MORE COUNTEREXAMPLES TO REGULARITY FOR MINIMIZERS OF THE AVERAGE-DISTANCE PROBLEM

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ABSTRACT. The average-distance problem, in the penalized formulation, involves minimizing

$$E_\mu^\lambda(\Sigma) := \int_{\mathbb{R}^d} d(x,\Sigma)d\mu(x) + \lambda\mathcal{H}^1(\Sigma), \tag{1}$$

among path-wise connected, closed sets Σ with finite $\mathcal{H}^1$-measure, where $d \geq 2$, μ is a given measure, λ is a given parameter and $d(x,\Sigma) := \inf_{y\in\Sigma}|x-y|$. The average-distance problem can be also considered among compact, convex sets with perimeter and/or volume penalization, i.e. minimizing

$$\mathcal{E}(\mu,\lambda_1,\lambda_2)(\cdot) := \int_{\mathbb{R}^d} d(x,\cdot)d\mu(x) + \lambda_1 \operatorname{Per}(\cdot) + \lambda_2 \operatorname{Vol}(\cdot), \tag{2}$$

where μ is a given measure, $\lambda_1, \lambda_2 \geq 0$ are given parameters with $\lambda_1 + \lambda_2 > 0$, and the unknown varies among compact, convex sets. Very little is known about the regularity of minimizers of (2). In particular it is unclear if minimizers of (2) are in general C^1 regular. The aim of this paper is twofold: first, we provide in $\mathbb{R}^2$ a second approach in constructing minimizers of (1) which are not C^1 regular; then, using the same technique, we provide an example of minimizer of (2) whose border is not C^1 regular, under perimeter penalization only.

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1. INTRODUCTION

The average-distance problem was proposed by Buttazzo, Oudet and Stepanov in [2]. To guarantee well-posedness, an a priori bound on the $\mathcal{H}^1$-measure of admissible minimizers was given, and this formulation is often referred as *"constrained formulation"*. To overcome the excessive rigidity imposed by hard constraints on the $\mathcal{H}^1$-measure, Buttazzo, Mainini and Stepanov proposed in [1] the *"penalized formulation"*:

Problem 1.1. *Given $d \geq 2$, a compactly supported, nonnegative measure μ, and $\lambda > 0$, minimize*

$$\int_{\mathbb{R}^d} d(x,\Sigma)d\mu(x) + \lambda\mathcal{H}^1(\Sigma), \qquad d(x,\Sigma) := \inf_{y\in\Sigma}|x-y|$$

with the unknown Σ varying in

$$\mathcal{A} := \{X \subseteq \mathbb{R}^d : X \text{ compact, path-wise connected, } \mathcal{H}^1(X) < \infty\}.$$

To simplify notations, for future reference let

$$F_\mu : \mathcal{A} \longrightarrow \mathbb{R}, \qquad F_\mu(\Sigma) := \int_{\mathbb{R}^d} d(x,\Sigma)d\mu$$

$$E_\mu^\lambda : \mathcal{A} \longrightarrow \mathbb{R}, \qquad E_\mu^\lambda(\Sigma) := F_\mu(\Sigma) + \lambda \mathcal{H}^1(\Sigma).$$

The functional F_μ will be often referred as *"average-distance functional"*. In the following, any considered measure will be assumed nonnegative, compactly supported probability measure. The choice of working with probability measure is for the sake of simplicity, and it is not restrictive since the main result is an existence result. Existence of minimizers follows from Blaschke and Gołab theorems.

In the following the expression *"average-distance problem"* will refer to Problem 1.1. Moreover, the $\mathcal{H}^1$-measure of a set will be often referred as "length". Originally this problem stemmed from mathematical modeling of optimization problems. A classic example can be found in urban planning: let

- μ be the distribution of passengers in a given region,
- Σ (the unknown) be the transport network to be built.

In this case $F_\mu(\Sigma)$ is the "average distance" of passengers from the network (thus smaller values of $F_\mu(\Sigma)$ imply that "on average, passengers are quite close to the network Σ", i.e. "Σ is easily accessible"), and $\lambda\mathcal{H}^1(\Sigma)$ is the cost to build such network. Thus minimizing E_μ^λ is determining the network which "optimizes accessibility" for passengers, under cost considerations.

A more recent application can be found in data approximation: let

- μ be the distribution of data points,
- Σ (the unknown) be a one dimensional object which approximates the data.

In this case $F_\mu(\Sigma)$ is the error of such approximation, while $\lambda\mathcal{H}^1(\Sigma)$ is the cost associated to its complexity. Thus minimizing E_μ^λ is equivalent to determine the "best" approximation, which balances approximation error and cost.

In applications, sometimes the integrand $d(x, \Sigma)$ in $F_\mu(\Sigma)$ can be replaced by $d(x, \Sigma)^p$ for some power $p \geq 1$ (the case $p = 2$ is most common). However for the purposes of this paper the exponent p is not relevant, and we will consider only the case $p = 1$. The regularity of minimizers of Problem 1.1 is quite a delicate problem: it is known that minimizers are union of at most $[1/\lambda]$ branches, and such branches are Lipschitz regular (Buttazzo, Oudet, Paolini and Stepanov [2, 3, 4, 13]), satisfying a curvature estimate (Slepčev et al. [11]), but can fail to be C^1 regular (Slepčev [15]). Other results were proven by Santambrogio, Tilli [14, 16] and Lemenant [8]. A review is available in [7].

Average distance problem among convex sets. As proposed by Lemenant and Mainini in [9], the average-distance problem can be also considered among compact, convex sets, under perimeter and/or volume penalization:

Problem 1.2. *Given $d \geq 2$, a measure μ, and parameters $\lambda_1, \lambda_2 \geq 0$ satisying $\lambda_1 + \lambda_2 > 0$, minimize*

$$\mathcal{E}(\cdot) = \mathcal{E}(\mu, \lambda_1, \lambda_2)(\cdot) := \int_{\mathbb{R}^d} d(x, \cdot) d\mu + \lambda_1 \operatorname{Per}(\cdot) + \lambda_2 \operatorname{Vol}(\cdot),$$

with the unknown varying in

$$\mathcal{C} := \{K \subseteq \mathbb{R}^d : K \text{ compact and convex}\}.$$

Here the "perimeter" *of a set* $E \subseteq \mathbb{R}^d$ *is defined as the total variation (in* $\mathbb{R}^d$*) of its characteristic function* χ_E*, and the* "volume" *as its* $\mathcal{L}^d$ *measure.*

The motivations to study this problem are mainly theoretical, although one could easily find some applications (see [9]). Some partial results about regularity have been proven in [9]. However it is unclear if minimizers of Problem 1.2 (under only perimeter or volume penalization, not both as this case has been discussed in [9]) have C^1 regular border. The main result is:

Theorem 1.3. *In* $\mathbb{R}^2$*, there exists a measure* μ *and* $\lambda_1 > 0$ *such that*

$$\mathcal{E}(\mu, \lambda_1, 0)(\cdot) = \int_{\mathbb{R}^2} d(x, \cdot) d\mu + \lambda_1 \operatorname{Per}(\cdot)$$

admits a minimizer $K \in \mathcal{C}$ *whose border is not* C^1 *regular.*

We will actually prove a stronger result (Theorem 4.8), with quantitative lower bound estimates on the size of the jump of the tangent derivate.

Our construction strongly exploits the geometric rigidity of two-dimensional domains in Lemmas 3.7, 3.8 and 4.7. Moreover we are unable to prove a similar result for the case of volume penalization only, since the crucial estimates of Lemma 4.5 do not hold in this case. This paper will be structured as follows:

- Section 2 will recall preliminary results,
- Section 3 will construct (in $\mathbb{R}^2$) an explicit example of minimizer of Problem 1.1 failing to be C^1 regular, using an approach different from that used in [15],
- Section 4 will construct, using techniques presented in Section 3, an explicit example of minimizer K of Problem 1.2 under perimeter penalization only, whose border ∂K is not C^1 regular.

The approach used in Section 3 uses some ideas from [15]: indeed we will approximate the reference measure μ with a sequence of discrete measures $\mu_k \overset{*}{\rightharpoonup} \mu$. We will use also a result similar to [15, Lemma 11] (although with a slightly different proof), and similarly to [15], we will use the same result (Lemma 2.5) to pass to the limit $k \to +\infty$. However the core arguments (Lemmas 3.5, 3.7 and 3.8), which prove that for infinitely many indices k there exists a minimizer $\Sigma_k \in \operatorname{argmin}\ E^{\lambda}_{\mu_k}$ containing a corner v_k with turning angle (see Definition 2.4) bounded from below (roughly corresponding to Steps 5, 6, 7 of in [15, Theorem 12]), are significantly different. These are specifically tailored for the reference measure considered in Section 3, and cannot be easily adapted for measures [15, Theorem 12].

It is worth noticing that this approach allows also to construct an example of minimizer of Problem 1.1 whose set of corners (i.e. points where C^1 regularity does not hold) is not closed ([10]).

2. Preliminary results

The main goal of this section is to introduce some notations and recall well known results which will be used in Section 3.

The average-distance functional satisfies the following well known properties:

(1) for any probability measure μ on $\mathbb{R}^d$, and $\lambda > 0$, the functional E_μ^λ is lower semicontinuous w.r.t. $d_{\mathcal{H}}$ (here, and in the following, $d_{\mathcal{H}}$ will denote the Hausdorff distance),
(2) given $\Sigma \in \mathcal{A}$, and $\lambda > 0$, the mapping $\mu \mapsto E_\mu^\lambda(\Sigma)$ is continuous with respect to weak-* convergence of measures,
(3) if $\mu_n \overset{*}{\rightharpoonup} \mu$, then for any $\lambda > 0$, the sequence $\{E_{\mu_n}^\lambda\}$ Γ-converges to E_μ^λ, i.e.
   - for any Σ and sequence $\Sigma_n \overset{d_{\mathcal{H}}}{\to} \Sigma$ it holds $\liminf_n E_{\mu_n}^\lambda(\Sigma_n) \geq E_\mu^\lambda(\Sigma)$,
   - for any Σ there exists a sequence $\Sigma_n' \overset{d_{\mathcal{H}}}{\to} \Sigma$ such that $\limsup_n E_{\mu_n}^\lambda(\Sigma_n') \leq E_\mu^\lambda(\Sigma)$,
(4) consider a sequence $\mu_n \overset{*}{\rightharpoonup} \mu$, and for any n choose $\Sigma_n \in \text{argmin } E_\mu^\lambda$. Then upon subsequence $\Sigma_n \overset{d_{\mathcal{H}}}{\to} \Sigma \in \text{argmin } E_\mu^\lambda$.

For further details, we refer to [2, 3, 4, 15].

We recall the notion of Steiner graphs:

Definition 2.1. *Given a finite set of points $\Pi := \{P_1, \cdots, P_j\} \subseteq \mathbb{R}^d$, a Steiner graph of Π is a path-wise connected set with minimal length among all compact, path-wise connected sets containing Π.*

With an abuse of notation, in the following we will refer to Steiner graphs of *finite* sets as "Steiner graphs". For future reference, given points $p, q \in \mathbb{R}^d$, the notation $[\![p, q]\!]$ will denote the segment $\{(1-s)p + sq : s \in [0, 1]\}$.

