Γ-CONVERGENCE ANALYSIS FOR DISCRETE TOPOLOGICAL SINGULARITIES: THE ANISOTROPIC TRIANGULAR LATTICE AND THE LONG RANGE INTERACTION ENERGY

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ABSTRACT. We consider 2D discrete systems, described by scalar functions and governed by periodic interaction potentials. We focus on anisotropic nearest neighbors interactions in the hexagonal lattice and on isotropic long range interactions in the square lattice. In both these cases, we perform a complete Γ-convergence analysis of the energy induced by a configuration of discrete topological singularities. This analysis allows to prove the existence of many metastable configurations of singularities in the hexagonal lattice.

CONTENTS

Introduction

This paper deals with the asymptotic behaviour of the energy stored in a lattice, induced by a configuration of discrete topological singularities, as the atomic scale goes to zero.

Given an open bounded set $\Omega \subset \mathbb{R}^2$, a complex lattice Λ in $\mathbb{R}^2$, and a parameter $\varepsilon > 0$, we consider $\varepsilon\Lambda \cap \Omega$, which represents the reference configuration of our physical system. We focus on scalar systems governed by periodic potentials $\{g_{i,j}\}_{i,j\in\Lambda}$ acting on pairs of atoms of our lattice and we define the energy associated to a scalar filed $u : \varepsilon\Lambda \cap \Omega \to \mathbb{R}$ as

$$F_{\varepsilon,\Lambda}(u,\Omega) := \sum_{\varepsilon i, \varepsilon j \in \varepsilon\Lambda\cap\Omega} g_{i,j}(u(\varepsilon i) - u(\varepsilon j)).$$

In [3] (see also [14, 1, 2]), the asymptotic expansion, as $\varepsilon \to 0$, of the energy $F_{\varepsilon,\Lambda}$ has been rigorously derived in terms of Γ-convergence for $\Lambda = \mathbb{Z}^2$ and assuming that $g_{i,i+e_1} = g_{i,i+e_2}$ and $g_{i,j} = 0$ otherwise. Here we present some generalizations of the result in [3] for energies accounting for isotropic long range interactions in the square lattice and anisotropic nearest neighbors interactions on the hexagonal lattice which is a very relevant structure appearing in many context of discrete systems. The general case of anisotropic long range interaction energies is a very challenging goal and it goes beyond the purposes of this paper.

To clarify our setting, for every complex lattice Λ it is convenient to fix a map $L_\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L_\Lambda(\Lambda) = \mathbb{Z}^2$ and to consider a family of potentials $\{f_\xi\}_{\xi\in\mathbb{Z}^2}$ defined by $g_{i,j} := f_{L_\Lambda(i-j)}$. With this notation, the energy associated to a scalar field $u : \Omega \cap \varepsilon\Lambda \to \mathbb{R}$ can be rewritten as

$$F_{\varepsilon,\Lambda}(u,\Omega) = \sum_{\varepsilon i, \varepsilon j \in \varepsilon\Lambda\cap\Omega} f_{L_\Lambda(i-j)}(u(\varepsilon i) - u(\varepsilon j)).$$

We assume that f_ξ are non-negative one-periodic potentials, vanishing on the integers and quadratic in a suitable neighborhood of 0 (see Subsection 1.5 for the precise properties of the functions f_ξ).

As mentioned above, we focus only on two special kinds of systems: Either we assume $f_\xi = 0$ for any $\xi \notin \{e_1, e_2, e_1 + e_2\}$ or $f_\xi = f_{|\xi|}$ for any $\xi \in \mathbb{Z}^2$. The former case accounts for anisotropic nearest neighbors interactions in the hexagonal lattice and the corresponding energy will be denoteb by $F^{an}_{\varepsilon,\Lambda}$. The latter corresponds to isotropic long range interaction energies and the corresponding functional will be denoted by $F^{lr}_{\varepsilon,\Lambda}$.

Following along the lines the formalism in [5], discrete topological singularities are introduced through a discrete notion of topological degree of the field $v = e^{2\pi i u}$, i.e., by giving a suitable definition of the discrete curl of the gradient of u; loosely speaking, discrete topological singularities are points around which the discrete gradient of u has non trivial circulation and their distribution can be identified with a *discrete vorticity measure*, denoted by $\mu(u)$. This is a finite sum of Dirac masses centered in the triangular cells of the lattice and with multiplicities which are $+1$ of -1.

The main example of topological singularities we are interested in is given by the screw dislocations in crystals. In this context, $\varepsilon\Lambda$ is the projection of a complex 3D lattice $\varepsilon\mathcal{L}$ on a plane ortoghonal to e_3, which is assumed to be one of the generators of $\mathcal{L}$, $\varepsilon\Lambda \cap \Omega$ is the horizontal section of an infinite cylindrical crystal, and u represents an anti-plane displacement in the direction e_3 (see [4] for more details).

In the framework of linearized elasticity, the stored energy in its basic form can be written as

$$SD_{\varepsilon,\Lambda}(u,\Omega)=\frac{1}{2}\sum_{i,j\in\Omega\cap\varepsilon\Lambda}c_{L_\Lambda(i-j)}\operatorname{dist}^2(u(\varepsilon i)-u(\varepsilon j),Z),$$

where $\{c_\xi\}_{\xi\in\mathbb{Z}^2}$ are non-negative constants. The choice $f_\xi(a)=\frac{c_\xi}{2}\operatorname{dist}^2(a,\mathbb{Z})$ is consistent with the fact that $SD_{\varepsilon,\Lambda}$ represents the elastic energy of the crystal and that integer jumps of the displacement u, corresponding to plastic deformations, do not store elastic energy (see [5, 2, 14] for more details).

We remark that in this framework, the choice of the potentials in the functional $F^{an}_{\varepsilon,\Lambda}$ is relevant in order to deal with anti-plane energies defined in the most common crystal structures. As for instance, it can be seen that for Body Centered Cubic crystals, the projection Λ of the 3D lattice on the plane orthogonal to a diagonal of the cube gives the 2D hexagonal lattice and that the anti-plane energy with nearest neighbors interactions has the form of $F^{an}_{\varepsilon,\Lambda}$ (see [11, 4]).

The goal of this paper is the asymptotic expansion by Γ-convergence of the discrete energies $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$ as $\varepsilon\to 0$. In order to obtain these results we adopt the following strategy: To each $u:\varepsilon\Lambda\cap\Omega\to\mathbb{R}$ we associate the function $\bar u$ defined on the nodes of $\varepsilon\mathbb{Z}^2$ by setting

$$\bar u := u\circ L^{-1}_{\varepsilon,\Lambda},$$

with $L_{\varepsilon,\Lambda}(\cdot):=\varepsilon L_\Lambda(\cdot/\varepsilon)$. It follows that

$$F_{\varepsilon,\Lambda}(u,\Omega)=F_{\varepsilon,\mathbb{Z}^2}(\bar u,L_{\varepsilon,\Lambda}(\Omega)).$$

First we prove the Γ-convergence expansion for the functionals $F^{an}_{\varepsilon,\mathbb{Z}^2}$ and $F^{lr}_{\varepsilon,\mathbb{Z}^2}$ (see Section 2 and 3 respectively) and, afterwards, in Section 4, we translate such results for obtaining the Γ-expansion for $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$.

Our Γ-convergence analysis also contains a compactness statement, which represents the main difficulty. Indeed, one can see that short dipoles cost finite energy so that sequence having logarithmic bounded energy do not have necessarily bounded discrete vorticity. Therefore, the compactness result fails in the sense of weak star convergence but holds in a topology with respect to which annihilating dipoles have vanishing norm. This is the flat topology, i.e., the dual of Lipschitz continuous compactly supported functions.

As for the Γ-expansion of $F^{an}_{\varepsilon,\mathbb{Z}^2}$ we prove that

$$F^{an}_{\varepsilon,\mathbb{Z}^2}(u_\varepsilon,O)-\lambda_{\mathrm{self}}|\mu|(O)|\log\varepsilon|\xrightarrow{\Gamma}\mathbb{W}^{an}(\mu)+|\mu|(O)\gamma^{an},\tag{0.1}$$

with respect to the flat convergence of $\mu(u_\varepsilon)$ to μ. Here λ_{self} is a number depending on (the behaviour close to the wells of) $f_{e_1}, f_{e_2}, f_{e_1+e_2}$, μ is a finite sum of weighted Dirac deltas with degrees $d_i=\pm1$, $\mathbb{W}^{an}$ is the anisotropic version of the *renormalized energy* studied within the Ginzburg-Landau framework (see [6, 17]) and γ^{an} can be viewed as a core energy depending on the specific choice of the potentials f_ξ. The proof of this result is obtained through slight modifications of the techniques used in [3], since in our case not only anisotropies are allowed but we have also to deal with the interactions along the direction e_1+e_2.

As for the energies $F^{lr}_{\varepsilon,\mathbb{Z}^2}$ we get an expansion analogous to the one in (0.1). Indeed, thanks to our isotropy assumption ($f_\xi = f_{|\xi|}$), we can write $F^{lr}_{\varepsilon,\mathbb{Z}^2}$ as a sum of isotropic energies that account for nearest neighbors interactions (as done in [1]) and apply at each of these functionals the previous analysis.

Finally, in Section 5, as a consequence of our Γ-convergence result, we show that in the anisotropic case discrete systems exhibit many metastable configurations. Analogous results relative to the existence of metastable configurations have been recently obtained for isotropic energies in the square lattice in [3] and in the hexagonal lattice in [11, 12].

Concerning the dynamics of dislocations, the analysis developed in this paper is instrumental for the analysis of discrete screw dislocations along glide directions done in the companion paper [4].

The analysis of metastable configurations and dynamics of discrete topological singulaties in discrete systems governed by general long range interaction potentials is a fascinated and challenging problem, which, to our knowledge, is still open.

1. The discrete model for topological singularities

In this Section we introduce the discrete formalism used in the analysis of the problem we deal with. We will follow the approach of [5]; specifically, we will use the formalism and the notations in [2] (see also [14]).

1.1. **The discrete lattice.**

Here we recall the basic definitions of Bravais and complex lattices in $\mathbb{R}^2$.

Let v_1, v_2 be two linearly independent vectors in $\mathbb{R}^2$, referred to as *primitive vectors*. The Bravais lattice generated by v_1, v_2 is given by

$$\Lambda_{\mathcal{B}} := \{z_1 v_1 + z_2 v_2 \,:\, z_1, z_2 \in \mathbb{Z}\}. \tag{1.1}$$

Let $M \in \mathbb{N}$ and $\tau_1, \dots, \tau_M$ be M given vectors in $\mathbb{R}^2$, the complex lattice $\Lambda_{\mathcal{C}}$ generated by v_1, v_2 and with translation vectors $\tau_1, \dots, \tau_M$ is defined by

$$\Lambda_{\mathcal{C}} := \bigcup_{k=1}^{M} \{z_1 v_1 + z_2 v_2 + \tau_k \,:\, z_1, z_2 \in \mathbb{Z}\}. \tag{1.2}$$

Trivially, a Bravais lattice is a particular case of complex lattice, corresponding to $M = 1$ and $\tau_1 = 0$.

It is easy see that for any complex lattice Λ, there exists a piecewise linear map $L_\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$L_\Lambda(\Lambda) = \mathbb{Z}^2. \tag{1.3}$$

Moreover, if Λ is a Bravais lattice, then the application L_Λ is linear.

1.2. **Reference configuration.**

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz continuous boundary, representing the horizontal section of an infinite cylindrical crystal. We will consider discrete lattices casted in Ω, representing our discrete reference configuration. Then, we will introduce the notion of discrete topological singularity and the energy functionals.

Let Λ be a complex lattice in $\mathbb{R}^2$, and let $\varepsilon > 0$ be a lattice spacing parameter. Let L_Λ be a piecewise affine (linear if Λ is a Bravais lattice) transformation as in (1.3). We set

$$L_{\varepsilon,\Lambda}(x) := \varepsilon L_\Lambda(x/\varepsilon), \tag{1.4}$$

and we notice that there exist a linear map $\bar{L}_\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ and a constant $\bar{C} > 0$ such that

$$\|L_{\varepsilon,\Lambda} - \bar{L}_\Lambda\|_{L^\infty(\Omega)} \leq \bar{C}\varepsilon. \tag{1.5}$$

We will introduce the notion of discrete lattice Ω^0_ε casted in Ω. To this purpose, we introduce the polygonal domain Ω_ε as union of ε-triangles contained in Ω. In this respect, the ε-triangles will represent the minimal area elements in our model.

Let $\{T^+, T^-\}$ be the partition of the unit square $Q := [0,1]^2$ into two dimensional simplices defined by

$$T^+ := \{(x_1, x_2) \in Q \,:\, x_1 \geq x_2\},$$
$$T^- := \{(x_1, x_2) \in Q \,:\, x_1 \leq x_2\}.$$

We set

$$\Omega_{\varepsilon,\Lambda} := \bigcup_{\substack{j\in\mathbb{Z}^2 \\ \varepsilon j + \varepsilon T^\pm \subset L_{\varepsilon,\Lambda}(\bar{\Omega})}} L^{-1}_{\varepsilon,\Lambda}(\varepsilon j + \varepsilon T^\pm).$$

The reference lattice is given by $\Omega^0_{\varepsilon,\Lambda} := \varepsilon\Lambda \cap \Omega_{\varepsilon,\Lambda}$. The class of bonds is given by $\Omega^1_{\varepsilon,\Lambda} := \{(\varepsilon i, \varepsilon j) \in \Omega^0_{\varepsilon,\Lambda} \times \Omega^0_{\varepsilon,\Lambda} \,:\, i \neq j\}$. Finally, the class of ε-triangles is defined by

$$\Omega^2_{\varepsilon,\Lambda} := \{L^{-1}_{\varepsilon,\Lambda}(\varepsilon j + \varepsilon T^\pm) : \varepsilon j + \varepsilon T^\pm \subset \Omega_{\varepsilon,\Lambda}\}.$$

We will denote by $T^\pm_{i,\varepsilon} := L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon T^\pm)$ the generic element in Ω^2_ε.

Finally we define the *discrete boundary* of Ω as

$$\partial_{\varepsilon,\Lambda}\Omega := \partial\Omega_{\varepsilon,\Lambda} \cap \varepsilon\Lambda. \tag{1.6}$$

In the following, we will extend the use of these notations to any given open subset of $\mathbb{R}^2$.

1.3. **Discrete displacements and discrete topological singularities.** Here we introduce the classes of discrete functions on Ω^0_ε, and a notion of discrete topological singularities. To this purpose, we first set

$$\mathcal{AF}_{\varepsilon,\Lambda}(\Omega) := \left\{u : \Omega^0_{\varepsilon,\Lambda} \to \mathbb{R}\right\},$$

which represents the class of admissible scalar functions on Ω^0_ε.

Moreover, we introduce the class of admissible vector fields from Ω^0_ε to the set $\mathcal{S}^1$ of unit vectors in $\mathbb{R}^2$, by setting

$$\mathcal{AXY}_{\varepsilon,\Lambda}(\Omega) := \{v : \Omega^0_{\varepsilon,\Lambda} \to \mathcal{S}^1\}.$$

Notice that, to any function $u \in \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$, we can associate a function $v \in \mathcal{AXY}_{\varepsilon,\Lambda}(\Omega)$ setting

$$v := v(u) = e^{2\pi i u}.$$

In this framework, discrete topological singularities are defined on the triangular cells $T^\pm_{i,\varepsilon}$, which in turns provide the minimal resolution for their positions. Other variants could be taken into account, as for instance to consider primitive unit cells instead of triangles, and the analysis developed in this paper would apply with minor notational changes.

In order to define precisely discrete topological singularities, we first introduce a notion of discrete vorticity corresponding to both scalar and $\mathcal{S}^1$ valued functions. Let $P : \mathbb{R} \to \mathbb{Z}$ be defined as follows

$$P(t) = \operatorname{argmin}\left\{|t - s| \,:\, s \in \mathbb{Z}\right\}, \tag{1.7}$$

with the convention that, if the argmin is not unique, then we choose the minimal one. Let $u \in \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ be fixed. For every $T^{\pm}_{i,\varepsilon} \in \Omega^2_{\varepsilon,\Lambda}$ we introduce the discrete vorticity

$$\begin{aligned}
\alpha_u(T^-_{i,\varepsilon}) :=& \quad P(u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_1) - u(\varepsilon i)) \\
&+ P(u \circ L^{-1}_{\varepsilon}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_1 + \varepsilon e_2) - u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_1)) \\
&+ P(u(\varepsilon i) - u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_1 + \varepsilon e_2)) \\
\alpha_u(T^+_{i,\varepsilon}) :=& - P(u(\varepsilon i) - u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_1 + \varepsilon e_2)) \\
&- P(u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_2) - u(\varepsilon i)) \\
&- P(u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_1 + \varepsilon e_2) - u \circ L^{-1}_{\varepsilon,\Lambda}(L_{\varepsilon,\Lambda}(\varepsilon i) + \varepsilon e_2)).
\end{aligned}$$

One can easily see that α_u takes values in $\{-1, 0, 1\}$ and that $\alpha_u(T^-_{i,\varepsilon}) + \alpha_u(T^+_{i,\varepsilon}) \in \{-1, 0, 1\}$ for any $\varepsilon i \in \Omega^0_{\varepsilon,\Lambda}$.

Finally, we define the discrete vorticity measure $\mu(u)$ as follows

$$\mu(u) := \sum_{T^-_{i,\varepsilon} \in \Omega^2_{\varepsilon,\Lambda}} \alpha_u(T^-_{i,\varepsilon})\delta_{b(T^-_{i,\varepsilon})} + \sum_{T^+_{i,\varepsilon} \in \Omega^2_{\varepsilon,\Lambda}} \alpha_u(T^+_{i,\varepsilon})\delta_{b(T^+_{i,\varepsilon})}, \tag{1.8}$$

where $b(T^{\pm}_{i,\varepsilon})$ is the barycenter of the of the triangle $T^{\pm}_{i,\varepsilon}$.

