector field X as in Theorem 1.3 for μ-a.e. arc γ the starting point $s([\gamma])$ is in one of the points x such that $d_x < 0$ in the expression of div$X \llcorner \mathbb{B}^5$ and the end point $e([\gamma])$ is either in one of the points x with $d_x > 0$ or on the boundary ∂B^5. We may then consider as in [5] the Borel sets of the form

$$C_{x,y} = \{[\gamma] \ \text{ such that } s([\gamma]) = x, e([\gamma]) = y\} \text{with } d_x < 0, d_y > 0 \tag{2.5}$$

and

$$C_{\partial,y} = \{[\gamma] \ \text{ such that } s([\gamma]) \in \partial\mathbb{B}^5, e([\gamma]) = y\}, \quad \text{with } d_y > 0 \ , \tag{2.6}$$

where d_x are the integers appearing in (1.5). We observe that the above sets $C_{*,*}$ form a finite partition of μ-almost all of $\mathrm{spt}(\mu)$ and that given a function

$$\alpha : \{C_{x,y}\} \cup \{C_{\partial,y}\} \to [-1,1]$$

the measure

$$\mu_\alpha := \sum_{x,y} \alpha(C_{x,y})\mu \llcorner C_{x,y} + \sum_{x} \alpha(C_{\partial,y})\mu \llcorner C_{\partial,y} \tag{2.7}$$

also gives an L^1 vector field X_α with finite variation divergence which is defined via an equation like (2.1) and saisfies (2.2), (2.3) but not necessarily (2.4). The measure μ_{X_α} obtained applying Theorem 2.1 to X_α is supported on curves which are concatenations of curves in the support of μ_α.

2.2 Combinatorial results

We now fix the notations for a combinatorial structure which will be associated to data X, μ given above.

Definition 2.2 (X-graph)**.** *Consider a finite set of vertices V of which a special vertex $\partial \in V$ is highlited, a graph on V, i.e. a subset $E \subset V \times V$ such that $(a,b) \in E \Rightarrow (b,a) \in E$, and a weight function $w : E \to \mathbb{R}$ such that $w(a,b) = -w(b,a)$ unless $a = b = \partial$, in which case we require $w(\partial,\partial) \geq 0$. We call such data (V,E,w,∂) an X-**graph** if*

1. *We have integer fluxes: $\forall v \in V \setminus \{\partial\},\ fl(v) := \sum_a w(a,v) \in \mathbb{Z}$, where the sum is taken over all $a \in V$ such that $(a,v) \in E$.*
2. *$fl(\partial) := \sum\{w(a,\partial) :\ (a,\partial) \in E \setminus (V \times \{\partial\})\} = -1$.*
3. *For all $v \in V \setminus \{\partial\}$ we have that all terms in the sum defining $fl(v)$ have the same sign.*

*We denote V^+ the vertices for which the sign in the last point is positive and V^- those for which it is negative. Given $x, y \in V$ such that $(x,y),(y,x) \in E$ we say that (x,y) is a **directed edge** if $w(x,y) > 0$.*

We associate an X-graph to a vector field X as in Theorem 1.3 by defining

$$\begin{aligned} V &:= \{x : d_x \neq 0\} \cup \{\partial\}\ , \\ E &:= \{(x,y),(y,x),(\partial,y),(y,\partial) :\ C_{x,y}, C_{\partial,y} \text{ are as in (2.5),(2.6)}\} \end{aligned}$$

and for $* \in V$

$$w(*,y) := \mu(C_{*,y}),\ w(y,*) := -\mu(C_{*,y}) \quad \text{if } C_{*,y} \text{ appears in (2.5) or (2.6)}\ .$$

We leave the verification of the properties as in Definition 2.2 to the reader. Note that in this case we obtain property (3) of Definition 2.2 also for $v = \partial$ and we have $\partial \in V^-, w(\partial, \partial) = 0$. We also need the following definition which is a modification of that of an X-graph.

