EXAMPLE OF MINIMIZER OF THE AVERAGE-DISTANCE PROBLEM WITH NON CLOSED SET OF CORNERS

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Abstract. The average-distance problem, in the penalized formulation, involves minimizing

$$E^\lambda_\mu(\Sigma) := \int_{\mathbb{R}^d} \inf_{y\in\Sigma} |x-y| \mathrm{d}\mu(x) + \lambda\mathcal{H}^1(\Sigma),$$

among compact, connected sets Σ, where $\mathcal{H}^1$ denotes the 1-Hausdorff measure, $d \geq 2$, μ is a given measure and λ a given parameter. Regularity of minimizers is a delicate problem. It is known that even if μ is absolutely continuous with respect to Lebesgue measure, C^1 regularity does not hold in general. An interesting question is whether the set of corners, i.e. points where C^1 regularity does not hold, is closed. The aim of this paper is to provide an example of minimizer whose set of corners is not closed, with reference measure μ absolutely continuous with respect to Lebesgue measure.

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1. Introduction

The average-distance problem, in the penalized formulation, was introduced by Buttazzo, Mainini and Stepanov in [1]:

Problem 1.1. *Given $d \geq 2$, a measure μ, and a parameter $\lambda > 0$, minimize*

$$E^\lambda_\mu : \mathcal{A} \longrightarrow \mathbb{R}, \qquad E^\lambda_\mu(\Sigma) := F_\mu(\Sigma) + \lambda\mathcal{H}^1(\Sigma),$$

where $\mathcal{H}^1$ denotes the 1-Hausdorff measure and

$$F_\mu : \mathcal{A} \longrightarrow \mathbb{R}, \qquad F_\mu(\Sigma) := \int_{\mathbb{R}^d} d(x,\Sigma)\mathrm{d}\mu, \quad d(x,\Sigma) := \inf_{y\in\Sigma} |x-y|,$$

$$\mathcal{A} := \{\Sigma \subseteq \mathbb{R}^d : \Sigma \text{ compact and connected}\}.$$

An earlier variant is the constrained formulation, first studied by Buttazzo, Oudet and Stepanov in [2]:

Problem 1.2. *Given $d \geq 2$, a measure μ, and a parameter $L \geq 0$, minimize*

$$\min_{\Sigma\in\mathcal{A},\ \mathcal{H}^1(\Sigma)\leq L} F_\mu(\Sigma).$$

Existence of minimizers (for both Problems 1.1 and 1.2) follows (see for instance [1, 2, 3]) from Blaschke selection theorem and Gołab theorem. The functional F_μ will be often referred as *"average-distance functional"*, and Problem 1.1 as *"average-distance problem"*. In the following, any considered measure will be assumed nonnegative, compactly supported, probability measures. The choice to work with probability measures is done for the sake of simplicity.

The average-distance problem originally stemmed from mathematical modeling of optimization problems. A classic application is found in urban planning: let

- μ be the distribution of passengers in a given region,
- Σ (the unknown) be the transport network to be built.

In this case $F_\mu(\Sigma)$ is the "average distance" of passengers from the network (thus smaller values of $F_\mu(\Sigma)$ imply that Σ is "easily accessible"), and $\lambda\mathcal{H}^1(\Sigma)$ is the cost to build such network. Thus minimizing E_μ^λ is determining the network which "best serves" the passengers, under cost considerations.

A recent application is found in data approximation: let

- μ be the distribution of data points,
- Σ (the unknown) be a one-dimensional set which approximates the data.

In this case $F_\mu(\Sigma)$ is the approximation error, while $\lambda\mathcal{H}^1(\Sigma)$ is the cost associated to its complexity. Thus minimizing E_μ^λ is determining the "best" approximation, which balances approximation error and cost. In data approximation the regularity of Σ is important: indeed it has been proven (Slepčev [9]) that, outside triple points, a positive amount of mass is projected on any point for which C^1 regularity fails. This corresponds to a loss of information, and is undesirable.

Regularity of minimizers of both Problems 1.1 and 1.2 is quite a delicate problem. It is known that:

- C^1 regularity is false in general (Slepčev [9]);
- minimizers are finite union of at most $[1/\lambda]$ (with $[\cdot]$ denoting the integer part mapping) Lipschitz regular curves, in any dimension (Slepčev et al. [8]);
- the sum of the total curvature measure of each branch is uniformly bounded from above by $|\mu|/\lambda$, with $|\mu|$ denoting the total mass of the reference measure μ (Slepčev et al. [8]).

For future reference, given $\Sigma \in \mathcal{A}$, a point $p \in \Sigma$ of degree two (i.e. $\Sigma\backslash\{p\}$ has exactly two connected components, see Definition 2.3) for which C^1 regularity fails will be referred as "corner". Since the approach used in [9] is only suited for constructing minimizers with finitely many corners, it is unclear if (for minimizers) the set of corners is generally closed, or even finite. The aim of this paper is to provide an example of minimizer whose set of corners is not closed. The main result is:

Theorem 1.3. *In $\mathbb{R}^2$, there exists a measure μ, a parameter λ, and a minimizer $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ satisfying:*

- *Σ is a simple curve,*
- *there exists a sequence $\{v_n\} \subseteq \Sigma$ such that C^1 regularity fails at v_n for any n (i.e. v_n is a corner for any n),*
- *there exists point $v \in \Sigma$ such that $\{v_n\} \to v \in \Sigma$, and Σ is C^1 regular in v (i.e. v is not a corner).*

We will prove a stronger result (Theorem 3.21), with quantitative estimates on the jump of the tangent derivative at v_n, $n = 1, 2, \cdots$. As corollary we have:

Corollary 1.4. *The minimizer* Σ *from Theorem 1.3 is also minimizer for the constrained problem*

$$\min_{\Sigma' \in \mathcal{A},\ \mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma)} F_\mu(\Sigma'). \tag{1}$$

We will use the construction from [7], i.e. we will approximate the reference measure with discrete measures, analyze minimizers in the discrete case (subsection 3.2), and pass to the limit (subsection 3.3). However, since the aim is to construct infinitely many corners, we need several additional estimates on the mutual distance between corners of minimizers of the discrete problem. This allows to deduce that minimizers of the discrete problem have infinitely many corners, and passing to the continuum limit, we still have infinitely many corners. This strongly exploits the two-dimensional structure (in particular in Lemmas 3.15 and 3.17), and cannot be extended to higher dimensional domains. Note that Theorem 1.3 states *only* the existence of a minimizer with the above properties. However it does *not* preclude the existence of other minimizers $\Sigma' \in \text{argmin } E^\lambda_\mu$ containing only finitely many (or even zero) points at which C^1 regularity fails. This paper will be structured as follows.

- In Section 2 we recall preliminary results.
- In Section 3 we construct an explicit example of minimizer of Problem 1.1 whose set of corners is not closed. In particular:
  - in subsection 3.1 we determine the main elements of our construction, including the reference measure μ and parameter λ,
  - in subsection 3.2 we approximate the reference measure with a sequence of discrete measures, and analyze minimizers of the discrete case,
  - in subsection 3.3 we pass to the continuum limit.
- In Section 4 we prove some technical lemmas used in Section 3.

2. Preliminary results

The main goal of this section is to introduce some notations and recall well known results used in Section 3. The average-distance functional satisfies the following well known properties:

(1) given a measure μ and $\lambda > 0$, the mapping $\Sigma \mapsto E^\lambda_\mu(\Sigma)$ is lower semicontinuous with respect to $d_{\mathcal{H}}$ (Hausdorff distance);
(2) given $\Sigma \in \mathcal{A}$ and $\lambda > 0$, the mapping $\mu \mapsto E^\lambda_\mu(\Sigma)$ is continuous with respect to weak-* convergence of measures,
(3) if $\{\mu_n\} \xrightharpoonup{*} \mu$, then for any $\lambda > 0$, it holds $\{E^\lambda_{\mu_n}\} \xrightarrow{\Gamma} E^\lambda_\mu$, that is
   - for any Σ and sequence $\{\Sigma_n\} \xrightarrow{d_{\mathcal{H}}} \Sigma$ it holds $\liminf_{n\to+\infty} E^\lambda_\mu(\Sigma_n) \geq E^\lambda_\mu(\Sigma)$,
   - for any Σ there exists a sequence $\{\Sigma_n\} \xrightarrow{d_{\mathcal{H}}} \Sigma$ such that $\limsup_{n\to+\infty} E^\lambda_\mu(\Sigma_n) \leq E^\lambda_\mu(\Sigma)$,
(4) consider a sequence $\{\mu_n\} \xrightharpoonup{*} \mu$ and for any n choose $\Sigma_n \in \text{argmin } E^\lambda_{\mu_n}$. Then there exists $\Sigma \in \text{argmin } E^\lambda_\mu$ such that (upon subsequence) $\{\Sigma_n\} \xrightarrow{d_{\mathcal{H}}} \Sigma$.

For further details (including proofs), we refer to [2, 3, 4, 9].

Recall that given a set of points $\Pi := \{P_1, \cdots, P_j\} \subseteq \mathbb{R}^2$, a Steiner graph for Π is a path-wise connected set with minimal length (among the family of path-wise connected sets containing Π). Steiner graphs are not unique in general. The next result proves an intrinsic connection between Steiner graphs and minimizers of the average distance functional.

Proposition 2.1. *Given a discrete probability measure $\mu := \sum_{i=1}^n a_i\delta_{x_i}$ on $\mathbb{R}^2$, with $a_1, \cdots, a_n \geq 0$ and δ denoting the Dirac measure supported on the subscripted point, a parameter $\lambda > 0$, then any minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$ is a Steiner graph.*

Proof. For the proof we refer to [9]. □

Definition 2.2. *Given a discrete probability measure $\mu := \sum_{i=1}^n a_i\delta_{x_i}$ on $\mathbb{R}^2$, $\lambda > 0$, and a minimizer $\Sigma \in \text{argmin}_{\mathcal{A}}\ E_\mu^\lambda$, a point $v \in \Sigma$ is a "vertex" if there exists $i \in \{1, \cdots, n\}$ such that $d(x_i, \Sigma) = |x_i - v|$.*

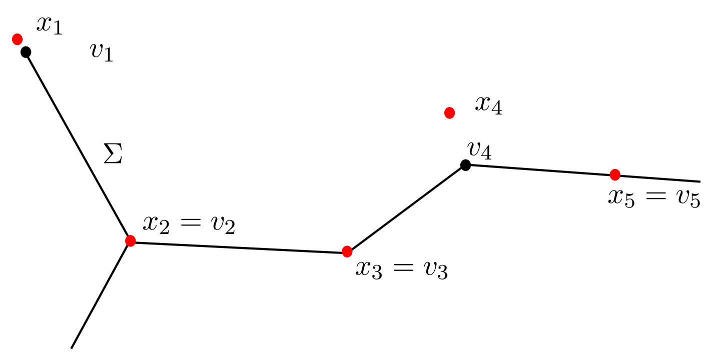

FIGURE 1. Some examples of vertices ($\{v_1, \cdots, v_5\}$ in figure). The red dots denote the support ($\{x_1, \cdots, x_5\}$ in figure) of the reference measure. Here $d(x_i, \Sigma) = |x_i - v_i|$, $i = 1, \cdots, 5$.

Next we define the notion of "degree" of a point.

