Global regularity and stability of solutions to the single and double obstacle problems with nonstandard growth

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Abstract

We study the regularity properties of solutions to the single and double obstacle problem with non standard growth. Our main results are a global reverse Hölder inequality, Hölder continuity up to the boundary, and stability of solutions with respect to continuous perturbations in the variable growth exponent.

Key words: global higher integrability, boundary regularity, obstacle problem, Hardy inequality
2000 MSC: 35J60, 35B20, 35J25, 46E35

1. Introduction

The obstacle problem is, roughly speaking, solving a partial differential equation with the additional constraint that the solution is required to stay above a given function, the obstacle. This leads to a variational inequality. From a minimization point of view, the problem is to find a minimizer with fixed boundary values in the set of functions lying above the obstacle function. Such a set is convex, and thus a unique minimizer exists under reasonable assumptions. The potential theoretic viewpoint to the obstacle problem is finding the smallest superharmonic function which lies above the obstacle. This is the balayage concept of potential theory. Finally, the double obstacle problem adds another constraint, the requirement that the solution must also stay below another given function.

In this paper we deal with the single and double obstacle problems associated to quasi-linear elliptic equations

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = 0 \tag{1.1}$$

with non-standard structural conditions. These conditions involve a variable growth exponent $p(\cdot)$. The prototype of such equations is the $p(\cdot)$-Laplacian

$$-\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = 0. \tag{1.2}$$

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Our interests are regularity up to the boundary, and stability with respect to perturbations in the growth exponent $p(\cdot)$. We work almost exclusively with the definition in terms of a variational inequality.

In the last fifteen years or so, there has been a growing interest in the calculus of variations and nonlinear partial differential equations with nonstandard growth conditions, and the related Lebesgue and Sobolev spaces with variable exponent. This is not only a matter of purely mathematical interest, but also applications to elasticity [42], non-Newtonian fluids [40], and image processing [4] have been proposed. The interested reader can find numerous further references in the overview paper [22] and the monograph [6].

Recently some papers appeared in the case of obstacle problems with non standard growth. See [38, 37], for existence and uniqueness of an entropy solution, in the framework of Lewy-Stampacchia inequalities. A treatment of the double obstacle problem in Orlicz-Sobolev spaces can be found in [39]. For work on regularity results, see [8, 9] for Hölder continuity results in the setting of Morrey and Campanato spaces, and [10] for gradient estimates of Calderón-Zygmund type. The balayage related to supersolutions of (1.2), and its relation to the obstacle problem is considered in [31]. Finally, we mention the paper [21] where the obstacle problem is employed as a tool to develop nonlinear potential theory for (1.2), in the spirit of [25].

The current paper aims at complementing the paper [21] by extending the results for weak solutions in [11] to cover the obstacle problem. More specifically, we prove global higher integrability of the gradient, Hölder continuity up to the boundary and stability of the solutions to the single and double obstacle problem with respect to continuous perturbations in the growth exponent $p(x)$ for solutions of single and double obstacle problems.

It is well known that regularity results for problems of $p(\cdot)$-growth require some assumption on the function $p(\cdot)$. We make the standard assumption on $p(\cdot)$, the so called logarithmic Hölder continuity condition. This condition was introduced by Zhikov [43] in the context of Lavrentiev phenomenon, and it has turned to be very useful in regularity problems and in other applications. Indeed, the condition turns up quite naturally in the estimates of the De Giorgi and Moser methods, and there are very few regularity results that do not assume logarithmic Hölder continuity.

In order to consider properties of solutions up to the boundary, we need a suitable hypothesis on the domain. Here we assume that the complement of the domain satisfies a measure density condition. This condition is rather standard in regularity theory. For our purposes, the most important consequences of it are that Sobolev–Poincaré inequalities can be applied up to the boundary, and that the boundary estimates from [32] become available.

Our stability results concerns perturbations in the growth exponent $p(\cdot)$. More specifically, we consider a sequence $p_i(\cdot)$ of variable exponents converging uniformly to the function $p(\cdot)$, and show that the corresponding sequence of solutions with fixed boundary values and obstacles converges, up to subsequences, to the solution of the limit problem. The chief technical problem here is that changing the growth exponent changes the underlying Sobolev space in which the solutions lie. Higher integrability is the key tool in dealing with this difficulty, as it allows working in a fixed Sobolev space.

The paper is organized as follows. In Section 2 we present the preliminaries about the obstacle problem. In Sections 3, 4 and 5 we consider the single obstacle problem results for the solution to the obstacle problem. More precisely, Section 3 contains a higher integrability result, Section 4 the Hölder continuity up to the boundary, and Section 5 deals with the stability. Finally, in Section 6 we discuss how these results are extended to the double obstacle problem.

2. The obstacle problem

In this section, we discuss the definition of solutions to the obstacle problem. To this end, we also introduce some notation, and recall a number of other definitions and facts.

We call a bounded measurable function $p : \mathbb{R}^n \to (1, \infty)$, $n \geq 2$, a variable exponent. We denote

$$p_E^- = \inf_{x \in E} p(x), \text{ and } p_E^+ = \sup_{x \in E} p(x),$$

where E is a measurable subset of $\mathbb{R}^n$. We assume that $1 < p_\Omega^- \leq p_\Omega^+ < \infty$, where Ω is an open, bounded subset of $\mathbb{R}^n$. We abbreviate $p^- := p_{\mathbb{R}^n}^-$ and $p^+ := p_{\mathbb{R}^n}^+$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f defined on Ω for which

$$\int_\Omega |f|^{p(x)} \, \mathrm{d}x < \infty.$$

The Luxemburg norm on this space is defined as

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space, see [30]. For a constant function $p(\cdot)$ the variable exponent Lebesgue space coincides with the standard Lebesgue space. The conjugate exponent $p'(\cdot)$ is defined pointwise by $1/p(x) + 1/p'(x) = 1$. The Hölder inequality

$$\int_\Omega fg \, \mathrm{d}x \leq C \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

holds for functions $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $f \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇f exists and satisfies $|\nabla f| \in L^{p(\cdot)}(\Omega)$. This space is a Banach space with the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{p(\cdot)} + \|\nabla f\|_{p(\cdot)}.$$

For basic properties of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, we refer to [6, 30].

Smooth functions are not dense in $W^{1,p(\cdot)}(\Omega)$ without additional assumptions on the exponent $p(\cdot)$. This was observed by Zhikov [43, 44] in the context of the Lavrentiev phenomenon, which means that minimal values of variational integrals may differ depending on whether one minimizes over smooth functions or Sobolev functions. Zhikov has also introduced the logarithmic Hölder continuity condition to rectify this. The condition is

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)} \tag{2.1}$$

for all $x, y \in \Omega$ such that $|x - y| \leq 1/2$. If the exponent is bounded and satisfies (2.1), smooth functions are dense in variable exponent Sobolev spaces and we can define the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. We refer to [5, 6, 20, 24, 41] for density results in variable exponent Sobolev spaces.

We will use logarithmic Hölder continuity in the form

$$R^{-(p_B^+ - p_B^-)} \leq C, \tag{2.2}$$

where $B = B(x_0, 2R) \Subset \Omega$. It is well-known that requiring (2.2) to hold for all such balls is equivalent with condition (2.1); a proof of this is given in [5, Lemma 3.2]. An elementary consequence of (2.2) is the inequality

$$C^{-1} R^{-p(y)} \leq R^{-p(x)} \leq C R^{-p(y)}, \tag{2.3}$$

which holds for any points $x, y \in B(x_0, 2R)$ with a constant depending only on the constant of (2.2). We use phrases like "by log-Hölder continuity" when applying either (2.2) or (2.3). Further, expressions such as "the constant depends on p" are taken to mean a dependency on p^+, p^-, and the log-Hölder constant of p.

We need the following assumptions, with strictly positive constants α and β, to hold for the operator $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$.

1. $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$,
2. $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for almost all $x \in \Omega$,
3. $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^{p(x)}$ for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
4. $|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p(x)-1}$ for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
5. $(\mathcal{A}(x, \eta) - \mathcal{A}(x, \xi)) \cdot (\eta - \xi) > 0$ for all $x \in \Omega$ and $\eta \neq \xi \in \mathbb{R}^n$.

We may assume that $\alpha \leq \beta$ by choosing β larger if necessary. These are called the structure conditions of $\mathcal{A}$.

In this article, we always assume that $p(\cdot)$ is log-Hölder continuous with $1 < p^- \leq p^+ < \infty$ and that Ω is a bounded open set in $\mathbb{R}^n$

The above structural conditions imply that we can define solutions in the weak sense in the space $W^{1,p(\cdot)}(\Omega)$. More precisely, a function $u \in W^{1,p(\cdot)}(\Omega)$ is a weak solution to

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = 0,$$

if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = 0$$

for all test functions $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Further, u is a weak supersolution if the above integral is nonnegative for all nonnegative test functions φ, and a weak subsolution if it is nonpositive. By regularity theory [1, 2, 3, 12], there is a locally Hölder continuous reprensentative of a weak solution.

Let $\psi : \Omega \to [-\infty, \infty)$ be a function, called *an obstacle*; let $f \in W^{1,p(\cdot)}(\Omega)$ be a function which gives the boundary values. Define

$$\mathcal{K}_\psi^{f,p(\cdot)}(\Omega) = \{u \in W^{1,p(\cdot)}(\Omega) : \ u - f \in W_0^{1,p(\cdot)}(\Omega), \ u \geq \psi, \ \text{a.e. in } \Omega\}.$$

We say that a function $u \in \mathcal{K}_{\psi}^{f,p(\cdot)}(\Omega)$ is a *solution to the obstacle problem* $\mathcal{K}_{\psi}^{f,p(\cdot)}(\Omega)$ if

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla(v-u) \, \mathrm{d}x \geq 0 \tag{2.4}$$

for every $v \in \mathcal{K}_{\psi}^{f,p(\cdot)}(\Omega)$.

All basic properties of solutions to the obstacle problem follow as in [25], Chapter 3. In particular, we note the fact that solutions to obstacle problems are always weak supersolutions.

Let us now take care of the matter of existence and uniqueness of the solution to the obstacle problem. If we make the mild additional assumption that the operator $\mathcal{A}$ comes from an Euler–Lagrange equation of a strictly convex variational integral, such as the $p(\cdot)$-Dirichlet integral, then existence and uniqueness follow from the fact that we are minimizing over the convex set $\mathcal{K}_{\psi}^{f,p(\cdot)}$. Alternatively, we may appeal to abstract functional analysis results about monotone operators as in [36].

Remark 2.1. Let us notice that, by replacing f by $f_1 = \max\{f, \psi\}$ we may assume that the boundary value function f satisfies $f \geq \psi$ in Ω. Indeed $f_1 = (\psi - f)^+ + f$ and since

$$0 \leq (\psi - f)^+ \leq (u-f)^+ \in W_0^{1,p}(\Omega),$$

the function $(\psi - f)^+$, and hence $u - f_1$ belongs to $W_0^{1,p}(\Omega)$. Similar considerations hold for the double obstacle problem. Here we denote $(f)^- := -\min\{f, 0\}$ and $(f)^+ := \max\{f, 0\}$.

3. Global higher integrability

In this section, we consider the higher integrability up to the boundary for the single obstacle problem. More precisely, we show that under some natural assumptions, the solution u to the $\mathcal{K}_{\psi}^{f,p(\cdot)}(\Omega)$ obstacle problem, of which we a priori only know that $|\nabla u|^{p(\cdot)} \in L^1(\Omega)$, actually satisfies $|\nabla u|^{p(\cdot)(1+\varepsilon)} \in L^1(\Omega)$, for a small $\varepsilon > 0$, assuming that the boundary values and the obstacle are sufficiently regular. This result can be used to study the corresponding stability problem.

