A MODICA-MORTOLA APPROXIMATION FOR THE STEINER PROBLEM

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Abstract. In this note we present a way to approximate the Steiner problem by a family of elliptic energies of Modica-Mortola type, with an additional term relying on the weighted geodesic distance which takes care of the connexity constraint.

Résumé

Dans cette note nous présentons une méthode d'approximation du problème de Steiner par une famille de fonctionnelles de type Modica-Mortola, avec un terme additionnel basé sur une distance géodésique à poids, pour prendre en compte la contrainte de connexité.

Titre : Une approximation à la Modica-Mortola pour le problème de Steiner.

Version française abrégée

Le problème bien connu dit "de Steiner" consiste à trouver un compact connexe de longueur minimal qui contient certains points du plan donnés au départ, en nombre fini. L'ensemble minimal est alors un arbre fini constitué de segments qui peuvent se joindre par nombre de 3 uniquement, formant des angles de 120° [6, 12]. L'un des aspects qui a rendu ce problème si célèbre réside dans sa complexité de calcul, malgré une formulation simple en apparence, faisant partie de la liste des 21 problèmes NP-complets de Karp [7] (le temps polynomial étant évalué par rapport au nombre de points).

Dans cette note nous proposons une méthode susceptible de donner lieu à des solutions approchées du Problème de Steiner. La stratégie repose sur l'emploi de fonctionnelles de type elliptique à la manière de Modica-Mortola [9], comme l'ont fait d'autres auteurs auparavant concernant des problèmes liés au périmètre ou longueur d'un fermé [2, 10, 13, 11, 1, 8]. La nouveauté dans notre approche est l'ajout d'un terme permettant de gérer la contrainte de connexité sur l'ensemble à minimiser.

Ce nouveau terme fait intervenir la fonction distance pondérée d_φ, définie en (2). Cette fonction peut être calculée numériquement sur une grille par une méthode, dite *fast-marching* [14], qui a été récemment améliorée dans [3] permettant le calcul à la fois de d_φ et de son gradient par rapport à φ. La fonctionnelle approximante que nous proposons est la suivante

$$S_\varepsilon(\varphi) := \frac{1}{4\varepsilon}\int_\Omega (1-\varphi)^2 dx + \varepsilon \int_\Omega \|\nabla \varphi\|^2 + \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{N} d_\varphi(x_i, x_1).$$

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Notre résultat principal stipule qu'étant donné une suite de minimiseurs φ_ε de S_ε et la suite de fonctions $d_{\varphi_\varepsilon}(\cdot, x_1)$ associée, ces fonctions convergent à une sous-suite près vers une fonction d, dont l'ensemble de niveau $\{d = 0\}$ est un minimiseur du Problème de Steiner associés aux points $\{x_i\}$ (Théorème 2.1).

Les deux premiers termes de la fonctionnelle rappellent la fonctionnelle de Modica-Mortola. Le point essentiellement nouveau demeure dans l'ajout du troisième terme de la fonctionnelle, basé sur le fait suivant : si $\sum_{i=1}^{N} d_\varphi(x_i, x_1) = 0$, alors l'ensemble $\{d_\varphi = 0\}$ doit être connexe par arcs et contenir les $\{x_i\}$.

Dans l'article [4], écrit conjointement avec M. Bonnivard, nous utilisons cette technique pour approcher également certaines variantes du problème de Steiner, comme par exemple la fonctionnelle de distance moyenne. On peut y trouver des preuves détaillées ainsi que des simulations numériques.

Les méthodes numériques envisagées, inspirées par le travail d'É. Oudet dans [10, 11], se basent sur une méthode de gradient appliquée à chaque fonctionnelle S_ε (qui est convexe pour ε grand), en diminuant par étapes la valeur de ε et prenant comme initialisation à chaque étape le point de minimum approché trouvé à l'étape précédente. Cela ne garantie pas de converger vers un minimum global, mais permet en général de choisir un "bon" minimum local.