Definition 2.3 ($\bar{X}$-graph)**.** *Consider $(V, E, w, \bar{\partial})$ as in Definition 2.2 except that $\bar{\partial} \subset V$ may now contain more than one vertex, and that we require $w(a, b) = -w(b, a)$ for all edges $(a, b) \in E$ with no exception. We say that $(V, E, w, \bar{\partial})$ form an $\bar{X}$-graph if*

1. $\forall v \in V \setminus \bar{\partial},\ fl(v) \in \mathbb{Z}$.
2. $\sum_{v \in \bar{\partial}} fl(v) = 0$.
3. *For $v \in V \setminus \bar{\partial}$ all terms in the sum defining $fl(v)$ have the same sign.*

Note that as a consequence of the fact that $w(a, b) = -w(b, a)$ even for $a, b \in \bar{\partial}$ it follows that a $\bar{X}$-graph has no loops, unlike X-graphs who were allowed to have loops on the boundary.

A tool in our combinatorial construction will be the maxflow-mincut theorem, to state which we recall the definition of a (combinatorial) flow.

Definition 2.4 (X-flows, $\bar{X}$-flows and cuts)**.** *Let (V, E, w, ∂) be an X-graph and fix a vertex $a^+ \in V^+$. A function $f : E \to \mathbb{R}$ such that $f(a, b) = -f(b, a)$ for $(a, b) \neq (\partial, \partial)$ and $f(\partial, \partial) \geq 0$ is a X**-flow** if the following properties hold:*

1. $|f(a, b)| \leq |w(a, b)|$ *for all* $(a, b) \in E$.
2. *For all $v \in V \setminus \{\partial, A^+\}$ there holds $\sum\{f(a, v) :\ (a, v) \in E\} = 0$.*
3. *For all $(x, \partial), (a^+, y) \in E$ there holds $sgn(f(x, \partial)) = sgn(w(x, \partial))$ and $sgn(f(a^+, y)) = sgn(w(a^+, y))$.*

*We call the vertex a^+ the **sink** of the X-flow f. The **value of the** X**-flow** f is by definition the number $val(f) := \sum\{f(\partial, y) :\ (\partial, y) \in E, y \neq \partial\}$. We say that the edge (a, b) is **saturated** by f if equality holds in point 1. above. If all edges with an end equal to $v \in V$ are saturated, we say that f **saturates** v.*

*If $(V, E, w, \bar{\partial})$ is a $\bar{X}$-graph then we define a $\bar{X}$**-flow** as above, except that $f(a, b) = -f(b, a)$ will be required to hold for all edges (a, b) with no exception.*

*For a given vertex $a^+ \in V^+$ a **cut between** ∂ **and** a^+ of the X-graph (V, E, w, ∂) is a subset $S \subset E$ such that for every path $(\partial := a_0, a_1)$, (a_1, a_2), $\ldots$, $(a_k, a_{k+1} := a^+)$ such that $(a_i, a_{i+1}) \in E$ for all $i = 0, \ldots, k$ there exists an index i such that either $(a_i, a_{i+1}) \in S$ or $(a_{i+1}, a_i) \in S$. The **value of a***

cut *S is by definition the number $val(S) := \sum\{|w(x,y)| : \ (x,y) \in S\}$.*

We say that a X-flow (or a $\bar{X}$-flow) f ***saturates the cut*** *S if it saturates all edges of S.*

Note that for a X-flow the following are equivalent: (a) f saturates ∂; (b) $f = w$ on all edges with an end in ∂; (c) f has value 1. Our main combinatorial result is the following:

Proposition 2.5 (existence of a saturating X-flow)**.** *Let (V, E, w, ∂) be an X-graph. Then we may find a X-flow f which saturates ∂.*

We will need the following result present in [5], of which we present a different proof:

Proposition 2.6 (existence of a saturating $\bar{X}$-flow, [5])**.** *Let $(V'', E'', w'', \bar{\partial})$ be a $\bar{X}$-graph with the bound*

$$\sum \Big\{|w''(a,b)| : \ a \in \bar{\partial}, \ (a,b) \in E\Big\} < 1 \ . \tag{2.8}$$

Then there exists a $\bar{X}$-flow f'' saturating $\bar{\partial}$.

Proof of Proposition 2.6: We proceed by induction on the number of non-boundary vertices $\#(V'' \setminus \bar{\partial})$. In the case $\#(V'' \setminus \bar{\partial}) = 0$ we may take $f'' = w''$ and we cnclude.