The next result (from [15]) proves an intrinsic connection between Steiner graphs and minimizers of Problem 1.1, when the reference measure is discrete:

Proposition 2.2. *Given $d \geq 2$, a discrete measure $\mu := \sum_{i=1}^n a_i \delta_{x_i}$ on $\mathbb{R}^d$, and $\lambda > 0$, then any minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$ is a Steiner graph.*

The following classic result (see for instance [5, 6]) proves several geometric properties about Steiner graphs:

Proposition 2.3. *Given a Steiner graph G, it holds:*
- *G is topologically a tree, i.e. it does not contain loops (subsets homeomorphic to S^1, the unit circle of $\mathbb{R}^2$),*
- *if $[\![u, v]\!]$ and $[\![v, w]\!]$ are edges, with a common vertex v, then $\widehat{uvw} \geq 2\pi/3$,*
- *the maximal degree of any vertex is 3,*
- *if v is a vertex of degree 3, let $[\![u_i, v]\!]$, $i = 1, 2, 3$ be the three different edges containing v, then the angle between any two such edges is $2\pi/3$, and these edges are coplanar.*

In view of Propositions 2.2 and 2.3, the following definition will be useful:

Definition 2.4. *Given a discrete measure μ, $\lambda > 0$, and $\Sigma \in \text{argmin}_{\mathcal{A}} E_\mu^\lambda$, a vertex $v \in \Sigma$ is called:*
- *"endpoint" if has degree 1,*
- *"corner" if has degree 2,*
- *"triple junction" if has degree 3.*

If v is a corner, denoting by w, z the two vertices for which $[\![w, v]\!]$ and $[\![v, z]\!]$ are edges, the "turning angle" *in v is:*

$$\mathrm{TA}(v) := \pi - \widehat{wvz}.$$

Similarly, given a subset $A \subseteq \Sigma$, the turning angle of A is defined as

$$\mathrm{TA}(A) := \sum_{u \in A,\ u \text{ corner}} \mathrm{TA}(u).$$

Given $v \in \Sigma$ and $x \in \mathrm{supp}(\mu)$, the following expressions will be used:

- *"x talks to v", "x projects on v", "v talks to x"*: all these mean $d(v, \Sigma) = |x - v|$;
- *"v receives mass from A"*, where $A \subseteq \mathrm{supp}(\mu)$: there exists $x \in A$ such that x talks to v;
- $TM(\mu, v, \Sigma)$ ($TM(v)$ when no risk of confusion arises) denotes the total mass of projecting on v, which we note may not coincide with the total mass supported on the points talking to v. The quantity $TM(\mu, v, \Sigma)$ will be often referred as "(amount of) mass projecting on v". For a detailed discussion see Lemma 2.1 in [11];
- *"H mass projects on v"*, where $H \geq 0$: this means $TM(\mu, v, \Sigma) = H$.

Given two points p, q the notation $[\![p, q]\!]$ will denote the straight segment $\{(1-t)p + tq : t \in [0, 1]\}$. Finally we recall a convergence result :

Lemma 2.5. *Given a sequence of curves $\{\gamma_k\} : [0, 1] \longrightarrow K$ ($k \in \mathbb{N}$), with $K \subseteq \mathbb{R}^2$ a given compact set, satisfying*

$$\sup_k \|\gamma_k'\|_{BV} < +\infty, \qquad \sup_k \mathcal{H}^1(\gamma_k([0, 1])) < +\infty,$$

then there exists a curve $\gamma : [0, 1] \longrightarrow K$, such that (upon subsequence) it holds:

(1) $\gamma_k \to \gamma$ in C^α, for any $\alpha \in [0, 1)$,
(2) $\gamma_k' \to \gamma'$ in L^p, for any $p \in [1, \infty)$,
(3) $\gamma_k'' \overset{*}{\rightharpoonup} \gamma''$ in the space of signed Borel measures.

For a detailed proof we refer to [15, Lemma 6].

3. Counterexample

The aim of this section is to present a different approach in constructing minimizers (of Problem 1.1) failing to be C^1 regular.

Endow $\mathbb{R}^2$ with the standard Cartesian coordinate system. Let

$$\mu = \mu(r, \eta) := \left(\frac{1 - \eta}{2\pi r^2} (\chi_{B'} + \chi_{B''}) + \frac{\eta}{\pi r^2} \chi_B \right) \cdot \mathcal{L}^2, \tag{3}$$

$$\lambda = \lambda(\eta) := \frac{1 - \eta}{2} - 10^{-100}, \tag{4}$$

where χ denotes the characteristic function (of the subscripted set), and

$$B' := B((-1, 0), r), \quad B'' := B((1, 0), r), \quad B := B((0, 1), r).$$

The constant 10^{-100} in (4) is chosen such that (upon further choosing $\eta < 1/6 - 10^{-100}$) it holds $1/3 < \lambda < (1 - \eta)/2$ (this is the crucial point used in the proof of statement *(i)* of Lemma 3.2 –

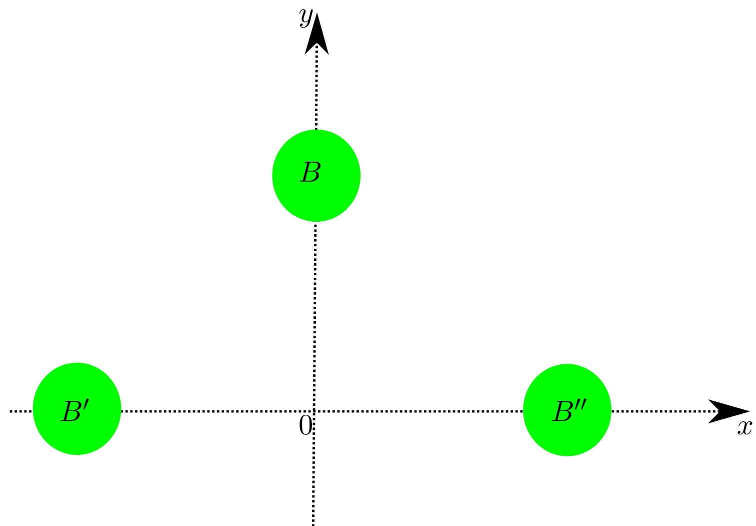

FIGURE 1. A schematic representation of the support of μ. For the sake of clarity the radius r has been chosen large.

the particular value 10^{-100} is not relevant, and chosen only to have an explicit constant to work with). Parameters η and r will be determined later (see conditions (C1) and (C2)). For the sake of brevity, we will omit writing dependencies on η and r if no risk of confusion arises.

The next result ([11, Lemma 3.1]) relates λ with the maximum topological complexity of minimizers of E^λ_μ.

Lemma 3.1. *Given a measure μ' and $\lambda' > 0$, then any minimizer of $E^{\lambda'}_{\mu'}$ contains at most $[1/\lambda']$ endpoints, with $[\cdot]$ denoting the integer part mapping.*

The detailed proof is available in [11, Lemma 3.1]. Here we present a sketch of the proof. In [2] it has been proven that for any compact, path-wise connected, Lipschitz regular set X (note that any minimizer of $E^{\lambda'}_{\mu'}$ satisfies such properties) it holds:

- for any sufficiently small $\varepsilon > 0$, endpoint $v \in X$, the set $X_\varepsilon := X \backslash B(v, \varepsilon)$ is also compact, path-wise connected and Lipschitz regular, and $\mathcal{H}^1(X_\varepsilon) \le \mathcal{H}^1(X) - \varepsilon$.

If there exists a minimizer $\Sigma \in \text{argmin } E^{\lambda'}_{\mu'}$ and an endpoint $w \in \Sigma$ such that $TM(\mu', w, \Sigma) < \lambda$, then (for sufficiently small ε) the competitor $\Sigma_\varepsilon := \Sigma \backslash B(w, \varepsilon)$ satisfies

- $\mathcal{H}^1(\Sigma_\varepsilon) \le \mathcal{H}^1(\Sigma) - \varepsilon$,
- since Σ_ε contains points on $\partial B(w, \varepsilon)$, it follows $\int_{\mathbb{R}^2} d(x, \Sigma_\varepsilon) d\mu' \le \int_{\mathbb{R}^2} d(x, \Sigma) d\mu' + \varepsilon TM(\mu', w, \Sigma)$.

Since $TM(\mu', w, \Sigma) < \lambda$, it follows $E^{\lambda'}_{\mu'}(\Sigma_\varepsilon) \le E^{\lambda'}_{\mu'}(\Sigma) - (\lambda - TM(\mu', w, \Sigma))\varepsilon < E^{\lambda'}_{\mu'}(\Sigma)$, contradicting the minimality of Σ.

A direct consequence is that if $\lambda > 1/3$, any minimizer $\Sigma \in \text{argmin } E^\lambda_{\mu(r,\eta)}$ is a simple curve (or a singleton) independently of r, η.

Lemma 3.2. *Let μ and λ be the quantities defined in* (3) *and* (4). *Then there exist $r_0, \eta_0 > 0$ such that for any $r \in (0, r_0)$, $\eta \in (0, \eta_0)$, any minimizer $\Sigma \in \operatorname{argmin}\ E^\lambda_\mu$ satisfies:*

(i) $\Sigma \cap B((-1,0), 0.01) \neq \emptyset$, $\Sigma \cap B((1,0), 0.01) \neq \emptyset$,

(ii) $\Sigma \subseteq \{y < 1/3\}$,

(iii) $\Sigma \subseteq \{y \geq -r\}$.

Note that *(i)* precludes the case of Σ being a singleton. The constant 0.01 in statement *(i)* is also quite arbitrary, and used to ensure estimate (5), and that any minimizer contains points "close to" $(\pm 1, 0)$ respectively.

Proof. For any η such that $\lambda > 1/3$, Σ is a simple curve (or a singleton) in view of [11, Lemma 3.1]. Choose $\eta_1 < 1/3$ such that $\lambda(\eta) > 1/3$ for any $\eta \in (0, \eta_1)$. To prove (i), note that passing to the limit $r \to 0$ the measure $\mu = \mu(r, \eta)$ converges (with respect to weak-* topology) to

$$\bar{\mu} := \frac{1-\eta}{2}(\delta_{(-1,0)} + \delta_{(1,0)}) + \eta \delta_{(0,1)}.$$

Since $\bar{\mu}((\pm 1, 0)) = (1-\eta)/2 \overset{(4)}{>} \lambda$, any minimizer $\bar{\Sigma} \in \operatorname{argmin}\ E^\lambda_{\bar{\mu}}$ contains $\{(\pm 1, 0)\}$: indeed, if there exists $\bar{\Sigma} \in \operatorname{argmin} E^\lambda_{\bar{\mu}}$, $\bar{\Sigma} \not\ni (1,0)$, choosing an arbitrary $\bar{w} \in \operatorname{argmin}_{y \in \bar{\Sigma}} |y - (1,0)|$, the competitor $\bar{\Sigma}' := \bar{\Sigma} \cup [\![\bar{w}, (1,0)]\!]$ satisfies

$$\mathcal{H}^1(\bar{\Sigma}') \leq \mathcal{H}^1(\bar{\Sigma}) + |\bar{w} - (1,0)|, \qquad \int_{\mathbb{R}^2} d(x, \bar{\Sigma}') d\bar{\mu} \leq \int_{\mathbb{R}^2} d(x, \bar{\Sigma}) d\bar{\mu} - |\bar{w} - (1,0)| \bar{\mu}((\pm 1, 0)),$$

hence $E^\lambda_{\bar{\mu}}(\bar{\Sigma}') < E^\lambda_{\bar{\mu}}(\bar{\Sigma})$ in view of $\bar{\mu}((\pm 1, 0)) = (1-\eta)/2 \overset{(4)}{>} \lambda$, contradicting the minimality of $\bar{\Sigma}$. Thus $\bar{\Sigma} \ni (1,0)$. The proof of $\bar{\Sigma} \ni (-1,0)$ is identical.

Since for sequences $r_k \to 0$, $\{\Sigma_k \in \operatorname{argmin}_{\mathcal{A}} E^\lambda_{\mu(r_k,\eta)}\}$ it holds (upon subsequence) $\Sigma_k \overset{d_{\mathcal{H}}}{\to} \Sigma \in \operatorname{argmin}_{\mathcal{A}} E^\lambda_{\bar{\mu}}$, there exists a value $r_1 > 0$ such that for any $r \in (0, r_1)$, $\eta \in (0, \eta_1)$, statement (i) holds.

To prove (ii) it suffices to note that any set X containing $\{p_1, p_2, q\}$ with $p_1 \in B((-1,0), 0.01)$, $p_2 \in B((1,0), 0.01)$ and $q \in \{y = 1/3\}$ satisfies

$$(\forall \alpha + \beta = 2) \quad \mathcal{H}^1(X) \geq \sqrt{(\alpha - 0.01)^2 + (1/3 - 0.01)^2} + \sqrt{(\beta - 0.01)^2 + (1/3 - 0.01)^2}, \tag{5}$$

hence (by taking $\alpha = \beta = 1$)

$$\mathcal{H}^1(X) \geq 2\sqrt{0.99^2 + (1/3 - 0.01)^2},$$

and

$$E^\lambda_\mu(X) \geq \lambda \mathcal{H}^1(X) \geq 2\lambda \sqrt{0.99^2 + (1/3 - 0.01)^2}.$$

The competitor $[\![(-1,0), (1,0)]\!]$ satisfies

$$E^\lambda_\mu([\![(-1,0), (1,0)]\!]) \leq \frac{1-\eta}{2} r + \eta + 2\lambda.$$

Passing to the limit $r, \eta \to 0$ gives

$$E^\lambda_\mu([\![(-1,0), (1,0)]\!]) \leq 2\lambda < 2\lambda \sqrt{0.99^2 + (1/3 - 0.01)^2} \leq \lambda \mathcal{H}^1(X).$$

Thus there exist $r_2, \eta_2 > 0$ such that for any $\eta \in (0, \eta_2), r \in (0, r_2)$ it holds

$$E_\mu^\lambda([\![(-1,0),(1,0)]\!]) < \lambda\mathcal{H}^1(X) \le E_\mu^\lambda(X),$$

i.e. $X \notin \text{argmin } E_\mu^\lambda$. The arbitrariness of X implies that for any such $\eta \in (0, \eta_2)$, $r \in (0, r_2)$, any minimizer of E_μ^λ (which intersects $\{y < 1/3\}$ in view of *(i)*) cannot intersect $\{y = 1/3\}$, hence statement *(ii)*.