The outline of the arguments is standard: a combination of a Caccioppoli inequality and a Sobolev-Poincaré inequality yields a reverse Hölder inequality. Higher integrability then follows from a suitable version of Gehring's lemma.

Since we are interested in regularity properties up to the boundary, we need to make some assumption about the domain. Here we assume that the complement of Ω satisfies the measure density condition. More precisely, this means that

$$|\Omega^c \cap B| \geq c|B| \tag{3.1}$$

whenever $B = B(x_0, r)$ is a ball centered at a point $x_0 \in \Omega^c$. This condition is widely used in regularity theory, and it is also fairly weak. For instance, all domains with a Lipschitz boundary clearly satisfy this condition.

In what follows, we choose a number M such that

$$\max\left\{\int_{\Omega} |\nabla u|^{p(x)} \, \mathrm{d}x, \int_{\Omega} |\nabla \psi|^{p(x)} \, \mathrm{d}x, \int_{\Omega} |\nabla f|^{p(x)} \, \mathrm{d}x\right\} \leq M.$$

The dependency of the constant in our reverse Hölder inequality on the norms of ∇u, $\nabla \psi$ and ∇f will be expressed by a dependency on M. By testing the definition with $v = \max\{\psi, f\}$ and using Young's inequality, we see that

$$\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \le C\left(\int_\Omega |\nabla \psi|^{p(x)} + |\nabla f|^{p(x)}\,\mathrm{d}x\right),$$

so any M larger than the right hand side will do.

Theorem 3.1. *Suppose that the complement of Ω satisfies the measure density condition* (3.1)*, and let u be the solution to the $\mathcal{K}_\psi^{f,p(\cdot)}(\Omega)$ obstacle problem, where $\psi, f \in W^{1,p(\cdot)}(\Omega)$ and $|\nabla\psi|, |\nabla f| \in L^{p(\cdot)(1+\delta)}(\Omega)$ for some $\delta > 0$. Assume that there exists a compact set $K \subset \Omega$ such that $f \ge \psi$ in $\Omega \setminus K$.*

Then there exist a positive number ε_0 and a constant C depending only on n, p, the structure of $\mathcal{A}$, M and the constant in the measure density condition, such that $|\nabla u| \in L^{p(\cdot)(1+\varepsilon)}(\Omega)$ whenever $0 < \varepsilon < \varepsilon_0$, and

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)(1+\varepsilon)}\,\mathrm{d}x \le C\bigg[&\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\ &+ \int_\Omega |\nabla\psi|^{p(x)(1+\varepsilon)}\,\mathrm{d}x + \int_\Omega |\nabla f|^{p(x)(1+\varepsilon)}\,\mathrm{d}x + 1\bigg].\end{aligned} \tag{3.2}$$

Proof. Let B_0 be a ball with $\Omega \subset \frac{1}{2}B_0$. Let $r_0 := \operatorname{dist}\{\partial\Omega, K\}$. Let $B \equiv B(x, r)$, $x \in \Omega$, and assume that $0 < r < \frac{1}{4}r_0$ and $4B \subset B_0$. The proof divides into two cases depending on whether we are near the boundary or not.

• **Case 1:** $2B \subset \Omega$**.** Let $\eta \in \mathcal{C}_0^\infty(2B)$ be a cut-off function such that $\eta = 1$ in $\bar{B}$, $0 \le \eta \le 1$ and $|\nabla\eta| \le C/\operatorname{diam}(B)$. We would like to test (2.4) with

$$v := u - c_u - \eta^{p_{2B}^+}\,(u - c_u - (\psi - c_\psi)),$$

where c_u and c_ψ denote the mean value of the functions u and ψ respectively in $2B$, i.e.

$$c_u := \fint_{2B} u\,\mathrm{d}x := \frac{1}{|2B|}\int_{2B} u\,\mathrm{d}x \qquad c_\psi := \fint_{2B} \psi\,\mathrm{d}x := \frac{1}{|2B|}\int_{2B} \psi\,\mathrm{d}x.$$

To this aim, we need to show that v is an admissible test function, for a suitable obstacle problem. We notice that $v \in \mathcal{K}_{\psi-c_u}^{f-c_u,p(\cdot)}$, $v - (f - c_u) \in W_0^{1,p(\cdot)}(\Omega)$ because $\eta \in \mathcal{C}_0^\infty(\Omega)$. Since $c_u \ge c_\psi$ we obtain

$$\begin{aligned} v &= (1 - \eta^{p_{2B}^+})(u - c_u) + \eta^{p_{2B}^+}\,(\psi - c_\psi) \\ &\ge (1 - \eta^{p_{2B}^+})(\psi - c_u) + \eta^{p_{2B}^+}(\psi - c_u) = \psi - c_u, \end{aligned}$$

a.e. in Ω. We calculate

$$\nabla v = (1 - \eta^{p_{2B}^+})\nabla(u - c_u) + \eta^{p_{2B}^+}\nabla(\psi - c_\psi) + p_{2B}^+\eta^{p_{2B}^+-1}\nabla\eta[(\psi - c_\psi) - (u - c_u)].$$

Since $u - c_u$ is a solution to the $\mathcal{K}_{\psi-c_u}^{f-c_u,p(\cdot)}$ obstacle problem and v is a test function, we have

$$0 \le \int_\Omega \mathcal{A}(x, \nabla u)\cdot\nabla(v - u)\,\mathrm{d}x = \int_{2B} \mathcal{A}(x, \nabla u)\cdot\nabla(v - u)\,\mathrm{d}x$$

and thus

$$
\begin{aligned}
\int_{2B} \mathcal{A}(x,\nabla u)\cdot\nabla u \,\mathrm{d}x &\le \int_{2B} \mathcal{A}(x,\nabla u)\cdot\nabla v \,\mathrm{d}x \\
&\le \int_{2B} (1-p_{2B}^{+})\,\mathcal{A}(x,\nabla u)\cdot\nabla u \,\mathrm{d}x \\
&+ \int_{2B} \eta^{p_{2B}^{+}}\mathcal{A}(x,\nabla u)\cdot\nabla\psi \,\mathrm{d}x \\
&+ p_{2B}^{+}\,\beta \int_{\Omega} |\nabla u|^{p(x)-1}\,\eta^{p_{2B}^{+}-1}\,|\nabla\eta|\,(|\psi-c_\psi|+|u-c_u|)\,\mathrm{d}x,
\end{aligned}
$$

where, in the last line, we used the structure conditions on $\mathcal{A}$. Simplifying and using again the structure conditions of $\mathcal{A}$, we have

$$
p_{2B}^{+}\int_{2B} \mathcal{A}(x,\nabla u)\cdot\nabla u \,\mathrm{d}x \ge \alpha\, p_{2B}^{+}\int_{2B} |\nabla u|^{p(x)} \,\mathrm{d}x.
$$

On the other hand, using Young's inequality, for some suitable $\zeta\in(0,1)$ we get

$$
\begin{aligned}
\int_{2B} \eta^{p_{2B}^{+}}\mathcal{A}(x,\nabla u)\cdot\nabla\psi \,\mathrm{d}x &\le \int_{2B} \beta|\nabla u|^{p(x)-1}|\nabla\psi| \,\mathrm{d}x \\
&\le \zeta\int_{2B} |\nabla u|^{p(x)} + c_\zeta\int_{2B} |\nabla\psi|^{p(x)} \,\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
&\beta\int_{2B} p_{2B}^{+}|\nabla u|^{p(x)-1}\,\eta^{p_{2B}^{+}-1}|\nabla\eta|\,(|\psi-c_\psi|+|u-c_u|)\,\mathrm{d}x \\
&\le \zeta\int_{2B} \eta^{p_{2B}^{+}}|\nabla u|^{p(x)} \,\mathrm{d}x + c_\zeta\int_{2B}\left(\left|\frac{u-c_u}{\mathrm{diam}(B)}\right|^{p(x)}+\left|\frac{\psi-c_\psi}{\mathrm{diam}(B)}\right|^{p(x)}\right)\mathrm{d}x.
\end{aligned}
$$

Observe that we used the definition of p_{2B}^{+} to get

$$
\tilde{p} := \frac{p(x)(p_{2B}^{+}-1)}{p(x)-1} \ge p_{2B}^{+} \qquad \forall\, x\in 2B \tag{3.3}
$$

and to estimate $\eta^{\tilde{p}} \le \eta^{p_{2B}^{+}}$ in the second inequality. Now, we choose ζ, which depends on $n, p^-, p^+, \alpha, \beta$, small enough to absorb the gradient of u to the left hand side. We connect all the previous estimates and take the mean values, and get the Caccioppoli type inequality

$$
\begin{aligned}
\fint_{B} |\nabla u|^{p(x)} \,\mathrm{d}x \quad &\le \quad C\fint_{2B}\left|\frac{u-c_u}{\mathrm{diam}(B)}\right|^{p(x)} \mathrm{d}x \\
&+C\fint_{2B}\left|\frac{\psi-c_\psi}{\mathrm{diam}(B)}\right|^{p(x)} \mathrm{d}x + C\fint_{2B}|\nabla\psi|^{p(x)}\,\mathrm{d}x,
\end{aligned}
$$

where C only depends on $n, p^-, p^+, \alpha, \beta$. Now arguing as in [44], Theorem 1.3, or [7], Theorem 3.1, we can use the usual constant exponent Sobolev-Poincaré inequality and log-Hölder continuity, which yield

$$
\fint_{2B}\left|\frac{u-c_u}{\mathrm{diam}(B)}\right|^{p(x)} \mathrm{d}x \le C\left(\fint_{2B}|\nabla u|^{\frac{p(x)}{\theta}}\,\mathrm{d}x\right)^{\bar{\theta}} + C
$$

and

$$\fint_{2B}\left|\frac{\psi-c_\psi}{\mathrm{diam}(B)}\right|^{p(x)}\,\mathrm{d}x \le C\left(\fint_{2B}|\nabla\psi|^{\frac{p(x)}{\theta}}\,\mathrm{d}x\right)^{\bar\theta}+C \le C\fint_{2B}(|\nabla\psi|^{p(x)}+1)\,\mathrm{d}x,$$

where we have chosen $\bar\theta := \min\left\{\sqrt{\frac{n+1}{n}},p^-\right\}$. From this we can deduce the following reverse Hölder estimate:

$$\fint_{B}|\nabla u|^{p(x)}\,\mathrm{d}x \le C\left(\fint_{2B}|\nabla u|^{\frac{p(x)}{\theta}}\,\mathrm{d}x\right)^{\bar\theta}+C\fint_{2B}|\nabla\psi|^{p(x)}\,\mathrm{d}x+C,$$

with $C\equiv C(n,p^-,p^+,\alpha,\beta,M)$.

• **Case 2:** $2B\setminus\Omega\neq\emptyset$**.** This case is more complicated, as the boundary of Ω will be involved in the analysis. We divide this case in three steps.

