An example of non-existence of plane-like minimizers for an almost-periodic Ising system

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Abstract

We give an example of a ferromagnetic spin system with uniformly almost-periodic coefficients whose ground states may not be confined in any finite strip, in contrast to what previously proved in the case of periodic coefficients by Caffarelli and de la Llave.

1 Introduction

This paper stems from a previous work by Caffarelli and de la Llave [8], where the authors studied properties of ground states for periodic *ferromagnetic lattice spin systems*. In that paper they proved that the energy of such ground states can be confined on a strip of finite width around a plane (*plane-like minimizers*). That property, in turn, gives the existence of a limit *surface tension*, and implies that the related scaled energies can be *homogenized*; i.e., that they can be approximated by properly defined macroscopic energies when the lattice spacing is scaled and tends to 0 (see Remark 4 below). Such macroscopic energies can be defined using the formalization of Γ-convergence, in analogy with the homogenization of surface energies in a continuous setting [2]. We note that the properties of ground states are a subtler issue than the computation of the limit energies, in that they may be important to determine the scale of the corrections in this limit passage. We refer to the paper by Caffarelli and de la Llave for a wider introduction to the subject.

Beyond the periodic context, homogenization results for ferromagnetic lattice spin energies have also been obtained for almost-periodic or random environments [4] (see [5] for dilute spin systems), as well as in some aperiodic settings [6]. In this paper we show that if the coefficients of the ferromagnetic system are *uniformly almost periodic* then there can be no ground state confined on a strip, in contrast to the periodic case. To that end, we provide an explicit two-dimensional example, where the coefficients are the uniform limit of periodic coefficients (with increasing period).

2 Setting of the problem

We consider a discrete system of nearest-neighbour interactions in dimension two with coefficients $c_{ij} \geq c > 0$, $i, j \in \mathbb{Z}^2$. The corresponding ferromagnetic spin energy is

$$F(u) = \sum_{ij} c_{ij}(u_i - u_j)^2, \tag{1}$$

where $u : \mathbb{Z}^2 \to \{-1, 1\}$, $u_i = u(i)$, and the sum runs over the set of *nearest neighbours* or *bonds* in $\mathbb{Z}^2$, which is denoted by

$$\mathcal{Z} = \{(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |i - j| = 1\}.$$

Such energies correspond to inhomogeneous surface energies on the continuum [1, 4].

Definition 1 *We say that u is a* ground state *if we have*

$$\sum_{ij} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big) \leq 0 \tag{2}$$

for all v such that $v_i = u_i$ except for a finite number of indices (so that actually the sum runs over a finite set of nearest neighbours).

Definition 2 *We say that u is a* plane-like ground state *or a* plane-like minimizer *for F in the direction ν if u is a ground state and there exists a number M such that (up to a change of sign of all values of u) we have*

$$u_i = \begin{cases} 1 & \text{if } \langle i, \nu\rangle \geq M \\ -1 & \text{if } \langle i, \nu\rangle \leq -M. \end{cases} \tag{3}$$

The relevance of this definition lies in a result by Caffarelli and de la Llave, who proved that if c_{ij} is periodic then for all directions ν there exists a plane-like minimizer of F in the direction ν [8], in analogy to what previously shown for continuous interfaces [7]. Related results and applications can be found, e.g., in [9, 10, 11, 12].

If we identify the function u with its piecewise-constant interpolation, then being a plane-like minimizer can be interpreted as the property that the interface $\partial\{u = 1\}$ lies in a strip around a line (or a hyperplane, in higher dimension, whence the name plane-like minimizer). Note that this interface cannot be periodic if ν is an 'irrational' direction (i.e., it is not a multiple of a vector in $\mathbb{Z}^2$).

3 The example

This section is devoted to an example of *uniformly almost-periodic* coefficients c_{ij} such that there exist no plane-like minimizer for the corresponding F for all directions ν.

We consider the following nested sets of bonds: for $n \geq 1$ we define

$$B_n = \Big\{(i, j) \in \mathcal{Z} : \frac{i_1 + j_1}{2} \text{ or } \frac{i_2 + j_2}{2} \in \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n \mathbb{Z}\Big\}$$

Since $2 \cdot 3^{n+1} + 4 \cdot 3^{n+1}\mathbb{Z} \subset 2 \cdot 3^n + 4 \cdot 3^n\mathbb{Z}$ we have $B_{n+1} \subset B_n$. We set $B_0 = \mathcal{Z}$.