To prove (iii), let

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \pi(x,y) := (x, \max\{y, -r\}).$$

Note that:

- any $X \in \mathcal{A}$ satisfies $\mathcal{H}^1(\pi(X)) \le \mathcal{H}^1(X)$, with equality holding only if $X \subseteq \{y \ge -r\}$,
- for any $X \in \mathcal{A}$, $x \in \text{supp}(\mu)$ it holds $d(x, X) = d(x, \pi(X))$.

Thus any minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$ satisfies $\pi(\Sigma) = \Sigma$. Note that the proof of statement (iii) does not impose any further condition on the "smallness" of η, r. Therefore for any $\eta \in (0, \min\{\eta_1, \eta_2\})$, $r \in (0, \min\{r_1, r_2\})$ all the previous conclusions hold, and letting $\eta_0 := \min\{\eta_1, \eta_2\}$, $r_0 := \min\{r_1, r_2\}$ concludes the proof. □

As consequence we have:

Corollary 3.3. *Let μ and λ be the quantities defined in* (3) *and* (4). *Then there exist $r_0', \eta_0' > 0$ such that for any $r \in (0, r_0')$, $\eta \in (0, \eta_0')$, and minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$, it holds:*

$$\begin{aligned}
(\forall z \in B) &\quad \text{argmin}_{w \in \Sigma} |z - w| \subseteq \{-0.01 \le x \le 0.01\},\\
(\forall z' \in B') &\quad \text{argmin}_{w' \in \Sigma} |z' - w'| \subseteq \{-1.01 \le x \le -0.99\},\\
(\forall z'' \in B'') &\quad \text{argmin}_{w'' \in \Sigma} |z'' - w''| \subseteq \{0.99 \le x \le 1.01\}.
\end{aligned}$$

Note that $\bigcup_{z \in B} \text{argmin}_{w \in \Sigma} |z - w|$ contains $\pi_\Sigma(B)$, where

$$\pi_\Sigma : \mathbb{R}^2 \longrightarrow \Sigma, \qquad \pi_\Sigma(x) := \text{the unique point of } \text{argmin}_{y \in \Sigma} |x - y|.$$

The "projection" map π_Σ is well defined $\mathcal{L}^2$-a.e. (for further details we refer to [12]). Thus this result states that the "projections" of B, B', B'' are mutually disjoint and "distant" (with respect to Hausdorff distance).

Proof. Consider an arbitrary minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$. Impose $\eta < \eta_0$, $r < r_0$ (with η_0, r_0 from Lemma 3.2) such that all three statements of Lemma 3.2 hold.

Statement *(i)* of Lemma 3.2 gives that Σ intersects both $B((-1,0), 0.01)$ and $B((1,0), 0.01)$, hence for any $a \in [-0.99, 0.99]$ the intersection $\Sigma \cap \{x = a\}$ is non empty. Combining statements *(ii)* and *(iii)* of Lemma 3.2 gives that for any $a \in [-0.99, 0.99]$ it holds $\Sigma \cap \{x = a\} \subseteq \{-r \le y < 1/3\}$. Choose arbitrary $z \in B$ and $w \in \Sigma \cap \{x = z_x\}$, with z_x denoting the x coordinate of z. Note that $|z - w| \le 1 + 2r$, hence for any $z^* \in B$ it holds $|z^* - w| \le 1 + 4r$. Thus such w satisfies (for any $r < \min\{r_0, (\sqrt{2} - 1)/6\}$, with r_0 from Lemma 3.2)

$$(\forall z \in B) \quad |z - w| \le 1 + 4r < \sqrt{2} - 2r = \inf_{x \in B,\ y \in B'} |x - y| = \inf_{x \in B,\ y \in B''} |x - y|.$$

The same arguments give the existence of $w', w'' \in \Sigma$ such that

$$(\forall z' \in B') \quad |z' - w'| \le r + \varepsilon(r), \qquad (\forall z'' \in B'') \quad |z'' - w''| \le r + \varepsilon(r),$$

where

$$\varepsilon(r) := \max\{d((-1,0),\Sigma), d((1,0),\Sigma)\}.$$

From the proof of statement *(i)* of Lemma 3.2 it follows $\lim_{r\to 0^+} \varepsilon(r) = 0$. Thus the proof is complete. □

Thus choose r, η such that:

(C1) $\eta \le 10^{-100}$, $r/\eta \le 1/48\pi$,

(C2) conclusions of Lemma 3.2 and Corollary 3.3 hold.

Note that the "smallness" of η and r is already hidden in (C2). However (C1) is useful when an explicit estimate is convenient. The value 10^{-100} is "highly non optimal", but sufficient for our purposes (its role is to ensure that for any $x \in [0, \eta/\lambda]$ it holds $\sin x \ge x/2$, $\tan x \le 2x$, which will be used in Lemma 3.8). Condition $r/\eta \le 1/48\pi$ will be used in the crucial Lemma 3.8.

Discrete measures. Similarly to [15], the first step involves approximating (in the weak-* topology) μ with a sequence of discrete measures. Given three points v_1, v_2, v_3, define the *"region of influence"* $V(v_2)$ as follows:

1. if v_1, v_2, v_3 are collinear, then $V(v_2)$ is the unique line passing through v_2 and orthogonal to $v_3 - v_2$,
2. otherwise, let $\theta_i := \frac{v_{i+1}-v_i}{|v_{i+1}-v_i|}$ $(i = 1, 2)$, $\xi := \frac{\theta_2+\theta_1}{|\theta_2+\theta_1|}$, $b := \frac{\theta_2-\theta_1}{|\theta_2-\theta_1|}$, $\beta := \mathrm{TA}(v_2)/2$, and
$$V(v_2) := v_2 + \{x \in \mathbb{R}^2 : |\langle \xi, x\rangle| \le \langle b, x\rangle \tan\beta\},$$
where $\langle , \rangle$ denotes the standard Euclidean scalar product of $\mathbb{R}^2$.

Note that if $\mathrm{TA}(v_2) > 0$, $V(v_2)$ is an angle with vertex v_2, of amplitude $\mathrm{TA}(v_2)$, and the border $\partial V(v_2)$ is union of two half-lines $l^\pm$ starting in v_2.

For $j = 1, 2, \cdots$, define

$$\mu_j := \frac{1-\eta}{2}\Big(\frac{1}{\sharp(B'\cap\frac{1}{j}\mathbb{Z}^2)}\sum_{i=1}^{\sharp(B'\cap\frac{1}{j}\mathbb{Z}^2)}\delta_{p'_i}\Big) + \frac{1-\eta}{2}\Big(\frac{1}{\sharp(B''\cap\frac{1}{j}\mathbb{Z}^2)}\sum_{i=1}^{\sharp(B''\cap\frac{1}{j}\mathbb{Z}^2)}\delta_{p''_i}\Big)$$
$$+\eta\Big(\frac{1}{\sharp(B\cap\frac{1}{j}\mathbb{Z}^2)}\sum_{i=1}^{\sharp(B\cap\frac{1}{j}\mathbb{Z}^2)}\delta_{p_i}\Big), \tag{6}$$

where

$$\{p'_i\} := B'\cap\frac{1}{j}\mathbb{Z}^2, \quad \{p''_i\} := B''\cap\frac{1}{j}\mathbb{Z}^2, \quad \{p_i\} := B\cap\frac{1}{j}\mathbb{Z}^2,$$

Geometrically, this means that the mass supported in B (resp. B', B'') is uniformly distributed on the uniform grid $B\cap\frac{1}{j}\mathbb{Z}^2$, (resp. $B'\cap\frac{1}{j}\mathbb{Z}^2$, $B''\cap\frac{1}{j}\mathbb{Z}^2$). Note that in Corollary 3.3 replacing the reference μ with μ_j, the same conclusion holds (with the same proof). In particular any point can receive mass from at most one of the balls B, B', B''. For the sake of brevity, in the following we will refer to Corollary 3.3 when using its conclusion, even if the reference measure of the context is μ_j instead of μ.

The next result proves that if a positive fraction of the mass supported in B projects on a point v, then $\text{TA}(v) > 0$.

Lemma 3.4. *Consider the family of measures* $\{\mu_j\}$ *defined in* (6). *Let* λ *be the parameter defined in* (4). *Then for any index* j *and minimizer* $\Sigma \in \text{argmin } E^\lambda_{\mu_j}$, *if a positive fraction of the mass supported in* B *projects on a point* $v \in \Sigma$, *then* $\text{TA}(v) > 0$.

Proof. Assume for the sake of contradiction there exists an index j, a minimizer $\Sigma \in \text{argmin } E^\lambda_{\mu_j}$, and a $v \in \Sigma$ such that $\text{TA}(v) = 0$ but $TM(\mu_j, v, \Sigma) > 0$. Let $E \subseteq B$ be the set of points talking to v. Simple geometric considerations give $E \subseteq V(v)$, which (since $\text{TA}(v) = 0$) is the line through v orthogonal to $v_1 - v_2$.

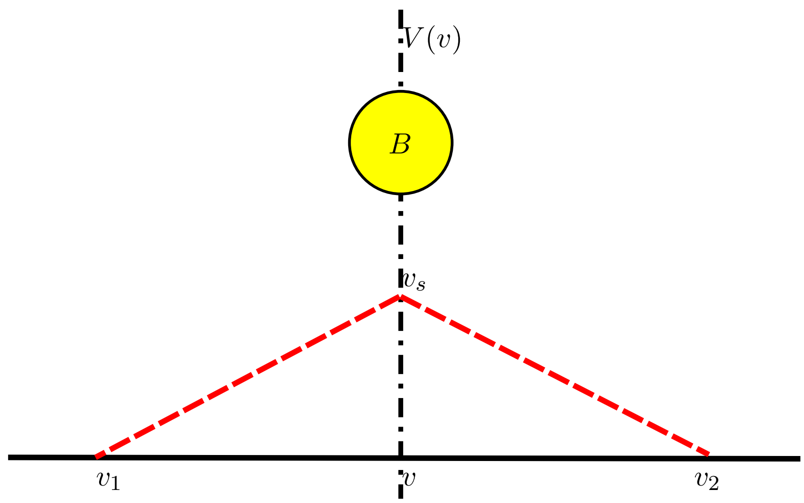

FIGURE 2. This is a schematic representation of the variation.

Consider the variation in Figure 2: for any $s < 1/3$ define the competitor Σ_s as

$$\Sigma_s := \Sigma \setminus [\![v_1, v_2]\!] \cup ([\![v_1, v_s]\!] \cup [\![v_2, v_s]\!]),$$

where v_s is the unique point of $V(v) \cap R^+$ (with R^+ denoting the half-plane – delimited by the line through v_1, v_2 – containing B) such that $|v_s - v| = s$. Choice $s < 1/3$ ensures $v_s \notin E$, since by construction $E \subseteq B \subseteq \{y \geq 1 - r\} \overset{(C1)}{\subseteq} \{y \geq 1 - 10^{-200}\}$, while $v \in \{y < 1/3\}$ (Lemma 3.2). Combining with $E \subseteq V(v)$ gives that for any $w \in E$ it holds $|w - v_s| = |w - v| - s$. Integrating over E gives

$$F_{\mu_j}(\Sigma) - F_{\mu_j}(\Sigma_s) \geq \mu_j(E)s = O(s),$$

since by hypothesis $\mu_j(E) = TM(\mu_j, v, \Sigma) > 0$. Note that for for sufficiently small s it holds

$$\mathcal{H}^1(\Sigma_s) - \mathcal{H}^1(\Sigma) = O(s^2).$$

Thus for sufficiently small s it holds $E^{\lambda}_{\mu_j}(\Sigma_s) \leq E^{\lambda}_{\mu_j}(\Sigma) - TM(\mu_j, v, \Sigma)s + O(s^2) < E^{\lambda}_{\mu_j}(\Sigma)$, i.e. the minimality of Σ is contradicted. □

This result has not been used in [15], and due to the very constructions therein, it is unclear if the proof we used is valid for the reference measure in of [15, Theorem 12]. The next result proves a relation between the turning angle of a given corner and the amount of mass projecting on it.