This definition of discrete vorticity extends to $\mathcal{S}^1$ valued fields in the obvious way, by setting $\mu(v) = \mu(u)$ where u is any function in $\mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ such that $v(u) = v$. Moreover, by the very definition of $\mu(u)$, we have that for every open subset A of Ω we have that $\mu(u)(A)$ depends only on $u \llcorner \partial_{\varepsilon,\Lambda} A$.

Let $\mathcal{M}(\Omega)$ be the space of Radon measures in Ω and set

$$X(\Omega) := \left\{ \mu \in \mathcal{M}(\Omega) \,:\, \mu = \sum_{i=1}^{N} d_i \delta_{x_i}\,,\ N \in \mathbb{N}, d_i = \pm 1, x_i \in \Omega \right\},$$

$$\begin{aligned}
X_{\varepsilon,\Lambda}(\Omega) := \Bigg\{ \mu \in X : \mu = &\sum_{T^-_{i,\varepsilon} \in \Omega^2_{\varepsilon,\Lambda}} \alpha(T^-_{i,\varepsilon})\delta_{b(T^-_{i,\varepsilon})} + \sum_{T^+_{i,\varepsilon} \in \Omega^2_{\varepsilon,\Lambda}} \alpha_u(T^+_{i,\varepsilon})\delta_{b(T^+_{i,\varepsilon})}, \\
&\alpha(T^-_{i,\varepsilon}), \alpha(T^+_{i,\varepsilon}) \in \{-1, 0, 1\} \Bigg\}.
\end{aligned} \tag{1.9}$$

We will denote by $\|\mu\|_{\mathrm{flat}}$ the norm of the dual of $W^{1,\infty}_0(\Omega)$, referred to as *flat norm*, and by $\mu_n \overset{\mathrm{flat}}{\to} \mu$ the flat convergence of μ_n to μ. Moreover, we will localize such notation on any open set A writing $\mu_n \overset{\mathrm{flat}(A)}{\longrightarrow} \mu$.

1.4. **Discrete vorticity measure and Jacobian.**

Here we show the link between the discrete vorticity measure introduced above and the Jacobian of a "continuous" field. To this aim, let $O \subset \mathbb{R}^2$ be open and bounded and let $\Lambda = \mathbb{Z}^2$. To each $v \in \mathcal{AXY}_{\varepsilon,\mathbb{Z}^2}(O)$ we can associate its piecewise affine interpolation $\tilde{v}$ according with the triangulation $\{T^{\pm}_{i,\varepsilon}\}_{i\in\mathbb{Z}^2}$, i.e., for any $\varepsilon i \in O^2_{\varepsilon,\mathbb{Z}}$ we set

$$\begin{aligned}(1.10)\quad \tilde v(x) &= v(\varepsilon i) + \frac{v(\varepsilon i+\varepsilon e_1)-v(\varepsilon i)}{\varepsilon|\xi|}((x-\varepsilon i)\cdot e_1)\\ &\quad + \frac{v(\varepsilon i+\varepsilon e_2)-v(\varepsilon i)}{\varepsilon}((x-\varepsilon i)\cdot e_2) \quad \text{for } x\in T^-_{i,\varepsilon}\\ \tilde v^{\xi,h}(x) &= v(\varepsilon i) + \frac{v(\varepsilon i+\varepsilon e_2)-v(\varepsilon i)}{\varepsilon}((x-i)\cdot e_2)\\ &\quad + \frac{v(\varepsilon i+\varepsilon e_1)-v(\varepsilon i)}{\varepsilon}((x-\varepsilon i)\cdot e_1) \quad \text{for } x\in T^+_{i,\varepsilon}.\end{aligned}$$

One can easily verify that if A is an open subset of O with smooth boundary and if $|\tilde v| > c > 0$ on $\partial A_{\varepsilon,\mathbb{Z}^2}$, then

$$\mu(v) = \deg(\tilde v, \partial A_{\varepsilon,\mathbb{Z}^2}), \tag{1.11}$$

where, given an open bounded set $V\subset\mathbb{R}^2$ with Lipschitz continuous boundary, the *degree* of a function $w=(w_1,w_2)\in H^{\frac12}(\partial V;\mathbb{R}^2)$ with $|w|\ge c>0$, is defined by

$$\deg(w,\partial V) := \frac{1}{2\pi}\int_{\partial V}\left(\frac{w_1}{|w|}\nabla\frac{w_2}{|w|} - \frac{w_2}{|w|}\nabla\frac{w_1}{|w|}\right)\cdot\tau \,\mathrm{d}s\,. \tag{1.12}$$

In [8] it is proved that the quantities above are well defined and that the definition in (1.12) is well posed. Note that $\mu(v)(T^{\pm}_{i,\varepsilon})=0$ whenever $|\tilde v|>0$ on $T^{\pm}_{i,\varepsilon}$.

Finally, we remark that, for every $w\in H^1(V;\mathbb{R}^2)$, by Stokes theorem, we have

$$\int_V J\frac{w}{|w|}\,\mathrm{d}x = \deg(w,\partial V),$$

where Jw is the *Jacobian* of w and it is the L^1 function defined by $Jw := \det\nabla w$.

Here we recall two results about the Jacobian and the discrete vorticity measure, that will be useful in the proof of our Γ-convergence theorems.

Proposition 1.1 (Proposition 5.2, [2])**.** *Let $\{v_\varepsilon\}\in\mathcal{AXY}_{\varepsilon,\mathbb{Z}^2}(O)$ be a sequence such that $XY_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O)\le C|\log\varepsilon|$ for some constant $C>0$; then*

$$\|\tfrac{J\tilde v_\varepsilon}{\pi} - \mu(v_\varepsilon)\|_{\mathrm{flat}}\to 0 \quad \text{as } \varepsilon\to 0.$$

Lemma 1.2 (Lemma 1, [1])**.** *Let $A\subset\mathbb{R}^2$ be an open bounded set and let $\{w_\varepsilon\}$ and $\{z_\varepsilon\}$ be two sequences in $W^{1,2}(A;\mathbb{R}^2)$. If there exists a constant $C>0$ such that*

$$(a)\int_A |w_\varepsilon - z_\varepsilon|^2\,\mathrm{d}x \le C\varepsilon^2|\log\varepsilon|, \quad (b)\int_A|\nabla w_\varepsilon-\nabla z_\varepsilon|^2\,\mathrm{d}x\le C|\log\varepsilon|, \tag{1.13}$$

then $\|Jw_\varepsilon - Jz_\varepsilon\|_{\mathrm{flat}}\to 0$ as $\varepsilon\to 0$.

1.5. **The discrete energies.** Here we introduce a class of energy functionals defined on $\mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$. To this end, we fix L_Λ as in (1.3) and we consider interaction potentials defined on $\mathbb{Z}^2$. More precisely, let $\{f_\xi\}_{\xi\in\mathbb{Z}^2}$ be a family of 1-periodic potentials satisfying the following assumption: There exists a family of non-negative constants $\{c_\xi\}_{\xi\in\mathbb{Z}^2}$ with $c_{e_1}, c_{e_2}>0$ such that

$$f_\xi(a)\ge\frac{c_\xi}{2}|e^{2\pi i a}-1|^2 = c_\xi(1-\cos 2\pi a), \tag{1.14}$$

$$f_\xi(a) = 2\pi^2 c_\xi a^2 + \mathrm{O}(a^3). \tag{1.15}$$

We will focus on two specific cases: the anisotropic energy in the triangular lattice and the isotropic long range interaction energy.

The first one is obtained by assuming that $f_\xi \equiv 0$ if $\xi \notin \{e_1, e_2, e_1+e_2\}$; we define the anisotropic energy in the triangular lattice as

$$F^{an}_{\varepsilon,\Lambda}(u,\Omega) := \sum_{\substack{(\varepsilon i,\varepsilon j)\in\Omega^1_{\varepsilon,\Lambda}\\ L_\Lambda(i-j)\in\{e_1,e_2,e_1+e_2\}}} f_{L_\Lambda(i-j)}(u(\varepsilon i)-u(\varepsilon j)). \tag{1.16}$$

As for the case of isotropic long range interaction energy, we assume that the constants c_ξ satisfy:

$$c_\xi = c_{\xi^\perp} \text{ for every } \xi\in\mathbb{Z}^2 \text{ (where } (\xi_1,\xi_2)^\perp = (-\xi_2,\xi_1)); \tag{1.17}$$

$$\sum_{\xi\in\mathbb{Z}^2} c_\xi|\xi|^2 < +\infty. \tag{1.18}$$

and we define

$$F^{lr}_{\varepsilon,\Lambda}(u,\Omega) := \sum_{(\varepsilon i,\varepsilon j)\in\Omega^1_{\varepsilon,\Lambda}} f_{L_\Lambda(i-j)}(u(\varepsilon i)-u(\varepsilon j)). \tag{1.19}$$

The main motivation for our analysis comes from the study discrete screw dislocations in crystals and XY spin systems. In the screw dislocations case, the potentials $f_\xi(a)$ are nothing but $c_\xi \mathrm{dist}^2(a,\mathbb{Z})$; as for the spin systems, for any $v\in\mathcal{AXY}_{\varepsilon,\Lambda}(\Omega)$, we define

$$XY^{an}_{\varepsilon,\Lambda}(v,\Omega) := \frac{1}{2}\sum_{\substack{(\varepsilon i,\varepsilon j)\in\Omega^1_{\varepsilon,\Lambda}\\ L_\Lambda(i-j)\in\{e_1,e_2,e_1+e_2\}}} c_{L_\Lambda(i-j)}|v(\varepsilon i)-v(\varepsilon j)|^2, \tag{1.20}$$

$$XY^{lr}_{\varepsilon,\Lambda}(v,\Omega) := \frac{1}{2}\sum_{(\varepsilon i,\varepsilon j)\in\Omega^1_{\varepsilon,\Lambda}} c_{L_\Lambda(i-j)}|v(\varepsilon i)-v(\varepsilon j)|^2. \tag{1.21}$$

Also these potentials fit with our framework, once we rewrite it in terms of the phase u of v. Indeed, setting $f_\xi(a) = 1-\cos(2\pi a)$, we have

$$XY^{an}_{\varepsilon,\Lambda}(v,\Omega) = \frac{1}{2}\sum_{\substack{(\varepsilon i,\varepsilon j)\in\Omega^1_{\varepsilon,\Lambda}\\ L_\Lambda(i-j)\in\{e_1,e_2,e_1+e_2\}}} f_{L_\Lambda(i-j)}(u(\varepsilon i)-u(\varepsilon j)),$$

$$XY^{lr}_{\varepsilon,\Lambda}(v,\Omega) := \frac{1}{2}\sum_{(\varepsilon i,\varepsilon j)\in\Omega^1_{\varepsilon,\Lambda}} f_{L_\Lambda(i-j)}(u(\varepsilon i)-u(\varepsilon j)).$$

We notice that assumption (1.15) on $F^{an}_{\varepsilon,\Lambda}$ (resp. $F^{lr}_{\varepsilon,\Lambda}$) reads as

$$F^{an}_{\varepsilon,\Lambda}(u,\Omega) \ge XY^{an}_{\varepsilon,\Lambda}(e^{2\pi i u},\Omega) \quad (\text{resp. } F^{lr}_{\varepsilon,\Lambda}(u,\Omega) \ge XY^{lr}_{\varepsilon,\Lambda}(e^{2\pi i u},\Omega)). \tag{1.22}$$

Remark 1.3. *Notice that the functionals $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$ can be seen as functionals defined on the square lattice $\varepsilon\mathbb{Z}^2$. More precisely, for any $u\in\mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ we have*

$$F^{an}_{\varepsilon,\Lambda}(u,\Omega) = F^{an}_{\varepsilon,\mathbb{Z}^2}(u\circ L^{-1}_{\varepsilon,\Lambda}, L_{\varepsilon,\Lambda}(\Omega)) \tag{1.23}$$

$$F^{lr}_{\varepsilon,\Lambda}(u,\Omega) = F^{lr}_{\varepsilon,\mathbb{Z}^2}(u\circ L^{-1}_{\varepsilon,\Lambda}, L_{\varepsilon,\Lambda}(\Omega)). \tag{1.24}$$

In the following we will prove the expansion by Γ-convergence for the energies $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$. As mentioned in the Introduction, we will adopt the following strategy: In Sections 2 and 3 we will prove the Γ-expansion for the functionals $F^{an}_{\varepsilon,\mathbb{Z}^2}$ and $F^{lr}_{\varepsilon,\mathbb{Z}^2}$ respectively. Afterwards, in Section 4 we will use the Γ-convergence results above in order to prove the Γ-expansion of the energies $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$.

2. The Γ-convergence analysis for $F^{an}_{\varepsilon,\mathbb{Z}^2}$

In this section we develop the Γ-convergence analysis of the functionals $F^{an}_{\varepsilon,\mathbb{Z}^2}$ as $\varepsilon \to 0$. Such analysis is closely related to the one given for the isotropic case in [3, Sections 3 and 4], but requires some cares due to the presence of the anisotropies and of the interaction along the direction $\frac{e_1+e_2}{\sqrt{2}}$.

2.1. **The zero order Γ-convergence result for $F^{an}_{\varepsilon,\mathbb{Z}^2}$.** The essential ingredient in order to obtain the Γ-expansion of the energies $F^{an}_{\varepsilon,\mathbb{Z}^2}$ is given by a localized Γ-liminf inequality for this energy.

Let $O \subset \mathbb{R}^2$ open and bounded with Lipschitz continuous boundary.

Theorem 2.1. *Set* $\lambda_{self} := \sqrt{c_{e_1}c_{e_2} + c_{e_1}c_{e_1+e_2} + c_{e_2}c_{e_1+e_2}}$.
The following Γ*-convergence result holds true.*

(i) *(Compactness) Let* $\{\bar u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ *be such that* $F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) \le C|\log\varepsilon|$ *for some positive* C*. Then, up to a subsequence,* $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu$*, for some* $\mu \in X(O)$*.*

(ii) *(Localized* Γ*-liminf inequality) Let* $\{\bar u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ *be such that* $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i\delta_{x_i}$*, with* $d_i \in \mathbb{Z}\setminus 0$ *and* $x_i \in O$*. Then, there exists a constant* $C \in \mathbb{R}$ *such that, for any* $i = 1,\dots,M$ *and for every* $\sigma < \frac12 dist(B^{-\frac12}x_i, B^{-\frac12}(\partial\Omega) \cup \bigcup_{j\ne i} B^{-\frac12}x_j)$*, we have*

$$\liminf_{\varepsilon\to 0} F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, E^B_\sigma(x_i)) - \pi\lambda_{self}|d_i|\log\frac{\sigma}{\varepsilon} \ge C, \tag{2.1}$$

where B *is defined in* (2.5)*. In particular*

$$\liminf_{\varepsilon\to 0} F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) - \pi\lambda_{self}|\mu|(O)\log\frac{\sigma}{\varepsilon} \ge C.$$

(iii) *(*Γ*-limsup inequality) For every* $\mu \in X(O)$*, there exists a sequence* $\{\tilde u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ *such that* $\mu(\tilde u_\varepsilon) \overset{\text{flat}}{\to} \mu$ *and*

$$\pi\lambda_{self}|\mu|(\Omega) \ge \limsup_{\varepsilon\to 0}\frac{F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,\Omega)}{|\log\varepsilon|}.$$

The theorem above has been proved in [3] for $c_{e_1} = c_{e_2} = 1$ and $c_{e_1+e_2} = 0$ by combining a sharp lower bound of the energy on annuli without singularities with (a discrete modification of) the ball construction technique introduced by Sandier [15] and Jerrard [13]. In this paper we will give only the anisotropic counterparts of these tools (see Subsection 2.2 and 2.3 below). Then, the proof closely follows the lines of the one of [3, Theorem 3.1] and it is omitted.

2.2. **Lower bound on elliptic annuli.** We notice that, as a consequence of (1.22), it is enough to prove the lower bound of the energy for the functional $XY^{an}_{\varepsilon,\mathbb{Z}^2}$.

First of all, let us consider the continuous energy associated to $XY^{an}_{\varepsilon,\mathbb{Z}^2}$. More precisely, for every $v \in \mathcal{AXY}_{\varepsilon,\mathbb{Z}^2}(O)$, let $\tilde v : O_{\varepsilon,\mathbb{Z}^2} \to \mathbb{R}^2$ be the piecewise affine interpolation of v according with the triangulation $\{T^{\pm}_{i,\varepsilon}\}_{i\in\mathbb{Z}^2}$ defined in Subsection 1.2 (see (1.10) for the definition of $\tilde v$).

Using that

$$\left|\frac{v(\varepsilon i+\varepsilon e_1+\varepsilon e_2) - v(\varepsilon i)}{\varepsilon}\right|^2 = 2|\partial_{\frac{e_1+e_2}{\sqrt2}}\tilde v(x)|^2, \text{ for any } x \in T^-_{i,\varepsilon}\cup T^+_{i,\varepsilon} \text{ with } T^{\pm}_{i,\varepsilon} \in O^2_{\varepsilon,\mathbb{Z}^2},$$

it is easy to show that for any open subset $A \subset O$, the following holds true

$$\frac{1}{2}\int_{A_{\varepsilon,\mathbb{Z}^2}} \langle Q\nabla\tilde{v}, \nabla\tilde{v}\rangle \,\mathrm{d}x + \frac{1}{2}\int_{B^A_{\varepsilon,\mathbb{Z}^2}} \langle Q\nabla\tilde{v}, \nabla\tilde{v}\rangle \,\mathrm{d}x \geq XY_{\varepsilon,\mathbb{Z}^2}(v, A) \geq \frac{1}{2}\int_{A_{\varepsilon,\mathbb{Z}^2}} \langle Q\nabla\tilde{v}, \nabla\tilde{v}\rangle \,\mathrm{d}x, \tag{2.2}$$

where $B^A_{\varepsilon,\mathbb{Z}^2} := \{x \in A_{\varepsilon,\mathbb{Z}^2} : \mathrm{dist}(x, \partial A_{\varepsilon,\mathbb{Z}^2}) \leq \varepsilon\}$ and

$$Q := \begin{pmatrix} c_{e_1} + c_{e_1+e_2} & c_{e_1+e_2} \\ c_{e_1+e_2} & c_{e_2} + c_{e_1+e_2} \end{pmatrix}. \tag{2.3}$$

For any $A \subset \mathbb{R}^2$ open and bounded and for any $w \in H^1(A; \mathbb{R}^2)$, we define

$$\mathcal{F}^{an}(w, A) := \frac{1}{2}\int_A \langle Q\nabla w, \nabla w\rangle \,\mathrm{d}x = \frac{\sqrt{\det Q}}{2}\int_A \langle B\nabla w, \nabla w\rangle \,\mathrm{d}x, \tag{2.4}$$

where we have set

$$B := \frac{Q}{\sqrt{\det Q}}. \tag{2.5}$$

Finally, we notice that

$$\begin{aligned} \mathcal{F}^{an}(w, A) &= \frac{\sqrt{\det Q}}{2}\int_A |B^{\frac{1}{2}}\ \nabla w|^2 \,\mathrm{d}x \\ &= \frac{\sqrt{\det Q}}{2}\int_A |\nabla(w \circ B^{\frac{1}{2}})(B^{-\frac{1}{2}}x)|^2 \,\mathrm{d}x \\ &= \frac{\sqrt{\det Q}}{2}\int_{B^{-\frac{1}{2}}(A)} |\nabla(w \circ B^{\frac{1}{2}})(y)|^2 \,\mathrm{d}y, \end{aligned} \tag{2.6}$$

where in the last line we have used the change of variable $y = B^{-\frac{1}{2}}x$ and the fact that $\det(B^{-\frac{1}{2}}) = 1$.