Definition 2.3. *Given $\Sigma \in \mathcal{A}$, consider a point $v \in \Sigma$ such that $\Sigma\backslash\{v\}$ has finitely many connected components. Then the "degree" of v is defined as the number of connected components of $\Sigma\backslash\{v\}$.*

Note that the degree of a v depends also on Σ. However for the sake of brevity we will omit writing such dependency if no risk of confusion arises. While it is possible to define the degree of v even when $\Sigma\backslash\{v\}$ has infinitely many connected components (see [4, Definition 2.2]), but for our purposes this is not required. For future reference, given two points p and q, let

$$[\![p, q]\!] := \{(1 - s)p + sq : s \in [0, 1]\}.$$

In view of Proposition 2.1, a segment with endpoints in two vertices and containing no other vertices will be referred as "edge". The following classical result (see for instance [5, 6]) proves several geometrical properties about Steiner graphs:

Proposition 2.4. *Given a Steiner graph G, it holds:*

- *G is a tree,*
- *if $[\![u, v]\!]$ and $[\![v, w]\!]$ are edges, with a common vertex v, then $\widehat{uvw} \geq 2\pi/3$,*
- *the maximal degree of any vertex is 3,*
- *if v is a vertex of degree 3, denoting by $[\![u_i, v]\!]$, $i = 1, 2, 3$ the 3 different edges containing v, then the angle between any two such edges is $2\pi/3$, and these edges are coplanar.*

Similarly to [9], in view of Propositions 2.1 and 2.4, the following definition will be useful:

Definition 2.5. *Given a discrete measure μ, a parameters $\lambda > 0$, and $\Sigma \in \operatorname{argmin}_{\mathcal{A}} E^{\lambda}_{\mu}$, a vertex $v \in \Sigma$ is called:*

- *"endpoint" if it has degree 1,*
- *"triple junction" if it has degree 3.*

If v is a vertex of degree 2, denoting by w, z the two vertices such that $[\![w, v]\!]$ and $[\![v, z]\!]$ are edges, the "turning angle" *in v is defined as:*

$$TA(v) := \pi - \widehat{wvz}.$$

Similarly, given a subset $A \subseteq \Sigma$, the turning angle of A is defined as

$$TA(A) := \sum_{u \in A,\ u \text{ vertex of degree 2}} TA(u).$$

For brevity, if $A = \{v\}$ is a singleton, we will write $\mathrm{TA}(v)$ instead of $\mathrm{TA}(\{v\})$. Note that the turning angle $\mathrm{TA}(v)$ describes the curvature of Σ at v. Lemma 3.7 proves a connection between vertices of degree 2 and corners. Given a discrete measure μ and $\Sigma \in \mathcal{A}$, for the sake of brevity, the following expressions will be used ($v \in \Sigma$ is a vertex, while x is a generic point):

- *"v is tied to x"*: the vertex v coincides with some point $x \in \operatorname{supp}(\mu)$,
- *"v is free"*: the vertex v coincides with no point $x \in \operatorname{supp}(\mu)$,
- *"v talks to x"* or *x projects on v*: both mean $d(x, \Sigma) = |x - v|$, with the former (resp. latter) used when v (resp. x) is the main object of analysis in the context,
- *"v talks to some mass"*: v talks to some point in $\operatorname{supp}(\mu)$,
- $TM(\mu, v, \Sigma)$ ($TM(v)$ when there is no risk of confusion) denotes the total mass of projecting on v — for a detailed discussion see [8, Lemma 2.1],
- *"H mass projects on v"*, where $H \geq 0$: this means $TM(\mu, v, \Sigma) = H$.

The last 4 expressions will be used even for non discrete measures μ. The following assertions are the main tools used to analyze minimizers, when the reference measure is discrete.

Proposition 2.6. *Given a discrete measure μ, a parameter $\lambda > 0$, and $\Sigma \in \operatorname{argmin} E^{\lambda}_{\mu}$, it holds:*

(1) if $v \in \Sigma$ is a triple junction, then $TM(\mu, v, \Sigma) = 0$,
(2) if different vertices v, v' talk to some point $y \in \operatorname{supp}(\mu)$, then there exist $x, x' \in \operatorname{supp}(\mu)$ such that v is tied to x and v' is tied to x',
(3) if $v \in \Sigma$ is an endpoint then $TM(\mu, v, \Sigma) \geq \lambda$,
(4) if $v \in \Sigma$ is a vertex of degree 2, denoting by w, z the two vertices such that $[\![w, v]\!]$ and $[\![v, z]\!]$ are edges, then

$$TA(v) \leq \frac{\pi}{2\lambda} TM(\mu, v, \Sigma). \tag{2}$$

For the proof we refer to Lemma 9, Corollary 10 and Lemma 11 of [9]. Note that given a subset $A \subseteq \Sigma$, inequality (2) holds for any vertex of $v \in A$ of degree 2, hence

$$\mathrm{TA}(A) \leq \frac{\pi}{2\lambda} \sum_{v \in A,\ v \text{ vertex of degree 2}} TM(\mu, v, \Sigma). \tag{3}$$

If Σ is itself a curve, then

$$\mathrm{TA}(\Sigma) \leq \frac{\pi}{2\lambda} \sum_{v \text{ vertex of degree 2}} TM(\mu, v, \Sigma);$$

using Proposition 2.6, zero mass projects on triple junctions, and zero mass projects on the interior of the edges, thus all the mass projects on endpoints or vertices of degree 2. Denoting by P_0 and P_1 the two endpoints of Σ (the case Σ being a singleton is trivial), it holds

$$\begin{aligned}\mathrm{TA}(\Sigma) &\leq \frac{\pi}{2\lambda} \sum_{v \text{ vertex of degree 2}} TM(\mu, v, \Sigma) \\ &\leq \frac{\pi}{2\lambda}(1 - TM(\mu, P_0, \Sigma) - TM(\mu, P_1, \Sigma)) \leq \frac{\pi}{2\lambda}(1 - 2\lambda),\end{aligned}$$

where the last inequality follows from Proposition 2.6.

A similar result has been proven (in [8], to which to refer for the proof) for generic measures:

Lemma 2.7. *Given a measure μ, a parameter $\lambda > 0$ and $\Sigma \in \operatorname{argmin} E_\mu^\lambda$, for any subset $A \subseteq \Sigma$ (A can be a singleton) it holds*

$$\sum_j \|\alpha_j'\|_{TV} \leq \frac{\pi}{2\lambda} TM(A),$$

with $TM(A)$ denoting the total mass projecting on points of A, and $\alpha_j : [0,1] \longrightarrow A$ denoting the arc-length parameterizations of branches making up A.

Finally we recall a classical convergence result:

Lemma 2.8. *Given a sequence of curves $\{\gamma_k\} : [0,1] \longrightarrow K$, with $K \subseteq \mathbb{R}^2$ a given compact set, satisfying*

$$\sup_k \|\gamma_k'\|_{BV} < +\infty, \qquad \sup_k \mathcal{H}^1(\gamma_k([0,1])) < +\infty,$$

then there exists a curve $\gamma : [0,1] \longrightarrow K$, such that (upon subsequence) it holds:

(1) $\{\gamma_k\} \to \gamma$ in C^α for any $\alpha \in [0,1)$,
(2) $\{\gamma_k''\} \overset{*}{\rightharpoonup} \gamma''$ in the space of signed Borel measures.

For the proof we refer to [9]. For the sake of brevity, we will never relabel subsequences if no risk of confusion arises.

3. Counterexample

The aim of this section is to construct an explicit example of minimizer whose set of corners is not closed. Our construction will require a lot of technicalities, and careful choice of constants. Many "strange looking" constants will appear through the section, and their choice is often very arbitrary, but acceptable for our purposes.

The reference measure will be:

$$\mu := \mu_{\text{heavy}} + \mu_{\text{light}} \tag{4}$$

where

$$\mu_{\text{heavy}} := \frac{1-\eta}{2\pi\rho^2}\Big(\mathcal{L}^2_{\llcorner B((-L,h),\rho)} + \mathcal{L}^2_{\llcorner B((L,h),\rho)}\Big), \tag{5}$$

$$\mu_{\text{light}} := \sum_{n=1}^{+\infty} \frac{m_n}{\pi\varrho_n^2}\mathcal{L}^2_{\llcorner B_n}, \tag{6}$$

and $B_n := B\big((c_n, 0), \varrho_n\big)$. Parameter ρ will be determined in subsection 3.1, while h, L, η, m_n, c_n, ϱ_n are chosen such that:

(C1) $h := 1$, $\eta := \sum_{n=1}^{\infty} m_n$, $\lambda \in \left(\frac{1-3\eta}{2}, \frac{1-2\eta}{2}\right)$, sufficiently small c_1, sufficiently large $L > 10^9$, ϱ_n, m_n, c_n defined inductively such that

$$\varrho_n \le 100^{-n} m_n,\ m_n \le 100^{-n} c_n,\ \varrho_{n+1} \le 100^{-n}\varrho_n,\ m_{n+1} \le 100^{-n} m_n,\ c_{n+1} \le 100^{-n} c_n,$$

and

(a) $\pi(1-2\lambda)/(2\lambda) \le 0.001$, hence $\pi\eta/\lambda \le 0.001$,
(b) for any n_0 it holds $\frac{1}{5h}\cdot\frac{c_{n_0}}{2} > \frac{\pi}{4\lambda}\sum_{n\ge n_0} m_n$, and $\frac{5h\pi}{4\lambda} m_{n_0} + \varrho_{n_0}(1+\tan 0.01) \le \frac{c_{n_0}}{10}$,
(c) for any $n_1 < n_2$ it holds $\frac{5h\pi}{2\lambda}\sum_{n=n_1}^{n_2} m_n < 0.01|c_{n_1} - c_{n_2}|$,
(d) $\pi - 2\arctan\dfrac{L+h/3}{2h/3 - 1/10} > \dfrac{\pi}{2\lambda}(1-2\lambda)$,
(e) $\inf\{n \in \mathbb{N} : 0.09/4m_s > 2\pi\varrho_s \text{ for any } s \ge n\} = 1$,
(f) $\dfrac{h}{10} + \dfrac{2h(L-1)}{L} > 2h$, $2h - \dfrac{2h(L-1)}{L} < \dfrac{h}{10}$.

Clearly ϱ_n, m_n, c_n can be easily chosen satisfying (a), (b), (c) and (e). Choosing sufficiently large L ensures (f). Finally note that the left-hand side term in (d) roughly corresponds to $2/L$ for very large L, while the right-hand side term $\pi(1-2\lambda)/(2\lambda)$ depends only on $\lambda \in \left(\frac{1-3\eta}{2}, \frac{1-2\eta}{2}\right)$, hence (d) can be ensured by further reducing the values of m_n (and consequently η, and eventually ϱ_n, c_n). These conditions, while quite "strange looking", will be used in many proofs:

- condition (a) will be used to ensure the "smallness" of several angles, so several technical results (such as Lemmas 3.16 and 3.8) are applicable (see for instance the last inequality in (16)),
- condition (b) will be used in Lemmas 3.13 (to deduce the contradiction after (11)) and 3.18 (last inequality in (29)),
- condition (c) will be used in Lemmas 3.14 to deduce the contradiction after (14), and in Lemma 3.19 (inequality (32)),
- condition (d) will be used in Lemma 3.3 to deduce the contradiction after (14),
- condition (e) will be used in Lemma 3.17, immediately after (18)
- condition (f) will be used in Lemma 3.10 (inequalities (25) and (26)).

The choice of using h (instead of simply "1") is done to make clearer where such quantity appears (mostly as length). The values 0.001, 100^{-n}, 10^9 are very arbitrary, but sufficient for the purposes of this paper. Moreover, with our construction we definitely need to choose some values such parameters (explicit values simplify proofs, and there is no point in determining the "optimal" values), to satisfy the (technical) results of this section. Moreover, we will often use non sharp (but formally simpler) estimates in the proofs when possible. Note that due to our choice of ϱ_n, m_n, c_n, for μ_{light} it holds:

- the distance between two distinct balls B_{n_1} and B_{n_2} (assume $n_2 > n_1$) is "much larger" (note the factor 100^{-n} in (C1)) than $\sum_{n=n_1}^{n_2} m_n$ (which is roughly "the combined masses of the balls in between");
- for each ball B_n, the mass supported on it (i.e. m_n) is much larger (by a factor least 100^n) than its own radius (i.e. ϱ_n). Hence the "density" of B_n is high.

This will be crucial for our construction, and it will result from the construction that corners arise exactly due to the presence of such "density peaks".

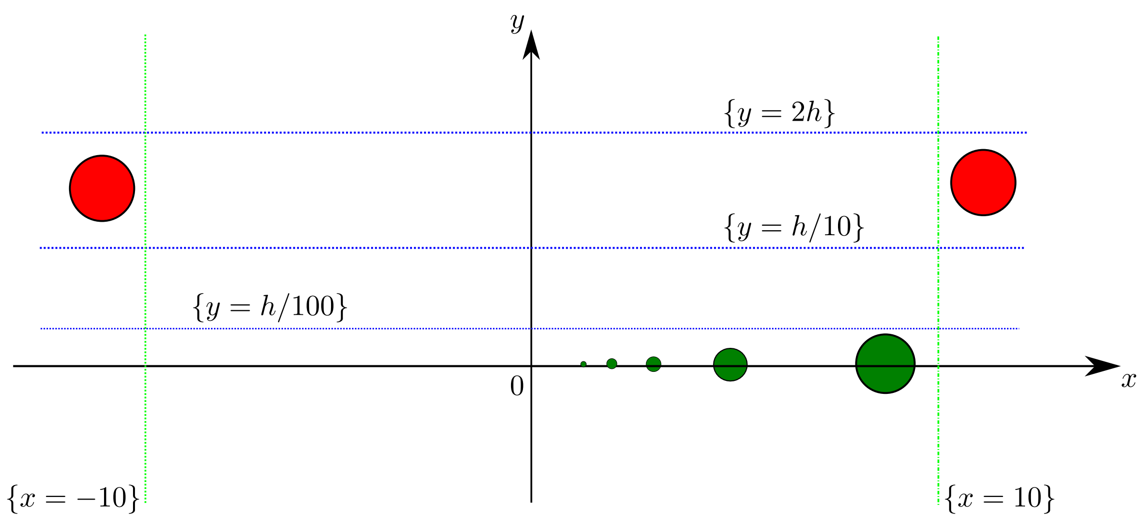

FIGURE 2. This is a representation (highly not to scale) of the supports of μ_{heavy} (red) and μ_{light} (green). The represented lines will be relevant for our construction.