STEP 1: CACCIOPPOLI TYPE INEQUALITY. Let $\eta\in\mathcal{C}_0^\infty(2B)$ be the cut-off function chosen in the previous case. This time we test (2.4) with $v := u-\eta^{p_D^+}(u-f)$, where $u-f\in W_0^{1,p(\cdot)}(\Omega)$ and where we write $D:=2B\cap\Omega$. Since $f\ge\psi$ in $\Omega\setminus K$ and the radius of B is small enough, it is not difficult to see that $v\in\mathcal{K}_\psi^{f,p(\cdot)}$ and therefore it is an admissible test function. Thus we have

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x\\
\le\;&\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla v\,\mathrm{d}x\\
\le\;&\int_\Omega(1-\eta^{p_D^+})\mathcal{A}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x-\int_D p_D^+\,\eta^{p_D^+-1}(u-f)\mathcal{A}(x,\nabla u)\cdot\nabla\eta\,\mathrm{d}x\\
&+\int_D\eta^{p_D^+}\mathcal{A}(x,\nabla u)\cdot\nabla f\,\mathrm{d}x.
\end{aligned}$$

We simplify the previous chain of inequalities, using the structure conditions on $\mathcal{A}$, and deduce

$$\begin{aligned}
\alpha\int_D\eta^{p_D^+}|\nabla u|^{p(x)}\,\mathrm{d}x \;\le\;& \int_D\eta^{p_D^+}\,\beta\,|\nabla u|^{p(x)-1}\,|\nabla f|\,\mathrm{d}x\\
&+p_D^+\int_D\eta^{p_D^+}\,\beta\,|\nabla u|^{p(x)-1}|u-f|\,|\nabla\eta|\,\mathrm{d}x.
\end{aligned}$$

Exploiting the Young inequality, as in the previous case, and taking the mean values, we obtain the Caccioppoli type inequality

$$\begin{aligned}
\frac{1}{|2B|}\int_D\eta^{p_D^+}|\nabla u|^{p(x)}\,\mathrm{d}x \;\le\;& C\,\frac{1}{|2B|}\int_D\left|\frac{u-f}{\mathrm{diam}(B)}\right|^{p(x)}\,\mathrm{d}x\\
&+C\,\frac{1}{|2B|}\int_{4B\cap\Omega}|\nabla f|^{p(x)}\,\mathrm{d}x.
\end{aligned}$$

Here the constants C only depend on n,p^-,p^+,α,β. The choice to replace D with $4B\cap\Omega$ in the last integral will be clear later.

STEP 2: CHOICE OF θ AND LOCALIZATION. In order to obtain a suitable reverse Hölder inequality, we have to choose a proper value of the parameter θ, which will be used in the application of Gehring lemma. First note that if p is defined only in Ω then we can extend it to the whole $\mathbb{R}^n$ with a same log-Hölder constant, p^- and p^+. Fix $1<\theta<\min\{n',p^-\}$, n' being the conjugate exponent of n. We choose an upper bound for the radius of B such that

$$p_{3B}^+ < n'\frac{p_{3B}^-}{\theta}\,.$$

STEP 3: APPLICATION OF SOBOLEV–POINCARÉ INEQUALITY. Since $u-f \in W_0^{1,p(\cdot)}(\Omega)$, $u-f$ has a zero extension belonging to $W^{1,p(\cdot)}(\mathbb{R}^n)$. Since the complement of Ω satisfies the measure density condition, we see that there exists a constant C, depending on the constant appearing in (3.1), such that $|\{x\in 3B : u-f=0\}| \geq C|3B|$.

Let now $q\in(1,n)$ such that $p^+\leq q^*$ and write $h:=\min\{q, p_{3B}^-/\theta\}$. We combine the standard constant exponent (p_{3B}^+,h)-Sobolev-Poincaré inequality in a ball [33, Corollary 1.64, p.38] and the inequality

$$\|v\|_{L^q(3B)} \leq 2\left(\frac{|3B|}{|\{x\in 3B : v(x)=0\}|}\right)^{1/q}\|v-v_{3B}\|_{L^q(3B)}$$

(see, e.g., [26, Lemma 2.3]) to obtain

$$\left(\fint_{3B}|v|^{p_{3B}^+}\,dy\right)^{\frac{1}{p_{3B}^+}} \leq C\,\mathrm{diam}(B)\left(\fint_{3B}|\nabla v|^h\,dy\right)^{\frac{1}{h}},$$

where the constant C depends on n, p_{3B}^+, h and the constant in the measure density condition (3.1). Using this inequality with $v=u-f$ in the second step and Hölder's inequality in the third step, we obtain

$$\begin{aligned}
&\fint_{3B}\left|\frac{u-f}{\mathrm{diam}(B)}\right|^{p(x)}\,\mathrm{d}x\\
\leq&\fint_{3B}\left|\frac{u-f}{\mathrm{diam}(B)}\right|^{p_{3B}^+}\,\mathrm{d}x+1\\
\leq&C\left(\fint_{3B}|\nabla(u-f)|^h\,\mathrm{d}x\right)^{p_{3B}^+/h}+1\\
\leq&C\left(\fint_{3B}|\nabla(u-f)|^{p_{3B}^-/\theta}\,\mathrm{d}x\right)^{\frac{p_{3B}^+\theta}{p_{3B}^-}}+C\\
\leq&C\left(\fint_{3B}|\nabla(u-f)|^{p(x)/\theta}+1\,\mathrm{d}x\right)^{\frac{\theta(p_{3B}^+-p_{3B}^-)}{p_{3B}^-}}\\
&\times\left(\fint_{3B}|\nabla(u-f)|^{p(x)/\theta}+1\,\mathrm{d}x\right)^{\theta}+C,
\end{aligned}$$

where the constant C depends only on n and q i.e. n and p^+. Since p is log-Hölder continuous $|3B|^{-[\theta(p_{3B}^+-p_{3B}^-)]/p_{3B}^-}$ is bounded, we obtain

$$\begin{aligned}
\fint_{3B}\left|\frac{u-f}{\mathrm{diam}(B)}\right|^{p(x)}\,\mathrm{d}x &\leq C\left(\fint_{3B}|\nabla u|^{p(x)/\theta}\,\mathrm{d}x\right)^{\theta}+C\left(\fint_{3B}|\nabla f|^{p(x)/\theta}\,\mathrm{d}x\right)^{\theta}+C\\
&\leq C\left(\fint_{3B}|\nabla u|^{p(x)/\theta}\,\mathrm{d}x\right)^{\theta}+C\fint_{3B}|\nabla f|^{p(x)}\,\mathrm{d}x+C,
\end{aligned}$$

where the integral of $|\nabla f|$ has been estimated by Hölder's inequality.

• **Gehring lemma and conclusion.** We set $\theta_1 := \min\{\bar\theta, \theta\}$, where $\bar\theta$ has been fixed in Case 1 and θ in Case 2. Summing up, we obtain by Cases 1 and 2 the following reverse Hölder estimate

$$\frac{1}{|B|}\int_{B\cap\Omega}|\nabla u|^{p(x)}\,\mathrm{d}x \le C\left(\frac{1}{|4B|}\int_{4B\cap\Omega}|\nabla u|^{\frac{p(x)}{\theta_1}}\,\mathrm{d}x\right)^{\theta_1}$$
$$+\frac{C}{|4B|}\int_{4B\cap\Omega}|\nabla f|^{p(x)}\,\mathrm{d}x+\frac{C}{|4B|}\int_{4B\cap\Omega}|\nabla\psi|^{p(x)}+C,$$

which holds for all sufficiently small balls with constants C depending on n, p, α, β, θ, M but independent of the radius of the ball. Let

$$g(x) := \begin{cases} |\nabla u|^{\frac{p(x)}{\theta_1}}, & \text{if } x\in\Omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(x) := \begin{cases} |\nabla f|^{\frac{p(x)}{\theta_1}} + |\nabla\psi|^{\frac{p(x)}{\theta_1}}, & \text{if } x\in\Omega \\ 0 & \text{otherwise.} \end{cases}$$

Then the previous reverse Hölder inequality reads

$$\fint_B g^{\theta_1}\,\mathrm{d}x \le C\left(\fint_{4B} g\,\mathrm{d}x\right)^{\theta_1} + C\fint_{4B} h^{\theta_1}\,\mathrm{d}x + C, \tag{3.4}$$

whenever $4B\subset B_0$ is sufficiency small. Now we can use a standard version of Gehring's lemma (see for example [17], Chap. V, or [18], Theorem 6.6), and find a number $\varepsilon > 0$ and a constant C such that

$$\begin{aligned}\fint_B |\nabla u|^{p(x)(1+\varepsilon)}\,\mathrm{d}x \quad &\le \quad C\Bigg[\left(\fint_{4B}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{1+\varepsilon} \\ &\qquad + \fint_{4B}|\nabla f|^{p(x)(1+\varepsilon)}\,\mathrm{d}x + \fint_{4B}|\nabla\psi|^{p(x)(1+\varepsilon)}\,\mathrm{d}x + 1\Bigg].\end{aligned}$$

The estimate (3.2) then follows from this by noting that due to the boundedness of Ω, $\overline{\Omega}$ can be covered by a finite number of balls such that the previous inequality holds. □

4. Continuity up to the boundary

In this section, we discuss continuity properties of solutions to the obstacle problem up to the boundary. We start by briefly stating what can be said about the obstacle problem by means of the Wiener criterion for weak solutions. The main effort is then to prove Hölder regularity up to the boundary.

For solutions, we recall the following definition.

Definition 4.1. Let u be an arbitrary weak solution to (1.1) such that $u - f \in W_0^{1,p(\cdot)}(\Omega)$, where $f \in W^{1,p(\cdot)}(\Omega)\cap\mathcal{C}(\bar\Omega)$ is the function which gives the boundary values. A boundary point $x_0\in\partial\Omega$ is regular, if

$$\lim_{x\to x_0} u(x) = f(x_0)$$

for all such functions u.

It is well known that for equations similar to the p-Laplacian, regular boundary points can be characterized in terms of the so-called Wiener criterion, see [13, 28, 34]. The variable exponent version of the Wiener criterion is Theorem 1.1 of [3]. See also [32].

If the obstacle ψ is continuous, the solution to the obstacle problem in $\mathcal{K}_{\psi}^{f,p(\cdot)}$ is continuous, and it is a weak solution in the open set $\{u > \psi\}$, see [21, Theorem 4.11]. Using this fact, boundary continuity for the obstacle problem is a straightforward application of the Wiener criterion for solutions, see [25, Theorem 6.29] and its $p(\cdot)$-adaptation [31, Theorem 6.1].

Theorem 4.2. *Let u be the solution of the $\mathcal{K}_{\psi}^{f,p(\cdot)}$-obstacle problem, where $\psi, f \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$. If $x_0 \in \partial\Omega$ is regular for solutions with respect to Ω, then*

$$\lim_{x \to x_0} u(x) = f(x_0).$$

Let us now switch our attention to the matter of Hölder regularity up to the boundary. This turns out to be a consequence of certain estimates extracted from the proof of the sufficiency of the Wiener criterion due to Gariepy and Ziemer [13]. More precisely, one can use the method of [13] to estimate the rate of convergence to the boundary value at a point in a precise manner. Further, since the key estimate is for subsolutions, this works also for the obstacle problem.

For our present purposes, the following local Hölder estimate is convenient.

Theorem 4.3. *Let u be a solution to the obstacle problem in $\mathcal{K}_{\psi}^{f,p(\cdot)}$, and let B_r be a ball such that $B_r \Subset \Omega$. If the obstacle ψ is Hölder continuous with the exponent α, then the local Hölder estimate*

$$\operatorname*{osc}_{B_\rho} u \leq C \left(\frac{\rho}{r}\right)^{\kappa} \left(\operatorname*{osc}_{B_r} u + c\, r^{\alpha}\right) \tag{4.1}$$

where $\rho \leq r$, holds for some $0 < \kappa < 1$.

Proof. This follows by the argument of Theorem 3.7 in [35]. The estimates necessary to run this argument are the supremum estimate for obstacle problems [21, Theorem 4.9], and the weak Harnack inequality for supersolutions [23, Theorem 3.7]. □

The following proposition provides the key estimate for Hölder continuity near the boundary.