We remark that by the very definition of Q in (2.3),

$$\lambda_{\mathrm{self}} = \sqrt{\det Q}. \tag{2.7}$$

Recalling the definition of B in (2.5), for any $\rho > 0$ and for any $x \in \mathbb{R}^2$, we set

$$E^B_\rho(x) := B^{\frac{1}{2}}(B_\rho(B^{-\frac{1}{2}}x)); \tag{2.8}$$

moreover we set $E^B_\rho := E^B_\rho(0)$.

We first give the lower bound of the energy $\mathcal{F}^{an}$ on elliptic annuli. Let $0 < r < R$ and let $w \in H^1(E^B_R \setminus E^B_r; \mathcal{S}^1)$ with $\deg(w, \partial E^B_R) = d$. Set $w^B(y) := w(B^{\frac{1}{2}}y)$, by (2.6) and Jensen's inequality, we get

$$\begin{aligned} \mathcal{F}^{an}(w, E^B_R \setminus E^B_r) &= \frac{\lambda_{\mathrm{self}}}{2}\int_{B_R\setminus B_r} |\nabla w^B(y)|^2 \,\mathrm{d}y \\ &\geq \frac{\lambda_{\mathrm{self}}}{2}\int_r^R\int_{\partial B_\rho} |(w^B \times \nabla w^B)\cdot\tau|^2 \,\mathrm{d}s\,\mathrm{d}\rho \geq \lambda_{\mathrm{self}}\int_r^R \frac{1}{\rho}\pi d^2 \,\mathrm{d}\rho \geq \lambda_{\mathrm{self}}\pi|d|\log\frac{R}{r}, \end{aligned} \tag{2.9}$$

where we have used that $\deg(w^B, \partial B_R) = \deg(w, \partial E^B_R) = d$.

In the following Proposition we show that also for the functionals $XY^{an}_{\varepsilon,\mathbb{Z}^2}$ an estimate analogous to (2.9) holds up to an error due to the discrete setting. We first notice that, by its very definition, B is symmetric and hence also $B^{-\frac{1}{2}}$ is. Using that $\det B^{-\frac{1}{2}} = 1$, we have that the eigenvalues of $B^{-\frac{1}{2}}$ are of the form λ, $-\frac{1}{\lambda}$. We set $m := \max\left\{\lambda^2, \frac{1}{\lambda^2}\right\}$.

Proposition 2.2. *Fix $\varepsilon > 0$ and let $m\sqrt{2}\varepsilon < r < R - m\sqrt{2}\varepsilon$. For any field $v : (E^B_R \setminus E^B_r) \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}^1$ with $|\tilde v| \geq \frac{1}{2}$ in $E^B_{R-\sqrt{2}\varepsilon} \setminus E^B_{r+\sqrt{2}\varepsilon}$, it holds*

$$XY^{an}_{\varepsilon,\mathbb{Z}^2}(v, E^B_R \setminus E^B_r) \geq \lambda_{self}\, \pi|\mu(v)(E^B_r)| \log \frac{R}{r + \varepsilon\big(\alpha|\mu(v)(E^B_r)| + m\sqrt{2}\big)}, \tag{2.10}$$

where $\alpha > 0$ is a universal constant.

Proof. By (2.2), using Fubini's theorem, we have that

$$XY^{an}_{\varepsilon,\mathbb{Z}^2}(v, E^B_R \setminus E^B_r) \geq \int_{r+m\sqrt{2}\varepsilon}^{R-m\sqrt{2}\varepsilon} \mathcal{F}^{an}(\tilde v, \partial E^B_\rho)\, \mathrm{d}\rho. \tag{2.11}$$

Fix $r + m\sqrt{2}\varepsilon < \rho < R - m\sqrt{2}\varepsilon$ and let T be a simplex of the triangulation of the ε-lattice. Set $\gamma_T(\rho) := \partial E^B_\rho \cap T$, let $\bar\gamma_T(\rho)$ be the segment joining the two extreme points of $\gamma_T(\rho)$ and let $\bar\gamma(\rho) = \bigcup_T \bar\gamma_T(\rho)$; then

$$\begin{aligned}
\mathcal{F}^{an}(\tilde v, \partial E^B_\rho) &= \frac{1}{2}\int_{\cup_T \gamma_T(\rho)} c_{e_1}|\partial_{e_1}\tilde v|^2 + c_{e_2}|\partial_{e_2}\tilde v|^2 + 2c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt{2}}}\tilde v|^2\, \mathrm{d}s \\
&= \tfrac{1}{2}\textstyle\sum_T (c_{e_1}|\partial_{e_1}\tilde v_{|T}|^2 + c_{e_2}|\partial_{e_2}\tilde v_{|T}|^2 + 2c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt{2}}}\tilde v_{|T}|^2)\mathcal{H}^1(\gamma_T(\rho)) \\
&\geq \tfrac{1}{2}\textstyle\sum_T (c_{e_1}|\partial_{e_1}\tilde v_{|T}|^2 + c_{e_2}|\partial_{e_2}\tilde v_{|T}|^2 + 2c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt{2}}}\tilde v_{|T}|^2)\mathcal{H}^1(\bar\gamma_T(\rho)) \\
&= \mathcal{F}^{an}(\tilde v, \bar\gamma_\rho).
\end{aligned} \tag{2.12}$$

Set $m(\rho) := \min_{\bar\gamma(\rho)} |\tilde v|$. Set $\tilde v^B(y) := \tilde v(B^{\frac{1}{2}}y)$. By (2.6), we have

$$\mathcal{F}^{an}(\tilde v, \bar\gamma(\rho)) = \frac{\lambda_{\rm self}}{2}\int_{B^{-\frac{1}{2}}(\bar\gamma(\rho))} |\nabla \tilde v^B(y)|^2\, \mathrm{d}y. \tag{2.13}$$

Using Jensen's inequality and the fact that $\mathcal{H}^1(\bar\gamma(\rho)) \leq \mathcal{H}^1(\partial E_\rho) = \mathcal{H}^1(B^{\frac{1}{2}}(\partial B_\rho))$, we get

$$\begin{aligned}
\frac{1}{2}\int_{\bar\gamma(\rho)} |\nabla \tilde v^B|^2\, \mathrm{d}s &\geq \frac{1}{2}\int_{B^{-\frac{1}{2}}(\bar\gamma(\rho))} m^2(\rho)\left|\left(\frac{\tilde v^B}{|\tilde v^B|} \times \nabla\frac{\tilde v^B}{|\tilde v^B|}\right)\cdot\tau\right|^2 \mathrm{d}s \\
&\geq \frac{1}{2}\frac{m^2(\rho)}{\mathcal{H}^1(L(\bar\gamma(\rho)))}\left|\int_{B^{-\frac{1}{2}}(\bar\gamma(\rho))}\left(\frac{\tilde v^B}{|\tilde v^B|} \times \nabla\frac{\tilde v^B}{|\tilde v^B|}\right)\cdot\tau\, \mathrm{d}s\right|^2 \\
&\geq \frac{m^2(\rho)}{\rho}\pi|d|^2
\end{aligned} \tag{2.14}$$

where we have set $d := \deg(\tilde v, \partial E^B_\rho) = \mu(v)(E^B_r)$, which does not depend on ρ (since $|\tilde v| \geq 1/2$) and coincides with $\deg(\tilde v^B, \partial B_\rho)$. Moreover, by elementary geometry arguments (see the proof of [3, Proposition 3.2] for more details), we have that there exists a universal constant $\bar\alpha$ such that

$$\mathcal{F}^{an}(\tilde v, \partial E^B_\rho) \geq \bar\alpha\frac{1 - m^2(\rho)}{\varepsilon}. \tag{2.15}$$

In view of (2.14) and (2.15) for any $r + m\sqrt{2}\varepsilon < \rho < R - m\sqrt{2}\varepsilon$ we have

$$\mathcal{F}^{an}(\tilde v, \partial E^B_\rho) \geq \frac{m^2(\rho)}{\rho}\pi|d| \vee \bar\alpha\frac{1 - m^2(\rho)}{\varepsilon} \geq \frac{\lambda_{\rm self}\pi|d|\bar\alpha}{\varepsilon\lambda_{\rm self}\pi|d| + \bar\alpha\rho}.$$

By this last estimate and (2.11) we get

$$XY^{an}_{\varepsilon,\mathbb{Z}^2}(v,E^B_R\setminus E^B_r)\ge\lambda_{\mathrm{self}}\pi|\mu(v)(E^B_r)|\log\frac{\varepsilon(\frac{\lambda_{\mathrm{self}}\pi}{\bar\alpha}|\mu(v)(E^B_r)|-m\sqrt{2})+R}{\varepsilon(\frac{\lambda_{\mathrm{self}}\pi}{\bar\alpha}|\mu(v)(E^B_r)|+m\sqrt{2})+r}.$$

Assuming without loss of generality that $\bar\alpha<1$, we immediately get (2.10) with $\alpha=\frac{\lambda_{\mathrm{self}}\pi}{\bar\alpha}$. □

2.3. **Ellipse Construction.** Here we introduce a slight modification of the *ball construction* introduced in [15, 13]. We follow the formalism of [3, Subsection 3.3], where this construction has been revisited in order to deal with isotropic discrete energies. Since the energies $XY^{an}_{\varepsilon,\mathbb{Z}^2}$ are anisotropic, we are led to consider ellipses in place of balls (as in [16]).

Let $G:\mathbb{R}^2\to\mathbb{R}^2$ be an isomorphism. For any $\rho>0$ and for every $x\in\mathbb{R}^2$, we set

$$E^G_\rho(x):=G^{-1}(B_\rho(Gx)). \tag{2.16}$$

Let $\mathcal{E}=\left\{E^G_{R_1}(x_1),\dots,E^G_{R_N}(x_N)\right\}$ be a finite family of pairwise disjoint ellipses in $\mathbb{R}^2$ of the type in (2.16) and let $\mu=\sum_{i=1}^N d_i\delta_{x_i}$ with $d_i\in\mathbb{Z}\setminus\{0\}$. Let F be a positive superadditive set function on the open subsets of $\mathbb{R}^2$, i.e., such that $F(A\cup B)\ge F(A)+F(B)$, whenever A and B are open and disjoint. We assume that there exist two constants $c,C>0$ such that

$$F(A^G_{r,R}(x))\ge C\pi|\mu(E^G_r(x))|\log\frac{R}{c+r}, \tag{2.17}$$

for any elliptic annulus $A^G_{r,R}(x)=E^G_R(x)\setminus E^G_r(x)$, with $A^G_{r,R}(x)\subset\Omega\setminus\bigcup_i E^G_{R_i}(x_i)$.

Let t be a parameter which represents an artificial time. For any $t>0$ one can construct (see [3]) a finite family of pairwise disjoint balls $\mathcal{B}(t)$ satisfying

(1) $\bigcup_{i=1}^N B_{R_i}(Gx_i)\subset\bigcup_{B\in\mathcal{B}(t)}B$,
(2) $\sum_{B\in\mathcal{B}(t)}R(B)\le(1+t)\sum_i R_i+(1+t)cN(N^2+N+1)$, where $R(B)$ denotes the radius of the ball B.

For every t let $\mathcal{E}^G(t):=\{G^{-1}(B)\}_{B\in\mathcal{B}(t)}$. Using the same arguments in [3], one can show that

$$F(E^G)\ge C\pi|\mu(E^G)|\log(1+t)\quad\text{for any } E^G\in\mathcal{E}^G(t)\text{ with } E^G\subset\Omega. \tag{2.18}$$

2.4. **The anisotropic renormalized energy and the first-order Γ-limit.** Here we recall and revisit the main definitions and results of [6] we need in order to state our Γ-expansion result (Theorem 2.5).

Fix $\mu=\sum_{i=1}^M d_i\delta_{x_i}$ with $d_i\in\{-1,+1\}$ and $x_i\in O$. In order to define the anisotropic renormalized energy, let $\Phi_{Q,O}$ the solution to the following problem

$$\begin{cases}\operatorname{div} Q\nabla\Phi_{Q,O}=\lambda_{\mathrm{self}}2\pi\mu & \text{in } O\\ \Phi_{Q,O}=0 & \text{on } \partial O,\end{cases}$$

and let $R_{Q,O}(x)=\Phi_{Q,O}(x)-\sum_{i=1}^M d_i\log|B^{-\frac12}(x-x_i)|$. Notice that $R_{Q,O}$ satisfies $\operatorname{div} Q\nabla R_{Q,O}=0$ in O and $R_{Q,O}(x)=-\sum_{i=1}^M d_i\log|B^{-\frac12}(x-x_i)|$ for any $x\in\partial O$. The *anisotropic renormalized energy* corresponding to the configuration μ is then defined by

$$\mathbb{W}^{an}_{O}(\mu) := -\pi\, \lambda_{\mathrm{self}}(\sum_{i\neq j} d_i d_j \log|B^{-\frac{1}{2}}(x_i - x_j)| + \pi\sum_{i=1}^{M} d_i R_{Q,O}(x_i)). \tag{2.19}$$

It is easy to see that if $Q = I$, then $\mathbb{W}^{an}_{O}(\mu) = \mathbb{W}_O(\mu)$ where $\mathbb{W}_O$ is the classical isotropic renormalized energy defined in the Ginzburg-Landau framework (see [6]) and given by

$$\mathbb{W}_O(\sum_{i=1}^{M} d_i\delta_{y_i}) := -\pi(\sum_{i\neq j} d_i d_j \log|y_i - y_j| + \pi\sum_{i=1}^{M} d_i R_{I,O}(y_i)). \tag{2.20}$$

In general, using the change of variable $B^{-\frac{1}{2}}$, we have

$$\mathbb{W}^{an}_{O}(\mu) = \lambda_{\mathrm{self}}\mathbb{W}_{B^{-\frac{1}{2}}(O)}(B^{-\frac{1}{2}}\mu), \tag{2.21}$$

where we have denoted by $B^{-\frac{1}{2}}\mu$ the push-forward of the measure μ through $B^{-\frac{1}{2}}$, i.e. $B^{-\frac{1}{2}}\mu := \sum_{i=1}^{M} d_i\delta_{B^{-\frac{1}{2}}x_i}$.

We show now that $\mathbb{W}^{an}_{A}(\mu)$ is continuous with respect to the Hausdorff convergence of the sets A. We recall that the *Hausdorff distance* among two closed subsets $C_1, C_2 \subset \mathbb{R}^2$ is defined as follows

$$\mathrm{d}_H(C_1, C_2) := \max\left\{\sup_{x\in C_1}\inf_{y\in C_2}\mathrm{dist}(x,y), \sup_{y\in C_2}\inf_{x\in C_1}\mathrm{dist}(x,y)\right\}.$$

Let $\{A^h\}$ be a sequence of open bounded subsets of A such that $\mathrm{supp}\,(\mu)\subset A^h$ for any $h\in\mathbb{N}$; then

$$\mathrm{d}_H((A^h)^c, A^c)\to 0 \,\mathrm{as}\, h\to\infty \Rightarrow \mathbb{W}^{an}_{A^h}(\mu) \text{ converges uniformly to } \mathbb{W}^{an}_{A}(\mu), \tag{2.22}$$

where for any $U\subset\mathbb{R}^2$, we have set $B^c := \mathbb{R}^2\setminus U$.

To this end, by (2.21), it is enough to prove that

$$\mathbb{W}_{B^{-\frac{1}{2}}(A^h)}(B^{-\frac{1}{2}}\mu) \text{ converges uniformly to } \mathbb{W}_{B^{-\frac{1}{2}}(A)}(B^{-\frac{1}{2}}\mu),$$

and, more precisely, that,

$$R_{I,B^{-\frac{1}{2}}(A^h)} \text{converges uniformly to } R_{I,B^{-\frac{1}{2}}(A)} \text{on the compact subsets of } A. \tag{2.23}$$

Set $y_i := B^{-\frac{1}{2}}x_i$ and $\nu := B^{-\frac{1}{2}}\mu = \sum_{i=1}^{M} d_i\delta_{y_i}$. For any $h\in\mathbb{N}$ we set $\mathcal{A}^h := B^{-\frac{1}{2}}(A^h)$ and $\mathcal{A} := B^{-\frac{1}{2}}(A)$. Trivially, $\mathrm{supp}\,(\nu)\subset\mathcal{A}^h$ and $\mathrm{dist}_H((\mathcal{A}^h)^c, \mathcal{A}^c)\to 0$ as $h\to\infty$. The interested reader can prove that such condition is equivalent to the assumption that for any compact subset $K\subset\subset\Omega$, $K\subset\mathcal{A}^h$ for h sufficiently large.