Supposing that the statement is true when $\#(V'' \setminus \bar{\partial}) < n$ we may prove it fo the case $\#(V'' \setminus \bar{\partial}) = n$ as follows.

Up to reducing to the connected components we may assume that the underlying graph (V'', E'') is connected.

Applying the maxflow-mincut theorem [2] we obtain the existence of a $\bar{X}$-flow of maximum value f and of a minimum value cut S saturated by f. Then S separates the graph (V'', E'') into two connected components $V_1 \supset \bar{\partial}^+, V_2 \supset \bar{\partial}^-$ which are both $\bar{X}$-graphs with $\#(V \setminus \bar{\partial}) < n$, if we take $V_i' := V_i \cup \{\text{ends of edges in } S\}$, $E_i := [E'' \cap (V_i \times V_i)] \cup S$, $w_i := w''|_{E_i}$ for $i = 1, 2$

$$\bar{\partial}_1 = \bar{\partial}^+ \cup \{\text{ends of edges in } S\} \cap V_2, \quad \bar{\partial}_2 = \bar{\partial}^- \cup \{\text{ends of edges in } S\} \cap V_1 \ .$$

The properties of a $\bar{X}$-graph are all easy to verify for the $(V_i, E_i, w, \bar{\partial}_i)$ except perhaps for property 2. in Definition 2.3, i.e. the fact that the total flux through the boundaries are zero. To prove this we use the bound (2.8) and the integrality condition in the definition of a $\bar{X}$-graph. Let $S\pm$ be the set of edges in S for which $f''(a,b) = \pm w''(a,b)$. We have $val(f) < 1/2$ as a consequence of the fact that in the original $\bar{X}$-graph the total flux through $\bar{\partial}$ was zero and of (2.8). Since f'' saturates S we have

$$1/2 > val(S) = \sum_{(a,b)\in S} |w''(a,b)| = val(S^+) + val(S^-) \ .$$

From the integrality condition 1. in Definition 2.3 and from the fact that S disconnects the underlying graph (V'', E'') we deduce that the total flux through $\bar{\partial}_i$ must be an integer for $i = 1, 2$. Ath the same time the absolute value of this flux is bounded by $val(\bar{\partial}^+) + val(S) < 1$. Therefore it must be zero, and condition 2. in Definition 2.3 is verified.

We may then apply the inductive hypotheses and obtain flows f_1, f_2 saturating $\bar{\partial}_1, \bar{\partial}_2$. In particular these flows coincide on S and they extend to $\bar{X}$-flow f'' over the initial $\bar{X}$-graph. The fact that f'' saturates $\bar{\partial}$ follows from the fact that f_i saturates $\bar{\partial}_i$ for $i = 1, 2$. $\square$

Proof of Proposition 2.5: We proceed by induction on the number of vertices $\#V$. For $\#V = 2$ it suffices to take $f = w$.

Let now $n > 2$, assume that the thesis is true whenever $\#V < n$, and consider the case $\#V = n$. Choose as a sink a vertex $a^+ \in V^+$. Consider a X-flow f with sink a^+ of maximal value. By the maxflow-mincut theorem [2] there exists a cut S realizing the minimum of possible values of cuts between ∂ and a^+ and such that f saturates S and the value of f equals the value of S.

If $val(f) = 1$ then f saturates ∂ and we conclude. If $val(f) = 0$ then S can be taken to be empty, thus a^+, ∂ are in different connected components of the X-graph. In this case we may remove the connected component of a^+ and reduce to the case $\#V < n$. We then conclude by inductive hypothesis.

Consider now the remeining case when the value of f is in $]0,1[$. Let $S^\pm$ be the sets of edges in S for which $\mathrm{sgn}(w) = \pm$ and let $s^\pm := \sum_{S^\pm} |w(e)|$. We then conclude from the definition of an X-graph and from the fact that f saturates S that

$$s^+ - s^- \in \mathbb{Z}, \quad val(f) = s^+ + s^- \in]0,1[\ \text{ thus } s^+ = s^- \ . \tag{2.9}$$

We then replace the X-graph (V, E, ∂, w) with the X-graph (V', E', ∂, w') defined as follows:

- $V' \subset V$ consists of all vertices in the connected component of ∂ with respect to the cut S.