Proposition 4.4. *Let $x_0 \in \partial\Omega$, assume that u is a function such that $(u - k)_+\zeta \in W_0^{1,p(\cdot)}(\Omega)$ whenever $\zeta \in C_0^{\infty}(B(x_0, r))$, and that*

$$u_k = \begin{cases} (u-k)_+ & \text{in } \Omega, \\ 0 & \text{otherwise} \end{cases}$$

is a subsolution. Set

$$m(r) = \sup_{B(x_0,r)} u_k.$$

If the complement of Ω satisfies the measure density condition (3.1), there is a number $\widetilde{\gamma}_0 < 1$ such that

$$(m(r) + r)\widetilde{\gamma}_0 \leq m(r) - m(r/2) + r. \tag{4.2}$$

Proof. Since the measure density condition implies the capacity density condition used in [32], (4.2) follows from the proof of [32, Theorem 3.3], the remark after it, and the observations in Section 4 of [32]. $\square$

We are now ready to estimate the convergence rate at a boundary point.

Proposition 4.5. *Let u be the solution to the obstacle problem in $\mathcal{K}_{\psi}^{f,p(\cdot)}$, where $\psi \in W^{1,p(\cdot)}(\Omega) \cap \mathcal{C}^{\alpha}(\overline{\Omega})$ and $f \in W^{1,p(\cdot)}(\Omega) \cap \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^{\alpha}(\partial\Omega)$ are such that $f \geq \psi$ on $\partial\Omega$.*

Assume that the complement of Ω satisfies the measure density condition, and define

$$v(r) = \sup_{B(x_0,r)\cap\Omega} u(x) - f(x_0).$$

Then either

$$v(r) \leq C\, r^{\alpha}, \tag{4.3}$$

or there is a number $0 < \beta < 1$ such that

$$v(\rho) \leq 2^{\beta} \left(\frac{\rho}{r}\right)^{\beta} (v(r) + cr^{\alpha}) \tag{4.4}$$

for all $\rho \leq r/2$.

Proof. We denote

$$D_r = \Omega \cap B(x_0, r) \quad \text{and} \quad S_r = \partial\Omega \cap B(x_0, r)$$

for short. Let

$$k(r) = \max\left\{\sup_{S_{2r}} f, \sup_{D_{2r}} \psi\right\}.$$

The proof now divides into two parts: if $\sup_{D_r} u \leq k(r)$, then we will show that (4.3) holds, and in the case $\sup_{D_r} u > k(r)$ we will prove (4.4).

We begin by assuming that $\sup_{D_r} u \leq k(r)$. If $k(r) = \sup_{S_{2r}} f$, then (4.3) holds trivially. If $k(r) = \sup_{D_{2r}} \psi$, we use the fact that $f(x_0) \geq \psi(x_0)$ to estimate

$$\sup_{D_r} u - f(x_0) \leq \sup_{D_r} u - \psi(x_0) \leq \sup_{D_{2r}} \psi - \psi(x_0) \leq C\, r^{\alpha},$$

so that (4.3) holds also in this case.

Assume then that

$$\sup_{D_r} u > k(r). \tag{4.5}$$

Since u is a solution in the set $\{u > \psi\}$, and $k(r) \geq \sup_{D_{2r}} \psi$, $(u - k(r))_+$ is a subsolution in D_{2r}. Further, since $k(r) \geq \sup_{S_{2r}} f$ and f gives the boundary values of u, $\zeta(u - k(r))_+ \in W_0^{1,p(\cdot)}(\Omega)$ whenever $\zeta \in \mathcal{C}_0^{\infty}(B(x_0, r))$. Thus we may apply (4.2) to get

$$\left(\sup_{D_r}(u - k(r))_+ + r\right)\widetilde{\gamma}_0 \leq \sup_{D_r}(u - k(r))_+ - \sup_{D_{r/2}}(u - k(r))_+ + r.$$

Now, by (4.5), we have

$$\sup_{D_r}(u - k(r))_+ = \sup_{D_r} u - k(r).$$

Further, we have either

$$\sup_{D_{r/2}} (u - k(r))_+ = \sup_{D_{r/2}} u - k(r),$$

or

$$-k(r) \leq - \sup_{D_{r/2}} u.$$

In both cases, we get the estimate

$$(\sup_{D_r} u - k(r) + r)\widetilde{\gamma}_0 \leq \sup_{D_r} u - \sup_{D_{r/2}} u + r.$$

Adding and substracting $f(x_0)$ on both sides, we get

$$(v(r) - \phi(r) + r)\widetilde{\gamma}_0 \leq v(r) - v(r/2) + r,$$

where we denoted

$$\phi(r) = k(r) - f(x_0).$$

Rearranging this leads to

$$v(r/2) \leq \gamma_0(v(r) + \phi(r) + r),$$

where

$$\gamma_0 = \max\{1 - \widetilde{\gamma}_0, \widetilde{\gamma}_0\} < 1.$$

To deal with $\phi(r)$, we note that if $k(r) = \sup_{S_{2r}} f$, we have

$$\phi(r) = \sup_{S_{2r}} f - f(x_0) \leq C\, r^\alpha,$$

and if $k(r) = \sup_{D_{2r}} \psi$, we have

$$\phi(r) \leq \sup_{D_{2r}} \psi - \psi(x_0) \leq C\, r^\alpha$$

since $f(x_0) \geq \psi(x_0)$.

We have arrived at the inequality

$$v(r/2) \leq \gamma_0(v(r) + Cr^\alpha + r).$$

The estimate (4.4) now follows by a standard iteration procedure. Indeed, setting $B(r) = Cr^\alpha + r$, we see that

$$\begin{aligned} v(r/2^m) \leq & \gamma_0^m \left(v(r) + \sum_{j=0}^{m-1} B(r/2^j)\gamma_0^{j-1} \right) \\ \leq & \gamma_0^m \left(v(r) + \frac{Cr^\alpha}{\gamma_0(1-\gamma_0)} \right) \end{aligned}$$

for $m = 1, 2, \ldots$, since $B(r) \leq Cr^\alpha$. Now, given $\rho \leq r/2$, we find m such that

$$\frac{r}{2^{m+1}} < \rho \leq \frac{r}{2^m},$$

or, it other words,

$$\log_2\left(\frac{r}{2\rho}\right) < m \le \log_2\left(\frac{r}{\rho}\right).$$

We set $\beta = \log_2(1/\gamma_0)$; then $0 < \beta \le 1$ and $\gamma_0 = 2^{-\beta}$. We have

$$\gamma_0^m \le 2^{-\beta\log_2(r/2\rho)} = 2^\beta\left(\frac{\rho}{r}\right)^\beta,$$

and the estimate

$$v(\rho) \le v(r/2^m) \le 2^\beta\left(\frac{\rho}{r}\right)^\beta (v(r) + Cr^\alpha)$$

follows. □

To glue together the local estimate of Theorem 4.3 and the boundary estimate of the previous proposition, we use the following elementary lemma. See [25, Lemma 6.47] for the proof.

Lemma 4.6. *Assume that u is a function such that the estimate (4.1) holds. If there are constants $L \ge 0$ and $0 < \gamma < 1$ such that*

$$|u(x) - u(x_0)| \le L|x - x_0|^\gamma,$$

for all $x \in \Omega$ and $x_0 \in \partial\Omega$, then

$$|u(x) - u(y)| \le L_1|x - y|^{\gamma_1}$$

for all $x, y \in \overline{\Omega}$. We can choose $\gamma_1 = \min\{\gamma, \kappa, \alpha\}$, where κ and α are the exponents in (4.1), and $L_1 = CL\max\{1, \operatorname{diam}\Omega^{\gamma-\gamma_1}\}$.

Hölder continuity up to the boundary now follows by combining Proposition 4.5 and Lemma 4.6. We choose a number $r_0 \le 1$ to function as a cutoff point for being near the boundary.

Theorem 4.7. *Let u be the solution to the obstacle problem in $\mathcal{K}_\psi^{f,p(\cdot)}$, where $\psi \in W^{1,p(\cdot)}(\Omega)\cap\mathcal{C}^\alpha(\overline{\Omega})$ and $f \in W^{1,p(\cdot)}(\Omega)\cap\mathcal{C}(\overline{\Omega})\cap\mathcal{C}^\alpha(\partial\Omega)$ are such that $f \ge \psi$ on $\partial\Omega$. Assume also that the complement of Ω satisfies the measure density condition.*

Then

$$|u(x) - u(y)| \le M_1|x - y|^\delta$$

for all $x, y \in \overline{\Omega}$, where δ is any number such that

$$0 < \delta \le \min\{\alpha/2, \beta/2, \kappa\}.$$

Here κ is the local Hölder exponent from (4.1) and β the exponent from Proposition 4.5, and $M_1 = C\sup_\Omega |u| r_0^{-2\delta} M \max\{1, \operatorname{diam}\Omega^2\}$.

Proof. The proof is a matter of verifying the assumption of Lemma 4.6 by means of Proposition 4.5. Let $x_0 \in \partial\Omega$, $x \in \Omega$, set $\rho = |x - x_0|$ and $r = 2\rho^{1/2}$, and assume first that $\rho \le r_0^2/10$. If (4.3) holds for this value of r, it is clear that

$$|u(x) - u(x_0)| \le C|x - x_0|^\delta.$$

On the other hand, if (4.4) holds for r, we have

$$|u(x)-u(x_0)| \le v(\rho) \le C\rho^{\beta/2}(v(r)+\rho^{\alpha/2}) \le C|x-x_0|^\delta.$$

Finally, if $\rho = |x-x_0| \ge r_0^2/10$, we have the trivial estimate

$$|u(x_0)-u(x)| \le C \sup_\Omega |u| r_0^{-2\delta}|x-x_0|^\delta,$$

so that the assumption holds also in this case. □

5. Stability of solutions to the obstacle problem

The aim of this section is to show that the solutions to the obstacle problem are stable under perturbations in the growth exponent given suitable assumptions. More precisely, let $p_i : \mathbb{R}^n \to (1,\infty)$ be continuous functions that converge pointwise to a function p, and assume that they satisfy the log-Hölder continuity condition

$$|p_i(x)-p_i(y)| \le \frac{C}{-\log|(x-y)|}$$

with a constant independent of i; we will assume that the convergence is uniform by Ascoli-Arzelà's Theorem. Further, let the vector fields $\mathcal{A}_i(x,\xi)$ have p_i-growth with structural constants α and β independent of i, and assume they converge to $\mathcal{A}_0(x,\xi)$ uniformly on compact subsets of $\mathbb{R}^n$. Suppose that u_i is the solution to the following $\mathcal{K}_\psi^{f,p_i(\cdot)}(\Omega)$-obstacle problem

$$\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla(v-u_i)\,\mathrm{d}x \ge 0 \tag{5.1}$$

for every $v \in \mathcal{K}_\psi^{f,p_i(\cdot)}(\Omega)$ and u_0 the solution to

$$\int_\Omega \mathcal{A}_0(x,\nabla u_0)\cdot\nabla(v-u_0)\,\mathrm{d}x \ge 0 \tag{5.2}$$

for all $v \in \mathcal{K}_\psi^{f,p(\cdot)}(\Omega)$.

If the boundary values f are sufficiently regular, we will extract a limit function u from the sequence (u_i) and then show that $u=u_0$. The main result is the following theorem. The proof consists in a proper combination of the proof of Theorem 7.1 in [11] together with the elements in [14].

As an example where the above conditions are satisfied, one can think of

$$\mathcal{A}_i(x,\xi) = |\xi|^{p_i(x)-2}\xi.$$

Then the uniform convergence of $\mathcal{A}_i(x,\xi)$ to $\mathcal{A}_0(x,\xi) = |\xi|^{p(x)-2}\xi$ follows from the uniform convergence of the functions p_i by an application of the mean value theorem.

Theorem 5.1. *Suppose that the complement of Ω satisfies the measure density condition* (3.1)*. Let (p_i), $\mathcal{A}_i(x,\xi)$, and $\mathcal{A}_0(x,\xi)$ be as described above.*

Let the boundary value function f and the obstacle function ψ be in $W^{1,p(\cdot)(1+\gamma)}(\Omega)$ for some $\gamma > 0$. Then the following are true.