By its very definition, $R_{I,\mathcal{A}^h}$ is the solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{A}^h \\ u(\cdot) = -\sum_{i=1}^{M} d_i\log|\cdot - y_i| & \text{on } \partial\mathcal{A}^h. \end{cases}$$

Proposition 2.3 below, applied with $u_h = R_{I,\mathcal{A}^h}$ and $u_0 = R_{I,\mathcal{A}}$, proves that (2.23) holds true, whence (2.22) follows

Proposition 2.3. *Let $\mathcal{A}\subset\mathbb{R}^2$ open bounded with Lipschitz boundary and let $\{\mathcal{A}^h\}$ be a sequence of open bounded Lipschitz subsets of $\mathcal{A}$ such that* $\mathrm{d}_H((\mathcal{A}^h)^c, \mathcal{A}^c)\to 0$

as $h \to \infty$. Furthermore, let $f \in C^\infty$ outside a compact subset of $\mathcal{A}$. For any $h \in \mathbb{N}$ let u_h be the solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{A}^h \\ u = f & \text{on } \partial\mathcal{A}^h \end{cases}$$

and let u_0 be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{A} \\ u = f & \text{on } \partial\mathcal{A} \end{cases}$$

Then u_h converges uniformly to u_0 on the compact subsets of $\mathcal{A}$.

Proof. First of all we notice that, by the classical theory on harmonic functions, $u_h \in C^\infty(\mathcal{A}^h) \cap C(\bar{\mathcal{A}}^h)$ and $u_0 \in C^\infty(\mathcal{A}) \cap C(\bar{\mathcal{A}})$. Fix now a compact $K \subset \mathcal{A}$. By the hypothesis, for h sufficiently large, $K \subset \mathcal{A}^h$. Moreover $v_h = u_h - u_0$ is solution of the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{A}^h \\ v = f - u_0 & \text{on } \partial\mathcal{A}^h. \end{cases}$$

By the maximum principle of harmonic functions, we have that

$$\max_K |v_h| \le \max_{A^h} |v_h| = \max_{\partial\mathcal{A}^h} |f - u_0|.$$

The claim follows noticing that u_0 is continuous up to the boundary. $\square$

Through this section and whenever the dependence on the domain is clear from the context, we will use $\mathbb{W}^{an}(\mu)$ in place of $\mathbb{W}^{an}_O(\mu)$.

Let $\sigma > 0$ be such that the ellipses $E^B_\sigma(x_i)$ are pairwise disjoint and contained in O and set $O^B_\sigma := O \setminus \cup_{i=1}^M E^B_\sigma(x_i)$. It is convenient to consider (as done in [6]) the following auxiliary minimum problems.

$$m^{an}(\sigma,\mu) := \min_{w \in H^1(O^B_\sigma;\mathcal{S}^1)} \left\{ \mathcal{F}^{an}(w) \,:\, \deg(w, \partial E^B_\sigma(x_i)) = d_i \right\}, \tag{2.24}$$

$$\tilde{m}^{an}(\sigma,\mu) := \min_{w \in H^1(O_\sigma;\mathcal{S}^1)} \Big\{ \mathcal{F}^{an}(w) \,: \\ w(\cdot) = \frac{\alpha_i}{\sigma^{d_i}} (B^{-\frac{1}{2}}(\cdot - x_i))^{d_i} \text{ on } \partial E^B_\sigma(x_i),\, |\alpha_i| = 1 \Big\}. \tag{2.25}$$

For any $y \in \mathbb{R}^2 \setminus \{0\}$, we define $\theta(y)$ as the polar coordinate $\arctan y_2/y_1$ and let $\theta^B(x) := \theta(B^{-\frac{1}{2}}x)$. Moreover, for any $i = 1, \dots, M$ we set

$$\theta^B_i(x) := \theta(B^{-\frac{1}{2}}(x - x_i)). \tag{2.26}$$

Given $\varepsilon > 0$, we introduce the discrete minimization problem in the ellipse E^B_σ

$$\gamma^{an}(\varepsilon,\sigma) := \min_{\bar{u} \in \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(E^B_\sigma)} \left\{ F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar{u}, E^B_\sigma) \,:\, 2\pi\,\bar{u}(\cdot) = \theta^B(\cdot) \text{ on } \partial_{\varepsilon,\mathbb{Z}^2} E^B_\sigma \right\}, \tag{2.27}$$

where the discrete boundary $\partial_{\varepsilon,\mathbb{Z}^2}$ is defined in (1.6).

Theorem 2.4. *It holds*

$$\lim_{\sigma\to 0} m^{an}(\sigma,\mu) - \pi\lambda_{self}|\mu|(\Omega)|\log\sigma| \\ = \lim_{\sigma\to 0} \tilde{m}^{an}(\sigma,\mu) - \pi\lambda_{self}|\mu|(\Omega)|\log\sigma| = \mathbb{W}^{an}(\mu). \tag{2.28}$$

Moreover, for any fixed $\sigma > 0$, the following limit exists finite

$$\lim_{\varepsilon\to 0}(\gamma^{an}(\varepsilon,\sigma) - \pi\lambda_{self}|\log\frac{\varepsilon}{\sigma}|) =: \gamma^{an} \in \mathbb{R} \tag{2.29}$$

The proof of (2.28) is a consequence of [3, Theorem 4.1] (see also [6]) and of the change of variable $y = B^{-\frac{1}{2}}x$. We briefly sketch it.

Sketch of the Proof. Set

$$m(\sigma, B^{-\frac{1}{2}}\mu) := \min_{z\in H^1(B^{-\frac{1}{2}}(O)\setminus B_\sigma(B^{-\frac{1}{2}}x_i);\mathcal{S}^1)} \Big\{ \int_{B^{-\frac{1}{2}}(O)\setminus B_\sigma(B^{-\frac{1}{2}}x_i)} |\nabla z|^2 \,\mathrm{d}x \ : \ \deg(z, \partial^B_\sigma(B^{-\frac{1}{2}}x_i)) = d_i \Big\},$$

$$\tilde m(\sigma, B^{-\frac{1}{2}}\mu) := \min_{z\in H^1(O^B_\sigma;\mathcal{S}^1)} \Big\{ \int_{B^{-\frac{1}{2}}(O)\setminus B_\sigma(B^{-\frac{1}{2}}x_i)} |\nabla z|^2 \,\mathrm{d}x \ : \ z(\cdot) = \frac{\alpha_i}{\sigma^{d_i}}(\cdot - x_i)^{d_i} \quad \text{on } \partial B_\sigma(B^{-\frac{1}{2}}x_i),\ |\alpha_i| = 1 \Big\}.$$

By [3, Theorem 4.1] we have that

$$\lim_{\sigma\to 0} m(\sigma, B^{-\frac{1}{2}}\mu) - \pi|B^{-\frac{1}{2}}\mu|(O)|\log\sigma| = \lim_{\sigma\to 0} \tilde m(\sigma, B^{-\frac{1}{2}}\mu) - \pi|B^{-\frac{1}{2}}\mu|(O)|\log\sigma| = \mathbb{W}(B^{-\frac{1}{2}}\mu).$$

It is easy to see that, if z_σ is a minimizer of the problem $m(\sigma, B^{-\frac{1}{2}}\mu)$ (resp. $\tilde m(\sigma, B^{-\frac{1}{2}}\mu)$), then $w_\sigma = z_\sigma \cdot B^{-\frac{1}{2}}$ is a minimizer of the problem $m^{an}(\sigma,\mu)$ (resp. $\tilde m^{an}(\sigma,\mu)$). Moreover, by (2.4),

$$m(\sigma, B^{-\frac{1}{2}}\mu) = m^{an}(\sigma,\mu) \quad (\text{resp. } \tilde m(\sigma, B^{-\frac{1}{2}}\mu) = \tilde m^{an}(\sigma,\mu)). \tag{2.30}$$

The claim follows combining (2.30) with (2.21).

As for (2.29), its proof is identical to the one of [3, formula (4.6)] and it is omitted. □

2.5. **The first-order Γ-convergence result for $F^{an}_{\varepsilon,\mathbb{Z}^2}$.** We are now in a position to state the first-order Γ-convergence result for the functionals $F^{an}_{\varepsilon,\mathbb{Z}^2}$.

Theorem 2.5. *The following* Γ*-convergence result holds true.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{\bar u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ be a sequence satisfying $F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) - M\pi\lambda_{self}|\log\varepsilon| \le C$. Then, up to a subsequence, $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in O$ and $\sum_i |d_i| \le M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-lim inf inequality) Let $\{\bar u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ be such that $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu$, with $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in O$ for every i. Then,*

$$\liminf_{\varepsilon\to 0} F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) - M\pi\lambda_{self}\,|\log\varepsilon| \ge \mathbb{W}^{an}(\mu) + M\gamma^{an}. \tag{2.31}$$

(iii) *(Γ-lim sup inequality) Given $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in O$ for every i, there exists $\{\bar u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ with $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu$ such that*

$$F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) - M\pi\lambda_{self}\,|\log\varepsilon| \to \mathbb{W}^{an}(\mu) + M\gamma^{an}.$$

Proof. The proof of Theorem 2.5 closely follows the proof of [3, Theorem 4.2] but for the reader's convenience we include it. Recalling that $F^{an}_{\varepsilon,\mathbb{Z}^2}(u) \ge XY^{an}_{\varepsilon,\mathbb{Z}^2}(e^{2\pi i u})$, the proof of the compactness property (i) will be done for $F^{an}_{\varepsilon,\mathbb{Z}^2} = XY^{an}_{\varepsilon,\mathbb{Z}^2}$. On the other hand, the constant γ^{an} depends on the potentials f_ξ ($\xi \in \{e_1, e_2, e_1+e_2\}$), so its derivation requires a specific proof.

Let us fix some notation we will use in this proof. We recall that $E^B_\rho(x)$ is an ellipse of the form (2.8). For any $0<r<R$ and $x\in\mathbb{R}^2$, set

$$A^B_{r,R}(x) := E^B_R(x)\setminus E^B_r(x). \tag{2.32}$$

Moreover, for any $\bar u_\varepsilon \in \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ we set $v_\varepsilon := v(\bar u_\varepsilon) = e^{2\pi i \bar u_\varepsilon}$ and we indicate with $\tilde v_\varepsilon$ the piecewise affine interpolation of v_ε defined in (1.10).

Proof of (i): Compactness. The fact that, up to a subsequence, $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $\sum_{i=1}^N |d_i| \le M$ is a direct consequence of the zero order Γ-convergence result stated in Theorem 2.1 (i). Assume now $\sum_{i=1}^N |d_i| = M$ and let us prove that $|d_i|=1$. Let $0<\sigma_1<\sigma_2$ be such that $E^B_{\sigma_2}(x_i)$ are pairwise disjoint and contained in O and let ε be small enough so that $E^B_{\sigma_2}(x_i)$ are contained in $O_{\varepsilon,\mathbb{Z}^2}$. Since $F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) \ge XY^{an}_{\varepsilon,\mathbb{Z}^2}(e^{2\pi i\bar u_\varepsilon}, O)$,

$$F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O) \ge \sum_{i=1}^N XY^{an}_{\varepsilon,\mathbb{Z}^2}(e^{2\pi i\bar u_\varepsilon}, E^B_{\sigma_1}(x_i)) + \sum_{i=1}^N XY^{an}_{\varepsilon,\mathbb{Z}^2}(e^{2\pi i\bar u_\varepsilon}, A^B_{\sigma_1,\sigma_2}(x_i)). \tag{2.33}$$

Moreover let t be a positive number and let ε be small enough so that $t > m\sqrt{2}\varepsilon$. Then, by (2.1) and (2.2), we get

$$F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O) \ge \pi\lambda_{\mathrm{self}} \sum_{i=1}^N |d_i| \log\frac{\sigma_1}{\varepsilon} + \lambda_{\mathrm{self}}\mathcal{F}^{an}(\tilde v_\varepsilon, A^B_{\sigma_1+t,\sigma_2-t}(x_i)) + C. \tag{2.34}$$

By the energy bound and by the definition of $\mathcal{F}^{an}$, we deduce that

$$\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x \le \frac{2}{\min\{c_{e_1},c_{e_2}\}}\mathcal{F}^{an}(\tilde v_\varepsilon, A^B_{\sigma_1+t,\sigma_2-t}(x_i)) \le C$$

and hence, up to a subsequence, $\tilde v_\varepsilon \rightharpoonup v_i$ in $H^1(A^B_{\sigma_1+t,\sigma_2-t}(x_i);\mathbb{R}^2)$ for some field v_i. Moreover, since

$$\frac{1}{\varepsilon^2}\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} (1-|\tilde v_\varepsilon|^2)^2\,\mathrm{d}x \le C XY^{an}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O) \le C|\log\varepsilon|,$$

(see [1, Lemma 2] for more details), we deduce that $|v_i|=1$ a.e..

Furthermore, by standard Fubini's arguments, for a.e. $\sigma_1+t<\sigma<\sigma_2-t$, up to a subsequence the trace of $\tilde v_\varepsilon$ is bounded in $H^1(\partial E^B_\sigma(x_i);\mathbb{R}^2)$, and hence it converges uniformly to the trace of v_i. By the very definition of degree it follows that $\deg(v_i,\partial E^B_\sigma(x_i)) = d_i$.

Hence, by (2.9), for every i we have

$$\mathcal{F}^{an}(v_i, A_{\sigma_1+t,\sigma_2-t}(x_i)) \ge \pi\lambda_{\mathrm{self}}|d_i|^2\log\frac{\sigma_2-t}{\sigma_1+t}. \tag{2.35}$$

By (2.34) and (2.35), we conclude that for ε smal enough

$$F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O) \ge \pi\lambda_{\mathrm{self}}\sum_{i=1}^N\left(|d_i|\log\frac{\sigma_1}{\varepsilon} + |d_i|^2\log\frac{\sigma_2-t}{\sigma_1+t}\right) + C$$

$$\ge \pi\lambda_{\mathrm{self}}\left(M|\log\varepsilon| + \sum_{i=1}^N(|d_i|^2-|d_i|)\log\frac{\sigma_2}{\sigma_1} + \sum_{i=1}^N|d_i|^2\log\frac{\sigma_1(\sigma_2-t)}{\sigma_2(\sigma_1+t)}\right) + C.$$

The energy bound yields

$$\sum_{i=1}^{N}(|d_i|^2-|d_i|)\log\frac{\sigma_2}{\sigma_1}+\sum_{i=1}^{N}|d_i|^2\log\frac{\sigma_1(\sigma_2-t)}{\sigma_2(\sigma_1+t)}\leq C;$$

therefore, letting $t\to 0$ and $\sigma_1\to 0$, we conclude $|d_i|=1$.

Proof of (ii): Γ-liminf inequality. Fix $r>0$ so that the ellipses $E^B_r(x_i)$ are pairwise disjoint and compactly contained in O. Let $\{O^h\}$ be an increasing sequence of open smooth sets compactly contained in O such that $\cup_{h\in\mathbb{N}}O^h=O$. Without loss of generality we can assume that $F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)\leq\pi\lambda_{\mathrm{self}}M|\log\varepsilon|+C$, which together with Theorem 2.1 yields

$$F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O\setminus\bigcup_{i=1}^{M}E^B_r(x_i))\leq C. \tag{2.36}$$

For every $r>0$, by (2.36) we deduce $XY^{an}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O\setminus\bigcup_{i=1}^{N}E^B_r(x_i))\leq C$. Fix $h\in\mathbb{N}$ and let ε be small enough so that $O^h\subset O_{\varepsilon,\mathbb{Z}^2}$. Since

$$\int_{O^h\setminus\bigcup_{i=1}^{N}E^B_r(x_i)}|\nabla\tilde v_\varepsilon|^2\leq\frac{2}{\min\{c_{e_1},c_{e_2}\}}XY^{an}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O\setminus\bigcup_{i=1}^{M}E^B_r(x_i))\leq C,$$

by a diagonalization argument, there exists a unitary field $v\in H^1(O\setminus E^B_r(x_i);\mathcal{S}^1)$ such that, up to a subsequence, $\tilde v_\varepsilon\rightharpoonup v$ in $H^1_{\mathrm{loc}}(O\setminus\cup_{i=1}^{M}\{x_i\};\mathbb{R}^2)$.

Let $\sigma>0$ be such that $E^B_\sigma(x_i)$ are pairwise disjoint and contained in O^h. Recalling the definition of $A^B_{r,R}(x)$ in (2.32), we set $A^B_{r,R}:=A^B_{r,R}(0)$. Let $t\leq\sigma$, and consider the minimization problem

$$\min_{w\in H^1(A^B_{t/2,t};\mathcal{S}^1)}\left\{\mathcal{F}^{an}(w,A^B_{t/2,t})\,:\,\deg(w,\partial E^B_{\frac{t}{2}})=1\right\}.$$

It is easy to see that the minimum is $\pi\lambda_{\mathrm{self}}\log 2$ and that the set of minimizers is given by (the restriction at $A^B_{t/2,t}$ of the functions in)

$$\mathcal{K}:=\left\{\alpha\frac{B^{-\frac12}z}{|B^{-\frac12}z|}\,:\,\alpha\in\mathbb{C},|\alpha|=1\right\}. \tag{2.37}$$

Set

$$\mathrm{d}_t(w,\mathcal{K}):=\min\left\{\mathcal{F}^{an}(w-v,A^B_{t/2,t})\,:\,v\in\mathcal{K}\right\}. \tag{2.38}$$

For any $v\in\mathcal{K}$ and $w\in H^1(A^B_{t/2,t};\mathbb{R}^2)$, by (2.6), we have

$$\mathcal{F}^{an}(w-v,A^B_{t/2,t})=\lambda_{\mathrm{self}}\int_{B_t\setminus B_{t/2}}|\nabla w^B-\nabla v^B|^2\,\mathrm{d}y,$$

where we have set $w^B(y):=w(B^{\frac12}y)$ and $v^B(y):=v(B^{\frac12}y)$. By this fact, it follows that (see [3] for further details) for any given $\delta>0$ there exists a positive $\omega(\delta)$ (independent of t) such that

$$\mathcal{F}^{an}(\tilde v_\varepsilon,A^B_{\frac{t}{2}+m\sqrt2\varepsilon,t-m\sqrt2\varepsilon})\geq\pi\lambda_{\mathrm{self}}\,\log 2+\omega(\delta), \tag{2.39}$$

whenever $\mathrm{d}_t(\tilde v_\varepsilon(\cdot),\mathcal{K}_i)\geq\delta$, where

$$\mathcal{K}_i:=\left\{\alpha\frac{B^{-\frac12}(z-x_i)}{|B^{-\frac12}(z-x_i)|}\,:\,\alpha\in\mathcal{C},|\alpha|=1\right\}.$$

Let $P\in\mathbb{N}$ be such that $P\,\omega(\delta)\geq \mathbb{W}^{an}(\mu)+M(\gamma^{an}-\pi\lambda_{\mathrm{self}}\log\sigma-C)$ where C is the constant in (2.1). For $p=1,\dots,P$, set $C^B_p(x_i):=E^B_{2^{1-p}\sigma}(x_i)\setminus E^B_{2^{-p}\sigma}(x_i)$. We distinguish among two cases.