Let a_n be a strictly decreasing sequence of real numbers with

$$a := \inf_n a_n > 0. \tag{4}$$

For all $i, j \in \mathbb{Z}^2$ with $|i-j| = 1$ we set

$$c_{ij} = a_n \text{ if } (i,j) \in B_n \setminus B_{n+1}, \quad n = 0, 1, \dots$$

Note that $B_n \cap [-3^n, 3^n] = \emptyset$. Hence, we deduce that $\bigcap_n B_n = \emptyset$ and the coefficients c_{ij} are well defined for all $(i,j) \in \mathcal{Z}$. Note moreover that we can write

$$c_{ij} = a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) \chi_{B_n}(i,j),$$

where χ_{B_n} is the characteristic function of B_n.

Remark 3 (almost-periodicity) Note that the coefficients c^k_{ij} defined by

$$c^k_{ij} = \max\{c_{ij}, a_k\}$$

are $4 \cdot 3^k$-periodic both in i and j; i.e., $c^k_{(i+4\cdot3^k)\,j} = c^k_{i\,(j+4\cdot3^k)} = c^k_{ij}$. Indeed,

$$c^k_{ij} = a_0 + \sum_{n=1}^{k} (a_n - a_{n-1}) \chi_{B_n}(i,j),$$

and each B_n is a $4 \cdot 3^k$-periodic (i.e., $B_n + 4 \cdot 3^k\mathbb{Z}^2 = B_n$) if $n \leq k$. Note also that c^k_{ij} converge uniformly to c_{ij} on $\mathcal{Z}$. Hence, the system of coefficients c_{ij} is *uniformly almost periodic*; more precisely, it is the uniform limit of a family of periodic coefficients of increasing periods. This is somewhat the strongest notion of almost periodicity, implying all other types of almost periodicity (see, e.g., [3]). The coefficients are not *quasi*periodic (i.e., diagonal functions of periodic functions in more variables) for which the existence of a counterexample is open. Related questions for that type of coefficients can be found in [13].

Remark 4 (homogenizability) Note that the set of coefficients c_{ij} is *homogenizable* (in the terminology of [4]): if we define the family of energies

$$F_\varepsilon(u) = \sum_{ij} \varepsilon c_{ij} (u_i - u_j)^2 \qquad \text{if } u : \varepsilon\mathbb{Z}^2 \to \{-1, 1\},$$

where $u_i = u(\varepsilon i)$, then, upon identifying each u with its piecewise-constant interpolation as a L^1-function, F_ε Γ-converge to the energy

$$F_0(u) = 8a \int_{\partial\{u=1\}} (|\nu_1| + |\nu_2|)\, d\mathcal{H}^1 \qquad \text{if } u \in BV(\mathbb{R}^2; \{-1, 1\})$$

(a given by (4)), where $\partial\{u = 1\}$ is understood as the reduced boundary of the set $\{u = 1\}$ and ν is its measure-theoretical normal. This can be proved using the results in [4] Section 2.1.2, or directly by comparison, on one hand remarking that, using that $c_{ij} \geq a$ for all i and j, we have

$$F_\varepsilon(u) \geq a \sum_{ij} \varepsilon(u_i - u_j)^2 \qquad \text{if } u : \varepsilon\mathbb{Z}^2 \to \{-1, 1\}$$

and the Γ-limit of the energies of this right-hand side is F_0 by [1]. On the other hand, by Remark 3, for all u we can find a sequences of functions $\{u_\varepsilon\}$ converging to u and such that

$$F_\varepsilon(u_\varepsilon) \leq a_n \sum_{ij} \varepsilon((u_\varepsilon)_i - (u_\varepsilon)_j)^2 \leq 8\, a_n \mathcal{H}^1(\partial\{u = 1\})$$

(the factor 8 comes from the fact that each nearest-neighbour pair is accounted for twice, and that $((u_\varepsilon)_i - (u_\varepsilon)_j)^2 = 4$ for non-zero interactions).

We now show that there exists no plane-like minimizer for the energy F in any direction ν. The argument we will follow will be to compare the energy of a competitor u lying in a strip with a function v suggested by the remark above, obtained by modifying u in a finite set and with the modified interface lying in B_n for n sufficently large.

We first consider the 'oblique' case; i.e., when ν is not a coordinate direction. By symmetry it is sufficient to consider the case

$$\nu_1 < 0, \qquad 0 < \nu_2 \leq -\nu_1;$$

i.e., the direction of the strip

$$S^M_\nu := \{x \in \mathbb{R}^2 : |\langle x, \nu\rangle| < M\}$$

is increasing and at an angle not less than 45 degrees (see the one in Fig. 1).

