An example of non-existence of plane-like minimizers for an almost-periodic Ising system

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1 Introduction

The homogenization of *ferromagnetic spin systems* in deterministic or random environments [3], as well as in some aperiodic settings [4], has been carried over in analogy with the homogenization of surface energies [2]. The computation of an effective surface energy for such systems relies on the characterization of those ground states that follow a planar interface, and the related homogenization formulas. For systems with periodic coefficients it has been shown that the energy of such ground states can be confined on a strip of finite width around a plane (*plane-like minimizers*) [5]. In this paper we show that this is not the case if the coefficients are *uniformly almost periodic* by giving an explicit two-dimensional example where there is no ground state confined on a strip. In this example the coefficents are the uniform limit of periodic coefficients (with increasing period).

2 Setting of the problem

We consider a discrete system of nearest-neighbour interactions in dimension two with coefficients $c_{ij} \geq c > 0$, $i, j \in \mathbb{Z}^2$. The corresponding ferromagnetic spin energy is

$$F(u) = \sum_{ij} c_{ij}(u_i - u_j)^2, \tag{1}$$

where $u : \mathbb{Z}^2 \to \{-1, 1\}$, $u_i = u(i)$, and the sum runs over the set of *nearest neighbours* or *bonds* in $\mathbb{Z}^2$, which is denoted by

$$\mathcal{Z} = \{(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |i - j| = 1\}.$$

Such energies correspond to inhomogeneous surface energies on the continuum [1, 3].

Definition 1 *We say that u is a* ground state *if we have*

$$\sum_{ij} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big)\le 0 \tag{2}$$

for all v such that $v_i=u_i$ except for a finite number of indices (so that actually the sum runs over a finite set).

Definition 2 *We say that u is a* plane-like ground state *or* plane-like minimizer *for F in the direction ν if u is a ground state and there exists a number M such that (up to a change of sign of all values of u) we have*

$$u_i=\pm 1 \qquad \textit{if } \pm\langle i,\nu\rangle\ge M. \tag{3}$$

The relevance of this definition lies in a result by Caffarelli and de la Llave, who proved that if c_{ij} is periodic then for all directions ν there exists a plane-like minimizer of F in the direction ν[5].

If we identify the function u with its piecewise-constant interpolation, then being a plane-like minimizer can be interpreted as the property that the interface $\partial\{u=1\}$ lies in a strip around a line (or a hyperplane in higher dimension, whence the name plane-like minimizer). Note that this interface cannot be periodic if ν is an 'irrational' direction.

3 The example

This section is devoted to an example of *uniformly almost-periodic* coefficients c_{ij} such that there exits plane-like minimizer for the corresponding F for all directions ν.

We consider the following nested sets: for $n\ge 1$ we define

$$B_n=\Big\{(i,j)\in\mathcal{Z}:\frac{i_1+j_1}{2}\text{ or }\frac{i_2+j_2}{2}\in\frac{1}{2}+2\cdot 3^n+4\cdot 3^n\mathbb{Z}\Big\}$$

Since $2\cdot 3^{n+1}+4\cdot 3^{n+1}\mathbb{Z}\subset 2\cdot 3^n+4\cdot 3^n\mathbb{Z}$ we have $B_{n+1}\subset B_n$. We set $B_0=\mathcal{Z}$.

For all $i,j\in\mathbb{Z}^2$ with $|i-j|=1$ we set

$$c_{ij}=\frac{1}{2}+\frac{1}{2^n}\text{ if }(i,j)\in B_n\setminus B_{n+1},\quad n=0,1,\dots \tag{4}$$

Remark 3 (almost-periodicity) Note that the coefficients c^n_{ij} defined by

$$c^n_{ij}=\max\Big\{c_{ij},\frac{1}{2}+\frac{1}{2^n}\Big\}$$

are $4\cdot 3^n$-periodic and converge uniformly to c_{ij} on $\mathcal{Z}$. Hence, the system of coefficients c_{ij} is *uniformly almost periodic*; more precisely, it is the uniform limit of a family of periodic coefficients of increasing periods.