Note that μ depends on several parameters appearing in (5) and (6). For the sake of brevity (unless otherwise specified) we omit writing such dependencies.

Intuitively:

- $\text{supp}(\mu_{\text{heavy}})$ is union of two "small and distant" balls, each of which contains "slightly less than one half" of the total mass;
- $\text{supp}(\mu_{\text{light}})$ is union of balls B_n, $n \geq 1$, each of which containing mass m_n.

As will be clear in the following, μ_{light} is the measure that "generates corners", while the role of μ_{heavy} is to force minimizers to have "large length" and "little curvature".

3.1. **Choosing parameters.** The aim of this subsection is to choose a suitable parameter ρ. The proofs of Lemmas 3.1, 3.2 and 3.3 are available in Section 4. This is done for reader's convenience, since those are mostly technical lemmas, whose proofs do not contain ideas significant to our main purposes.

Lemma 3.1. *There exists $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$, any minimizer of E_μ^λ is a simple curve with positive length.*

Lemma 3.2. *For any $\varepsilon > 0$ there exists $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$, any minimizer of E_μ^λ contains points p, q such that*

$$\max\{|p - (-L, h)|, |q - (L, h)|\} < \varepsilon.$$

Lemma 3.3. *There exists $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$, any minimizer of E_μ^λ is contained in the half-plane $\{y > h/10\}$.*

The choice $\{y > h/10\}$ is quite arbitrary, but since by construction (and (C1)) it holds $\mathrm{supp}(\mu_{\mathrm{light}}) \subseteq \{y < h/100\}$, a crucial consequence is:

(C2) for any minimizer $\Sigma \in \mathrm{argmin}\ E_\mu^\lambda$ it holds

$$\inf_{z \in \mathrm{supp}(\mu_{\mathrm{light}})} d(z, \Sigma) \overset{(C1)}{\geq} \frac{1}{10} - \frac{1}{100} = 0.09 > 0.$$

The same argument (the proof is available in Section 4) also proves that for such λ and ρ, any minimizer is contained in the half-plane $\{y < 2h\}$. Choose $\rho \leq 1$ such that the conclusions of Lemmas 3.1, 3.2 and 3.3 hold. Moreover we impose

(C3) for any minimizer $\Sigma \in \mathrm{argmin}\ E_\mu^\lambda$, there exist points $p, q \in \Sigma$ such that $|p-(-L, h)| \leq 1/4$, $|q - (L, h)| \leq 1/4$.

This is possible in view of Lemma 3.2. Note that after choosing ρ, the reference measure μ is uniquely determined. Thus we have proven:

Lemma 3.4. *There exist ρ such that any minimizer of E_μ^λ is contained in the strip $\{h/10 < y < 2h\}$ and (C2) holds.*

Note that (in view of Lemma 3.2 and for suitable choice of ρ) since L has been chosen sufficiently large, there exists a vertical strip Ξ such that points in $B((-L, h), \rho)$ cannot project on any point $z \in \Sigma \cap \Xi$: indeed letting $\Xi := \{-10 \leq x \leq 10\}$ (here the values -10 and 10 are quite arbitrary, but acceptable for the purposes of this paper), it holds

$$(\forall x \in B((-L, h), \rho))(\forall z \in \Xi) \quad |x - z| \geq L - h - 10 \overset{(C1)}{>} 10^9 - 11 > 5/4 \geq \rho + h/4 \geq |x - p|$$

for some p given by Lemma 3.2. The same argument proves that points in $B((L, h), \rho)$ cannot project to any point in Ξ.

Until now we have proven (for our choice of parameters):

- for any minimizer Σ, any point in $B((-L, h), \rho) \cup B((L, h), \rho)$ cannot project on $\Sigma \cap \Xi$,
- any minimizer contains points p, q satisfying

$$|p - (-L, h)| \leq h/4, \qquad |q - (L, h)| \leq h/4,$$

- any minimizer is contained in the strip $\{h/10 < y < 2h\}$.

Combining these facts with (C3), only points in $\mathrm{supp}(\mu_{\mathrm{light}})$ can project on $\Sigma \cap \Xi$. Recall that by construction the total mass in $\mathrm{supp}(\mu_{\mathrm{light}})$ is η.

Definition 3.5. *Let v_1, v_2 be non zero vectors of $\mathbb{R}^2$. The "angle between" v_1 and v_2, which we will denote by $\angle(v_1, v_2)$, is defined as*

$$\angle(v_1, v_2) := \arccos \frac{\langle v_1, v_2 \rangle}{|v_1||v_2|} \in [0, \pi],$$

where $\langle , \rangle$ denotes the standard Euclidean scalar product of $\mathbb{R}^2$. Given segments/half-lines/lines l_1 and l_2, the "angle between l_1 and l_2" (which we denote by $\angle(l_1, l_2)$) is defined as

$$\angle(l_1, l_2) := \min_{v_1 \| l_1,\ v_2 \| l_2} \angle(v_1, v_2).$$

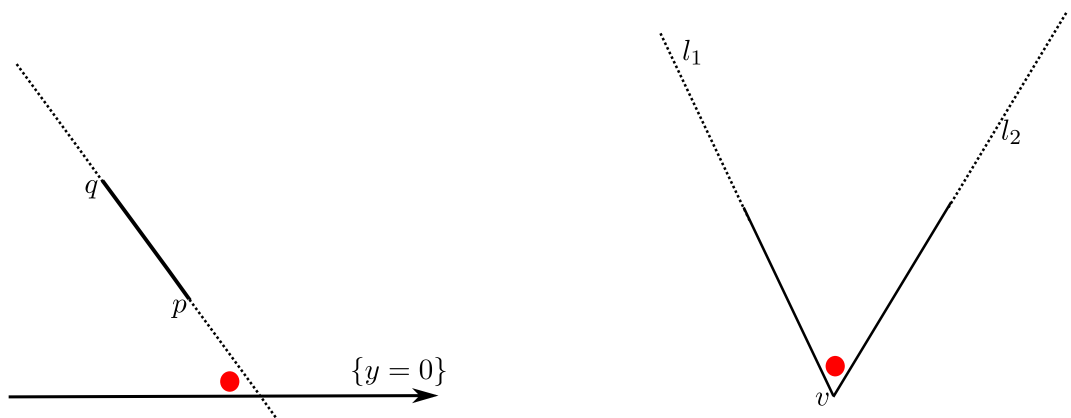

FIGURE 3. Angle between a segment $[\![p, q]\!]$ and $\{y = 0\}$ (left, the dotted lines represent the line containing $[\![p, q]\!]$); and between two half-lines l_1 and l_2 with common vertex v (right).

We will often use the angle between a segment/half-line/line and $\{y = 0\}/\{x = 0\}$, and expressions such as

$$\angle(l, \{y = 0\}), \quad \angle(l, \{x = 0\}), \quad \angle([\![p_1, p_2]\!], \{y = 0\}), \cdots .$$

The parameter ρ will have little importance in the following, as its "role" is to ensure that minimizers contain points "close to" $(\pm L, h)$ (i.e. p and q from Lemma 3.2). In the following, it will be clear that corners will arise due to measure μ_{light}. Since $\text{supp}(\mu_{\text{light}})$ (along with all points talking to points in $\text{supp}(\mu_{\text{light}})$) is contained in the strip Ξ, we will tacitly assume (unless explicitly stated) that we are working only in Ξ, and all statements will tacitly assume that quantities involved are contained in Ξ.

3.2. **Discrete measures.** The first step involves approximating μ with discrete measures. Similarly to [9], given three points v_1, v_2, v_3, define the *"wedge"* $V(v_2)$ as follows:

(1) if v_1, v_2, v_3 are collinear, then $V(v_2)$ is the unique line passing through v_2 and orthogonal to $v_3 - v_2$,

(2) otherwise, let $\theta_i := \frac{v_{i+1}-v_i}{|v_{i+1}-v_i|}$ $(i = 1, 2)$, $\xi := \frac{\theta_2+\theta_1}{|\theta_2+\theta_1|}$, $b := \frac{\theta_2-\theta_1}{|\theta_2-\theta_1|}$, $\beta := \mathrm{TA}(v_2)/2$, and

$$V(v_2) := v_2 + \{x \in \mathbb{R}^2 : |\langle \xi, x\rangle| \le \langle b, x\rangle \tan\beta\},$$

where $\langle , \rangle$ denotes the standard Euclidean scalar product of $\mathbb{R}^2$.

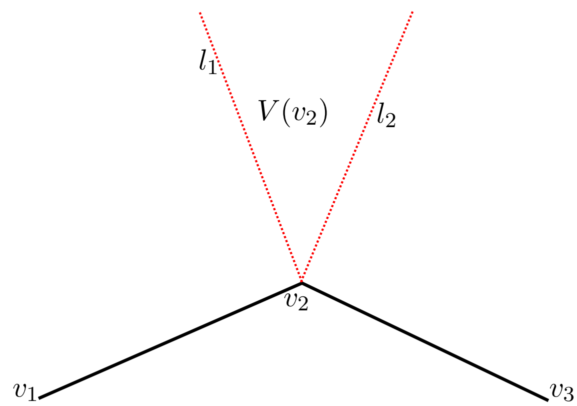

FIGURE 4. An example of wedge $V(v_2)$. Here $l_1 \perp [\![v_1, v_2]\!]$, $l_2 \perp [\![v_2, v_3]\!]$.

Note that if $\mathrm{TA}(v) > 0$, by definition the wedge $V(v)$ is itself an angle (intended as part of the plane contained between two half-lines starting at the same point). Thus expressions like *"bisector of* $V(v)$*"*, *"amplitude of* $V(v)$*"*, etc. will be used. Note also that its border $\partial V(v)$ is the union of two half-lines; although $\partial V(v)$ will play an important role in many proofs, it is almost never important to "distinguish" the half-lines forming it, thus in the following we will often use expressions like *"*$\partial V(v)$ *is the union of two half-lines* $l_\pm$*"*, without stating precisely which half-line corresponds to l_- (nor which half-line corresponds to l_+).

Let

$$\mu_j := \sum_i \frac{1-\eta}{2\cdot \sharp\Big(B((-L,h),\rho)\cap \frac{\varrho_n}{j}\mathbb{Z}^2\Big)}\delta_{q_i^j} + \sum_i \frac{1-\eta}{2\cdot \sharp\Big(B((L,h),\rho)\cap \frac{\varrho_n}{j}\mathbb{Z}^2\Big)}\delta_{\tilde{q}_i^j} + \sum_{n=1}^{\infty}\sum_i \frac{m_n}{\sharp(B_n \cap \frac{\varrho_n}{j}\mathbb{Z}^2)}\delta_{p_{i,n}^j}, \tag{7}$$

where $\{p_{i,n}^j\}$ (resp. $\{q_i^j\}$, $\{\tilde{q}_i^j\}$) are the (finitely many) points of the lattice $B_n \cap \frac{\varrho_n}{j}\mathbb{Z}$ (resp. $B((-L,h),\rho)\cap\frac{1}{j}\mathbb{Z}^2$, $B((L,h),\rho)\cap\frac{1}{j}\mathbb{Z}^2$). Intuitively, the mass supported in B_n (resp. $B((-L,h),\rho)$, $B((L,h),\rho)$) is being uniformly distributed on the (uniform) lattice $B_n\cap\frac{\varrho_n}{j}\mathbb{Z}^2$ (resp. $B((-L,h),\rho)\cap\frac{1}{j}\mathbb{Z}^2$, $B((L,h),\rho)\cap\frac{1}{j}\mathbb{Z}^2$).

The particular form of discretization in (7) has no relevant role, but we just need one to work with. It suffices that $\mathrm{supp}(\mu_j) \subseteq \mathrm{supp}(\mu)$. For future reference, any measure μ_j will refer to the (family of) measures defined in (7). Recall that μ was fixed towards the end of subsection 3.1. We will first work with discrete measures μ_j, then take the limit $j \to +\infty$. The key points of our proof are the following:

(1) vertices of degree 2 are corners (Lemma 3.7),
(2) for any j, any minimizer $\Sigma \in \mathrm{argmin}\ E^{\lambda}_{\mu_j}$ is a graph in Ξ, with an upper bound on its curvature (Lemmas 3.9 and 3.10)
(3) for any j, any minimizer $\Sigma \in \mathrm{argmin}\ E^{\lambda}_{\mu_j}$ contains infinitely many corners (Lemmas 3.13 and 3.14); moreover we will give a *lower* bound estimate on the turning angle of such corners (Lemma 3.17, the most crucial result of subsection 3.2),
(4) these corners are "distant" (Lemma 3.19).

All results are valid with both discrete (i.e. μ_j) and non-discrete reference measures (i.e. μ). However the proofs of Lemmas 3.7 and 3.17 require to work with discrete measures, hence for simplicity we will always work with discrete measures.