(a) *First case:* For ε small enough and for every fixed $1\leq p\leq P$, there exists at least one i such that $\mathrm{d}_{2^{1-p}\sigma}(\tilde v_\varepsilon,\mathcal{K}_i)\geq\delta$, then by (2.1), (2.39) and the lower semicontinuity of the functional $\mathcal{F}^{an}$, we conclude

$$
\begin{aligned}
F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O^h)&\geq\sum_{i=1}^M XY^{an}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,E^B_{2^{-P}\sigma}(x_i))+\sum_{p=1}^P\sum_{i=1}^M XY^{an}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,C_p(x_i))\\
&\geq\lambda_{\mathrm{self}}\,M(\pi\log\frac{\sigma}{2^P}+\pi|\log\varepsilon|+C)+P(M\lambda_{\mathrm{self}}\,\pi\log2+\omega(\delta))+\mathrm{o}(\varepsilon)\\
&\geq\pi M\lambda_{\mathrm{self}}\,|\log\varepsilon|+M\gamma^{an}+\mathbb{W}^{an}(\mu)+\mathrm{o}(\varepsilon).
\end{aligned}
$$

(b) *Second case:* Up to a subsequence, there exists $1\leq\bar p\leq P$ such that for every i we have $\mathrm{d}_{\bar\sigma}(\tilde v_\varepsilon,\mathcal{K}_i)\leq\delta$, where $\bar\sigma:=2^{1-\bar p}\sigma$. Let $\alpha_{\varepsilon,i}$ be the unitary vector such that $\mathcal{F}^{an}(\tilde v_\varepsilon-\alpha_{\varepsilon,i}\frac{B^{-\frac12}(x-x_i)}{|B^{-\frac12}(x-x_i)|},C_{\bar p}(x_i));\mathbb{R}^2)=\mathrm{d}_{\bar\sigma}(\tilde v_\varepsilon,\mathcal{K}_i)$.

One can construct a function $\hat u_\varepsilon\in\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ such that
(i) $\hat u_\varepsilon=\bar u_\varepsilon$ on $\partial_{\varepsilon,\mathbb{Z}^2}(\mathbb{R}^2\setminus E^B_{2^{-\bar p}\sigma}(x_i))$;
(ii) $e^{2\pi i\hat u_\varepsilon(j)}=\alpha_{\varepsilon,i}\frac{B^{-\frac12}(j-x_i)}{|B^{-\frac12}(j-x_i)|}$ for any $j\in\partial_{\varepsilon,\mathbb{Z}^2}E^B_{2^{1-\bar p}\sigma}(x_i)$;
(iii) $F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,E^B_{\bar\sigma}(x_i))\geq F^{an}_{\varepsilon,\mathbb{Z}^2}(\hat u_\varepsilon,E^B_{\bar\sigma}(x_i))+r(\varepsilon,\delta)$ with $\lim_{\delta\to0}\lim_{\varepsilon\to0}r(\varepsilon,\delta)=0$.

The proof of (i)-(iii) is quite technical, and consists in adapting standard cut-off arguments to our discrete setting. For the reader convenience we skip the details of the proof, and assuming (i)-(iii) we conclude the proof of the lower bound.

By Theorem (2.4), we have that

$$
\begin{aligned}
F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)&\geq XY^{an}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O^h\setminus\bigcup_{i=1}^M E^B_{\bar\sigma}(x_i))+\sum_{i=1}^M F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,E^B_{\bar\sigma}(x_i))\\
&\geq\mathcal{F}^{an}(\tilde v_\varepsilon,O^h\setminus\cup_{i=1}^M E^B_{\bar\sigma}(x_i))+\sum_{i=1}^M F^{an}_\varepsilon(\hat u_\varepsilon,E^B_{\bar\sigma}(x_i))+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)\\
&\geq\mathcal{F}^{an}(\tilde v_\varepsilon,O^h\setminus\cup_{i=1}^M E^B_{\bar\sigma}(x_i))+M(\gamma^{an}-\pi\lambda_{\mathrm{self}}\log\frac{\varepsilon}{\bar\sigma})+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)\\
&\geq\mathcal{F}^{an}(v,O\setminus\cup_{i=1}^M E^B_{\bar\sigma}(x_i))+M(\gamma^{an}-\pi\lambda_{\mathrm{self}}\log\frac{\varepsilon}{\bar\sigma})+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)+\mathrm{o}(1/h)\\
&\geq M\pi\lambda_{\mathrm{self}}\,|\log\varepsilon|+M\gamma^{an}+\mathbb{W}^{an}(\mu)+r(\varepsilon,\delta)+\mathrm{o}(\varepsilon)+\mathrm{o}(\bar\sigma)+\mathrm{o}(1/h).
\end{aligned}
$$

The proof follows sending $\varepsilon\to0$, $\delta\to0$, $\sigma\to0$ and $h\to\infty$.

Proof of (iii): Γ-limsup inequality. This proof in analogue to the one given in [3] for the isotropic case. We only sketch its anisotropic counterpart. Let w_σ be a function that agrees with a minimizer of (2.25) in $O\setminus\bigcup_{i=1}^M E^B_\sigma(x_i)=O^B_\sigma$. Then, $w_\sigma=\alpha_i e^{i\theta^B_i}$ on $\partial E^B_\sigma(x_i)$ for some $|\alpha_i|=1$ (θ^B_i is defined in (2.26)).

For every $\rho>0$ we can always find a function $w_{\sigma,\rho}\in C^\infty(\overline{O}_\sigma;\mathcal{S}^1)$ such that $w_{\sigma,\rho}=\alpha_i e^{i\theta^B_i}$ on $\partial E^B_\sigma(x_i)$, and

$$\mathcal{F}^{an}(w_{\sigma,\rho},O_\sigma)-\mathcal{F}^{an}(w_\sigma,O_\sigma)\leq\rho.$$

Moreover, for every $i = 1, \dots, M$ let $w_i \in \mathcal{AXY}_{\varepsilon,\mathbb{Z}^2}(E^B_\sigma(x_i))$ be a function which agrees with $\alpha_i e^{i\theta^B_i}$ on $\partial_\varepsilon E^B_\sigma(x_i)$ and such that its phase minimizes problem (2.27). If necessary, we extend w_i to $(\overline{E}^B_\sigma(x_i) \cap \varepsilon\mathbb{Z}^2) \setminus (E^B_\sigma(x_i))^0_{\varepsilon,\mathbb{Z}^2}$ to be equal to $\alpha_i e^{i\theta^B_i}$. Finally, define the function $w_{\varepsilon,\sigma,\rho} \in \mathcal{AXY}_{\varepsilon,\mathbb{Z}^2}(O)$ which coincides $w_{\sigma,\rho}$ on $O_\sigma \cap \varepsilon\mathbb{Z}^2$ and with w_i on $\overline{E^B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2$. In view of assumption (3) on f, a straightforward computation shows that any phase $\bar{u}_{\varepsilon,\sigma,\rho}$ of $w_{\varepsilon,\sigma,\rho}$ is a recovery sequence, i.e.,

$$\lim_{\varepsilon\to 0} F^{an}_{\varepsilon,\mathbb{Z}^2}(\bar{u}_{\varepsilon,\sigma,\rho}, O) - M\pi\lambda_{\mathrm{self}}|\log\varepsilon| = \mathbb{W}^{an}(\mu) + M\gamma^{an} + o(\rho,\sigma),$$

with $\lim_{\sigma\to 0}\lim_{\rho\to 0} o(\rho,\sigma) = 0$. $\square$

Remark 2.6. We notice that in the case of isotropic nearest neighbors interaction on the square lattice, i.e., if $c_{e_1} = c_{e_2} = 1$ and $c_{e_1+e_2} = 0$, Theorem 2.5 coincides with Theorem 4.2 in [3]. In this case $Q = B = I$, $E^B_\sigma(x) = B_\sigma(x)$ for every $x \in \mathbb{R}^2$ and for every $\sigma > 0$, and $\lambda_{\mathrm{self}} = 1$. In this case we set

$$F_{\varepsilon,\mathbb{Z}^2}(\cdot, O) := F^{an}_{\varepsilon,\mathbb{Z}^2}(\cdot, O) \text{ and } XY_{\varepsilon,\mathbb{Z}^2}(\cdot, O) := XY^{an}_{\varepsilon,\mathbb{Z}^2}(\cdot, O). \tag{2.40}$$

3. The Γ-convergence analysis for $F^{lr}_{\varepsilon,\mathbb{Z}^2}$

Here we give the asymptotic expansion by Γ-convergence of the functional $F^{lr}_{\varepsilon,\mathbb{Z}^2}$. The main idea is to decompose the energy $F^{lr}_{\varepsilon,\mathbb{Z}^2}$ in the sum of isotropic $F_{\varepsilon,\mathbb{Z}^2}$ energies and to use for each of these energies the Γ-convergence analysis developed in Section 2.

To this purpose, let us first introduce the main notation we will use throughout this section.

3.1. **Notation.** For any $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2$, we set $\xi^\perp := (-\xi_2, \xi_1)$ and we notice that $\mathbb{Z}^2$ may be partitioned as follows

$$\mathbb{Z}^2 = \bigcup_{h=1}^{|\xi|^2} \mathbb{Z}^2_{h,\xi}, \tag{3.1}$$

where $\mathbb{Z}^2_{h,\xi} := z_h + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp$ with $\{z_h\}_{h=1,\dots,|\xi|^2} := \{x \in \mathbb{Z}^2 : 0 \le x\cdot\xi < |\xi|, 0 \le x\cdot\xi^\perp < |\xi|\}$ (here $\cdot$ denotes the standard scalar product in $\mathbb{R}^2$).

We define the ξ-cube as

$$Q_\xi := \{a\xi + b\xi^\perp : 0 \le a, b \le 1\}.$$

Let $\{T^+_\xi, T^-_\xi\}$ be the partition of the ξ-cube Q_ξ into the 2-dimensional simplices defined by

$$T^+_\xi := \{x \in Q_\xi : x\cdot\xi^\perp \le x\cdot\xi\},$$
$$T^-_\xi := \{x \in Q_\xi : x\cdot\xi \le x\cdot\xi^\perp\}.$$

For every $\varepsilon > 0$, $\xi \in \mathbb{Z}^2$, $h \in \{1, \dots, |\xi|^2\}$ and for every $i \in \mathbb{Z}^2_\xi$, we set $T^\pm_{i,\varepsilon,\xi} := \varepsilon i + \varepsilon T^\pm_\xi$.

Let O be an open bounded subset of $\mathbb{R}^2$ with Lipschitz continuous boundary.

We set

$$O_{\varepsilon,\xi,h} := \bigcup_{i\in\mathbb{Z}^2_{\xi,h} : T^\pm_{i,\varepsilon,\xi} \subset \overline{O}} T^\pm_{i,\varepsilon,\xi}. \tag{3.2}$$

The reference lattice and the class of bonds in $\mathbb{Z}^2_{h,\xi}$ are given by

$$\begin{aligned} O^0_{\varepsilon,\xi,h} &:= \varepsilon\mathbb{Z}^2_{\xi,h}\cap O_{\varepsilon,\xi,h}\\ O^1_{\varepsilon,\xi,h} &:= \{(i,j)\in O^0_{\varepsilon,\xi,h}\times O^0_{\varepsilon,\xi,h}\,:\, i\neq j\}, \end{aligned}$$

Moreover, the class of $\varepsilon\xi$-triangular cells contained in Ω is defined by

$$O^2_{\varepsilon,\xi,h} := \{T^{\pm}_{i,\varepsilon,\xi}\,:\, i\in\varepsilon\mathbb{Z}^2_{\xi,h},\, T^{\pm}_{i,\varepsilon,\xi}\subset O_{\varepsilon,\xi,h}\}.$$

Let $\bar u\in\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$. Recalling the definition of P in (1.7), for every $T^{\pm}_{i,\varepsilon,\xi}\in O_{\varepsilon,\xi,h}$ we set

$$\begin{aligned} \alpha_{\bar u}(T^-_{i,\varepsilon,\xi}) :=& P(\bar u(\varepsilon i+\varepsilon\xi)-\bar u(\varepsilon i)) + P(\bar u(\varepsilon i+\varepsilon\xi+\varepsilon\xi^{\perp})-\bar u(\varepsilon i+\varepsilon\xi))\\ &+P(\bar u(\varepsilon i)-\bar u(\varepsilon i+\varepsilon\xi+\varepsilon\xi^{\perp}))\\ \alpha_{\bar u}(T^+_{i,\varepsilon}) :=& -P(\bar u(\varepsilon i)-\bar u(\varepsilon i+\varepsilon\xi+\varepsilon\xi^{\perp})) - P(\bar u(\varepsilon i+\varepsilon\xi^{\perp})-\bar u(\varepsilon i))\\ &-P(\bar u(\varepsilon i+\varepsilon\xi+\varepsilon\xi^{\perp})-\bar u(\varepsilon i+\varepsilon\xi^{\perp})), \end{aligned}$$

and we define the discrete vorticity measure for each cell $T^{\pm}_{i,\varepsilon,\xi}\in O_{\varepsilon,\xi,h}$ as

$$\mu^{\xi,h}(\bar u) := \sum_{T^{\pm}_{i,\varepsilon,\xi}\in O^2_{\varepsilon,\xi,h}} \alpha_{\bar u}(T^-_{i,\varepsilon,\xi})\delta_{b(T^-_{i,\varepsilon,\xi})} + \sum_{T^{+}_{i,\varepsilon,\xi}\in O^2_{\varepsilon,\xi,h}} \alpha_{\bar u}(T^+_{i,\varepsilon})\delta_{b(T^+_{i,\varepsilon,\xi})}, \tag{3.3}$$

where $b(T^{\pm}_{i,\varepsilon,\xi})$ is the barycenter of the of the triangle $T^{\pm}_{i,\varepsilon,\xi}$.

Once again, this definition of discrete vorticity extends to $\mathcal{S}^1$ valued fields in the obvious way, i.e., by setting $\mu^{\xi,h}(v)=\mu^{\xi,h}(\bar u)$ where $\bar u$ is any function in $\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ such that $v(\bar u)=v$.

We notice that for any $\bar u\in\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$, $F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u,O)$ can be rewritten as follows

$$F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u,O) = \sum_{\xi\in\mathbb{Z}^2}\sum_{h=1}^{|\xi|^2} F^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(\bar u,O), \tag{3.4}$$

where

$$F^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(\bar u,O) := \sum_{\substack{(\varepsilon i,\varepsilon i+\varepsilon\xi)\in O^1_{\varepsilon,\xi,h}\\ (\varepsilon i,\varepsilon i+\varepsilon\xi)\in O^1_{\varepsilon,\xi,h}}} f_\xi(\bar u(\varepsilon i+\varepsilon\xi)-\bar u(\varepsilon i)) + f_{\xi^{\perp}}(\bar u(\varepsilon i+\varepsilon\xi^{\perp})-\bar u(\varepsilon i)). \tag{3.5}$$

Fon any $v:\Omega^0_{\varepsilon,\mathbb{Z}^2}\to\mathcal{S}^1$, we set

$$XY^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(v,O) := \frac12 \sum_{\substack{(\varepsilon i,\varepsilon j)\in O^1_{\varepsilon,\xi,h}\\ |i-j|=|\xi|}} \left|\frac{v(\varepsilon i)-v\varepsilon(j)}{|\xi|}\right|^2 \tag{3.6}$$

By assumptions (1.14) and (1.17) on the potentials f_ξ, we have immediately that

$$F^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(\bar u,O) \ge c_\xi|\xi|^2 XY^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(e^{2\pi i\bar u},O). \tag{3.7}$$

Finally, we define the piecewise affine interpolations according with the triangulation $\{T^{\pm}_{i,\varepsilon,\xi}\}_{i\in\mathbb{Z}^2_{h,\xi}}$ since it will be useful in the proof of our results. Fix $\xi\in\mathbb{Z}^2$ and $h\in\{1,\dots,|\xi|^2\}$. For any $v:\Omega^0_{\varepsilon,\mathbb{Z}^2}\to\mathcal{S}^1$, let $\tilde v^{\xi,h}:\Omega_{\varepsilon,\xi,h}\to\mathbb{R}^2$ be the piecewise affine interpolation of v, according with the triangulation $\left\{T^{\pm}_{i,\varepsilon,\xi}\right\}_{i\in\varepsilon\mathbb{Z}^2_{\xi,h}}$, i.e., for any $i\in\mathbb{Z}^2_{\xi,h}$ we set

$$\begin{aligned}\tilde v^{\xi,h}(x)&=v(\varepsilon i)+\frac{v(\varepsilon i+\varepsilon\xi)-v(\varepsilon i)}{\varepsilon|\xi|}\left((x-\varepsilon i)\cdot\frac{\xi}{|\xi|}\right)\\&\quad+\frac{v(\varepsilon i+\varepsilon\xi^\perp)-v(\varepsilon i)}{\varepsilon|\xi|}\left((x-\varepsilon i)\cdot\frac{\xi^\perp}{|\xi|}\right)\quad\text{for } x\in T^-_{i,\varepsilon,\xi}\\ \tilde v^{\xi,h}(x)&=v(\varepsilon i)+\frac{v(\varepsilon i+\varepsilon\xi^\perp)-v(\varepsilon i)}{\varepsilon|\xi|}\left((x-\varepsilon i)\cdot\frac{\xi^\perp}{|\xi|}\right)\\&\quad+\frac{v(\varepsilon i+\varepsilon\xi)-v(\varepsilon i)}{\varepsilon|\xi|}\left((x-\varepsilon i)\cdot\frac{\xi}{|\xi|}\right)\quad\text{for } x\in T^+_{i,\varepsilon,\xi}.\end{aligned}\tag{3.8}$$

Remark 3.1. *Notice that if $\xi=e_1$, then $h=1$, and for any $\bar u\in\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}$ we have $\mu^{e_1,1}(\bar u)\equiv\mu(\bar u)$ and $F^{e_1,1}_{\varepsilon,\mathbb{Z}^2}(\bar u,O)\equiv F_{\varepsilon,\mathbb{Z}^2}(\bar u,O)$, with $F_{\varepsilon,\mathbb{Z}^2}$ defined as in Remark 2.6 (see formula (2.40)). Moreover, set $v:=e^{2\pi i\bar u}$; then $XY^{e_1,1}_{\varepsilon,\mathbb{Z}^2}(v,O)\equiv XY_{\varepsilon,\mathbb{Z}^2}(v,O)$, and the definition of $\tilde v^{e_1,1}$ coincides wih the definition of $\tilde v$ in* (1.10).