La preuve du théorème décrit plus haut est de type Γ-convergence. Plus précisément, différemment de ce qui a été fait dans [10, 11] ainsi que dans les autres cas étudiés dans [4], il n'est pas possible de manière évidente d'exprimer notre résultat sous la forme d'un énoncé de $\Gamma-$convergence d'une suite de fonctionnelles vers une autre. Cependant, la démonstration en suit le même schéma. La Γ-limsup découle de techniques classiques que l'on peut trouver dans [2]. En revanche la Γ-liminf est plus délicate. L'un des points difficiles à montrer est la rectifiabilité d'une limite Hausdorff d'ensembles de niveaux de fonctions d_{φ_ε} associées à des φ_ε d'énergies uniformément bornées. L'argument original de Modica-Mortola [9] est bien sûr essentiel, mais de nouvelles techniques nécessitent d'être introduites.

1. Introduction

Given a finite number of points $D := \{x_i\}_{i=1,\dots,N} \subset \Omega \subset \mathbb{R}^2$, the so-called Steiner problem consists in solving

$$\min\left\{\mathcal{H}^1(K) \quad ; \; K \subset \mathbb{R}^2 \text{ compact, connected, and containing } D\right\}. \tag{1}$$

Here, $\mathcal{H}^1(K)$ stands for the one-dimensional Hausdorff measure of K. It is known that minimizers for (1) do exist, need not to be unique, and are trees composed by a finite number of segments joining with only triple junctions at 120°, whereas computing a minimizer is very hard (some versions of the Steiner Problem belong to the original list of NP-complete problems by Karp, [7]). We refer for instance to [6] for a history of the problem and to [12] for recent mathematical results about it.

In this note we propose a way to approximate the problem, and we prove convergence to an exact solution as some parameter ε goes to zero. Our strategy is to approximate the length by an elliptic energy of Modica-Mortola [9] type. This strategy was pursued before by many authors for similar problems involving the perimeter or the length of a closed set (see e.g. [2, 10, 13, 11, 1, 8]), but the novelty here is that we are able to add a term taking care of the connexity constraint. This term relies on the weighted geodesic distance d_φ, defined as follows. Given $\Omega \subset \mathbb{R}^2$, for any non-negative function $\varphi \in C^0(\overline{\Omega})$, we define the corresponding weighted geodesic distance through

$$d_\varphi(x,y) := \inf\left\{\int_\gamma \varphi(x)d\mathcal{H}^1(x);\ \gamma \text{ curve in } \Omega \text{ connecting } x \text{ and } y\right\}. \tag{2}$$

Given a function φ and a point x_1, the distance $d_\varphi(\cdot, x_1)$ can be treated numerically by the so-called *fast-marching* method [14] since it is a solution of $\|\nabla u\| = \varphi$ with $u(x_1) = 0$ in the viscosity sense. A recent improvement of this algorithm (see [3]) is now able to compute at the same time d_φ and its gradient with respect to φ, which is useful every time one needs to optimize w.r.t. φ a functional involving d_φ. Our proposal to approximate the problem (1) is then to minimize

$$S_\varepsilon(\varphi) := \frac{1}{4\varepsilon}\int_\Omega (1-\varphi)^2 dx + \varepsilon\int_\Omega \|\nabla\varphi\|^2 + \frac{1}{\sqrt{\varepsilon}}\sum_{i=1}^N d_\varphi(x_i, x_1),$$

among all functions $\varphi \in \mathcal{A} := H^1(\Omega)\cap C^0(\overline{\Omega})\cap\{\varepsilon \le \varphi \le 1 \text{ and } \varphi = 1 \text{ on } \partial\Omega\}$.

The first two terms are a simple variant of the standard Modica-Mortola functional, already used in [2]: as $\varepsilon \to 0$, they force φ to tend to 1 a.e. and pay the transition between the value 1 and the value ε by means of the length of the transition set, while the last term tends to enforce connectedness. The key point is that whenever $\sum_{i=1}^N d_\varphi(x_i, x_1) = 0$, the set $\{d_\varphi = 0\}$ must be path-connected, must contain all the points $\{x_i\}$, and the path connecting them inside this set are such that $\varphi = 0$ $\mathcal{H}^1$−a.e. on them.