The first result is an analogous of Lemma 3.4 for minimizers of $E^{\lambda}_{\mu_j}$:

Lemma 3.6. *For any index j, any minimizer of $E^{\lambda}_{\mu_j}$ is contained in the strip $\{h/10 < y < 2h\}$.*

Proof. The same arguments used in the proof of Lemma 3.4 can be applied without any modification to minimizers of $E^{\lambda}_{\mu_j}$. □

The next result proves that vertices of degree 2 have positive turning angle.

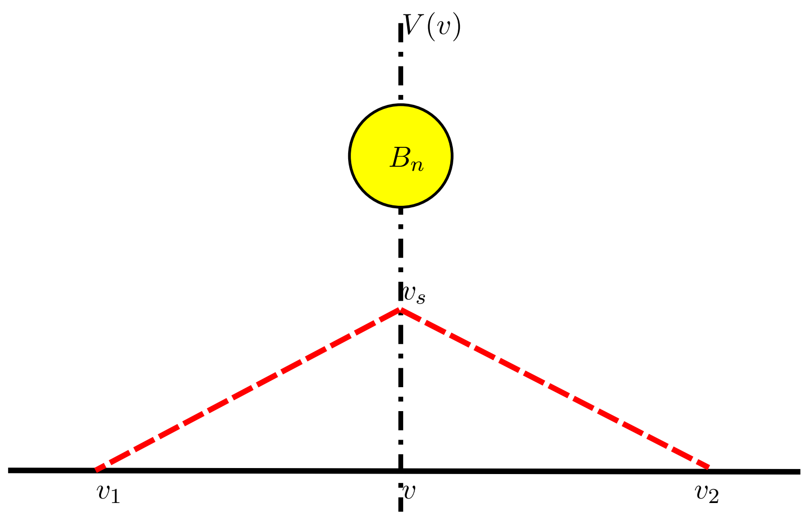

FIGURE 5. This is a schematic representation of the variation used in the proof of Lemma 3.7. Here $(v_1 - v_2) \perp (v_s - v)$, and $|v_s - v| = s$.

Lemma 3.7. *For any index j, minimizer $\Sigma \in \operatorname{argmin} E^{\lambda}_{\mu_j}$, if a point $v \in \Sigma$ satisfies $TM(\mu_j, v, \Sigma) > 0$, then $TA(v) > 0$. In particular v is a corner.*

Proof. Note that Lemma 3.6 implies $\Sigma \subseteq \{y \geq h/10\}$, while $\operatorname{supp}(\mu_{\text{light}}) \subseteq \{y \leq h/100\}$.

Assume $\text{TA}(v) = 0$, hence $V(v)$ is a line. Hypothesis $TM(\mu_j, v, \Sigma) > 0$ implies the existence of an index n such that v receives mass from B_n. Let

$$\Sigma_s := (\Sigma \backslash [\![v_1, v_2]\!]) \cup ([\![v_1, v_s]\!] \cup [\![v_2, v_s]\!]).$$

Then the same argument from [7, Lemma 3.4] holds: direct computation gives that for $s \to 0$ it holds

$$F_{\mu_j}(\Sigma) - F_{\mu_j}(\Sigma_s) \geq sTM(\mu_j, v, \Sigma) = O(s), \quad \mathcal{H}^1(\Sigma_s) - \mathcal{H}^1(\Sigma) = O(s^2),$$

thus the minimality of Σ is contradicted. □

The proofs of the next five lemmas (Lemmas 3.8, 3.9, 3.10, 3.11 and 3.12) are quite technical, and do not contain ideas relevant to our main construction. Thus for reader's convenience, these proofs are presented in Section 4.

Lemma 3.8. *Let p, p', p'' be a triple of points satisfying:*

(1) $p', p'' \in \{y = 0\}$, $p \in \{0 < y \leq 2h\}$,

(2) $\widehat{p'pp''} = 2\theta$, with $\theta \leq 0.01$,

(3) $\angle(\beta, \{x = 0\}) \leq \tau \leq 0.01$, where β denotes the bisector of $\widehat{p'pp''}$).

Then it holds $|p' - p''| \leq 5h\theta$.

Here the value 0.01 is arbitrary, and chosen to ensure that θ and τ are "small". This is sufficient, since we will mostly work with angles whose value does not exceed 0.01.

Lemma 3.9. *For any index j, minimizer $\Sigma \in \operatorname{argmin} E^{\lambda}_{\mu_j}$, $l \in [-L/2, L/2]$, the intersection $\Sigma \cap \{x = l\}$ is a singleton.*

Lemma 3.10. *For any index j, minimizer $\Sigma \in \operatorname{argmin} E^{\lambda}_{\mu_j}$, there exists no couple of points v_1, $v_2 \in \Sigma \cap \{-L/2 \leq x \leq L/2\}$ such that $\angle([\![v_1, v_2]\!], \{y = 0\})$ is greater than $4h/L + \pi\eta/\lambda$. Moreover for all $s \in [t_0, t_1]$ such that $f'(s)$ is well defined it holds $|\arctan f'(s)| \leq 4h/L + \pi\eta/\lambda$, where $f : [0, 1] \longrightarrow \Sigma$ is an arbitrary bijective, constant speed parameterization, and t_0, t_1 are the unique times such that $f(t_0) \in \{x = -L/2\}$, $f(t_1) \in \{x = L/2\}$.*

The bound $4h/L + \pi\eta/\lambda$ is "small", since $4h/L + \pi\eta/\lambda \overset{(C1)}{<} 4/10^9 + 0.001$. The quantity $4h/L + \pi\eta/\lambda$ will often appear as angle, and for future reference let

$$\vartheta^* := 4h/L + \pi\eta/\lambda. \tag{8}$$

Moreover, given a point p, the notation p_x (resp. p_y), or $(p)_x$ (resp. $(p)_y$) if p has subscripted indices, will denote the x (resp. y) coordinate of p.

Lemma 3.11. *For any index j, minimizer $\Sigma \in \operatorname{argmin} E^{\lambda}_{\mu_j}$ and corner $v \in \Sigma$, it holds $V(v) \cap \Sigma = \{v\}$.*

Lemma 3.12. *Let j be a given index, and $\Sigma \in \operatorname{argmin}\ E^\lambda_{\mu_j}$ a given minimizer. Let $v_1, v_2 \in \Sigma$ be corners such that v_i talks to points in B_{n_i} $(i = 1, 2)$ with $n_1 > n_2$. Then $(v_1)_x < (v_2)_x$.*

Lemma 3.13. *For any index j and minimizer $\Sigma \in \operatorname{argmin}\ E^\lambda_{\mu_j}$, no corner $v \in \Sigma$ satisfies $V(v) \ni (0, 0)$.*

Proof. Assume the opposite, i.e. there exist j, $\Sigma \in \operatorname{argmin}\ E^\lambda_{\mu_j}$ and a corner $v \in \Sigma$ satisfying $(0, 0) \in V(v)$. Proposition 2.6 forces $TM(\mu_j, v, \Sigma) > 0$, hence $V(v) \cap B_n \neq \emptyset$ for some n, and let

$$n_0 := \inf\{n : V(v) \cap B_n \neq \emptyset\} < +\infty.$$

Lemma 3.4 gives $v_y \leq 2h$, and let $\tau := \angle(\beta, \{y = 0\})$, with β denoting the bisector of $V(v)$. Note that $\tau \in [\frac{\pi}{2} - \vartheta^*, \frac{\pi}{2} + \vartheta^*]$ in view of Lemma 3.10. Since $V(v) \cap B_n \neq \emptyset$, $V(v) \cap \{y = 0\}$ contains a point with x coordinate at least $c_{n_0}/2$ (the factor 1/2 is not optimal, but acceptable for the purposes of the proof). Let $\theta := \mathrm{TA}(v)/2$, Lemma 3.8 gives $\frac{c_{n_0}}{2} \leq 5h\theta$, i.e.

$$\theta = \frac{\mathrm{TA}(v)}{2} \geq \frac{1}{5h} \cdot \frac{c_{n_0}}{2}. \tag{9}$$

By construction, v can talk only to points in the union $\bigcup_{n \geq n_0} B_n$, which satisfies

$$\mu_j\Big(\bigcup_{n \geq n_0} B_n\Big) \leq \sum_{n \geq n_0} m_n. \tag{10}$$

Combining estimate (9), (10) with Proposition 2.6 gives

$$\frac{1}{5h} \cdot \frac{c_{n_0}}{2} \leq \theta \leq \frac{\pi}{4\lambda} \sum_{n \geq n_0} m_n, \tag{11}$$

which is a contradiction (independently of n_0) in view of (C1). ☐

The next result proves that no corner receives mass from distinct balls B_{n_1}, B_{n_2}, $n_1 \neq n_2$.

Lemma 3.14. *For index j and minimizer $\Sigma \in \operatorname{argmin}\ E^\lambda_{\mu_j}$, there exists no corner $v \in \Sigma$ and indices $n_1 < n_2$ such that both $V(v) \cap B_{n_1}$ and $V(v) \cap B_{n_2}$ are non empty.*

Proof. Assume the opposite, i.e. there exists a corner $v \in \Sigma$ and indices $n_1 < n_2$ such that

$$V(v) \cap B_{n_1} \neq \emptyset, \quad V(v) \cap B_{n_2} \neq \emptyset.$$

Let

$$n_- := \inf\{n : V(v) \cap B_n \neq \emptyset\}, \qquad n_+ := \sup\{n : V(v) \cap B_n \neq \emptyset\}.$$

The contradiction assumption ensures $n_- < n_+$. Note that this gives $\mathcal{L}^1(V(v) \cap \{y = 0\}) \geq (c_{n_-} - c_{n_+})/2$ (again factor 1/2 is not optimal but acceptable for the purposes of the proof) since:

- $V(v)$ intersects both B_{n_-} and B_{n_+},
- $\angle(\beta, \{x = 0\}) \leq \vartheta^*$ (with β defined as the bisector of $V(v)$) in view of Lemma 3.10 (hence β is "almost orthogonal" to $\{y = 0\}$).

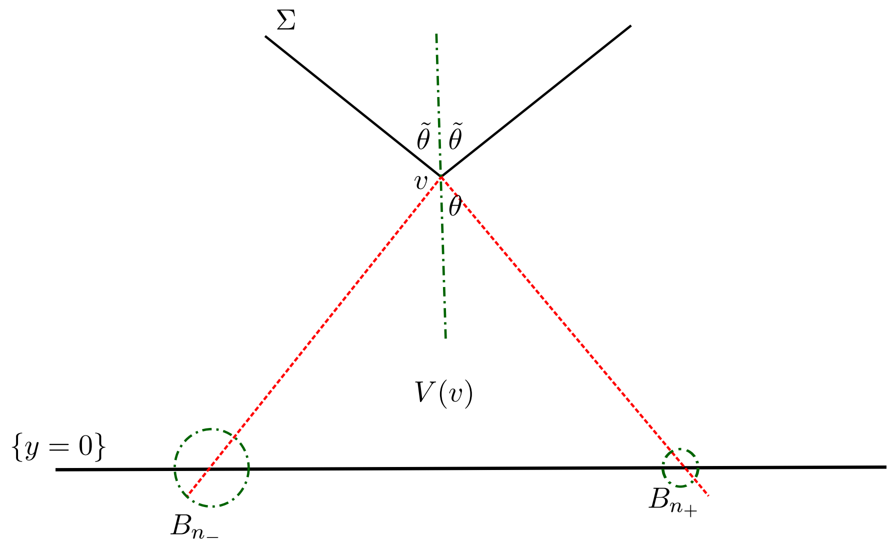

FIGURE 6. If v talks to points in two distinct balls, $\mathcal{L}^1(V(v)\cap\{y=0\})$ is "large", thus the turning angle $\mathrm{TA}(v)$ is "large". But there is not enough mass to allow for such large turning angle. Here we omitted representing the balls (if these exist) B_n with $n_- < n < n_+$. The relation between θ and $\tilde\theta$ is $2\theta = \pi - 2\tilde\theta = \pi - \mathrm{TA}(v)$.

Lemma 3.8 gives

$$4h \cdot \frac{\mathrm{TA}(v)}{2} \geq \frac{4}{5}\mathcal{L}^1(V(v)\cap\{y=0\}) \geq \frac{2}{5}(c_{n_-} - c_{n_+}). \tag{12}$$

However, since by construction v can talk only to points in $\bigcup_{n=n_-}^{n_+} B_n$, Proposition 2.6 gives

$$\mathrm{TA}(v) \leq \frac{\pi}{2\lambda}\mu_j\Big(\bigcup_{n=n_-}^{n_+} B_n\Big) = \frac{\pi}{2\lambda}\sum_{n=n_-}^{n_+} m_n, \tag{13}$$

thus

$$\frac{\pi}{\lambda}\sum_{n=n_-}^{n_+} m_n = 2h\cdot\frac{\pi}{2\lambda}\sum_{n=n_-}^{n_+} m_n \overset{(13)}{\geq} 2h\mathrm{TA}(v) \overset{(12)}{\geq} \frac{2}{5}(c_{n_-} - c_{n_+}). \tag{14}$$

This is a contradiction (in view of (C1)) for any n_- and n_+. □

Combining Lemmas 3.13 and 3.14, we obtain:

- for any index j, any minimizer $\Sigma \in \operatorname{argmin}\, E^{\lambda}_{\mu_j}$ contains infinitely many corners.