3.2. **The zero-order Γ-convergence result for $F^{lr}_{\varepsilon,\mathbb{Z}^2}$.** We start this section by stating the zero-order Γ-convergence result for the functionals $F^{lr}_{\varepsilon,\mathbb{Z}^2}$. This result has been proved in [1] for the XY^{lr}_ε.

Theorem 3.2. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $\{\bar u_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ be such that $F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)\le C|\log\varepsilon|$ for some positive constant C. Then, up to a subsequence, $\mu(\bar u_\varepsilon)\overset{\text{flat}}{\to}\mu$, for some $\mu\in X(O)$.*

(ii) *(Localized Γ-liminf inequality) Let $\{\bar u_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ be such that $\mu(\bar u_\varepsilon)\overset{\text{flat}}{\to}\mu=\sum_{i=1}^M d_i\delta_{x_i}$ with $d_i\in\mathbb{Z}\setminus\{0\}$ and $x_i\in O$. Then, there exists a constant $C\in\mathbb{R}$ such that, for any $i=1,\dots,M$ and for every $\sigma<\frac12 dist(x_i,\partial O\cup\bigcup_{j\neq i}x_j)$, we have*

$$\liminf_{\varepsilon\to0}F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,B_\sigma(x_i))-\pi\sum_{\xi\in\mathbb{Z}^2}c_\xi|\xi|^2|d_i|\log\frac{\sigma}{\varepsilon}\ge C.\tag{3.9}$$

In particular

$$\liminf_{\varepsilon\to0}F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)-\pi\sum_{\xi\in\mathbb{Z}^2}c_\xi|\xi|^2|\mu|(\Omega)\log\frac{\sigma}{\varepsilon}\ge C.$$

(iii) *(Γ-limsup inequality) For every $\mu\in X(O)$, there exists a sequence $\{\bar u_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ such that $\mu(\bar u_\varepsilon)\overset{\text{flat}}{\to}\mu$ and*

$$\sum_{\xi\in\mathbb{Z}^2}c_\xi|\xi|^2\pi|\mu|(O)\ge\limsup_{\varepsilon\to0}\frac{F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)}{|\log\varepsilon|}.$$

The proof of this result is result is a consequence of the following lemma.

Lemma 3.3. *Let $\{\bar u_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ be such that $F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)\le C|\log\varepsilon|$ for some positive constant C. Then for every $\xi\in\mathbb{Z}^2$ and for every $h\in\{1,\dots,|\xi|^2\}$*

$$\|\mu^{\xi,h}(\bar u_\varepsilon)-\mu(\bar u_\varepsilon)\|_{\text{flat}}\to0.$$

Proof. Set $v_\varepsilon := e^{2\pi i \bar u_\varepsilon}$ and let $\tilde v_\varepsilon$ and $\tilde v_\varepsilon^{\xi,h}$ be defined as in (1.10) and (3.8) respectively. Fix $\xi \in \mathbb{Z}^2$ and $h \in \{1, \dots, |\xi|^2\}$. By triangular inequality, we have

$$\|\mu^{\xi,h}(\bar u_\varepsilon) - \mu(\bar u_\varepsilon)\|_{\mathrm{flat}} \le \|\mu^{\xi,h}(\bar u_\varepsilon) - J(\tilde v_\varepsilon^{\xi,h})\|_{\mathrm{flat}} + \|J(\tilde v_\varepsilon^{\xi,h}) - J(\tilde v_\varepsilon)\|_{\mathrm{flat}} + \|J(\tilde v_\varepsilon) - \mu(\bar u_\varepsilon)\|_{\mathrm{flat}}.$$

By Proposition 1.1, we have that the first and the third terms on the rhs of the inequality below vanish as $\varepsilon \to 0$; therefore, in order to prove the claim, it is enough to show that for every open set $U \subset\subset O$

$$\|J(\tilde v_\varepsilon^{\xi,h}) - J(\tilde v_\varepsilon)\|_{\mathrm{flat}(U)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.10}$$

To this end we will show that the sequences $\{\tilde v_\varepsilon^{\xi,h}\}$ and $\{\tilde v_\varepsilon\}$ satisfy the assumptions of Lemma 1.2. This fact has been proved in [1] (see proof of Theorem 4.8(i)) but, for the sake of completeness, we present the proof here.

Let U' be such that $U \subset\subset U' \subset\subset O$. For ε small enough we have that $U' \subset O_{\varepsilon,\xi,h}$, with $O_{\varepsilon,\xi,h}$ defined as in (3.2), and

$$\begin{aligned}\int_{U'} (|\nabla \tilde v_\varepsilon^{\xi,h}|^2 + |\nabla \tilde v_\varepsilon|^2)\,\mathrm{d}x &\le XY_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon, O) + XY_\varepsilon^{\xi,h}(v_\varepsilon, O) \\ &\le F_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) + \frac{1}{c_\xi|\xi|^2} F^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) \le C|\log\varepsilon|,\end{aligned} \tag{3.11}$$

and hence

$$\int_{U'} |\nabla \tilde v_\varepsilon^{\xi,h} - \nabla \tilde v_\varepsilon|^2\,\mathrm{d}x \le C|\log\varepsilon|. \tag{3.12}$$

Set $g_\varepsilon := \tilde v_\varepsilon^{\xi,h} - \tilde v_\varepsilon$; since $g_\varepsilon(\varepsilon i) \equiv 0$ for every $i \in \mathbb{Z}^2_{\xi,h}$, we have that for every $x \in \varepsilon i + \varepsilon Q_\xi$,

$$g_\varepsilon(x) = \int_0^1 \nabla g_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)\,\mathrm{d}t \tag{3.13}$$

and hence, by Jensen's inequality, we get

$$|g_\varepsilon(x)|^2 \le \int_0^1 |\nabla g_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)|^2\,\mathrm{d}t. \tag{3.14}$$

Set $t_0 = \frac{1}{\sqrt{2}|\xi|}$. For any given $\varepsilon i + \varepsilon Q_\xi$, if $t \le t_0$, we find $|t(x - \varepsilon i)| \le \varepsilon$, which yields, by construction of the piecewise affine interpolations, that $\nabla g_\varepsilon(\varepsilon i + t(x-\varepsilon i)) \cdot (x - \varepsilon i)$ is constant on $(0, t_0)$. Then the following estimate holds true

$$\int_0^1 |\nabla g_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)|^2\,\mathrm{d}t \le 2\int_{\frac{t_0}{2}}^1 |\nabla g_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)|^2\,\mathrm{d}t.$$

Integrating (3.14) over $\varepsilon i + \varepsilon Q_\xi$, and using the previous estimate, we get

$$\int_{\varepsilon i + \varepsilon Q_\xi} |g_\varepsilon(x)|^2\,\mathrm{d}x \le \varepsilon^2|\xi|^2 2\int_{\frac{t_0}{2}}^1 \int_{\varepsilon i + \varepsilon Q_\xi} |\nabla g_\varepsilon(\varepsilon i + t(x - \varepsilon i))|^2\,\mathrm{d}x\,\mathrm{d}t,$$

which, by the change of variable $y = \varepsilon i + t(x - \varepsilon i)$, yields

$$\int_{\varepsilon i + \varepsilon Q_\xi} |g_\varepsilon(x)|^2\,\mathrm{d}x \le \frac{8\varepsilon^2|\xi|^2}{t_0^2} \int_{\varepsilon i + \varepsilon Q_{t\xi}} |\nabla g_\varepsilon(y)|^2\,\mathrm{d}y \le C\varepsilon^2|\xi|^2 \int_{\varepsilon i + \varepsilon Q_\xi} |\nabla g_\varepsilon|^2\,\mathrm{d}x.$$

Finally, summing over $\varepsilon i \in U^{\xi,h}_{\varepsilon,\mathbb{Z}^2}$, by (3.13), we obtain

$$\int_U |g_\varepsilon|^2 \ \mathrm{d}x \ \le \ \sum_{\varepsilon i \in U^{\xi,h}_{\varepsilon,\mathbb{Z}^2}} \int_{\varepsilon i+\varepsilon Q_\xi} |g_\varepsilon|^2 \ \mathrm{d}x \ \le \ C\varepsilon^2 \int_{U'} |\nabla g_\varepsilon|^2 \ \mathrm{d}x \ \le \ C\varepsilon^2 |\log \varepsilon|.$$

□

Since the proof of Theorem 3.2 is based essentially on Theorem 2.1 and on the proof of Theorem 4.8 in [1] we briefly sketch it.

Proof of Theorem 3.2. Since $c_{e_1} = c_{e_2} > 0$ the compactness property is a direct consequence of Theorem 2.1(i).

As for the proof of Γ-liminf inequality, fix $i \in \{1, \dots, M\}$. Without loss of generality, we can assume that

$$\begin{aligned}\liminf_{\varepsilon\to 0} F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, B_\sigma(x_i)) - \pi \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 |d_i| \log\frac{\sigma}{\varepsilon} \\ = \lim_{\varepsilon\to 0} F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, B_\sigma(x_i)) - \pi \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 |d_i| \log\frac{\sigma}{\varepsilon} < +\infty.\end{aligned}$$

Fix $\xi \in \mathbb{Z}^2$ and $h \in \{1, \dots, |\xi|^2\}$. By Lemma 3.3, we get $\mu^{\xi,h}(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu$. Therefore, by (3.7) and by Theorem 2.1(ii) applied with $F^{an}_{\varepsilon,\mathbb{Z}^2} = XY^{\xi,h}_{\varepsilon,\mathbb{Z}^2}$ we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} F^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, B_\sigma(x_i)) - c_\xi |\xi^2| \pi |d_i| \log\frac{\sigma}{\varepsilon} \\ \ge c_\xi |\xi|^2 (\liminf_{\varepsilon\to 0} XY^{\xi,h}_{\varepsilon,\mathbb{Z}^2}(e^{2\pi i \bar u_\varepsilon}, B_\sigma(x_i)) - \pi |d_i| \log\frac{\sigma}{\varepsilon}) \ge C.\end{aligned}$$

By summing over $h = 1, \dots, |\xi|^2$ and over ξ we get (3.9).

The proof of the Γ-limsup inequality is standard and left to the reader. □

3.3. **The first-order Γ-convergence result for $F^{lr}_{\varepsilon,\mathbb{Z}^2}$.** Finally, we state the first order Γ-convergence result for $F^{lr}_{\varepsilon,\mathbb{Z}^2}$. To this purpose we need to introduce the following discrete minimum problem

$$\gamma^{lr}(\varepsilon,\sigma) := \min_{\bar u \in \mathcal{AF}_{\varepsilon,\mathbb{Z}\in}(B_\sigma)} \{F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u, B_\sigma) \, : \, 2\pi \bar u(\cdot) = \theta(\cdot) \ \text{ on } \partial_{\varepsilon,\mathbb{Z}^2} B_\sigma\},$$

where the discrete boundary $\partial_{\varepsilon,\mathbb{Z}^2}$ is defined in in (1.6) and $\theta(y)$ is the polar coordinate $\arctan y_2/y_1$.

The following proposition is the long range counterpart of Proposition 2.28.

Proposition 3.4. *For any fixed $\sigma > 0$, the following limit exists finite*

$$\lim_{\varepsilon\to 0} (\gamma^{lr}(\varepsilon,\sigma) - \pi \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 |\log\tfrac{\varepsilon}{\sigma}|) =: \gamma^{lr} \in \mathbb{R}. \tag{3.15}$$

Proof of (3.15). First, by scaling, it is easy to see that $\gamma(\varepsilon,\sigma)^{lr} = I^{lr}(\frac{\varepsilon}{\sigma})$ where $I^{lr}(t)$ is defined by

$$I^{lr}(t) := \min\left\{ F^{lr}_{1,\mathbb{Z}^2}(\bar u, B_{\frac{1}{t}}) \mid 2\pi \bar u(\cdot) = \theta(\cdot) \text{ on } \partial_{1,\mathbb{Z}^2} B_{\frac{1}{t}} \right\}.$$

We aim at proving that

$$0 < t_1 \le t_2 \ \Rightarrow \ I^{lr}(t_1) \le \pi \sum_{\xi\in\mathbb{Z}^2} \log\frac{t_2}{t_1} + I^{lr}(t_2) + O(t_2). \tag{3.16}$$

By (3.16) and by Theorem 3.2(ii), it follows that

$$\exists \lim_{t\to 0^+}(I^{lr}(t) - \pi \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 |\log t|) > -\infty.$$

We prove now that (3.16) holds true. First we notice that for every $x \in A = B_R \setminus B_r$ and for every $\xi \in \mathbb{Z}^2$

$$|\nabla\theta(x)| = \sqrt{\Big|\partial_{\frac{\xi}{|\xi|}}\theta(x)\Big|^2 + \Big|\partial_{\frac{\xi^\perp}{|\xi|}}\theta(x)\Big|^2} \le \frac{c}{r},$$

for some constant $c > 0$. Therefore, by standard interpolation estimates (see for instance [9] and [1]) and using assumption (3) on f, we have that, as $0 < r < R \to \infty$,

$$\begin{aligned}(3.17)\quad F^{lr}_{1,\mathbb{Z}^2}(\theta/2\pi, A) &= \sum_{\xi\in\mathbb{Z}^2} c_\xi \sum_{h=1}^{|\xi|^2} F^{\xi,h}_{1,Z^2}(\theta/2\pi, A)\\ &\le \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2} c_\xi \sum_{h=1}^{|\xi|^2} \sum_{\substack{(\varepsilon i,\varepsilon i+\varepsilon\xi)\in A^1_{1,\xi,h}\\ (\varepsilon i,\varepsilon i+\varepsilon\xi^\perp)\in A^1_{1,\xi,h}}} |\theta(\varepsilon i+\varepsilon\xi) - \theta(\varepsilon i)|^2 + \big|\theta(\varepsilon i+\varepsilon\xi^\perp) - \theta(\varepsilon i)\big|^2\\ &\le \pi \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 \log\frac{R}{r} + O(1/r).\end{aligned}$$

Let u_2 be a minimizer for $I^{lr}(t_2)$ and for any $i \in \mathbb{Z}^2$ define

$$u_1(i) := \begin{cases} u_2(i) & \text{if } |i| \le \frac{1}{t_2}\\ \frac{\theta(i)}{2\pi} & \text{if } \frac{1}{t_2} \le |i| \le \frac{1}{t_1}, \end{cases}$$

By (3.17) we have

$$I^{lr}(1/R) \le I^{lr}(1/r) + \pi \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 \log\frac{r}{R} + O(1/r),$$

which yields (3.16) for $r = \frac{1}{t_2}$ and $R = \frac{1}{t_1}$. □

To ease the notation, for any $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $d_i \in \{-1,+1\}$ and $x_i \in O$, we set

$$\mathbb{W}^{lr}(\mu) := \sum_{\xi\in\mathbb{Z}^2} c_\xi |\xi|^2 \mathbb{W}(\mu). \tag{3.18}$$

where $\mathbb{W}$ is defined in (2.20).

Theorem 3.5. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{\bar u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,Z^2}(O)$ be a sequence satisfying $F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O) - M\pi\sum_{\xi\in\mathbb{Z}^2} c_\xi|\xi|^2|\log\varepsilon| \le C$. Then, up to a subsequence, $\mu(\bar u_\varepsilon) \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \le M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-$\liminf$ inequality) Let $\{\bar u_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ be such that $\mu(\bar u_\varepsilon)\overset{\mathrm{flat}}{\to}\mu$, with $\mu=\sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i|=1$ and $x_i\in O$ for every i. Then,*

$$\liminf_{\varepsilon\to0}F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)-M\pi\sum_{\xi\in\mathbb{Z}^2}c_\xi|\xi|^2|\log\varepsilon|\geq\mathbb{W}^{lr}(\mu)+M\gamma^{lr}.$$

(iii) *(Γ-$\limsup$ inequality) Given $\mu=\sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i|=1$ and $x_i\in O$ for every i, there exists $\{\bar u_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(O)$ with $\mu(\bar u_\varepsilon)\overset{\mathrm{flat}}{\to}\mu$ such that*

$$F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)-M\pi\sum_{\xi\in\mathbb{Z}^2}c_\xi|\xi|^2|\log\varepsilon|\to\mathbb{W}^{lr}(\mu)+M\gamma^{lr}.$$

Proof. The proof of the Theorem closely follows the one of Theorem 2.5. In particular, as for the proof of Γ-liminf inequality we sketch only the main differences, whereas the proof of Γ-limsup inequality is the same of Theorem 2.5(iii) and it is omitted.