1. *There is a small number δ_0 such that sequence (u_i) of solutions to the obstacle problems* (5.1) *has a subsequence which converges in $W^{1,p(\cdot)(1+\delta)}(\Omega)$ for any $\delta < \delta_0$ to the solution u_0 of* (5.2)*.*
2. *If the obstacle is locally Hölder continuous, we also get convergence in $\mathcal{C}^{\alpha}_{\text{loc}}(\Omega)$.*
3. *Finally, if the obstacle ψ is Hölder continuous up to the boundary, and the boundary values f are Hölder continuous on the boundary, the subsequence can be taken to converge in $\mathcal{C}^{\alpha}(\bar{\Omega})$.*

Before proceeding with the proof, we note some results which will be used to verify that the limit function attains the right boundary values in Sobolev's sense.

Lemma 5.2. *Suppose that the complement of Ω satisfies the measure density condition* (3.1)*. Then the variable exponent Hardy inequality*

$$\left\| \frac{u}{\operatorname{dist}(x, \partial\Omega)} \right\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \tag{5.3}$$

holds for all functions $u \in W_0^{1,p(\cdot)}(\Omega)$. The constant C depends only on the dimension, $p(\cdot)$, and the constant in (3.1)*.*

Proof. The measure density condition implies that the pointwise Hardy inequality

$$\frac{u}{\operatorname{dist}(x, \partial\Omega)} \leq C\mathcal{M}|\nabla u|$$

holds; see [19]. Now the claim follows by integrating this and using the fact that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\Omega)$. □

Applying the above Hardy inequality and [11, Lemma 4.5] as in [11, Lemma 4.7], we get the following result.

Lemma 5.3. *Let (p_i) be a sequence of* log*-Hölder continuous variable exponents with $1 < \inf_i p_i^- \leq \sup_i p_i^+ < \infty$ and with uniformly bounded* log*-Hölder constants so that $p_i \to p$ almost everywhere in Ω. Suppose that Ω is bounded, and its complement satisfies the measure density condition.*

Let $u \in W^{1,p(\cdot)}(\Omega)$ and $u_i \in W^{1,p_i(\cdot)}(\Omega)$ for every i be such that $u_i \to u$ almost everywhere in Ω. If $f \in W^{1,p(\cdot)}(\Omega) \cap \bigcap_i W^{1,p_i(\cdot)}(\Omega)$ and $u_i - f \in W_0^{1,p_i(\cdot)}(\Omega)$ for every i with

$$\int_\Omega |\nabla(u_i - f)|^{p_i(x)} \, \mathrm{d}x \leq M, \tag{5.4}$$

where M is finite and independent of i, then $u - f \in W_0^{1,p(\cdot)}(\Omega)$.

Proof of Theorem 5.1. We divide the proof in several step and we just sketch the main points, referring to [11] and [14] for the missing details.

• STEP 1: By the assumption on the boundary data, and choosing the index i large enough, we may suppose that

$$f \in W^{1,p_i(\cdot)(1+\gamma/2)}(\Omega). \tag{5.5}$$

We first need to prove that

$$\int_\Omega |\nabla u_i|^{p(x)(1+\gamma/4)}\ \mathrm{d}x \le C,$$

for large i and with the constant $C < \infty$ independent of i. By virtue of Remark 2.1, we can use directly f as a test function in (5.1), so that we have

$$\int_\Omega \mathcal{A}_i(x, \nabla u_i) \cdot \nabla(f - u_i)\ \mathrm{d}x \ge 0.$$

Using the structure conditions on $\mathcal{A}_i$ and Young's inequality, it is not difficult to get

$$\int_\Omega |\nabla u_i|^{p_i(x)}\ \mathrm{d}x \le C \int_\Omega |\nabla f|^{p_i(x)}\ \mathrm{d}x$$

and this together with Hölder's inequality, (5.5) and the global higher integrability result obtained in Theorem 3.1 brings the desired result. Note that we may choose

$$M = C\left(1 + \int_\Omega |\nabla\psi|^{p(x)(1+\gamma)} + |\nabla f|^{p(x)(1+\gamma)}\,\mathrm{d}x\right)$$

to ensure that the constant in Theorem 3.1 is independent of i for large i.

• STEP 2: Working as in [11], it is still possible to establish the bound

$$||u_i||_{W^{1,p(\cdot)(1+\gamma/4)}(\Omega)} \le C. \tag{5.6}$$

Let us set $\delta := \gamma/4$. Having established (5.6), compactness arguments allow us to extract a subsequence, still denoted by (u_i), such that

$$u_i \rightharpoonup u \ \text{ weakly in } W^{1,p(\cdot)(1+\delta)}(\Omega) \tag{5.7}$$

$$u_i \to u \ \text{ in } L^{p(\cdot)(1+\delta)}(\Omega) \tag{5.8}$$

$$u_i \to u \ \text{ pointwise a.e. in } \Omega. \tag{5.9}$$

• STEP 3: We need to show now that $u \in \mathcal{K}_\psi^{f,p(\cdot)}(\Omega)$. Indeed, $u_i \ge \psi$ and therefore, using (5.8)-(5.9), it turns out that $u \ge \psi$ as well. On the other hand, $u_i - f \in W_0^{1,p_i(\cdot)}(\Omega)$ and we can use Lemma 5.3, (5.8)-(5.9), and (5.6) to conclude that also $u - f \in W_0^{1,p(\cdot)}(\Omega)$.

• STEP 4: The next step is to extract a further subsequence so that

$$\nabla u_i \to \nabla u \ \text{ pointwise a.e. in } \Omega.$$

This can be done using a refinement of the method introduced by Kilpeläinen and Malý [27], following the lines of [14], see also [11]. The only point to check is to notice that, since u_i, $i = 1, 2, \dots$ is the solution to the $\mathcal{K}_\psi^{f,p_i(\cdot)}(\Omega)$-obstacle problem (5.1), it is also an $\mathcal{A}_i$-supersolution, i.e. it holds

$$\int_\Omega \mathcal{A}_i(x, \nabla u_i) \cdot \nabla\varphi\ \mathrm{d}x \ge 0,$$

for each $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \ge 0$. This allows to use the same test function as in the case of equations and continue in the same way with the rest of the computations. This is an important point, because we will see that for the double obstacle problem this fact fails to be true and we will need a different argument to obtain the result.

• STEP 5: From the pointwise convergence of the gradients established in step 4, it follows that $u_i \to u$ in $W^{1,p(\cdot)(1+\delta')}(\Omega)$ for all $\delta' < \delta$.

• STEP 6: To conclude we just need to check that $u = u_0$. We do this along the lines of [14]. We know that $u_0 \in \mathcal{K}_\psi^{f,p(\cdot)}(\Omega)$ and hence

$$\int_\Omega \mathcal{A}_0(x, \nabla u_0) \cdot (\nabla u - \nabla u_0)\, \mathrm{d}x \geq 0. \tag{5.10}$$

In order to obtain the inequality

$$\int_\Omega \mathcal{A}_0(x, \nabla u) \cdot (\nabla u_0 - \nabla u)\, \mathrm{d}x \geq 0, \tag{5.11}$$

we use an approximation method; the difficulty is that we do not know that $u_0 - f \in W_0^{1,p(\cdot)(1+\delta)}(\Omega)$ although $u_0, f \in W^{1,p(\cdot)(1+\delta)}(\Omega)$. For (5.11) we may again assume that $f \geq \psi$. Now $u_0 - f \in W_0^{1,p(\cdot)}(\Omega)$ and hence there is a sequence $\varphi_i \in \mathcal{C}_0^\infty(\Omega)$ such that $\varphi_i \to u_0 - f$ in $W^{1,p(\cdot)}(\Omega)$. Since $\psi - f \leq 0$ and since φ_i has compact support, we can say that

$$\max(\varphi_i, \psi - f) \in W_0^{1,p(\cdot)(1+\delta)}(\Omega)$$

and then $v_i - f \in W_0^{1,p(\cdot)(1+\delta)}(\Omega)$, where $v_i = \max\{\varphi_i, \psi - f\} + f$. On the other hand, $v_i \geq \psi - f + f = \psi$ a.e. and thus $v_i \in \mathcal{K}_\psi^{f,p(\cdot)(1+\delta)}(\Omega)$. If $v \in \mathcal{K}_\psi^{f,p(\cdot)(1+\delta)}(\Omega)$, then we may assume that $v \in \mathcal{K}_\psi^{f,p_i(\cdot)}(\Omega)$ for large i. Hence

$$\int_\Omega \mathcal{A}_i(x, \nabla u_i) \cdot (\nabla v - \nabla u_i)\, \mathrm{d}x \geq 0,$$

and letting $i \to \infty$ we obtain from (5.8) that

$$\int_\Omega \mathcal{A}_0(x, \nabla u) \cdot (\nabla v - \nabla u)\, \mathrm{d}x \geq 0.$$

Set $v = v_i$ in the previous inequality; now as long as $u_0 \geq \psi$ a.e., we have $v_i \to u_0$ in $W^{1,p(\cdot)}(\Omega)$ and therefore (5.11) is obtained. Now the conclusion of the proof comes easily: from the structure conditions on the operator $\mathcal{A}_0$, using (5.10) and (5.11), we deduce

$$0 \leq \int_\Omega (\mathcal{A}_0(x, \nabla u) - \mathcal{A}_0(x, \nabla u_0)) \cdot (\nabla u - \nabla u_0)\, \mathrm{d}x \leq 0$$

and this is possible only if $u = u_0$.

Finally, the convergences in $\mathcal{C}^\alpha_{loc}(\Omega)$ and in $\mathcal{C}^\alpha(\overline{\Omega})$ follow from the fact that the respective estimates are uniform in i. □

6. The double obstacle problem

In this section we are going to complete the results obtained so far by considering some of the same questions for the double obstacle problem.

Let Ω be as in Section 2; let $\psi : \Omega \to [-\infty, \infty)$ and $\varphi : \Omega \to (-\infty, \infty]$ be functions, called *obstacles* and let $f \in W^{1,p(\cdot)}(\Omega)$ be a function which gives the boundary values, such that $\psi \le f \le \varphi$ a.e. in Ω. Define

$$\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega) := \{u \in W^{1,p(\cdot)}(\Omega) : \, u - f \in W^{1,p(\cdot)}_0(\Omega), \; \psi \le u \le \varphi \text{ a.e. in } \Omega\}.$$

We say that a function $u \in \mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$ is a *solution to the* $\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$ *double obstacle problem* if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(v - u) \, \mathrm{d}x \, \ge \, 0$$

for all $v \in \mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$; here the vector field $\mathcal{A}$ is as in Section 2. Note that we allow the cases $\psi \equiv -\infty$ and $\varphi \equiv \infty$, so that the single upper and lower obstacle problems are a special case. This is useful when combined with the comparison lemma below.

The question of existence and uniqueness of solutions can be dealt in a similar way as for the case of the single obstacle problem. So let us devote our attention to the question of regularity.

6.1. Interior regularity and continuity up to the boundary

First of all we deal with the problem of the interior regularity for solutions to the double obstacle problems. In particular, we show that the solution is continuous if both the obstacles are continuous. The corresponding theorems have been established in the case of the single obstacle in [21]. We start with a couple of preliminary lemmas, the first of them being a comparison result. It allows us to transfer certain facts known for the single obstacle problems, for instance boundary continuity, over to the double obstacle case.