- E' consists of all edges in $E \cap (V' \times V')$ and of new edges of the form $(v', \partial), (\partial, v')$ where $(v', x) \in S, v' \in V'$.
- w' is defined to be equal to w on $E \cap V' \times V'$, while for $(v', x) \in S, v' \in V'$ we define $w'(v', \partial) = -w'(\partial, v') = w(v', x)$.

We see that the properties as in the definition of an X-graph are trivially valid at vertices $v' \in V' \setminus \{\partial\}$ while at ∂ they are still valid due to (2.9).

Since $\#V' < \#V$ we may apply the inductive hypothesis to (V', E', ∂, w') and find a X-flow f' with sink $b^+ \in (V')^+$ which saturates ∂. We then extend it to a X-flow $\bar{f}$ on the original X-graph (V, E, ∂, w) as follows.

Note that we may define a $\bar{X}$-graph $(V'', E'', w'', \bar{\partial})$ by defining

$$\begin{aligned} V'' &:= (V \setminus V') \cup \{\text{ends of edges in } S\} , \\ E'' &:= [E \cap (V'' \times V'')] \cup S , \\ w'' &:= w|_{E''} , \\ \bar{\partial} &:= \{\text{ends of edges in } S\} \setminus V' . \end{aligned}$$

By applying Proposition 2.6 to this $\bar{X}$-graph we may find a flow f'' on it saturating $\bar{\partial}$. In particular this flow conicides with f' on $\bar{\partial}$ and the extension $\bar{f}$ of f' via f'' is well-defined and is an X-flow, saturating ∂, as desired. $\square$

See Figure 2 for the corresponding picture for vector fields X as in Theorem 1.3.

2.3 Proof of Theorem 1.3

Proof. We apply Proposition 2.5 to the X-graph associated to a vector field X as in Theorem 1.3 and to the associated measure on arcs μ given by Theorem 2.1 applied to X.

From the flow f as given in Proposition 2.5 if $\alpha = \chi_{\{w \neq 0\}} f/w$ we construct a measure μ_α as in (2.7). This measure on arcs gives an L^1 vector field X_α which in turn decomposes via Smirnov's Theorem 2.1 via a different measure μ', this time into a superposition of arcs γ such that

$$\begin{aligned} s([\gamma]) &\in \partial \mathbb{B}^5 , \\ e([\gamma]) &= a^+ , \end{aligned}$$

where the point a^+ is the one where the charge corresponding to the sink of the flow f is located.

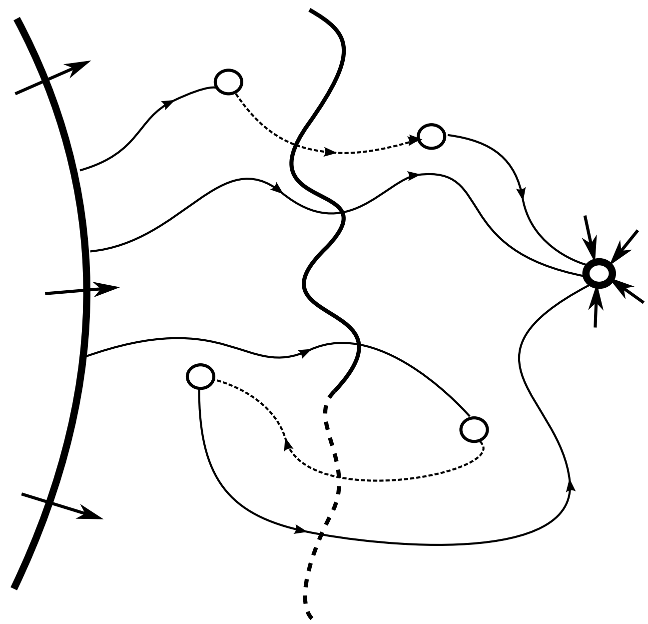

Figure 2: *We represent a possible situation in the last step of the proof of Proposition 2.5, seen at the level of the vector fields X as in Theorem 1.3. On the left we represent the boundary $\partial\mathbb{B}^5$ and on the right we have a positive charge corresponding to a possible sink of the saturating flow of Proposition 2.5. Some arcs corresponding to the saturating flow are represented by directed lines. These arcs are obtained as concatenations of arcs corresponding to the original vector field X, with the same orientation (continuous parts) or reversed orientation (dashed parts). The continuous vertical wiggly line in the center of the figure represents a minimal cut. We enlarging the cut with the dashed part of that line would make it non-minimal.*