Proof of (i) The fact that, up to a subsequence, $\mu(\bar u_\varepsilon)\overset{\mathrm{flat}}{\to}\mu=\sum_{i=1}^N d_i\delta_{x_i}$, with $\sum_{i=1}^N d_i\leq M$ is a direct consequence of the compactness result stated in Theorem 3.9(i). Assume now $\sum_{i=1}^N|d_i|=M$ and let us prove that $|d_i|=1$. By (3.4) and by assumption we have

$$c_{e_1}(F^{e_1,1}_\varepsilon(\bar u_\varepsilon,O)\,-\,M\pi|\log\varepsilon|)\;\leq\;\sum_{\xi\in\mathbb{Z}^2}c_\xi\sum_{h=1}^{|\xi|^2}(F^{\xi,h}_\varepsilon(\bar u_\varepsilon,O)\,-\,M\pi|\log\varepsilon|)\;\leq\;C;$$

then, recalling that $F^{e_1,1}_\varepsilon\equiv F_{\varepsilon,\mathbb{Z}^2}$ by Theorem 2.1(i) and Remark 2.6, we obtain the claim.

Proof of (ii) Let $r>0$ be such that the balls $B_r(x_i)$ are pairwise disjoint and contained in O. Let $\{O^n\}$ be an increasing sequence of open smooth sets compactly contained in O such that $\cup_{n\in\mathbb{N}}O^n=O$. Without loss of generality we can assume that $F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O)-M\pi\sum_{\xi\in\mathbb{Z}^2}c_\xi|\xi|^2|\log\varepsilon|\leq C$, which together with Theorem 3.2 yields

$$F^{lr}_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon,O\setminus\cup_{i=1}^M B_r(x_i))\leq C. \tag{3.19}$$

Set $v_\varepsilon:=e^{2\pi i\bar u_\varepsilon}$ and let $\tilde v_\varepsilon$ be the piecewise affine interpolation of v_ε defined in (1.10); for every $r>0$, by (3.19) we deduce that $XY^{lr}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O\setminus\cup_{i=1}^M B_r(x_i))\leq C$. Fix $n\in\mathbb{N}$ and let ε be small enough so that $O^n\subset O_{\varepsilon,\mathbb{Z}^2}$. Since

$$\begin{aligned}\int_{O^n\setminus\cup_{i=1}^r B_r(x_i)}|\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x&\leq\frac{2}{c_{e_1}}XY^{e_1,1}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O\setminus\cup_{i=1}^M B_r(x_i))\\&\leq\frac{2}{c_{e_1}}XY^{lr}_{\varepsilon,\mathbb{Z}^2}(v_\varepsilon,O\setminus\cup_{i=1}^M B_r(x_i))\leq C,\end{aligned}$$

by a diagonalization argument , there exists a unitary field $v\in H^1(O\setminus\cup_{i=1}^M B_r(x_i);\mathcal{S}^1)$, such that, up to a subsequence, $\tilde v_\varepsilon\rightharpoonup v$ in $H^1_{\mathrm{loc}}(O\setminus\cup_{i=1}^M B_r(x_i);\mathbb{R}^2)$. Moreover, by the proof of Lemma 3.3, it follows that for every $\xi\in\mathbb{Z}^2$ and for every $h\in\{1,\dots,|\xi|^2\}$, $\|\tilde v^{\xi,h}_\varepsilon-\tilde v_\varepsilon\|_{L^2}\to0$ and hence

$$\tilde v^{\xi,h}_\varepsilon\rightharpoonup v\text{ in }H^1_{\mathrm{loc}}(O\setminus\cup_{i=1}^M B_r(x_i);\mathbb{R}^2).$$

Let $\sigma>0$ be such that $B_\sigma(x_i)$ are pairwise disjoint and contained in O^n. For any $0<r<R$, we set $A_{r,R}(x):=B_R(x)\setminus B_r(x)$. Recalling the definition of $\mathcal{K}$ in (2.37) and of d_t in (2.38) and arguing as in the proof of Theorem 2.5, one can show that

for any given $\delta > 0$ there exists a positive $\omega(\delta)$ such that for every $t \le \sigma$, for every $\xi \in \mathbb{Z}^2$ and for every $h \in \{1, \dots, |\xi|^2\}$

$$\int_{A_{\frac{t}{2}+\sqrt{2}|\xi|\varepsilon, t-\sqrt{2}|\xi|\varepsilon}} |\nabla \tilde{v}_\varepsilon^{\xi,h}| \,\mathrm{d}x \ge \pi \log 2 + \omega(\delta) \tag{3.20}$$

whenever $\mathrm{d}_t(\tilde{v}_\varepsilon^{\xi,h}(\cdot + x_i), \mathcal{K}) \ge \delta$.

Let $P \in \mathbb{N}$ be such that

$$P\omega(\delta) \sum_{\xi \in \mathbb{Z}^2} c_\xi |\xi|^2 \ge \mathbb{W}^{lr}(\mu) + M(\gamma^{lr} - \pi \sum_{\xi \in \mathbb{Z}^2} c_\xi |\xi|^2 \log \sigma - C)$$

where C is the constant in (3.9). For $p = 1, \dots, P$, set $C_p(x_i) := E^B_{2^{1-p}\sigma}(x_i) \setminus E^B_{2^{-p}\sigma}(x_i)$. Then, arguing as in the proof of Theorem 2.5 (ii), one can prove the claim. □

4. The Γ-convergence analysis for $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$

In this section we will develop the Γ-convergence expansion for the energies $F^{an}_{\varepsilon,\Lambda}$ and $F^{lr}_{\varepsilon,\Lambda}$. Before stating the first order Γ-convergence result for such functionals we need to introduce the required notation.

Fix $L_{\varepsilon,\Lambda}$ as in (1.4) and let $\bar{L}_\Lambda$ be as in (1.5), i.e., there exists a positive constant $\bar{C}$ such that

$$\|L_{\varepsilon,\Lambda} - \bar{L}_\Lambda\|_{L^\infty(\Omega)} \le \bar{C}\varepsilon. \tag{4.1}$$

For every $\mu = \sum_{i=1}^M d_i \delta_{x_i}$, with $d_i \in \{-1, +1\}$ and $x_i \in \Omega$, we set

$$\mathbb{W}^{an}_\Lambda(\mu) := \mathbb{W}^{an}_{\bar{L}_\Lambda(\Omega)}(\bar{L}_\Lambda \mu) \quad \text{and} \quad \mathbb{W}^{lr}_\Lambda(\mu) := \mathbb{W}^{lr}_{\bar{L}_\Lambda(\Omega)}(\bar{L}_\Lambda \mu), \tag{4.2}$$

where $\mathbb{W}^{an}_{\bar{L}_\Lambda(\Omega)}$ and $\mathbb{W}^{lr}_{\bar{L}_\Lambda(\Omega)}$ are defined in (2.19) and (3.18) respectively and $\bar{L}_\Lambda \mu = \sum_{i=1}^M d_i \delta_{\bar{L}_\Lambda x_i}$.

Theorem 4.1. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ be a sequence satisfying $F^{an}_{\varepsilon,\Lambda}(u_\varepsilon, \Omega) - M\pi\lambda_{self}|\log \varepsilon| \le C$. Then, up to a subsequence, $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^N d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \le M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-lim inf inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ be such that $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$, with $\mu = \sum_{i=1}^M d^i \delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*

$$\liminf_{\varepsilon \to 0} F_{\varepsilon,\Lambda}(\tilde{u}_\varepsilon, \Omega) - M\pi\lambda_{self}|\log \varepsilon| \ge \mathbb{W}^{an}_\Lambda(\mu) + M\gamma^{an}. \tag{4.3}$$

(iii) *(Γ-lim sup inequality) Given $\mu = \sum_{i=1}^M d_i \delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ with $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ such that*

$$F_{\varepsilon,\Lambda}(u_\varepsilon, \Omega) - M\pi\lambda_{self}|\log \varepsilon| \to \mathbb{W}^{an}_\Lambda(\mu) + M\gamma. \tag{4.4}$$

In order to prove Theorem (4.1) above, we need the following result.

Lemma 4.2. *Let $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ be such that $|\mu(u_\varepsilon)|(\Omega) \le C'|\log \varepsilon|$ for some constant $C' > 0$, then*

$$\mu(u_\varepsilon) \overset{\text{flat}(\Omega')}{\longrightarrow} \mu \text{ if and only if } \mu(u_\varepsilon \circ L^{-1}_{\varepsilon,\Lambda}) \overset{\text{flat}(\bar{L}_\Lambda(\Omega'))}{\longrightarrow} \bar{L}^{-1}_\Lambda \mu \tag{4.5}$$

for every $\Omega' \subset\subset \Omega$ and for every $\mu = \sum_{i=1}^N d_i\delta_{x_i}$, with $d_i \in \mathbb{Z}\setminus\{0\}$ and $x_i \in \Omega'$.

Proof. Fix $\Omega' \subset\subset \Omega$. We first show that if $\mu(u_\varepsilon) \overset{\mathrm{flat}(\Omega')}{\longrightarrow} \mu$ with μ as in the statement, then

$$\|\mu(u_\varepsilon \circ L^{-1}_{\varepsilon,\Lambda}) - \bar L^{-1}_\Lambda \mu\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} \le \mathrm{o}(1) + |\bar L^{-1}_\Lambda| \|\mu(u_\varepsilon) - \mu\|_{\mathrm{flat}(\Omega')}, \tag{4.6}$$

where $\lim_{\varepsilon\to 0}\mathrm{o}(1) = 0$ and $|\bar L^{-1}_\Lambda| := \sup_{|x|=1} |\bar L^{-1}_\Lambda x|$. The proof of the opposite implication is fully analogous and left to the reader.

Set $\bar u_\varepsilon := u_\varepsilon \circ L^{-1}_{\varepsilon,\Lambda}$; by the triangular inequality

$$\begin{aligned}\|\mu(\bar u_\varepsilon) - \bar L^{-1}_\Lambda \mu\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} &\le \|\mu(\bar u_\varepsilon) - \bar L^{-1}_\Lambda \mu(u_\varepsilon)\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} \\ &\quad + \|\bar L^{-1}_\Lambda \mu(u_\varepsilon) - \bar L^{-1}_\Lambda \mu(u)\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))}, \\ &\le \|\mu(\bar u_\varepsilon) - \bar L^{-1}_\Lambda \mu(u_\varepsilon)\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} + |\bar L^{-1}_\Lambda| \|\mu(u_\varepsilon) - \mu(u)\|_{\mathrm{flat}(\Omega')}\end{aligned}$$

and therefore it is enough to show that

$$\|\mu(\bar u_\varepsilon) - \bar L^{-1}_\Lambda \mu(u_\varepsilon)\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} \to 0 \tag{4.7}$$

to prove the claim.

Let $\mu(u_\varepsilon) := \sum_{i=1}^{M_\varepsilon} d_{i,\varepsilon}\delta_{x_{i,\varepsilon}} \in X_{\varepsilon,\Lambda}(\Omega)$. Then

$$\mu(\bar u_\varepsilon) = L^{-1}_{\varepsilon,\Lambda}\mu(u_\varepsilon) = \sum_{i=1}^{M_\varepsilon} d_{i,\varepsilon}\delta_{L^{-1}_{\varepsilon,\Lambda}x_{i,\varepsilon}} \in X_{\varepsilon,\mathbb{Z}^2}(L_{\varepsilon,\Lambda}(\Omega))$$

whence

$$\begin{aligned}\|\mu(\bar u_\varepsilon) - \bar L^{-1}_\Lambda \mu(u_\varepsilon)\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} &= \|\sum_{i=1}^{M_\varepsilon} d_{i,\varepsilon}(\delta_{L^{-1}_{\varepsilon,\Lambda}x_{i,\varepsilon}} - \delta_{\bar L^{-1}_\Lambda x_{i,\varepsilon}}\|_{\mathrm{flat}(\bar L_\Lambda(\Omega'))} \\ &\le M_\varepsilon \sup_{y\in \bar L_\Lambda(\Omega')} |L^{-1}_{\varepsilon,\Lambda}y - \bar L^{-1}_\Lambda y| \le |L^{-1}_{\varepsilon,\Lambda}|\,\bar C\, C'\,\varepsilon|\log\varepsilon|,\end{aligned}$$

where in the last inequality we have used (4.1) and the fact that $M_\varepsilon = |\mu(u_\varepsilon)|(\Omega) \le C'|\log\varepsilon|$. ☐

We now are ready to the prove Theorem 4.1.

Proof of Theorem 4.1. Proof of (i). Let $\{O^h\}$ be an increasing sequence of open smooth sets such that $\cup_{h\in\mathbb{N}} O^h = \bar L_\Lambda(\Omega)$. Fix $h \in \mathbb{N}$, let $\varepsilon > 0$ be small enough so that $O^h \subset L_{\varepsilon,\Lambda}(\Omega)$. Set $\bar u_\varepsilon := u_\varepsilon \circ L^{-1}_{\varepsilon,\Lambda}$; by combining (1.23) with the upper bound in the assumption, we get

$$F_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, O^h) \le F_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, L_{\varepsilon,\Lambda}(\Omega)) \le M\pi\lambda_{self}|\log\varepsilon| + C; \tag{4.8}$$

therefore, by applying Theorem 2.5 and using a diagonal argument, we have that, up to a subsequence, $\mu(\bar u_\varepsilon) \overset{\mathrm{flat}(\bar L_\Lambda(\Omega))}{\longrightarrow} \bar\mu$, for some measure $\bar\mu = \sum_{i=1}^N d_i\delta_{y_i}$, with $d_i \in \mathbb{Z}\setminus\{0\}$ and $y_i \in \bar L_\Lambda(\Omega)$ and $\sum_{i=1}^N |d_i| \le M$.

Let us assume that $\sum_{i=1}^N |d_i| = M$. Trivially, for h sufficiently large, $\mathrm{supp}(\bar\mu) \subset O^h$. By Theorem 2.5(i), we have that $N = M$ and hence $|d_i| = 1$ for any i. By combining (4.8) with the fact that

$$|\mu(\bar u_\varepsilon)|(L_{\varepsilon,\Lambda}(\Omega)) \le F_{\varepsilon,\mathbb{Z}^2}(\bar u_\varepsilon, L_{\varepsilon,\Lambda}(\Omega)), \tag{4.9}$$

for ε sufficiently small, we get

$$|\mu(\bar u_\varepsilon)|(\bar L_\Lambda(\Omega)) = |\mu(\bar u_\varepsilon)|(O_h) \le M\pi\lambda_{\mathrm{self}}|\log\varepsilon| + C$$

and hence the claim follows by Lemma 4.2 with $\mu := \bar{L}_\Lambda^{-1}\bar{\mu}$.

Proof of (ii). We can assume without loss of generality that $F_{\varepsilon,\Lambda}(u_\varepsilon,\Omega) \le M\pi\lambda_{self}|\log\varepsilon| + C$. Set $\bar{u}_\varepsilon := u_\varepsilon \circ L_{\varepsilon,\Lambda}^{-1}$. By (4.9), it follows that

$$|\mu(\bar{u}_\varepsilon)|(L_{\varepsilon,\Lambda}(\Omega)) \le \lambda_{\mathrm{self}} M\pi|\log\varepsilon| + C. \tag{4.10}$$

Let $\{\Omega^h\}_{h\in\mathbb{N}}$ be a sequence of open bounded smooth subsets of Ω such that $\mathrm{supp}\,(\mu) \subset \Omega^h$ for any h, $\cup_{h\in\mathbb{N}}\Omega^h = \Omega$ and $\mathrm{d}_H((\Omega^h)^c,\Omega^c) \to 0$ as $h \to \infty$.

Fix $h \in \mathbb{N}$. Then, for ε small enough $L_{\varepsilon,\Lambda}(\Omega) \supset \bar{L}_\Lambda(\Omega^h)$.

By (4.10) and Lemma (4.2), we get $\mu(\bar{u}_\varepsilon) \overset{\mathrm{flat}(\bar{L}_\Lambda(\Omega^h))}{\longrightarrow} \bar{L}_\Lambda\mu$. Then, by Theorem 2.5(ii), applied with $\bar{\mu} = \bar{L}_\Lambda\mu$ and $O = \bar{L}_\Lambda(\Omega^h)$ we get

$$\begin{aligned}\liminf_{\varepsilon\to0} F_{\varepsilon,\mathbb{Z}^2}(\bar{u}_\varepsilon, L_{\varepsilon,\Lambda}(\Omega)) - M\pi\lambda_{\mathrm{self}}|\log\varepsilon| \\ \ge \liminf_{\varepsilon\to0} F_{\varepsilon,\mathbb{Z}^2}(\bar{u}_\varepsilon, \bar{L}_\Lambda(\Omega^h)) - M\pi\lambda_{\mathrm{self}}|\log\varepsilon| \ge \mathbb{W}^{an}_{\bar{L}_\Lambda(\Omega^h)}(\bar{\mu}) + M\gamma^{an}.\end{aligned}$$

The claim follows immediately by (4.2) and (2.22).

Proof of (iii). Let $\{\Omega^n\}_{n\in\mathbb{N}}$ be a sequence of open bounded smooth subsets of $\mathbb{R}^2$ such that $\cap_{n\in\mathbb{N}}\Omega^n = \Omega$ and $\mathrm{d}_H((\Omega^n)^c,\Omega^c) \to 0$ as $n \to \infty$.