In the paper [4], the authors together with M. Bonnivard used this idea to approximate some variant of the Steiner Problem, as the Average distance and p-Compliance problem. One can find therein detailed proofs and numerical experiments. The main idea for numerics is based on the work by É. Oudet in [10, 11]: for every ε one can run a gradient descent for S_ε (which is convex for large ε), and a candidate minimizer for the limit problem is obtained by reducing at each step the value ε and initializing the gradient with the critical point obtained at the previous step. There is no guarantee that this converges to a global minimum, but at least a "well-chosen" local minimum is chosen.

Existence of minimizers for S_ε. The existence of minimizers for the functional S_ε is a delicate matter. This depends on the fact that H^1 does not inject into C^0 and on the behavior of the map $\varphi \mapsto d_\varphi$. First, notice that we only restricted our attention to $\varphi \in C^0(\overline{\Omega})$ for the sake of simplicity. Indeed, it is possible to define d_φ as a continuous function as soon as $\varphi \in L^p$ for an exponent p larger than the dimension (here, $p > 2$, see [5]). The difficult question is which kind of convergence on φ provides pointwise convergence for d_φ. If one wanted upper semi-continuity of the map $\varphi \mapsto d_\varphi(x, x_1)$ (for fixed x and x_1), this would be easy, thanks to the concave behavior of d_φ, and any kind of weak convergence would be enough. Yet, in this case we would like lower semi-continuity, which is more delicate. An easy result is the following: if $\varphi_n \to \varphi$ uniformly and a uniform lower bound $\varphi_n \ge c > 0$ holds, then $d_{\varphi_n}(x, x_1) \to d_\varphi(x, x_1)$. Counterexamples are known if the lower bound is omitted. On the contrary, replacing the uniform convergence with a weak H^1 convergence (which would be natural in the minimization of S_ε) is a delicate matter (by the way, the continuity seems to be true and it is not known whether the lower bound is necessary or not).

For the sake of our paper, one could consider adding an extra term of the form $\varepsilon^{10}\int ||\nabla\varphi||^p$ with $p > 2$, which enforces continuity and uniform convergence, or just think that the results are given "provided a minimizer exist". From the point of view of the approximation result and of the numerical applications this is not crucial.

2. The main result

Theorem 2.1. *Let Ω be a bounded open convex set containing the convex hull of the $\{x_i\}$. For all $\varepsilon > 0$ let φ_ε be a minimizer of S_ε among all $\varphi \in \mathcal{A}$. Consider the sequence of functions $d_{\varphi_\varepsilon}(\,\cdot\,, x_1)$, which are $1-$ Lipshitz and converge, up to subsequences, to a certain function d. Then the set $K := \{d = 0\}$ is compact, connected and is a solution to the Steiner Problem* (1).

Proof. We first extract a subsequence such that the sequence of 1-Lipschitz functions $d_{\varphi_{\varepsilon_n}}(x, x_1)$ converges uniformly to some function $d(x)$. It is easy to see that the set $K = \{d(x) = 0\}$, is a compact and connected set as a Hausdorff limit of sub level sets of $d_{\varphi_{\varepsilon_n}}(\,\cdot\,, x_1)$, which are all compact connected sets.

Let now K' be any competitor in the Steiner Problem, that we can assume contained in Ω . By using a variant of [2, Theorem 3.1.], it is not difficult to construct a sequence of functions $\psi_\varepsilon \in H^1(\Omega)\cap C^0(\overline{\Omega})$, satisfying $\varepsilon \le \psi_\varepsilon \le 1$, $\psi_\varepsilon = 1$ on $\partial\Omega$ and $\limsup_n S_{\varepsilon_n}(\psi_{\varepsilon_n}) \le \mathcal{H}^1(K')$. In particular, following the construction of [2] it is easy to make the last term $\frac{1}{\sqrt{\varepsilon}}\sum_{i=1}^N d_{\varphi_\varepsilon}(x_i, x_1)$ tend to zero since φ_ε is very small close to K' (a careful look at the proof reveals that $1/\sqrt{\varepsilon}$ is needed in front of this term, or any other coefficient of the form $o((\varepsilon \ln \varepsilon)^{-1})$).