Suppose that such a plane-like minimizer u existed, and let ν, M be given by its definition. Up to changing the sign to u we may suppose that (3) holds.

With fixed n, let k_n be the minimal integer k such that the horizontal line

$$x_2 = \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n k$$

intersects

$$S^M_\nu \cap \Big\{(x_1, x_2) : x_1 > \frac{1}{2} + 2 \cdot 3^n\Big\};$$

i.e., the intersection of the strip with the half-plane on the right-hand side of the first vertical line of B_n in the half-plane $x_1 > 0$.

We consider the function v defined as

$$v_i = \begin{cases} 1 & \text{if } i_1 < \frac{1}{2} + 2 \cdot 3^n \text{ and } i_2 > \frac{1}{2} + 2 \cdot 3^n + 4 \cdot 3^n(k_n - 2) \\ u_i & \text{otherwise.} \end{cases} \tag{5}$$

If we identify the discrete function u with its piecewise-constant interpolation

$$u(x) = u\Big(\Big\lfloor x_1 - \frac{1}{2}\Big\rfloor, \Big\lfloor x_2 - \frac{1}{2}\Big\rfloor\Big) \tag{6}$$

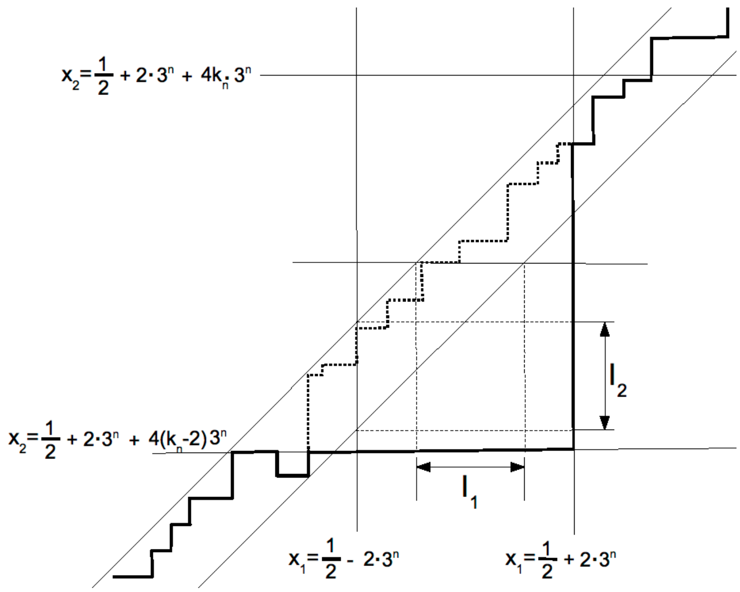

Figure 1: construction of a competitor v (oblique case).

then u can be pictured through the interface $\partial\{u = 1\}$ of this interpolation, and likewise v. In Fig. 1 the solid line represents the interface $\partial\{v = 1\}$ and the dotted line the part of the interface $\partial\{u = 1\}$ not included in $\partial\{v = 1\}$. The vertical and horizontal lines represent the bonds in B_n.

We now estimate the variation of the energy

$$\sum_{ij} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big),$$

where the sum can be limited to nearest neighbours i, j such that $i_1 < \frac{1}{2} + 2\cdot 3^n$ and $i_2 > \frac{1}{2} + 2\cdot 3^n + 4\cdot 3^n(k_n - 2)$ or the same holds for j_1 and j_2, since $u_i = v_i$ and $u_i = v_i$ otherwise. We note that on for such pairs we have $v_i \neq v_j$ only on (i, j) satisfying either of the two

(a) $i_1 = j_1$ and $\frac{i_2+j_2}{2} = \frac{1}{2} + 2\cdot 3^n + 4\cdot 3^n(k_n - 2)$;

(b) $i_2 = j_2$ and $\frac{i_1+j_1}{2} = \frac{1}{2} + 2\cdot 3^n$.

In case (a) there exists at least a pair of nearest neighbours (i', j') such that

(a') $i'_1 = j'_1 = i_1 (= j_1)$, $\frac{i_2+j_2}{2} \geq \frac{1}{2} + 2\cdot 3^n + 4\cdot 3^n(k_n - 2)$, and $u_{i'} \neq u_{j'}$.