Remark 4 (homogenizability) Note that the set of coefficients c_{ij} is *homogenizable* (in the terminology of [3]). If we define the family of energies

$$F_\varepsilon(u) = \sum_{ij} \varepsilon c_{ij}(u_i - u_j)^2 \qquad \text{if } u : \varepsilon\mathbb{Z}^2 \to \{-1, 1\}$$

where $u_i = u(\varepsilon i)$, then, upon identifying each u with its piecewise-constant interpolation as a L^1-function, F_ε Γ-converge to the energy

$$F_0(u) = 4 \int_{\partial\{u=1\}} (|\nu_1| + |\nu_2|) d\mathcal{H}^1 \qquad \text{if } u \in BV(\mathbb{R}^2; \{-1, 1\})$$

where $\partial\{u = 1\}$ is understood as the reduced boundary of the set $\{u = 1\}$ and ν its measure-theoretical normal.

This can be proved using the results in [3], or directly by comparison, on one side remarking that, using that $c_{ij} \geq 1/2$ for all i and j, we have

$$F_\varepsilon(u) \geq \frac{1}{2} \sum_{ij} \varepsilon(u_i - u_j)^2 \qquad \text{if } u : \varepsilon\mathbb{Z}^2 \to \{-1, 1\}$$

and the Γ-limit of the energies of this right-hand side is F_0 by [1]. On the other side, by the remark above, for all u we can find a sequences of functions $\{u_\varepsilon\}$ converging to u and such that

$$F_\varepsilon(u_\varepsilon) \leq \Big(\frac{1}{2} + \frac{1}{2^n}\Big) \sum_{ij} \varepsilon((u_\varepsilon)_i - (u_\varepsilon)_j)^2 \leq \Big(4 + \frac{8}{2^n}\Big) \mathcal{H}^1(\partial\{u = 1\})$$

(the factor 8 comes from the fact that each nearest-neighbour pair is accounted for twice, and that $((u_\varepsilon)_i - (u_\varepsilon)_j)^2 = 4$ for non-zero interactions).

We now show that there exists no plane-like minimizer for the energy F in any direction ν. We first consider the case when ν is not any coordinate direction. By symmetry it is sufficient to consider the case

$$\nu_1 < 0, \qquad 0 < \nu_2 \leq -\nu_1;$$

i.e., the direction of the strip

$$S^M_\nu := \{x \in \mathbb{R}^2 : \langle x, \nu\rangle| \leq M\}$$

is increasing and at an angle not less than 45 degrees.

Suppose that such a plane-like minimizer u existed, and let ν, M be given by its definition. Up to changing the sign to u we may suppose that (3).

With fixed n, let k_n be the minimal k such that the horizontal line

$$x_2 = \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n k$$

intersects

$$S_\nu^M \cap \Big\{(x_1, x_2) : x_1 > \frac{1}{2} + 2 \cdot 3^n\Big\};$$

i.e., the intersection of the strip with the half-plane on the right-hand side of the first vertical line of B_n.

We consider the function v defined as

$$v_i = \begin{cases} 1 & \text{if } i_1 < \frac{1}{2} + 2 \cdot 3^n \text{ and } i_2 > \frac{1}{2} - 2 \cdot 3^n + 4 \cdot 3^n k_n \\ -1 & \text{if } i_1 > \frac{1}{2} + 2 \cdot 3^n \text{ and } i_2 < \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n k_n \\ u_i & \text{otherwise.} \end{cases} \tag{5}$$

Note that by (3) the set $\{I : u_i \neq v_i\}$ is finite and contained in the horizontal strip defined by

$$S_n = \Big\{(x_1, x_2) : \frac{1}{2} - 2 \cdot 3^n + 4 \cdot 3^n k_n < x_2 < \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n k_n\Big\}.$$

If we identify the discrete function u with its piecewise-constant interpolation

$$u(x) = u\Big(\Big\lfloor x_1 - \frac{1}{2}\Big\rfloor, \Big\lfloor x_2 - \frac{1}{2}\Big\rfloor\Big) \tag{6}$$

then u can be pictured through the interface $\partial\{u = 1\}$, and likewise v. In Fig. 1 the solid line represents the interface $\partial\{v = 1\}$ and the dotted line the part of the interface $\partial\{u = 1\}$ not included in $\partial\{v = 1\}$. The vertical and horizontal lines represent the interactions in B_n

We now compute the variation of the energy

$$\sum_{ij} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big),$$

which we estimate separately on the sets

$$I_1 = \Big\{(i, j) : \frac{i_2 + j_2}{2} = \frac{1}{2} - 2 \cdot 3^n + 4 \cdot 3^n k_n, (i_1, j_1) \text{ or } (i_2, j_2) \in S_\nu^M\Big\}$$

$$I_2 = \Big\{(i, j) : \frac{i_2 + j_2}{2} = \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n k_n, (i_1, j_1) \text{ or } (i_2, j_2) \in S_\nu^M\Big\}$$

$$I_3 = \Big\{(i, j) : \frac{i_1 + j_1}{2} = \frac{1}{2} + 2 \cdot 3^n, (i_1, j_1) \text{ or } (i_2, j_2) \in S_\nu^M\Big\}$$