Lemma 3.5. *Consider the family of measures* $\{\mu_j\}$ *defined in* (6)*. Let* λ *be the parameter defined in* (4)*. Then for any index* j*, minimizer* $\Sigma \in \operatorname{argmin} E^{\lambda}_{\mu_j}$*, and corner* $v \in \Sigma$*, it holds:*

(i) upper bound estimate on the turning angle:

$$\mathrm{TA}(v) \leq \frac{\pi}{2\lambda} TM(v),$$

(ii) estimates on the curvature $\kappa(I)$ *of an arbitrary subset* $I \subseteq \Sigma$*:*

$$\kappa(I) \leq \frac{\pi}{2\lambda} \sum_{v \in I,\ v \text{ corner}} TM(v),$$

(iii) bounds for small turning angles:

$$\mathrm{TA}(v) \to 0^+ \implies \frac{\mathrm{TA}(v)}{TM(v)/\lambda} \to 1. \tag{7}$$

Moreover, if $\mathrm{TA}(v) \leq 0.01$ *then* $\dfrac{\mathrm{TA}(v)}{TM(v)/\lambda} \geq \dfrac{1}{2}$.

The value 0.01 is very arbitrary, and we only use the fact that for any $x \in [0, 0.01]$ it holds $\tan x \leq 2x$. Statements *(i)* and *(ii)* have been proven (or follow easily from) in [15]. However statement *(iii)*, which will play a crucial role in the following arguments, has not been used in [15], and it has not been proven explicitly. Although it may follow from [15, Lemma 9], our proof is somewhat easier.

Proof. Statements *(i)* and *(ii)* have been proven in [15]. To prove *(iii)*, note that upon scaling the configuration is that in Figure 3.

Consider the variations in Figure 3. For any $s < 1/3$ define the competitor

$$\Sigma^+_s := \Sigma \backslash ([\![v_1, v]\!] \cup [\![v_2, v]\!]) \cup ([\![v_1, v^+_s]\!] \cup [\![v_2, v^+_s]\!]), \qquad \{v^+_s\} := \beta \cap B(v, s).$$

Note that

- the same argument from the proof Lemma 3.2 gives $\inf_{y \in \Sigma^+_s, z \in B} |y - z| > 0$ (i.e. Σ^+_s and B are "distant") for any $s < 1/3$,
- simple geometric considerations give that $\min_{z \in V(v)}(|z - v| - |z - v^+_s|)$ is achieved for points $z \in \partial V(v)$, which satisfy

$$|z - v^+_s|^2 = |z - v|^2 + s^2 - 2\cos(\mathrm{TA}(v)/2)|z - v|s.$$

For small values of s, in first order approximation, this reads

$$|z - v^+_s|^2 \geq |z - v|^2 - 2\cos(\mathrm{TA}(v)/2)|z - v|s + O(s^2),$$

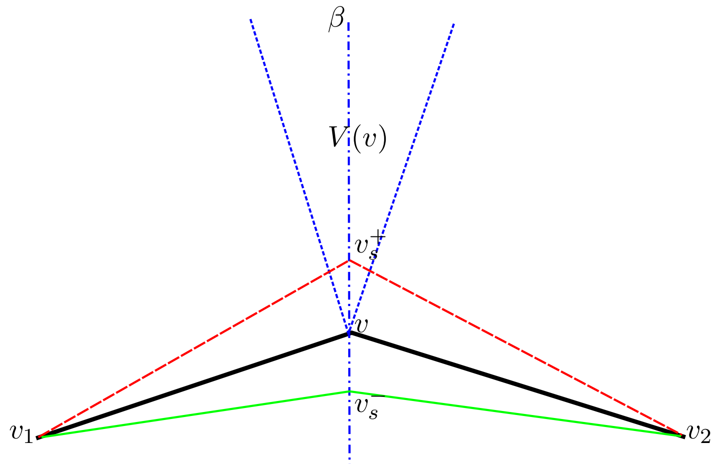

FIGURE 3. This is a schematic representation of the configuration. Here β is the bisector of $V(v)$, and $|v_1 - v| = |v_2 - v|$.

i.e.

$$|z-v|-|z-v_s^+|\geq\frac{2\cos(\mathrm{TA}(v)/2)|z-v|s+O(s^2)}{|z-v|+|z-v_s^+|}=s\cos(\mathrm{TA}(v)/2)+O(s^2).$$

Thus

$$F_{\mu_j}(\Sigma)-F_{\mu_j}(\Sigma_s^+)\geq TM(v)s\cos(\mathrm{TA}(v)/2)+O(s^2). \tag{8}$$

- For length, direct computation gives

$$|v_1-v_s^+|^2=|v_1-v|^2+s^2-2\cos\frac{\pi-\mathrm{TA}(v)}{2}s|v_1-v|,$$

which for small values of s gives

$$\mathcal{H}^1(\Sigma_s^+)-\mathcal{H}^1(\Sigma)=2\big(|v_1-v|-|v_1-v_s^+|\big)+O(s^2)=2\cos\frac{\pi-\mathrm{TA}(v)}{2}s+O(s^2)=2\sin\frac{\mathrm{TA}(v)}{2}s+O(s^2). \tag{9}$$

Combining estimates (8), (9) and minimality condition $E^\lambda_{\mu_j}(\Sigma)\leq E^\lambda_{\mu_j}(\Sigma_s^+)$ (for any $s>0$) yields

$$TM(v)\cos\frac{\mathrm{TA}(v)}{2}\leq 2\lambda\sin\frac{\mathrm{TA}(v)}{2}. \tag{10}$$

The competitor

$$\Sigma_s^- := \Sigma\backslash([\![v_1,v]\!]\cup[\![v_2,v]\!])\cup([\![v_1,v_s^-]\!]\cup[\![v_2,v_s^-]\!]),$$

where v_s^- is the point on $B(v,s)$ antipodal with respect to v_s^+, satisfies:

- for any z it holds $|z-v_s^-|\leq|z-v|+s$, i.e.

$$F_{\mu_j}(\Sigma_s^-)-F_{\mu_j}(\Sigma)\leq TM(v)s. \tag{11}$$

- For length, direct computation gives

$$|v_1 - v_s^+|^2 = |v_1 - v|^2 + s^2 - 2\cos\frac{\mathrm{TA}(v)}{2} s|v_1 - v|,$$

i.e.

$$\mathcal{H}^1(\Sigma_s^+) - \mathcal{H}^1(\Sigma) = 2\sin\frac{\mathrm{TA}(v)}{2} s + O(s^2). \tag{12}$$

Combining estimates (11), (12) and minimality condition $E^\lambda_{\mu_j}(\Sigma) \le E^\lambda_{\mu_j}(\Sigma_s^-)$ yields

$$TM(v) \ge 2\lambda \sin\frac{\mathrm{TA}(v)}{2}. \tag{13}$$

Combining (10) and (13) proves (7). The implication

$$\mathrm{TA}(v) \le \frac{1}{100} \Longrightarrow \frac{\mathrm{TA}(v)}{TM(v)/\lambda} \ge \frac{1}{2}$$

follows immediately from (10): under assumption $\mathrm{TA}(v) \le 0.01$, since for any $x \in [0, 0.01]$ it holds $\tan x \le 2x$, inequality (10) reads

$$TM(v) \le 2\lambda \tan\frac{\mathrm{TA}(v)}{2} \overset{\mathrm{TA}(v)\le 0.01}{\le} 2\lambda\,\mathrm{TA}(v),$$

hence $\frac{1}{2} \le \frac{\mathrm{TA}(v)}{TM(v)/\lambda}$, and the proof is complete. □

Lemma 3.6. *Consider the family of measures* $\{\mu_j\}$ *defined in* (6). *Let* λ *be the parameter defined in* (4). *Then for any index* j*, minimizer* $\Sigma \in \operatorname{argmin} E^\lambda_{\mu_j}$*, and corner* $v \in \Sigma$ *receiving mass from* B*, it holds* $V(v) \cap \Sigma = \{v\}$.

For future reference, the notation $\|\cdot\|_{TV}$ will denote the total variation semi-norm. We will omit writing the domain if no risk of confusion arises.

Proof. Let $f : [0,1] \longrightarrow \Sigma$ be a constant speed bijective parameterization, and denote with $t_v := f^{-1}(v)$. Assume there exists another point $w := f(t_w) \in V(v) \cap \Sigma$, $w \neq v$. Recall that by construction, the border $\partial V(v)$ is union of half-lines $l^\pm$ starting in v and orthogonal to the left/right tangent vector $\tau^\pm := \lim_{t \to t_v^\pm} f'(t)$. Since the amplitude of $V(v)$ is $\mathrm{TA}(v) \le \frac{\pi\eta}{2\lambda}$ (in view of Corollary 3.3 and Lemma 3.5), it follows $\angle(w-v)l^- \le \mathrm{TA}(v)$ (here $\angle(w-v)l^-$ denotes the angle between $w - v$ and an arbitrary vector of the form $v' - v$, with $v' \in l^- \backslash v$), i.e. $\angle(w-v)\tau^- \in [\pi/2 - \mathrm{TA}(v), \pi/2 + \mathrm{TA}(v)]$, thus $\|f'\|_{TV} \ge \pi/2 - \mathrm{TA}(v)$. Since

$$\|f'\|_{TV} \le \frac{\pi}{2\lambda}(1-2\lambda) \overset{(4),\,(C1)}{\le} \frac{2\pi \cdot 10^{-100}}{1 - 3\cdot 10^{-100}} < \frac{\pi}{2} - \frac{\pi \cdot 10^{-100}}{1 - 3 \cdot 10^{-100}} \overset{(4),\,(C1)}{\le} \frac{\pi}{2} - \frac{\pi\eta}{2\lambda} \le \frac{\pi}{2} - \mathrm{TA}(v),$$

with the first inequality due to Lemma 3.5, a contradiction has been achieved, concluding the proof. □

The next result proves that given distinct corners $v_1 \neq v_2$, then the intersection $V(v_1) \cap V(v_2)$ is empty.

Lemma 3.7. *Consider the family of measures $\{\mu_j\}$ defined in* (6)*. Let λ be the parameter defined in* (4)*. Then for any index j, minimizer $\Sigma \in \text{argmin } E^\lambda_{\mu_j}$, and distinct corners v_i, v_i' receiving mass from B, it holds $V(v_i) \cap V(v_i') = \emptyset$.*

The arguments we use in this proof strongly rely on Lemma 3.2, whose proof uses the particular construction of μ, and cannot be extended (at least without very significant modifications) to measures considered in [15, Theorem 12].

Proof. For the sake of brevity, given a point p, the notations p_x (resp. p_y) will denote the x (resp. y) coordinate of p. Assume for the sake of contradiction there exist distinct corners v_1, v_2 such that $V(v_1) \cap V(v_2) \ni v$.

Lemma 3.6 implies $v \notin \{v_1, v_2\}$, $V(v_1) \not\subseteq V(v_2)$ and $V(v_2) \not\subseteq V(v_1)$. Lemma 3.2 gives $\Sigma \subseteq \{y < 1/3\}$, while $B \subseteq \{y > 2/3\}$.

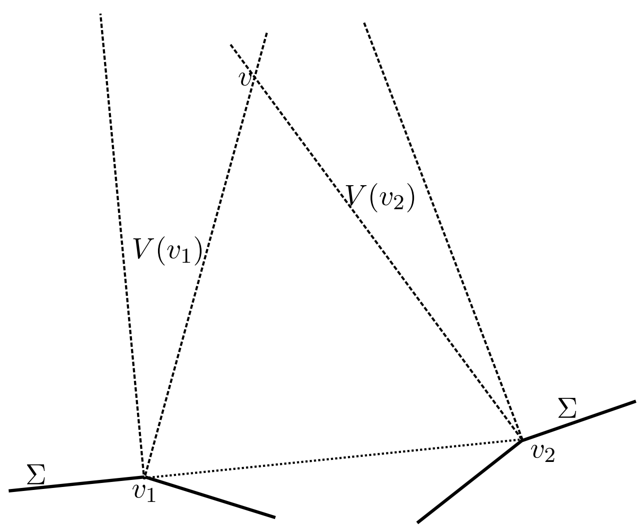

FIGURE 4. This is a schematic representation of the configuration.