Consider an index j and a minimizer $\Sigma \in \operatorname{argmin}\, E^{\lambda}_{\mu_j}$: let C_n be the set of corners (of Σ) talking to points in B_n. Combining Lemmas 3.12 and 3.14 gives:

- for any indices n_-, n_+ with $n_- \leq n_+$, any point in $\bigcup_{n=n_-}^{n_+} C_n$ can talk only to points in $\bigcup_{n=n_-}^{n_+} B_n$.

The next result proves that given two corners $v_1 \neq v_2$, then their wedges are disjoint.

Lemma 3.15. *For any index j and minimizer $\Sigma \in \text{argmin}\ E^{\lambda}_{\mu_j}$, and distinct corners $v_1, v_2 \in \Sigma$, the intersection $V(v_1) \cap V(v_2)$ is empty.*

Proof. The proof follows by applying the exact same arguments from [7, Lemma 3.7] to corners of Σ. □

The next result estimates the optimal turning angle in relation to the mass projecting on a corner. In particular it gives a crucial lower bound estimate.

Lemma 3.16. *For any index j, minimizer $\Sigma \in \text{argmin}\ E^{\lambda}_{\mu_j}$ and corner $v \in \Sigma$, let $TM(v) := TM(\mu_j, v, \Sigma)$. Then it holds*

$$2\lambda \sin \frac{TA(v)}{2} \leq TM(v) \leq 2\lambda \tan \frac{TA(v)}{2}.$$

Moreover, if $TA(v) \leq 0.01$ then

$$\frac{TA(v)}{TM(v)/\lambda} \geq \frac{1}{2}.$$

Note that we will mostly consider corners v talking to points in $\text{supp}(\mu_{\text{light}})$, and Proposition 2.6 gives that the amplitude of the wedge $V(v)$ does not exceed $\frac{\pi\eta}{2\lambda} \leq 0.01$.

Proof. The proof uses the exact same arguments from [7, Lemma 3.5]. □

The next result proves that there exist infinitely many indices n for which there exists a corner v_n receiving a positive fraction of the mass supported in B_n.

Lemma 3.17. *For any index j and $\Sigma \in \text{argmin}\ E^{\lambda}_{\mu_j}$ it holds:*

- *for any n there exists a corner $v_n \in \Sigma$ satisfying $TM(\mu_j, v_n, \Sigma) \geq m_n/4$. Moreover, $TA(v_n) \geq m_n/4$.*

Proof. Let Σ be an arbitrary minimizer of $E^{\lambda}_{\mu_j}$. The proof follows by applying the exact same construction (and related arguments) from [7, Lemma 3.8] to corners of Σ_j. Here we present a brief sketch. Let $\{v_i\}$ be the corners talking to points in B_n, f a parameterization of Σ, $t_i := f^{-1}(v_i)$ and $M_i := TM(\mu_j, v_i, \Sigma)$. Assume

$$(\forall i_1, i_2 \in \{1, \cdots, H\},\ i_1 \neq i_2) \quad M_{i_1} + M_{i_2} \leq m_n/2. \tag{15}$$

The goal is to prove that assumption (15) cannot hold for sufficiently large n.

- Claim: for any index i, except at most two, both $l_i^{\pm}$ must intersect the border ∂B_n.

This is proven by using the exact same arguments from the proof of [7, Lemma 3.8]. Elementary geometric arguments (for further details we refer to [7, Lemma 3.8]) give

$$d(v_i, B_n) \geq h/10 - h/100 = 0.09h\ (i = 1, \cdots, H), \qquad \min_{z \in l_i^-,\ |z - v_i| \geq 0.09h} d(z, l_i^+) \geq 0.09h \sin \text{TA}(v_i),$$

hence

$$\min_{z \in l_i^-,\ |z - v_i| \geq 0.09h} d(z, l_i^+) \geq 0.09h \sin \text{TA}(v_i) \geq \frac{0.09}{2} h\text{TA}(v_i). \tag{16}$$

The last inequality holds since (C1) gives $\pi m_n/(2\lambda) \leq \pi\eta/(2\lambda) \leq 0.001/2$, and for any $x \in [0, 0.001/2]$ it holds $\sin x \geq x/2$. Since for any index i, except at most two (which we denote by i' and i''), both $l_i^{\pm}$ intersect ∂B_n, choose points

$$w_i^{\pm} \in l_i^{\pm} \cap \partial B_n \qquad i = 1, \cdots, H, i \notin \{i', i''\}.$$

Clearly $V(v_i) \cap \partial B_n$ contains an arc connecting w_i^- and w_i^+. Thus

$$\mathcal{H}^1(V(v_i) \cap \partial B_n) \geq \min_{z \in l_i^-,\ |z-v_i| \geq 0.09h} d(z, l_i^+) \overset{(16)}{\geq} \frac{0.09}{2} h \mathrm{TA}(v_i)$$
$$\overset{\text{Lemma 3.16}}{\geq} \frac{0.09h}{2} \frac{M_i}{2\lambda} \overset{\lambda<1/2,\ h=1}{\geq} \frac{0.09}{2} M_i, \tag{17}$$

Lemma 3.15 gives $V(v_{i_1}) \cap V(v_{i_2}) = \emptyset$ whenever $i_1 \neq i_2$. Summing over indices $i \in \{1, \cdots, H\} \backslash \{i', i''\}$ gives

$$\mathcal{H}^1(\partial B_n) \geq \sum_{i=1,\ i \neq i', i''}^{H} \mathcal{H}^1(V(v_i) \cap \partial B_n) \overset{(17)}{\geq} \sum_{i=1,\ i \neq i', i''}^{H} \frac{0.09}{2} h M_i \overset{(15)}{\geq} \frac{0.09 m_n}{4}. \tag{18}$$

Note that (C1) gives $\inf\{n \in \mathbb{N} : \frac{0.09}{4} m_s > 2\pi\varrho_s \text{ for any } s \geq n\} = 1$. Thus

$$(\forall n) \quad \mathcal{H}^1(\partial B_n) \geq \sum_{i=1,\ i \neq i', i''}^{H} \mathcal{H}^1(V(v_i) \cap \partial B_n) \overset{(18)}{\geq} \frac{0.09}{4} h m_n > 2\pi\varrho_n = \mathcal{H}^1(\partial B_n),$$

which is a contradiction. Thus (for any $n \geq 1$) assumption (15) cannot hold, and (for any n) there exist indices i_n^*, i_n^{**} such that $M_{i_n^*} + M_{i_n^{**}} \geq m_n/2$, i.e. $\max\{M_{i_n^*}, M_{i_n^{**}}\} \geq m_n/4$. Since $\lambda < 1/2$, using Lemma 3.16 gives

$$\max\{\mathrm{TA}(v_{i_n^*}), \mathrm{TA}(v_{i_n^{**}})\} \geq \max\{M_{i_n^*}, M_{i_n^{**}}\}/2\lambda \geq m_n/4,$$

and the proof is complete. □

3.3. **Passing to the limit.** Now we have to pass to the limit $j \to +\infty$. The crucial step is to prove that corners are "far apart". This will be achieved over two lemmas. For reader's convenience, the proofs of Lemmas 3.18 and 3.19 are presented in Section 4.

Lemma 3.18. *For any index j, minimizer $\Sigma \in \operatorname{argmin} E_{\mu_j}^{\lambda}$, index n and corner v talking to some point in B_n, it holds:*

- *$V(v) \cap \{y = 0\}$ does not contain points q with $|q_x - c_n| > c_n/10$.*

Here the constant 1/10 (appearing in $|q_x - c_n| > c_n/10$) is quite arbitrary, and its role is to ensure that $V(v)$ contains only points with x coordinate "close to" c_n.

Lemma 3.19. *For any index j, minimizer $\Sigma \in \operatorname{argmin} E_{\mu_j}^{\lambda}$, and corners v_{n_i} $(i = 1, 2)$ talking to points in B_{n_i} $(i = 1, 2)$, it holds*

$$|v_{n_1} - v_{n_2}| \geq 0.8 c_{\min\{n_1, n_2\}}.$$

The constant 0.8 (appearing in $0.8c_{\min\{n_1,n_2\}}$) is quite arbitrary, but acceptable for our purposes since it ensures that v_{n_1} and v_{n_2} are "far apart" for any j.

Now we can pass to the limit $j \to +\infty$: for any index j choose a minimizer $\Sigma_j \in \text{argmin } E^\lambda_{\mu_j}$, and let $f_j : [0,1] \longrightarrow \Sigma_j$ a constant speed bijective parameterization. Since $\{\mu_j\} \stackrel{*}{\rightharpoonup} \mu$, upon subsequence it holds (using Lemma 2.8) $\{f_j\} \to f$ uniformly, for some $\Sigma \in \text{argmin } E^\lambda_\mu$ and parameterization $f : [0,1] \longrightarrow \Sigma$. Thus

$$\{\Sigma_j\} \stackrel{d_{\mathcal{H}}}{\to} \Sigma \in \text{argmin } E^\lambda_\mu.$$

Lemma 3.17 proves the existence of n_0 (independent of j) such that for any $n \geq n_0$, each minimizer Σ_j contains a corner v^j_n satisfying $\text{TA}(v^j_n) \geq m_n/4$. In other words, the measure f''_j has an atom of measure at least $m_n/4$ at time $t^j_n := f_j^{-1}(v^j_n)$. Again passing to the limit $j \to +\infty$, it holds (upon subsequence) $\{t^j_n\} \to t_n$ and $\{f''_j\} \stackrel{*}{\rightharpoonup} f''$, thus f'' has an atom of measure at least $m_n/4$ in t_n. Note that an atom for the measure f'' corresponds to a jump for the tangent derivative f', i.e. a corner for Σ.

Lemma 3.19 ensures that $v_{n_1} \neq v_{n_2}$ whenever $n_1 \neq n_2$, hence Σ has infinitely many corners. Consider a sequence $\{v_n\} \subseteq \Sigma$ of corners, not definitely constant. Let v be an accumulation point of $\{v_n\}$. It remains to prove that (upon choosing suitable sequence $\{v_n\}$) such v is not a corner itself.

Lemma 3.20. *Consider an arbitrary sequence $\{v_s\}$ such that v_s talks to points in B_s, i.e. $|v_s - z_s| = d(z_s, \Sigma)$. Then it admits an accumulation point v which is not a corner itself.*

Proof. The same argument from Lemma 3.9 proves that the intersection $\Sigma \cap \{x = l\}$ is a singleton for any $l \in [-L/2, L/2]$. The same argument from Lemma 3.12 proves that if there exist $v_1, v_2 \in \Sigma$ and $z_1 \in B_{n_1}$, $z_2 \in B_{n_2}$ with $n_1 > n_2$ such that $|v_i - z_i| = d(z_i, \Sigma)$ $(i = 1, 2)$, then $(v_1)_x < (v_2)_x$. Thus it is possible to choose a sequence $\{v_s\} \subseteq \Sigma$ such that v_s talks to some $z_s \in B_s$ (i.e. $|v_s - z_s| = d(z_s, \Sigma)$), hence $\{(v_s)_x\}$ is *strictly* decreasing. Let v be an accumulation point of $\{v_s\}$. It remains to prove that such v is not a corner itself. It suffices to show that there exist no index n and point $z \in B_n$ such that $|v - z| = d(z, \Sigma)$: if such couple n, z exists, then choose an index $N > n$ and it holds $|v_N - z_N| = d(z_N, \Sigma)$ but $(v_N)_x > v_x$, which (by the same argument from Lemma 3.12) is a contradiction. □

Thus we have proven:

Theorem 3.21. *In $\mathbb{R}^2$, there exists a measure μ, a parameter λ, and a minimizer $\Sigma \in \text{argmin } E^\lambda_\mu$ satisfying:*

- *Σ is a simple curve,*
- *there exists a sequence $\{v_n\} \subseteq \Sigma$ such that $TA(v_n) \geq m_n/4$ for any n,*
- *there exists point $v \in \Sigma$ such that $\{v_n\} \to v \in \Sigma$, and $TA(v) = 0$.*

As consequence, Theorem 1.3 is proven. Finally, we prove Corollary 1.4.