On the other hand it is clear from the minimizing property of φ_ε that

$$\liminf_n S_{\varepsilon_n}(\varphi_{\varepsilon_n}) \le \limsup_n S_{\varepsilon_n}(\psi_{\varepsilon_n}), \tag{3}$$

thus the proof will be finished provided that we show the following claim

$$\mathcal{H}^1(K) \le \liminf_{n\to+\infty} S_{\varepsilon_n}(\varphi_{\varepsilon_n}). \tag{4}$$

The full details of this fact can be found in [4, Lemma 3.1.]. We shall describe here only the ideas of proof, which is achieved within two main steps. The first one consists in finding a bound $\mathcal{H}^1(K) \le C$ when $\liminf S_{\varepsilon_n}(\varphi_{\varepsilon_n}) < +\infty$ (which is obviously the case here).

The main tool is the definition of the following geometric quantity: for each set $\Gamma \subset \mathbb{R}^2$, each unit vector $\nu \in \mathbb{S}^1$ and each $\lambda > 0$ we set

$$\Gamma_{\lambda,\nu} := \{x \in \mathbb{R}^2 \; : \; \text{there exists } t \in [-\lambda, \lambda] \text{ with } x - t\nu \in \Gamma\}$$

and we define

$$I_\lambda(\Gamma) := \frac{1}{2\pi\lambda}\int_{\mathbb{S}^1} (\mathscr{L}^2((\Gamma)_{\lambda,\nu})d\nu.$$

The following geometrical estimate [4, Lemma 2.6.] is one of our key ingredients and is of independent interest: whenever Γ_ε are compact connected sets converging to Γ as $\varepsilon \to 0$ in the Hausdorff distance, then

$$\exists \lambda, \varepsilon_0 > 0\,; \quad I_\lambda(\Gamma_\varepsilon) \ge C\mathcal{H}^1(\Gamma_0), \quad \forall \varepsilon \le \varepsilon_0, \tag{5}$$

where the constant C is universal.

Now, fix $\delta_0, \tau_0 > 0$, and let $\{z_1, z_2, \dots, z_N\} \subseteq K$ be a τ_0-network in K, i.e. $K \subseteq \bigcup_{1\le i\le N} B(z_i, \tau_0)$. Due to the convergence $d_{\varphi_\varepsilon}(z_i, x_\varepsilon) \to d(z_i) = 0$, for small

ε we can build a set $\Gamma_\varepsilon = \bigcup_{1\leq i\leq N} \Gamma_i^\varepsilon$ where each Γ_i^ε is a C^1 curve connecting z_i to x_1 and satisfying $\int_{\Gamma_i^\varepsilon} \varphi_\varepsilon(s) d\mathcal{H}^1(s) < \delta_0$.

We use the usual estimate $\frac{1}{4\varepsilon}(1-\varphi_\varepsilon)^2 + \varepsilon\|\nabla\varphi_\varepsilon\|^2 \geq \|\nabla(P(\varphi_\varepsilon))\|$ where $P(t) = t - t^2/2$ is a primitive of $(1-t)$, and compute the total variation of $P(\varphi_\varepsilon)$ in the direction ν on a set $(\Gamma_\varepsilon)_{\lambda,\nu}$. Using that $P(\varphi_\varepsilon)$ is almost 0 on Γ_ε (by definition of Γ_ε and using $P(t) \leq t$) and that, on the contrary, $P(\varphi_\varepsilon) \to P(1) = 1/2$ a.e., we get an estimate on $I_\lambda(\Gamma_\varepsilon)$. Thanks to (5) this turns into an estimate on the $\mathcal{H}^1$ measure of the Hausdorff limit of Γ_ε. By taking then the limit $\delta_0 \to 0$, and finally $\tau_0 \to 0$ one gets an estimate on $\mathcal{H}^1(K)$ and concludes the first step.

The second step is a refinement of the first: once we have established the rectifiability of K, we can use the existence of tangent line $\mathcal{H}^1$-a.e. on K. Using a similar argument as the one above but adapted locally around each point of K (i.e. choosing the direction ν orthogonal to the tangent to K instead of taking an average over all directions) we are able to prove the better estimate (4) and this finishes the proof. □

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