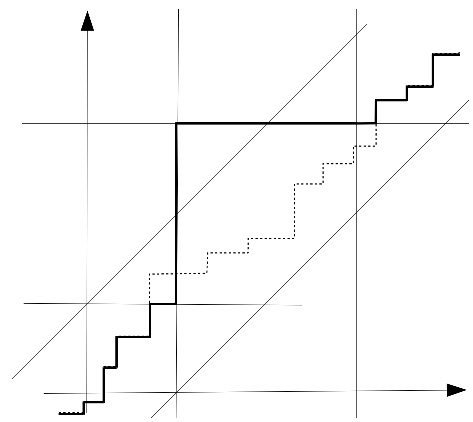

Figure 1: Construction of a competitor v (oblique case)

$$I_u = \Big\{(i,j) : i,j \in S_\nu^M \cap S_n, (i,j) \notin I_1 \cup I_2 \cup I_3\Big\}$$

$$I_v = \Big\{(i,j) : v_i \neq v_j, (i,j) \notin I_1 \cup I_2 \cup I_3; i \text{ or } j \in S_n\Big\}.$$

Note that outside the union of these sets $u_i = v_i$ and $u_j = v_j$; note moreover that

$$c_{ij} \leq \frac{1}{2} + \frac{1}{2^n} \text{ on } I_1 \cup I_2 \cup I_3 \cup I_v$$

$$c_{ij} \geq \frac{1}{2} + \frac{1}{2^{n-1}} \text{ on } I_u.$$

Up to taking a larger M we can suppose that

• u_i has the same value of u_j if $(i,j) \in \mathcal{Z}$ and $i \notin S_\nu^M$ (i.e., we have no interactions on the boundary of S_ν^M);

• the number of interactions in I_1 and I_2 (respectively, I_3) can be estimated by $4M/|\nu_1|$ (respectively, by $4M/|\nu_2|$). Note that $2M/|\nu_1|$ (respectively, $2M/|\nu_2|$) is the length of the intersection of an horizontal (respectively, vertical) line with S_ν^M.

We then have

$$\begin{aligned}\sum_{(i,j)\in I_1\cup I_2} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big) &\geq -\sum_{(i,j)\in I_1\cup I_2} c_{ij}(v_i-v_j)^2 \\ &\geq -\Big(\frac{1}{2}+\frac{1}{2^n}\Big)4\cdot\frac{8M}{|\nu_1|} \qquad (7)\end{aligned}$$

$$\begin{aligned}\sum_{(i,j)\in I_3} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big) &\geq -\sum_{(i,j)\in I_3} c_{ij}(v_i-v_j)^2 \\ &\geq -\Big(\frac{1}{2}+\frac{1}{2^n}\Big)4\cdot\frac{4M}{|\nu_2|} \qquad (8)\end{aligned}$$

$$\sum_{(i,j)\in I_u} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big) = \sum_{(i,j)\in I_u} c_{ij}(u_i-u_j)^2$$
$$\geq \Big(\frac{1}{2}+\frac{1}{2^{n-1}}\Big)\Big(4\cdot 3^n+4\cdot 3^n\frac{|\nu_2|}{|\nu_1|}+\frac{8M}{|\nu_2|}\Big) \qquad (9)$$

$$\sum_{(i,j)\in I_v} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big) = -\sum_{(i,j)\in I_v} c_{ij}(v_i-v_j)^2$$
$$\geq -\Big(\frac{1}{2}+\frac{1}{2^n}\Big)\Big(4\cdot 3^n-\frac{4M}{|\nu_2|}+4\cdot 3^n\frac{|\nu_2|}{|\nu_1|}-\frac{8M}{|\nu_2|}\Big). \qquad (10)$$

From estimates (7)–(10) we obtain

$$\sum_{(i,j)\in I_u} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big) \geq \frac{3^n}{2^n}\cdot 4\Big(1+\frac{|\nu_2|}{|\nu_1|}\Big)-32M\Big(\frac{1}{|\nu_1|}+\frac{1}{|\nu_2|}\Big). \qquad (11)$$

If n is large enough the right-hand side of this expression is positive, contradicting (2).