Proof. (of Corollary 1.4) In [2] it has been proven that any minimizer $\tilde{\Sigma}$ of (1) satisfies $\mathcal{H}^1(\tilde{\Sigma}) = \mathcal{H}^1(\Sigma)$, thus if Σ is not a minimizer of (1), choosing Σ^* minimizer of (1) would give

$$F_\mu(\Sigma^*) < F_\mu(\Sigma), \qquad \mathcal{H}^1(\Sigma^*) = \mathcal{H}^1(\Sigma),$$

contradicting $\Sigma \in \text{argmin } E^\lambda_\mu$. □

4. Appendix: proofs of lemmas from Section 3

4.1. **Lemmas from subsection 3.1.**

Proof. (of **Lemma 3.1**) The proof will be split in two parts.

- Claim 1: any minimizer has at most 2 endpoints.

Proposition 2.6 proves that any minimizer contains at most $[1/\lambda]$ endpoints, and (C1) implies $[1/\lambda] < 3$, thus the claim is proven.

- Claim 2: for sufficiently small ρ, any minimizer of $E^\lambda_{\mu_\rho}$ has positive length.

Consider the measure

$$\mu_0 := \mu_{\text{light}} + \frac{1-\eta}{2}(\delta_{(-L,h)} + \delta_{(L,h)}), \tag{19}$$

and clearly $\{\mu_\rho\} \overset{*}{\rightharpoonup} \mu_0$ as $\rho \to 0$ (here we highlighted the dependency on ρ). For any arbitrary point $P := (x_0, y_0)$ it holds

$$E^\lambda_{\mu_0}(\{P\}) \geq \frac{1-\eta}{2}(|P - (-L,h)| + |P - (L,h)|) \geq (1-\eta)L.$$

Let

$$\Lambda := [\![(-L,h),(L,h)]\!], \tag{20}$$

and

$$E^\lambda_{\mu_0}(\Lambda) \leq 2\lambda L + 2h\eta \overset{(C1)}{\leq} (1-2\eta)L + 2\eta \overset{(C1)}{<} (1-\eta)L \leq E^\lambda_{\mu_0}(\{P\}).$$

Thus any minimizer of $E^\lambda_{\mu_0}$ has positive length.

Since $\{\mu_\rho\} \overset{*}{\rightharpoonup} \mu_0$ as $\rho \to 0$, for sequences $\{\rho_n\} \to 0$, $\{\Sigma_n \in \text{argmin } E^\lambda_{\mu_{\rho_n}}\}$, there exists $\Sigma_\infty \in \text{argmin } E^\lambda_{\mu_0}$ such that (upon subsequence) $\{\Sigma_n\} \overset{d_\mathcal{H}}{\to} \Sigma_\infty$, and we just proved that such a Σ_∞ has positive length. Thus the proof is complete. □

Proof. (of **Lemma 3.2**) Let μ_0 be the measure defined in (19), and let Σ be an arbitrary minimizer of $E^\lambda_{\mu_0}$.

- Claim: any minimizer $\Sigma \in \text{argmin } E^\lambda_{\mu_0}$ contains $\{(\pm L, h)\}$.

Choose an arbitrary point $p' \in \text{argmin}_{z\in\Sigma} |z - (-L,h)|$, and consider the competitor

$$\tilde{\Sigma} := \Sigma \cup [\![p', (-L,h)]\!].$$

By construction

$$F_{\mu_0}(\Sigma) - F_{\mu_0}(\tilde{\Sigma}) \geq |p' - (-L,h)|\frac{1-\eta}{2}, \qquad \mathcal{H}^1(\tilde{\Sigma}) \leq \mathcal{H}^1(\Sigma) + |p' - (-L,h)|.$$

The minimality of Σ implies $(1-\eta)|p'-(-L,h)|/2 \le \lambda|p'-(-L,h)|$, and since $\lambda \overset{(C1)}{<} 1/2-\eta < (1-\eta)/2$, it follows $|p'-(-L,h)|=0$. Thus the claim is proven.

Assume the thesis is false, i.e. there exists $\varepsilon>0$ and a sequence $\{\rho_n\}\to 0$ such that for any n there exists a minimizer $\Sigma_n \in \text{argmin } E^\lambda_{\mu_{\rho_n}}$ satisfying $d((-L,h),\Sigma_n)\ge\varepsilon$.

Since $\{\mu_{\rho_n}\}\overset{*}{\rightharpoonup}\mu_0$ as $n\to\infty$, there exists $\Sigma_\infty \in \text{argmin } E^\lambda_{\mu_0}$ such that (upon subsequence) $\{\Sigma_n\}\overset{d_{\mathcal{H}}}{\to}\Sigma_\infty$. Thus we have:

- $d((-L,h),\Sigma_n)\ge\varepsilon$ for any n,
- $(-L,h)\in\Sigma_\infty$, but $\{\Sigma_n\}\overset{d_{\mathcal{H}}}{\to}\Sigma_\infty$.

This is a contradiction. The proof for (L,h) is analogous. □

Proof. (of **Lemma 3.3**) Lemmas 3.1 and 3.2 give the existence of $\rho_0>0$ such that for any $\rho\in(0,\rho_0)$ and $\lambda\in(\frac{1-3\eta}{2},\frac{1-2\eta}{2})$, any minimizer $\Sigma\in\text{argmin } E^\lambda_{\mu_\rho}$ satisfies:

- Σ is a simple curve with positive length,
- upon reducing the value of ρ_0, Σ contains points p,q with $|p-(-L,h)|$, $|q-(L,h)|\le h/4$.

Choose an arbitrary minimizer Σ. Let $f:[0,1]\longrightarrow\Sigma$ be a constant speed bijective parameterization, and let $t_p:=f^{-1}(p)$, $t_q:=f^{-1}(q)$. Since the mass projecting on each endpoint (i.e. $f(0)$ and $f(1)$) is at least λ, the mass projecting on $f\big((0,1)\big)$ is at most $1-2\lambda$. Moreover, the existence of p implies that any point $z\in B((-L,h),\rho)$ satisfies $|z-p|\le 2\rho+h/4$. Since at least λ mass projects on $f(0)$, this forces (upon using parameterization $g:[0,1]\longrightarrow\Sigma$, $g(t):=f(1-t)$ instead of f) $|f(0)-(-L,h)|\le h/4+2\rho$, thus (upon further imposing $\rho_0\le h/12$)

$$|f(0)-(-L,h)|\le h/4+\rho\le h/3.$$

Analogously $|f(1)-(L,h)|\le h/3$, hence $f(0)$ and $f(1)$ belong to the half-plane $\{y\ge 2h/3\}$.

If $f\big((0,1)\big)$ contains a point $f(T)\in\{y=h/10\}$, then

$$\|f'\|_{TV}\ge\pi-f(0)\widehat{f(T)}f(1).$$

Combining with conditions

$$|f(0)-(-L,h)|\le h/3,\qquad |f(1)-(L,h)|\le h/3,$$

elementary geometry gives that the amplitude of angle $f(0)\widehat{f(T)}f(1)$ is bounded from above by the amplitude of angle $\widehat{p_-p_0p_+}$ where

$$p_-:=(-L-h/3,2h/3)\,,\qquad p_0:=(0,1/10)\,,\qquad p_+:=(L+h/3,2h/3)\,.$$

Direct computation gives

$$\widehat{p_-p_0p_+}=2\arctan\frac{L+h/3}{2h/3-1/10}\Longrightarrow\|f'\|_{TV((0,1))}\ge\pi-f(0)\widehat{f(T)}f(1)\ge\pi-2\arctan\frac{L+h/3}{2h/3-1/10}.$$

Proposition 2.6 gives

$$\|f'\|_{TV((0,1))}\le\frac{\pi}{2\lambda}(1-2\lambda),$$

thus a necessary condition (for $\Sigma \cap \{y = h/10\} \neq \emptyset$) is

$$\pi - 2\arctan\frac{L + h/3}{2h/3 - 1/10} \leq \frac{\pi}{2\lambda}(1 - 2\lambda), \tag{21}$$

which contradicts (C1). As $f(0), f(1) \in \{y > h/10\}$, this ensures $f([0,1]) = \Sigma \subseteq \{y > h/10\}$. $\square$

4.2. **Lemmas from subsection 3.2.**

Proof. (of **Lemma 3.8**) Assume without loss of generality $p \in \{x = 0\}$. Simple geometric considerations give that $|p' - p''|$ is maximized (see Figure 7) when $p \in \{y = 2h\}$.

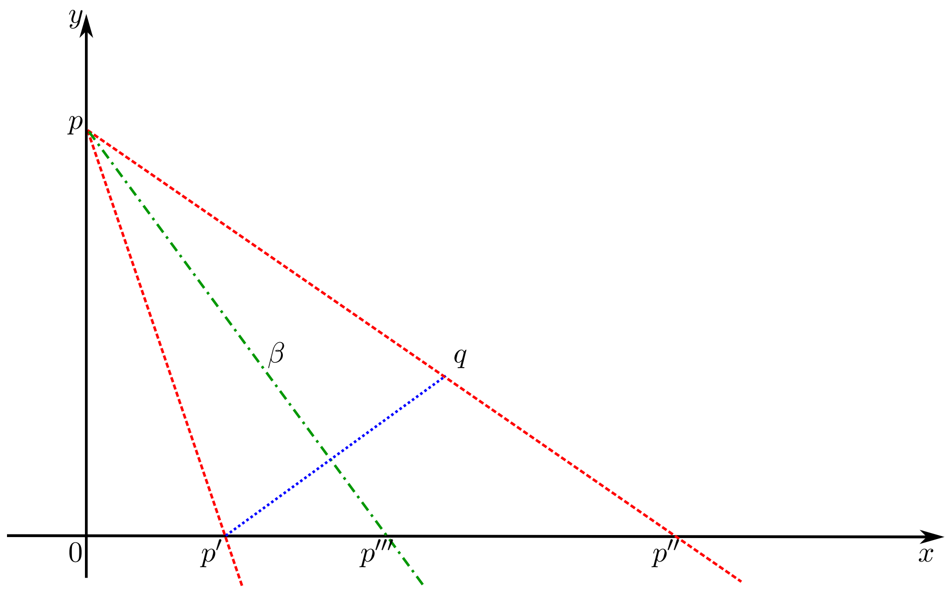

FIGURE 7. This is a schematic representation of the configuration if $\tau \geq \theta$. The proof for case $\tau \leq \theta$ is identical.

Assume $\tau \geq \theta$. Let $q \in [\![p, p'']\!]$ satisfying $|p - p'| = |p - q|$, and denote by p''' the intersection $[\![p, p']\!] \cap \beta$. Direct computation gives:

$$\widehat{qp'p} = \pi/2 - \theta, \quad \widehat{pp'''p'} = \pi/2 - \tau, \quad \widehat{p'qp''} = \pi/2 + \theta, \quad \widehat{p''p'p} = \pi/2 - \tau - \theta,$$
$$\widehat{pp'''p'} = \pi/2 - \tau, \quad \widehat{p'''p'q} = \tau, \quad \widehat{qp'p} = \pi/2 - \theta, \quad |p - p'''| = 2h/\cos\tau.$$

Applying sine theorem to triangles $\triangle pp'p'''$ and $\triangle pp'p''$ gives

$$|p - p'| = |p - p'''| \cdot \frac{\sin\widehat{pp'''p'}}{\sin\widehat{p'''p'p}} = \frac{2h}{\cos(\theta - \tau)}$$

and

$$|p - p''| = |p - p'| \cdot \frac{\sin 2\theta}{\cos(\theta + \tau)} = \frac{2h\sin 2\theta}{\cos(\theta - \tau)\cos(\theta + \tau)}$$

concluding the proof for case $\tau \geq \theta$. Case $\tau \leq \theta$ is solved with the same arguments.

Since $\tau, \theta \leq 0.01$, inequality

$$\frac{2h \sin 2\theta}{\cos(\theta - \tau)\cos(\theta + \tau)} \leq 5h\theta$$

holds (although this estimate is clearly not sharp, it is sufficient for the purposes of this paper), concluding the proof. □

Proof. (of **Lemma 3.9**) Consider arbitrary index j and minimizer $\Sigma \in \text{argmin } E^\lambda_{\mu_j}$. Let $f : [0,1] \longrightarrow \Sigma$ be a bijective, constant speed parameterization.