Lemma 6.1. *Let $f_1, f_2 \in W^{1,p(\cdot)}(\Omega)$. Assume that $\psi_1 \le \psi_2$, $\varphi_1 \le \varphi_2$ a.e. in Ω and that $(f_1 - f_2)_+ \in W^{1,p(\cdot)}_0(\Omega)$. Let u_1 be the solution to the $\mathcal{K}^{f_1,p(\cdot)}_{\psi_1,\varphi_1}$-obstacle problem and u_2 be the solution to the $\mathcal{K}^{f_2,p(\cdot)}_{\psi_2,\varphi_2}$-obstacle problem. Then $u_1 \le u_2$ a.e. in Ω.*

Proof. The proof is divided in two steps.
• STEP 1: Let us introduce the functions

$$v := \min\{u_1, u_2\}, \qquad w := \max\{u_1, u_2\}.$$

In this first step we prove that

$$v \in \mathcal{K}^{f_1,p(\cdot)}_{\psi_1,\varphi_1}(\Omega), \qquad w \in \mathcal{K}^{f_2,p(\cdot)}_{\psi_2,\varphi_2}(\Omega). \tag{6.1}$$

If $\min\{u_1, u_2\} = u_1$ then clearly $\psi_1 \le u_1 \le \varphi_1$, if instead $\min\{u_1, u_2\} = u_2$ then $\psi_1 \le \psi_2 \le u_2 \le u_1 \le \varphi_1$, due to the assumptions. The counterpart for w is analogous. Now, let $h := u_1 - f_1 - (u_2 - f_2) \in W^{1,p(\cdot)}_0(\Omega)$. It follows that

$$h \ge \min\{f_2 - f_1, h\} \ge -(f_2 - f_1)_- - h_- = (f_1 - f_2)_+ - h_- \in W^{1,p(\cdot)}_0(\Omega).$$

Elementary facts allow us to conclude that also $\min\{f_2 - f_1, h\} \in W^{1,p(\cdot)}_0(\Omega)$ and therefore

$$v - f_1 = \min\{u_2 - f_1, u_1 - f_1\} = u_2 - f_2 + \min\{f_2 - f_1, h\} \in W^{1,p(\cdot)}_0(\Omega)$$

and

$$
\begin{aligned}
w - f_2 &= \max\{u_2 - f_2, u_1 - f_2\} = u_1 - f_1 + \max\{-h, f_1 - f_2\} \\
&= u_1 - f_1 - \min\{f_2 - f_1, h\} \in W_0^{1,p(\cdot)}(\Omega).
\end{aligned}
$$

This proves (6.1).

• STEP 2: In this step we would like to show that

$$
\int_\Omega (\mathcal{A}(x, \nabla u_1) - \mathcal{A}(x, \nabla v)) \cdot (\nabla u_1 - \nabla v) \, \mathrm{d}x = 0; \tag{6.2}
$$

if this holds, then from the structure conditions on $\mathcal{A}$, we would have that $v := \min\{u_1, u_2\} = u_1$ and therefore $u_1 \leq u_2$, the desired result.

First of all, u_1 is the solution to the $\mathcal{K}^{f_1,p(\cdot)}_{\psi_1,\varphi_1}$-obstacle problem and $v \in \mathcal{K}^{f_1,p(\cdot)}_{\psi_1,\varphi_1}(\Omega)$, so it is a good test function, i.e.

$$
\int_\Omega \mathcal{A}(x, \nabla u_1) \cdot (\nabla v - \nabla u_1) \, \mathrm{d}x \geq 0. \tag{6.3}
$$

If we would be able to prove that

$$
-\int_\Omega \mathcal{A}(x, \nabla v) \cdot (\nabla u_1 - \nabla v) \, \mathrm{d}x \leq 0, \tag{6.4}
$$

then from the monotonicity condition coming from the structure conditions on $\mathcal{A}$ we would come to (6.2).

On the other hand, u_2 is the solution to the $\mathcal{K}^{f_2,p(\cdot)}_{\psi_2,\varphi_2}$-obstacle problem and $w \in \mathcal{K}^{f_2,p(\cdot)}_{\psi_2,\varphi_2}(\Omega)$, so it is a good test function and this gives

$$
\int_\Omega \mathcal{A}(x, \nabla u_2) \cdot (\nabla w - \nabla u_2) \, \mathrm{d}x \geq 0. \tag{6.5}
$$

But $w := \max\{u_2, u_1\}$, and thus

$$
\begin{aligned}
0 &\leq \int_\Omega \mathcal{A}(x, \nabla u_2) \cdot (\nabla w - \nabla u_2) \, \mathrm{d}x \\
&= \int_{\{u_1 < u_2\}} \mathcal{A}(x, \nabla u_2) \cdot (\nabla w - \nabla u_2) \, \mathrm{d}x + \int_{\{u_1 > u_2\}} \mathcal{A}(x, \nabla u_2) \cdot (\nabla w - \nabla u_2) \, \mathrm{d}x \\
&= 0 + \int_{\{u_1 > u_2\}} \mathcal{A}(x, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2) \, \mathrm{d}x.
\end{aligned} \tag{6.6}
$$

On the other hand, as $v := \min\{u_1, u_2\}$

$$
\begin{aligned}
\int_\Omega \mathcal{A}(x, \nabla v) \cdot (\nabla u_1 - \nabla v) \, \mathrm{d}x &= \int_{\{u_1 < u_2\}} \mathcal{A}(x, \nabla v) \cdot (\nabla u_1 - \nabla v) \, \mathrm{d}x \\
&\quad + \int_{\{u_1 > u_2\}} \mathcal{A}(x, \nabla v) \cdot (\nabla u_1 - \nabla v) \, \mathrm{d}x \\
&= 0 + \int_{\{u_1 > u_2\}} \mathcal{A}(x, \nabla u_2) \cdot (\nabla u_1 - \nabla u_2) \, \mathrm{d}x \overset{(6.6)}{\geq} 0.
\end{aligned}
$$

which is nothing but (6.4). This finishes the proof of this comparison lemma. □

The following lemma is a locality property of the double obstacle problem.

Lemma 6.2. *Let $\psi, \varphi : \Omega \to [-\infty, \infty)$ and $f \in W^{1,p(\cdot)}(\Omega)$. Let u be the solution to the $\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$-obstacle problem and let $\Omega' \subset \Omega$ be open. Then u is the solution to the $\mathcal{K}^{u,p(\cdot)}_{\psi,\varphi}(\Omega')$-obstacle problem.*

Proof. Let $v \in \mathcal{K}^{u,p(\cdot)}_{\psi,\varphi}(\Omega')$; then we have to show that

$$\int_{\Omega'} \mathcal{A}(x, \nabla u) \cdot \nabla(v - u)\, \mathrm{d}x \geq 0. \tag{6.7}$$

Since $v - u \in W^{1,p(\cdot)}_0(\Omega') \subset W^{1,p(\cdot)}_0(\Omega)$ and $v = (v - u) + u \in W^{1,p(\cdot)}(\Omega)$, we can define $v(x) = u(x)$ when $x \in \Omega \setminus \Omega'$. It follows that $\psi \leq v \leq \varphi$ a.e. in Ω since $\psi \leq v \leq \varphi$ a.e. in Ω' and $v = u$ in $\Omega \setminus \Omega'$.

We also have

$$v - f = (v - u) + (u - f) \in W^{1,p(\cdot)}_0(\Omega)$$

thus $v \in \mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$ and using the fact that u is the solution to the $\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$-obstacle problem, we get

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla(v - u)\, \mathrm{d}x \geq 0.$$

As long as $v(x) = u(x)$ when $x \in \Omega \setminus \Omega'$, we obtain (6.7) and we are done. □

The main result about continuity can be formulated as follows.

Theorem 6.3. *Let u be the solution to the $\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$-double obstacle problem, where we assume that φ is continuous and ψ is locally bounded. Then the function*

$$u^*(x) := \operatorname*{ess\,liminf}_{y \to x} u(y) := \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u$$

is lower semicontinuous in Ω and it is such that $u^*(x) = u(x)$ a.e. in Ω. Moreover, if ψ is also continuous, then the same holds for u^*.

Proof. Since the upper obstacle is locally bounded from above and the lower obstacle is locally bounded from below, the solution of the double obstacle problem is locally bounded, and u^* is finite at each point.

Now let $\alpha \in \mathbb{R}$, set $A := \{x \in \Omega : u^*(x) > \alpha\}$ and $x_0 \in A$. We have

$$u^*(x_0) = \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x_0,r)} u > \alpha,$$

hence there exists $\delta > 0$ such that $\operatorname{ess\,inf}_{B(x_0,\delta)} u > \alpha$. As for all $y \in B(x_0, \delta)$ there is $\delta_y > 0$ such that $B(y, \delta_y) \subset B(x_0, \delta)$, we have

$$u^*(y) = \operatorname*{ess\,liminf}_{z \to y} u(z) \geq \operatorname*{ess\,inf}_{B(y,\delta_y)} u \geq \operatorname*{ess\,inf}_{B(x_0,\delta)} u > \alpha.$$

This shows that the set A is open and that u^* is lower semicontinuous in Ω.

To show that $u^*(x) = u(x)$ a.e. in Ω, let $\varepsilon > 0$. By the continuity of φ we find for every $x \in \Omega$ a ball $B_x := B(x, r_x)$ such that

$$\sup_{B_x} \varphi \leq \inf_{B_x} \varphi + \varepsilon.$$

We can cover Ω by countably many such balls. Let further v be the lower semicontinuously regularized solution of the $\mathcal{K}_{\psi}^{u,p(\cdot)}(B_x)$-obstacle problem provided by [23], Theorem 4.1 (see also [21], Theorem 10).

Since u is a solution to the $\mathcal{K}_{\psi,\varphi}^{u,p(\cdot)}(B_x)$- obstacle problem, due to Lemma 6.2, then the comparison Lemma 6.1 implies that

$$u \leq v \text{ a.e. in } B_x. \tag{6.8}$$

Next, as $\psi \leq u \leq \varphi \leq \sup_{B_x} \varphi =: r$ a.e. in B_x, we have by the comparison principle (due to the fact that $v(x) = u(x)$ on ∂B_x) that $v \leq r$ a.e. in B_x. Thus v is a solution to the $\mathcal{K}_{\psi,r}^{u,p(\cdot)}(B_x)$-obstacle problem, which implies that $v - \varepsilon$ is a solution to the $\mathcal{K}_{\psi-\varepsilon,r-\varepsilon}^{u-\varepsilon,p(\cdot)}(B_x)$-obstacle problem. As $\psi - \varepsilon \leq \psi$, $r - \varepsilon \leq \inf_{B_x} \varphi \leq \varphi$ and $u - \varepsilon \leq u$ in B_x, another application of comparison Lemma 6.1 implies that $v - \varepsilon \leq u$ a.e. in B_x. Together with (6.8) we get

$$v - \varepsilon \leq u \leq v \text{ a.e. in } B_x \tag{6.9}$$

and thus (passing to the semicontinuously regularizations)

$$v - \varepsilon = v^* - \varepsilon \leq u^* \leq v^* = v$$

everywhere in B_x. This and (6.9) imply that

$$|u^* - u| < \varepsilon \text{ a.e. in } B_x \tag{6.10}$$

Hence $|u^* - u| \leq \varepsilon$ a.e. in Ω, since for a given $\varepsilon > 0$ we can cover Ω by countably many balls satisfying (6.10). Letting $\varepsilon \to 0$, we obtain that $u^* = u$ a.e. in Ω.