It remains the case when $\nu_1\nu_2=0$. By symmetry it suffices to consider the case $\nu_1=0$; i.e., when we suppose that u is a ground state such that there exists M such that

$$u_i=1 \quad \text{if } i_2>M, \qquad u_i=-1 \quad \text{if } i_2<-M.$$

Let $S_M=\{x:|x_1|\leq M\}$, and let n be such that

$$2\cdot 3^n > M+2. \qquad (12)$$

In this case there is no pair $(i,j)\in B_n\cap S_M$ with $i_2=j_2$ (i.e., there is no 'horizontal' bond in B_n lying in the strip S_M).

With fixed $k\in\mathbb{N}$ we define a test function v as follows:

$$v_i=\begin{cases}-1 & \text{if } 2\cdot 3^n<i_1\leq 2(1+2k)3^n,\ i_2<2\cdot 3^n\\ u_i & \text{otherwise.}\end{cases}$$

We can picture the functions u and v through the interfaces related to their piecewise-constant interpolations as done in the oblique case above. In Fig.2 the boldface solid line represents the interface related to v, the boldface dotted line represents the part of the interface related to u not included in that of v, the other solid lines represent the location of the bonds in B_n.

Let

$$I_1=\Big\{(i,j)\in\mathcal{Z}\cap B_n:\frac{i_1+j_1}{2}=\frac{1}{2}+2\cdot 3^n\Big\}$$

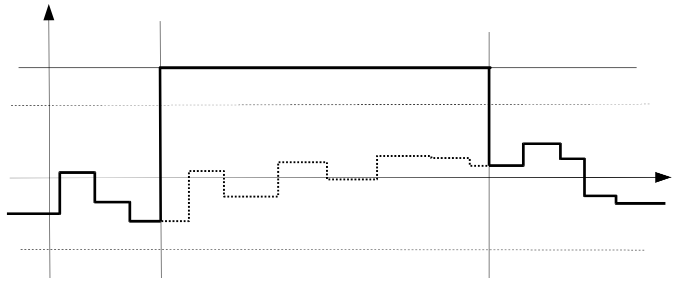

Figure 2: Construction of a competitor v (horizontal case)

$$I_2 = \Big\{(i,j) \in \mathcal{Z} \cap B_n : \frac{i_1 + j_1}{2} = \frac{1}{2} + 2(1+2k)3^n\Big\}$$

$$I_3 = \Big\{(i,j) \in \mathcal{Z} \cap B_n : \frac{i_2 + j_2}{2} = \frac{1}{2} + 2 \cdot 3^n\Big\}$$

$$I_u = \Big\{(i,j) \in \mathcal{Z} \cap B_n : i,j \in S_M, 2 \cdot 3^n < \min\{i_1, j_1\},\ \max\{i_1, j_1\} \leq 2(1+2k)3^n\Big\}$$

We can then estimate

$$\begin{aligned} & \sum_{ij} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big) \\ = & \sum_{(i,j)\in I_1\cup I_2\cup I_3\cup I_u} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big) \qquad (13) \\ \geq & -\sum_{(i,j)\in I_1\cup I_2\cup I_3} \Big(\frac{1}{2} + \frac{1}{2^n}\Big)(v_i - v_j)^2 + \sum_{(i,j)\in I_u} \Big(\frac{1}{2} + \frac{1}{2^{n-1}}\Big)(u_i - u_j)^2 \\ \geq & -8\Big(\frac{1}{2} + \frac{1}{2^n}\Big)\#(I_1 \cup I_2 \cup I_3) + 8\Big(\frac{1}{2} + \frac{1}{2^{n-1}}\Big)\#I_3, \qquad (14) \end{aligned}$$

where in the estimate for the sum on I_u we have taken into account only horizontal bonds where $u_i \neq u_j$ (whose number is greater than $\#I_3$). We can then estimate

$$\begin{aligned} \sum_{ij} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big) & \geq -8\Big(\frac{1}{2} + \frac{1}{2^n}\Big)(8 \cdot 3^n + 4k\,3^n) + 8\Big(\frac{1}{2} + \frac{1}{2^{n-1}}\Big)4k\,3^n \\ & \geq -64 \cdot 3^n + 4k\frac{3^n}{2^n}. \qquad (15) \end{aligned}$$

By taking k large enough (recall that now n is fixed by (12)) the last expression is positive, again contradicting (2).

Acknowledgments

The example in this paper has been constructed after a question by Matteo Novaga at a Oberwolfach Workshop on “Interplay of Analysis and Probability in Physics” in January 2012.

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