Since Σ is a simple curve, let $f : [0,1] \longrightarrow \Sigma$ be a constant speed bijective parameterization. Let

$$t_1 = f^{-1}(v_1), \qquad t_2 = f^{-1}(v_2),$$

and assume $t_1 < t_2$. Note that the triangle $\triangle v_1 v v_2$ is non degenerate, thus $\min\{\widehat{v_2 v_1 v}, \widehat{v v_2 v_1}\} < \pi/2$. Assume (by symmetry) $\widehat{v_2 v_1 v} < \pi/2$. Thus

$$\{\varepsilon > 0 : (\forall t \in (t_1, t_1 + \varepsilon))(\exists z \in [\![v_1, v_2]\!] \cap \{x = \big(f(t)\big)_x\}) : z_y > \big(f(t)\big)_y\} \neq \emptyset, \tag{14}$$

and let

$$\varepsilon^* := \sup\{\varepsilon > 0 : (\forall t \in (t_1, t_1 + \varepsilon))(\exists z \in [\![v_1, v_2]\!] \cap \{x = \big(f(t)\big)_x\}) : z_y > \big(f(t)\big)_y\}.$$

Clearly $\varepsilon^* \leq t_2 - t_1$. Consider the competitor

$$\tilde{\Sigma} := \Sigma \backslash f([t_1, t_1 + \varepsilon^*]) \cup [\![f(t_1), f(t_1 + \varepsilon^*)]\!],$$

constructed by replacing $f([t_1, t_1 + \varepsilon^*])$ with the straight segment $[\![f(t_1), f(t_1 + \varepsilon^*)]\!]$. By construction it holds $\mathcal{H}^1(\tilde{\Sigma}) < \mathcal{H}^1(\Sigma)$. Let $q \in B$ be an arbitrary point. Choose an arbitrary $t_w \in (t_1, t_1 + \varepsilon^*)$ such that $|q - f(t_w)| = d(q, \Sigma)$, and by definition there exists $\tilde{w} \in [\![f(t_1), f(t_1 + \varepsilon^*)]\!]$ satisfying $\tilde{w}_x = \big(f(t_w)\big)_x$, $\tilde{w}_y > \big(f(t_w)\big)_y$. Thus

$$z \in \{y > (\tilde{w}_y + \big(f(t_w)\big)_y)/2\} \Longrightarrow |z - \tilde{w}_y| < |z - f(t_w)|.$$

Since any point of $f([t_1, t_1 + \varepsilon^*])$ can only talk to masses supported in $B \subseteq \{y > 2/3\}$, while $[\![f(t_1), f(t_1 + \varepsilon^*)]\!] \subseteq \{y < 1/2\}$, it follows

$$(\forall z \in B)(\forall t \in [t_1, t_1 + \varepsilon^*]) \quad |z - f(t)| \geq |z - \tilde{w}_t|,$$

where $\tilde{w}_t$ is the unique point satisfying

$$(\tilde{w}_t)_x = \big(f(t)\big)_x, \quad (\tilde{w}_t)_y > \big(f(t)\big)_y, \qquad \tilde{w}_t \in [\![f(t_1), f(t_1 + \varepsilon^*)]\!].$$

Thus it follows $F_\mu(\tilde{\Sigma}) \leq F_\mu(\Sigma)$. Since $\mathcal{H}^1(\tilde{\Sigma}) < \mathcal{H}^1(\Sigma)$, the minimality of Σ is contradicted. Thus such a point v cannot exist. ☐

The next result is the core argument of our construction.

Lemma 3.8. *Consider the family of measures* $\{\mu_j\}$ *defined in* (6). *Let* λ *be the parameter defined in* (4). *Then for any sufficiently large index* j *and* $\Sigma_j \in \operatorname{argmin} E^\lambda_{\mu_j}$, *there exists a corner* $v_j \in \Sigma_j$ *such that* $TM(\mu_j, v_j, \Sigma_j) \geq \eta/4$. *Moreover,* $\mathrm{TA}(v_j) \geq \eta/6$.

Since we will use Lemma 3.7, this proof cannot be used for measures considered in [15, Theorem 12]. Note also that the choice of the denominator in $\mathrm{TA}(v_j) \geq \eta/6$ is quite arbitrary (and certainly not optimal), but acceptable for the purposes of this section (*any* lower bound estimate on $\mathrm{TA}(v_j)$ independent of j is sufficient for our purposes).

Proof. Fix an index j, and choose a minimizer $\Sigma \in \operatorname{argmin} E^\lambda_{\mu_j}$. Let $f : [0,1] \longrightarrow \Sigma$ be a constant speed bijective parameterization, and let $\{v_i\}_{i=1}^H$ be the set of corners receiving positive mass from B. Recall that Corollary 3.3 implies that such $\{v_i\}$ can talk only to mass supported in B.

Let $t_i := f^{-1}(v_i)$ and $M_i := TM(\mu_j, v_i, \Sigma)$. If there exist two indices i_1, i_2 such that $M_{i_1} + M_{i_2} \geq \eta/2$, then the proof is complete. Thus in the following we will assume

$$(\forall i_1, i_2,\ i_1 \neq i_2) \quad M_{i_1} + M_{i_2} \leq \eta/2. \tag{15}$$

The goal is to prove that this assumption leads to a contradiction.

Lemma 3.5 gives

$$\frac{M_i}{2\lambda} \leq \mathrm{TA}(v_i) \leq \frac{M_i}{\lambda}, \qquad i = 1, \cdots, H,$$

and combining with Lemma 3.2 gives

$$d(v_i, B) \geq \frac{1}{3} \sin \mathrm{TA}(v_i) \geq \frac{1}{6}\,\mathrm{TA}(v_j). \tag{16}$$

The last inequality holds since $\mathrm{TA}(v_j) \leq \eta/\lambda$, and with (C1) and (4) ensure that for any $x \in [0, \eta/\lambda]$ it holds $\sin x \geq x/2$. Let $l_i^\pm$ be the two half-lines forming the border $\partial V(v_i)$, Lemma 3.7 proves that $V(v_{i_1}) \cap V(v_{i_2}) = \emptyset$ whenever $i_1 \neq i_2$.

- Claim: for any corner v_i, except at most two, both half-lines $l_i^\pm$ must intersect the border ∂B.

Let v_{i_1}, v_{i_2} be the two corners for which (upon renaming) $l_{i_1}^+ \cap \partial B = l_{i_2}^+ \cap \partial B = \emptyset$ (clearly if such a couple v_{i_1}, v_{i_2} does not exist, then the claim is true). The goal is to prove that it does not exist a third corner v_{i_3} for which (upon renaming) $l_{i_3}^+ \cap \partial B = \emptyset$.

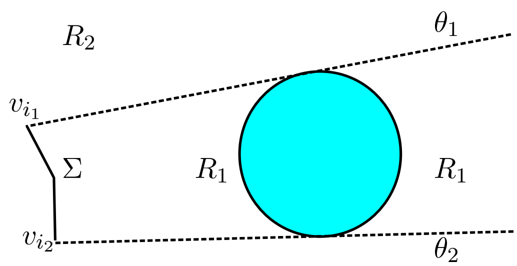

FIGURE 5. This is a schematic representation of the configuration.

Assume (upon renaming) $i_1 < i_2$, and both $l_{i_1}^-$ and $l_{i_2}^-$ intersect ∂B since:

- v_{i_1} and v_{i_2} receive mass from B, thus $V(v_{i_1}) \cap B$ and $V(v_{i_2}) \cap B$ are both non empty,
- $l_{i_1}^+$ and $l_{i_2}^+$ do not intersect ∂B,
- if $l_{i_1}^-$ (resp. $l_{i_2}^-$) does not intersect ∂B, then $B \subseteq V(v_{i_1})$ (resp. $B \subseteq V(v_{i_2})$) and Lemma 3.7 implies $B \cap V(v_{i_2}) = \emptyset$ (resp. $B \cap V(v_{i_1}) = \emptyset$). This is a contradiction.

Thus there exist half-lines $\theta_1 \subseteq V(v_{i_1})$ (resp. $\theta_2 \subseteq V(v_{i_2})$) starting in v_1 (resp. v_2) and tangent to ∂B. Note that

$$\mathbb{R}^2 \backslash (f([t_{i_1}, t_{i_2}]) \cup \theta_1 \cup \theta_2)$$

is divided in two connected components R_1 and R_2, of which (upon renaming) R_1 contains B. Note also that any half-line contained in R_1 must intersect ∂B.

Choose another corner v_{i_3}: since it talks to some mass in B, the intersection $V(v_{i_3}) \cap B$ is not empty, thus there exists a half-line $\phi \subseteq V(v_{i_3})$. Lemma 3.6 implies $V(v_{i_3}) \cap \Sigma = \{v_3\}$, and since $V(v_{i_3})$ is connected, it intersects B, but not $\theta_1 \cup \theta_2$ (Lemma 3.7). Thus it holds $V(v_{i_3}) \backslash \{v_{i_3}\} \subseteq R_1$. Since any half-line contained in R_1 must intersect ∂B, we conclude that both $l_{i_3}^\pm$ intersect ∂B, and the claim is proven.

Using (16) gives that any corner v_i such that both $l_i^\pm$ intersect ∂B satisfies

$$\min_{z \in l_i^-,\ |z - v_i| \geq 1/3} d(z, l_i^+) \geq \frac{1}{3} \sin \mathrm{TA}(v_i) \geq \frac{1}{6} \mathrm{TA}(v_i), \tag{17}$$

since Lemma 3.5 gives $\mathrm{TA}(v_i) \le \pi\eta/(2\lambda) \overset{(4),\,(C1)}{\le} \pi \cdot 10^{-100}/(1-3\cdot 10^{-100})$, and for any $x \in [0, \pi\eta/(2\lambda)]$ it holds $\sin x \ge x/2$. Since for any index i except at most two (which will be denoted by i' and i''), both $l_i^{\pm}$ intersect ∂B, choose $w_i^{\pm} \in l_i^{\pm} \cap \partial B$, and clearly $V(v_i) \cap \partial B$ is an arc connecting w_i^- and w_i^+. Combining with (17) gives

$$\mathcal{H}^1(V(v_i)\cap\partial B) \ge \min_{z\in l_i^-,\ |z-v_i|\ge 1/3} d(z, l_i^+) \ge \frac{1}{6}\,\mathrm{TA}(v_i),$$

and using Lemma 3.5 gives $\mathrm{TA}(v_i) \ge \dfrac{M_i}{2\lambda}$, i.e.

$$\mathcal{H}^1(V(v_i)\cap\partial B) \ge \frac{1}{6}\,\mathrm{TA}(v_i) \ge \frac{1}{6}\frac{M_i}{2\lambda} \ge \frac{1}{12}M_i. \tag{18}$$

Recalling that $V(v_{i_1}) \cap V(v_{i_2}) = \emptyset$ whenever $i_1 \ne i_2$, summing over indices $i \in \{1,\cdots,H\}\setminus\{i',i''\}$ gives

$$\mathcal{H}^1(\partial B) \ge \sum_{\substack{i=1\\ i\ne i',i''}}^{H} \mathcal{H}^1(V(v_i)\cap\partial B) \overset{(18)}{\ge} \sum_{\substack{i=1\\ i\ne i',i''}}^{H} \frac{1}{12}M_i \overset{(15)}{\ge} \frac{\eta}{24} \overset{(C1)}{>} 2\pi r = \mathcal{H}^1(\partial B),$$

which is a contradiction.