Next we prove that u^* is continuous if ψ is continuous. We already know that u^* is lower semicontinuous. To show that it is upper semicontinuous let $\varepsilon > 0$, $x \in \Omega$ and choose B_x as above. Let v be the continuous solution to the $\mathcal{K}_{\psi}^{u,p(\cdot)}(B_x)$-obstacle problem provided by Theorem 10 in [21]. It is shown above that

$$v(z) - \varepsilon \leq u^*(z) \leq v(z) \text{ for all } z \in B_x. \tag{6.11}$$

Thus using the fact that v is continuous we obtain

$$v(z) - \varepsilon = \limsup_{y \to z} v(y) - \varepsilon \leq \limsup_{y \to z} u^*(y) \leq \limsup_{y \to z} v(y) = v(z)$$

for all $z \in B_x$. This and (6.11) give

$$\left| \limsup_{y \to z} u^*(y) - u^*(z) \right| \leq \varepsilon \text{ for alll } z \in B_x$$

and hence

$$\left| \limsup_{y \to z} u^*(y) - u^*(z) \right| \leq \varepsilon \text{ for all } z \in \Omega.$$

Letting $\varepsilon \to 0$, we get that

$$\limsup_{y \to z} u^*(y) = u^*(z), \text{ for all } z \in \Omega.$$

This means that u^* is continuous in Ω. $\square$

We conclude the subsection with the following important result, which shows that the continuous solution to the continuous double obstacle problem is a weak solution in the open set where it does not touch either one of the two obstacles.

Theorem 6.4. *Let $\psi, \varphi : \Omega \to [-\infty, \infty)$ be continuous and $f \in W^{1,p(\cdot)}(\Omega)$. Let u be the continuous solution to the $\mathcal{K}_{\psi,\varphi}^{f,p(\cdot)}$-obstacle problem. Let also*

$$\Omega' := \{x \in \Omega : u(x) < \varphi(x)\}.$$

Then u is the solution to the $\mathcal{K}_{\psi}^{u,p(\cdot)}(\Omega')$-obstacle problem. Moreover u is a weak solution in the open set $\{x \in \Omega : \psi(x) < u(x) < \varphi(x)\}$ (with boundary values u).

Proof. Let $v \in \mathcal{K}_{\psi}^{u,p(\cdot)}(\Omega')$; our aim is to show that

$$\int_{\Omega'} \mathcal{A}(x, \nabla u) \cdot (\nabla v - \nabla u) \, \mathrm{d}x \geq 0. \tag{6.12}$$

First of all, notice that $w := \min\{u, v\} \in \mathcal{K}_{\psi,\varphi}^{u,p(\cdot)}(\Omega')$. Using the fact that u is the solution to the $\mathcal{K}_{\psi,\varphi}^{u,p(\cdot)}(\Omega')$-obstacle problem, we get

$$\int_{\Omega'} \mathcal{A}(x, \nabla u) \cdot (\nabla w - \nabla u) \, \mathrm{d}x = \int_{\{u>v\}} \mathcal{A}(x, \nabla u) \cdot (\nabla v - \nabla u) \, \mathrm{d}x \geq 0.$$

Thus, in order to conclude, we also need to show that

$$\int_{\{u \leq v\}} \mathcal{A}(x, \nabla u) \cdot (\nabla v - \nabla u) \, \mathrm{d}x \geq 0;$$

since

$$\int_{\{u \leq v\}} \mathcal{A}(x, \nabla u) \cdot (\nabla v - \nabla u) \, \mathrm{d}x = \int_{\Omega'} \mathcal{A}(x, \nabla u) \cdot (\nabla \max\{u, v\} - \nabla u) \, \mathrm{d}x,$$

this is the same thing as showing that (6.12) holds with the additional assumption $v \geq u$.

Let $\varepsilon, \varepsilon' > 0$ be given. Using the density results mentioned in Section 2 and the fact that $0 \leq v - u \in W_0^{1,p(\cdot)}(\Omega')$, we can conclude that there exists a function $0 \leq \tilde{\varphi} \in \mathcal{C}_0^{\infty}(\Omega')$ such that

$$||\tilde{\varphi} - (v - u)||_{W_0^{1,p(\cdot)}(\Omega')} < \varepsilon'. \tag{6.13}$$

Let $\tilde{v} := \tilde{\varphi} + u$. Taking possibly a smaller ε', it is not hard to show that, using (6.13) and the assumptions on the operator $\mathcal{A}$

$$\int_{\Omega'} \mathcal{A}(x, \nabla u) \cdot (\nabla \tilde{v} - \nabla v) \, \mathrm{d}x \leq \varepsilon. \tag{6.14}$$

On the other hand, u and φ are continuous on the compact set $\operatorname{supp} \tilde{\varphi}$ and $u(x) < \varphi(x)$ for every $x \in \operatorname{supp} \tilde{\varphi}$. Therefore we can conclude that there exists $\sigma > 0$ such that $u + \sigma \leq \varphi$ on $\operatorname{supp} \tilde{\varphi}$. Let $0 < t < 1$ be such that $t \max_{\Omega'} \varphi \leq \sigma$. Then

$$\psi(x) \leq z(x) := u(x) + t(\tilde{v}(x) - u(x)) = u(x) + t\tilde{\varphi}(x) \leq \varphi$$

for every $x \in \Omega'$. Since $z - u = t\tilde{\varphi} \in W_0^{1,p(\cdot)}(\Omega')$ and $\psi \le z \le \varphi$ in Ω', we obtain that $z \in \mathcal{K}_{\psi,\varphi}^{u,p(\cdot)}(\Omega')$. Elementary computations together with the fact that u is a solution to the $\mathcal{K}_{\psi,\varphi}^{u,p(\cdot)}(\Omega')$-obstacle problem, imply that

$$t \int_{\Omega'} \mathcal{A}(x, \nabla u) \cdot (\nabla \tilde{v} - \nabla u) \, \mathrm{d}x \ge 0.$$

Putting together this last inequality with (6.14) we get (6.12), therefore u is the solution to the $\mathcal{K}_{\psi}^{u,p(\cdot)}(\Omega')$-obstacle problem.

At this point, since

$$\{x \in \Omega' : u(x) > \psi(x)\} = \{x \in \Omega : \psi(x) < u(x) < \varphi(x)\},$$

it follows from Theorem 4.11 in [21] that u is also a solution in the open set $\{x \in \Omega : \psi(x) < u(x) < \varphi(x)\}$. This finishes the proof. □

6.2. Global higher integrability

The next result we are going to achieve is higher integrability for the solutions to the double obstacle problem. Now an appropriate choice of M is

$$M = C\left(\int_\Omega |\nabla f|^{p(x)} + |\nabla \varphi|^{p(x)} + |\nabla \psi|^{p(x)} \, \mathrm{d}x\right)$$

Theorem 6.5. *Suppose that the complement of Ω satisfies the measure density condition* (3.1)*, and let u be the solution to the $\mathcal{K}_{\psi,\varphi}^{f,p(\cdot)}(\Omega)$ double obstacle problem, where $\psi, \varphi, f \in W^{1,p(\cdot)}(\Omega)$ and $|\nabla\psi|$, $|\nabla\varphi|$, $|\nabla f| \in L^{p(\cdot)(1+\delta)}(\Omega)$. Assume that there exists a compact $K \subset \Omega$ such that $\psi \le f \le \varphi$ in $\Omega \setminus K$. Then there exist a positive number ε_0 and a constant C depending only on n, p, the structure of $\mathcal{A}$, M, and the constant in the measure density condition, such that $|\nabla u| \in L^{p(\cdot)(1+\varepsilon)}(\Omega)$ whenever $0 < \varepsilon < \varepsilon_0$, and*

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)(1+\varepsilon)} \, \mathrm{d}x \le C\bigg[&\int_\Omega |\nabla u|^{p(x)} \, \mathrm{d}x + \int_\Omega |\nabla \psi|^{p(x)(1+\varepsilon)} \, \mathrm{d}x \\ &+ \int_\Omega |\nabla \varphi|^{p(x)(1+\varepsilon)} \, \mathrm{d}x + \int_\Omega |\nabla f|^{p(x)(1+\varepsilon)} \, \mathrm{d}x + 1\bigg].\end{aligned} \tag{6.15}$$

Proof. The proof of this result goes along the lines of the proof of Theorem 3.1; we will sketch here the main differences for the reader's convenience. The most important point is the choice of a suitable test function in the first case: unlike the single obstacle problem, this choice is less apparent here. We will employ an idea from [16].

Let B_0 be a ball with $\Omega \subset \frac{1}{2}B_0$. Let $r_0 := \mathrm{dist}\{\partial\Omega, K\}$. Let $B \equiv B(x, r)$, $x \in \Omega$, and assume that $0 < r < \frac{1}{4}r_0$ and $4B \subset B_0$.

- **Case 1:** $2B \subset \Omega$

Let $\eta \in \mathcal{C}_0^\infty(2B)$ be a cut-off function such that $\eta = 1$ in $\bar{B}$, $0 \le \eta \le 1$ and $|\nabla \eta| \le C/\operatorname{diam}(B)$. Consider the function

$$v := (1 - \eta^{p_{2B}^+})(u - c_u) + \eta^{p_{2B}^+} w$$

where

$$w = (\varphi - c_u)^- + \min((\psi - c_u)^+, (\varphi - c_u)^+).$$

We notice that $w \in W^{1,p(\cdot)}(\Omega)$ because of the assumptions on the obstacles $\nabla\psi, \nabla\varphi \in L^{p(\cdot)(1+\delta)}$ for some $\delta > 0$. Moreover we see that $v \in \mathcal{K}^{f-c_u,p(\cdot)}_{\psi-c_u,\varphi-c_u}(\Omega)$ because we have that

$$w = \begin{cases} (\psi - c_u)^+ & \varphi \geq c_u \\ \varphi - c_u & \varphi < c_u \end{cases}$$

and this implies that

$$\psi - c_u \leq v \leq \varphi - c_u \qquad \text{a.e. in } \Omega.$$

At this point we exploit the fact that $u - c_u$ is a solution to the $\mathcal{K}^{f-c_u,p(\cdot)}_{\psi-c_u,\varphi-c_u}(\Omega)$ double obstacle problem and therefore we can write

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot (\nabla v - \nabla u) \, dx \geq 0.$$

We proceed as in Section 3 and exploit the structure conditions of $\mathcal{A}$ and Young's inequality, together with (3.3) and the fact that

$$|w| \leq \begin{cases} |\psi - c_\varphi| & \varphi \geq c_u \\ |\varphi - c_\psi| & \varphi < c_u \end{cases}$$

This way, we finally deduce

$$\begin{aligned} \fint_B |\nabla u|^{p(x)} \, dx \quad &\leq \quad C\fint_{2B} \left| \frac{u - u_{2B}}{\operatorname{diam}(B)} \right|^{p(x)} dx + C\fint_{2B} |\nabla\psi|^{p(x)} \, dx \\ &\quad + C\fint_{2B} |\nabla\varphi|^{p(x)} \, dx, \end{aligned}$$

where all constants C only depend on $n, p^-, p^+, \alpha, \beta$. At this point, arguing in a standard way, we get the following reverse Hölder estimate

$$\begin{aligned} \fint_B |\nabla u|^{p(x)} \, dx \quad &\leq \quad C \left(\fint_{2B} |\nabla u|^{\frac{p(x)}{\theta}} \, dx \right)^{\bar{\theta}} \\ &\quad + C\fint_{2B} |\nabla\psi|^{p(x)} \, dx + C\fint_{2B} |\nabla\varphi|^{p(x)} \, dx + C, \end{aligned}$$

with $C \equiv C(n, p^-, p^+, \alpha, \beta, M)$.