Note that from the remarks at the beginning of Section 2.1 the lenghts of such arcs give the L^1 norm of the vector field X_α and we have

$$\int_{\mathbb{B}^5} |X_\alpha| = \int \ell(\gamma)d\mu'(\gamma) \leq \int \ell(\gamma)d\mu(\gamma) = \int |X|. \tag{2.10}$$

The inequality is due to the fact that since we only decreased the weights of curves the new vector field X_α satisfies pointwise a.e. x the inequality $|X_\alpha|(x) \leq |X|(x)$. Note however that while $\mathrm{div}(X_\alpha) = \delta_{a^+} + |\mathbb{S}^4|^{-1}\mathcal{H}^4 \ll \mathbb{S}^4$ remains valid in the sense of distributions, we don't have anymore the information that X_α is locally smooth on $\mathbb{B}^5 \setminus \{a^+\}$.

For μ'-a.e. $\gamma \in \mathrm{spt}(\mu')$ we have

$$\ell(\gamma) \geq |s([\gamma]) - e([\gamma])| \ ,$$

thus if we denote the arc corresponding to a segment from a to b by $[\gamma]=[a,b]$ then we may write

$$\begin{aligned}\int_{\mathbb{B}^5}|X_\alpha| &= \int \ell(\gamma)d\mu'(\gamma)\\ &\geq \int |s([\gamma])-e([\gamma])|d\mu'(\gamma)\\ &= \int_{\mathbb{S}^4}|x-a^+|d\mathcal{H}^4(x)\ . \qquad (2.11)\end{aligned}$$

Note that

$$\left.\frac{\partial}{\partial t}\right|_{t=0}\int_{\mathbb{S}^4}|x-(a^++tb)|d\mathcal{H}^4(x)=-b\cdot\int_{\mathbb{S}^4}\frac{x-a^+}{|x-a^+|}d\mathcal{H}^4(x)\ ,$$

thus the minimum of (2.11) is achieved when a^+ satisfies

$$\fint_{\mathbb{S}^4}\frac{x-a^+}{|x-a^+|}=0\ ,$$

i.e. precisely for $a^+=0$. The thesis of Theorem 1.3 now follows from (2.10), (2.11). $\square$

3 Proof of Theorem 1.2

Proof. We see from (1.3), (1.4) that for the PoincaÈ-Hodge dual vector field X to $\mathrm{tr}(F\wedge F)$ there holds $\|X\|_{L^1}\leq\|F\|^2_{L^2}$ with equality in the case of the curvature form F_{rad} as in Theorem 1.2. From the fact [6] that $\mathcal{R}^\infty(\mathbb{B}^5)$ is dense in $\mathcal{A}_{SU(n)}(\mathbb{B}^5)$ it follows that the infimum of the L^2 norm of the curvature given by Theorem 1.1 in the case of the boundary trace $A_{\mathbb{S}^4}$, is equal to the infimum of the same functional over $[A]\in\mathbb{R}^\infty(\mathbb{B}^5)$ with trace $\sim A_{\mathbb{S}^4}$. In particular we have that

$$\begin{aligned}\int_{\mathbb{B}^5}|F_{rad}|^2 &= \int_{\mathbb{B}^5}|X_{rad}|\\ &= \inf\left\{\int|X|:\ X \text{ as in Theorem 1.3}\right\}\\ &\leq \inf\left\{\int|F_A|^2:\ [A]\in\mathcal{R}^\infty(\mathbb{B}^5),\ i^*_{\partial\mathbb{B}^5}A\sim A_{\mathbb{S}^4}\right\}\\ &= \min\left\{\int|F_A|^2:\ [A]\in\mathcal{A}^{A^4_{\mathbb{S}}}_{SU(n)}(\mathbb{B}^5)\right\}\ .\end{aligned}$$

In particular all inequalities above must be equalities and Theorem 1.2 follows. $\square$

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