Thus there exist indices i', i'' such that $M_{i'} + M_{i''} \ge \eta/2$, i.e. $\max\{M_{i'}, M_{i''}\} \ge \eta/4$ independently of j. Using Lemma 3.5, we conclude that $\max\{\mathrm{TA}(v_{i'}), \mathrm{TA}(v_{i''})\} \ge \frac{\eta}{8\lambda}$, and since $8\lambda < 6$ in view of (4) and (C1), the proof is complete. ☐

Passing to the limit. Now we can pass to the limit $j \to +\infty$. The arguments we use are quite standard, and similar to those used in [15] (mainly Step 8 of Theorem 12). For any j choose $\Sigma_j \in \text{argmin } E^{\lambda}_{\mu_j}$, and since $\mu_j \overset{*}{\rightharpoonup} \mu$, upon subsequence it holds $\Sigma_j \overset{d_{\mathcal{H}}}{\to} \Sigma \in \text{argmin } E^{\lambda}_{\mu}$. Let

$$(j = 1, 2, \cdots) \quad f_j : [0,1] \longrightarrow \Sigma_j, \qquad f : [0,1] \longrightarrow \Sigma$$

be constant speed bijective parameterizations, such that $f_j \to f$ uniformly. Lemma 3.8 proves that for any j there exists a corner $v_j = f_j^{-1}(t_j)$ such that $\mathrm{TA}(v_j) \ge \eta/6$. In other words, the measure f_j'' has an atom of measure at least $\eta/6$ in t_j. Note that conditions of Lemma 2.5 are satisfied:

- $\sup_j \|f_j'\|_{TV} \le 1/\lambda$ in view of [11, Theorem 5.1],
- the minimality condition $\Sigma_j \in \text{argmin } E^{\lambda}_{\mu_j}$, $j = 1, 2, \cdots$ implies:
  - $\sup_j \mathcal{H}^1(\Sigma_j) < +\infty$, since the opposite would imply the existence of a subsequence $\{\Sigma_{j(k)}\}$ satisfying $\lambda\mathcal{H}^1(\Sigma_{j(k)}) \to +\infty$,
  - there exists a compact set K containing $\bigcup_j \Sigma_j$, since the opposite, i.e. there exists a subsequence $\{\Sigma_{j(k)}\}$ and a sequence $h_k \to +\infty$ such that $\Sigma_{j(k)} \cap \big(\mathbb{R}^2 \setminus B((0,0), h_k)\big) \ne \emptyset$ for any k, would imply

$$\lim_{k\to+\infty} \Big(\inf_{y\in\Sigma_{j(k)},\ z\in \mathrm{supp}(\mu_{j(k)})} |y-z| \Big) \ge \lim_{k\to+\infty} h_k - \sup_j \mathcal{H}^1(\Sigma_j) = +\infty,$$

hence $F_{\mu_{j(k)}}(\Sigma_{j(k)}) \to +\infty$.

Thus upon subsequence $t_j \to t$, and the convergence $f''_j \overset{*}{\rightharpoonup} f''$ (given by Lemma 2.5), implies that the measure f'' has an atom of size at least $\eta/6$ in t. Corollary 3.3 gives $f(t) \in \{-0.01 \le x \le 0.01\}$, thus $f(t)$ is not an endpoint. Since an atom for the measure f'' corresponds to a jump for the tangent derivative f', we conclude that Σ admits a corner in $f(t)$, with $\mathrm{TA}(f(t)) \ge \eta/6$.

Thus we have proven:

Theorem 3.9. *Let μ be the measure defined in* (3) *and λ the parameter defined in* (4). *Then there exists a minimizer $\Sigma \in \operatorname{argmin}\ E^\lambda_\mu$ containing a corner v with $\mathrm{TA}(v) \ge \eta/6$.*

Corollary 3.10. *The minimizer Σ from Theorem 3.9 is also minimizer for the constrained problem*

$$\min_{\mathcal{H}^1(\cdot)\le\mathcal{H}^1(\Sigma)} \int_{\mathbb{R}^2} d(x,\cdot)d\mu. \tag{19}$$

Proof. In [2] it has been proven that any minimizer $\tilde\Sigma$ of (19) satisfies $\mathcal{H}^1(\tilde\Sigma) = \mathcal{H}^1(\Sigma)$, thus if Σ is not a minimizer of (19), choosing Σ^* minimizer of (19) would give

$$\int_{\mathbb{R}^2} d(x,\Sigma^*)d\mu < \int_{\mathbb{R}^2} d(x,\Sigma)d\mu, \qquad \mathcal{H}^1(\Sigma^*) = \mathcal{H}^1(\Sigma),$$

contradicting $\Sigma \in \operatorname{argmin}\ E^\lambda_\mu$. □

4. Average distance problem among convex sets

The aim of this section is to analyze regularity properties of minimizers of Problem 1.2. In particular we construct a minimizer failing to be C^1 regular, under perimeter penalization only. Unfortunately, the arguments we use cannot be extended to the case of volume penalization. We recall if both perimeter and volume are penalized, it has been proven in [9] that minimizers can fail to be C^1.

The considered energy will be

$$\mathcal{E} = \mathcal{E}(\mu,\lambda) : \mathcal{C} \longrightarrow [0,\infty), \qquad \mathcal{E}(\mu,\lambda)(K) := \int_{\mathbb{R}^2} d(x,K)d\mu + \lambda\,\mathrm{Per}(K),$$

where $\mathcal{C}$ and $\mathrm{Per}(\cdot)$ have been defined in Problem 1.2, μ is a given measure and $\lambda > 0$ a given parameter. For the sake of brevity, we will omit writing the dependencies on $\mu,\ \lambda$ when no risk of confusion arises. Existence of minimizers, as proven in [9], follows from Blaschke and Gołab theorems.

Let

$$p_1 := (-\delta/2, 0), \quad p_2 := (\delta/2, 0), \quad p := (0,a), \quad \delta := 10^{-100}$$

$$\mu_{r,a,\eta} := \begin{cases} \dfrac{1-\eta}{2}\left(\dfrac{1}{\pi r^2}\mathcal{L}^2_{\llcorner B(p_1,r)} + \dfrac{1}{\pi r^2}\mathcal{L}^2_{\llcorner B(p_2,r)}\right) + \eta\left(\dfrac{1}{\pi r^2}\mathcal{L}^2_{\llcorner B(p,r)}\right) & \text{if } r>0 \\ \dfrac{1-\eta}{2}\left(\delta_{p_1}+\delta_{p_2}\right) + \eta\delta_p & \text{if } r=0. \end{cases} \tag{20}$$

Here for given point q, the notation "δ_q" denotes the Dirac measure in q. The exact value of δ is not relevant, but required to be "small" (its "smallness" will be used Lemmas 4.2 and 4.4, allowing for λ to be chosen such that $\lambda\delta$ is "small"). Parameters a, η, r will be determined later. Note that for any $r < \delta/4$, the balls $B(p_1, r)$, $B(p_2, r)$, $B(p, r)$ are mutually disjoint. The construction of the counterexample will be achieved over three steps:

(1) first, prove the existence of parameters λ, η, a, such that any minimizer of $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ contains $\{p_1, p_2\}$ but not p (Lemma 4.1),
(2) then, choose suitable r, approximate $\mu_{r,a,\eta}$ with a sequence of discrete measures $\mu_j \overset{*}{\rightharpoonup} \mu_{r,a,\eta}$, and prove that minimizers of $\mathcal{E}(\mu_j, \lambda)$ contain a corner with uniformly bounded amplitude (Lemma 4.7),
(3) finally, take the limit $j \to +\infty$.

The choice to approximate $\mu_{r,a,\eta}$ is advantageous since:

(i) given an atomic measure ν (i.e. ν is sum of finitely many Dirac measures) and parameter λ, there exists a *polygon* $K \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\nu, \lambda)$,
(ii) given sequences $\nu_j \overset{*}{\rightharpoonup} \nu$, $\{C_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\nu_j, \lambda)\}$, it holds (upon subsequence)

$$C_j \overset{d_{\mathcal{H}}}{\to} C \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\nu, \lambda).$$

The proof is identical to the case of average distance problem among trees (Problem 1.1), noting that the convex hull of finitely many points is a convex polygon.

Basic configuration. A key result is:

Lemma 4.1. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in* (20)*. Then there exist $\lambda, \eta > 0$ and $a > 1$ such that the* unique *minimizer of $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is an isosceles triangle $\triangle p_1 p_2 q$, with base $[\![p_1, p_2]\!]$ and $q = (0, q_y) \in \mathbb{R}^2$, $q_y \in (0, a)$.*

The proof will be split over several lemmas.

Lemma 4.2. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in* (20)*. Then for any $a \geq 1$, there exist $\lambda_0, \eta_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, $\eta \in (0, \min\{\eta_0, \lambda/2\})$, any minimizer $K \in$ argmin $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ contains $\{p_1, p_2\}$.*

Condition $\lambda > \eta/2$ will be crucial for the proof of Lemma 4.4.

Proof. Choose arbitrary $a \geq 1$. Note that

$$\mathcal{E}(\mu_{0,a,\eta}, \lambda)([\![p_1, p_2]\!]) = a\eta + \lambda\delta,$$

thus for any minimizer K it holds

$$\lambda \operatorname{Per}(K) \leq \mathcal{E}(\mu_{0,a,\eta}, \lambda)(K) \leq \mathcal{E}(\mu_{0,a,\eta}, \lambda)([\![p_1, p_2]\!]) = a\eta + \lambda\delta. \tag{21}$$

Let $\pi : \mathbb{R}^2 \longrightarrow K$ be the projection map, and assume (for the sake of contradiction) $p_1 \notin K$, i.e. $\pi(p_1) \neq p_1$. Let $e_1 := \dfrac{p_1 - \pi(p_1)}{|p_1 - \pi(p_1)|}$, and let e_2 be a unit vector orthogonal to e_1. Consider the family of linear applications

$$T_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad T_\varepsilon e_i = (1 + \varepsilon) e_i, \quad i = 1, 2.$$

By construction it holds

$$(\forall \varepsilon > 0)(\forall E \subseteq \mathbb{R}^2) \quad E \text{ convex} \Longrightarrow T_\varepsilon E \text{ convex}.$$

Moreover

$$F_{\mu_0}(K) - F_{\mu_0}(T_\varepsilon K) \geq \mu_0(\{p_1\})\varepsilon = \frac{1-\eta}{2}\varepsilon, \qquad \mathrm{Per}(T_\varepsilon K) - \mathrm{Per}(K) = \varepsilon\,\mathrm{Per}(K) + o(\varepsilon) \tag{22}$$

This yields

$$\lambda\,\mathrm{Per}(T_\varepsilon K) - \lambda\,\mathrm{Per}(K) \leq \lambda\varepsilon\,\mathrm{Per}(K) \leq (a\eta + \lambda\delta)\varepsilon,$$

thus there exist λ^*, $\eta^* = \eta^*(a)$ such that for any $\lambda \in (0, \lambda^*)$, $\eta \in (0, \eta^*)$ it holds

$$(a\eta + \lambda\delta)\varepsilon \leq 0.3\varepsilon. \tag{23}$$

Let $\lambda_0 := \lambda^*$ and $\eta_0 := \min\{\eta^*, 0.6\}$. Combining (23) and (22) (for $\lambda \in (0, \lambda_0)$, $\eta \in (0, \eta_0)$ satisfying $\lambda > \eta/2$) yields

$$\lambda\,\mathrm{Per}(T_\varepsilon K) - \lambda\,\mathrm{Per}(K) \leq (a\eta + \lambda\delta)\varepsilon \leq 0.3\varepsilon < \frac{1-\eta}{2}\varepsilon \leq F_{\mu_0}(K) - F_{\mu_0}(T_\varepsilon K),$$

contradicting the minimality of K, and concluding the proof. □

The values 0.3 and 0.6 appearing in the proof are arbitrary, the key point is that by choosing suitable λ and η it holds $a\eta + \lambda\delta \leq (1-\eta)/2$.

Corollary 4.3. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in* (20)*. Then there exist η, λ, a, satisfying $\lambda > \eta/2$, such that any minimizer $K \in$ argmin $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is an isosceles triangle with base $[\![p_1, p_2]\!]$.*

Proof. Lemma 4.2 implies that for suitable choice of parameters η, λ, a, any minimizer $K \in$ argmin $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ contains $\{p_1, p_2\}$. Since for any convex set E the projection map $\pi_E : \mathbb{R}^2 \longrightarrow E$ is well defined, it follows that any minimizer $K \in$ argmin $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ should be the convex hull of three points (namely p_1, p_2 and $\pi_K(p)$), i.e. a triangle with an edge $[\![p_1, p_2]\!]$. Since p lies on the axis of $[\![p_1, p_2]\!]$, and for any triangle with fixed base and height the isosceles one minimizes the perimeter, the proof is complete. □

Lemma 4.4. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in* (20)*. Then there exist η, λ, a, satisfying $\lambda > \eta/2$ and* (23)*, such that any minimizer $K \in$ argmin $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is a non degenerate triangle not containing the point $p = (0, a)$.*

Before the proof, note that a sufficient condition for (23) is $a\eta < 0.15$ and $\lambda\delta < 0.15$. Since $\delta = 10^{-100}$, for any $\lambda < 1$, condition $\lambda\delta < 0.15$ is satisfied (both values 0.15 are used here simply because $0.15 + 0.15 \leq 0.3$, with 0.3 appearing in (23)). A potential issue can be present when choosing η and a, since it is required that "choosing η small does not force to choose a large". This will be essentially the main point of this lemma.