This follows from the fact that (assuming $j_2 = i_2 + 1$) we have $u_i (= v_i) = -1$ and $u_{(i_1,m)} = 1$ for m large enough so that (m, i_2) lies above the strip S^M_ν. Note that we may assume, up to taking a slightly larger M, that both i' and j' lie in S^M_ν.

Similarly, in case (b) there exists at least a pair of nearest neighbours (i', j') such that

(b') $i'_2 = j'_2 = i_2 (= j_2)$, $\frac{i_1+j_1}{2} \leq \frac{1}{2} + 2\cdot 3^n$, and $u_{i'} \neq u_{j'}$.

We subdivide further our analysis by considering the sets of indices defined as follows:

• I_1 is the set of all i_1 such that there exist a pair (i',j') with $i'_1 = j'_1 = i_1$, $\frac{i_2+j_2}{2} \geq \frac{1}{2} + 2\cdot 3^n + 4\cdot 3^n(k_n-2)$, and $(i',j') \in B_n \cap S_\nu^M$;

• I_2 is the set of all i_2 such that there exists (i',j') with $i'_2 = j'_2 = i_2$, $\frac{i_1+j_1}{2} \leq \frac{1}{2} + 2\cdot 3^n$, and $(i,j) \in B_n \cap S_\nu^M$.

The interfaces of v corresponding to such sets are highlighted in Fig. 1. These are the sets such that the coefficients $c_{i'j'}$ corresponding, respectively, to pairs satisfying (a') and (b') may have a lower value than the corresponding c_{ij}.

In order to finally estimate the variation of the energy, we note that if (i,j) is such that $v_i \neq v_j$ and (a) is satisfied then

• if $i_1 \in I_1$ and (i',j') satisfies (a') then we have

$$c_{i'j'} - c_{ij} \geq a - a_n \tag{7}$$

• if $i_1 \notin I_1$ and (i',j') satisfies (a') then we have

$$c_{i'j'} - c_{ij} \geq a_{n-1} - a_n \tag{8}$$

(here we have used that when $i_1 \notin I_1$ all such (i',j') do not belong to B_n).

Analogously, if (i,j) is such that $v_i \neq v_j$ and (b) is satisfied then

• if $i_2 \in I_2$ and (i',j') satisfies (b') then we have

$$c_{i'j'} - c_{ij} \geq a - a_n \tag{9}$$

• if $i_2 \notin I_2$ and (i',j') satisfies (b') then we have

$$c_{i'j'} - c_{ij} \geq a_{n-1} - a_n. \tag{10}$$

It remains to note that

$$\#\{(i,j) \text{ satisfying (b) with } v_i \neq v_j\} \geq 8\cdot 3^n \tag{11}$$

(the factor 8 instead of 4 comes from double counting). and that

$$\#I_1 \leq \frac{2M}{|\nu_1|}, \qquad \#I_2 \leq \frac{4M}{|\nu_2|} \tag{12}$$

In (12) we used that $2M/|\nu_1|$ (respectively, $2M/|\nu_2|$) is the length of the intersection of an horizontal (respectively, vertical) line with S_ν^M. The factor 2 in the second estimate in (12) comes from the fact that the strip can intersect two lines in B_n if the slope is close to 45 degrees.

Taking (7)–(11) into account we infer the estimate

$$\sum_{ij} c_{ij}\Big((u_i-u_j)^2 - (v_i-v_j)^2\Big) \geq 2\Big(\frac{2M}{|\nu_1|} + \frac{4M}{|\nu_2|}\Big)(a-a_n) + 8\cdot 3^n(a_{n-1}-a_n). \tag{13}$$

In order to conclude, it suffices to show that the right-hand side in this formula is strictly positive for n large enough. This holds, provided that

$$a_{n-1} - a_n >> 3^{-n}(a_n - a).$$

To check this, we write $b_n = a_n - a$, which is a sequence decreasing to 0, and argue by contradiction, supposing that there exists C such that for all n we have

$$b_{n-1} - b_n \leq C3^{-n} b_n.$$

Summing up, we obtain

$$b_{n-1} = \sum_{k=n}^{\infty} (b_{k-1} - b_k) \leq C3^{-n} \sum_{j=0}^{\infty} 3^{-j} b_{n+j} \leq C3^{-n} \frac{3}{2} b_n.$$

By choosing n large enough so that $C3^{-n}\frac{3}{2} < 1$ we obtain $b_{n-1} < b_n$, which is a contradiction, since b_n is decreasing.