Assume the opposite, i.e. there exist distinct times $t_0 < t_1$ and $l \in [-L/2, L/2]$ such that $\{f(t_0), f(t_1)\} \subseteq \{x = l\}$. Thus the tangent derivative f' turns by at least π in the time interval $[t_0, t_1]$. Since Lemma 2.6 gives $\|f'_{|[t_0,t_1]}\|_{TV} \leq \frac{\pi\eta}{2\lambda}$, this implies the existence of $t \in [t_0, t_1]$ such that $f(t)$ talks to points in $\text{supp}(\mu_{\text{heavy}})$. Note that (C3) gives the existence of p_1, p_2 such that $\max\{|p_1 - (-L, h)|, |p_2 - (L, h)|\} \leq 1$, hence for any point $z \in \text{supp}(\mu_{\text{heavy}})$ it holds

$$\min\{|z - p_1|, |z - p_2|\} \leq 1 + 2\rho.$$

If $f(t)$ talks to points in $\text{supp}(\mu_{\text{heavy}})$, then there exists $w \in \text{supp}(\mu_{\text{heavy}})$ such that $|w - f(t)| \leq 1 + 2\rho$. Without loss of generality assume $w \in B\big((-L, h), \rho\big)$, and $f(0)$ is the endpoint talking to points in $B\big((-L, h), \rho\big)$. This gives $f(t) \in \{x \leq -L + 1 + 2\rho\}$. Consider the competitor Σ'

$$\Sigma' := \big(\Sigma \backslash f([t_0, t_1])\big) \cup [\![f(t_0), f(t_1)]\!].$$

Since any point $z \in B\big((-L, h), \rho\big)$ satisfies $d(z, \Sigma) \leq |z - f(0)| \leq 1 + 2\rho$, it follows that any such $z \in B\big((-L, h), \rho\big)$ projecting on $f([t_0, t_1]) \subseteq \Sigma$ can now project on $f(0) \in \Sigma'$, hence $d(z, \Sigma') \leq d(z, \Sigma) + 1 + 2\rho$ and

$$F_{\mu_j}(\Sigma') - F_{\mu_j}(\Sigma) \leq \mu_j\Big(B\big((-L, h), \rho\big)\Big)(1 + 2\rho) < \frac{1}{2} + \rho. \tag{22}$$

The last inequality is due to

$$\mu_j\Big(B\big((-L, h), \rho\big)\Big) \overset{(7)}{=} \mu\Big(B\big((-L, h), \rho\big)\Big) < \frac{1}{2}.$$

Since $f(t_0), f(t_1) \in \{x = l\} \subseteq \{x \geq -L/2\}$, $f(t) \in \{x \leq -L + 1 + 2\rho\}$, and f is injective, it follows

$$\mathcal{H}^1(f([t_0, t_1])) = \mathcal{H}^1(f([t_0, t])) + \mathcal{H}^1(f([t, t_1])) \geq 2(L/2 - 1 - 2\rho),$$

hence

$$\mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma') \geq 2\left(\frac{L}{2} - 1 - 2\rho\right) - |f(t_0) - f(t_1)| \geq L - (2 + 4\rho) - \left(2 - \frac{1}{10}\right)h. \tag{23}$$

Combining (22), (23) and (C1) gives $E^\lambda_{\mu_j}(\Sigma') < E^\lambda_{\mu_j}(\Sigma)$, contradicting the minimality of Σ and concluding the proof. □

Proof. (of **Lemma 3.10**) Consider arbitrary index j and minimizer $\Sigma \in \text{argmin } E^\lambda_{\mu_j}$. Let $f : [0,1] \longrightarrow \Sigma$ be a bijective, constant speed parameterization. Lemma 3.4 gives $\Sigma \subseteq \{-h/10 < y < 2h\}$, and Lemma 3.2 implies that $\Sigma \cap \{x = -L/2\}$ and $\Sigma \cap \{x = L/2\}$ are both singletons.

Lemma 3.9 gives that any point $f(t)$, $t \in [t_0, t_1]$ can only talk to points in $\mathrm{supp}(\mu_{\mathrm{light}})$. Since f is C^1 regular outside a countable set (since any corner receives a positive amount of mass, see [9]), there exists a sequence $\{s_k\} \searrow t_0$ such that:

- $|f(s_k) - f(t_0)| \leq 1$ for any k,
- f is C^1 regular at s_k for any k.

- Claim: for any k it holds $\angle(f'(s_k), \{y = 0\}) \leq \vartheta^*/2$.

We proof the claim by a contradiction argument. Proposition 2.6 gives that the total variation of $f'_{|[s_k,t_1]}$ satisfies

$$\|f'_{|[s_k,t_1]}\|_{TV} \leq \frac{\pi\eta}{2\lambda}, \tag{24}$$

and if there exists s_k such that $\angle(f'(s_k), \{y = 0\}) > \vartheta^*/2$, then it follows:

- $\angle(f'(s), \{y = 0\}) \geq 2h/L$, i.e. $|\arctan f'(s)| \geq 2h/L$ for any $s \in [s_k, t_1]$ where $f'(s)$ is well defined,
- for any $s \in [s_k, t_1]$ where $f'(s)$ is not well defined, let $f'_-(s)$ (resp. $f'_+(s)$) be the left (resp. right) tangent derivative, and both $\angle(f'_-(s), \{y = 0\})$ and $\angle(f'_+(s), \{y = 0\})$ exceed $2h/L$ in view of (24) and $|\arctan f'(s_k)| \geq \vartheta^*/2$,
- $\arctan f'(s)$ cannot change sign, in view of (24) and $|\arctan f'(s_k)| \geq \vartheta^*/2$.

Since Lemma 3.4 gives $f(s_k) \in \{h/10 < y < 2h\}$, the following dichotomy arises:

- if $\arctan f'(s_k) \geq \vartheta^*/2$, then:
  - $\arctan f'(s) \geq 2h/L$ for all $s \in [s_k, t_1]$ where $f'(s)$ is well defined,
  - for any $s \in [s_k, t_1]$ where $f'(s)$ is not well defined it holds

  $$\arctan f'_-(s) \geq 2h/L, \qquad \arctan f'_+(s) \geq 2h/L.$$

  Since $|f(s_k) - f(t_0)| \leq 1$, $f(t_0) \in \{x = -L/2\}$, it follows

  $$f(t_1)_y \geq \frac{h}{10} + \frac{2h}{L}(L-1) \overset{(C1)}{>} 2h, \tag{25}$$

  contradicting Lemma 3.4.
- Similarly, if $\arctan f'(s_k) \leq -\vartheta^*/2$, then the same argument gives

  $$f(t_1)_y \leq 2h - \frac{2h(L-1)}{L} \overset{(C1)}{<} \frac{h}{10}, \tag{26}$$

  again contradicting Lemma 3.4.

Thus in both cases we obtain a contradiction, proving the claim. If there exist $v_1, v_2 \in \Sigma \cap \{-L/2 \leq x \leq L/2\}$ such that $\angle([\![v_1, v_2]\!], \{y = 0\}) \geq \vartheta^*$, then there exists a time t' such that the $\angle(f'(t'), \{y = 0\}) > \vartheta^*/2$, contradicting the arguments above. Thus the proof is complete. $\square$

Proof. (of **Lemma 3.11**) Consider arbitrary j and $\Sigma \in \mathrm{argmin}\ E^{\lambda}_{\mu_j}$. Let $f : [0,1] \longrightarrow \Sigma$ be a bijective, constant speed parameterization. Note that since any point of $\Sigma \cap \{-L/2 \leq x \leq L/2\}$ can talk only to points in $\bigcup_{n=1}^{\infty} B_n$, and $\mu_j\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \eta$, hence Lemma 2.6 gives $\mathrm{TA}(v) \leq \frac{\pi\eta}{2\lambda}$. Let $t_v := f^{-1}(v)$, and for any $w \in V(v)$, $\angle([\![w, v]\!], f'_+(v))$ (with $f'_+(v)$ defined as the right tangent derivative at v) is at least $\frac{\pi}{2} - \frac{\pi\eta}{2\lambda}$. Let t_0, t_1 be the unique times such that $f(t_0) \in \{x = -L/2\}$. $f(t_1) \in \{x = L/2\}$. Lemma 3.10 gives:

- for any $s \in [t_0, t_1]$ where $f'(s)$ is well defined it holds $|\arctan f'(s)| \le \vartheta^*$, i.e. $\angle(f'(s), \{y = 0\}) \le \vartheta^*$,
- for any $s \in [t_0, t_1]$ where $f'(s)$ is not well defined, it holds $|\arctan f'_-(s)| \le \vartheta^*$ and $|\arctan f'_+(s)| \le \vartheta^*$, where $f'_-(s)$ denotes the left tangent derivative, i.e. $\angle(f'_-(s), \{y = 0\}) \le \vartheta^*$ and $\angle(f'_+(s), \{y = 0\}) \le \vartheta^*$.

Thus
$$\angle([\![w, v]\!], \{y = 0\}) \ge \frac{\pi}{2} - \frac{\pi\eta}{2\lambda} - \vartheta^* > \vartheta^*,$$
contradicting Lemma 3.10, and concluding the proof. □

Proof. (of **Lemma 3.12**) Consider arbitrary j and $\Sigma \in \text{argmin } E^\lambda_{\mu_j}$. Let $f : [0,1] \longrightarrow \Sigma$ be a constant speed bijective parameterization. Assume the thesis is false, i.e. there exist corners $v_1, v_2 \in \Sigma$ such that v_i talks to points in B_{n_i} $(i = 1, 2)$ with $n_1 > n_2$ and $(v_1)_x \le (v_2)_x$. This implies the existence of points $z_1 \in B_{n_1}$, $z_2 \in B_{n_2}$ (thus hypothesis $n_1 > n_2$ gives $(z_2)_x > (z_1)_x$), such that $d(z_i, \Sigma) = |z_i - v_i|$, $i = 1, 2$.

Case $(v_1)_x = (v_2)_x$. This implies that Σ intersects $\{x = (v_1)_x\}$ in at least two points (v_1 and v_2), contradicting Lemma 3.9.

Case $(v_1)_x > (v_2)_x$. Lemma 3.11 gives $[\![z_1, v_1]\!] \cap \Sigma = \{v_1\}$ and $[\![z_2, v_2]\!] \cap \Sigma = \{v_2\}$. Since v_i talks to z_i $(i = 1, 2)$ it follows $z_i \in V(v_i)$ $(i = 1, 2)$. Let l be the axis of $[\![v_1, v_2]\!]$, and let R_i be the half-plane (with boundary l) containing v_i $(i = 1, 2)$.

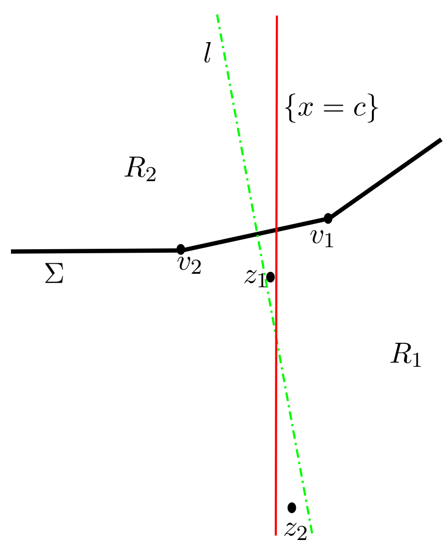

FIGURE 8. A schematic representation of case $l \not\parallel \{x = c\}$.

Since

$$z_1 \in B_{n_1} = B\big((c_{n_1}, 0), \varrho_{n_1}\big), \quad z_2 \in B_{n_2} = B\big((c_{n_2}, 0), \varrho_{n_2}\big), \quad n_1 > n_2,$$

there exists $c \in (c_{n_1} + \varrho_{n_1}, c_{n_2} - \varrho_{n_2})$ ((C1) gives $c_{n_1} + \varrho_{n_1} < c_{n_2} - \varrho_{n_2}$ for any $n_1 > n_2$) such that $z_1 \in \{x < c\}$, $z_2 \in \{x > c\}$. Lemma 3.10 implies $\angle([\![v_1, v_2]\!], \{y = 0\}) \leq \vartheta^*$, hence $\angle(l, \{x = c\}) \leq \vartheta^*$.