Proof. Corollary 4.3 implies that there exist η, λ, a, such that any minimizer $K \in$ argmin $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is a non degenerate isosceles triangle with base $[\![p_1, p_2]\!]$. Let h be its height (relative to the base $[\![p_1, p_2]\!]$), and the third vertex has the form $q_h := (0, h) \in \mathbb{R}^2$. Direct computation gives

$$\psi(h) := \mathcal{E}(\mu_{0,a,\eta}, \lambda, \alpha)(\triangle p_1 p_2 q_h) = (a-h)\eta + \lambda(\delta + \sqrt{\delta^2 + 4h^2})$$

and

$$\frac{d}{dh}\psi(h) = -\eta + \frac{4h\lambda}{\sqrt{\delta^2 + 4h^2}}, \tag{24}$$

thus the optimal value for h is

$$h^* = 4\delta\left(\frac{4\lambda^2}{\eta^2} - 1\right).$$

Choosing $\lambda > \eta/2$ guarantees $h^* > 0$. Note that further imposing $\lambda/\eta \le 10^3$ yields $h^* \le 4\delta\sqrt{4\cdot 10^6 - 1}$ (10^3 is again an arbitrary value, and the particular form of the upper bound for h^* is not relevant, the key point is that h^* can be just bounded from above) , thus choosing $a \in [3,5]$ (the extremes 3 and 5 are arbitrary, the key point is $a > h^*$, which is satisfied since $h^* \le 400\delta < 3$), $\eta < 0.15/5$, $\lambda \le 1$ gives

$$a\eta < 0.15, \qquad \lambda\delta < 0.15,$$

hence the compatibility with (23). Since the unique minimizer of $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is a triangle with vertices p_1, p_2, $q_{h^*} := (0, h^*)$, and we just proved $h^* < a$, it follows $p = (0,a) \notin \triangle p_1p_2q_{h^*}$, concluding the proof. □

Proof. (of Lemma 4.1) The proof follows by combining Lemmas 4.2, 4.4 and Corollary 4.3, and noting that these are valid if λ, η, a satisfy:

$$a\eta < 0.15, \quad \lambda\delta < 0.15, \quad 3 \le a \le 5, \quad 0.7 \le \frac{\lambda}{\eta} \le 10^3 < \frac{1}{2}\sqrt{1 + \frac{1}{2\delta}}.$$

Here the value 0.7 is used only to ensure $\lambda > 2\eta$, while $\lambda/\eta \le 10^3$ is used to ensure $h^* < 3 \le a$ (note that with such bounds we get $h^* = 4\delta(4\lambda^2/\eta^2 - 1) \overset{\delta=10^{-100}}{\le} 4\sqrt{4\cdot 10^6 - 1}\cdot 10^{-100} < 3 \le a$). Since there exist triplets (λ, η, a) satisfying these conditions, the proof is complete. □

Note that it is possible to further impose

$$\frac{\eta}{\lambda} \le 0.01.$$

This condition will be used in Lemma 4.7. The value 0.01 is arbitrary here, and we will only use the fact that for any $x \in [0, 0.01]$ it holds $\sin x \ge x/2$.

Construction of the counterexample. Choose parameters λ, η, a such that any minimizer $K \in \text{argmin } \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ satisfies $0 < d(p, K) =: b$ (this choice is possible due to Lemma 4.1). Note there exists r such that for any minimizer $K \in \text{argmin } \mathcal{E}(\mu_{r,a,\eta}, \lambda)$ it holds:

- distance estimate:
$$d_{\mathcal{H}}(B(p,r), K) \ge b/4, \tag{25}$$
- for any minimizer $C_j \in \text{argmin}_{\mathcal{C}}\, \mathcal{E}(\mu_j, \lambda)$, the "projection" sets
$$\bigcup_{z\in B(p,r)} \text{argmin}_{w\in\partial K} |z - w|, \qquad \bigcup_{z'\in B(p',r)} \text{argmin}_{w\in\partial K} |z' - w|, \qquad \bigcup_{z''\in B(p'',r)} \text{argmin}_{w\in\partial K} |z'' - w|$$
  are mutually disjoint. This is possible in view of Lemma 4.1,
- and
$$2\pi r < \frac{b\eta}{48\lambda}. \tag{26}$$
  This condition will be used in Lemma 4.7.

Choose λ, η, a, r such that all the conditions and results mentioned (until now) in this section hold. From now parameters λ, η, a, r will be fixed.

Similarly to (6), let

$$\mu_j := \frac{1-\eta}{2}\Big(\frac{1}{\sharp(B(p',r)\cap\frac{1}{j}\mathbb{Z}^2)}\sum_{x\in B(p',r)\cap\frac{1}{j}\mathbb{Z}^2}\delta_x\Big)+\frac{1-\eta}{2}\Big(\frac{1}{\sharp(B(p'',r)\cap\frac{1}{j}\mathbb{Z}^2)}\sum_{x\in B(p'',r)\cap\frac{1}{j}\mathbb{Z}^2}\delta_x\Big)$$
$$+\eta\Big(\frac{1}{\sharp(B(p,r)\cap\frac{1}{j}\mathbb{Z}^2)}\sum_{x\in B(p,r)\cap\frac{1}{j}\mathbb{Z}^2}\delta_x\Big). \tag{27}$$

The results we use to analyze minimizers of $\mathcal{E}(\mu_j, \lambda)$ are adapted versions of Lemmas 3.5, 3.7 and 3.8.

Lemma 4.5. *Consider the family of measures* $\{\mu_j\}$ *defined in* (27). *Then for any index* j*, there exists a convex* polygon K *minimizing* $\mathcal{E}(\mu_j, \lambda)$ *and satisfying*

$$(\forall v \in \bigcup_{z\in B(p,r)} \operatorname{argmin}_{w\in\partial K}|z-w|)\quad \frac{M_v}{2\lambda} \le \mathrm{TA}(v) \le \frac{\pi M_v}{2\lambda},$$

where $M_v := TM(\mu, v, \partial K)$*.*

Proof. The proof is done by adapting the arguments from Lemma 3.5, to deal with the convexity constraint. Let K be a convex polygon minimizing $\mathcal{E}(\mu_j, \lambda)$, $v \in K$ be an arbitrary corner receiving mass from $B(p, r)$. Choose $v_1, v_2 \in \partial K$ such that $[\![v_1, v]\!]$, $[\![v_2, v]\!]$ are straight segments and $|v_1 - v| = |v_2 - v| > 0$. Note that it is possible to choose such points v_1, v_2 exactly because K is a convex polygon.

- Upper bound estimate $\mathrm{TA}(v) \le \dfrac{\pi M_v}{2\lambda}$.

Consider the modification in Figure 6. The point $v_s^- \in K$ is chosen on the bisector of the angle $\widehat{v_1vv_2}$ such that $|v - v_s^-| = s$ (s is a free parameter, and we will be interested in the behavior for small s). Define the competitor

$$K_s := \operatorname{conv}\Big(\big(\partial K\backslash([\![v_1,v]\!]\cup[\![v_2,v]\!])\big)\cup[\![v_1,v_s^-]\!]\cup[\![v_2,v_s^-]\!]\Big),$$

where $\operatorname{conv}(\cdot)$ denotes the convex hull. By construction

$$\partial K_s = \partial K\backslash([\![v_1,v]\!]\cup[\![v_2,v]\!])\cup([\![v_1,v_s^-]\!]\cup[\![v_2,v_s^-]\!]). \tag{28}$$

Direct computation gives

$$\mathrm{Per}(K)-\mathrm{Per}(K_s)=s\sin\frac{\mathrm{TA}(v)}{2}+o(s),\qquad \int_{\mathbb{R}^2} d(x,K_s)d\mu_j \le \int_{\mathbb{R}^2} d(x,K)d\mu_j + M_v s,$$

and using the minimality of K gives the upper bound estimate.

- Lower bound estimate $\mathrm{TA}(v) \ge \dfrac{M_v}{2\lambda}$.

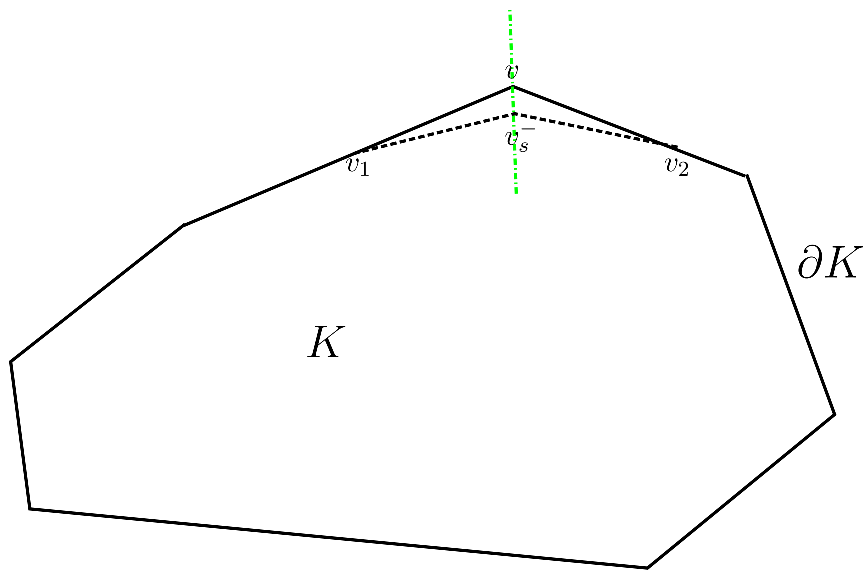

FIGURE 6. This is a schematic representation of the considered variation. The green dash-dotted line is the bisector of the angle $\widehat{v_1 v v_2}$.

Consider the modification in Figure 7. The point $v_s^+ \notin K$ is chosen on the bisector of the angle $\widehat{v_1 v v_2}$ such that $|v - v_s^+| = s$ (s is a free parameter, and we will be interested in the behavior for small s). Let

$$\tilde{K}_s := \text{conv}\left(\left(\partial K \backslash ([\![v_1, v]\!] \cup [\![v_2, v]\!])\right) \cup [\![v_1, v_s^+]\!] \cup [\![v_2, v_s^+]\!] \right).$$

For the sake of brevity, given points $w, z \in \partial K$, the notation $[\![w, z]\!]_{\partial K}$ will denote the unique *clockwise* (this is well defined since we endowed $\mathbb{R}^2$ with an orthogonal coordinate system, and ∂K is homeomorphic to the unit circle S^1) path in ∂K with endpoints in w and z. In this case

$$\partial \tilde{K}_s = \partial K \backslash ([\![w_1, v]\!]_{\partial K} \cup [\![v, w_2]\!]_{\partial K}) \cup [\![w_1, v_s^+]\!]_{\partial K} \cup [\![v_s^+, w_2]\!]_{\partial K},$$

where w_1 and w_2 are the intersections between ∂K and the two half-lines lines starting in v_s^+ and tangent to K. Direct computation gives

$$\int_{\mathbb{R}^2} d(x, \tilde{K}_s) d\mu_j \leq \int_{\mathbb{R}^2} d(x, K) - M_v s \cos \frac{\text{TA}(v)}{2}. \tag{29}$$

By construction it holds

$$\text{Per}(\tilde{K}_s) \leq \mathcal{H}^1\left(\left(\partial K \backslash ([\![v_1, v]\!] \cup [\![v_2, v]\!])\right) \cup [\![v_1, v_s^+]\!] \cup [\![v_2, v_s^+]\!] \right), \tag{30}$$

and direct computation gives

$$\mathcal{H}^1\left(\left(\partial K \backslash ([\![v_1, v]\!] \cup [\![v_2, v]\!])\right) \cup [\![v_1, v_s^+]\!] \cup [\![v_2, v_s^+]\!] \right) - \text{Per}(K) = s \sin \frac{\text{TA}(v)}{2} + o(s). \tag{31}$$