• **Case 2:** $2B \setminus \Omega \neq \emptyset$

In this case we can proceed as in the proof of Theorem 3.1 with the obvious modifications, since, due to Remark 2.1, we can still use $v := u - \eta^{p_D^+}(u - f)$ as a test function. Therefore we obtain a proper Caccioppoli inequality, and then localize and choose a suitable exponent θ so that we can apply a standard Sobolev-Poincaré inequality. We get the following reverse Hölder estimate (where $\theta_1 := \min\{\bar\theta, \theta\}$)

$$\begin{aligned}&\frac{1}{|B|}\int_{B\cap\Omega}|\nabla u|^{p(x)}\,\mathrm{d}x \le C\left(\frac{1}{|4B|}\int_{4B\cap\Omega}|\nabla u|^{\frac{p(x)}{\theta_1}}\,\mathrm{d}x\right)^{\theta_1}\\&+\frac{C}{|4B|}\int_{4B\cap\Omega}|\nabla f|^{p(x)}\,\mathrm{d}x\\&+\frac{C}{|4B|}\int_{4B\cap\Omega}|\nabla\psi|^{p(x)}\,\mathrm{d}x+\frac{C}{|4B|}\int_{4B\cap\Omega}|\nabla\varphi|^{p(x)}\,\mathrm{d}x+C,\end{aligned}$$

which holds for sufficiently small balls with constants C depending on n, p, α, β, θ, M but independent of the radius of the ball. Now we set

$$g(x) := \begin{cases} |\nabla u|^{\frac{p(x)}{\theta_1}}, & \text{if } x \in \Omega\\ 0 & \text{otherwise}\end{cases}$$

and

$$h(x) := \begin{cases} |\nabla f|^{\frac{p(x)}{\theta_1}} + |\nabla\psi|^{\frac{p(x)}{\theta_1}} + |\nabla\varphi|^{\frac{p(x)}{\theta_1}}, & \text{if } x \in \Omega\\ 0 & \text{otherwise.}\end{cases}$$

With this notation, the current reverse Hölder inequality becomes the same as (3.4). Thus, as before, an application of Gehring's lemma and a covering argument conclude the proof. □

6.3. Continuity up to the boundary

Regularity up to the boundary for the double obstacle problem is a consequence of the comparison Lemma 6.1, and the corresponding results for suitable single obstacle problems. More specifically, we have the following theorem.

Theorem 6.6. *Let u be the solution to the obstacle problem $\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$. Assume that $\psi, \varphi \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ and $f \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$.*

1. *If $x_0 \in \partial\Omega$ is regular for solutions, then*

$$\lim_{x\to x_0} u(x) = f(x_0).$$

2. *If $\psi, \varphi \in \mathcal{C}^\alpha(\Omega)$, then the local Hölder estimate* (4.1) *holds for u.*
3. *If the complement of Ω satisfies the measure density condition, and $\psi, \varphi \in \mathcal{C}^\alpha(\overline{\Omega})$, $f \in \mathcal{C}^\alpha(\partial\Omega)$, then u is Hölder continuous in $\overline{\Omega}$.*

Proof. Let v and w be the solutions to the $\mathcal{K}^{f,p(\cdot)}_{\psi,\infty}$ and $\mathcal{K}^{f,p(\cdot)}_{-\infty,\varphi}$ obstacle problems, respectively. The conclusion about attaining the right boundary value at regular boundary points follows from the comparison Lemma 6.1 and the fact that Theorem 4.2 holds for v and w. See [29, Theorem 4.1 and Corollary 4.2] for how to prove the local Hölder estimate (4.1) for double obstacle problems. For Hölder continuity up to the boundary, we note that since the second assumption in Lemma 4.6 holds for v and w, the comparison lemma 6.1 again implies that it holds also for u. More specifically, if $x_0 \in \partial\Omega$ and $x \in \Omega$, we have

$$|u(x) - u(x_0)| = u(x) - u(x_0) \le v(x) - v(x_0) \le C|x - x_0|^\delta$$

if $u(x) > u(x_0)$, and

$$|u(x) - u(x_0)| = u(x_0) - u(x) \le w(x_0) - w(x) \le C|x - x_0|^\delta$$

otherwise, since u, v, and w attain the same value $f(x_0)$ at x_0. □

6.4. Stability

In the last subsection, we establish the stability result corresponding to Theorem 5.1 for the double obstacle problem. Assume that the functions p_i and the fields $\mathcal{A}_i(x,\xi)$ converge to p and $\mathcal{A}_0$ respectively in the same senses as in Section 5. Then the conclusion is analoguos; in other words if u_i is the solution to the $\mathcal{K}^{f,p_i(\cdot)}_{\psi,\varphi}(\Omega)$ double obstacle problem

$$\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla(v-u_i)\;\mathrm{d}x \geq\; 0 \tag{6.16}$$

for every $v\in\mathcal{K}^{f,p_i(\cdot)}_{\psi,\varphi}(\Omega)$ and u_0 the solution to

$$\int_\Omega \mathcal{A}_0(x,\nabla u_0)\cdot\nabla(v-u_0)\;\mathrm{d}x \geq\; 0 \tag{6.17}$$

for all $v\in\mathcal{K}^{f,p(\cdot)}_{\psi,\varphi}(\Omega)$, we can extract a a limit function u from the sequence (u_i) and then show that $u=u_0$, provided that the boundary values and the obstacles are sufficiently regular.

Theorem 6.7. *Suppose that the complement of Ω satisfies the measure density condition* (3.1)*. If the boundary value function f and the obstacles ψ,φ are in $W^{1,p(\cdot)(1+\gamma)}(\Omega)$ for some $\gamma>0$, then the following are true.*

1. *The sequence (u_i) of solutions to the double obstacle problems* (6.16) *has a subsequence which converges in $W^{1,p(\cdot)(1+\delta)}(\Omega)$ for any $\delta<\delta_0$ for some small δ_0 to the solution u_0 to* (6.17)*.*
2. *If both of the obstacles are locally Hölder continuous, then the subsequence can be taken to converge also in $\mathcal{C}^\alpha_{\mathrm{loc}}(\Omega)$*
3. *Further, if the obstacles ψ and φ are Hölder continuous up to the boundary, and the boundary values f are Hölder continuous on the boundary, we have converge in $\mathcal{C}^\alpha(\bar{\Omega})$.*

Proof. It is not difficult to see that the main difference in the proof with respect to Theorem 5.1 is in Step 4, i.e. obtaining that $\nabla u_i\to\nabla u$ a.e. This is due to the fact that a solution to the double obstacle problem is no longer a supersolution and therefore a different approach must by employed. We will adapt the idea contained in [16]. We may proceed as in the proof of Theorem 5.1 to find subsequences such that (5.6) and (5.8)-(5.9) still hold. We fix $G\Subset G'\Subset\Omega$ and $\eta\in\mathcal{C}^\infty_0(\Omega)$ such that $0\leq\eta\leq 1$, $\operatorname{supp}\eta\Subset G'$ and $\eta=1$ on G. Let $v_i=u_i+\eta(u-u_i)$. Since $u\in\mathcal{K}^{f,p_i(\cdot)}_{\psi,\varphi}(\Omega)$, we have

$$v_i\geq\;(1-\eta)\psi+\eta\psi=\psi,\qquad v_i\leq\;(1-\eta)\varphi+\eta\varphi=\varphi,$$

and since $v_i-f\in W^{1,p_i(\cdot)}_0(\Omega)$, $v_i\in\mathcal{K}^{f,p_i(\cdot)}_{\psi,\varphi}(\Omega)$, we can write

$$\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot(\nabla v_i-\nabla u_i)\;\mathrm{d}x\geq\;0.$$

Thus we have

$$\begin{aligned}
&\int_\Omega(\mathcal{A}_i(x,\nabla u_i)-\mathcal{A}_i(x,\nabla u))\cdot(\nabla u_i-\nabla u)\,\eta\;\mathrm{d}x\\
\leq\;&\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla\eta(u-u_i)\;\mathrm{d}x-\int_\Omega\eta\mathcal{A}_i(x,\nabla u)\cdot(\nabla u_i-\nabla u)\;\mathrm{d}x\\
=:\;&I_i^1+I_i^2.
\end{aligned}$$

At this point, using the structure conditions of $\mathcal{A}_i$, we see that

$$\begin{aligned} I_i^1 &\le \beta C \int_\Omega |\nabla u_i|^{p_i(x)-1} |u-u_i| \,\mathrm{d}x \\ &\le C\,||u_i-u||_{L^{p(\cdot)(1+\delta)}(\Omega)} \to 0 \end{aligned}$$

as $i \to \infty$, because of (5.6) and (5.8)-(5.9); here we set $\delta = \gamma/4$.

On the other hand

$$\begin{aligned} |I_i^2| &= \Big| \int_\Omega [\eta(\mathcal{A}_i(x,\nabla u) - \mathcal{A}_0(x,\nabla u)) \cdot (\nabla u_i - \nabla u) \\ &\quad + \eta \mathcal{A}_0(x,\nabla u) \cdot (\nabla u_i - \nabla u)] \,\mathrm{d}x \Big| \\ &\le \Big| \int_\Omega \eta(\mathcal{A}_i(x,\nabla u) - \mathcal{A}_0(x,\nabla u)) \cdot (\nabla u_i - \nabla u) \,\mathrm{d}x \Big| \\ &\quad + \Big| \int_\Omega \eta \mathcal{A}_0(x,\nabla u) \cdot (\nabla u_i - \nabla u) \,\mathrm{d}x \Big| \end{aligned}$$

The second term goes to zero as $i \to \infty$ due to (5.8). Moreover

$$\Big| \int_\Omega \eta(\mathcal{A}_i(x,\nabla u) - \mathcal{A}_0(x,\nabla u)) \cdot (\nabla u_i - \nabla u) \,\mathrm{d}x \Big| \le C||\mathcal{A}_i(x,\nabla u) - \mathcal{A}_0(x,\nabla u)||_{L^{p_i'(\cdot)}(\Omega)}$$

due to (5.6); here p_i' is the conjugate exponent of p_i. The fact that

$$\int_\Omega |\mathcal{A}_i(x,\nabla u) - \mathcal{A}_0(x,\nabla u)|^{p_i'(x)} \,\mathrm{d}x \to 0$$

as $i \to \infty$, then follows by Lebesgue's dominate convergence theorem: indeed, by assumption we have that

$$\mathcal{A}_i(x,\xi) \to \mathcal{A}_0(x,\xi) \qquad \text{for a.a. } x \in \Omega \text{ locally uniformly in } \mathbb{R}^n$$

and this implies

$$|\mathcal{A}_i(x,\nabla u(x) - \mathcal{A}_0(x,\nabla u(x))|^{p_i'(x)} \,\mathrm{d}x \to 0 \qquad \text{a.e. in } \Omega;$$

and the conclusion comes out by taking into account the estimate

$$\begin{aligned} |\mathcal{A}_i(x,\nabla u) - \mathcal{A}(x,\nabla u)|^{p_i'(x)} &\le C|\nabla u|^{p_i(x)} + C|\nabla u|^{(p(x)-1)p_i'(x)} \\ &\le C + C|\nabla u|^{p(x)(1+\delta)} + C|\nabla u|^{p(x)(1+\delta)}. \end{aligned}$$

Summing up, we have shown that

$$\int_G (\mathcal{A}_i(x,\nabla u_i) - \mathcal{A}_i(x,\nabla u)) \cdot (\nabla u_i - \nabla u) \,\mathrm{d}x \to 0,$$

as $i \to \infty$. This yields that

$$\lim_{i\to\infty} (\mathcal{A}_i(x,\nabla u_i) - \mathcal{A}_i(x,\nabla u)) \cdot (\nabla u_i - \nabla u) = 0.$$

At this point the conclusion of Step 4 is reached by contradiction as in [11]. The rest of the proof can be completed in a similar way, and we omit the details. □

Acknowledgments

The first author wishes to thank the kind hospitality of J. Kinnunen and the Nonlinear PDE research group at the Institute of Mathematics (Helsinki University of Technology) for the nice and friendly atmosphere there. The third author is also grateful to the University of Trento for the kind hospitality during his visit in autumn 2009. This project has been financially supported by the Academy of Finland and University of Trento. The third author was supported by the Norwegian Research Council project "Nonlinear Problems in Mathematical Analysis."

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