- Case $l \parallel \{x = c\}$. This implies that the line through v_1, v_2 is parallel to $\{y = 0\}$, then $z_1 \in \{x < c\}$ and $(v_1)_x > (v_2)_x$ imply $|z_1 - v_2| < |z_1 - v_1|$, which is a contradiction.
- Case $l \not\parallel \{x = c\}$. Conditions $|z_1 - v_1| \leq |z_1 - v_2|$ and $|z_2 - v_2| \leq |z_2 - v_1|$ imply $z_1 \in R_1$, and $z_2 \in R_2$. Hence $z_1 \in R_1 \cap \{x < c\}$, $z_2 \in R_2 \cap \{x > c\}$. Note that since $\angle(l, \{x = c\}) \leq \vartheta^*$, it follows $\angle([\![z_1, z_2]\!], \{x = c\}) \leq \vartheta^*$, hence $\angle([\![z_1, z_2]\!], \{y = 0\}) \in [\pi/2 - \vartheta^*, \pi/2 + \vartheta^*]$. This is a contradiction since

  $$B_{n_i} := B\big((c_{n_i}, 0), \varrho_{n_i}\big) \subseteq [c_{n_i} - \varrho_{n_i}, c_{n_i} + \varrho_{n_i}] \times [-\varrho_{n_i}, \varrho_{n_i}], \qquad i = 1, 2,$$

  and direct computations (using elementary analytic geometry) gives that the angle between any line intersecting both $[c_{n_i} - \varrho_{n_i}, c_{n_i} + \varrho_{n_i}] \times [-\varrho_{n_i}, \varrho_{n_i}]$ $(i = 1, 2)$ and $\{y = 0\}$ does not exceed

  $$\frac{\varrho_{n_2} + \varrho_{n_1}}{c_{n_2} - \varrho_{n_2} - (c_{n_1} + \varrho_{n_1})} \overset{(C1)}{<} \frac{\pi}{2} - \vartheta^* < \angle([\![z_1, z_2]\!], \{y = 0\}).$$

Thus in both cases a contradiction arises, and the proof is complete. □

4.3. **Lemmas from subsection 3.3.**

Proof. (of **Lemma 3.18**) Consider arbitrary index j, minimizer $\Sigma \in \operatorname{argmin} E^\lambda_{\mu_j}$ and corner $v \in \Sigma$. Since v talks to points in B_n, it follows $V(v) \cap B_n \neq \emptyset$. Lemma 3.14 gives that v talks *only* to points in B_n, and denoting by $\tau_v^\pm$ the left/right tangent derivative in v (the exact order is not relevant), Lemma 3.10 gives $\angle(\tau_v^-, \{y = 0\}) \leq \vartheta^*$, $\angle(\tau_v^+, \{y = 0\}) \leq \vartheta^*$, hence the bisector of $V(v)$ (which we denote by β) is "almost orthogonal" to $\{y = 0\}$, that is $\angle(\beta, \{x = 0\}) \leq \vartheta^*$.

Since v talks only to points B_n, Lemma 3.16 implies that the amplitude of $V(v)$ does not exceed $\frac{\pi m_n}{2\lambda}$. Lemma 3.8 gives

$$\mathcal{L}^1\big(V(v) \cap \{y = 0\}\big) \leq \frac{5h\pi}{4\lambda} m_n. \tag{27}$$

Since $V(v) \cap B_n \neq \emptyset$, and by construction

$$B_n = B\big((c_n, 0), \varrho_n\big) \subseteq [c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n]$$

it follows $V(v) \cap [c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n] \neq \emptyset$, and elementary geometry gives that

$$\inf_{x \in V(v) \cap \{y=0\},\ z \in [c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n]} |x - y|$$

is maximum (i.e. $V(v) \cap \{y = 0\}$ is farthest from $[c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n]$, and $V(v) \cap \{y = 0\}$ can contain points farthest away from $(c_n, 0)$) when $V(v) \cap [c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n]$ is a singleton, either $\{(c_n - \varrho_n, -\varrho_n)\}$ or $\{(c_n + \varrho_n, -\varrho_n)\}$. Consider the case $V(v) \cap [c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n] = \{(c_n - \varrho_n, -\varrho_n)\}$. Denote by $\Lambda \subseteq V(v)$ the half-line (starting in v) through $(c_n - \varrho_n, -\varrho_n)$, and

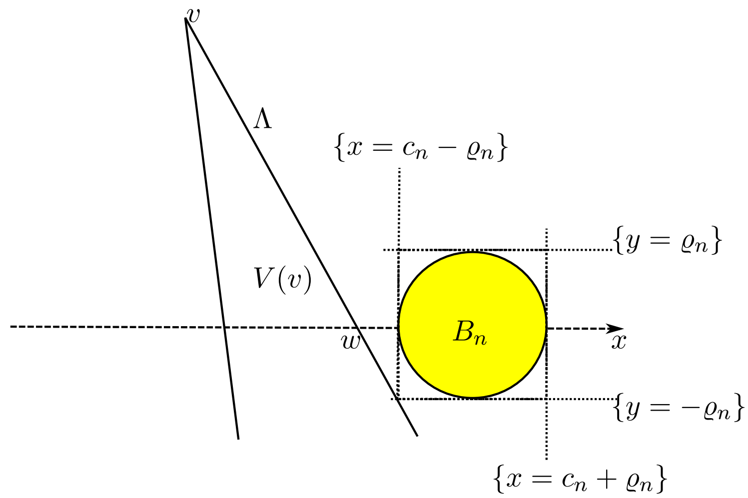

FIGURE 9. Representation of an extremal case, when $V(v) \cap \{y = 0\}$ can contain points farthest from $(c_n, 0)$.

let $w := \Lambda \cap \{y = 0\}$ ($\Lambda \not\parallel \{y = 0\}$ since $\angle(\Lambda, \{x = 0\}) \le \vartheta^* < \pi/2$). Synthetic geometric considerations give $|w - (c_n - \varrho_n, 0)| \le \varrho_n \tan\vartheta^*$, hence

$$|w - (c_n, 0)| \le \varrho_n(1 + \tan\vartheta^*), \tag{28}$$

and for any $z \in V(v) \cap \{y = 0\}$ it holds

$$\begin{aligned}|z - (c_n, 0)| &\le |z - w| + |w - (c_n, 0)| \le \mathcal{L}^1(V(v) \cap \{y = 0\}) + |w - (c_n, 0)| \\ &\overset{(27),\ (28)}{\le} \frac{5h\pi}{4\lambda} m_n + \varrho_n(1 + \tan\vartheta^*).\end{aligned}$$

Note that $\vartheta^* \le 0.01$, hence

$$(\forall n) \quad \frac{5h\pi}{4\lambda} m_n + \varrho_n(1 + \tan\vartheta^*) \le \frac{5h\pi}{4\lambda} m_n + \varrho_n(1 + \tan 0.01) \overset{(C1)}{\le} \frac{c_n}{10}. \tag{29}$$

The proof for case $V(v) \cap [c_n - \varrho_n, c_n + \varrho_n] \times [-\varrho_n, \varrho_n] = \{(c_n + \varrho_n, -\varrho_n)\}$ is analogous. □

Proof. (of **Lemma 3.19**) Assume $n_1 < n_2$, let $f : [0, 1] \longrightarrow \Sigma$ be a constant speed, bijective parameterization, and let $t_{n_i} := f^{-1}(v_{n_i})$, $i = 1, 2$. Proposition 2.6 gives

$$\|f'_{|[t_{n_1}, t_{n_2}]}\|_{TV} \le \frac{\pi}{2\lambda} \sum_{n=n_1}^{n_2} m_n.$$

Let β_i be the bisector of $V(v_{n_i})$, $i = 1, 2$. It holds $\angle(\beta_1, \beta_2) \le \frac{\pi}{2\lambda} \sum_{n=n_1}^{n_2} m_n$ since any point $f(t)$ ($t \in (t_{n_1}, t_{n_2})$) can only talk to points in $\bigcup_{n=n_1}^{n_2} B_n$. Let:

- β_3 the half-line starting in v_{n_1} and parallel to β_2;
- $q_i := \beta_1 \cap \{y = 0\}$, $i = 1, 2, 3$. Such q_1 exists since $\beta_1 \not\parallel \{y = 0\}$ as $\angle(\beta_1, \{x = 0\}) \le \vartheta^* < \pi/2$; similarly for the existence of q_2 and q_3;
- $w \in \beta_3$ such that $[\![v_{n_2}, w]\!] \parallel \{y = 0\}$, hence the quadrilateral $q_2 q_3 v_{n_2} w$ is a parallelogram.

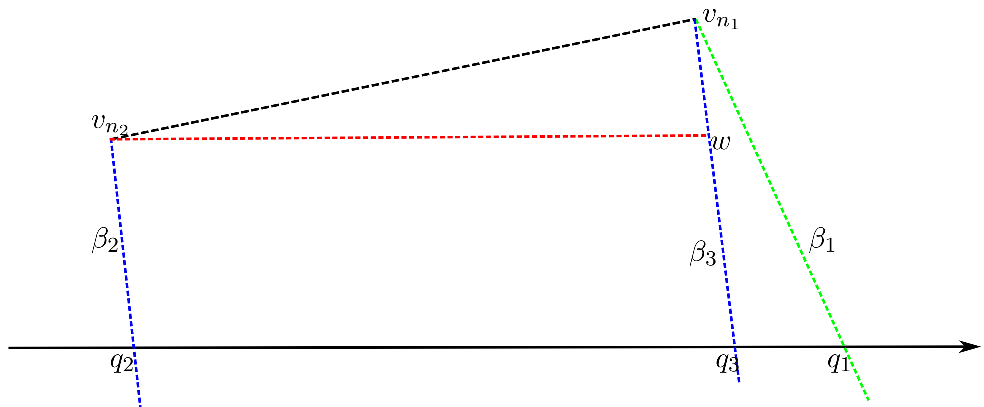

FIGURE 10. The construction and relevant quantities.

By construction $|w - v_{n_2}| = |q_2 - q_3|$, and

$$\widehat{v_{n_1} w v_{n_2}} \in [\pi/2 - \vartheta^*, \pi/2 + \vartheta^*] \tag{30}$$

since $[\![v_{n_2}, w]\!] \parallel \{y = 0\}$ and $\angle(\beta_3, \{x = 0\}) \leq \vartheta^*$. Thus $\widehat{v_{n_2} v_{n_1} w} \in [\pi/2 - 2\vartheta^*, \pi/2 + \vartheta^*]$. Applying sine theorem to triangle $\triangle v_{n_1} v_{n_2} w$ gives

$$\frac{|v_{n_2} - v_{n_1}|}{\sin \widehat{v_{n_1} w v_{n_2}}} = \frac{|v_{n_2} - w|}{\sin \widehat{v_{n_2} v_{n_1} w}} = \frac{|q_2 - q_3|}{\sin \widehat{v_{n_2} v_{n_1} w}}. \tag{31}$$

We need a lower bound estimate on $|q_2 - q_3|$. Note that $\widehat{q_3 v_{n_1} q_1} = \angle(\beta_1, \beta_2) \leq \frac{\pi}{2\lambda} \sum_{n=n_1}^{n_2} m_n$. Since $\angle(\beta_3, \{x = 0\}) \leq \vartheta^*$, Lemma 3.8 gives

$$|q_3 - q_1| \leq 5h\widehat{q_3 v_{n_1} q_1} \leq \frac{5h\pi}{2\lambda} \sum_{n=n_1}^{n_2} m_n \overset{(C1)}{\leq} 0.01|c_{n_1} - c_{n_2}|. \tag{32}$$

Note that (C1) gives $c_{n_2} \leq 0.01 c_{n_1}$. Since by definition $q_i \in V(v_{n_i}) \cap \{y = 0\}$ $(i = 1, 2)$, Lemma 3.18 gives $|(q_i)_x - c_{n_i}| \leq c_{n_i}/10$ $(i = 1, 2)$, thus

$$(q_1)_x \geq 0.9 c_{n_1}, \quad (q_2)_x \leq 1.1 c_{n_2} \overset{(C1)}{\leq} 0.011 c_{n_1}, \Longrightarrow |q_1 - q_2| \geq (0.9 - 0.011) c_{n_1} = 0.889 c_{n_1}.$$

Combining with (32) gives

$$\frac{|q_1 - q_2|}{|q_1 - q_3|} \geq \frac{0.889 c_{n_1}}{0.01 c_{n_1}} = 88.9 \Longrightarrow |q_2 - q_3| \geq \frac{88.9}{89.9} |q_1 - q_2|,$$

since clearly the value of $|q_2 - q_3|$ is minimum when $q_3 \in [\![q_1, q_2]\!]$. Combining with (31) gives

$$\frac{|v_{n_2} - v_{n_1}|}{\sin \widehat{v_{n_1} w v_{n_2}}} = \frac{|q_2 - q_3|}{\sin \widehat{v_{n_2} v_{n_1} w}} \geq \frac{88.9}{89.9} \cdot \frac{|q_1 - q_2|}{\sin \widehat{v_{n_2} v_{n_1} w}},$$

and since $\widehat{v_{n_1} w v_{n_2}}$ (resp. $\widehat{v_{n_2} v_{n_1} w}$) is valued in $[\pi/2 - \vartheta^*, \pi/2 + \vartheta^*]$ (resp. $[\pi/2 - 2\vartheta^*, \pi/2 + \vartheta^*]$), while $\vartheta^* = 4h/L + \pi\eta/\lambda \leq 4/10^9 + 0.001$, it follows

$$\frac{88.9}{89.9} \cdot \frac{\sin \widehat{v_{n_1} w v_{n_2}}}{\sin \widehat{v_{n_2} v_{n_1} w}} \cdot 0.889 \geq \frac{88.9}{89.9} \sin\left(\frac{\pi}{2} - 2(4 \cdot 10^{-9} + 0.001)\right) \cdot 0.889 \geq 0.8,$$

hence

$$|v_{n_2} - v_{n_1}| \geq \frac{88.9}{89.9} \cdot \frac{\sin \widehat{v_{n_1} w v_{n_2}}}{\sin \widehat{v_{n_2} v_{n_1} w}} \cdot |q_1 - q_2| \geq 0.8 c_{n_1}.$$

Thus the proof is complete. □

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