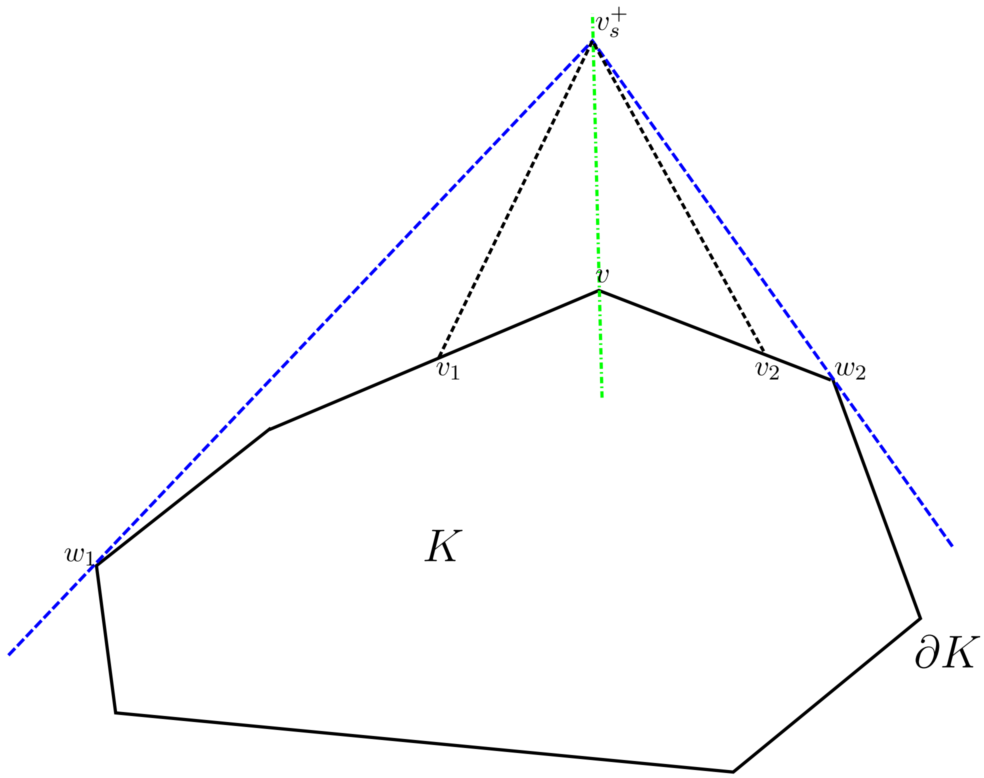

FIGURE 7. This is a schematic representation of the considered variation. The quantity $|v_s^+ - v| = s$ has been purposely exaggerated for the sake of clarity. The green dash-dotted line is the bisector of the angle $\widehat{v_1 v v_2}$.

Combining (29), (30), (31) with the minimality of K (compared against $\tilde{K}_s$) gives the desired inequality. □

Lemma 4.6. *Consider the family of measures $\{\mu_j\}$ defined in* (27). *Then for any sufficiently large index j, there exists a polygon $K_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)$ satisfying:*

- *given distinct corners $v_1, v_2 \in \bigcup_{z \in B(p,r)} \operatorname{argmin}_{w \in \partial C_j} |z - w|$, i.e. v_1, v_2 receive mass only from $B(p,r)$, it holds $V(v_1) \cap V(v_2) = \emptyset$.*

Proof. Note that

- r has been chosen such that (25) holds,
- for any sequence of minimizers $\{C_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)\}$ it holds (upon subsequence)

$$C_j \overset{d_{\mathcal{H}}}{\to} C \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_{r,a,\eta}, \lambda).$$

Thus for sufficiently large j, any minimizer $C_j \in \operatorname{argmin} \mathcal{E}(\mu_j, \lambda)$ satisfies $d_{\mathcal{H}}(C_j, B(p,r)) \geq b/8$ (for the definition of b, see the arguments immediately before (25)). Note also that (upon choosing sufficiently large index j), for any minimizer $C_j \in \text{argmin } \mathcal{E}(\mu_j, \lambda)$, the sets

$$\bigcup_{z\in B(p,r)} \text{argmin}_{w\in\partial C_j} |z-w|, \qquad \bigcup_{z'\in B(p',r)} \text{argmin}_{w\in\partial C_j} |z'-w|, \qquad \bigcup_{z''\in B(p'',r)} \text{argmin}_{w\in\partial C_j} |z''-w|$$

are mutually disjoint. Intuitively, this implies that any point of ∂C_j receives mass from at most one of the balls $B(p,r)$, $B(p',r)$, $B(p'',r)$. Then the conclusion follows by using the same construction from the proof of Lemma 3.7, which can be applied without modification since it preserves convexity. □

Lemma 4.7. *Consider the family of measures $\{\mu_j\}$ defined in* (27)*. Then for any sufficiently large index j, there exists a minimizer $K_j \in$ argmin $\mathcal{E}(\mu_j, \lambda)$ satisfying:*

- *there exists a corner $v_j \in \partial K_j$, receiving mass from $B(p,r)$, such that* $\text{TA}(v_j) \geq \eta/(8\lambda)$.

Again the denominator 8λ is quite arbitrary, but sufficient for the purposes of this section (indeed any positive lower bound to $\text{TA}(v_j)$ independent of j, and valid for any sufficiently large j, is sufficient). The proof follows by applying straightforwardly the same argument from the proof of Lemma 3.8, with the roles of Lemmas 3.5 and 3.7 replaced by Lemmas 4.5 and 4.6. However, since this result is crucial for the purposes of this section, we will report its proof.

Proof. Let $B := B(p,r)$. Consider an index j, a polygon $K_j \in \text{argmin}_{\mathcal{C}}\, \mathcal{E}(\mu_j, \lambda)$, and let $\{v_i\}_{i\in\mathcal{I}} \subseteq K_j$ be the (finite) set of corners receiving mass from B, with $\mathcal{I}$ a suitable set of indices. Similarly to the proof of Lemma 3.8, for any index $i \in \mathcal{I}$ let $V(v_i)$ be the wedge of v_i, and let $l_i^\pm$ the two half-lines (the order is not relevant) forming the border $\partial V(v_i)$.

Again it holds (with the same proof from Lemma 3.8):

- for any index $i \in \mathcal{I}$, except at most two, both half-lines $l_i^\pm$ must intersect ∂B.

Let $M_i := TM(\mu_j, v_i, \partial K_j)$ ($i \in \mathcal{I}$). If there exists a couple of indices $i', i'' \in \mathcal{I}$ such that $M_{i'} + M_{i''} \geq \eta/2$, then the proof is complete. Thus assume:

$$(\forall i', i'' \in \mathcal{I},\ i'' \neq i'') \quad M_{i'} + M_{i''} \leq \eta/2. \tag{32}$$

The goal is to prove that (32) gives a contradiction.

Let $\mathcal{J} \subseteq \mathcal{I}$ be the set of indices i such that both half-lines $l_i^\pm$ intersect ∂B. This implies that there exist points $p_i^\pm \in l_i^\pm \cap \partial B$; for any index $i \in \mathcal{J}$ choose an arc of minimal length $\phi_i \subseteq \partial B \cap V(v_i)$ connecting p_i^- and p_i^+. Clearly $\mathcal{H}^1(\phi_i) \geq |p_i^- - p_i^+|$. However, since $d_{\mathcal{H}}(K_j, B) \geq b/8$ and $\text{TA}(v_i) \geq M_i/(2\lambda)$ (Lemma 4.5), elementary geometry (combined with the fact that M_i and $\text{TA}(v_i)$ are very small) gives

$$|p_i^- - p_i^+| \geq |v_i - p_i^-| \sin \text{TA}(v_i) \geq \frac{b}{8} \sin \text{TA}(v_i) \geq \frac{b}{12}\, \text{TA}(v_i) \geq \frac{bM_i}{24\lambda}.$$

The last inequality hold since $\text{TA}(v_i) \leq M_i/\lambda \leq \eta/\lambda \leq 0.01$, hence Lemma 4.5 can be applied. The second-to-last inequality holds since we imposed $\eta/\lambda \leq 0.01$, for any $x \in [0, \eta/\lambda]$ it holds $\sin x \geq x/2$.

Lemma 4.6 gives that the wedge of distinct corners are disjoint, and in particular the arcs ϕ_i ($i \in \mathcal{J}$) are mutually disjoint. Thus summing over indices $i \in \mathcal{J}$ gives

$$\sum_{i\in\mathcal{J}} \mathcal{H}^1(\phi_i) \geq \frac{b}{24\lambda}\sum_{i\in\mathcal{J}} M_i \overset{(32)}{\geq} \frac{b\eta}{48\lambda}, \tag{33}$$

while by construction it holds $\phi_i \subseteq \partial B$ ($i \in \mathcal{J} \subseteq \mathcal{I}$), yielding

$$\sum_{i\in\mathcal{J}} \mathcal{H}^1(\phi_i) \leq 2\pi r. \tag{34}$$

Combining inequalities (33) and (34) gives

$$2\pi r = \mathcal{H}^1(\partial B) \geq \sum_{i\in\mathcal{J}} \mathcal{H}^1(\phi_i) \geq \frac{b\eta}{48\lambda},$$

which contradicts condition (26). Thus there exists a couple of indices $i', i'' \in \mathcal{I}$ such that $M_{i'} + M_{i''} \geq \eta/2$, hence $\max\{M_{i'}, M_{i''}\} \geq \eta/4$, and using Lemma 4.5 gives $\max\{\mathrm{TA}(v_{i'}), \mathrm{TA}(v_{i''})\} \geq \eta/(8\lambda)$, concluding the proof. □

Now it is possible to pass to the limit: for any j choose a minimizer $K_j \in$ argmin $\mathcal{E}(\mu_j, \lambda)$ such that the conclusion of Lemma 4.7 holds. Let $\varphi_j : [0,1] \longrightarrow \partial K_j$ be a constant speed parameterization, and it is clear that

$$\sup_j \|\varphi_j\|_{L^1} < +\infty, \qquad \sup_j \|\varphi_j\|_{TV} < +\infty,$$

since the former (which is a uniform bound on perimeter) follows from the minimality of K_j, and the latter (which is a uniform bound on curvature) follows from the convexity of K_j.Note also

- there exists a compact set containing $\bigcup_j K_j$.

Indeed the uniform bound on perimeters $\sup_j \mathcal{H}^1(K_j) < +\infty$ gives also a uniform bound on diameters (since for any convex set its diameter does not exceed its perimeter), hence if there exists a subsequence (which we do not relabel) $\{\Sigma_j\}$ and a sequence $\{w_j\}$ satisfying

$$w_j \in K_j, \qquad |w_j| \to +\infty,$$

then

$$\inf_{w\in K_j,\, z\in \mathrm{supp}(\mu_j)} |w-z| \to +\infty \Longrightarrow E^\lambda_{\mu_j}(K_j) \geq \int_{\mathbb{R}^2} d(x, K_j)d\mu_j \geq \inf_{w\in K_j,\, z\in \mathrm{supp}(\mu_j)} |w-z| \to +\infty.$$

This contradicts $K_j \in$ argmin $E^\lambda_{\mu_j}$ for any sufficiently large j, since supp$(\mu_j) \subseteq B \cup B' \cup B''$ gives

$$\sup_j \min E^\lambda_{\mu_j} \leq \sup_j E^\lambda_{\mu_j}(\{(0,0)\}) \leq \int_{\mathbb{R}^2} d(x, \{(0,0)\})d\mu_j \leq 1 + r.$$

Upon subsequence $K_j \overset{d_\mathcal{H}}{\to} K$, with K convex, thus $\partial K_j \overset{d_\mathcal{H}}{\to} \partial K$. Lemma 2.5 gives (upon subsequence) the existence of a limit curve $\varphi = \lim_j \varphi_j$ (this limit is taken in the C^0 topology) parameterizing ∂K. Since for any j, the measure φ''_j (which does not change sign due to the convexity of K_j) has an atom of measure at least $\eta/(8\lambda)$ at some time t_j, and (upon subsequence) $t_j \to t$, the convergence (upon subsequence) $\varphi''_j \overset{*}{\rightharpoonup} \varphi''$ implies that the measure φ'' has an atom of size at least $\eta/(8\lambda)$ at time t. Since an atom for the curvature measure φ'' is equivalent to a jump for the tangent derivative φ', it follows:

Theorem 4.8. *Let $\mu_{r,a,\eta}$ be the measure defined in* (20). *Then for suitable choice of parameters λ, a, η, r, there exists a minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{r,a,\eta}, \lambda)$ whose border ∂K is not C^1 regular.*

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