It remains the case when $\nu_1\nu_2 = 0$. By symmetry, it suffices to consider the case $\nu_1 = 0$; i.e., when we suppose that u is a ground state such that there exists M such that

$$u_i = 1 \quad \text{if } i_2 > M, \qquad u_i = -1 \quad \text{if } i_2 < -M.$$

Let $S_M = \{x : |x_2| \leq M\}$, and let n be such that

$$2 \cdot 3^n > M + 2. \tag{14}$$

In this case there is no pair $(i, j) \in B_n \cap S_M$ with $i_2 = j_2$ (i.e., there is no 'horizontal' bond in B_n lying in the strip S_M).

With fixed $k \in \mathbb{N}$ we define a test function v as follows:

$$v_i = \begin{cases} -1 & \text{if } 2 \cdot 3^n < i_1 \leq 2(1+2k)3^n,\ i_2 < 2 \cdot 3^n \\ u_i & \text{otherwise.} \end{cases}$$

We can picture the functions u and v through the interfaces related to their piecewise-constant interpolations as done in the oblique case above. In Fig. 2 the boldface solid line represents the interface related to v, the boldface dotted line represents the part of the interface related to u not included in that of v, the other solid lines represent the location of the bonds in B_n.

Let

$$I_1 = \Big\{(i, j) \in \mathcal{Z} \cap B_n : \frac{i_1 + j_1}{2} = \frac{1}{2} + 2 \cdot 3^n\Big\}$$

$$I_2 = \Big\{(i, j) \in \mathcal{Z} \cap B_n : \frac{i_1 + j_1}{2} = \frac{1}{2} + 2(1+2k)3^n\Big\}$$

$$I_3 = \Big\{(i, j) \in \mathcal{Z} \cap B_n : \frac{i_2 + j_2}{2} = \frac{1}{2} + 2 \cdot 3^n\Big\}$$

$$I_u = \Big\{(i, j) \in \mathcal{Z} \cap B_n : i, j \in S_M, 2 \cdot 3^n < \min\{i_1, j_1\},\ \max\{i_1, j_1\} \leq 2(1+2k)3^n\Big\}$$

We can then estimate

$$\begin{aligned} & \sum_{ij} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big) \\ = & \sum_{(i,j)\in I_1\cup I_2\cup I_3\cup I_u} c_{ij}\Big((u_i - u_j)^2 - (v_i - v_j)^2\Big) \\ \geq & -\sum_{(i,j)\in I_1\cup I_2\cup I_3} a_n (v_i - v_j)^2 + \sum_{(i,j)\in I_u\setminus B_n} a_{n-1}(u_i - u_j)^2. \end{aligned}$$

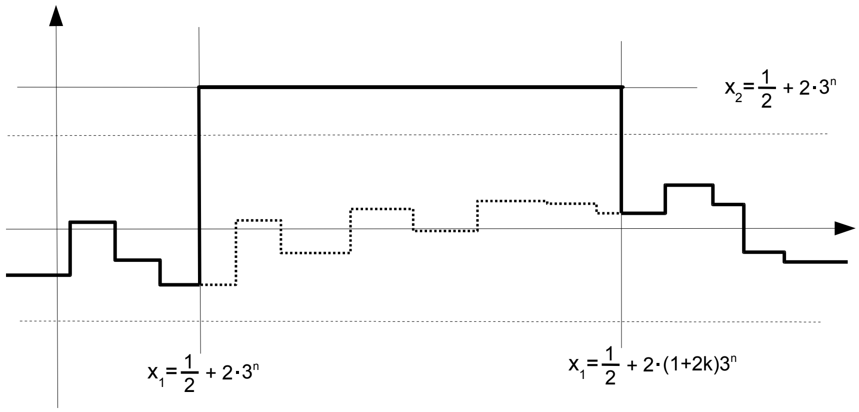

Figure 2: Construction of a competitor v (horizontal case)

Estimating the sum on $I_u \setminus B_n$ with only horizontal bonds where $u_i \neq u_j$ (whose number is greater than $\#I_3$), we then have

$$\begin{aligned}\sum_{ij} c_{ij}\Big((u_i-u_j)^2-(v_i-v_j)^2\Big) &\geq -8a_n(8\cdot 3^n+4k\,3^n)+8a_{n-1}4k\,3^n\\ &= 32k(a_{n-1}-a_n)-64a_n3^n.\end{aligned}$$

By taking k large enough (recall that now n is fixed by (14)) the last expression is positive, since $a_{n-1}>a_n$, again contradicting (2).

Acknowledgments

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