Fix $n \in \mathbb{N}$. Then, for ε small enough $L_{\varepsilon,\Lambda}(\Omega) \subset \bar{L}_\Lambda(\Omega^n)$. By Theorem 2.5(iii) applied with $\bar{\mu} = \bar{L}_\Lambda\mu$ and $O = \bar{L}_\Lambda(\Omega^n)$, there exists $\bar{u}^n_\varepsilon \in \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(\bar{L}_\Lambda(\Omega^n))$ such that $\mu(\bar{u}^n_\varepsilon) \overset{\mathrm{flat}(\bar{L}_\Lambda(\Omega^n))}{\to} \bar{\mu}$ and

$$\begin{aligned}\limsup_{\varepsilon\to0} F_{\varepsilon,\mathbb{Z}^2}(\bar{u}^n_\varepsilon, L_{\varepsilon,\Lambda}(\Omega)) - M\pi\lambda_{\mathrm{self}}|\log\varepsilon| \\ \le \limsup_{\varepsilon\to0} F_{\varepsilon,\mathbb{Z}^2}(\bar{u}^n_\varepsilon, \bar{L}_\Lambda(\Omega^n)) - M\pi\lambda_{\mathrm{self}}|\log\varepsilon| \le \mathbb{W}^{an}(\bar{\mu}) + M\gamma^{an}.\end{aligned}$$

By a standard diagonal argument there exists a sequence $\{\bar{u}_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\mathbb{Z}^2}(L_{\varepsilon,\Lambda}(\Omega))$ ($\bar{u}_\varepsilon := \bar{u}^{n_\varepsilon}_\varepsilon$) such that $\mu(\bar{u}_\varepsilon) \overset{\mathrm{flat}(\bar{L}_\Lambda(\Omega'))}{\longrightarrow} \bar{\mu}$ and

$$\limsup_{\varepsilon\to0} F_{\varepsilon,\mathbb{Z}^2}(\bar{u}_\varepsilon, L_{\varepsilon,\Lambda}(\Omega)) - M\pi\lambda_{\mathrm{self}}|\log\varepsilon| \le \mathbb{W}^{an}(\bar{L}_\Lambda\mu) + M\gamma^{an}.$$

Set $u_\varepsilon := \bar{u}_\varepsilon \circ L_{\varepsilon,\Lambda}$; by Lemma 4.2 $\mu(u_\varepsilon)) \overset{\mathrm{flat}(\Omega')}{\longrightarrow} \mu$ and by (1.23) and (4.2), it satisfies (4.4). $\square$

Remark 4.3. *Set* $A := \det \bar{L}_\Lambda \bar{L}_\Lambda^{-1} Q(\bar{L}_\Lambda^{-1})^T$ *and let* Φ_A *be the solution to*

$$\begin{cases} \mathrm{div}\ A\nabla\Phi_A = \lambda_{self} 2\pi\mu & \text{in } \Omega \\ \Phi_A = 0 & \text{on } \partial A, \end{cases}$$

Set $C := \frac{A}{\sqrt{\det A}}$ *and* $R_A(x) := \Phi(x) - \sum_{i=1}^M d_i \log|C^{-\frac12}(x-x_i)|$*, a straightforward computation shows that*

$$\mathbb{W}^{an}_\Lambda(\mu) := -\pi\,\lambda_{self}(\sum_{i\neq j} d_i d_j \log|C^{-\frac12}(x_i-x_j)| + \pi\sum_{i=1}^M d_i R_A(x_i)). \tag{4.11}$$

By using Theorem 3.5 and Lemma 4.2, arguing as in the proof of Theorem 4.1, one can prove the Γ-convergence expansion for the functionals $F^{lr}_{\varepsilon,\Lambda}$.

Theorem 4.4. *The following* Γ*-convergence result holds true.*

(i) *(Compactness) Let* $M \in \mathbb{N}$ *and let* $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ *be a sequence satisfying* $F^{lr}_{\varepsilon,\Lambda}(u_\varepsilon,\Omega) - M\pi\sum_{\xi\in\mathbb{Z}^2} c_\xi|\xi|^2|\log\varepsilon| \le C$*. Then, up to a subsequence,* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ *for some* $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ *with* $d_i \in \mathbb{Z}\setminus\{0\}$*,* $x_i \in \Omega$ *and* $\sum_i |d_i| \le M$*. Moreover, if* $\sum_i |d_i| = M$*, then* $\sum_i |d_i| = N = M$*, namely* $|d_i| = 1$ *for any* i*.*

(ii) *(Γ-$\liminf$ inequality) Let* $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ *be such that* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$*, with* $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ *with* $|d_i| = 1$ *and* $x_i \in \Omega$ *for every* i*. Then,*

$$\liminf_{\varepsilon\to 0} F^{lr}_{\varepsilon,\Lambda}(u_\varepsilon,\Omega) - M\pi\sum_{\xi\in\mathbb{Z}^2} c_\xi|\xi|^2|\log\varepsilon| \ge \mathbb{W}^{lr}_\Lambda(\mu) + M\gamma^{lr}.$$

(iii) *(Γ-$\limsup$ inequality) Given* $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ *with* $|d_i| = 1$ *and* $x_i \in \Omega$ *for every* i*, there exists* $\{u_\varepsilon\} \subset \mathcal{AF}_{\varepsilon,\Lambda}(O)$ *with* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ *such that*

$$F^{lr}_{\varepsilon,\Lambda}(u_\varepsilon,\Omega) - M\pi\sum_{\xi\in\mathbb{Z}^2} c_\xi|\xi|^2|\log\varepsilon| \to \mathbb{W}^{lr}_\Lambda(\mu) + M\gamma^{lr}.$$

5. Existence of metastable configurations of screw dislocations in the triangular lattice

Here we will prove the existence of many local minimizers for the functionals $F^{an}_{\varepsilon,\Lambda}$. Through this section, we will assume that $f_\xi(a) = c_\xi f(a)$ for every $\xi \in \{e_1, e_2, e_1+e_2\}$ where f satisfies (1f)-(3f) Let $f:\mathbb{R}\to\mathbb{R}$ be such that

(1f) $f \in C^0\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right) \cap C^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)$;

(2f) There exists $\delta > 0$ such that for every $t \in [\frac{1}{2}-\delta, \frac{1}{2}+\delta]$ we have $C_1(\frac{1}{2}-t)^2 < f(\frac{1}{2}) - f(t)$ for some $C_1 > 0$ and

$$S := \sup_{t\in(-\frac{1}{2},\frac{1}{2})} f''(t) < \frac{\min\{c_\xi\ :\ \xi\in\{e_1,e_2,e_1+e_2\}\}}{2\sum_{\xi\in\{e_1,e_2,e_1+e_2\}} c_\xi} C_1;$$

(3f) f is increasing in $[0,\frac{1}{2}]$ and even.

We remark that the assumptions above are satisfied by the energy density of the screw dislocations functionals, $f(a) = \operatorname{dist}^2(a,\mathbb{Z})$, while they are not satisfied by the spin functional potential of the XY model.

Lemma 5.1. *There exists* $\alpha > 0$ *and* $E > 0$ *such that the following holds true: Let* $u \in \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ *such that dist*$(u(i)-u(j),\mathbb{Z}) > \frac{1}{2} - \alpha$ *for some* $(i,j) \in \Omega^1_{\varepsilon,\Lambda}$*. Then there exists a function* $w \in \mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ *such that* $w \equiv u$ *in* $\Omega^0_{\varepsilon,\Lambda}$ *and* $F^{an}_{\varepsilon,\Lambda}(w,\Omega) \le F^{an}_{\varepsilon,\Lambda}(u,\Omega) - E$*.*

Proof. As a consequence of assumption (2f), it is easy to see that there exists $\gamma > 0$ and a positive constant C_2 such that

$$f(\tfrac{1}{2}) - f(\gamma) - f(\tfrac{1}{2}-\gamma) > C_2. \tag{5.1}$$

First, we prove the statement assuming $f \in C^2(\mathbb{R})$. In this case, assumption (5) reads as $f'(\frac{1}{2}) = 0$ and $|f''(\frac{1}{2})| > 2C_1$.

To ease the notation, we will assume that $\Lambda = \mathbb{Z}^2$.

Set $N := \{e_2, e_1+e_2, e_1, -e_2, -e_1-e_2, -e_1\}$. We will assume that $i \notin \partial_{\varepsilon,\mathbb{Z}^2}\Omega$ so that $i + \varepsilon\xi \in \Omega^0_{\varepsilon,\mathbb{Z}^2}$ for any $\xi \in N$.

The case $i \in \partial_{\varepsilon,\mathbb{Z}^2}\Omega$ is fully analogous and it is left to the reader. Without loss of generality, we can assume that $u(i) = 0$. For sake of notation, we set

$$E^i(u) := \sum_{\xi\in N} c_\xi f(u(i + \varepsilon\xi)).$$

Let N_c be the set of the vectors ξ satisfying $\mathrm{dist}(u(i + \varepsilon\xi), \mathbb{Z}) > \frac{1}{2} - \alpha$, with α to be selected.

We distinguish among three cases.

Case 1: $\sum_{\xi\in N_c} c_\xi > \sum_{\xi\in N\setminus N_c} c_\xi$.

In this case, we set $w(i) := \frac{1}{2}$ and we get

$$\begin{aligned} E^i(u) - E^i(w) &\geq \textstyle\sum_{\xi\in N_c} c_\xi (f(\frac{1}{2} + \alpha) - f(\alpha)) - \sum_{\xi\in N\setminus N_c} c_\xi f(\frac{1}{2} + \alpha) \\ &= \textstyle(\sum_{\xi\in N_c} c_\xi - \sum_{\xi\in N\setminus N_c} c_\xi) f(\frac{1}{2}) + \mathrm{o}(1), \end{aligned}$$

where $\mathrm{o}(1) \to 0$ as $\alpha \to 0$.

Case 2: $\sum_{\xi\in N_c} c_\xi = \sum_{\xi\in N\setminus N_c} c_\xi$.

Set

$$a := \left(2\frac{\sum_{\xi\in N_c} c_\xi}{\min_{\xi\in N\setminus N_c} c_\xi}\right)^{\frac{1}{2}}. \tag{5.2}$$

There are two possibilities: either $\max_{\xi\in N\setminus N_c} \mathrm{dist}(u(i+\varepsilon\xi), \mathbb{Z}) \geq a\alpha$ or $\max_{\xi\in N\setminus N_c} \mathrm{dist}(u(i+\varepsilon\xi), \mathbb{Z}) < a\alpha$.

In the first case, let $\bar\xi$ be a vector which realizes the maximum in the problem above. Then we set $w(i) = \frac{1}{2}$ and we get

$$E^i(w) \leq \sum_{\xi\in N_c} c_\xi f(\alpha) + c_{\bar\xi} f(\tfrac{1}{2} - a\alpha) + \textstyle\sum_{\xi\in N\setminus(N_c\cup\{\bar\xi\})} f(\frac{1}{2}); \tag{5.3}$$

moreover by definition of $E^i(u)$, we have that

$$E^i(u) \geq \sum_{\xi\in N_c} c_\xi f(\tfrac{1}{2} - \alpha) + c_{\bar\xi} f(a\alpha). \tag{5.4}$$

Combining (5.3) with (5.4) and by the definition of a in (5.2) we get

$$\begin{aligned} E^i(u)-E^i(w) &\geq \textstyle(\sum_{\xi\in N_c} c_\xi - \sum_{\xi\in N\setminus N_c} c_\xi) f(\frac{1}{2}) + (c_{\bar\xi} a^2 - \sum_{\xi\in N_c} c_\xi) \frac{f''(0)-f''(\frac{1}{2})}{2}\alpha^2 + \mathrm{o}(\alpha^2) \\ &= \frac{\sum_{\xi\in N_c} c_\xi}{\min_{\xi\in N\setminus N_c}\{c_\xi\}} \frac{f''(0) - f''(\frac{1}{2})}{2}\alpha^2 + \mathrm{o}(\alpha^2). \end{aligned}$$

We assume now that $\mathrm{dist}(u(i + \varepsilon\xi), \mathbb{Z}) < a\alpha$ for every $\xi \in N \setminus N_c$. In this case we set $w(i) = \gamma$ with γ given in (5.1). Then, by continuity,

$$E^i(w) = \sum_{\xi\in N_c} c_\xi f(\tfrac{1}{2} - \gamma) + \sum_{\xi\in N\setminus N_c} c_\xi f(\gamma) + \mathrm{o}(1).$$

Since $E^i(u) \geq \sum_{\xi\in N_c} c_\xi f(\frac{1}{2} - \alpha)$, we get

$$E^i(u)-E^i(w) \geq \sum_{\xi\in N_c} c_\xi (f(\tfrac{1}{2}) - f(\tfrac{1}{2} - \gamma)) - \sum_{\xi\in N\setminus N_c} c_\xi f(\gamma) + \mathrm{o}(\alpha) \geq \sum_{\xi\in N_c} c_\xi C_2 + \mathrm{o}(\alpha),$$

where the last inequality follows by (5.1) and by the assumption.

Case 3: $\sum_{\xi\in N_c} c_\xi < \sum_{\xi\in N\setminus N_c} c_\xi$.

Let

$$A := 3\frac{\frac{\sum_{\xi\in N} c_\xi \sum_{\xi\in N_c} c_\xi}{\min_{\xi\in N} c_\xi}}{\frac{\sum_{\xi\in N} c_\xi}{\min_{\xi\in N} c_\xi}\sum_{\xi\in N_c} c_\xi - \sum_{\xi\in N\setminus N_c} c_\xi}.$$

We set $w(i)=\eta$ with $|\eta| = A\alpha$ and $\eta\sum_{\xi\in N\setminus N_c} c_\xi f'(u(i+\varepsilon\xi))\geq 0$.

Then

$$\begin{aligned}
E^i(u) - E^i(w) &= \sum_{\xi\in N_c} c_\xi (f(\tfrac{1}{2}+\alpha) - f(\tfrac{1}{2}+\alpha - A\alpha))\\
&\quad + \sum_{\xi\in N\setminus N_c} c_\xi (f(u(i+\varepsilon\xi)) - f(u(i+\varepsilon\xi)-\eta)\\
&= \frac{A(A-2)}{2}\alpha^2 |f''(\tfrac{1}{2})|\textstyle\sum_{\xi\in N_c} c_\xi + \eta\sum_{\xi\in N\setminus N_c} c_\xi f'(u(i+\varepsilon\xi))\\
&\quad - \frac{A^2}{2}\alpha^2 \sum_{\xi\in N\setminus N_c} c_\xi f''(u(i+\varepsilon\xi)) + \mathrm{o}(\alpha^2)\\
&\geq \frac{A}{2}\alpha^2 S((A-2)\frac{\sum_{\xi\in N} c_\xi}{\min_{\xi\in N} c_\xi}\sum_{\xi\in N_c} c_\xi - A\sum_{\xi\in N\setminus N_c} c_\xi) + \mathrm{o}(\alpha^2)\\
&= \frac{A}{2}\alpha^2 S\frac{\sum_{\xi\in N} c_\xi \sum_{\xi\in N_c} c_\xi}{\min \xi\in N c_\xi} + \mathrm{o}(\alpha^2).
\end{aligned}$$

By combining Case 1, Case 2 and Case 3, choosing α small enough, the claim easily follows.

The general case can be recovered by approximating f in a neighborhood of $\frac{1}{2}$ with C^2 functions still satisfying assumptions (1f)-(3f). $\square$

As a consequence of Lemma 5.1, we obtain the existence of a minimimizer for the energy F^{an}_ε assuming, in addition to (1)-(3), that $f_\xi(\cdot) = c_\xi f(\cdot)$, with f satisfying (1f)-(3f).

Theorem 5.2. *Given $\mu_0 = \sum_{i=1}^M d_i\delta_{x_i}$ with $x_i\in\Omega$ and $d_i\in\{1,-1\}$ for $i = 1,\dots,M$, there exists a constant $K\in\mathbb{N}$ such that, for ε small enough, there exists $k_\varepsilon\in\{1,\dots,K\}$ such that the following minimum problem is well-posed*

$$\min\{F^{an}_{\varepsilon,\Lambda}(u,\Omega)\,:\,\|\mu(u)-\mu_0\|_{\mathrm{flat}} < k_\varepsilon\varepsilon\}. \tag{5.5}$$

Moreover, let α be given by Lemma 5.1; then, any minimizer u_ε of the problem in (5.5) *satisfies*

$$\begin{aligned}
&\mathit{dist}(u_\varepsilon(i) - u_\varepsilon(j),\mathbb{Z}) \leq \frac{1}{2} - \alpha \\
&\quad \textit{for every } (i-j)\in\Omega^1_{\varepsilon,\Lambda},\ \textit{with } i-j\in\{\pm L^{-1}_{\varepsilon,\Lambda}e_1, \pm L^{-1}_{\varepsilon,\Lambda}e_2, \pm L^{-1}_{\varepsilon,\Lambda}(e_1+e_2)\}
\end{aligned}\tag{5.6}$$

and it is a local minimizer for $F^{an}_{\varepsilon,\Lambda}$.

Moreover, let $\{u^0_\varepsilon\}\subset\mathcal{AF}_{\varepsilon,\Lambda}(\Omega)$ be such that

$$\lim_{\varepsilon\to 0} F^{an}_{\varepsilon,\Lambda}(u^0_\varepsilon,\Omega) - M\pi\lambda_{self}|\log\varepsilon| = \mathbb{W}^{an}_\Lambda(\mu_0) + M\gamma^{an}.$$

Then, for ε small enough, the following fact hold true:

(i) *u^0_ε satisfies the condition* (5.6)*;*

(ii) *The solution* $u_\varepsilon = u_\varepsilon(t)$ *to the gradient flow of* $F^{an}_{\varepsilon,\Lambda}$ *from* u^0_ε*, i.e.,*

$$\begin{cases} \frac{\dot{u}_\varepsilon}{|\log\varepsilon|} = -\nabla F^{an}_{\varepsilon,\Lambda}(u_\varepsilon) & \textit{in } (0,+\infty)\times\Omega^0_{\varepsilon,\Lambda} \\ u_\varepsilon(0) = u^0_\varepsilon & \textit{in } \Omega^0_\varepsilon, \end{cases}$$

satisfies $\mu(u_\varepsilon(t)) = \mu(u^0_\varepsilon)$ *for every* $t > 0$.

(iii) *There exists* $\bar{u}^0_\varepsilon$ *such that* $\bar{u}^0_\varepsilon \in \operatorname{argmin}\{F^{an}_{\varepsilon,\Lambda}(u) \,:\, \mu(u) = \mu(u^0_\varepsilon)\}$ *satisfies* (5.6) *and it is a local minimizer for* $F^{an}_{\varepsilon,\Lambda}$.

Theorem 5.2 is a consequence of Lemma 5.1 and its proof follows closely the ones of Theorems 5.5 and 5.6 in [3].

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